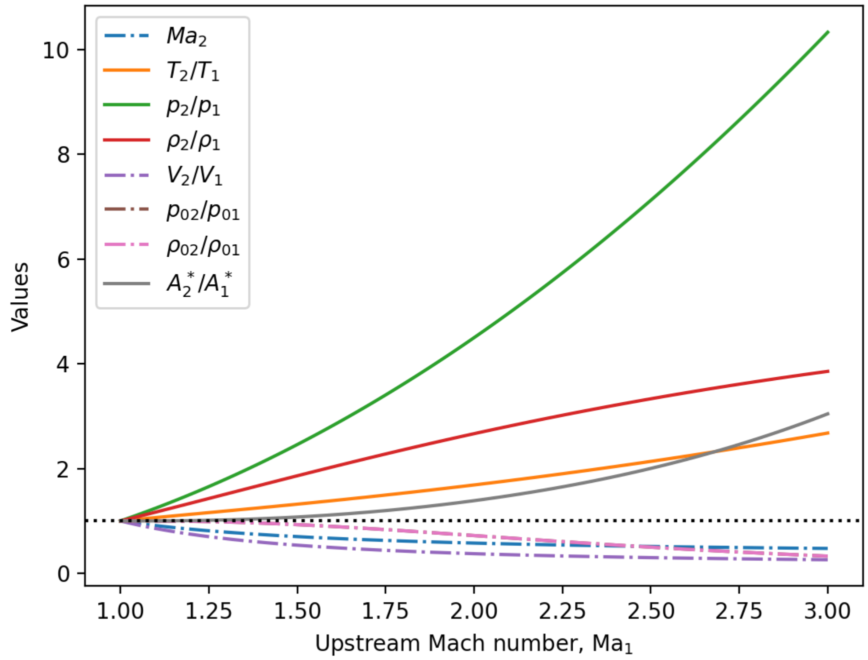


Figure 13.30. Plots of the various property ratios as functions of the upstream Mach number, Ma_1, for $k = 1.4$. Note that $T_{02}/T_{01} = 1$ and $p_{02}/p_{01} = \rho_{02}/\rho_{01}$.

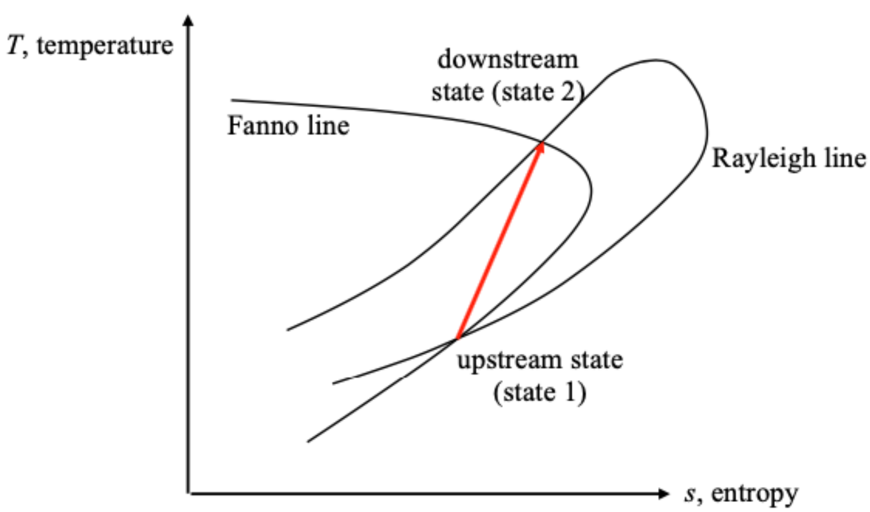


Figure 13.31. On a T-s plot, the states across a normal shock occur at the intersection of the Fanno and Rayleigh lines.

13.18. Flow through Converging-Diverging Nozzles

Consider flow through a converging-diverging nozzle (aka a de Laval nozzle) as shown in Figure 13.32. Let's hold the stagnation pressure, p_0, fixed and vary the back pressure, p_B. The plot shown in Figure 13.33 shows

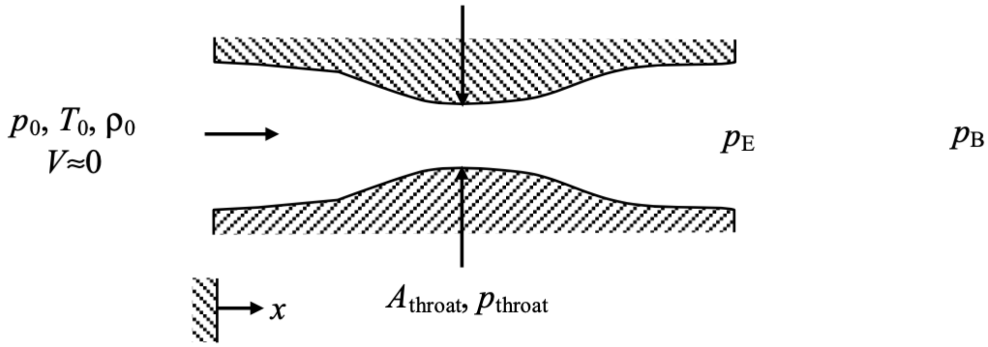


Figure 13.32. A schematic of a converging-diverging nozzle.

how the pressure ratio, p/p_0, varies with the location, x, in the nozzle for various values of the back pressure ratio, p_B/p_0.

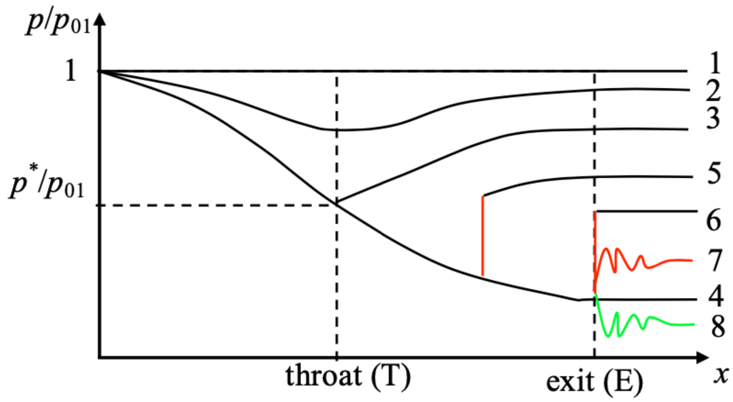


Figure 13.33. The pressure ratio p/p_0 plotted as a function of location x in a converging-diverging nozzle for different back pressure ratios p_B/p_0. The different cases, identified by the numbers on the right-side of the plot, are described in the text.

Cases (identified by the numbers on the right side in the plot):

(1) There is no flow through the device since $p_B = p_0$.
(2) There is subsonic flow throughout the device. The exit pressure equals the back pressure, i.e., $p_E = p_B$, since the exit Mach number is subsonic. Also, $\text{Ma}_T < 1, A_T > A^*, \text{Ma}_E < 1, \dot{m} < \dot{m}_{\text{choked}}$. The flow everywhere is isentropic.
(3) There is subsonic flow throughout the device except at the throat where $p_T = p^*$ ($\text{Ma}_T = 1, A_T = A^*$). The flow is now choked since downstream pressure changes won't make it upstream of the throat. The mass flow rate is now $\dot{m} = \dot{m}_{\text{choked}}$. Further decreases in p_B will not affect the flow upstream of the throat and the mass flow rate will remain at the choked mass flow rate value. The exit pressure equals the back pressure for this case $p_E = p_B$, since the flow is subsonic at the exit ($\text{Ma}_E < 1$). The flow everywhere is isentropic.
(4) Subsonic flow will occur in the converging section, sonic flow will occur at the throat ($p_T = p^*, \text{Ma}_T = 1, A_T = A^*$), and supersonic flow will occur in the diverging section ($\text{Ma}_E > 1$). This type of flow is called correctly expanded, perfectly expanded, or design flow since no shock waves form anywhere in the device and $p_E = p_B$. Note that p_E does not equal p_B because the flow is subsonic at the exit, but it's because the flow is at design conditions (a special case).

(5) Subsonic flow will occur in the converging section and sonic flow will occur at the throat ($p_T = p^*, \mathrm{Ma}_T = 1, A_T = A_1^*$). A portion of the diverging section will be supersonic with a normal shock wave occurring at a location such that the subsonic flow downstream of the shock will have an exit pressure equal to the back pressure: $p_E = p_B$ since $\mathrm{Ma}_E < 1$. As the back pressure decreases, the shock wave moves downstream of the throat and toward the exit. The pressure rise across the shock wave also increases as the back pressure decreases. There is isentropic flow upstream of the shock and downstream of it, but across the shock the flow is non-isentropic.

(6) This case is similar to Case 5 except that the shock wave is precisely at the nozzle exit. The pressure just downstream of the shock wave equals the back pressure since the flow is subsonic there ($p_{E2} = p_B, \mathrm{Ma}_{E2} < 1$). The flow everywhere within the converging-diverging nozzle is isentropic except right at the exit.

(7) The flow within the converging-diverging nozzle (and the exit) is isentropic ($\mathrm{Ma}_E > 1$). The normal shock that was located at the exit for Case 6 has moved outside the device to form a complicated sequence of oblique shock waves alternating with expansion fans. These are two-dimensional phenomena to be discussed in a following section of notes. This case is called the over-expanded case since the diverging section of the device has an area that over-expands the flow to a pressure that is lower than the back pressure ($p_E < p_B$). External shock waves are required to compress the flow to match the back pressure.

(8) This case is similar to Case 7 except that the flow outside of the device forms a sequence of expansion fans alternating with oblique shock waves (a sequence out of phase with the sequence mentioned in Case 7). This case is called the under-expanded case since the diverging section of the device has an area that is not large enough to drop the exit pressure to the back pressure ($p_E > p_B, \mathrm{Ma}_E > 1$). External expansion waves are required expand the flow to match the back pressure.

Notes:

(1) The critical back pressure ratio corresponding to Case 3 can be found from the isentropic relations (the flow throughout the entire device is isentropic). Assume that the geometry, and hence the exit-to-throat area ratio, A_E/A_T, is given. Since for Case 3 the flow is choked we know that $A_T = A^*$. Furthermore, since the exit flow is subsonic we also know that $p_E = p_B$. From the area ratio we can determine the exit Mach number, Ma_E,

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{(where the subsonic } \mathrm{Ma}_E \text{ is found).} \tag{13.195}$$

The back pressure ratio, p_B/p_0, for Case 3 can be determined given the exit Mach number,

$$\frac{p_B}{p_0} = \frac{p_E}{p_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}_E^2\right)^{\frac{k}{1-k}}. \tag{13.196}$$

(2) The critical back pressure ratio corresponding to Case 4 can be determined in a manner similar to that described previously in Note 1. For Case 4 however, the supersonic value for Ma_E should be used when determining the exit Mach number from the area ratio.

(3) The critical back pressure ratio corresponding to Case 6 can be found by combining the isentropic relations with the normal shock wave relations. When the shock wave occurs right at the exit of the device, the flow just upstream of the exit can be found from the isentropic relations,

$$\frac{A_E}{A_1^*} = \frac{A_E}{A_T} = \frac{1}{\mathrm{Ma}_{E1}}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_{E1}^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{(where the supersonic } \mathrm{Ma}_{E1} \text{ is found),} \tag{13.197}$$

$$\frac{p_E}{p_{01}} = \left(1+\frac{k-1}{2}\mathrm{Ma}_{E1}^2\right)^{\frac{k}{1-k}}. \tag{13.198}$$

Note that the subscript "1" denotes the conditions just upstream of the shock wave. To determine the conditions just downstream of the shock we use the normal shock wave relations,

$$\frac{p_{E2}}{p_{E1}} = \frac{2k}{k+1}\mathrm{Ma}_{E1}^2 - \frac{k-1}{k+1}, \tag{13.199}$$

where p_{E2} is the pressure just downstream of the shock. Since the downstream flow is subsonic and because we're at the exit of the device, the downstream pressure, p_{E2}, must also equal the back pressure, p_B. Thus,

$$\frac{p_B}{p_{01}} = \frac{p_{E2}}{p_{01}} = \frac{p_{E2}}{p_{E1}}\frac{p_{E1}}{p_{01}}, \tag{13.200}$$

where the different ratios are given by Eqs. (13.198) and (13.199).

(4) The location of a shock wave for a back pressure in the range between Case 3 and Case 5 can be determined through iteration. For example:
   (a) Assume a location for the shock wave, e.g., pick a value for A/A_T since the geometry is known.
   (b) Determine the Mach number and pressure just upstream of the shock, Ma_1 and p_1, using the isentropic relations as discussed in Note 3,

$$\frac{A}{A_1^*} = \frac{A}{A_T} = \frac{1}{\mathrm{Ma}_1}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{(where the supersonic } \mathrm{Ma}_1 \text{ is found)}, \tag{13.201}$$

$$\frac{p_1}{p_{01}} = \left(1+\frac{k-1}{2}\mathrm{Ma}_1^2\right)^{\frac{k}{1-k}}. \tag{13.202}$$

   (c) Calculate the stagnation pressure ratio and sonic area ratio across the shock using the normal shock relations,

$$\frac{p_{02}}{p_{01}} = \frac{A_1^*}{A_2^*} = \left(\frac{\frac{k+1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_1^2}\right)^{\frac{k}{k-1}}\left(\frac{2k}{k+1}\mathrm{Ma}_1^2 - \frac{k-1}{k+1}\right)^{\frac{1}{1-k}}. \tag{13.203}$$

   (d) Determine the exit Mach number and exit pressure ratio using the isentropic relations and the downstream sonic area and stagnation pressure,

$$\frac{A_E}{A_2^*} = \frac{A_E}{A_T}\frac{A_1^*}{A_2^*} = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{(where the subsonic } \mathrm{Ma}_E \text{ is chosen)}. \tag{13.204}$$

   Note that since the flow is choked, the throat area is equal to the upstream sonic area, i.e., $A_T = A_1^*$. The exit pressure ratio is found from the isentropic relations,

$$\frac{p_E}{p_{02}} = \left(1+\frac{k-1}{2}\mathrm{Ma}_E^2\right)^{\frac{k}{1-k}}. \tag{13.205}$$

   Note that since the exit Mach number is subsonic, the exit pressure will equal the back pressure, i.e., $p_E = p_B$.
   (e) Calculate the ratio of the back pressure to the upstream stagnation pressure,

$$\frac{p_B}{p_{01}} = \frac{p_E}{p_{02}}\frac{p_{02}}{p_{01}}. \tag{13.206}$$

   (f) Check to see if the back pressure ratio calculated in Step (e) matches with the given back pressure ratio. If so, then the assumed location of the shock is correct. If not, then the go back to Step (a) and repeat. If the back pressure ratio calculated in Step (e) is less than the given back pressure ratio, then the assumed shock location is too far downstream. If the back pressure ratio calculated in Step (e) is greater than the given back pressure ratio, then the assumed shock location is too far upstream. The logic for this step is illustrated in Figure 13.34.
   (g) Photographs for the various converging-diverging nozzle cases are shown in Figure 13.35.

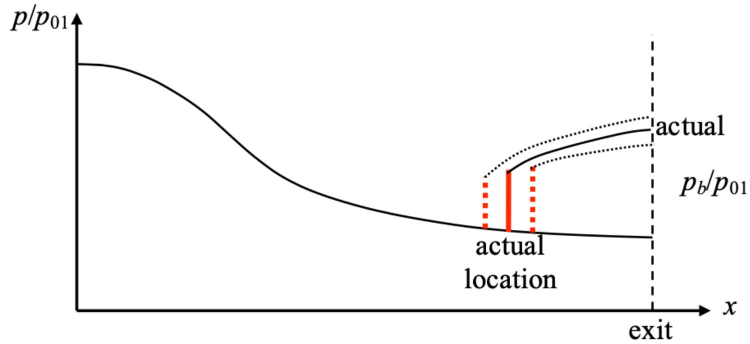


Figure 13.34. A plot illustrating in what direction to change the shock location during iteration. The calculated back pressure is compared to the actual back pressure. If the calculated back pressure is larger than the actual back pressure, then the shock should be moved further downstream. Alternately, if the calculated back pressure is smaller than the actual back pressure, then the shock should be moved further upstream.

(h) In real nozzles flows, the flow will typically separate from the nozzle walls as a result of the large adverse pressure gradient occurring across a shock wave. Interaction of the shock with the separated boundary layer results in a more gradual pressure rise than what is expected for the ideal, normal shock analysis.
It is also possible that downstream pressure information can propagate upstream in the diverging section even when the core flow is supersonic. In a real flow, a boundary layer will form along the wall with the flow in part of this boundary layer being subsonic. Thus, pressure information can propagate upstream within the subsonic part of the boundary layer and affect the flow in the diverging section. When the back pressure is in the range corresponding to Case 7 (back pressure less than the exit pressure when a shock stands at the exit, and greater than the isentropic case corresponding to supersonic diverging section flow), oblique shocks will typically form within the diverging section and flow separation occurs as shown in Figure 13.36. The exact pressure and location of the separation point are dependent on the boundary layer flow.

(i) Experimental pressure measurement data within a converging-diverging nozzle are shown in Figure 13.37. Also included in the plot are predictions using the analysis described in this section (a combination of isentropic flow relations and normal shock wave relations). As can be observed in the plot, the real data are predicted well by the models.

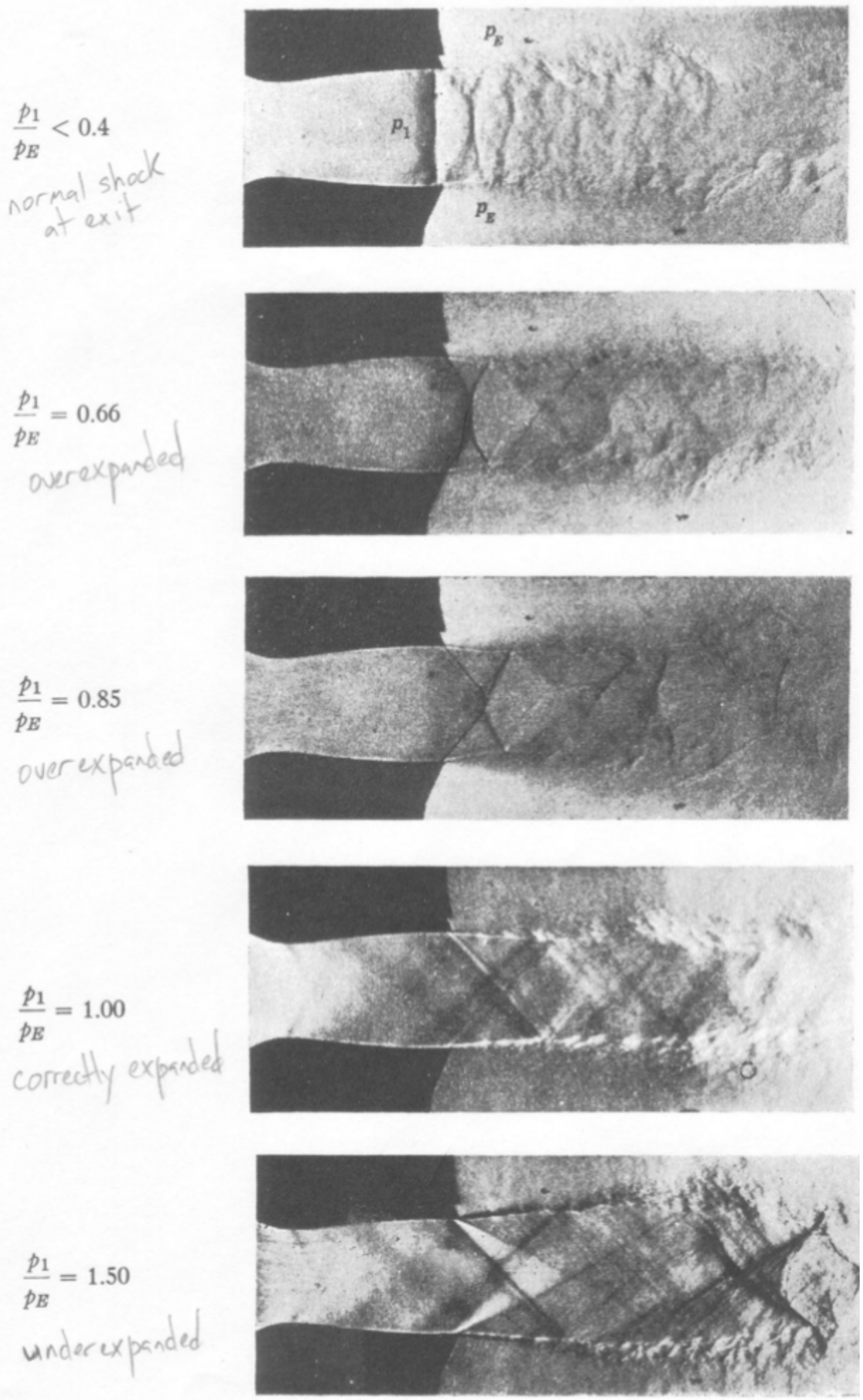


FIG. 5·4 Schlieren photographs of flow from a supersonic nozzle at different back pressures. **(Figure from: Liepmann, H.W. and Roshko, A., *Elements of Gasdynamics*, Wiley.)**

FIGURE 13.35. Photographs corresponding to the different converging-diverging nozzle cases shown in Figure 13.33.

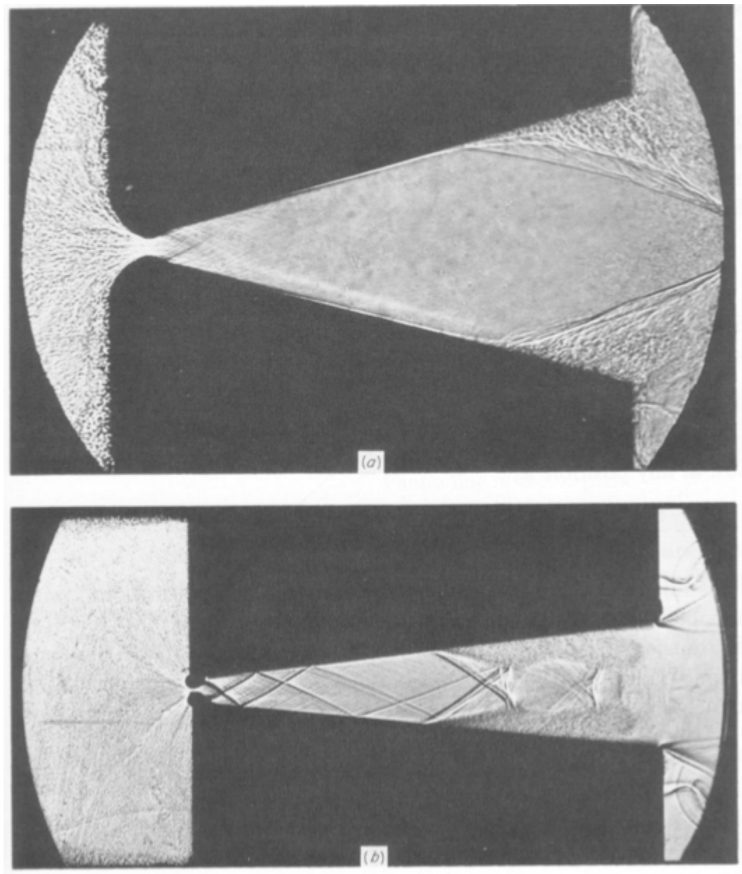
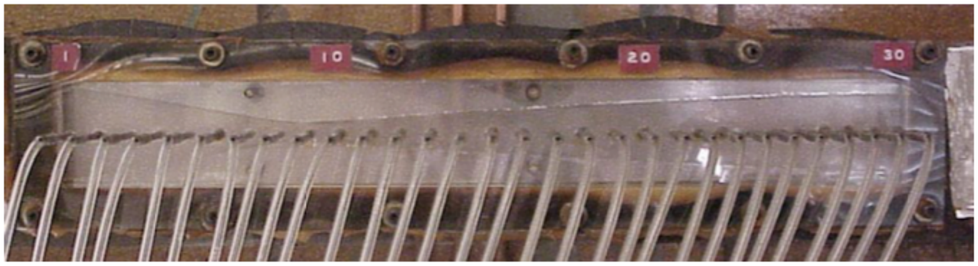


Figure 6.16
Shadowgraphs showing flow separation in supersonic nozzles. (*Courtesy of H. O. Amann, Ernst Mach Institut.*)

Figure 13.36. Photographs showing separated flow in supersonic flow in the diverging section of a converging-diverging nozzle.

A converging-diverging nozzle with pressure taps along the length of the device. The flow is from left to right.



The pressure ratio as a function of the axial distance in the CD nozzle for various back pressures.

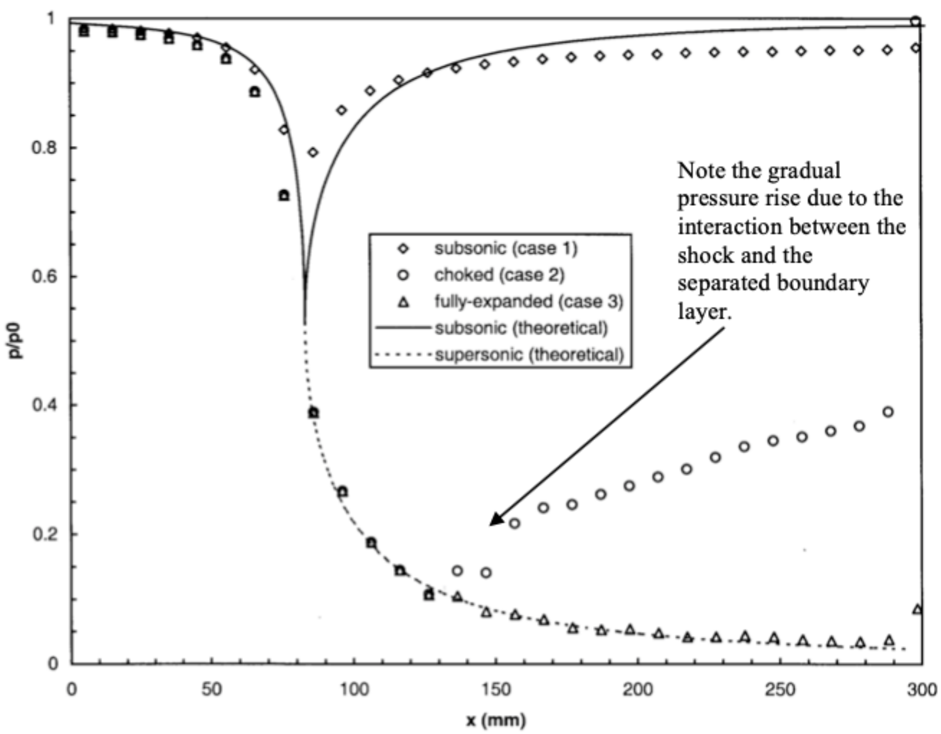
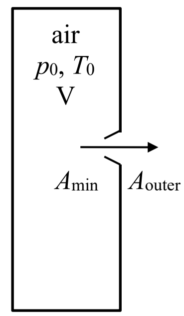


Figure 13.37. A photograph of a converging-diverging nozzle and corresponding pressure data shown in a plot.

During a test docking of the Progress M-34 supply ship with the Mir space station in 1997, a collision occurred which punctures the hull of Spektr Module of Mir. Assume the puncture hole had a minimum area of 1.0 cm^2 and an outer area of 1.5 cm^2 (the size of the hole was not directly measured). The volume of the Spektr module was 61.9 m^3 and had an initial interior pressure of 100 kPa (abs) and temperature of 34 °C.

1. Determine the mass flow rate of air from the capsule when the hole initially occurred.
2. Write an equation relating how the mass of air inside the module changed with time. You may assume that the air behaved as a perfect gas throughout the entire discharge process and that the temperature remained constant inside the space station (thanks to the small discharge rate and onboard heaters).
3. Calculate the thrust acting the space station for the initial conditions.



SOLUTION:

Since the air in the space station is discharging into space, the back pressure is essentially zero and the flow will always be choked with a mass flow rate of,

$$\dot{m} = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} \rho_0 \sqrt{\gamma R T_0}\, A^* \tag{1}$$

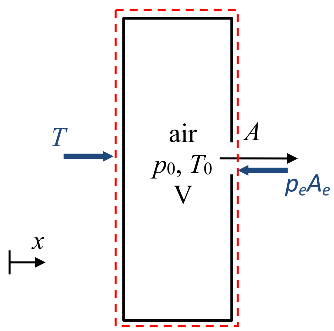
where,

$$\rho_0 = \frac{p_0}{R T_0} = \frac{M}{V} \tag{2}$$

where M is the mass of air within the space station and V is the interior volume of the station. Using the given data:

γ = 1.4
R = 287 J/(kg.K)
$p_{0,t=0}$ = 100*10^3 Pa (abs)
T_0 = 34 + 273 = 307 K
$A^* = A_{\text{min}}$ = 1 cm^2 = 1*10^{-4} m^2
V = 61.9 m^3
$\Rightarrow \rho_0$ = 1.135 kg/m^3
$\Rightarrow M_{t=0}$ = 70.25 kg
$\Rightarrow \boxed{\dot{m}_{t=0} = 2.90 \times 10^{-2} \text{ kg/s}}$

The mass in the space station may be found as a function of time by applying conservation of mass to a control volume surrounding the station as shown in the figure below.



$$\frac{d}{dt}\int_{\text{CV}}\rho dV+\int_{\text{CS}}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=0 \tag{3}$$

where,

$$\frac{d}{dt}\int_{\text{CV}}\rho dV=\frac{dM}{dt} \tag{4}$$

$$\int_{\text{CS}}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=\dot{m} \tag{5}$$

Note that since the back pressure is always zero, the mass flow rate out of the space station will always be choked. Substitute and simplify.

$$\frac{dM}{dt}=-\dot{m}=-\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\rho_0\sqrt{\gamma RT_0}A^*=-\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{M}{V}\right)\sqrt{\gamma RT_0}A^* \tag{6}$$

where Eqs. (1) and (2) have been used. Solve the differential equation given in Eq. (6).

$$\int_{M=M_{t=0}}^{M=M}\frac{dM}{M}=-\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{\sqrt{\gamma RT_0}A^*}{V}\right)\int_{t=0}^{t=t}dt \quad \text{(Note that } T_0 = \text{constant.)} \tag{7}$$

$$\ln\left(\frac{M}{M_{t=0}}\right)=-\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{\sqrt{\gamma RT_0}A^*}{V}\right)t \tag{8}$$

$$\boxed{M=M_{t=0}\exp\left[-\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{\sqrt{\gamma RT_0}A_{\min}}{V}\right)t\right]} \quad \text{where } A^*=A_{\min} \tag{9}$$

The thrust acting on the space station may be found by applying the Linear Momentum Equation to the same control volume,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{10}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(The thrust is the force required to hold Mir stationary.)} \tag{11}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m} V_e \tag{12}$$

$$F_{B,x} = 0 \tag{13}$$

$$F_{S,x} = T - p_e A_e \quad (\text{where } A_e = A_{\text{outer}}) \tag{14}$$

Substitute and simplify,

$$T = \dot{m} V_e + p_e A_e \tag{15}$$

The exit conditions may be found using isentropic relations since the flow through the hole is under-expanded.

$$\frac{A_e}{A^*} = \frac{A_{\text{outer}}}{A_{\text{min}}} = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_e = 1.8541 \tag{16}$$

$$\frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_e/p_0 = 0.1602 \Rightarrow p_e = 16.02 \text{ kPa (abs)} \quad (p_0 = 100 \text{ kPa abs}) \tag{17}$$

$$\frac{T_e}{T_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{-1} \Rightarrow T_e/T_0 = 0.5926 \Rightarrow T_e = 181.9 \text{ K} \quad (T_0 = 307 \text{ K}) \tag{18}$$

$$V_e = \text{Ma}_e\sqrt{\gamma R T_e} \Rightarrow V_e = 501.3 \text{ m/s} \tag{19}$$

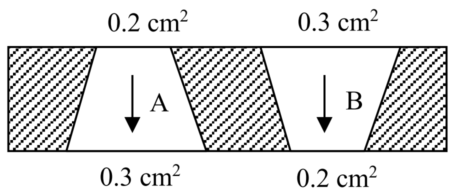
Now calculate the thrust using Eq. (15) and the mass flow rate found in the first part of this problem.

$$\boxed{T_{t=0} = 16.94 \text{ N}} \tag{20}$$

Note that this is the thrust at $t = 0$. The thrust will vary with time since the stagnation pressure, and thus exit pressure, will vary with time as mass discharges from the space station.

The orientation of a hole can make a difference. Consider holes A and B in the figure below which are identical but reversed. For the given air properties on either side, compute the mass flow rate through each hole and explain why they are different.

p_0 = 150 kPa (abs)
T_0 = 20 °C



p_{back} = 100 kPa (abs)

SOLUTION:

First consider flow through hole B which can be considered a converging nozzle. First check to see if the flow is choked.

$$\frac{p_B}{p_0} = \frac{100 \text{ kPa}}{150 \text{ kPa}} = 0.6667 > \frac{p^*}{p_0} = 0.5283 \Rightarrow \text{The flow is not choked. (Note that } \gamma_{air} = 1.4.) \tag{1}$$

The mass flow rate can be found from the conditions at the hole exit.

$$\dot{m} = \rho_E V_E A_E \tag{2}$$

where,

$$\rho_E = \rho_0\left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{1}{1-\gamma}} \tag{3}$$

$$\rho_0 = \frac{p_0}{RT_0} \tag{4}$$

$$\frac{p_E}{p_0} = \frac{p_B}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \tag{5}$$

$$V_E = \text{Ma}_E\sqrt{\gamma R T_E} \tag{6}$$

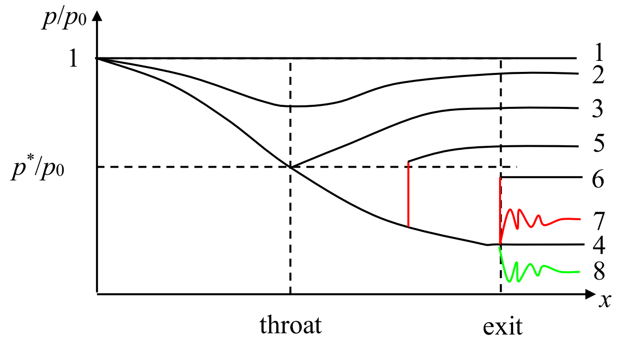
$$T_E = T_0\left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{-1} \tag{7}$$

Using the given data:

γ = 1.4
R = 287 J/(kg·K)
p_0 = 150e3 Pa (abs)
T_0 = 20 °C = 293 K
p_E = 100e3 Pa (abs) (Note that since the exit flow is subsonic, $p_E = p_B$.)
A_E = 0.2 cm^2 = 2.0e-5 m^2
Ma$_E$ = 0.784
ρ_0 = 1.784 kg/m^3
ρ_E = 1.335 kg/m^3
T_E = 261 K
V_E = 254 m/s

$$\boxed{\therefore\ \dot{m}_B = 6.78\text{e-3 kg/s}}$$

Now consider hole A which can be modeled as a converging-diverging nozzle. Check to see what p_B/p_0 ratio will result in choked flow (case 3 in the figure below).



$$\frac{A_{E,\mathrm{crit}}}{A^*} = \frac{A_E}{A_T} = \frac{1}{\mathrm{Ma}_{E,\mathrm{crit}}}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_{E,\mathrm{crit}}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_{E,\mathrm{crit}} = 0.43 \tag{8}$$

using $A_E = 0.3\ \mathrm{cm}^2$ and $A_T = 0.2\ \mathrm{cm}^2$.

$$\frac{p_{E,\mathrm{crit}}}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_{E,\mathrm{crit}}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \frac{p_{E,\mathrm{crit}}}{p_0} = 0.8805 \tag{9}$$

For the given situation, $p_B/p_0 = 0.6667$ (refer to Eq. (1)) $< p_{E,\mathrm{crit}}/p_0 = 0.8805$ so the flow for hole A must be choked! The mass flow rate through the hole can be found using the (sonic) conditions at the throat.

$$\dot{m} = \rho_T V_T A_T \tag{10}$$

where

$$\rho_T = \rho^* = \rho_0\left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{1-\gamma}} = 0.6339 \quad (\text{using } \gamma_{\mathrm{air}} = 1.4) \tag{11}$$

$$\rho_0 = \frac{p_0}{RT_0} \tag{12}$$

$$\frac{p_E}{p_0} = \frac{p_B}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \tag{13}$$

$$V_T = c^* = \sqrt{\gamma R T^*} \tag{14}$$

$$T^* = T_0\left(1+\frac{\gamma-1}{2}\right)^{-1} = 0.8333 \tag{15}$$

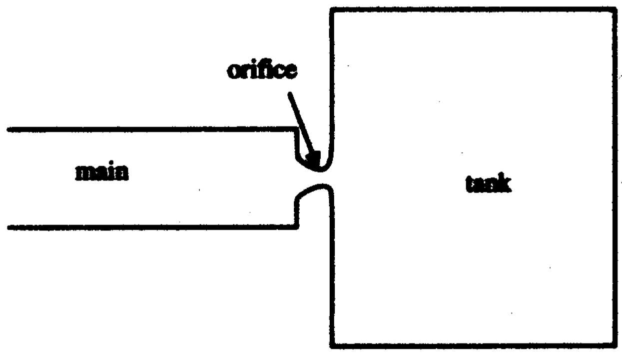
Using the given data:

$A_T = 0.2\ \mathrm{cm}^2 = 2.0\mathrm{e}{-5}\ \mathrm{m}^2$
$\rho^* = 1.131\ \mathrm{kg/m}^3$
$T^* = 244.2\ \mathrm{K}$
$V_T = 313.2\ \mathrm{m/s}$
$\boxed{\therefore\ \dot{m}_A = 7.08\mathrm{e}{-3}\ \mathrm{kg/s}}$

The different mass flow rates through holes A and B are because the flow through hole A is choked (the hole acts as a converging-diverging nozzle) while through hole B the flow is not choked (the hole acts as a converging nozzle).

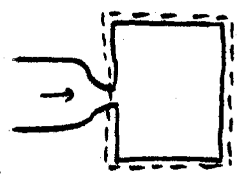
A large main is connected to an evacuated tank with a volume of 10 ft^3 by means of a rounded-entrance, converging nozzle having a diameter of 0.01 in. Initially, a diaphragm over the orifice seals the tank from the main. The air in the main is at 100 psia and 70 °F. The diaphragm is suddenly broken and air rushes into the tank. Estimate the time required for the pressure in the tank to reach 25psia, based on the following assumptions:

a. **The flow is quasi-static.**
b. **There is no heat conduction from the tank to the air.**
c. **The pressure and temperature in the main remain constant.**



Solution:

- Apply COE to the following CV:



$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h+\tfrac{1}{2}V^2\right)\rho(\underline{u}_{rel}\cdot\hat{n})\,dS = \dot{Q}_{\substack{into\\CV}} + \dot{W}_{\substack{other\\on\ CV}}$$

where $\displaystyle\frac{d}{dt}\int_{CV} e\rho\, dV = \frac{d}{dt}(Mu)_{tank}$ — specific internal energy

$$\int_{CS}\left(h+\tfrac{1}{2}V^2\right)\rho(\underline{u}_{rel}\cdot\hat{n})\,dS = -\dot{m}\left(h+\tfrac{1}{2}V^2\right)$$

$$\dot{Q}_{\substack{into\\CV}} = 0$$

$$\dot{W}_{\substack{other\\on\ CV}} = 0$$

- Also treat the air as a perfect gas:

$$u = c_v T \qquad p = \rho R T$$
$$h = c_p T$$

$$\Rightarrow \frac{d}{dt}\left(c_v M T\right)_{tank} - \dot{m}\left(c_p T_e + \tfrac{1}{2}V_e^2\right) = 0$$

- the subscript "e" refers to the conditions at the orifice exit

$$\Rightarrow \underline{M_{tank}\left.\frac{dT}{dt}\right|_{tank} + T_{tank}\left.\frac{dM}{dt}\right|_{tank} = \dot{m}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)}$$

1 of 3

<u>SOLUTION...</u>

- Now apply COM to the same CV:

$$\frac{d}{dt}\int_{CV}\rho\, dV + \int_{CS}\rho(\mathbf{u}_{rel}\cdot\hat{n})\,dS = 0$$

where $\frac{d}{dt}\int_{CV}\rho\, dV = \left.\frac{dM}{dt}\right|_{tank}$

$$\int_{CS}\rho(\mathbf{u}_{rel}\cdot\hat{n})\,dS = -\dot{m}$$

$$\Rightarrow \underline{\left.\frac{dM}{dt}\right|_{tank} = -\dot{m}}$$

Also,

$$\dot{m} = \rho_e V_e A_e$$

where the subscript "e" refers to the orifice exit

- Substitute into the COE eqn:

$$M_{tank}\left.\frac{dT}{dt}\right|_{tank} - \dot{m}T_{tank} = \dot{m}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)$$

- Now consider the conditions within the tank:

$$(p = \rho RT)_{tank} \text{ where } \rho_{tank} = \frac{M_{tank}}{V_{tank}}$$

$$\Rightarrow T_{tank} = \frac{pV_{tank}}{M_{tank}R} \Rightarrow \frac{dT_{tank}}{dt} = \frac{dp_{tank}}{dt}\left(\frac{V_{tank}}{M_{tank}R}\right) - \underbrace{\frac{dM_{tank}}{dt}}_{=-\dot{m}}\left(\frac{pV_{tank}}{RM_{tank}^2}\right)$$

- Substitute and simplify:

$$\frac{dp_{tank}}{dt}\frac{V_{tank}}{R} + \dot{m}\,\cancel{\frac{p_{tank}V_{tank}}{RM_{tank}}} - \dot{m}\,\cancel{\frac{p_{tank}V_{tank}}{RM_{tank}}} = \dot{m}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)$$

$$\Rightarrow \underline{\frac{dp_{tank}}{dt} = \frac{\rho_e V_e A_e R}{V_{tank}}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)}$$

- Now determine the conditions at the orifice exit. Note that the flow will be choked until the tank pressure rises above:

$$\frac{p_{tank}}{p_0} = \frac{p_e}{p_0} = \frac{p^*}{p_0} = 0.5283 \Rightarrow p_{tank} = 52.83\text{ psia}$$

where $p_0 = 100$ psia

2 of 3

SOLUTION...

- Thus, the flow into the tank for ~~tank~~ tank pressures less than 52.83 psia will be choked. Since we're only interested in the pressure up to 25 psia, the flow will remain choked throughout the entire filling process.

$$\Rightarrow Ma_e = 1 \Rightarrow \begin{aligned} \rho_e &= \rho^* \\ T_e &= T^* \\ V_e &= c^* = \sqrt{\gamma R T^*} \end{aligned}$$

where
$$\rho^* = \rho_0 \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{1}{1-\gamma}} = \frac{p_0}{RT_0}\left(1 + \frac{\gamma-1}{2}\right)^{\frac{1}{1-\gamma}}$$
$$T^* = T_0\left(1 + \frac{\gamma-1}{2}\right)^{-1}$$

- For
$$\begin{aligned} \gamma &= 1.4 \\ R &= 1716 \frac{ft^2}{s^2 \cdot {}^\circ R} \\ p_0 &= 100 \text{ psia} = 1.44 \times 10^4 \frac{lb_f}{ft^2} \\ T_0 &= 70^\circ F = 529.67\,{}^\circ R \end{aligned}$$

$$\Rightarrow \begin{aligned} \rho^* &= 1.00 \times 10^{-2} \frac{slug}{ft^3} \\ T^* &= 440\,{}^\circ R \\ V^* &= 1030 \text{ ft/s} \end{aligned}$$

- Additionally,
$$\begin{aligned} \mathcal{V}_{tank} &= 10 ft^3 \\ c_v &= 4280 \frac{ft^2}{s^2 \cdot {}^\circ R} \\ A_e &= \frac{\pi (0.01 in)^2}{4} = 7.85 \times 10^{-5} in^2 = 5.45 \times 10^{-7} ft^2 \end{aligned}$$

$$\Rightarrow p_{tank} = \int \frac{dp_{tank}}{dt} dt = \frac{\rho_e V_e A_e R}{\mathcal{V}_{tank}}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)\Delta t$$

- For $p_{tank} = 25$ psia $= 3600 \frac{lb_f}{ft^2}$, $\boxed{\Delta t = 5050 s = 1.4 \text{ hrs}}$

Air flows isentropically from a large reservoir where the pressure and temperature are 1.0 MPa (abs) and 350 K, respectively, through a converging-diverging nozzle, with exit area of 0.001 m^2. The design back pressure is 87.5 kPa (abs) but the nozzle operates at a back pressure of 50.0 kPa (abs). Determine the exit Mach number and mass flow rate.

SOLUTION:

$A_e = 0.001\,\text{m}^2$

$p_{b,design} = 87.5\,\text{kPa}$
$p_b = 50.0\,\text{kPa}$

- Since the nozzle operates at a pressure below the design pressure, the flow within the nozzle will remain isentropic and choked.

$$\underline{p_e = p_{b,design} = 87.5\,\text{kPa}}$$

$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma-1}{2}Ma_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\Rightarrow \boxed{Ma_e = 2.24} \quad \text{using } p_e = 87.5\,\text{kPa},\ p_0 = 1.0\,\text{MPa},\ \gamma = 1.4$$

$$\frac{A_e}{A^*} = \frac{1}{Ma_e}\left(\frac{1+\frac{\gamma-1}{2}Ma_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\Rightarrow \underline{A^* = 4.80\times10^{-4}\,\text{m}^2} \quad \text{using } Ma_e = 2.24,\ A_e = 0.001\,\text{m}^2$$

$$\dot{m} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}\,A^*$$

$$\therefore \boxed{\dot{m} = 1.04\,\text{kg/s}} \quad \text{using } \gamma = 1.4,\ p_0 = 1.0\times10^6\,\text{Pa},\ R = 287\,\text{J/kg}\cdot\text{K},\ T_0 = 350\,\text{K},\ A^* = 4.80\times10^{-4}\,\text{m}^2$$

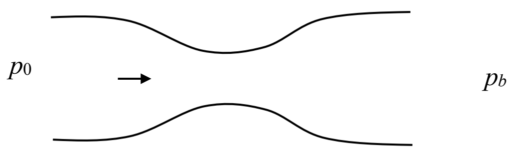
1 of 1

A rocket engine is designed to operate at a pressure ratio (inlet reservoir pressure/back pressure) of 37. Find:

a. the ratio of the exit area to the throat area which is necessary for the supersonic exhaust to be correctly expanded,
b. the Mach number of the exit flow under correctly expanded conditions,
c. the lowest pressure ratio (p_0/p_b) at which the same nozzle would be choked, and
d. the pressure ratio (p_0/p_b) at which there would be a normal shock wave at the exit.

Assume the specific heat ratio of the gas is 1.4.

SOLUTION:



$p_0/p_e = 37$

The area ratio may be found from the isentropic sonic area ratio and the isentropic pressure ratio.

$$\frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{\mathrm{Ma}_e = 3.0} \text{ (since at design conditions, the flow is isentropic)} \tag{1}$$

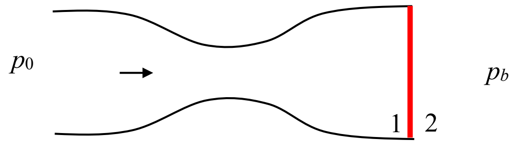
$$\frac{A_e}{A^*} = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{A_e/A_t = 4.3} \text{ (since } A_t = A^*\text{)} \tag{2}$$

The lowest pressure ratio for which the nozzle will be choked may be found Eqns. (2) and (1), but using the subsonic Mach number.

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_e = 0.14 \tag{3}$$

$$\frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_0/p_b = 1.01} \text{ Note that } p_e = p_b \text{ when the flow just becomes choked.} \tag{4}$$

Now consider a case where a shock wave occurs at the exit of the device.



From Eqn. (1), $\text{Ma}_{e1} = 3.0$ and $p_{01}/p_{e1} = 37$. From the normal shock relations,

$$\text{Ma}_2^2 = \frac{(\gamma-1)\text{Ma}_1^2 + 2}{2\gamma\text{Ma}_1^2 - (\gamma-1)} \Rightarrow \text{Ma}_2 = 0.475 \tag{5}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\text{Ma}_1^2}{2+(\gamma-1)\text{Ma}_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \Rightarrow p_{02}/p_{01} = 0.327 \tag{6}$$

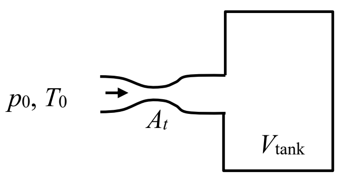
and the isentropic relations:

$$\frac{p_2}{p_{02}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_2/p_{02} = 0.857 \tag{7}$$

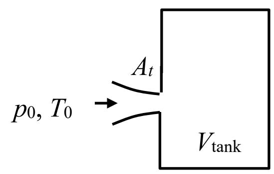
Since the flow downstream of the shock is subsonic, $p_2 = p_b$. Thus,

$$\frac{p_{01}}{p_b} = \left(\frac{p_{01}}{p_{02}}\right)\left(\frac{p_{02}}{p_2}\right) \Rightarrow \boxed{p_{01}/p_b = 3.6} \tag{8}$$

Which nozzle will fill the tank faster (or will they fill at the same rate), assuming that the tank is initially evacuated? Justify your answer. The upstream stagnation properties, throat areas, and tank volumes are identical in both cases.



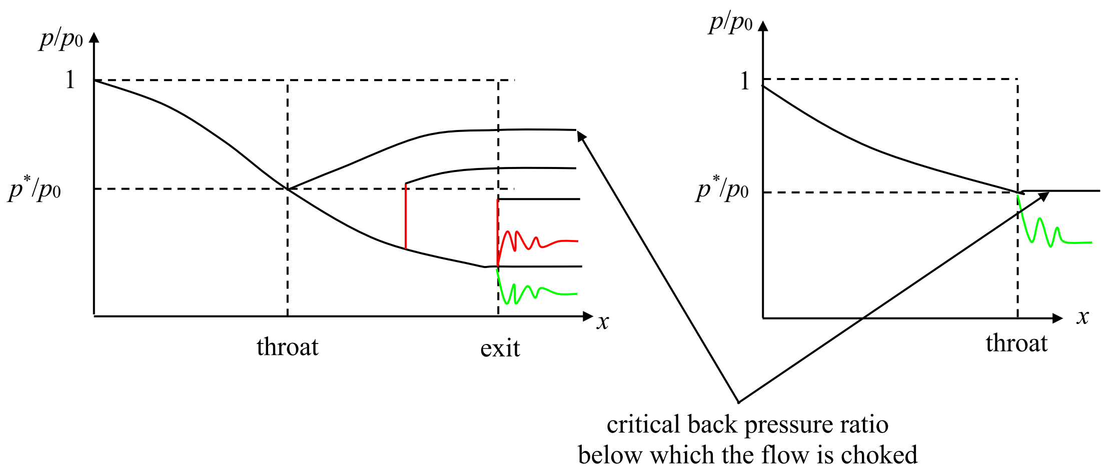
converging-diverging nozzle



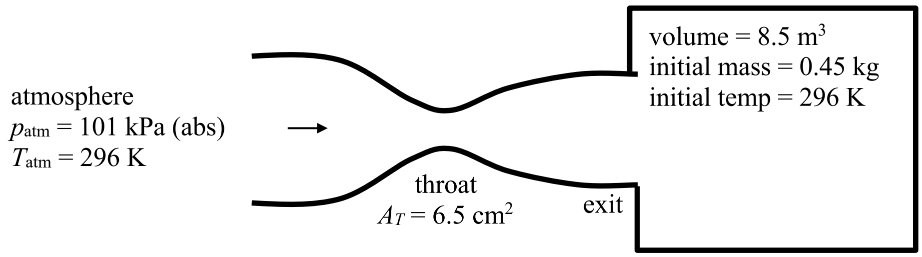
converging nozzle

SOLUTION:

The converging-diverging nozzle will fill the tank faster. Since the tank is initially evacuated, the flow will start at choked conditions in each case. Hence, the mass flow rate into each tank will be the choked flow mass flow rate (i.e., the maximum mass flow rate), which will be identical in both cases since the throat areas and stagnation properties are identical. However, the converging-diverging nozzle will remain choked for a wider range of back pressure ratios than the converging nozzle. Hence, converging-diverging nozzle will fill the tank more rapidly.



An 8.5 m^3 vacuum tank is to be used to create a flow at an exit Mach number of $\text{Ma}_E = 2.0$ (refer to the figure below). A plug is put into the nozzle and the tank is evacuated until it contains 0.45 kg of air at a temperature of 296 K. When the plug is removed, air flows from the atmosphere into the tank through the converging-diverging nozzle. The throat area is $A_T = 6.5$ cm^2.



a. Determine the design exit area.
b. Determine the initial pressure in the tank.
c. Determine the initial mass flow rate through the nozzle.
d. Determine the exit pressure, p_E, immediately after the flow begins.
e. Determine the tank pressure at which a normal shock wave will stand in the nozzle exit plane.

SOLUTION:

The design exit area may be found from the design exit Mach number, $\text{Ma}_{E,d} = 2.0$, and the isentropic flow relations.

$$\frac{A_{E,d}}{A^*} = \frac{1}{\text{Ma}_{E,d}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_{E,d}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow A_{E,d}/A^* = 1.6875 \Rightarrow \boxed{A_{E,d} = 11.0\text{ cm}^2} \quad (1)$$

where the sonic area is equal to the throat area, $A^* = A_T = 6.5$ cm^2, since the flow goes from stagnation conditions to supersonic conditions.

The initial pressure in the tank may be found using the ideal gas law,

$$p = \rho RT = \left(\frac{M}{V}\right)RT \Rightarrow \boxed{p_{\text{tank}}(t=0) = 4.50\text{ kPa (abs)}} \quad (2)$$

where $M = 0.45$ kg, $V = 8.5$ m^3, $R = 287$ J/(kg.K), and $T = 296$ K.

To determine the exit plane pressure and initial mass flow rate through the nozzle, first determine whether or not the flow is choked. Determine the pressure at the exit plane when the flow first becomes choked (*i.e.*, $\text{Ma}_T = 1$) by first determining the exit Mach number when the flow first becomes choked, then using this Mach number and the isentropic relations to determine the exit pressure ratio.

$$\frac{A_E}{A^*} = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_E^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_E = 0.372 \quad (3)$$

where $A_E/A^* = 1.6875$ from Eq. (1) (note that when the flow is choked, $A^* = A_T$). The pressure at the exit for this condition is found from the isentropic flow relation.

$$\frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_E/p_0 = 0.9088 \Rightarrow p_E = 91.8\text{ kPa (abs)} \quad (4)$$

where $p_0 = p_{\text{atm}} = 101$ kPa (abs). Since this exit pressure is larger than the initial tank pressure, the flow must be choked and the mass flow rate is then,

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \;\Rightarrow\; \boxed{\dot{m}_{\text{choked}} = 0.154 \text{ kg/s}} \tag{5}$$

where $p_0 = 101$ kPa (abs), $T_0 = 296$ K, $R = 287$ J/(kg.K), $\gamma = 1.4$, and $A^* = A_T = 6.5$ cm^2.

The design pressure for the nozzle is found using the isentropic relations and the design Mach number.

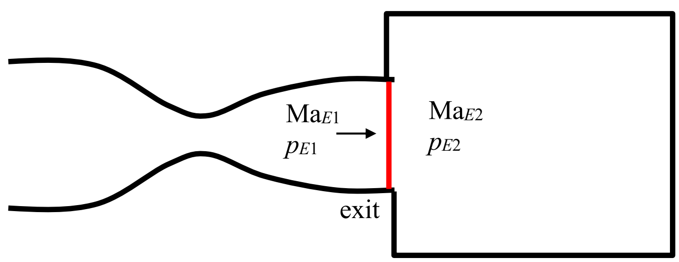
$$\frac{p_{E,d}}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_{E,d}^2\right)^{\frac{\gamma}{1-\gamma}} \;\Rightarrow\; p_{E,d}/p_0 = 0.1278 \;\Rightarrow\; p_{E,d} = 12.9 \text{ kPa (abs)} \tag{6}$$

Since the exit pressure at design is larger than the initial tank pressure, the flow must be underexpanded and the exit pressure will be equal to the design exit pressure of $\boxed{p_{E,d} = 12.9 \text{ kPa (abs)}}$.

The tank pressure at which a normal shock stands in the exit plane is found by using the design Mach number and exit pressure found in Eq. (6) just upstream of the shock, then applying the normal shock relations across the exit shock wave.

$$\text{Ma}_{E1} = 2.0 \;\Rightarrow\; p_{E2}/p_{E1} = 4.500 \text{ (from the normal shock relations)} \;\Rightarrow\; p_{E2} = 58.1 \text{ kPa (abs)} \tag{7}$$

where $p_{E1} = 12.9$ kPa (abs) from Eq. (6). Since the flow just downstream of the shock is subsonic, the downstream exit pressure will equal the back pressure. Thus, the tank pressure at which a normal shock just stands at the exit is $\boxed{58.1 \text{ kPa (abs)}}$.



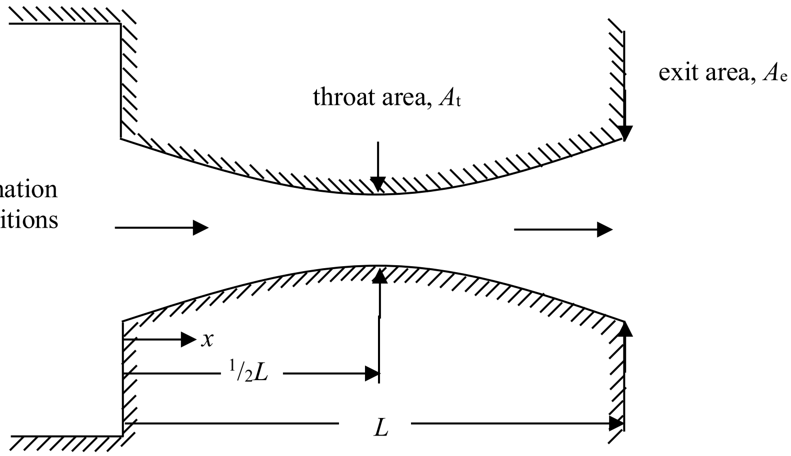
A converging-diverging nozzle, with an exit to throat area ratio, A_e/A_t, of 1.633, is designed to operate with atmospheric pressure at the exit plane, $p_e = p_{atm}$.

a. Determine the range(s) of stagnation pressures for which the nozzle will be free from normal shocks.

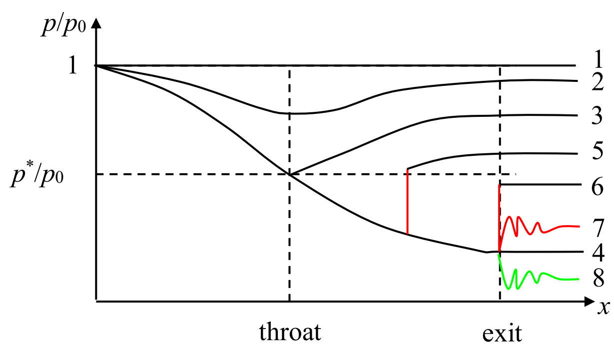
b. If the stagnation pressure is $1.5p_{atm}$, at what position, x, will the normal shock occur?

The converging-diverging nozzle area, A, varies with position, x, as:

$$\frac{A(x)}{A_E} = \left(\frac{A_E}{A_T} - 1\right)\left(2\frac{x}{L} - 1\right)^2 + 1$$



SOLUTION:



If there are no shocks, then the flow is assumed to remain isentropic. Determine the back pressure corresponding to isentropic sonic area ratio. Consider, for the moment, only the subsonic condition (case 3 shown in the figure above).

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = 1.633 = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_E^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_E = 0.39 \quad \text{(isentropic flow relations)} \tag{1}$$

$$\Rightarrow \frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} = 0.9016 \quad \text{(isentropic flow relations)} \tag{2}$$

Hence, for $p_{atm} \le p_0 \le p_{atm}/0.9016 = 1.11\ p_{atm}$ the flow throughout the nozzle will be subsonic and, as a result, there will be no shocks within the nozzle.

It's also possible to have isentropic flow within the nozzle, yet have a shock wave at the nozzle exit (case 6 in the figure). For the case when a normal shock wave is stationed at the nozzle exit:

$$\frac{A_E}{A^*}=\frac{A_E}{A_T}=1.633=\frac{1}{\mathrm{Ma}_{E1}}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_{E1}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_E=1.96 \text{ (isentropic flow relations)} \tag{3}$$

$$\Rightarrow \frac{p_{E1}}{p_{01}}=\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_{E1}^2\right)^{\frac{\gamma}{1-\gamma}}=0.1359 \text{ (isentropic flow relations)} \tag{4}$$

$$\text{and } \frac{p_{E2}}{p_{E1}}=\frac{p_{\text{atm}}}{p_{E1}}=\frac{2\gamma}{\gamma+1}\mathrm{Ma}_{E1}^2-\frac{\gamma-1}{\gamma+1}=4.3152 \text{ (normal shock relations)} \tag{5}$$

Note that since downstream of the shock the flow is subsonic and at the exit, $p_{E2} = p_{\text{atm}}$.

Now determine the upstream stagnation pressure corresponding to the given conditions.

$$\frac{p_{E2}}{p_{01}}=\frac{p_{\text{atm}}}{p_{01}}=\left(\frac{p_{E2}}{p_{E1}}\right)\left(\frac{p_{E1}}{p_{01}}\right)=(4.3152)(0.1359)=0.5864 \tag{6}$$

$$\therefore p_{01}=\frac{p_{\text{atm}}}{0.5864}=1.7052p_{\text{atm}} \tag{7}$$

Therefore, the device will not contain normal shocks for the following range of stagnation conditions:

$$\boxed{1\le\frac{p_0}{p_{\text{atm}}}\le 1.11 \text{ and } \frac{p_0}{p_{\text{atm}}}>1.71} \tag{8}$$

Based on Eq. (8), a normal shock will occur somewhere within the diverging portion of the nozzle if the stagnation pressure is $p_{01} = 1.5p_{\text{atm}}$. Use an iterative approach to determine the location of the shock as given below.

a. Assume a location for the shock wave (e.g., pick a value for A/A_t since the geometry is known).
b. Determine the Mach number and pressure just upstream of the shock, Ma_1 and p_1, using the isentropic relations as discussed in Note 2.

$$\frac{A}{A_1^*}=\frac{A}{A_t}=\frac{1}{\mathrm{Ma}_1}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \text{ (where the supersonic } \mathrm{Ma}_1 \text{ is chosen)} \tag{9}$$

$$\frac{p_1}{p_{01}}=\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \tag{10}$$

c. Calculate the stagnation pressure ratio and sonic area ratio across the shock using the normal shock relations:

$$\frac{p_{02}}{p_{01}}=\frac{A_1^*}{A_2^*}=\left[\frac{\frac{\gamma+1}{2}\mathrm{Ma}_1^2}{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}\right]^{\gamma/\gamma-1}\left[\frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2-\frac{\gamma-1}{\gamma+1}\right]^{1/1-\gamma} \tag{11}$$

d. Determine the exit Mach number and exit pressure ratio using the isentropic relations and the downstream sonic area and stagnation pressure:

$$\frac{A_e}{A_2^*}=\frac{A_e}{A_t}\frac{A_1^*}{A_2^*}=\frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \text{ (where the subsonic } \mathrm{Ma}_e \text{ is chosen)} \tag{12}$$

Note that since the flow is choked, the throat area is equal to the upstream sonic area, i.e., $A_t = A_1^*$.

$$\frac{p_e}{p_{02}} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \tag{13}$$

Note that since the exit Mach number is subsonic, the exit pressure will equal the back pressure, i.e., $p_e = p_b$.

e. Calculate the ratio of the back pressure to the upstream stagnation pressure:

$$\frac{p_b}{p_{01}} = \frac{p_e}{p_{02}}\frac{p_{02}}{p_{01}} \tag{14}$$

f. Check to see if the back pressure ratio calculated in step (e) matches with the given back pressure ratio. If so, then the assumed location of the shock is correct. If not, then the go back to step (a) and repeat. If the back pressure ratio calculated in part (e) is less than the given back pressure ratio, then the assumed shock location is too far upstream. If the back pressure ratio calculated in part (e) is greater than the given back pressure ratio, then the assumed shock location is too far downstream.

Apply this algorithm using the given data and summarize in the following table. (Note that the correct position is found manually in this case, but the method could easily be made into a computer program and the correct position could be found using an approach such as a bisection method.)

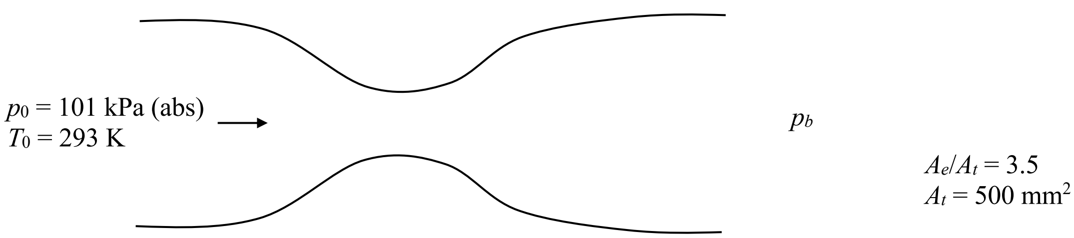
γ =	1.4
A_E/A_T =	1.633
p_{01}/p_{atm} =	1.5

x/L	**$A(x)/A_T$**	**Ma_1**	**p_1/p_{01}**	**p_{02}/p_{01}**	**A_1^*/A_2^***	**A_E/A_2^***	**Ma_E**	**p_E/p_{02}**	**p_B/p_{01}**	**p_{01}/p_B**	**comment**
0.8000	1.2279	1.5716	0.2454	0.9056	0.9056	1.4788	0.4382	0.8764	0.7937	1.2600	shock too far upstream
0.9000	1.4051	1.7681	0.1827	0.8267	0.8267	1.3500	0.4948	0.8459	0.6993	1.4299	shock too far upstream
0.9500	1.5127	1.8649	0.1575	0.7835	0.7835	1.2794	0.5343	0.8234	0.6451	1.5502	shock too far downstream
0.9250	1.4573	1.8167	0.1696	0.8052	0.8052	1.3150	0.5134	0.8354	0.6727	1.4865	shock too far upstream
0.9375	1.4846	1.8408	0.1635	0.7944	0.7944	1.2973	0.5235	0.8296	0.6591	1.5173	shock too far downstream
0.9313	1.4709	1.8288	0.1665	0.7998	0.7998	1.3061	0.5184	0.8326	0.6659	1.5017	shock too far downstream
0.9281	1.4641	1.8228	0.1681	0.8025	0.8025	1.3105	0.5159	0.8340	0.6693	1.4941	shock too far upstream
0.9297	1.4675	1.8258	0.1673	0.8012	0.8012	1.3083	0.5172	0.8333	0.6676	1.4980	shock too far upstream
0.9305	1.4692	1.8273	0.1669	0.8005	0.8005	1.3072	0.5178	0.8329	0.6667	1.4998	shock too far upstream
0.9309	1.4700	1.8280	0.1667	0.8002	0.8002	1.3067	0.5181	0.8327	0.6663	1.5007	shock too far downstream
0.9307	1.4696	1.8276	0.1668	0.8004	0.8004	1.3070	0.5179	0.8329	0.6666	1.5002	shock too far downstream
0.9306	**1.46947**	**1.82753**	**0.1669**	**0.8004**	**0.8004**	**1.30702**	**0.51791**	**0.8329**	**0.6666**	**1.5001**	**just about right!**

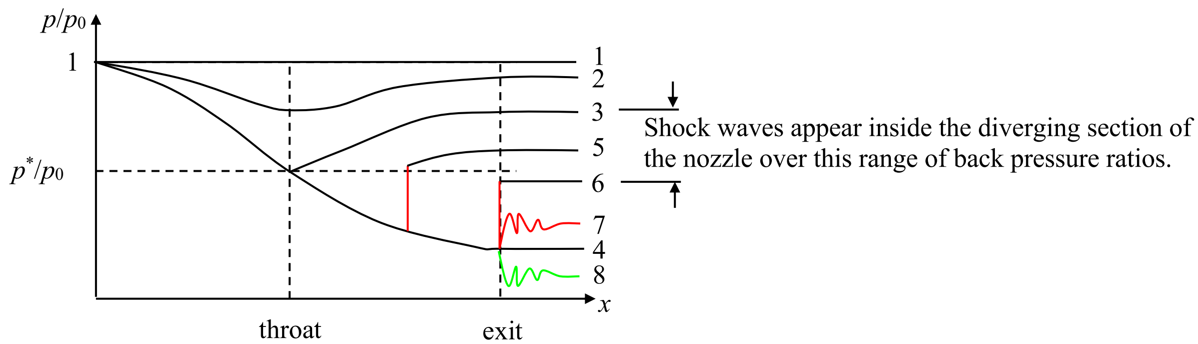
Thus, the shock is located at $\boxed{x/L = 0.9306}$.

Air flows through a converging-diverging nozzle, with $A_e/A_t = 3.5$ where $A_t = 500$ mm^2. The upstream stagnation conditions are atmospheric; the back pressure is maintained by a vacuum system. Determine the range of back pressures for which a normal shock will occur within the nozzle and the corresponding mass flow rate.

SOLUTION:



A shock wave will appear within the nozzle for the range of back pressures indicated in the figure shown below.



The back pressure ratio corresponding to case 3 may be found from the isentropic relations:

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_e = 0.168 \tag{1}$$

(Note that for case 3, $A_t = A^*$ since the flow is choked.)

$$\frac{p_b}{p_0} = \frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_0 = 0.980 \tag{2}$$

(Note that since the flow is subsonic at the exit, $p_e = p_b$.)

The back pressure ratio corresponding to case 6 may be found by combining the isentropic relations with the normal shock relations.

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\text{Ma}_{e1}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_{e1}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad \Rightarrow \text{Ma}_{e1} = 2.80 \tag{3}$$

$$\text{Ma}_{e2}^2 = \frac{(\gamma-1)\text{Ma}_{e1}^2+2}{2\gamma\text{Ma}_{e1}^2-(\gamma-1)} \quad \Rightarrow \text{Ma}_{e2} = 0.488 \tag{4}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\text{Ma}_{e1}^2}{2+(\gamma-1)\text{Ma}_{e1}^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_{e1}^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \quad \Rightarrow p_{02}/p_{01} = 0.389 \tag{5}$$

$$\frac{p_b}{p_{02}} = \frac{p_{e2}}{p_{02}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_{e2}^2\right)^{\frac{\gamma}{1-\gamma}} \quad \Rightarrow p_b/p_{02} = 0.850 \tag{6}$$

$$\frac{p_b}{p_{01}} = \left(\frac{p_b}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right) \quad \Rightarrow p_b/p_{01} = 0.331 \tag{7}$$

Thus, a normal shock wave will appear in the diverging portion of the converging-diverging nozzle over the range:

$$\boxed{0.331 < p_b/p_0 < 0.980} \tag{8}$$

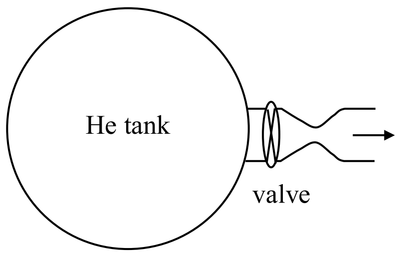
where $p_0 = 1$ atm = 101 kPa (abs).

The mass flow rate when the flow is choked is:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \quad \Rightarrow \boxed{\dot{m}_{\text{choked}} = 0.119 \text{ kg/s}} \tag{9}$$

where $\gamma = 1.4$, $p_0 = 101$ kPa, $R = 287$ J/(kg·K), $T_0 = 293$ K, and $A^* = A_t = 500$ mm^2.

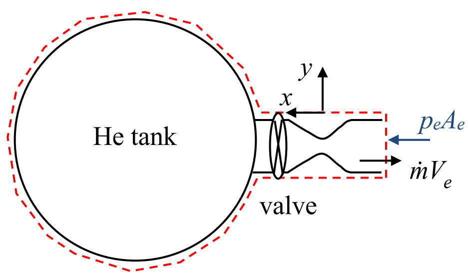
A satellite includes a correctional propulsive unit consisting of a tank that is 3 ft^3 in volume and contains helium initially at 2000 psia. Heaters on the satellite maintain the tank temperature at 0 °F. The tank is connected to a short, insulated, convergent-divergent nozzle having a throat area of 1 in^2 and an exit area of 3 in^2. The mass of the satellite, exclusive of the helium, is 50 lb_m. Plot the acceleration of the satellite as a function of time if the valve is left open.



SOLUTION:

The pressure in space is nearly zero so the flow from the nozzle will always be underexpanded.

Apply the LME in the x-direction to the CV shown below. Use a frame of reference fixed to the satellite.



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{CV} a_x \rho dV \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(The velocity of He in tank is zero in the given FOR.)} \tag{2}$$

$$\int_{CS} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}V_e \tag{3}$$

$$F_{B,x} = 0 \tag{4}$$

$$F_{S,x} = p_e A_e \tag{5}$$

$$\int_{CV} a_x \rho dV = M_{\text{sat}} a_{\text{sat}} \tag{6}$$

Substitute and simplify.

$$M_{\text{sat}} a_{\text{sat}} = \dot{m}V_e + p_e A_e \tag{7}$$

Since the flow within the nozzle will always be choked and isentropic (the back pressure is zero), the mass flow rate is:

$$\dot{m} = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \quad \text{where } A^* = A_{\text{throat}} \tag{8}$$

The exit velocity and pressure may be expressed in terms of the exit Mach number:

$$V_e = \text{Ma}_e\left(\frac{c_e}{c_0}\right)c_0 = \text{Ma}_e\left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{-1/2}\sqrt{\gamma R T_0} \tag{9}$$

$$p_e = p_0\left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \tag{10}$$

Substitute and simplify.

$$M_{\text{sat}}a_{\text{sat}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\gamma p_0 A_t \text{Ma}_e\left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{-1/2} + p_0\left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} A_e \tag{11}$$

The Mach number at the exit is found using the given area ratio:

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{3\text{ in.}^2}{1\text{ in.}^2} = 3.0 = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_e = 3.0 \quad (\text{Note: } \gamma_{\text{He}} = 1.66) \tag{12}$$

Using the given data:

p_0 = 2000 psia
A_t = 1 in^2
A_e = 3 in^2
Ma_e = 3.0
γ_{He} = 1.66
R_{He} = 386.1 (ft.lb$_f$)/(lb$_m$.°R)
V_{tank} = 3 ft^3
T_0 = 460 °R

$$\Rightarrow M_{\text{sat}}(t=0) = 50\text{ lb}_\text{m} + \underbrace{\left(\frac{p_0 V_{\text{tank}}}{RT_0}\right)}_{\text{initial mass of He}} = 54.9\text{ lb}_\text{m} \tag{13}$$

$$\Rightarrow a_{\text{sat}}(t=0) = 54.7\text{ ft/s}^2 = 1.7g \quad (\text{where } g \text{ is } 32.2\text{ ft/s}^2) \tag{14}$$

Note that the pressure within the tank will decrease with time as helium is discharged from the tank (the tank temperature remains constant due to the heaters).

$$p_0 = \rho_0 R T_0 = \left(\frac{M_{\text{He}}}{V_{\text{tank}}}\right)RT_0 \tag{15}$$

$$\frac{dp_0}{dt} = \frac{dM_{\text{He}}}{dt}\left(\frac{RT_0}{V_{\text{tank}}}\right) = -\dot{m}\left(\frac{RT_0}{V_{\text{tank}}}\right) \quad \text{from conservation of mass} \tag{16}$$

Substitute Eqn. (8) and simplify.

$$\frac{dp_0}{dt} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A_t\left(\frac{RT_0}{V_{\text{tank}}}\right) = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\frac{p_0\sqrt{\gamma RT_0}A_t}{V_{\text{tank}}} \tag{17}$$

$$\int_{p_0(t=0)}^{p_0(t)}\frac{dp_0}{p_0} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\frac{\sqrt{\gamma RT_0}A_t}{V_{\text{tank}}}\int_{t=0}^{t=t}dt \tag{18}$$

$$p_0 = p_{0,t=0}\exp\left[-\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\frac{\sqrt{\gamma RT_0}A_t t}{V_{\text{tank}}}\right] \tag{19}$$

Substitute Eq. (19) into Eq. (13) to determine how the satellite mass changes with time.

$$M_{\text{sat}}(t) = 50\ \text{lb}_\text{m} + \frac{p_{0,t=0}V_{\text{tank}}}{RT_0}\exp\left[-\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\frac{\sqrt{\gamma RT_0}A_t t}{V_{\text{tank}}}\right] \tag{20}$$

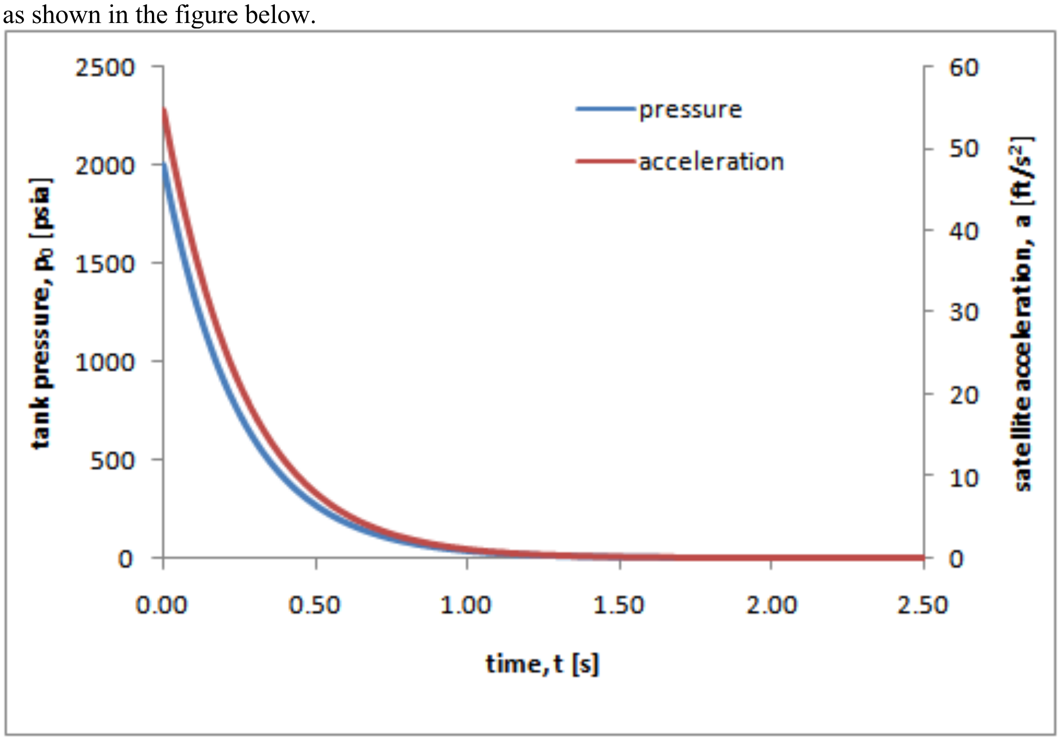
Summarizing:

$$p_0 = p_{0,t=0}\exp\left[-\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\frac{\sqrt{\gamma RT_0}A_t t}{V_{\text{tank}}}\right] \tag{21}$$

$$M_{\text{sat}} = 50\ \text{lb}_\text{m} + \frac{p_0 V_{\text{tank}}}{RT_0} \tag{22}$$

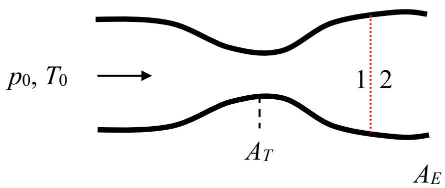
$$M_{\text{sat}}a_{\text{sat}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\gamma p_0 A_t \text{Ma}_e\left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{-1/2} + p_0\left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}}A_e \tag{23}$$

Using the given initial data, the satellite tank pressure and acceleration may be plotted as a function of time as shown in the figure below.



Air flows through a frictionless, adiabatic converging-diverging nozzle. The air in the reservoir feeding the nozzle has a pressure and temperature of 700 kPa (abs) and 500 K, respectively. The ratio of the nozzle exit to throat area is 11.91. A normal shock wave stands where the upstream Mach number is 3.0. Calculate the Mach number, the static temperature, and static pressure at the nozzle exit plane.

SOLUTION:



p_0 = 700 kPa (abs)
T_0 = 500 K
A_E/A_T = 11.91
Ma_1 = 3.0

Using the normal shock relations:

$$\mathrm{Ma}_1 = 3.0 \Rightarrow \quad \mathrm{Ma}_2 = 0.4752 \tag{1}$$
$$T_{02}/T_{01} = 1 \tag{2}$$
$$p_{02}/p_{01} = 0.3283 \tag{3}$$

The flow is isentropic from the reservoir to just upstream of the shock (location 1) so that:

$$p_{01} = p_0 \tag{4}$$
$$T_{01} = T_0 \tag{5}$$
$$A_1/A_1^* = 4.2346 \text{ (using } \mathrm{Ma}_1 = 3.0) \tag{6}$$

The flow is also isentropic from just downstream of the shock (location 2) to the exit so that:

$$p_{0E} = p_{02} \tag{7}$$
$$T_{0E} = T_{02} \tag{8}$$
$$A_2/A_2^* = 1.390 \text{ (using } \mathrm{Ma}_2 = 0.4752) \tag{9}$$

Combine the previous equations to get the exit stagnation conditions.

$$p_{0E} = p_{02} = \left(\frac{p_{02}}{p_{01}}\right)p_{01} = \left(\frac{p_{02}}{p_{01}}\right)p_0 = (0.3283)(700 \text{ kPa}) = 229.8 \text{ kPa} \tag{10}$$

$$T_{0E} = T_{02} = \left(\frac{T_{02}}{T_{01}}\right)T_{01} = \left(\frac{T_{02}}{T_{01}}\right)T_0 = (1)(500 \text{ K}) = 500 \text{ K} \tag{11}$$

Now determine the exit sonic area ratio (A_E/A^*) so that it can be used to determine the exit Mach number.

$$\frac{A_E}{A_2^*} = \left(\frac{A_E}{A_T}\right)\left(\frac{A_1^*}{A_1}\right)\left(\frac{A_2}{A_2^*}\right) = (11.91)\left(\frac{1}{4.2346}\right)(1.390) = 3.9094 \quad \text{(Note that } A_T = A_1^*.) \tag{12}$$

Use this area ratio and the isentropic flow sonic area relation to determine the exit Mach number. Note that the exit Mach number will be subsonic since the flow downstream of the shock wave is subsonic.

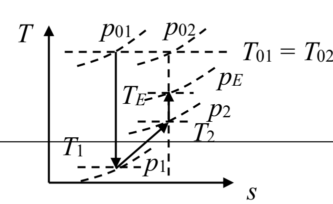
$$\frac{A_E}{A_2^*} = 3.9094 \Rightarrow \boxed{\mathrm{Ma}_E = 0.15} \tag{13}$$

Use the isentropic flow relations with the exit Mach number to determine the stagnation temperature and pressure ratios.

$$\frac{T_E}{T_{0E}} = 0.9955 \quad \text{and} \quad \frac{p_E}{p_{0E}} = 0.9844 \tag{14}$$

Combine Eqns. (14) with Eqs. (10) and (11) to determine the exit static temperature and pressure.

$\boxed{T_E = 498 \text{ K}}$ and $\boxed{p_E = 226 \text{ kPa (abs)}}$



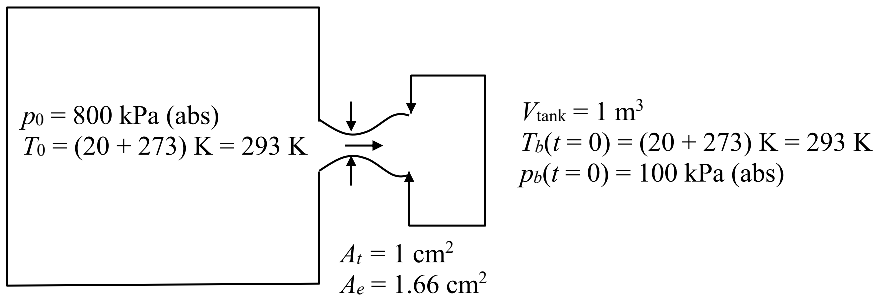
A large reservoir at 20 °C and 800 kPa (abs) is used to fill a small tank through a converging-diverging nozzle with 1 cm^2 throat area and 1.66 cm^2 exit area. The small tank has a volume of 1 m^3 and is initially at 20 °C and 100 kPa (abs). Estimate the elapsed time when:

a. shock waves begin to appear inside the nozzle, and
b. the mass flow rate begins to drop below its maximum value.

You may assume that the tank filling process occurs isothermally.

c. Describe (but you need not work out) how your solution approach would change if the tank is well insulated so that the filling process occurs adiabatically.

SOLUTION:



First check to see where the flow is on the diagram below. At $t = 0$:

$$p_b/p_0 = (100\text{ kPa})/(800\text{ kPa}) = 0.125 \tag{1}$$

The back pressure ratios corresponding to cases 3 (onset of choked flow) and 4 (design conditions) – refer to the plot below – may be found from the isentropic relations.

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_e = 0.3796,\ \text{Ma}_e = 1.9802 \tag{2}$$

using A_e = 1.66 cm^2 and $A^* = A_t$ = 1 cm^2 ($\Rightarrow A_e/A^* = 1.66$).

$$\frac{p_b}{p_0} = \frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_0 = 0.9053,\ 0.1318 \tag{3}$$

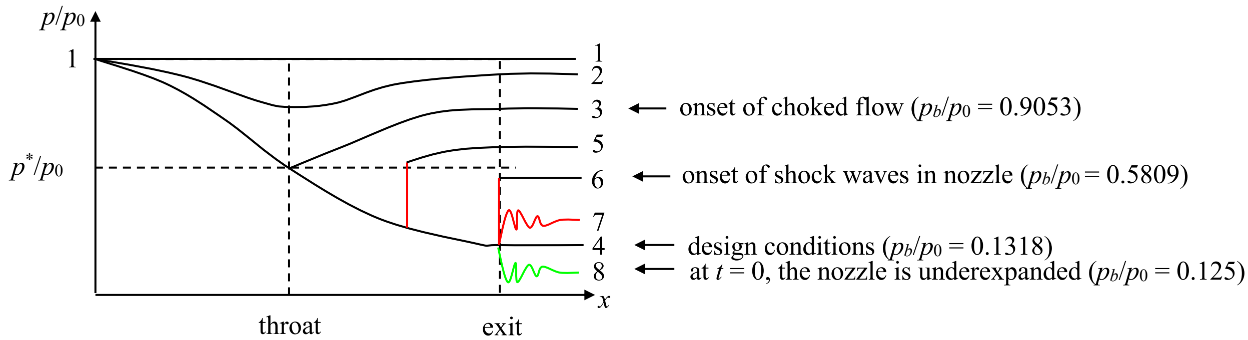
The back pressure ratio when a normal shock wave stands at the nozzle exit may be found by combining the isentropic and normal shock wave relations.

$$\text{Ma}_{e2}^2 = \frac{(\gamma-1)\text{Ma}_{e1}^2+2}{2\gamma\text{Ma}_{e1}^2-(\gamma-1)} \Rightarrow \text{Ma}_{e2} = 0.5808 \text{ (using Ma}_{e1} = 1.9802 \text{ – from Eq. (2))} \tag{4}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\text{Ma}_{e1}^2}{2+(\gamma-1)\text{Ma}_{e1}^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_{e1}^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \Rightarrow p_{02}/p_{01} = 0.7301 \tag{5}$$

$$\frac{p_b}{p_{02}} = \frac{p_{e2}}{p_{02}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_{e2}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_{02} = 0.7957 \tag{6}$$

$$\frac{p_b}{p_{01}} = \left(\frac{p_b}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right) \Rightarrow p_b/p_{01} = 0.5809 \tag{7}$$



The mass flow rate into the tank will be choked until $p_b/p_0 \geq 0.9053$. The choked mass flow rate into the tank is given by:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \;\Rightarrow\; \dot{m}_{\text{choked}} = 0.1889 \text{ kg/s} \tag{8}$$

The (back) pressure in the tank may be found by applying conservation of mass to a control volume surrounding the tank and making use of the ideal gas law.

$$M_{\text{tank}} = \rho_b V_{\text{tank}} = \frac{p_b V_{\text{tank}}}{RT_b} = \dot{m}t + M_0 \;\Rightarrow\; p_b = \frac{RT_b}{V_{\text{tank}}}\left(\dot{m}t + M_0\right) \tag{9}$$

Note that the mass flow rate into the tank is the choked mass flow rate (Eq. (8), which remains constant up until case 3 is reached), and M_0 is the mass inside the tank at $t = 0$.

$$M_0 = M_{\text{tank},t=0} = \left(\frac{p_b V_{\text{tank}}}{RT_b}\right)_{t=0} \;\Rightarrow\; \underline{M_0 = 1.189 \text{ kg}} \tag{10}$$

where $p_{b,t=0} = 100$ kPa, $T_{b,\,t=0} = 293$ K, $R = 287$ J/(kg·K), and $V_{\text{tank}} = 1$ m^3.

Thus, the time for the onset of shock waves in the nozzle (case 6) is:

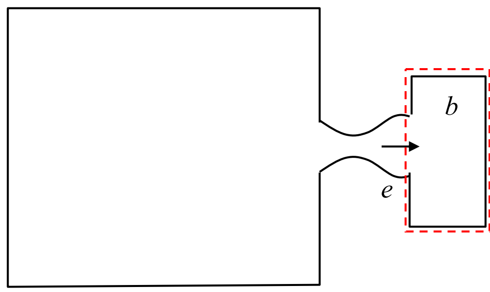
$$\frac{p_b}{p_0} = 0.5809 = \frac{RT_b}{p_0 V_{\text{tank}}}\left(\dot{m}t_{\text{shocks}} + M_0\right) \tag{11}$$

$$t_{\text{shocks}} = \frac{1}{\dot{m}}\left[0.5809\left(\frac{p_0 V_{\text{tank}}}{RT_b}\right) - M_0\right] \;\Rightarrow\; \boxed{t_{\text{shocks}} = 23.0 \text{ s}} \tag{12}$$

The time for when the flow is no longer choked (case 3) is:

$$t_{\text{unchoked}} = \frac{1}{\dot{m}}\left[0.9053\left(\frac{p_0 V_{\text{tank}}}{RT_b}\right) - M_0\right] \;\Rightarrow\; \boxed{t_{\text{unchoked}} = 39.3 \text{ s}} \tag{13}$$

If we assume that the tank fills adiabatically (likely a more realistic scenario), then the calculations become much more involved since the temperature in the tank will also vary as the pressure varies. The (back) pressure in the tank will increase as additional mass enters the tank. We can determine how the pressure varies by applying conservation of energy and conservation of mass to a control volume surrounding the tank as shown below.



Applying conservation of energy to a control volume surrounding the tank gives:

$$\frac{d}{dt}\left(M_{\text{tank}} c_V T_b\right) - \dot{m}_e\left(c_P T_e + \tfrac{1}{2}V_e^2\right) = 0 \tag{14}$$

$$c_V\left(T_b \frac{dM_{\text{tank}}}{dt} + M_{\text{tank}}\frac{dT_b}{dt}\right) - \dot{m}_e\left(c_P T_e + \tfrac{1}{2}V_e^2\right) = 0 \tag{15}$$

$$c_V\left[\dot{m}_e T_b + \left(\dot{m}_e t + M_0\right)\frac{dT_b}{dt}\right] - \dot{m}_e\left(c_P T_e + \tfrac{1}{2}V_e^2\right) = 0 \quad \text{(where conservation of mass has been used)} \tag{16}$$

$$c_V\left(\dot{m}_e t + M_0\right)\frac{dT_b}{dt} + \dot{m}_e c_V T_b = \dot{m}_e\left(c_P T_e + \tfrac{1}{2}V_e^2\right) \tag{17}$$

where the mass flow rate entering the tank is given by Eq. (8) (the flow is choked for the conditions we're interested in so that mass flow rate will remain constant). Note that we are assuming that the tank is well insulated indicating that the filling process occurs adiabatically ($\dot{Q}_{\text{into tank}} = 0$). The temperature and velocity of the air entering the tank (T_e and V_e) may be found following an approach similar to the ones used previously to determine the exit pressure. For back pressures less than the value corresponding to case 6 (normal shock at the exit plane), the exit temperature and velocity are given by:

$$\frac{T_e}{T_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_e^2\right)^{-1} \Rightarrow T_e/T_0 = 0.5605 \Rightarrow \underline{T_e = 164.2\text{ K}} \tag{18}$$

$$V_e = \text{Ma}_e c_e = \text{Ma}_e\sqrt{\gamma R T_e} \Rightarrow \underline{V_e = 508.7\text{ m/s}} \tag{19}$$

The (back) pressure in the tank may be found by applying conservation of mass to the same control volume and making use of the ideal gas law.

$$M_{\text{tank}} = \rho_b V_{\text{tank}} = \frac{p_b V_{\text{tank}}}{R T_b} = \dot{m}t + M_0 \Rightarrow p_b = \frac{R T_b}{V_{\text{tank}}}\left(\dot{m}t + M_0\right) \tag{20}$$

where T_b is found from the (numerical) solution of Eq. (17). Note that the mass flow rate into the tank is the choked mass flow rate (Eqn. (8), which remains constant up until case 3 is reached), and M_0 is the mass inside the tank at $t = 0$ (Eq. (10)).

When $p_b/p_0 \geq 0.5806$ (corresponding to case 6), then the temperature and velocity of the air entering the tank (T_e and V_e) must be found by taking into consideration a normal shock wave located somewhere within the diverging portion of the nozzle. The exit temperature and velocity will depend upon the location of the shock wave, which in turn will depend upon the back pressure. Hence, the shock finding algorithm described in the course notes (it won't be repeated here) must be used for a given back pressure to determine the location of the normal shock wave. Once this location has been found (and hence, Ma_1 is known) , the exit temperature and velocity may be found by combining the isentropic and normal shock relations:

$$\frac{T_e}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_e^2\right)^{-1} \quad \text{(Note that } T_0 = T_{01} = T_{02}\text{)} \tag{21}$$

$$V_e = \mathrm{Ma}_e c_e = \mathrm{Ma}_e \sqrt{\gamma R T_e} \tag{22}$$

where the exit Mach number is found from the area ratio:

$$\frac{A_e}{A_2^*} = \left(\frac{A_e}{A_1^*}\right)\left(\frac{A_1^*}{A_2^*}\right) = \left(\frac{A_e}{A_t}\right)\left(\frac{A_1^*}{A_2^*}\right) = \frac{1}{\mathrm{Ma}_e}\left(\frac{1 + \frac{\gamma - 1}{2}\mathrm{Ma}_e^2}{1 + \frac{\gamma - 1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{23}$$

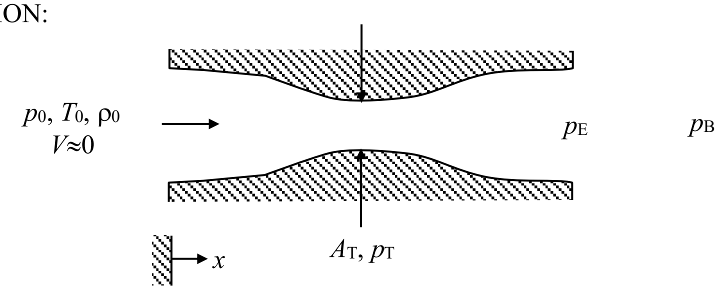
where

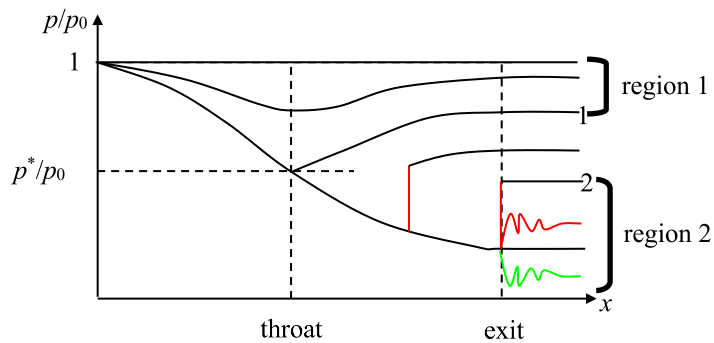
$$\frac{A_1^*}{A_2^*} = \left[\frac{(\gamma + 1)\mathrm{Ma}_1^2}{2 + (\gamma - 1)\mathrm{Ma}_1^2}\right]^{\gamma/(\gamma-1)}\left[\frac{\gamma + 1}{2\gamma \mathrm{Ma}_1^2 - (\gamma - 1)}\right]^{1/(\gamma-1)} \tag{24}$$

where Ma_1 is the Mach number just upstream of the shock wave. With the calculated T_e and V_e, Eqn. (17) may be solved numerically simultaneously with Eq. (20) so that p_b may be determined.

A converging-diverging nozzle, with $A_e/A_t = 1.633$, is designed to operate with atmospheric pressure at the exit plane. Determine the range(s) of stagnation pressures for which the nozzle will be free from normal shocks.

SOLUTION:





There will be two ranges of back pressures that will not produce shock waves *within* the C-D nozzle. In region 1 shown above, the entire flow remains subsonic (with possible sonic flow at the throat). In region 2 the flow is subsonic in the converging section, sonic at the throat, then subsonic throughout the diverging section. Shock waves and expansion fans may occur *outside* of the C-D nozzle in region 2.

Consider pressure curve 1 indicated in the figure above. For this case the exit Mach number is given by:

$$\frac{A_E}{A_T} = \frac{A_E}{A^*} = 1.633 = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{1}$$

Solve for the subsonic exit Mach number to get:

$\mathrm{Ma}_E = 0.387$

Now use the isentropic stagnation pressure ratio to determine the reservoir stagnation pressure for these conditions.

$$\frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_0 = 1.11 p_E = 112 \text{ kPa} \quad (\text{where } p_E = p_{atm} = 101 \text{ kPa}) \tag{2}$$

Hence, the nozzle will be shock free for:

$\boxed{p_{atm} \le p_0 \le 1.11 p_{atm} \text{ or } 101 \text{ kPa} \le p_0 \le 112 \text{ kPa}}$

Now consider pressure curve 2 indicated in the figure above. For this case a normal shock wave occurs at the nozzle exit plane. Just upstream of the shock wave the Mach number can be found using the sonic area ratio.

$$\frac{A_{E1}}{A^*} = \frac{A_E}{A_T} = 1.633 \quad \Rightarrow \quad \text{Ma}_{E1} = 1.96 \ \text{(using the isentropic flow relations)}$$

$\Rightarrow$ $p_{E2}/p_{E1} = 4.3152$ (using the normal shock relations with $\text{Ma}_{E1} = 1.96$)

$\Rightarrow$ $p_{E1}/p_{01} = 0.1359$ (using the isentropic flow relations with $\text{Ma}_{E1} = 1.96$)

Now solve for p_{E2}/p_{01}.

$$\frac{p_{E2}}{p_{01}} = \left(\frac{p_{E2}}{p_{E1}}\right)\left(\frac{p_{E1}}{p_{01}}\right) = (4.3152)(0.1359) = 0.5864$$

Note that $p_{01} = p_0$ (the reservoir pressure) and $p_{E2} = p_{\text{atm}}$ (since the flow downstream of the shock is subsonic).

$\Rightarrow$ $p_0 = 1.7052\, p_{\text{atm}}$

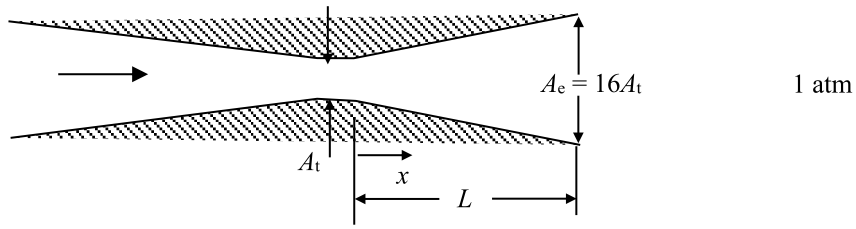
Thus, normal shocks will not form in the C-D nozzle when:

$\boxed{p_0 > 1.71\, p_{\text{atm}} \ \text{or} \ p_0 > 172 \text{ kPa}}$

To summarize, the C-D nozzle will remain shock-free for the following range of stagnation pressures:

$\boxed{p_{\text{atm}} \le p_0 \le 1.11 p_{\text{atm}} \ \text{and} \ p_0 > 1.71\, p_{\text{atm}}}$

A crude converging-diverging nozzle with an exit-to-throat area ratio of $A_e/A_t = 16$ is built using a straight-sided conical diffuser as shown in the figure below.



The nozzle is supplied by an air reservoir of pressure, p_{res}, and temperature, T_{res}. The nozzle discharges into atmospheric conditions (p_{atm} = 1 atm).

a. If a shock wave forms half-way along the diffuser, i.e., $x/L = 0.5$, determine the reservoir pressure, p_{res}.
b. Determine over what range of reservoir pressures the flow will be choked.

SOLUTION:

First determine the area in the straight-sided nozzle as a function of position in the nozzle.

$$r = (r_e - r_t)\left(\frac{x}{L}\right) + r_t \tag{1}$$

$$A = \pi r^2 \tag{2}$$

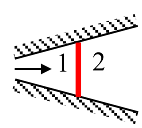
$$\frac{A}{A_t} = \left(\frac{r}{r_t}\right)^2 = \left[\left(\frac{r_e}{r_t} - 1\right)\left(\frac{x}{L}\right) + 1\right]^2 \quad \text{where} \quad \frac{r_e}{r_t} = \sqrt{\frac{A_e}{A_t}} \tag{3}$$

$$\therefore \frac{A}{A_t} = \left[\left(\sqrt{\frac{A_e}{A_t}} - 1\right)\left(\frac{x}{L}\right) + 1\right]^2 \tag{4}$$

For $x/L = 1/2$ and $A_e/A_t = 16$, $A/A_t = 6.25$. (5)

Using the isentropic flow relations (or tables) for air ($\gamma = 1.4$) and noting that the throat is also the sonic area since there is a shock wave in the diverging section:

$$\frac{A_1}{A^*} = 6.25 \Rightarrow \text{Ma}_1 = 3.411 \text{ and } \frac{p_1}{p_{01}} = 0.0149 \tag{6}$$



Using the normal shock relations (or tables) for air:

$$\text{Ma}_1 = 3.411 \Rightarrow \text{Ma}_2 = 0.4547, \frac{p_{02}}{p_{01}} = 0.2300, \frac{A_2^*}{A_1^*} = 4.3474 \tag{7}$$

Now determine the sonic area ratio at the exit, downstream of the shock wave.

$$\frac{A_e}{A_2^*} = \left(\frac{A_e}{A_1^*}\right)\left(\frac{A_1^*}{A_2^*}\right) = \left(\frac{16}{1}\right)\left(\frac{1}{4.3474}\right) = 3.6804 \tag{8}$$

Using the isentropic flow relations (or tables) for air:

$$\frac{A_e}{A_2^*} = 3.6804 \Rightarrow \text{Ma}_e = 0.1597, \frac{p_e}{p_{02}} = 0.9824 \tag{9}$$

Now determine the upstream stagnation pressure using the pressure ratios. Note that $p_e = p_{atm} = 1$ atm since the exit Mach number is subsonic.

$$p_{01} = \left(\frac{p_{01}}{p_{02}}\right)\left(\frac{p_{02}}{p_e}\right)p_e = \left(\frac{1}{0.2300}\right)\left(\frac{1}{0.9824}\right)(1\text{ atm}) \tag{10}$$

$$\boxed{\therefore p_{01} = 4.43\text{ atm}} \tag{11}$$

For a flow that just becomes choked:

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = 16 \Rightarrow \text{Ma}_e = 0.0362, \frac{p_e}{p_0} = 0.9991 \tag{12}$$

$$p_0 = \left(\frac{p_0}{p_e}\right)p_e = \left(\frac{1}{0.9991}\right)(1\text{ atm}) \tag{13}$$

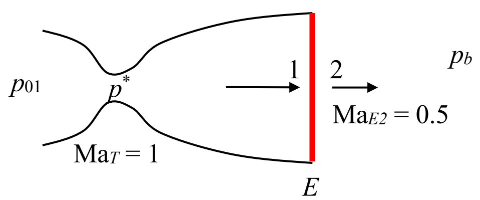
$$\therefore p_0 = 1.001\text{ atm} \tag{14}$$

Therefore, $\boxed{\text{the flow will be choked for } p_0 \geq 1.001\text{ atm}}$.

For the purposes of an experiment, we wish to design a de Laval nozzle which will be supplied from a compressed air reservoir (specific heat ratio of 1.4). It is required that:
1. there is a normal shock across the exit of the diffuser, and
2. the jet emerging downstream of the shock should have a Mach number of 0.5.

Find:
a. the ratio of the cross-sectional area at the diffuser exit to the cross-sectional area of the throat,
b. the ratio of the ambient pressure downstream of the shock to the pressure in the compressed air reservoir, and
c. the ratio of the ambient pressure downstream of the shock to the throat pressure.

SOLUTION:



The Mach number just upstream of the shock wave at the exit may be found using the normal shock relations,

$$\mathrm{Ma}_{E2}^2 = \frac{(k-1)\mathrm{Ma}_{E1}^2 + 2}{2k\mathrm{Ma}_{E1}^2 - (k-1)} \Rightarrow \underline{\mathrm{Ma}_{E1} = 2.6457} \tag{1}$$

The ratio of the cross-sectional area at the diffuser exit to the cross-sectional area of the throat may be found using the isentropic sonic area ratio and the Mach number just upstream of the shock,

$$\frac{A_E}{A_T} = \frac{A_E}{A^*} = \frac{1}{\mathrm{Ma}_{E1}}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_{E1}^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \boxed{A_E/A_T = 3.0236} \tag{2}$$

Note that since the flow at the exit is supersonic, the throat must be at a sonic Mach number.

The pressure ratio, p_b/p_{01}, is given by,

$$\frac{p_b}{p_{01}} = \left(\frac{p_b}{p_{E2}}\right)\left(\frac{p_{E2}}{p_{E1}}\right)\left(\frac{p_{E1}}{p_{01}}\right) \Rightarrow \boxed{p_b/p_{01} = 0.3736} \tag{3}$$

where

$$\frac{p_b}{p_{E2}} = 1 \text{ (since } \mathrm{Ma}_{E2} < 1) \tag{4}$$

$$\frac{p_{E2}}{p_{E1}} = \frac{2k}{k+1}\mathrm{Ma}_{E1}^2 - \frac{k-1}{k+1} \Rightarrow p_{E2}/p_{E1} = 7.9997 \text{ (normal shock relations)} \tag{5}$$

$$\frac{p_{E1}}{p_{01}} = \left(1+\frac{k-1}{2}\mathrm{Ma}_{E1}^2\right)^{\frac{k}{1-k}} \Rightarrow p_{E1}/p_{01} = 0.0467 \text{ (isentropic stagnation pressure ratio)} \tag{6}$$

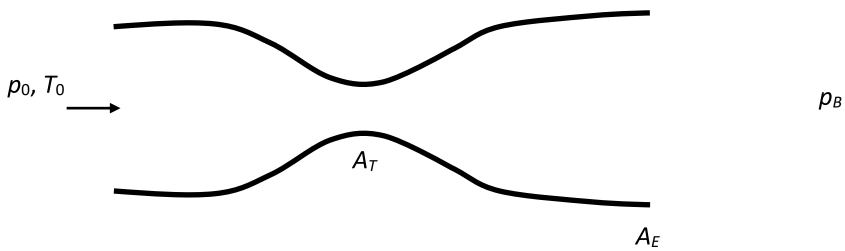
The pressure ratio, p_b/p^*, is given by,

$$\frac{p_b}{p^*} = \left(\frac{p_b}{p_{01}}\right)\left(\frac{p_{01}}{p^*}\right) \Rightarrow \boxed{p_b/p^* = 0.7071} \tag{7}$$

where

$$\frac{p^*}{p_{01}} = \left(1+\frac{k-1}{2}\right)^{\frac{k}{1-k}} \Rightarrow p^*/p_{01} = 0.5283 \text{ (isentropic stagnation pressure ratio)} \tag{8}$$

Consider the flow of air through the converging-diverging nozzle shown in the figure below. The flow begins at stagnation conditions with p_0 = 100 kPa (abs) and T_0 = 300 K. The nozzle exit-to-throat area ratio is A_E/A_T = 1.688 with a throat area of $A_T = 1.0*10^{-4}$ m^2.



a. Determine the back pressure at which the flow first becomes choked.
b. Determine the *range* of back pressures at which the flow at the exit is supersonic.
c. Determine the mass flow rate through the nozzle when the exit Mach number is 0.2.

SOLUTION:
The flow first becomes choked when the Mach number at the throat is equal to one ($A_T = A^*$) then goes back to subsonic conditions. The area ratio for this case is

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = 1.688 = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{k-1}{2}\text{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k-1}{2(k+1)}} \underset{\text{Ma}_E<1}{\Rightarrow} \text{Ma}_E = 0.3721 \,. \tag{1}$$

where k = 1.4. Since the flow is entirely isentropic, the back pressure ratio corresponding to this Mach number may be found using,

$$\frac{p_E}{p_0} = \left(1+\tfrac{k-1}{2}\text{Ma}_E^2\right)^{\frac{k}{1-k}} \quad \Rightarrow \; p_E/p_0 = 0.9088 \tag{2}$$

Since the exit Mach number is subsonic, the exit pressure and back pressure are the same. Hence,

$$p_B = p_E. \tag{3}$$

Using the given inlet stagnation pressure and Eqs. (2) and (3),

$$\boxed{p_B = 90.8 \text{ kPa (abs)}} \tag{4}$$

The flow at the exit will be supersonic for back pressures less than the case when a normal shock wave stands at the nozzle exit. The back pressure at which a normal shock stands at the exit may be found by noting that the flow upstream of the exit will be entirely isentropic (and choked), with the Mach number just upstream of the shock at the exit being supersonic. Hence,

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = 1.688 = \frac{1}{\text{Ma}_{E1}}\left(\frac{1+\frac{k-1}{2}\text{Ma}_{E1}^2}{1+\frac{k-1}{2}}\right)^{\frac{k-1}{2(k+1)}} \underset{\text{Ma}_{E1}>1}{\Rightarrow} \text{Ma}_{E1} = 2.000 \,. \tag{5}$$

The pressure ratio just upstream of the shock at the exit may be found from the isentropic relations,

$$\frac{p_{E1}}{p_{01}} = \left(1+\tfrac{k-1}{2}\text{Ma}_{E1}^2\right)^{\frac{k}{1-k}} \;\Rightarrow\; p_{E1}/p_{01} = 0.1278 \;\Rightarrow\; p_{E1} = 12.78 \text{ kPa (abs)} \tag{6}$$

where p_{01} is the upstream stagnation pressure (the stagnation pressure decreases across the shock).

The static pressure ratio across the shock may be found using the normal shock relations,

$$\frac{p_{E2}}{p_{E1}} = \frac{2k}{k+1}\text{Ma}_{E1}^2 - \frac{k-1}{k+1} \;\Rightarrow\; p_{E2}/p_{E1} = 4.500 \tag{7}$$

so that the pressure just downstream of the shock is,

$$p_{E2} = \left(\frac{p_{E2}}{p_{E1}}\right)\left(\frac{p_{E1}}{p_{01}}\right)p_{01} \;\Rightarrow\; p_{E2} = 57.51 \text{ kPa (abs)}. \tag{8}$$

Note that the Mach number just downstream of the exit will be subsonic, so the downstream pressure will be equal to the back pressure. Hence,

$$p_B = p_{E2} \,. \tag{9}$$

Thus, the range of back pressures for which the exit Mach number will be supersonic is,

$$\boxed{p_B < 57.51 \text{ kPa (abs).}} \tag{10}$$

Note that the Mach number downstream of the shock wave is,

$$\text{Ma}_{E2}^2 = \frac{(k-1)\text{Ma}_{E1}^2 + 2}{2k\text{Ma}_{E1}^2 - (k-1)} \quad \Rightarrow \text{Ma}_{E2} = 0.5774 \tag{11}$$

and the stagnation pressure ratio across the shock wave is,

$$\frac{p_{02}}{p_{01}} = \left[\frac{(k+1)\text{Ma}_{E1}^2}{2+(k-1)\text{Ma}_{E1}^2}\right]^{k/(k-1)} \left[\frac{k+1}{2k\text{Ma}_{E1}^2 - (k-1)}\right]^{1/(k-1)} \Rightarrow p_{02}/p_{01} = 0.7209 \Rightarrow p_{02} = 72.1 \text{ kPa (abs)} \tag{12}$$

The isentropic stagnation pressure ratio at the downstream side of the shock is,

$$\frac{p_{E2}}{p_{02}} = \left(1 + \tfrac{k-1}{2}\text{Ma}_{E2}^2\right)^{\frac{k}{1-k}} \Rightarrow p_{E2}/p_{02} = 0.7978, \tag{13}$$

The back pressure for this case can be found by combining relations in the following manner,

$$p_b = p_{E2} = \left(\frac{p_{E2}}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right)p_{01} = (0.7978)(0.7209)(100 \text{ kPa}) = 57.51 \text{ kPa} \,, \tag{14}$$

which is exactly the same result found in Eqs. (8) and (9).

Given that the flow chokes at an exit Mach number of 0.3721 (found from Eq. (1)), the flow in the device must be entirely subsonic when the exit Mach number is 0.2. Thus, the mass flow rate may be found from the isentropic relations evaluated at the exit,

$$\dot{m} = \rho_E V_E A_E \,, \tag{15}$$

where,

$$\rho_E = \rho_0 \left(1 + \tfrac{k-1}{2}\text{Ma}_E^2\right)^{\frac{1}{1-k}} \text{ and } \rho_0 = \frac{p_0}{RT_0} \,, \tag{16}$$

$$V_E = c_E \text{Ma}_E = \sqrt{kRT_E}\,\text{Ma}_E \,, \tag{17}$$

$$T_E = T_0 \left(1 + \tfrac{k-1}{2}\text{Ma}_E^2\right)^{-1} , \tag{18}$$

$$A_E = \left(\frac{A_E}{A_T}\right) A_T \,. \tag{19}$$

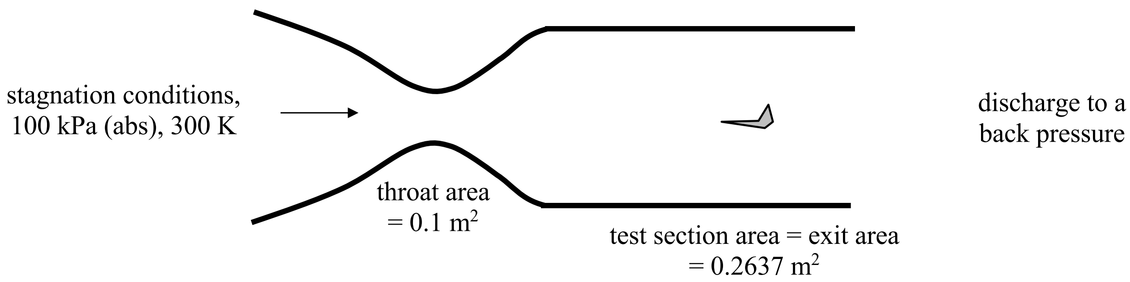
Using the given data,

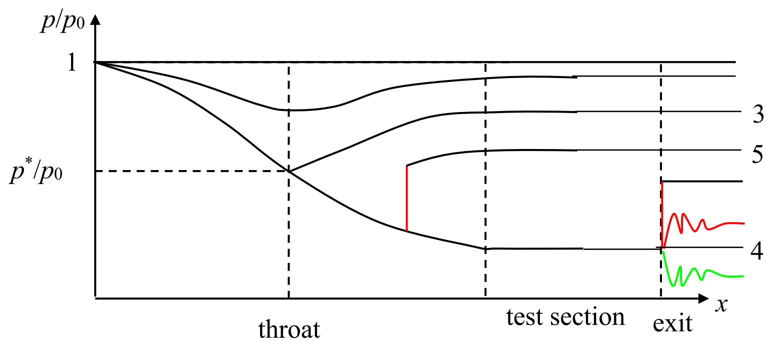
$\Rightarrow$ $\rho_0 = 1.161 \text{ kg/m}^3$
$\rho_E = 1.139 \text{ kg/m}^3$
$T_E = 297.6$ K
$c_E = 345.8$ m/s
$V_E = 69.16$ m/s
$A_E = 1.69 \times 10^{-4} \text{ m}^2$

$$\Rightarrow \boxed{\dot{m} = 1.33 \times 10^{-2} \text{ kg/s}} \tag{20}$$

Note that this mass flow rate is less than the choked flow mass flow rate (since the flow isn't choked).

Consider the supersonic wind tunnel shown in the following schematic. Air is the working fluid and the test section area is constant.





a. What is the design Mach number of the test section?

SOLUTION:

The test section design Mach number may be found using the isentropic sonic area ratio and choosing the supersonic test section Mach number (case 4 in the diagram above),

$$\frac{A_{TS}}{A_T} = \frac{0.2637\text{ m}^2}{0.1\text{ m}^2} = 2.637 = \frac{A_{TS}}{A^*} = \frac{1}{\text{Ma}_{TS}}\left(\frac{1+\frac{k-1}{2}\text{Ma}_{TS}^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \boxed{\text{Ma}_{TS} = 2.50}. \quad (1)$$

Note that at design conditions, the throat Mach number is one.

b. What is the mass flow rate through the wind tunnel at design conditions?

SOLUTION:

The flow through the wind tunnel will be choked at design conditions, with a mass flow rate of,

$$\dot{m}_{\text{choked}} = \left(1+\frac{k-1}{2}\right)^{\frac{k+1}{2(1-k)}} p_0\sqrt{\frac{k}{RT_0}}A^* \Rightarrow \boxed{\dot{m} = 23.3\text{ kg/s}}, \quad (2)$$

where $A^* = A_T$.

c. What is the maximum back pressure at which the throat will reach sonic conditions?

SOLUTION:

When the throat just reaches sonic conditions (case 3 in the diagram above), the throat area will equal the sonic area ($A^* = A_T$) and the exit Mach number may be found using the isentropic sonic area ratio since the flow through the entire converging-diverging nozzle will be subsonic (no shock waves),

$$\frac{A_E}{A_T} = \frac{0.2637\text{ m}^2}{0.1\text{ m}^2} = 2.637 = \frac{A_E}{A^*} = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{k-1}{2}\text{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \text{Ma}_E = 0.2263. \tag{3}$$

The exit pressure may be found from this Mach number using the isentropic stagnation pressure ratio,

$$\frac{p_E}{p_0} = \left(1+\frac{k-1}{2}\text{Ma}_E^2\right)^{\frac{k}{1-k}} \Rightarrow p_E/p_0 = 0.9650 \Rightarrow p_E = 96.5\text{ kPa (abs)}, \tag{4}$$

using $p_0 = 100$ kPa (abs). Since the exit Mach number is subsonic, the exit and back pressures are equal. Hence,

$$\boxed{p_B = p_E = 96.5\text{ kPa (abs)}}. \tag{5}$$

d. Assume a shock wave stands in the diverging section where the area is 0.1688 m^2. What is the back pressure at these conditions?

SOLUTION:

The Mach number just upstream of the shock wave may be found using the isentropic sonic area ratio since the flow leading up to the shock wave is isentropic and the throat area is at sonic conditions (since shock waves only form in supersonic flows, case 5 in the diagram shown above),

$$\frac{A_1}{A_T} = \frac{0.1688\text{ m}^2}{0.1\text{ m}^2} = 1.688 = \frac{A_1}{A^*} = \frac{1}{\text{Ma}_1}\left(\frac{1+\frac{k-1}{2}\text{Ma}_1^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \text{Ma}_1 = 2.00. \tag{6}$$

The stagnation pressure ratio and sonic area ratio across the shock are,

$$\frac{p_{02}}{p_{01}} = \frac{A_1^*}{A_2^*} = \left[\frac{(k+1)\text{Ma}_1^2}{2+(k-1)\text{Ma}_1^2}\right]^{k/(k-1)}\left[\frac{k+1}{2k\text{Ma}_1^2-(k-1)}\right]^{1/(k-1)} \Rightarrow p_{02}/p_{01} = A^*_1/A^*_2 = 0.7209. \tag{7}$$

The flow downstream of the shock wave is isentropic and subsonic. Thus, the pressure at the exit may be found

$$\frac{A_E}{A_2^*} = \frac{A_E}{A_1^*}\frac{A_1^*}{A_2^*} = \frac{A_E}{A_T}\frac{A_1^*}{A_2^*} = \left(\frac{0.2637\text{ m}^2}{0.1\text{ m}^2}\right)(0.7209) = 1.901 = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{k-1}{2}\text{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \text{Ma}_E = 0.3240. \tag{8}$$

The exit pressure may be found from the isentropic stagnation pressure ratio downstream of the shock and the exit Mach number,

$$\frac{p_{TS}}{p_{02}} = \left(1+\frac{k-1}{2}\text{Ma}_{TS}^2\right)^{\frac{k}{1-k}} \Rightarrow p_{TS}/p_{02} = 0.9299. \tag{9}$$

Accounting for the change in stagnation pressure ratio across the shock,

$$p_{TS} = \left(\frac{p_{TS}}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right)p_{01} = (0.9299)(0.7209)(100\text{ kPa}) \Rightarrow p_{TS} = 67.03\text{ kPa (abs)} \tag{10}$$

Since the exit is at a subsonic Mach number, the exit and back pressures are equal,

$$\boxed{p_B = p_E = 67.0\text{ kPa (abs)}}. \tag{11}$$