## 2.4. Buoyant Force and Center of Buoyancy

When an object is submerged in a fluid, the pressure acting on the object deeper in the fluid (i.e., in the direction of gravity) will be larger than the pressure acting on the object shallower in the fluid. As a result, there will be a net pressure force acting on the object. This net pressure force is known as the buoyant force.

To derive the value of the buoyant force, consider the vertical pressure forces acting on a narrow cylinder with cross-sectional area $dA$ within a fully-submerged object as shown in Figure 2.17.

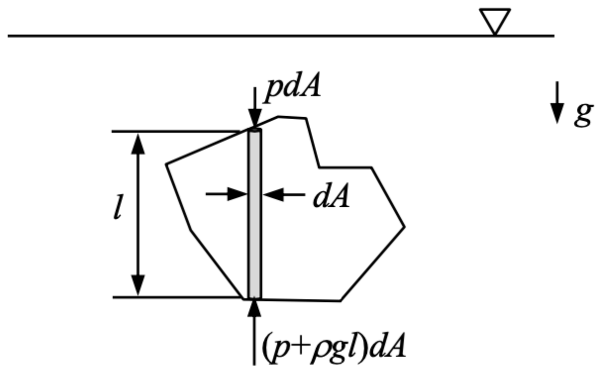



FIGURE 2.17. Pressure forces on a thin cylinder of cross-sectional area $dA$ and height $l$ from within a fully-submerged object.

The net pressure force in the vertical direction on the narrow cylinder, assuming an incompressible fluid, is,

$$dF_{p,\text{net}} = (p + \rho_{\text{fluid}} g l)dA - pdA = \rho_{\text{fluid}} g l dA \quad \text{(acting opposite to gravity).} \tag{2.78}$$

The total net pressure force acting on the object is found by integrating these small bits of pressure force over the entire cross-sectional area of the object,

$$F_{p,\text{net}} = \int_A dF_{p,\text{net}} = \int_A \rho_{\text{fluid}} g l dA \quad \text{(acting opposite to gravity).} \tag{2.79}$$

Since the density and gravity are assumed constant here, they may be pulled outside the integral,

$$F_{p,\text{net}} = \rho_{\text{fluid}} g \int_A l dA \quad \text{(acting opposite to gravity),} \tag{2.80}$$

$$= \rho_{\text{fluid}} g \int_V dV \quad \text{(acting opposite to gravity),} \tag{2.81}$$

$$\boxed{F_{p,\text{net}} = F_B = \rho_{\text{fluid}} g V_{\substack{\text{submerged}\\ \text{object}}}} \quad \text{(acting opposite to gravity),} \tag{2.82}$$

where the integral is simply the volume of the submerged object. This net pressure force acting on the object is referred to as the buoyant force.

*Notes:*

(1) Equation (2.82) states that the buoyant force is equal to the weight of the fluid that's been displaced by the submerged object. This relationship is also known as Archimede's Principle.

(2) The same analysis can be used for partially submerged objects. In that case, the pressure acting on the top of the object is atmospheric pressure while the pressure at the bottom is $p_{\text{atm}} + \rho_{\text{fluid}} g l'$, where $l'$ is the length of the narrow cylinder that's submerged in the fluid (Figure 2.18). After integrating over the objects cross-sectional area (similar to Eq. (2.80), we would arrive at exactly the same relation as in Eq. (2.82) except that the $V_{\text{submerged object}}$ refers to just that volume that is submerged in the fluid.

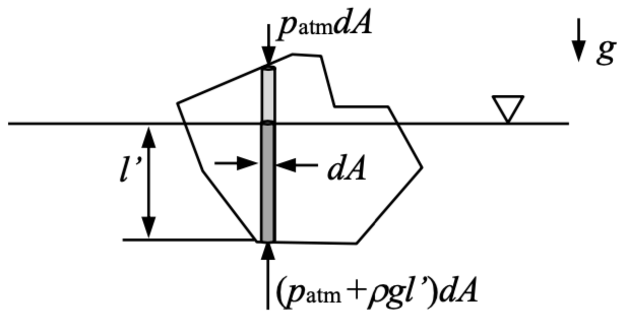



FIGURE 2.18. Pressure forces on a thin cylinder of cross-sectional area $dA$. The depth of the cylinder below the free surface is $l'$.

(3) There is no net pressure force on the object in the directions perpendicular to gravity since the pressure only varies parallel to the gravitational vector.

The resultant buoyant force acts at the center of buoyancy. The center of buoyancy is found by equating the moment caused by the resultant buoyant force acting at the center of buoyancy to the distributed moment caused by the distributed pressure forces. Consider moments about the $z$ axis in Figure 2.19.

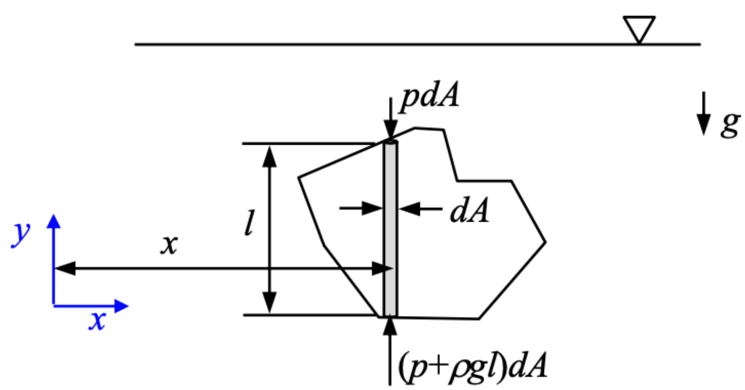



FIGURE 2.19. Moments about the $z$ axis due to the pressure forces acting on a thin cylinder of cross-sectional area $dA$ and height $l$ from within a fully-submerged object.

$$x_{CB}\hat{\boldsymbol{i}} \times \underbrace{\rho g V \hat{\boldsymbol{j}}}_{\substack{\text{buoyant}\\ \text{force}}} = \int_A x\hat{\boldsymbol{i}} \times \underbrace{\rho g l dA \hat{\boldsymbol{j}}}_{\substack{\text{net pressure force}\\ \text{on cylinder}}}, \tag{2.83}$$

$$x_{CB}\rho g V\hat{\boldsymbol{k}} = \rho g\hat{\boldsymbol{k}} \int_A x dV \quad (dV = l dA), \tag{2.84}$$

$$x_{CB} = \frac{1}{V}\int_V x dV, \tag{2.85}$$

which is the center of displaced volume. Performing similar analyses about the $x$ and $y$ axes produces similar results. Thus, *the center of buoyancy is located at the center of the displaced volume.* This is true for both fully submerged and partially submerged objects.

### 2.5. Stable Orientation of a Submerged Object

Submerged objects will be in an equilibrium orientation when the forces acting on the object are such that there is no net moment on the object. Considering only the object weight and a buoyant force, an equilibrium orientation will only occur when the two forces are co-linear, as shown in Figure 2.20. Neither object experiences a net moment.

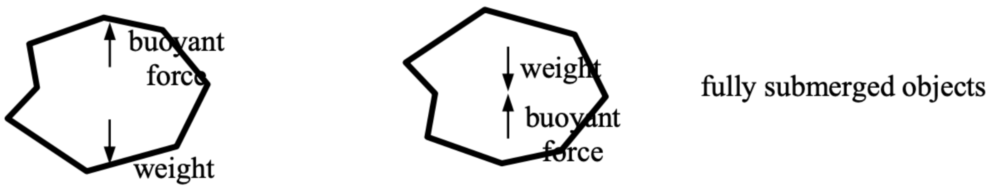



Figure 2.20. The buoyant force, acting at the center of buoyancy, and weight, acting at the center of gravity, for two fully-submerged objects in equilibrium. The object on the left is stable, but the object on the right is unstable.

The object on the left is in a stable equilibrium while the object on the right is in an unstable equilibrium. The reason for the difference is that if each object is rotated slightly, the object on the left will experience a moment that restores it back to its original configuration. However, a small perturbation to the right-hand object will result in a moment that will cause the object to move away from its initial configuration.

The stability of partially submerged objects is a particularly important topic when considering the design of ships. The Swedish ship *Vasa* is a famous example of a ship that was unstable and "turtled" shortly after setting sail for the first time. Unfortunately, stability analysis of partially submerged objects can be complicated since the submerged volume and center of buoyancy changes as the orientation of the object rotates. For example, consider the stability of the simple shape shown in Figure 2.21 (CG is the center of gravity and CB is the center of buoyancy). The initial configuration of the object (on the left) appears to be

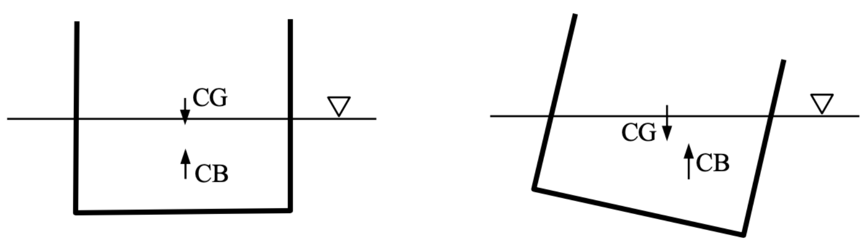



Figure 2.21. The center of buoyancy and center of gravity for a partially-submerged object. The center of buoyancy changes location as the submerged volume changes.

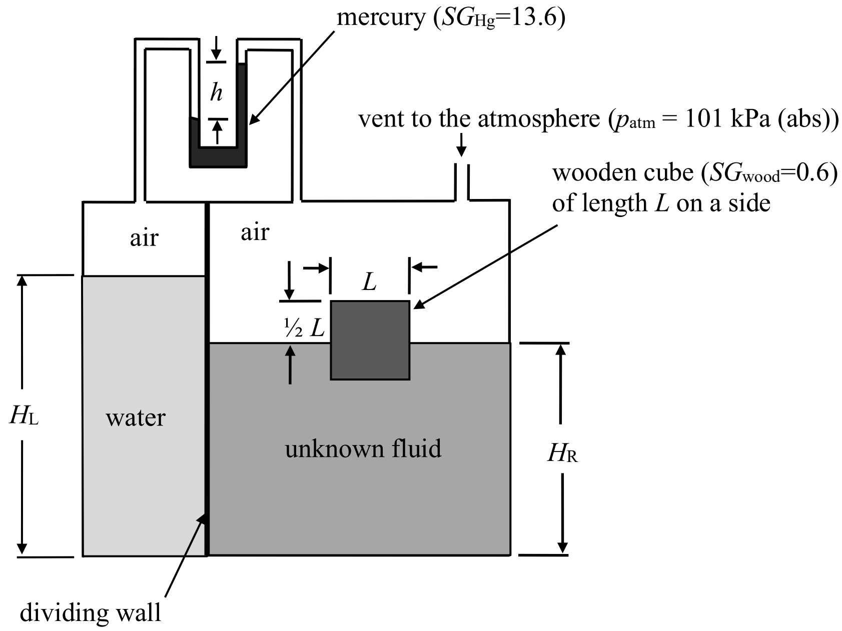
in unstable equilibrium with the center of gravity above the center of buoyancy. However, when the object is tilted (on the right), the center of buoyancy shifts to one side such that it acts to restore the object to its initial configuration. Hence, the object is actually initially in stable equilibrium.

A tank is divided by a wall into two independent chambers. The left chamber is filled to a depth of $H_L$=6m with water ($\rho_{H20}$=1000 kg/m$^3$) and the right side if filled to a depth of $H_R$=5m with an unknown fluid. A wooden cube ($SG_{wood}$=0.6) with a length of $L$=0.20m on each side floats half submerged in the unknown fluid. Air ($\rho_{air}$=1.2 kg/m$^3$) fills the remainder of the container above each fluid. The right container has a pipe that is vented to the atmosphere while the left container is sealed from the atmosphere. A manometer using mercury as the gage fluid ($SG_{Hg}$=13.6) connects the two chambers and indicates that $h$=0.150 m.

a. Determine the density of the unknown fluid.

b. Determine the magnitude of the force (per unit depth into the page) acting on the dividing wall due to the unknown fluid.

c. Determine the magnitude of the force (per unit depth into the page) acting on the dividing wall due to the water.




SOLUTION:

Balance forces on the wooden cube.

$$\sum F_y = 0 = \rho_{\text{fluid}} g \left(\tfrac{1}{2} L\right) L^2 - \rho_{\text{wood}} g L^3 \tag{1}$$

$$\boxed{\therefore \rho_{\text{fluid}} = 2\rho_{\text{wood}} = 2SG_{\text{wood}}\rho_{H_2O}} \tag{2}$$

Using the given data:

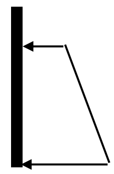
$SG_{wood}$ = 0.6

$\rho_{H2O}$ = 1000 kg/m$^3$

$\Rightarrow$ $\boxed{\rho_{fluid} = 1200 \text{ kg/m}^3}$

Now determine the force acting on the wall due to the unknown fluid.

$$F_{p,R} = \int_{y=0}^{y=H_R} \underbrace{\left(p_{\text{atm}} + \rho_{\text{fluid}} g y\right)}_{=p\ (\text{abs})} \underbrace{dy(1)}_{=dA} \tag{3}$$



$$\boxed{\therefore F_{p,R} = p_{\text{atm}} H_R + \tfrac{1}{2}\rho_{\text{fluid}} g H_R^2} \tag{4}$$

Using the given data:

$p_{\text{atm}}$ = 101 kPa (abs)
$H_R$ = 5 m
$\rho_{\text{fluid}}$ = 1200 kg/m$^3$
$g$ = 9.81 m/s$^2$
$\Rightarrow$ $\boxed{F_{p,R} = 506\ \text{kN/m}}$

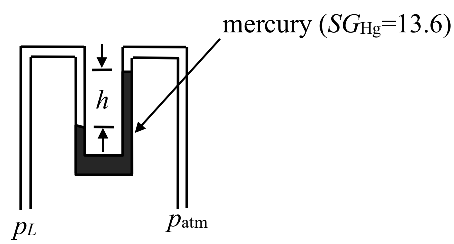
Now find the pressure force due to the water.

$$F_{p,L} = \int_{y=0}^{y=H_L} \underbrace{\left(p_L + \rho_{H_2O} g y\right)}_{=p\ (\text{abs})} \underbrace{dy(1)}_{=dA} \tag{5}$$

$$\therefore F_{p,L} = p_L H_L + \tfrac{1}{2}\rho_{H_2O} g H_L^2 \tag{6}$$

where $p_L$ is the (absolute) pressure acting on the free surface of the water. This pressure may be found using the manometer.

$$p_L = p_{\text{atm}} + \rho_{\text{Hg}} g h = p_{\text{atm}} + SG_{\text{Hg}} \rho_{H_2O} g h \tag{7}$$



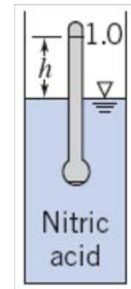

Substitute Eqn. (7) into Eqn. (6).

$$\boxed{\therefore F_{p,L} = \left(p_{\text{atm}} + SG_{\text{Hg}} \rho_{H_2O} g h\right) H_L + \tfrac{1}{2}\rho_{H_2O} g H_L^2} \tag{8}$$

Using the given data:

$p_{\text{atm}}$ = 101 kPa (abs)
$SG_{\text{Hg}}$ = 13.6
$\rho_{H2O}$ = 1000 kg/m$^3$
$g$ = 9.81 m/s$^2$
$h$ = 0.150 m
$H_L$ = 6 m
$\Rightarrow$ $\boxed{F_{p,L} = 903\ \text{kN/m}}$

A hydrometer is a specific gravity indicator, the value being indicate by the level at which the free surface intersects the stem when floating in a liquid. The 1.0 mark is the level when in distilled water. For the unit shown, the immersed volume in distilled water is 15 cm$^3$. The stem is 6 mm in diameter. Find the distance, $h$, from the 1.0 mark to the surface when the hydrometer is placed in a nitric acid solution of specific gravity 1.5.




SOLUTION:
Since the hydrometer is in equilibrium, its weight and the buoyant force should equal each other. When submerged in distilled water,

$$W = \rho_{H2O} g V_{disp,H2O} = \rho_{H2O} g h \frac{\pi}{4} d^2 \;\Rightarrow\; h = \frac{W}{\rho_{H2O} g \frac{\pi}{4} d^2}, \quad (1)$$

where $A$ is the hydrometer's cross-sectional area. The height $h$ is the location where the mark is made for distilled water.

When submerged in nitric acid,

$$W = \rho_{HNO_3} g \frac{\pi}{4} d^2 (h + \Delta h) \;\Rightarrow\; h + \Delta h = \frac{W}{\rho_{HNO_3} g \frac{\pi}{4} d^2} = \frac{W}{SG_{HNO_3} \rho_{H2O} g \frac{\pi}{4} d^2}. \quad (2)$$

Combining Eqs. (1) and (2),

$$\frac{W}{\rho_{H2O} g \frac{\pi}{4} d^2} + \Delta h = \frac{W}{SG_{HNO_3} \rho_{H2O} g \frac{\pi}{4} d^2}, \quad (3)$$

$$\Delta h = \frac{W}{SG_{HNO_3} \rho_{H2O} g \frac{\pi}{4} d^2} - \frac{W}{\rho_{H2O} g \frac{\pi}{4} d^2}, \quad (4)$$

$$\Delta h = \frac{W}{\rho_{H2O} g \frac{\pi}{4} d^2} \left( \frac{1}{SG_{HNO_3}} - 1 \right), \quad (5)$$

$$\Delta h = \frac{\rho_{H2O} g V_{disp,H2O}}{\rho_{H2O} g \frac{\pi}{4} d^2} \left( \frac{1}{SG_{HNO_3}} - 1 \right), \quad (6)$$

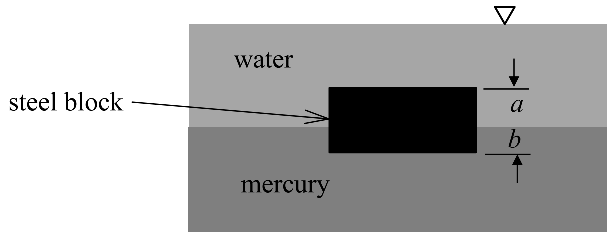
$$\Delta h = \frac{V_{disp,H2O}}{\frac{\pi}{4} d^2} \left( \frac{1}{SG_{HNO_3}} - 1 \right). \quad (7)$$

Using the given data,

$V_{disp,H2O}$ = 15 cm$^3$,
$d$ = 0.6 cm,
$SG_{HNO3}$ = 1.5,
=> $\boxed{\Delta h = -17.7 \text{ cm}}$.

The hydrometer moves upward a distance of 17.7 cm from where the distilled water mark is located.

A uniform block of steel (with a specific gravity of 7.85) will "float" at a mercury-water interface as shown in the figure. What is the ratio of the distances *a* and *b*?




SOLUTION:

Balance forces in the vertical direction,

$$\sum F_V = 0 = -W_{\text{block}} + F_{B,Hg} + F_{B,H_2O} = -\rho_{\text{block}} V_{\text{block}} g + \rho_{Hg} V_{\substack{\text{block,}\\ \text{in } Hg}} g + \rho_{H_2O} V_{\substack{\text{block,}\\ \text{in } H_2O}} g \,, \tag{1}$$

where the buoyant forces are equal to the weights of the displaced fluids.

Re-writing in terms of the lengths *a* and *b* and the block's cross-sectional area $A_{\text{block}}$,

$$-\rho_{\text{block}} A_{\text{block}} (a+b) + \rho_{Hg} A_{\text{block}} b + \rho_{H_2O} A_{\text{block}} a = 0 \,, \tag{2}$$

$$-\rho_{\text{steel}} (a+b) + \rho_{Hg} b + \rho_{H_2O} a = 0 \,, \tag{3}$$

$$-\rho_{H_2O} SG_{\text{steel}} b \left( \frac{a}{b} + 1 \right) + \rho_{H_2O} SG_{Hg} + \rho_{H_2O} b \frac{a}{b} = 0 \,, \tag{4}$$

$$-SG_{\text{steel}} \left( \frac{a}{b} + 1 \right) + SG_{Hg} + \frac{a}{b} = 0 \,, \tag{5}$$

$$\boxed{\frac{a}{b} = \frac{SG_{Hg} - SG_{\text{steel}}}{SG_{\text{steel}} - 1}} \,. \tag{6}$$

Using the given data,

$SG_{Hg}$ = 13.6

$SG_{\text{steel}}$ = 7.85

$\Rightarrow$ $\boxed{a/b = 0.83}$

Note that we could also solve this problem by balancing the block's weight with the pressure forces acting on the top and bottom block surfaces.

$$\sum F_V = 0 = -W_{\text{block}} + F_{p,H_2O} + F_{p,Hg} = -\rho_{\text{block}} A_{\text{block}} (a+b) g - \rho_{H_2O} g (H-a) A_{\text{block}} + \left( \rho_{H_2O} g H + \rho_{Hg} g b \right) A_{\text{block}} \,, \tag{7}$$

where *H* is the depth of the water-mercury interface. Simplifying this equation gives,

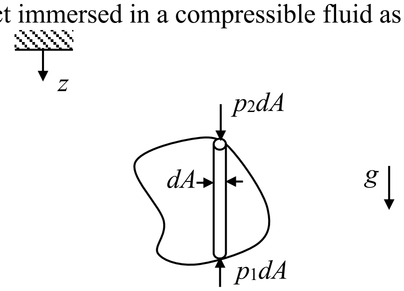
$$-\rho_{\text{block}} (a+b) - \rho_{H_2O} (H-a) + \rho_{H_2O} H + \rho_{Hg} b = 0 \,, \tag{8}$$

$$-\rho_{\text{block}} (a+b) + \rho_{H_2O} a + \rho_{Hg} b = 0 \,, \tag{9}$$

which is exactly the same as Eq. (3).

Archimedes principle states that the buoyant force acting on a submerged object is equal to the weight of the fluid displaced by that object. Is this true for *compressible* fluids?

SOLUTION:
Consider an arbitrary object immersed in a compressible fluid as shown in the figure below.




Determine the net pressure force acting on a parallelpiped of the material with a differential cross-sectional area,

$$dF_P = \left(p_1 - p_2\right)dA, \tag{1}$$

where,

$$p_1 = p_{z=0} + \int_{z=0}^{z=z_1} \rho g\, dz\,, \tag{2}$$

and,

$$p_2 = p_{z=0} + \int_{z=0}^{z=z_2} \rho g\, dz\,, \tag{3}$$

where $\rho$ is the density of the fluid (not the object).

Equation (1) becomes,

$$dF_P = \left( p_{z=0} + \int_{z=0}^{z=z_1} \rho g\, dz - p_{z=0} - \int_{z=0}^{z=z_2} \rho g\, dz \right) dA\,, \tag{4}$$

$$dF_P = dA \int_{z=z_2}^{z=z_1} \rho g dz\,. \tag{5}$$

The net pressure force acting over the entire object, i.e., the buoyant force, is,

$$F_P = \int_A dF_P = \int_A \int_{z=z_2}^{z=z_1} \rho g dz dA\,. \tag{6}$$

Assuming that the gravitational acceleration is constant (usually a good assumption),

$$F_P = g \int_A \int_{z=z_2}^{z=z_1} \rho\, dz dA\,, \tag{7}$$

Note that the integrals in the previous equation give the mass of the fluid displaced by the object, i.e.,

$$M_{\substack{\text{fluid displaced}\\ \text{by object}}} = \int_A \int_{z=z_2}^{z=z_1} \rho\, dz dA\,. \tag{8}$$

Thus, just as with the incompressible case, <u>the buoyant force in a compressible fluid is equal to the weight of the fluid displaced by the object</u>,

$$F_p = M_{\substack{\text{fluid displaced}\\ \text{by object}}} g\,. \tag{9}$$

Consider an ice cube with initial volume $V_{\text{ice},0}$ floating in a cup of water of initial volume $V_{\text{water},0}$. The specific gravity of ice is $SG_{\text{ice}}$. Show mathematically that, as the ice cube melts, the water level in the cup remains unchanged.

SOLUTION:
If a mass of ice, $\Delta m_{\text{ice}}$, melts ($\Delta m_{\text{ice}} < 0$), it will correspond to an equal increase in water, $\Delta m_{\text{water}}$, i.e.,

$$\Delta m_{\text{water}} = -\Delta m_{\text{ice}} \,. \tag{1}$$

Expressed in terms of volumes,

$$\rho_{\text{water}} \Delta V_{\text{water}} = -\rho_{\text{ice}} \Delta V_{\text{ice}} = -SG_{\text{ice}} \rho_{\text{water}} \Delta V_{\text{ice}} \,, \tag{2}$$

$$\Delta V_{\text{water}} = -SG_{\text{ice}} \Delta V_{\text{ice}} \,. \tag{3}$$

The volume of water displaced by the ice is found by equating the weight of the displaced water to the weight of the ice (Archimedes Law),

$$\rho_{\text{water}} V_{\text{water,disp}} g = \rho_{\text{ice}} V_{\text{ice}} g = SG_{\text{ice}} \rho_{\text{water}} V_{\text{ice}} g \,, \tag{4}$$

$$V_{\text{water,disp}} = SG_{\text{ice}} V_{\text{ice}} \,. \tag{5}$$

Thus, if a volume of ice melts, $\Delta V_{\text{ice}}$, then the amount of water displaced, in order to balance the new ice weight, is,

$$\Delta V_{\text{water,disp}} = SG_{\text{ice}} \Delta V_{\text{ice}} \,. \tag{6}$$

Note that if the ice melts ($\Delta V_{\text{ice}} < 0$) , less water needs to be displaced to support the (smaller) ice weight ($\Delta V_{\text{water,disp}} < 0$).

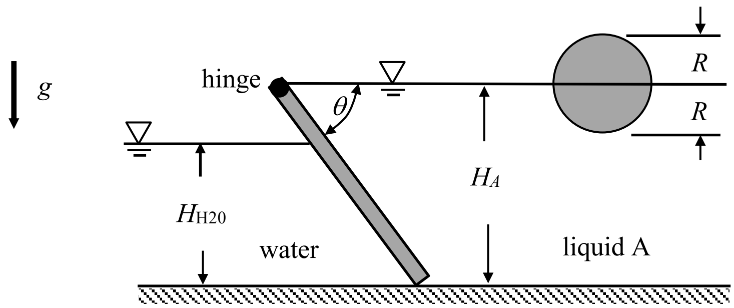
Thus, the sum of the volume of water added due to melting and the change in displaced water volume due to a change in the weight of the ice is,

$$\Delta V_{\text{water}} + \Delta V_{\text{water,disp}} = -SG_{\text{ice}} \Delta V_{\text{ice}} + SG_{\text{ice}} \Delta V_{\text{ice}} = 0 \,. \tag{7}$$

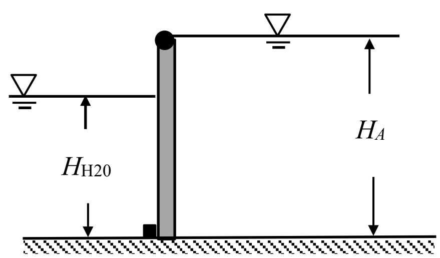
The increase in water volume is exactly balanced by a decrease in the displaced water volume, which means that the water level height won't change!

This fact has important implications regarding the rise in sea level due to melting ice. Melting free-floating ice, e.g., icebergs, won't result in an increase in sea level. However, ice that was originally supported by land, e.g., glaciers, will contribute to an increase in sea levels.

Consider the system shown below. A wooden sphere of radius $R$ and specific gravity $SG_{\text{wood}}$ is half submerged in an unknown liquid, referred to as liquid A. Liquid A, which has a depth $H_A$, is separated from a pool of water, which has a depth $H_{\text{H2O}}$, by a hinged gate tilted at an angle $\theta$ with respect to the horizontal. The gate has a width $b$ into the page.




a. What is the density of liquid A, $\rho_A$, in terms of the specific gravity of the wooden sphere ($SG_{\text{wood}}$) and the density of water ($\rho_{\text{H20}}$)?
b. What is the pressure force liquid A exerts on the inclined gate in terms of (a subset of) $\rho_A$, $H_A$, $g$, $b$, and $\theta$? Write this force as a vector.
c. Assuming the gate has negligible mass and the angle $\theta$ is 90° so the gate is vertical (figure shown below), at what height $H_{\text{H20}}$ will the gate just start to rotate about its hinge? Write this height in terms of (a subset of) $\rho_A$, $\rho_{\text{H20}}$, $H_A$, $g$, and $b$.



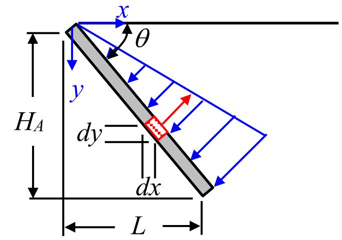

SOLUTION:

The density of liquid A may be found by balancing the weight of the wooden sphere with the buoyant force acting on it,

$$F_W = F_B \Rightarrow \rho_{\text{wood}} \tfrac{4}{3}\pi R^3 g = \rho_A \underbrace{\tfrac{1}{2}\tfrac{4}{3}\pi R^3}_{\substack{\text{half of the}\\\text{sphere is}\\\text{submerged}}} g \Rightarrow \rho_A = 2\rho_{\text{wood}}\,, \tag{1}$$

$$\boxed{\rho_A = 2SG_{\text{wood}}\rho_{\text{H20}}}. \tag{2}$$

The force that liquid A exerts on the gate may be found by integrating pressure forces along the length of the gate,




$$\mathbf{F}_{A\text{ on gate}} = \int_A -p\,d\mathbf{A}\,, \tag{3}$$

where,

$$p = \rho_A g y \text{ (gage pressure)}, \tag{4}$$

$$d\mathbf{A} = bdy\hat{\mathbf{i}} - bdx\hat{\mathbf{j}}\,, \tag{5}$$

so that,

$$\mathbf{F}_{A\text{ on gate}} = \int_A -(\rho_A g y)\left(bdy\hat{\mathbf{i}} - bdx\hat{\mathbf{j}}\right) = \rho_A g b\left(-\hat{\mathbf{i}}\int_{y=0}^{y=H_A} y\,dy + \hat{\mathbf{j}}\int_{x=0}^{x=L} y\,dx\right). \tag{6}$$

Note that,

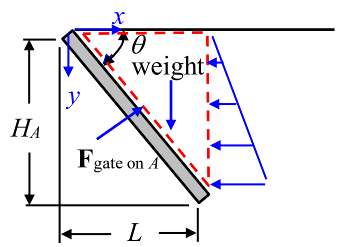
$$y = x\tan\theta \quad \text{and} \quad H_A = L\tan\theta, \tag{7}$$

so that Eq. (6) becomes,

$$\mathbf{F}_{A\text{ on gate}} = \rho_A g b\left(-\hat{\mathbf{i}}\int_{y=0}^{y=H_A} y\,dy + \hat{\mathbf{j}}\int_{x=0}^{x=H_A/\tan\theta} x\tan\theta\,dx\right), \tag{8}$$

$$\boxed{\mathbf{F}_{A\text{ on gate}} = -\tfrac{1}{2}\rho_A g b H_A^2\hat{\mathbf{i}} + \tfrac{1}{2}\rho_A g b\frac{H_A^2}{\tan\theta}\hat{\mathbf{j}}}. \tag{9}$$

Another approach to calculating the force on the gate is to balance forces on the triangular block of liquid shown in the figure below.

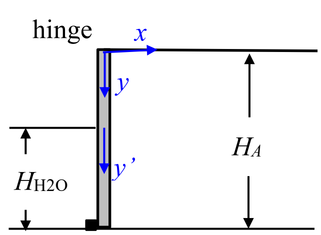



$$\sum\mathbf{F} = \mathbf{0} = \underbrace{\int_{y=0}^{y=H_A} -(\rho_A g y)\left(dyb\hat{\mathbf{i}}\right)}_{\text{pressure force on side of fluid block}} \underbrace{+\rho_A g\tfrac{1}{2}LH_A b\hat{\mathbf{j}}}_{\text{weight of fluid block}} + \underbrace{\mathbf{F}_{\text{gate on }A}}_{\text{force gate exerts on block}}, \tag{10}$$

$$\mathbf{F}_{\text{gate on }A} = \tfrac{1}{2}\rho_A g b H_A^2\hat{\mathbf{i}} - \tfrac{1}{2}\rho_A g\frac{H_A^2}{\tan\theta}b\hat{\mathbf{j}}, \tag{11}$$

where Eq. (7) has been used. Note that since $\mathbf{F}_{A\text{ on gate}} = -\mathbf{F}_{\text{gate on }A}$, the final result is the same as what was found in Eq. (9)!

For the specific case when $\theta = 90°$ (figure shown below), the moments about the hinge are,




$$\sum M_{\text{hinge},z} = 0 = -\int_{y'=0}^{y'=H_{H2O}}\left[y' + \left(H_A - H_{H2O}\right)\right]\left(\rho_{H2O} g y'\right)\left(bdy'\right) + \int_{y=0}^{y=H_A} y\left(\rho_A g y\right)\left(bdy\right), \tag{12}$$

$$0 = gb\left(-\rho_{H2O}\int_{y'=0}^{y=H_{H2O}}\left[y'^2 + \left(H_A - H_{H2O}\right)y'\right]dy' + \rho_A\int_{y=0}^{y=H_A} y^2\,dy\right), \tag{13}$$

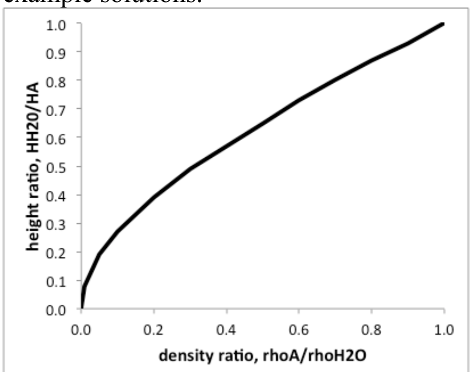
$$\rho_{H2O}\left[\tfrac{1}{3}H_{H2O}^3 + \tfrac{1}{2}\left(H_A - H_{H2O}\right)H_{H2O}^2\right] = \tfrac{1}{3}\rho_A H_A^3, \tag{14}$$

$$\left(\frac{H_{\text{H20}}}{H_A}\right)^3+\frac{3}{2}\left[1-\left(\frac{H_{\text{H20}}}{H_A}\right)\right]\left(\frac{H_{\text{H20}}}{H_A}\right)^2=\frac{\rho_A}{\rho_{\text{H20}}}, \tag{15}$$

$$\left(\frac{H_{\text{H20}}}{H_A}\right)^3+\frac{3}{2}\left(\frac{H_{\text{H20}}}{H_A}\right)^2-\frac{3}{2}\left(\frac{H_{\text{H20}}}{H_A}\right)^3=\frac{\rho_A}{\rho_{\text{H20}}}, \tag{16}$$

$$\boxed{\left(\frac{H_{\text{H20}}}{H_A}\right)^3-3\left(\frac{H_{\text{H20}}}{H_A}\right)^2+2\left(\frac{\rho_A}{\rho_{\text{H20}}}\right)=0}. \tag{17}$$

This equation could be solved numerically for $H_{\text{H20}}/H_A$ given a value for $\rho_A/\rho_{\text{H20}}$. The following plot shows example solutions.




An alternate approach for deriving Eq. (17) is to sum moments about the hinge, but make note of the fact that the center of pressure on each wall is one-third of the liquid depth from the bottom of the wall,

$$\sum M_{\text{hinge},z}=0=-\left[\left(H_A-H_{\text{H2O}}\right)+\tfrac{2}{3}H_{\text{H2O}}\right]\left(\tfrac{1}{2}\rho_{\text{H2O}}gbH_{\text{H2O}}^2\right)+\left(\tfrac{2}{3}H_A\right)\left(\tfrac{1}{2}\rho_A gbH_A^2\right), \tag{18}$$

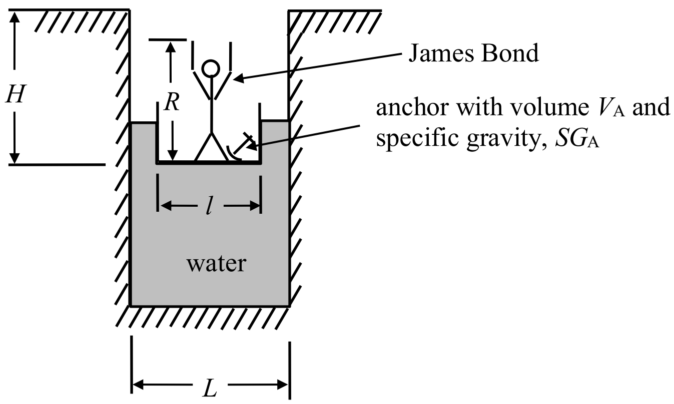
$$\tfrac{1}{2}\rho_{\text{H2O}}H_{\text{H2O}}^2H_A-\tfrac{1}{6}\rho_{\text{H2O}}H_{\text{H2O}}^3=\tfrac{1}{3}\rho_A H_A^3, \tag{19}$$

$$3\left(\frac{H_{\text{H2O}}}{H_A}\right)^2-\left(\frac{H_{\text{H2O}}}{H_A}\right)^3=2\left(\frac{\rho_A}{\rho_{\text{H2O}}}\right), \tag{20}$$

$$\left(\frac{H_{\text{H2O}}}{H_A}\right)^3-3\left(\frac{H_{\text{H2O}}}{H_A}\right)^2+2\left(\frac{\rho_A}{\rho_{\text{H2O}}}\right)=0, \tag{21}$$

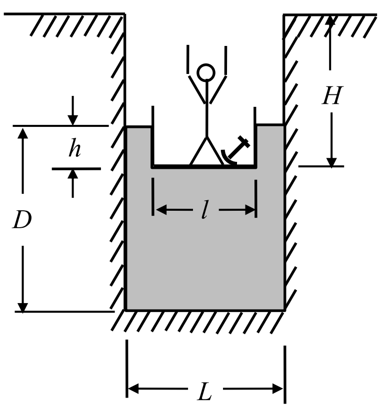
which is the same as Eq. (17).

James Bond is trapped on a small raft in a steep walled pit filled with water as shown in the figure. Both the raft and pit have square cross-sections with a side length of $l$=3 ft for the raft and $L$=4 ft for the pit. In the raft there is a steel anchor ($SG_A$=7.85) with a volume of $V_A$=1 ft$^3$. In the current configuration, the distance from the floor of the raft to the top of the pit is $H$=7.5 ft. Unfortunately, Bond can only reach a distance of $R$=7 ft from the floor of the raft. In order for Bond to escape, would it be helpful for him to toss the anchor overboard? Justify your answer with calculations. (Hint: The mass of water is conserved in this problem.)

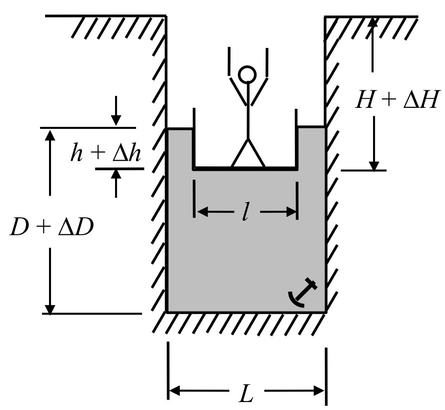



SOLUTION:

Consider the cases when the anchor is in the raft and out of the raft as shown in the figures below.




(a)




(b)

First consider the change in the position of the raft floor relative to the free surface of the water.

Case (a):
$$\underbrace{\left(m_{\text{raft+Bond}} + m_{\text{anchor}}\right)g}_{\text{weight of raft \& contents}} = \underbrace{\rho_{H_2O} g l^2 h}_{\text{weight of displaced water}} \tag{1}$$

Case (b):
$$\underbrace{\left(m_{\text{raft+Bond}}\right)g}_{\text{weight of raft \& contents}} = \underbrace{\rho_{H_2O} g l^2 \left(h + \Delta h\right)}_{\text{weight of displaced water}} \tag{2}$$

Subtract Eqn. (2) from Eqn. (1) and simplify.

$$\left(m_{\text{raft+Bond}} + m_{\text{anchor}}\right)g - \left(m_{\text{raft+Bond}}\right)g = \rho_{H_2O}gl^2h - \rho_{H_2O}gl^2\left(h+\Delta h\right) \tag{3}$$

$$m_{\text{anchor}} = -\rho_{H_2O}l^2\Delta h \tag{4}$$

$$\Delta h = -\frac{m_{\text{anchor}}}{\rho_{H_2O}l^2} \tag{5}$$

$$\therefore \Delta h = -\frac{SG_{\text{anchor}}V_{\text{anchor}}}{l^2} \tag{6}$$

Note that since $V_{\text{anchor}} > 0$, $\Delta h < 0$ and thus the raft moves up relative to the free surface. However, the free surface will also move so we still don't yet know whether Bond moves up or down relative to the surface of the pit.

We must now consider the movement of the free surface of the water.

Case (a): $$V_{H_2O} = \underbrace{L^2D}_{\text{volume of H}_2\text{O in pit}} - \underbrace{l^2h}_{\text{volume of raft in H}_2\text{O}} \tag{7}$$

Case (b): $$V_{H_2O} = \underbrace{L^2\left(D+\Delta D\right)}_{\text{volume of H}_2\text{O in pit}} - \underbrace{l^2\left(h+\Delta h\right)}_{\text{volume of raft in H}_2\text{O}} - V_{\text{anchor}} \tag{8}$$

Since the volume of water is conserved, Eqns. (7) and (8) must be equal.

$$L^2\left(D+\Delta D\right) - l^2\left(h+\Delta h\right) - V_{\text{anchor}} = L^2D - l^2h \tag{9}$$

$$L^2\Delta D - l^2\Delta h - V_{\text{anchor}} = 0$$

$$\therefore \Delta D = \frac{l^2\Delta h + V_{\text{anchor}}}{L^2} \tag{10}$$

$$\therefore \Delta D = \frac{\left(1-SG_{\text{anchor}}\right)V_{\text{anchor}}}{L^2} \quad \text{(where Eqn. (6) has been utilized)} \tag{11}$$

Note that since $SG_{\text{anchor}} > 1$, $\Delta D < 0$, i.e. the free surface moves downward.

Combine the expressions for $\Delta h$ and $\Delta D$ to determine the movement of the raft bottom relative to the pit walls.

$$D + H - h = \left(D+\Delta D\right) + \left(H+\Delta H\right) - \left(h+\Delta h\right) \tag{12}$$

$$\Delta H = -\Delta D + \Delta h \tag{13}$$

$$\Delta H = -\frac{\left(1-SG_{\text{anchor}}\right)V_{\text{anchor}}}{L^2} - \frac{SG_{\text{anchor}}V_{\text{anchor}}}{l^2} \tag{14}$$

$$\boxed{\therefore \Delta H = \frac{V_{\text{anchor}}}{L^2}\left[SG_{\text{anchor}}\left(1-\frac{L^2}{l^2}\right)-1\right]} \tag{15}$$

Use the given data to determine $\Delta H$.

$V_{\text{anchor}}$ = 1 ft$^3$
$L$ = 4 ft
$SG_{\text{anchor}}$ = 7.85
$l$ = 3 ft
$\Rightarrow$ $\boxed{\Delta H = \text{-0.44 ft (The raft moves closer to the top of the pit.)}}$

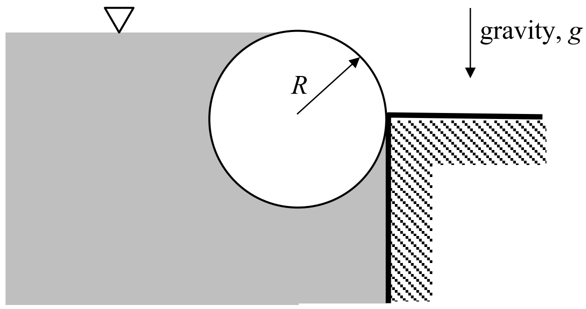
Recall that $H$ = 7.5 ft and Bond can only reach $R$ = 7 ft. After tossing the anchor overboard, the bottom of the raft is $H + \Delta H$ = 7.06 ft > $R$ = 7 ft. $\boxed{\text{Hence, Bond still can't reach the top of the pit.}}$

Goodbye, Mr. Bond.

A cylindrical log of radius $R$ and length $L$ rests against the top of a dam. The water is level with the top of the log and the center of the log is level with the top of the dam. You may assume that the contact point with the dam is frictionless. Obtain expressions for

a. the mass of the log, and
b. the contact force between the log and dam.

Express your answers in terms of (a subset of) $\rho_{H2O}$, $g$, $L$, and $R$.

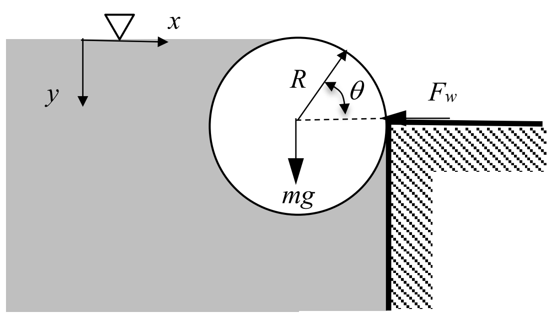



SOLUTION:

The mass of the log, $m$, may be found by performing a force balance in the vertical direction,

$$\sum F_y = 0 = mg + F_{P,y}\ , \tag{1}$$

where $g$ is the acceleration due to gravity. Note that the point of contact with the dam is assumed to be frictionless.




The net vertical pressure force, $F_{P,y}$, is found by integrating the vertical component of the pressure force around the log,

$$F_{P,y} = \int_{\theta=\pi/2}^{\theta=2\pi} p\sin\theta\, dA = \int_{\theta=\pi/2}^{\theta=2\pi} \underbrace{\rho g y}_{=p}\sin\theta \underbrace{R d\theta(L)}_{=dA}, \tag{2}$$

$$F_{P,y} = \int_{\theta=\pi/2}^{\theta=2\pi} \rho g\underbrace{(R - R\sin\theta)}_{=y}\sin\theta R\, d\theta(L) = \rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} (1-\sin\theta)\sin\theta\, d\theta\ , \tag{3}$$

$$F_{P,y} = \rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} \left(\sin\theta - \sin^2\theta\right) d\theta\ , \tag{4}$$

where $\rho$ is the density of the water. Evaluating the integral in Eq. (4) gives,

$$F_{P,y} = \rho g R^2 L\left\{-\cos\theta\Big|_{\theta=\pi/2}^{\theta=2\pi} - \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta)\right]_{\theta=\pi/2}^{\theta=2\pi}\right\} = \rho g R^2 L\left[-1 - \tfrac{1}{2}\left(2\pi - \tfrac{\pi}{2}\right)\right], \tag{5}$$

$$F_{P,y} = -\left(1 + \tfrac{3\pi}{4}\right)\rho g R^2 L\ . \tag{6}$$

Substituting into Eq. (1) and solving for $m$ gives,

$$\boxed{m = \left(1 + \frac{3\pi}{4}\right)\rho R^2 L}. \tag{7}$$

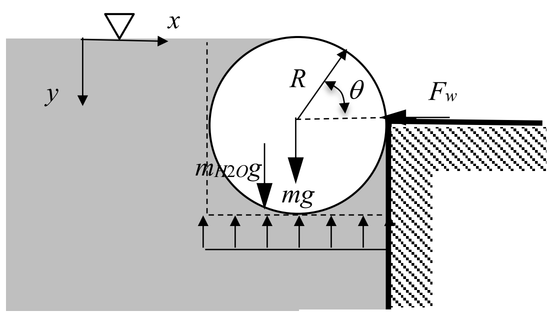
An alternate, easier method for determining the vertical pressure force acting on the log is to note that the vertical surface forces acting along a horizontal plane at the bottom of the log is,

$$\sum F_y = 0 = \underbrace{\rho g(2R)(2RL)}_{\text{pressure force at bottom}} \underbrace{-mg}_{\text{log weight}} \underbrace{-\rho g L \tfrac{3}{4}\left(4R^2 - \pi R^2\right)}_{\text{weight of water}} = 4\rho g L R^2 - mg - \rho g L \tfrac{3}{4}\left(4R^2 - \pi R^2\right), \tag{8}$$

$$mg = \rho g L R^2\left[4 - \tfrac{3}{4}(4 - \pi)\right] = \rho g L R^2\left[4 - 3 + \tfrac{3}{4}\pi\right], \tag{9}$$

$$m = \rho R^2 L\left(1 + \frac{3\pi}{4}\right), \tag{10}$$

which is the same result found in Eq. (7).




An even easier method is to use a buoyant force, although one must recognize the appropriate volume to use to determine the displaced volume. A vertical force balance for the log gives,

$$\sum F_y = 0 = -mg + F_B \quad \Rightarrow \quad m = \frac{F_B}{g}, \tag{11}$$

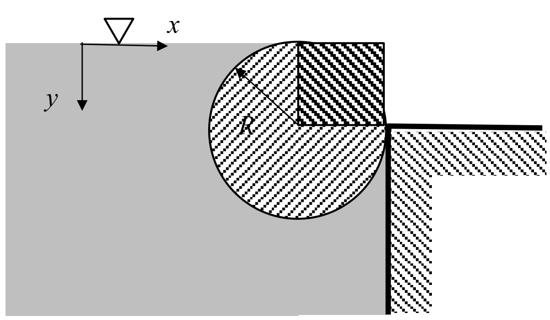
where $F_B$ is the buoyant force, which is the weight of the displaced fluid. Note that in this case, the displaced volume of fluid is the volume of the log, plus the volume above the right, upper quadrant of the log as shown in the figure below,

$$F_B = \rho g V_{\text{displaced}} = \rho g\left(\tfrac{3}{4}\pi R^2 + R^2\right)L = \rho g R^2 L\left(\tfrac{3\pi}{4} + 1\right), \tag{12}$$

Combining Eqs. (11) and (12) gives the mass of the log,

$$m = \rho R^2 L\left(\tfrac{3\pi}{4} + 1\right), \tag{13}$$

which is exactly the same result as found in the previous two methods.



Now consider a horizontal force balance for the log.

$$\sum F_x = 0 = -F_w + F_{P,x}\,, \tag{14}$$

where $F_w$ is the horizontal force exerted by the wall on the wall and $F_{P,x}$ is the horizontal component of the net pressure force acting on the log due to the water. The net horizontal pressure force is given by,

$$F_{P,x} = \int_{\theta=\pi/2}^{\theta=2\pi} -p\cos\theta\, dA = \int_{\theta=\pi/2}^{\theta=2\pi} \underbrace{(-\rho g y)}_{=p}\cos\theta \underbrace{(R d\theta L)}_{=dA} = \int_{\theta=\pi/2}^{\theta=2\pi} \left[-\rho g \underbrace{(R - R\sin\theta)}_{=y}\right]\cos\theta (R d\theta L)\,, \tag{15}$$

$$F_{P,x} == -\rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} (1-\sin\theta)\cos\theta\, d\theta = -\rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} (\cos\theta - \sin\theta\cos\theta)\, d\theta\,. \tag{16}$$

Evaluate the integrals in Eq. (16),

$$F_{P,x} = -\rho g R^2 L\left[\sin\theta\Big|_{\theta=\pi/2}^{\theta=2\pi} - \tfrac{1}{2}\sin^2\theta\Big|_{\theta=\pi/2}^{\theta=2\pi}\right] = -\rho g R^2 L\left[\sin\theta\Big|_{\theta=\pi/2}^{\theta=2\pi} - \tfrac{1}{2}\sin^2\theta\Big|_{\theta=\pi/2}^{\theta=2\pi}\right], \tag{17}$$

$$F_{P,x} = -\rho g R^2 L\left[-1 + \tfrac{1}{2}\right] = \tfrac{1}{2}\rho g R^2 L\,. \tag{18}$$

Substitute into Eq. (14) and solve for the wall force.

$$\boxed{F_w = \tfrac{1}{2}\rho g R^2 L}. \tag{19}$$

Another, much simpler method for finding the wall force is to note that the horizontal pressure force acting on the log will simply be the pressure force acting on the horizontally projected area.

$$F_{P,x} = \int_{y=0}^{y=R} p\, dA = \int_{y=0}^{y=R} \underbrace{(\rho g y)}_{=p}\underbrace{dyL}_{=dA} = \rho g L\int_{y=0}^{y=R} y\, dy = \tfrac{1}{2}\rho g R^2 L\,, \tag{20}$$

which is precisely the same result found in Eq. (18). Note that the horizontal pressure force is only evaluated from $y = 0$ to $y = R$ since on the bottom half of the log, the pressure forces from either side of the log cancel each other out.