9.6. The Kármán Momentum Integral Equation (KMIE)

So far we've only examined boundary layer flows that lend themselves to similarity solutions. This, of course, is very restrictive. There are many non-similar boundary layer flows that we would also like to investigate. Since the majority of fluid mechanics problems are complex, we often have to resort to empirical or semi-empirical methods for investigating the flows in greater detail. Here we'll discuss one such semi-empirical method used for investigating boundary layers called the Kármán Momentum Integral Equation (KMIE). The idea is straightforward and relies on the Linear Momentum Equation.

Consider a differential control volume as shown in Figure 9.11. The top of the control volume is defined by the line separating the boundary layer region from the outer flow region (this is *not* a streamline). Apply the

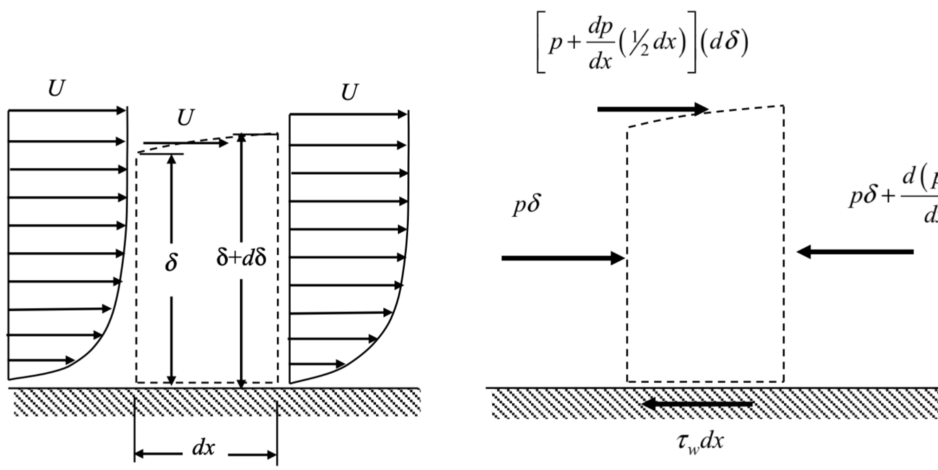


Figure 9.11. Schematics showing the control volume and free body diagram used in deriving the KMIE.

Linear Momentum Equation in the x-direction to the control volume, assuming unit depth,

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}, \tag{9.123}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady state)}, \tag{9.124}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \underbrace{-\int_{y=0}^{y=\delta} \rho u^2 dy}_{\text{left}} + \underbrace{\left[\int_0^\delta \rho u^2 dy + \frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right) dx\right]}_{\text{right}} \underbrace{-U\frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right) dx}_{\text{top}}, \tag{9.125}$$

$$F_{B,x} = 0, \quad \text{(body forces are negligibly small in boundary layers compared to other terms)}, \tag{9.126}$$

$$F_{S,x} = \underbrace{p\delta}_{\text{left}} \underbrace{-\left[p\delta + \frac{d}{dx}(p\delta)(dx)\right]}_{\text{right}} + \underbrace{\left[p + \frac{dp}{dx}(\frac{1}{2}dx)\right](d\delta)}_{\text{top}} \underbrace{-\tau_w dx}_{\text{bottom}}. \tag{9.127}$$

The mass flow rate through the top is found via Conservation of Mass on the same control volume,

$$\underbrace{-\dot{m}_{\text{top}}}_{\text{top}} \underbrace{-\int_0^\delta \rho u dy}_{\text{left}} \underbrace{+\left[\int_0^\delta \rho u dy + \frac{d}{dx}\left(\int_0^\delta \rho u dy\right) dx\right]}_{\text{right}} = 0, \tag{9.128}$$

$$\therefore \dot{m}_{\text{top}} = \frac{d}{dx}\left(\int_0^\delta \rho u dy\right) dx. \tag{9.129}$$

Substituting and simplifying, neglecting higher order terms,

$$-\frac{dp}{dx}\delta dx - \tau_w dx = \frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right) dx - U\frac{d}{dx}\left(\int_0^\delta \rho u dy\right) dx, \tag{9.130}$$

$$-\frac{dp}{dx}\delta - \tau_w = \frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right) - U\frac{d}{dx}\left(\int_0^\delta \rho u dy\right). \tag{9.131}$$

Recall that the pressure at a given x location remains constant with y position so we can find dp/dx in terms of the outer (potential) flow velocity using Bernoulli's equation outside of the boundary layer,

$$p + \frac{1}{2}\rho U^2 = \text{constant} \implies \frac{dp}{dx} + \rho U\frac{dU}{dx} = 0, \tag{9.132}$$

$$\frac{dp}{dx} = -\rho U\frac{dU}{dx}. \tag{9.133}$$

In addition, we'll re-write the boundary layer thickness in terms of an integral so that,

$$-\frac{dp}{dx}\delta = \left(\rho U\frac{dU}{dx}\right)\int_0^\delta dy = \frac{dU}{dx}\int_0^\delta \rho U dy. \tag{9.134}$$

Additional re-arranging gives,

$$U\frac{d}{dx}\left(\int_0^\delta \rho u dy\right) = \frac{d}{dx}\left(\int_0^\delta \rho u U dy\right) - \frac{dU}{dx}\int_0^\delta \rho u dy. \tag{9.135}$$

Substituting Eqs. (9.134) and (9.135) into Eq. (9.131) gives,

$$\frac{dU}{dx}\left(\int_0^\delta \rho U dy\right) - \tau_w = \frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right) - \frac{d}{dx}\int_0^\delta \rho u U dy + \frac{dU}{dx}\left(\int_0^\delta \rho u dy\right). \tag{9.136}$$

Additional re-arranging and simplifying gives,

$$\tau_w = \frac{d}{dx}\left[\int_0^\delta \rho u(U-u)dy\right] + \frac{dU}{dx}\int_0^\delta \rho(U-u)dy, \tag{9.137}$$

$$= \frac{d}{dx}\left[\rho U^2 \underbrace{\int_0^\delta \rho\frac{u}{U}\left(1-\frac{u}{U}\right)dy}_{=\delta_D}\right] + \frac{dU}{dx}\rho U \underbrace{\int_0^\delta \left(1-\frac{u}{U}\right)dy}_{=\delta_D}. \tag{9.138}$$

$$\tag{9.139}$$

Thus, if the fluid has constant density,

$$\boxed{\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\delta_M\right) + \delta_D U\frac{dU}{dx}}. \tag{9.140}$$

This equation is known as the Kármán Momentum Integral Equation (KMIE).

Notes:

(1) If the pressure remains constant, then $dU/dx = 0$ and,

$$\tau_w = \rho U^2 \frac{d\delta_m}{dx}. \tag{9.141}$$

(2) The typical methodology for using the KMIE is as follows.
   (a) Obtain an approximate expression for $U = U(x)$ from inviscid flow theory, e.g., potential flow theory. Recall that Bernoulli's equation can be used to relate the pressure and U.
   (b) Assume a velocity profile in the boundary layer subject to the appropriate boundary conditions, i.e., assume a form for,

$$\frac{u}{U} = f\left(\frac{y}{\delta}\right), \tag{9.142}$$

   subject to the boundary conditions,

$$\frac{u}{U}\left(\frac{y}{\delta} = 0\right) = 0 \quad \text{and} \quad \frac{u}{U}\left(\frac{y}{\delta} = 1\right) = 1. \tag{9.143}$$

   The form of the approximate velocity profile is typically found based on curve fits to experimental measurements of the boundary layer velocity profile. Higher order profiles will have additional boundary conditions. For example, a cubic curve fit will also have a boundary condition that matches the slope of the velocity profile at the free stream boundary.
   (c) The shear stress at the wall for a *laminar* flow can also be determined from the Newtonian stress-strain rate constitutive relations to be,

$$\tau_w = \mu\left(\frac{U}{\delta}\right) \left.\frac{d(u/U)}{d(y/\delta)}\right|_{\frac{y}{\delta}=0}. \tag{9.144}$$

   For a turbulent flow, experimental data for the wall shear stress are used instead since turbulent flows use time-averaged velocity profiles. This issue is discussed in greater detail later in these notes. The laminar wall shear stress must be the same shear stress as that found using the KMIE (Eq. (9.140)). Thus, we can equate the two shear stress expressions. The resulting differential equation can then be solved for the boundary layer thickness, δ, as a function of x.

(3) This approximate technique can be used for either laminar or turbulent flows. In fact, this method is especially useful for analyzing turbulent boundary layer profiles (discussed later in these notes).

Consider laminar flow over a flat plate (U = constant). Approximate the boundary layer velocity profile using a parabolic shape,

$$\frac{u}{U} = \begin{cases} 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 & 0 \le \frac{y}{\delta} < 1 \\ 1 & \frac{y}{\delta} \ge 1 \end{cases}$$

Using the KMIE, determine the dimensionless 99% boundary layer thickness, δ/x, as a function of Reynolds number based on the distance from the leading edge, $\mathrm{Re}_x = Ux/\nu$. Compare your result to the Blasius solution.

SOLUTION:
Evaluate the momentum thickness, δ_M,

$$\delta_M = \delta \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2) d\eta \quad \text{(where } \eta = y/\delta\text{)}, \tag{1}$$

$$\delta_M = \frac{2}{15}\delta. \tag{2}$$

Now substitute this momentum thickness into the KMIE. Note that $dU/dx = 0$ since U = constant,

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\delta_M) + \delta_D U \underbrace{\frac{dU}{dx}}_{=0} = U^2 \frac{d\delta_M}{dx}, \tag{3}$$

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} = \frac{2}{15}\rho U^2 \frac{d\delta}{dx}. \tag{4}$$

This shear stress should be the same as the shear stress found via,

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0} = \frac{\mu U}{\delta}\left.\frac{d(u/U)}{d\eta}\right|_{\eta=0} = \frac{\mu U}{\delta}(2 - 2\eta)_{\eta=0} = 2\frac{\mu U}{\delta}. \tag{5}$$

Equating the two shear stresses and solving the resulting differential equation,

$$\frac{2}{15}\rho U^2 \frac{d\delta}{dx} = 2\frac{\mu U}{\delta}, \tag{6}$$

$$\int_0^\delta \delta d\delta = 15\frac{\mu}{\rho U}\int_0^x dx \quad \text{(assuming } \delta = 0 \text{ when } x = 0\text{)}, \tag{7}$$

$$\frac{1}{2}\delta^2 = \frac{15\nu x}{U}, \tag{8}$$

$$\frac{\delta}{x} = \sqrt{\frac{30\nu x}{Ux^2}} = \sqrt{\frac{30}{\mathrm{Re}_x}} \quad \text{(using } \mathrm{Re}_x = Ux/\nu\text{)}, \tag{9}$$

$$\boxed{\frac{\delta}{x} \approx \frac{5.5}{\mathrm{Re}_x^{1/2}}}. \tag{10}$$

This approximate expression is only 10% different from the exact Blasius expression,

$$\frac{\delta}{x} \approx \frac{5.0}{\mathrm{Re}_x^{1/2}} \quad \text{(Blasius)}.$$

Using the momentum integral theorem, determine the friction coefficient, c_f, dimensionless boundary layer momentum thickness, δ_M/x, and the dimensionless boundary layer displacement thickness, δ_D/x, for laminar flat plate flow with no pressure gradient assuming a sinusoidal velocity profile:

$$\frac{u}{U} \approx \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right),$$

where δ is the 99% boundary layer thickness, y is the distance from the plate surface, and U is the outer flow speed. Compare your answers with the Blasius' exact laminar boundary layer solution.

SOLUTION:

Use the Kármán Momentum Integral Equation (KMIE),

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(\delta_M U^2\right) + \delta_D U \frac{dU}{dx} \tag{1}$$

Assuming a flat plate flow with no pressure gradient,

$$U = \text{constant} \Rightarrow \frac{dU}{dx} = 0 \text{ (from Bernoulli's equation applied outside the boundary layer)} \tag{2}$$

Simplifying Eqn. (1) gives,

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} \tag{3}$$

The momentum thickness is given by,

$$\begin{aligned}
\delta_M &= \int_{y=0}^{y=\delta} \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \delta \int_{y/\delta=0}^{y/\delta=1} \frac{u}{U}\left(1-\frac{u}{U}\right)d\left(y/\delta\right) \\
&= \delta\int_0^1 \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\left[1-\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]d\left(y/\delta\right) \\
&= \delta\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\Big|_{y/\delta=0}^{y/\delta=1} - \frac{1}{2}\frac{y}{\delta}\Big|_{y/\delta=0}^{y/\delta=1} + \frac{1}{2\pi}\sin\left(\pi\frac{y}{\delta}\right)\Big|_{y/\delta=0}^{y/\delta=1}\right]
\end{aligned}$$

$$\therefore \delta_M = \delta\left(\frac{2}{\pi}-\frac{1}{2}\right) \approx 0.1367\delta \tag{4}$$

Substitute Eq. (4) into Eq. (3),

$$\tau_w = 0.1367\rho U^2 \frac{d\delta}{dx} \tag{5}$$

For a laminar flow, the shear stress can also be expressed as,

$$\tau_w = \mu \frac{du}{dy}\Big|_{y=0}$$

$$\tau_w = \frac{\pi}{2}\frac{\mu U}{\delta} \tag{6}$$

Equate Eqs. (5) and (6) and solve for δ,

$$0.1367\rho U^2 \frac{d\delta}{dx} = \frac{\pi}{2}\frac{\mu U}{\delta}$$

$$\int_{\delta=0}^{\delta=\delta} \delta\, d\delta = 11.4908 \frac{\mu}{\rho U} \int_{x=0}^{x=x} dx$$

$$\tfrac{1}{2}\delta^2 = 11.4908 \frac{\mu}{\rho U} x$$

$$\boxed{\therefore \frac{\delta}{x} = 4.7939\sqrt{\frac{\mu}{\rho U x}} = \frac{4.7939}{\mathrm{Re}_x^{1/2}}} \qquad (7)$$

Equation (7) is only 4% different from the exact Blasius solution of $\delta/x = 5.0/\mathrm{Re}_x^{1/2}$.

From Eq. (4) the momentum thickness is,

$$\boxed{\frac{\delta_M}{x} = \frac{0.6553}{\mathrm{Re}_x^{1/2}}} \qquad (8)$$

This result is 1% different from the Blasius solution of $\delta_M/x = 0.664/\mathrm{Re}_x^{1/2}$.

The displacement thickness is given by,

$$\delta_D = \int_{y=0}^{y=\delta}\left(1-\frac{u}{U}\right)dy = \delta \int_{y/\delta=0}^{y/\delta=1}\left(1-\frac{u}{U}\right)d\left(y/\delta\right)$$

$$= \delta\int_0^1\left[1-\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]d\left(y/\delta\right)$$

$$= \delta\left[\left.\frac{y}{\delta}\right|_{y/\delta=0}^{y/\delta=1} + \frac{2}{\pi}\cos\left.\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right|_{y/\delta=0}^{y/\delta=1}\right]$$

$$\therefore \delta_D = \delta\left(1-\frac{2}{\pi}\right) \approx 0.3634\delta \qquad (9)$$

so that, when combined with Eq. (7),

$$\boxed{\frac{\delta_D}{x} = \frac{1.7420}{\mathrm{Re}_x^{1/2}}} \qquad (10)$$

This result is 1% different from the Blasius solution of $\delta_D/x = 1.72/\mathrm{Re}_x^{1/2}$.

The friction coefficient can be found using Eq. (6),

$$C_f \equiv \frac{\tau_w}{\tfrac{1}{2}\rho U^2} = \frac{\frac{\pi}{2}\frac{\mu U}{\delta}}{\tfrac{1}{2}\rho U^2} = \pi\frac{\mu}{\rho U\delta} = \pi\frac{\mu}{\rho U x}\frac{x}{\delta}$$

$$\boxed{C_f = \frac{0.6553}{\mathrm{Re}_x^{1/2}}} \qquad (11)$$

This result is 1% different form the Blasius solution of $C_f = 0.664/\mathrm{Re}_x^{1/2}$.

A measured dimensionless laminar boundary layer profile for flow past a flat plate is given in the table below. Use the momentum integral equation to determine the 99% boundary layer thickness. Compare your result with the exact (Blasius) result.

***y*/δ**	***u*/*U***
0.00	0.00
0.08	0.133
0.16	0.265
0.24	0.394
0.32	0.517
0.40	0.630
0.48	0.729
0.56	0.811
0.64	0.876
0.72	0.923
0.80	0.956
0.88	0.976
0.96	0.988
1.00	1.000

SOLUTION:

Apply the Kármán Momentum Integral Equation:

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(\delta_M U^2\right) + \delta_D U \frac{dU}{dx} \tag{1}$$

Assuming a flat plate flow with no pressure gradient:

$$U = \text{constant} \Rightarrow \frac{dU}{dx} = 0 \tag{2}$$

Simplifying Eqn. (1) gives:

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} \tag{3}$$

The momentum thickness is given by:

$$\delta_M = \int_{y=0}^{y=\delta} \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \delta \int_{y/\delta=0}^{y/\delta=1} \frac{u}{U}\left(1-\frac{u}{U}\right)d\left(y/\delta\right)$$

Integrating the data numerically using the trapezoidal rule gives:

$$\delta_M \approx 0.131\delta \tag{4}$$

Substitute into Eqn. (3).

$$\tau_w = 0.131\rho U^2 \frac{d\delta}{dx} \tag{5}$$

For a laminar flow, the shear stress can also be expressed as:

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0} = \mu \frac{U}{\delta} \left.\frac{d\left(u/U\right)}{d\left(y/\delta\right)}\right|_{y/\delta=0} \tag{6}$$

Differentiating the data numerically using a 1st order finite difference scheme:

$$\tau_w \approx 1.66\mu \frac{U}{\delta} \tag{7}$$

Equating Eqns. (5) and (7) gives:

$$0.131\rho U^2 \frac{d\delta}{dx} = 1.66\mu \frac{U}{\delta}$$

$$\int_{\delta=0}^{\delta=\delta} \delta d\delta = 12.67 \frac{\mu}{\rho U} \int_{x=0}^{x=x} dx$$

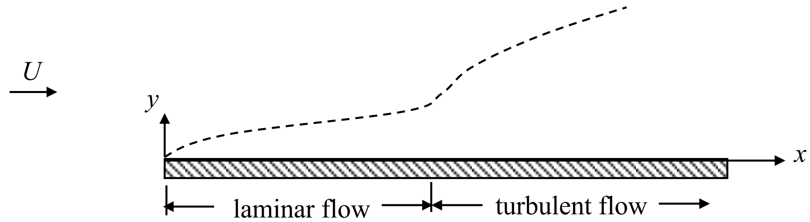
$$\tfrac{1}{2}\delta^2 = 12.67 \frac{\mu x}{\rho U}$$

$$\boxed{\therefore \frac{\delta}{x} = 5.034\sqrt{\frac{\mu}{\rho U x}} = \frac{5.034}{\mathrm{Re}_x^{1/2}}} \tag{8}$$

Equation (8) is within 1% of the exact Blasius solution of $\delta/x = 5.0/\mathrm{Re}_x^{1/2}$.

Another approach to this problem is to fit a polynomial curve to the given data rather than numerically differentiating and integrating the data.

The flat plate formulas for turbulent flow over a flat plate assume that turbulent flow begins at the leading edge ($x = 0$). In reality there is an initial region of laminar flow as shown in the figure.



1. Derive an expression for the 99% boundary layer thickness in the turbulent region by accounting for the laminar part of the flow.
2. Plot the dimensionless boundary layer thickness, δ/x, as a function of Reynolds number ($10^4 \leq \mathrm{Re}_x \leq 10^8$, use a log scale for the Re_x axis) for your derived relation and for the turbulent relation that does not consider the laminar part.

Assume a 1/7th power law velocity profile for the turbulent boundary layer and an experimental friction coefficient correlation of $C_f \approx 0.020\,\mathrm{Re}_\delta^{-1/6}$.

SOLUTION:

First determine the boundary layer thickness in the laminar flow region using the Blasius solution:

$$\frac{\delta}{x} = \frac{5.0}{\mathrm{Re}_x^{1/2}} \quad (\mathrm{Re}_x < 500{,}000) \tag{1}$$

Assume that the transition to turbulence occurs at a Reynolds number of 500,000 so that condition at the transition point is:

$$\delta_{\text{trans}} = 3.536 * 10^3 \left(\frac{\nu}{U}\right) \tag{2}$$

where

$$x_{\text{trans}} = \frac{500{,}000\nu}{U} \tag{3}$$

Now use the Karman Momentum Integral Equation to determine the boundary layer characteristics for the turbulent region. Assume that the velocity profile follows the following form:

$$\frac{u}{U_\infty} = \begin{cases} \left(\dfrac{y}{\delta}\right)^{1/7} & \dfrac{y}{\delta} < 1 \\ 1 & \dfrac{y}{\delta} \geq 1 \end{cases} \tag{4}$$

Using this velocity profile, the momentum thickness is:

$$\delta_M = \int_{y=0}^{y=\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_{y=0}^{y=\delta} \left(\frac{y}{\delta}\right)^{1/7} \left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right] dy = \frac{7}{72}\delta \tag{5}$$

To determine the shear stress, recall that from the Karman Momentum Integral Equation, with a constant outer velocity:

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} = \frac{7}{72}\rho U^2 \frac{d\delta}{dx} \tag{1.6}$$

so that the friction coefficient is:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{7}{36}\frac{d\delta}{dx} \tag{1.7}$$

Using the given experimental wall friction correlation:

$$C_f \approx 0.020\,\mathrm{Re}_\delta^{-1/6} \tag{1.8}$$

where $\mathrm{Re}_\delta = (U\delta/\nu)$, equate the two friction coefficients to give:

$$\frac{7}{36}\frac{d\delta}{dx} = 0.020\left(\frac{U\delta}{\nu}\right)^{-1/6} \tag{9}$$

$$\int_{\delta=\delta_{\text{trans}}}^{\delta=\delta} \delta^{1/6}\,d\delta = 0.103\left(\frac{U}{\nu}\right)^{-1/6}\int_{x=x_{\text{trans}}}^{x=x} dx \tag{10}$$

$$\delta^{7/6} - \delta_{\text{trans}}^{7/6} = 0.120\left(\frac{U}{\nu}\right)^{-1/6}\left(x - x_{\text{trans}}\right) \tag{11}$$

where Eqns. (2) and (3) are used for δ_{trans} and x_{trans}, respectively. Substituting and simplifying results in:

$$\delta^{7/6} - 1.380*10^4\left(\frac{\nu}{U}\right)^{7/6} = 0.120\left(\frac{U}{\nu}\right)^{-1/6}\left(x - 500,000\frac{\nu}{U}\right) \tag{12}$$

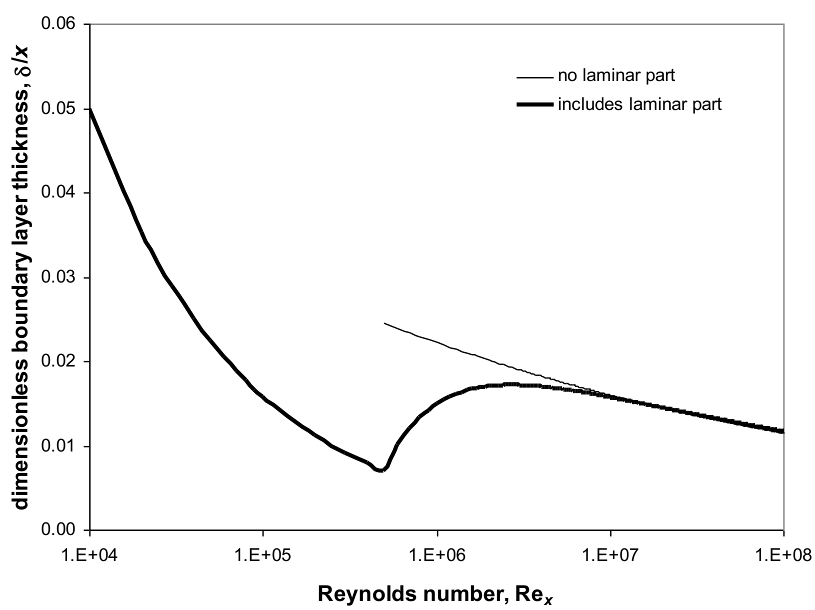
$$\delta^{7/6} = 0.120\left(\frac{\nu}{U}\right)^{1/6}x - 6.000*10^4\left(\frac{\nu}{U}\right)^{7/6} + 1.380*10^4\left(\frac{\nu}{U}\right)^{7/6} \tag{13}$$

$$\left(\frac{\delta}{x}\right)^{7/6} = 0.120\left(\frac{\nu}{Ux}\right)^{1/6} - 4.620*10^4\left(\frac{\nu}{Ux}\right)^{7/6} \tag{14}$$

$$\boxed{\frac{\delta}{x} = \left(\frac{0.120}{\mathrm{Re}_x^{1/6}} - \frac{4.620*10^4}{\mathrm{Re}_x^{7/6}}\right)^{6/7} \qquad \mathrm{Re}_x > 500,000} \tag{15}$$

Compare this result to one that assumes that the turbulent boundary layer starts from the leading edge:

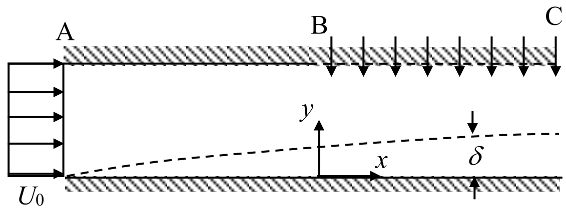
$$\frac{\delta}{x} \approx \frac{0.16}{\mathrm{Re}_x^{1/7}} \quad (\mathrm{Re}_x > 500,000) \tag{16}$$



Air flows between two parallel flat plates as shown in the figure below. The upper plate is porous from point B to point C and additional air is injected through this surface. As a result, the free stream speed, $U(x)$, varies as:

$$U(x) = U_0 + \alpha x$$

where U_0 is the air speed entering the channel (at point A), α is a constant, and x is the distance downstream of the point B. A boundary layer develops along the lower surface. Assuming a linear velocity distribution in the boundary layer, estimate the rate of boundary layer growth, $d\delta/dx$, in terms of δ, x, U_0, α, and the air properties.



SOLUTION:

Assuming a linear profile in the boundary layer means:

$$\frac{u}{U} = \frac{y}{\delta} \tag{1}$$

Note that with this velocity profile:

$$\frac{u}{U}\left(\frac{y}{\delta} = 0\right) = 0 \quad \text{and} \quad \frac{u}{U}\left(\frac{y}{\delta} = 1\right) = 1 \tag{2}$$

To determine the rate at which the boundary layer thickness grows with x, begin with the Karman momentum integral equation:

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2 \delta_M\right) + \delta_D U \frac{dU}{dx} \tag{3}$$

where

$$\delta_D = \int_{y=0}^{y=\delta} \left(1 - \frac{u}{U}\right) dy \tag{4}$$

$$\delta_M = \int_{y=0}^{y=\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \tag{5}$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{6}$$

$$U(x) = U_0 + \alpha x \Rightarrow \frac{dU}{dx} = \alpha \tag{7}$$

Substitute Eqn. (1) into Eqns. (4) - (6).

$$\delta_D = \int_{y=0}^{y=\delta} \left(1 - \frac{y}{\delta}\right) dy = \delta - \tfrac{1}{2}\delta = \tfrac{1}{2}\delta \tag{8}$$

$$\delta_M = \int_{y=0}^{y=\delta} \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \tfrac{1}{2}\delta - \tfrac{1}{3}\delta = \tfrac{1}{6}\delta \tag{9}$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\mu U}{\delta} = \frac{\mu\left(U_0 + \alpha x\right)}{\delta} \tag{10}$$

Substitute Eqns. (7), (8) - (10) into Eqn. (3) and simplify.

$$\frac{\nu\left(U_0 + \alpha x\right)}{\delta} = \frac{1}{6}\frac{d}{dx}\left[\left(U_0 + \alpha x\right)^2 \delta\right] + \frac{1}{2}\delta\left(U_0 + \alpha x\right)\alpha \tag{11}$$

$$\frac{\nu\left(U_0 + \alpha x\right)}{\delta} = \frac{1}{6}\left(U_0 + \alpha x\right)^2 \frac{d\delta}{dx} + \frac{1}{3}\delta\left(U_0 + \alpha x\right)\alpha + \frac{1}{2}\delta\left(U_0 + \alpha x\right)\alpha \tag{12}$$

$$\frac{\nu\left(U_0 + \alpha x\right)}{\delta} = \frac{1}{6}\left(U_0 + \alpha x\right)^2 \frac{d\delta}{dx} + \frac{5}{6}\delta\left(U_0 + \alpha x\right)\alpha \tag{13}$$

$$\boxed{\frac{d\delta}{dx} = \frac{6\nu}{\left(U_0 + \alpha x\right)}\frac{1}{\delta} - \frac{5\alpha}{\left(U_0 + \alpha x\right)}\delta} \tag{14}$$