(b) *Isentropic case:* Recall that for an ideal gas undergoing an isentropic process,

$$0 = c_p(T)\frac{dT}{T} - R\frac{dp}{p}, \tag{5.352}$$

$$dp = \frac{c_p(T)}{R}(\rho RT)\frac{dT}{T}, \tag{5.353}$$

$$dp = \rho c_p(T)dT. \tag{5.354}$$

Thus,

$$\int \frac{dp}{\rho} = \int c_p(T)dT = \int dh = \Delta h, \tag{5.355}$$

where h is the specific enthalpy. If the ideal gas has constant specific heats, i.e., is a "perfect" gas, then, $\Delta h = c_p\Delta T$ and,

$$\int \frac{dp}{\rho} = c_p\Delta T. \tag{5.356}$$

5.13.1. Another Approach to Deriving Bernoulli's Equation

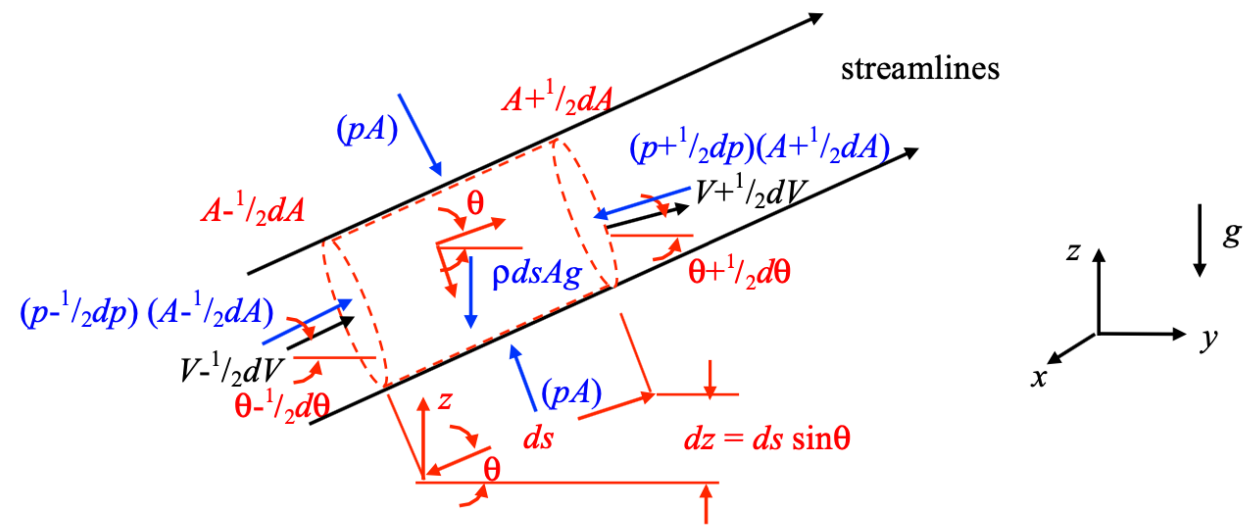


Figure 5.32. The differential control volume used to derive Bernoulli's Equation.

We can also derive Bernoulli's Equation using the Linear Momentum Equations and Conservation of Mass applied to a differential control volume as shown in Figure 5.32. Note that the control volume shown in the figure follows the streamlines. In the following analysis, we'll make the following simplifying assumptions:

(1) steady flow,
(2) inviscid flow, and
(3) incompressible fluid.

First apply Conservation of Mass to the control volume,

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0, \tag{5.357}$$

where,

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}, \tag{5.358}$$

$$\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \rho \left(V + \frac{1}{2}dV\right)\left(A + \frac{1}{2}dA\right) - \rho \left(V - \frac{1}{2}dV\right)\left(A - \frac{1}{2}dA\right), \tag{5.359}$$

$$= \rho V dA + \rho A dV + \text{H.O.T.s} \tag{5.360}$$

Note that there's no flow across the streamlines. Substituting these expressions into Conservation of Mass gives,

$$V dA = -A dV. \tag{5.361}$$

Now apply the Linear Momentum Equation to the same control volume in the streamline direction,

$$\frac{d}{dt}\int_{CV} u_s \rho dV + \int_{CS} u_s (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = F_{B,s} + F_{S,s}, \tag{5.362}$$

where,

$$\frac{d}{dt}\int_{CV} u_s \rho dV = 0 \quad \text{(steady flow)}, \tag{5.363}$$

$$\int_{CS} u_s (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = -\rho \left(V - \frac{1}{2}dV\right)^2\left(A - \frac{1}{2}dA\right) + \rho \left(V + \frac{1}{2}dV\right)^2\left(A + \frac{1}{2}dA\right), \tag{5.364}$$

$$= 2\rho V A dV + \rho V^2 dA + \text{H.O.T.s}, \tag{5.365}$$

$$F_{B,s} = \rho ds A(-g \sin\theta) = -\rho A g \underbrace{ds \sin\theta}_{=dz} = -\rho A g dz, \tag{5.366}$$

$$F_{S,s} = \left(p - \frac{1}{2}dp\right)\left(A - \frac{1}{2}dA\right) - \left(p + \frac{1}{2}dp\right)\left(A + \frac{1}{2}dA\right) + p dA, \tag{5.367}$$

$$= -A dp + \text{H.O.T.s}, \tag{5.368}$$

Combining these terms together into the Linear Momentum Equation,

$$2\rho V A dV + \rho V^2 dA = -\rho A g dz - A dp. \tag{5.369}$$

Now substitute the result from Conservation of Mass into the result from the Linear Momentum Equation and simplify,

$$2\rho V A dV + \underbrace{\rho V^2 dA}_{=-\rho V A dV} = -\rho A g dz - A dp, \tag{5.370}$$

$$\frac{dp}{\rho} + V dV + g dz = 0. \tag{5.371}$$

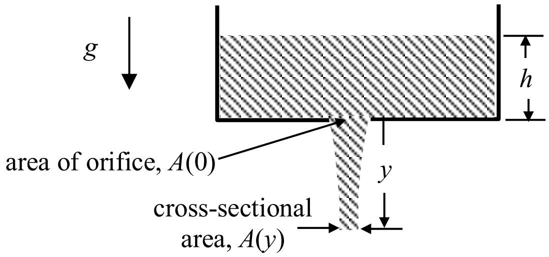
We can integrate this equation along the streamline to get,

$$\boxed{\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{constant}} \tag{5.372}$$

Again, it's important to review the assumptions builtin to the derivation of Eq. (5.372):

(1) steady flow,
(2) inviscid flow,
(3) incompressible fluid, and
(4) flow along a streamline.

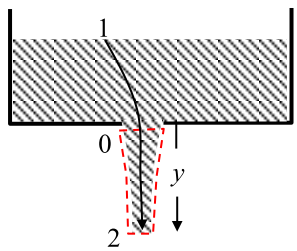
A water tank has an orifice in the bottom of the tank:



The height, h, of water in the tank is kept constant by a supply of water which is not shown. A jet of water emerges from the orifice; the cross-sectional area of the jet, $A(y)$, is a function of the vertical distance, y. Neglecting viscous effects and surface tension, find an expression for $A(y)$ in terms of $A(0)$, h, and y.

SOLUTION:

Apply Conservation of Mass to the following CV:



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0$$

where,

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(The flow is steady.)}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho V_0 A_0 + \rho V_2 A_2$$

Substitute and simplify,

$$V_2 = V_0 \frac{A_0}{A_2} \tag{1}$$

Now apply Bernoulli's Equation from point 1 to point 0 and from point 1 to point 2,

$$\left(p + \tfrac{1}{2}\rho V^2 - \rho g y\right)_1 = \left(p + \tfrac{1}{2}\rho V^2 - \rho g y\right)_0 = \left(p + \tfrac{1}{2}\rho V^2 - \rho g y\right)_2$$

where,

$p_1 = p_0 = p_2 = p_{atm}$ (These points are all at free surfaces.)

$V_1 = 0$ and V_0 and V_2 are related through Eq. (1).

$y_1 = -h, y_0 = 0, y_2 = y$

Substitute and simplify,

$$\rho g h = \tfrac{1}{2}\rho V_0^2 = \tfrac{1}{2}\rho V_2^2 - \rho g y$$

$$\rho g h = \tfrac{1}{2}\rho V_0^2 = \tfrac{1}{2}\rho V_0^2 \left(\frac{A_0}{A_2}\right)^2 - \rho g y$$

The first two equations in the previous expression state that,

$$V_0 = \sqrt{2gh} \tag{2}$$

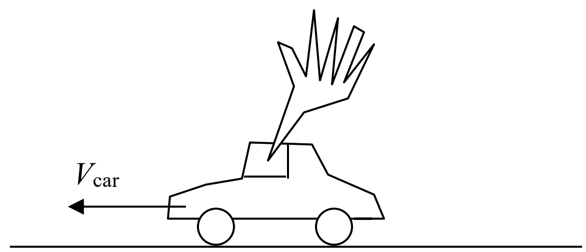
Equation (2) combined with the second two equations gives,

$$\left(\frac{A_0}{A_2}\right)^2 = 1 + \frac{\rho g y}{\tfrac{1}{2}\rho V_0^2}$$

$$\left(\frac{A_0}{A_2}\right)^2 = 1 + \frac{\rho g y}{\rho g h} = 1 + \frac{y}{h}$$

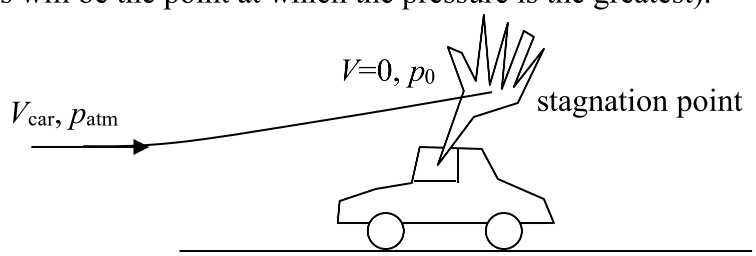
$$\boxed{\frac{A_2}{A_0} = \frac{1}{\sqrt{1 + \frac{y}{h}}}}$$

A person holds their hand out of a car window while driving through still air at a speed of V_{car}. What is the maximum pressure on the person's hand?



SOLUTION:

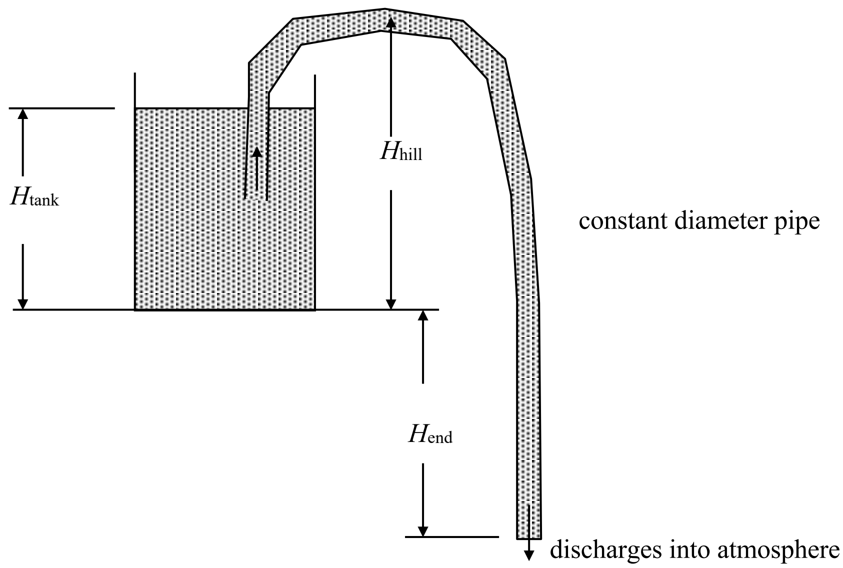
Change the frame of reference so that the car is stationary and the air approaches the car at a velocity, V_{car}. Apply Bernoulli's equation, neglecting elevation differences, along a streamline from a point far upstream of the car to the stagnation point on the person's hand (this will be the point at which the pressure is the greatest).



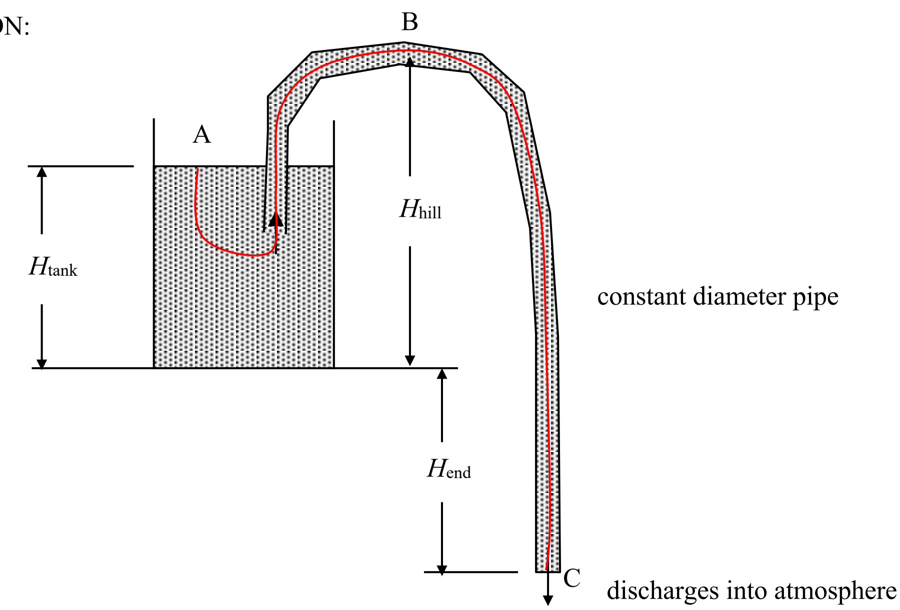
$$p_{atm} + \tfrac{1}{2}\rho V_{car}^2 = p_0 + \tfrac{1}{2}\rho \underbrace{V_0^2}_{=0}$$

$$\boxed{p_0 = p_{max} = p_{atm} + \tfrac{1}{2}\rho V_{car}^2} \tag{1}$$

Water is siphoned from a large tank through a constant diameter hose as shown in the figure. Determine the maximum height of the hill, H_{hill}, over which the water can be siphoned without cavitation occurring. Assume that the vapor pressure of the water is p_v, the height of the water free surface in the tank is H_{tank}, and the vertical distance from the end of the hose to the base of the tank is H_{end}.



SOLUTION:



Apply Bernoulli's equation along a streamline from the tank free surface (point A) to the end of the tube (point C).

$$\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_A=\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_C \quad (1)$$

where

$p_A = p_C = p_{\text{atm}}$

$V_A \approx 0$ (free surface of a large tank)

$z_A - z_C = H_{\text{tank}} + H_{\text{end}}$

Solving Eqn. (1) for V_C gives:

$$V_C=\sqrt{2g\left(H_{\text{tank}}+H_{\text{end}}\right)} \quad (2)$$

Now apply Bernoulli's equation along a streamline from the tank free surface (point A) to the top of the tube (point B). Note that the velocity everywhere within the tube will be equal to V_C (from conservation of mass).

$$\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_A=\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_B \quad (3)$$

where

$p_A = p_{\text{atm}}$

$p_B = p_{\text{v}}$

(From Eqn. (3) we see that the pressure at point B will decrease as H_{hill} increases so we should use the smallest allowable pressure at point B to determine the maximum H_{hill}.)

$V_A \approx 0$ (free surface of a large tank)

$V_B = V_C = \sqrt{2g\left(H_{\text{tank}}+H_{\text{end}}\right)}$ (from conservation of mass)

$z_A - z_B = H_{\text{tank}} - H_{\text{hill}}$

Substituting into Eqn. (3) and solving for H_{hill} gives:

$$\frac{p_{\text{atm}}}{\rho g}+H_{\text{tank}}-H_{\text{hill}}=\frac{p_v}{\rho g}+H_{\text{tank}}+H_{\text{end}}$$

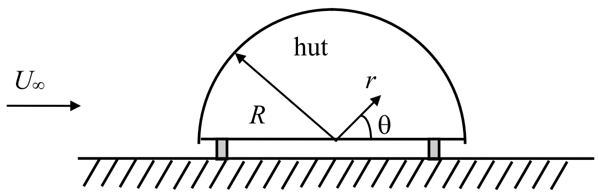
$$\boxed{H_{\text{hill}}=\frac{p_{\text{atm}}-p_v}{\rho g}-H_{\text{end}}} \quad (4)$$

You are to design Quonset huts for a military base. The design wind speed is $U_\infty = 30$ m/s and the free-stream pressure and density are $p_\infty = 101$ kPa and $\rho_\infty = 1.2$ kg/m^3, respectively. The Quonset hut may be considered to be a closed (no leaks) semi-cylinder with a radius of $R = 5$ m which is mounted on tie-down blocks as shown in the figure. The flow is such that the velocity distribution over the top of the hut is approximated by:

$$u_r\left(r = R\right) = 0$$

$$u_\theta\left(r = R\right) = -2U_\infty \sin\theta$$

The air under the hut is at rest.



a. What is the pressure distribution over the top surface of the Quonset hut?
b. What is the net lift force acting on the Quonset hut due to the air? Don't forget to include the effect of the air under the hut.
c. What is the net drag force acting on the hut? (Hint: A calculation may not be necessary here but some justification is required.)

SOLUTION:

Apply Bernoulli's equation over a streamline adjacent to the upper surface of the hut to determine the pressure distribution. Neglect elevation effects since the fluid is a gas and the elevation differences are small.

$$\left(p + \tfrac{1}{2}\rho V^2\right)_\infty = \left(p + \tfrac{1}{2}\rho V^2\right)_{\text{surface}} \tag{1}$$

where

$$p_\infty = 101 \text{ kPa}$$

$$V_\infty^2 = U_\infty^2 = \left(30 \text{ m/s}\right)^2 = 900 \text{ m}^2/\text{s}^2$$

$$p_{\text{surface}} = ?$$

$$V_{\text{surface}}^2 = u_\theta^2 = 4U_\infty^2 \sin^2\theta \quad (0 \le \theta \le \pi)$$

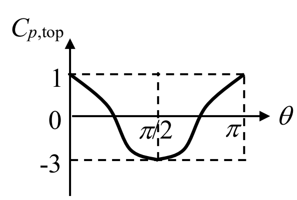
$$\rho = 1.2 \text{ kg/m}^3$$

Substitute and solve for the pressure on the hut's upper surface.

$$p_{\text{surface}} = p_\infty + \tfrac{1}{2}\rho\left(V_\infty^2 - V_{\text{surface}}^2\right)$$

$$p_{\text{surface}} = p_\infty + \tfrac{1}{2}\rho U_\infty^2\left(1 - 4\sin^2\theta\right)$$

$$\boxed{C_{p,\text{top}} = \frac{p_{\text{surface}} - p_\infty}{\tfrac{1}{2}\rho U_\infty^2} = 1 - 4\sin^2\theta} \tag{2}$$

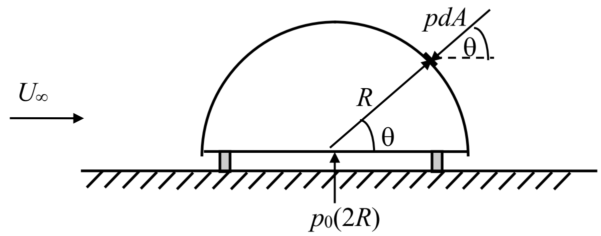


where C_p is known as a "pressure coefficient."

The pressure under the hut will be the stagnation pressure. It can also be found by applying Bernoulli's equation and noting that under the hut the velocity is zero.

$$C_{p,\text{bottom}} = \frac{p_0 - p_\infty}{\tfrac{1}{2}\rho U_\infty^2} = 1 \tag{3}$$

The net lift force is determined by integrating the vertical component of the pressure forces over the entire surface of the hut.



$$L = \underbrace{\int_{\theta=0}^{\theta=\pi} (-p\sin\theta)\underbrace{(Rd\theta)}_{=dA}}_{\text{top}} + \underbrace{p_0(2R)}_{\text{bottom}} \quad \text{(Note that positive lift is directed upwards.)} \tag{4}$$

where p_0 is the stagnation pressure.

$$L = -\int_{\theta=0}^{\theta=\pi} \left[p_\infty + \tfrac{1}{2}\rho U_\infty^2\left(1-4\sin^2\theta\right)\right]\sin\theta R d\theta + \left(p_\infty + \tfrac{1}{2}\rho U_\infty^2\right)(2R)$$

$$C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2(2R)} = -\frac{1}{2}\int_{\theta=0}^{\theta=\pi}\left[\frac{p_\infty}{\frac{1}{2}\rho U_\infty^2} + \left(1-4\sin^2\theta\right)\right]\sin\theta d\theta + \left(\frac{p_\infty}{\frac{1}{2}\rho U_\infty^2}+1\right)$$

where C_L is a "lift coefficient."

$$C_L = -\frac{1}{2}\frac{p_\infty}{\frac{1}{2}\rho U_\infty^2}\underbrace{\int_{\theta=0}^{\theta=\pi}\sin\theta\, d\theta}_{=-\cos\theta|_0^\pi=1+1=2} - \frac{1}{2}\int_{\theta=0}^{\theta=\pi}\left(\sin\theta - 4\sin^3\theta\right)d\theta + \frac{p_\infty}{\frac{1}{2}\rho U_\infty^2}+1$$

$$= -\frac{p_\infty}{\frac{1}{2}\rho U_\infty^2} - \frac{1}{2}\underbrace{\int_{\theta=0}^{\theta=\pi}\sin\theta\, d\theta}_{=-\cos\theta|_0^\pi=1+1=2} + 2\underbrace{\int_{\theta=0}^{\theta=\pi}\sin^3\theta\, d\theta}_{=-\frac{1}{3}\left(2+\sin^2\theta\right)\cos\theta|_0^\pi=\frac{2}{3}+\frac{2}{3}=\frac{4}{3}} + \frac{p_\infty}{\frac{1}{2}\rho U_\infty^2}+1$$

$$\boxed{C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2(2R)} = \frac{8}{3}} \tag{5}$$

The net drag force is determined by integrating the horizontal component of the pressure forces over the entire surface of the hut.

$$D = \int_{\theta=0}^{\theta=\pi}(-p\cos\theta)(Rd\theta) \tag{6}$$

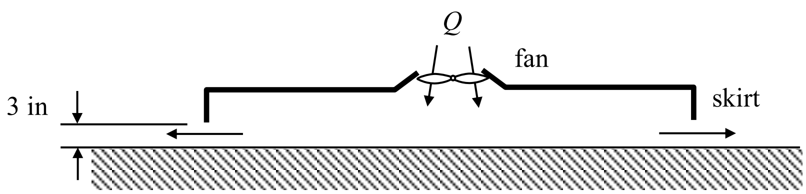
$$D = -\int_{\theta=0}^{\theta=\pi}\left[p_\infty + \tfrac{1}{2}\rho U_\infty^2\left(1-4\sin^2\theta\right)\right]\cos\theta R\, d\theta$$

$$= -Rp_\infty\underbrace{\int_{\theta=0}^{\theta=\pi}\cos\theta\, d\theta}_{=\sin\theta|_0^\pi=0} - \tfrac{1}{2}\rho U_\infty^2 R\left[\underbrace{\int_{\theta=0}^{\theta=\pi}\cos\theta\, d\theta}_{=\sin\theta|_0^\pi=0} - 4\underbrace{\int_{\theta=0}^{\theta=\pi}\sin^2\theta\cos\theta\, d\theta}_{=\frac{1}{3}\sin^3\theta|_0^\pi=0}\right]$$

$$\boxed{\therefore D = 0} \tag{7}$$

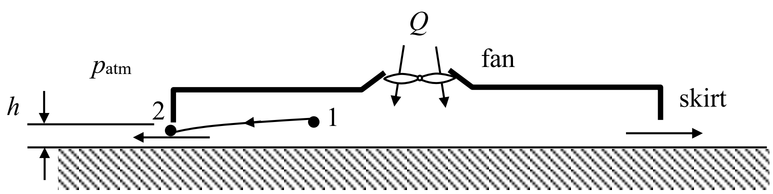
We could have also anticipated that the drag would be zero since the velocity field is symmetric between the upstream and downstream sides of the hut.

An air cushion vehicle is supported by forcing air into the chamber created by a skirt around the periphery of the vehicle as shown. The air escapes through the 3 in. clearance between the lower end of the skirt and the ground (or water). Assume the vehicle weighs 10,000 lb_f and is essentially rectangular in shape, 30 by 50 ft. The volume of the chamber is large enough so that the kinetic energy of the air within the chamber is negligible. Determine the flowrate, Q, needed to support the vehicle.



SOLUTION:

The weight of the vehicle is supported by the increased pressure within the chamber.



A simple force balance gives:

$$W = \left(p_1 - p_{\text{atm}}\right) A_{\text{projected}} \quad (1)$$

Note that we have neglected the downward momentum flux of the air caused by the fan since it will be negligible when compared to the weight of the vehicle.

The pressure within the chamber, p_1, can be found using Bernoulli's equation applied along the streamline shown in the previous figure.

$$\left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_2 = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_1 \quad (2)$$

where

$p_2 = p_{\text{atm}}$ $\qquad$ $p_1 = ?$

$V_2 = \dfrac{Q}{A_{\text{skirt}}}$ $\qquad$ $V_1 \approx 0$ (large chamber)

$z_2 \approx z_1$ (Elevation differences are negligible, especially since a gas is being considered.)

Substitute and simplify.

$$p_1 - p_{\text{atm}} = \tfrac{1}{2}\rho\left(\frac{Q}{A_{\text{skirt}}}\right)^2 \quad (3)$$

Substitute Eqn. (3) into Eqn. (1) and solve for the flow rate Q.

$$W = \tfrac{1}{2}\rho\left(\frac{Q}{A_{\text{skirt}}}\right)^2 A_{\text{projected}}$$

$$\boxed{Q = \sqrt{\frac{2WA_{\text{skirt}}^2}{\rho A_{\text{projected}}}}} \quad (4)$$

Substitute the given parameters.

W = 10000 lb_f = 322,000 $lb_m ft/s^2$

A_{skirt} = (3 in.)(ft/12 in.)[2(30 ft + 50 ft)] = 40 ft^2 (rectangular cross-section)

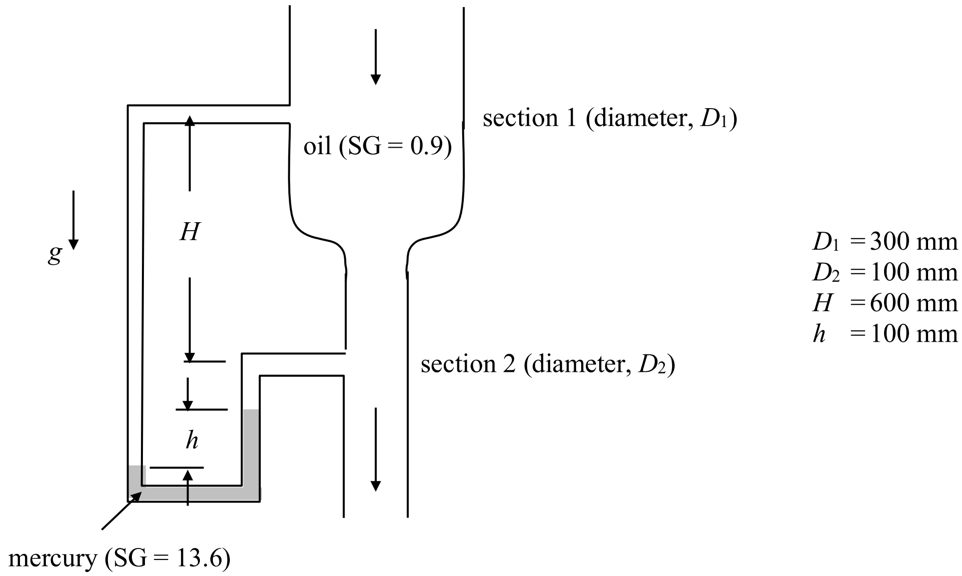
ρ = 7.68e-2 lb_m/ft^3

$A_{projected}$ = (30 ft)(50 ft) = 1500 ft^2 (rectangular cross-section)

$\Rightarrow$ $\boxed{Q = 2990 \text{ ft}^3/\text{s}}$

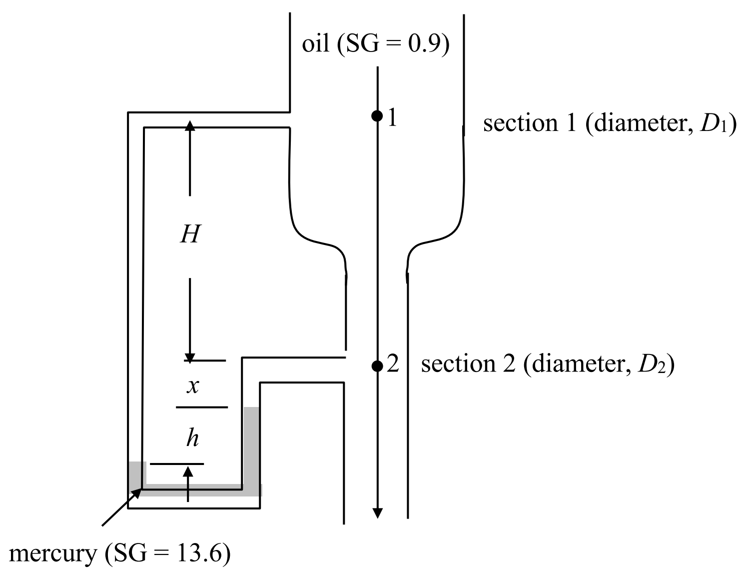
Oil flows through a contraction with circular cross-section as shown in the figure below. A manometer, using mercury as the gage fluid, is used to measure the pressure difference between sections 1 and 2 of the pipe. Assuming frictionless flow, determine:

a. the pressure difference, p_1-p_2, between sections 1 and 2, and
b. the mass flow rate through the pipe.



SOLUTION:

First determine the pressure difference using the manometer.



$$p_2 = p_1 + \rho_{\text{oil}} g(H + x + h) - \rho_{\text{Hg}} gh - \rho_{\text{oil}} gx$$
$$p_2 = p_1 + SG_{\text{oil}} \rho_{\text{H20}} g(H + h) - SG_{\text{Hg}} \rho_{\text{H20}} gh$$
$$\boxed{p_1 - p_2 = \rho_{\text{H20}} g\left[SG_{\text{Hg}} h - SG_{\text{oil}}(H + h)\right]} \quad (1)$$

Use the given parameters.

ρ_{H20} = 1000 kg/m^3
g = 9.81 m/s^2
SG_{Hg} = 13.6
h = 100e-3 m
SG_{oil} = 0.9
H = 600e-3 m
$\Rightarrow$ $\boxed{p_1 - p_2 = 7.2 \text{ kPa}}$

Now apply Bernoulli's equation along a streamline from 1 to 2 to determine the mass flow rate.

$$\left(\frac{p}{\rho_{oil}g}+\frac{V^2}{2g}+z\right)_2=\left(\frac{p}{\rho_{oil}g}+\frac{V^2}{2g}+z\right)_1$$

where

$$p_2 - p_1 = 7200 \text{ N/m}^2 \quad \text{(found previously)}$$

$$V_2=\frac{Q}{\frac{\pi D_2^2}{4}}=\frac{4Q}{\pi D_2^2} \qquad V_1=\frac{Q}{\frac{\pi D_1^2}{4}}=\frac{4Q}{\pi D_1^2}$$

$$z_1 - z_2 = H$$

Substitute and simplify.

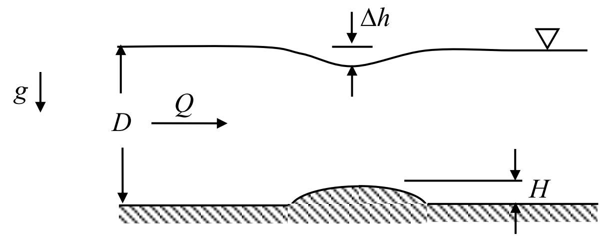
$$\frac{p_2-p_1}{\rho_{oil}g}-H=\frac{8Q^2}{\pi^2 g}\left(\frac{1}{D_1^4}-\frac{1}{D_2^4}\right)$$

$$\boxed{\dot{m}_{oil}=\rho_{oil}Q=\rho_{oil}\sqrt{\frac{\pi^2 g}{8}\left(\frac{D_1^4 D_2^4}{D_2^4-D_1^4}\right)\left(\frac{p_2-p_1}{\rho_{oil}g}-H\right)}} \quad (2)$$

Use the given parameters.

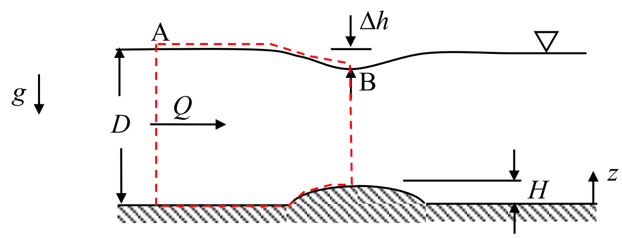
ρ_{H20} = 1000 kg/m^3
SG_{oil} = 0.9
g = 9.81 m/s^2
H = 600e-3 m
D_1 = 300e-3 m
D_2 = 100e-3 m
$p_1 - p_2$ = 7200 N/m^2
$\Rightarrow$ $\boxed{\dot{m} = 37.5 \text{ kg/s}}$

If the approach velocity is not too large, a hump of height, H, in the bottom of a water channel will cause a dip of magnitude Δh in the water level. This depression in the water can be used to determine the flow rate of the water. Assuming no losses and that the incoming flow has a depth, D, determine the volumetric flow rate, Q, as a function of Δh, H, D, and g (the acceleration due to gravity).



SOLUTION:

Assume steady, incompressible, inviscid flow with uniform velocity profiles at the inlet and outlet of the control volume.



Apply Bernoulli's Equation along a streamline on the free surface from point A to point B.

$$\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_B=\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_A \tag{1}$$

where

$$p_B=p_A=p_{\text{atm}} \tag{2}$$

$$V_B=\frac{Q}{D-H-\Delta h} \quad \text{and} \quad V_A=\frac{Q}{D} \tag{3}$$

$$z_B=D-\Delta h \quad \text{and} \quad z_A=D \tag{4}$$

Substitute and simplify.

$$\frac{p_{\text{atm}}}{\rho g}+\frac{1}{2g}\left(\frac{Q}{D-H-\Delta h}\right)^2+D-\Delta h=\frac{p_{\text{atm}}}{\rho g}+\frac{1}{2g}\left(\frac{Q}{D}\right)^2+D \tag{5}$$

$$\left(\frac{Q}{D-H-\Delta h}\right)^2-2g\Delta h=\left(\frac{Q}{D}\right)^2$$

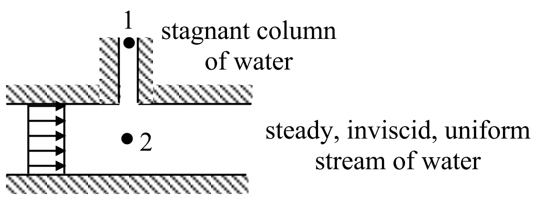
$$\boxed{\therefore Q=\sqrt{\frac{2g\Delta h}{\left(\frac{1}{D-H-\Delta h}\right)^2-\left(\frac{1}{D}\right)^2}}} \tag{6}$$

In which of the following scenarios is applying the following form of Bernoulli's equation:

$$\frac{p}{\rho g}+\frac{V^2}{2g}+z=\text{constant}$$

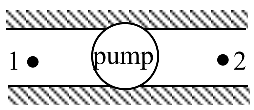
from point 1 to point 2 valid?

a.

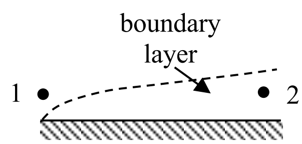


b.

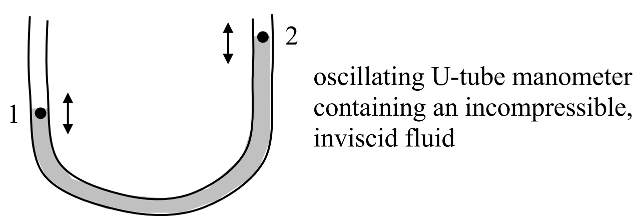
c.



d.



e.



SOLUTION:

Bernoulli's equation, as written in the problem statement, can be used in NONE of the scenarios presented.

a. The flow is rotational at the interface between the vertical and horizontal channels and, hence, Bernoulli's equation cannot be applied across the flow streamlines.

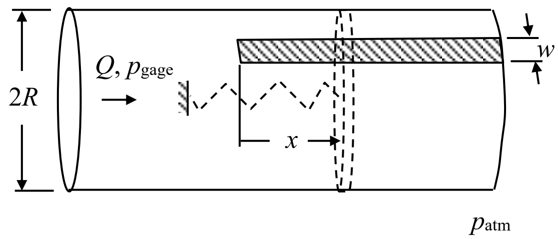
b. Since Ma > 0.3, the flow should be considered compressible. The given form of Bernoulli's equation is valid only for incompressible flows. An alternate form of Bernoulli's equation that takes compressibility effects into account could be used, however.

c. The pump between points 1 and 2 adds energy to the flow and, hence, the constant in Bernoulli's equation changes across the pump. The Extended Bernoulli's Equation (aka energy equation) could be used in this scenario instead of the given form of Bernoulli's equation.

d. Bernoulli's equation assumes inviscid flow. Viscous effects are significant in boundary layers and thus Bernoulli's equation may not be used.

e. The given form of Bernoulli's equation assumes steady flow. The oscillating U-tube is unsteady and the given Bernoulli's equation cannot be used. Note that it is possible to derive an unsteady form of Bernoulli's equation that could be used in the given situation.

The device shown in the figure below is proposed for measuring the exhalation pressure and volume flow rate of a person (the device is known as a "peak flow meter"). A circular tube, with inside radius R, has a slit of width w running down the length of it (a cut-out in the cylinder). Inside the tube is a lightweight, freely moving piston attached to a linear spring (with spring constant k). The equilibrium position of the piston is at $x = 0$ where the slit begins.



Derive equations for:

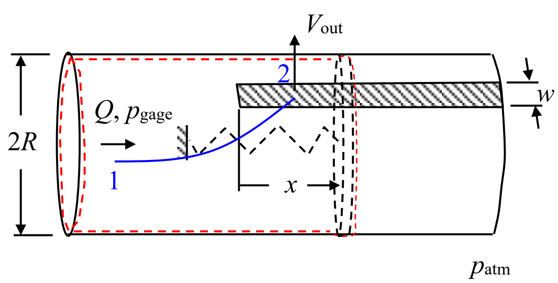
a. the volumetric flow rate, Q, and

b. the gage pressure in the tube, p_{gage},

in terms of (a subset of) the piston displacement, x, as well as the tube radius, R, slit width, w, spring constant, k, and the properties of air. Assume that the slit width, w, is so small that the outflow area is much smaller than the tube's cross-sectional area, πR^2, even at the piston's full extension.

SOLUTION:

Apply conservation of mass to the control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(at steady state)} \tag{2}$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho Q + \rho V_{\text{out}} wx \tag{3}$$

Substitute and simply to get:

$$-\rho Q + \rho V_{\text{out}} wx = 0 \tag{4}$$

$$Q = V_{\text{out}} wx \tag{5}$$

where V_{out} is the speed of the air flowing out of the slit. This speed may be found by applying Bernoulli's equation from a point located within the tube (1) and a point just at the slit exit (2).

$$\left(p + \tfrac{1}{2}\rho V^2\right)_1 = \left(p + \tfrac{1}{2}\rho V^2\right)_2 \tag{6}$$

where

$$p_1 = p_{\text{gage}} \tag{7}$$

$$p_2 = 0 \; (p_{\text{atm,gage}} = 0) \tag{8}$$

$$V_1 = Q/(\pi R^2) \tag{9}$$

$$V_2 = V_{\text{out}} \tag{10}$$

Since the slit area is much smaller than the outlet area, $V_1 << V_2$, Eqn. (6) becomes

$$V_{\text{out}} = \sqrt{\frac{2p_{\text{gage}}}{\rho}} . \tag{11}$$

Substituting into Eqn. (5) gives:

$$Q = wx\sqrt{\frac{2p_{\text{gage}}}{\rho}} \tag{12}$$

The pressure, p_{gage}, may be found by balancing forces on the piston:

$$p_{\text{gage}}\pi R^2 - kx = 0 \tag{13}$$

$$\boxed{p_{\text{gage}} = \frac{kx}{\pi R^2}} \tag{14}$$

Note that we could have used the linear momentum equation in the x-direction on the same control volume to arrive at this expression (see below).

Combining Eqns. (12) and (14) gives:

$$\boxed{Q = wx^{3/2}\sqrt{\frac{2k}{\rho\pi R^2}}} \tag{15}$$

Thus, by measure the displacement of the piston on the simple device shown in the figure, lung functions such as pressure and volumetric flow rate can be easily determined.

Note that we could have also worked out Eq. (14) of this problem using the linear momentum equation in the x-direction applied to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} u_x\rho\, dV + \int_{\text{CS}} u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{16}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x\rho\, dV = 0 \quad \text{(steady flow)} \tag{17}$$

$$\int_{\text{CS}} u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \frac{Q}{\pi R^2}\left(-\rho\frac{Q}{\pi R^2}\pi R^2\right) = -\rho\frac{Q^2}{\pi R^2} \quad \text{(no } x\text{-momentum flux out through side)} \tag{18}$$

$$F_{B,x} = 0 \tag{19}$$

$$F_{S,x} = p_{\text{gage}}\pi R^2 - kx \tag{20}$$

Substitute and simplify.

$$-\rho\frac{Q^2}{\pi R^2} = p_{\text{gage}}\pi R^2 - kx \tag{21}$$

Substituting from Eq. (12) ,

$$-\rho\frac{(wx)^2\left(2p_{\text{gage}}/\rho\right)}{\pi R^2} = p_{\text{gage}}\pi R^2 - kx \tag{22}$$

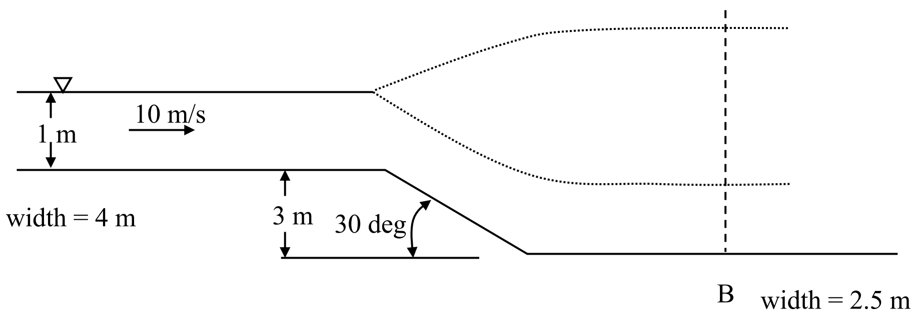
$$-\left[2\left(\frac{wx}{\pi R^2}\right)^2 + 1\right]\pi R^2 p_{\text{gage}} = -kx \tag{23}$$

But since $wx \ll \pi R^2$,

$$p_{\text{gage}} = \frac{kx}{\pi R^2} \quad \text{which is the same as Eq. (14)!} \tag{24}$$

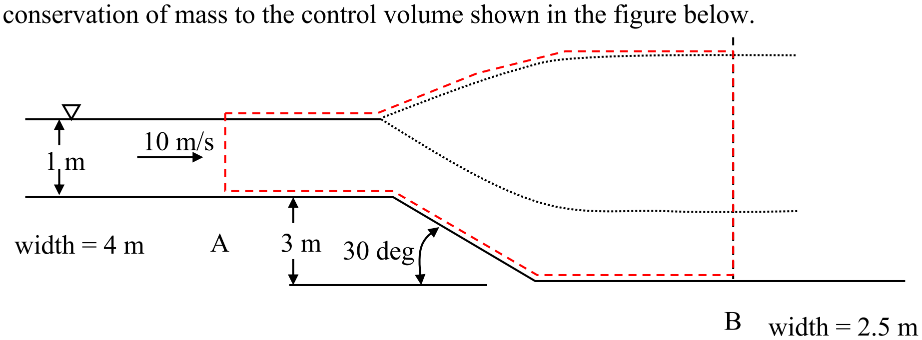
Water 1 m deep is flowing steadily at 10 m/s in a channel 4 m wide. The channel drops 3 m at 30 deg, and simultaneously narrows to 2.5 m as shown in the accompanying sketch.

Determine the two possible water depths at downstream station B. Neglect all losses.



SOLUTION:

Apply conservation of mass to the control volume shown in the figure below.



$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)} \tag{2}$$

$$\int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = -(\rho V z w)_A + (\rho V z w)_B \tag{3}$$

Substitute and simplify noting that the water density remains constant.

$$-(\rho V z w)_A + (\rho V z w)_B = 0 \Rightarrow (Vzw)_B = (Vzw)_A \Rightarrow V_B = V_A\left(\frac{z_A}{z_B}\right)\left(\frac{w_A}{w_B}\right) \tag{4}$$

Now apply Bernoulli's equation along the free surface of the stream from point A to point B.

$$\left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_B = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_A \tag{5}$$

where

$$p_A = p_B = p_{atm} \tag{6}$$

$$\frac{V_B^2}{2g} + z_B = \frac{V_A^2}{2g} + z_A \tag{7}$$

Substitute Eq. (4) and solve for z_B.

$$\frac{V_A^2 \left(\frac{z_A}{z_B}\right)^2 \left(\frac{w_A}{w_B}\right)^2}{2g} + z_B = \frac{V_A^2}{2g} + z_A \tag{8}$$

$$V_A^2 \left(\frac{z_A}{z_B}\right)^2 \left(\frac{w_A}{w_B}\right)^2 + 2gz_B = V_A^2 + 2gz_A \tag{9}$$

$$V_A^2 z_A^2 \left(\frac{w_A}{w_B}\right)^2 + 2gz_B^3 = \left(V_A^2 + 2gz_A\right) z_B^2 \tag{10}$$

$$z_B^3 - \left(\frac{V_A^2}{2g} + z_A\right) z_B^2 + \frac{V_A^2}{2g} z_A^2 \left(\frac{w_A}{w_B}\right)^2 = 0 \tag{11}$$

Using the given parameters:

V_A = 10 m/s
g = 9.81 m/s^2
z_A = 1 m
w_A = 4 m
w_B = 2.5 m

Eq. (11) may be written as,

$$z_B^3 - (6.097 \text{ m}) z_B^2 + (13.05 \text{ m}) = 0 \tag{12}$$

Solve Eq. (12) numerically to get,

$$z_B = 1.728 \text{ m}, 5.695 \text{ m}, -1.326 \text{ m} \tag{13}$$

Thus, the two possible depths at location B are: **1.7 m and 5.7 m**.

A Venturi pump is used in the design of a carburetor, a device used to create a fuel-air mixture to be fed into the cylinder of an internal combustion engine. Simplified schematics of a carburetor are shown in the following figures.

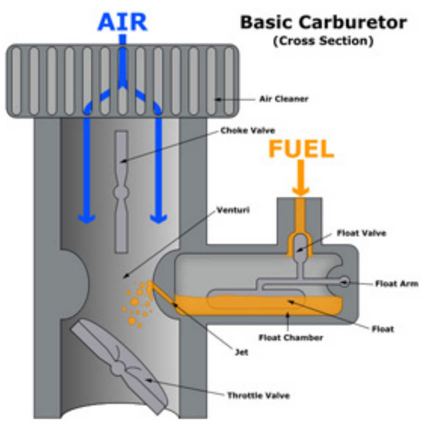
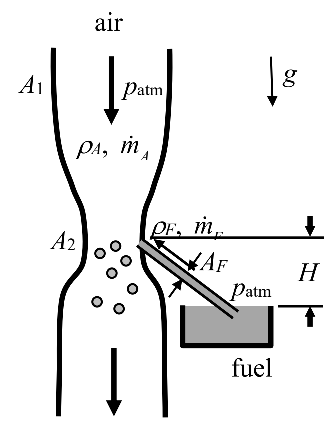


Image from: http://hdabob.com/wp-content/uploads/2009/10/carburetor.jpg

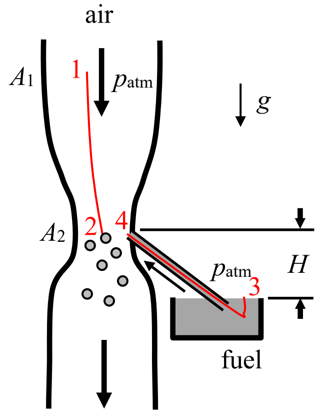


The air, which may reasonably be assumed to be incompressible, has a density ρ_A and the liquid fuel has density ρ_F. The fuel reservoir is located a distance H below the inlet port into the Venturi. The inlet air is at atmospheric pressure as is the free surface of the fuel reservoir. The air inlet cross-sectional area is A_1 and the Venturi throat area is A_2. The fuel line cross-sectional area is A_F.

If the desired air-to-fuel mass flow rate ratio at the outlet of the carburetor is R (= $\dot{m}_A / \dot{m}_F$), determine the required ratio A_1/A_2 in terms of (a subset of) the air-to-fuel ratio R, air density ρ_A, the fuel density ρ_F, the inlet air mass flow rate $\dot{m}_A$, the acceleration due to gravity g, the height from the fuel reservoir to the Venturi throat H, the fuel pipe area A_F, and the air inlet area A_1.

SOLUTION:

Apply Bernoulli's equation from 1 to 2.



$$\left(\frac{p}{\rho_A g}+\frac{V^2}{2g}+z\right)_2=\left(\frac{p}{\rho_A g}+\frac{V^2}{2g}+z\right)_1 \tag{1}$$

where

$p_1 = p_{atm}$ and $p_2 = ?$ (2)

$V_1=\frac{\dot{m}_A}{\rho_A A_1}$ and $V_2=\frac{\dot{m}_A}{\rho_A A_2}$ (3)

Δz is negligible compared to the other terms in B.E. since the fluid is a gas (4)

Substitute and simplify.

$$p_2-p_{atm}=\frac{1}{2}\frac{\dot{m}_A^2}{\rho_A}\left(\frac{1}{A_1^2}-\frac{1}{A_2^2}\right) \tag{5}$$

Apply Bernoulli's equation from 3 to 4.

$$\left(\frac{p}{\rho_F g}+\frac{V^2}{2g}+z\right)_4=\left(\frac{p}{\rho_F g}+\frac{V^2}{2g}+z\right)_3 \tag{6}$$

where

$p_3 = p_{atm}$ and $p_4 = p_2$ (7)

$V_3 \approx 0$ and $V_4=\frac{\dot{m}_F}{\rho_F A_F}$ (8)

$\Delta z = z_4 - z_3 = H$ (9)

Substitute and simplify.

$$\underbrace{p_4}_{=p_2}-p_{atm}=-\rho_F gH-\frac{1}{2}\frac{\dot{m}_F^2}{\rho_F A_F^2} \tag{10}$$

Combine Eqs. (5) and (10) and solve for A_1/A_2.

$$\frac{1}{2}\frac{\dot{m}_A^2}{\rho_A}\left(\frac{1}{A_1^2}-\frac{1}{A_2^2}\right)=-\rho_F gH-\frac{1}{2}\frac{\dot{m}_F^2}{\rho_F A_F^2} \tag{11}$$

$$\left(\frac{1}{A_1^2}-\frac{1}{A_2^2}\right)=-2\rho_A\rho_F\frac{gH}{\dot{m}_A^2}-\frac{1}{A_F^2}\frac{\rho_A}{\rho_F}\frac{\dot{m}_F^2}{\dot{m}_A^2} \tag{12}$$

$$\left(1-\frac{A_1^2}{A_2^2}\right)=-2\rho_A\rho_F\frac{gHA_1^2}{\dot{m}_A^2}-\frac{A_1^2}{A_F^2}\frac{\rho_A}{\rho_F}\frac{\dot{m}_F^2}{\dot{m}_A^2} \tag{13}$$

$$\left(\frac{A_1}{A_2}\right)^2=1+2\rho_A\rho_F\frac{gHA_1^2}{\dot{m}_A^2}+\left(\frac{A_1}{A_F}\right)^2\left(\frac{\rho_A}{\rho_F}\right)\left(\frac{\dot{m}_F}{\dot{m}_A}\right)^2 \tag{14}$$

$$\boxed{\frac{A_1}{A_2}=\sqrt{1+2\rho_A\rho_F\frac{gHA_1^2}{\dot{m}_A^2}+\left(\frac{A_1}{A_F}\right)^2\left(\frac{\rho_A}{\rho_F}\right)\left(\frac{\dot{m}_F}{\dot{m}_A}\right)^2}} \text{ or } \boxed{\frac{A_1}{A_2}=\sqrt{1+2\rho_A\rho_F\frac{gHA_1^2}{\dot{m}_A^2}+\left(\frac{A_1}{A_F}\right)^2\left(\frac{\rho_A}{\rho_F}\right)\frac{1}{R^2}}} \tag{15}$$

For a typical carburetor,

ρ_F = 770 kg/m^3 (gasoline)
ρ_A = 1.23 kg/m^3 (air)
A_1 = 1.34*10^{-3} m^2 (D_1 = 4.13cm = 1 5/8 in.)
A_F = 1.70*10^{-6} m^2 (D_F = 1.47 mm = 0.058 in.)
R = 14.7 (ideal fuel to air ratio for gasoline)
g = 9.81 m/s^2
H = 2.00*10^{-2} m (= 2 cm)
$\dot{m}_A$ = 0.290 kg/s (500 cfm @ 1.23 kg/m^3)
$\Rightarrow$ A_1/A_2 = 2.36 $\Rightarrow$ D_2 = 2.69 cm