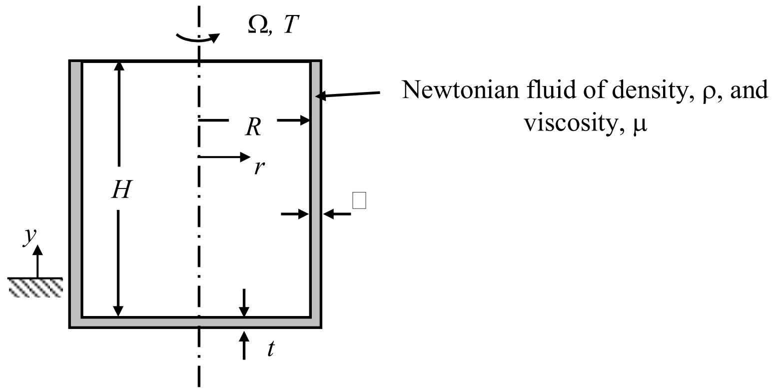
A rotating cylinder viscometer is shown in the figure below. The inner cylinder has radius, R, and height, H. An incompressible, viscous, Newtonian fluid of density, ρ, and viscosity, μ, is contained between the cylinders. The narrow gap between the cylinders has width, t ($\ll R$ and H). A torque, T, is required to rotate the inner cylinder at constant speed Ω. Determine the fluid viscosity, μ, in terms of the other system parameters.



SOLUTION:

Assume that the velocity profiles in the narrow gaps are linear since the gap size is small and curvature may be neglected ($t/R << 1$) . In the circumferential gap the fluid velocity gradient will be:

$$\left.\frac{du_\theta}{dr}\right|_{\text{circumference}} = \frac{0-R\Omega}{\left(R+t\right)-R} = -\frac{R\Omega}{t} \tag{1}$$

In the gap at the bottom of the cylinder the fluid velocity gradient will be:

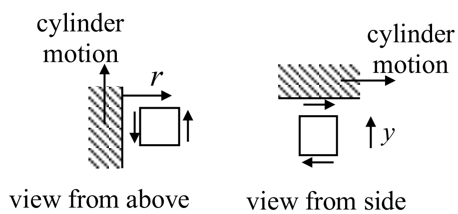
$$\left.\frac{du_\theta}{dy}\right|_{\text{bottom}} = \frac{r\Omega-0}{t-0} = \frac{r\Omega}{t} \tag{2}$$

The shear forces acting on the fluid at the inner cylinder wall and base will be:

$$dF_{\text{circumference}} = \tau\Big|_{\text{circumference}}\left(2\pi R dy\right) = \mu\left.\frac{du_\theta}{dr}\right|_{\text{circumference}}\left(2\pi R dy\right) = -\mu\frac{R\Omega}{t}\left(2\pi R dy\right)$$

$$dF_{\text{bottom}} = \tau\Big|_{\text{bottom}}\left(2\pi r dr\right) = \mu\left.\frac{du_\theta}{dr}\right|_{\text{bottom}}\left(2\pi r dr\right) = \mu\frac{r\Omega}{t}\left(2\pi r dr\right)$$

positive shear stresses on a fluid element adjacent to the cylinder



Note that the force acting on the cylinder will be equal in magnitude but in the opposite direction to the force acting on the fluid (Newton's 3rd Law).

The total torque acting on the cylinder (neglected the contribution from the corner regions) is:

$$\begin{aligned}
T &= \int_{y=0}^{y=H} dT_{\text{circumference}} + \int_{r=0}^{r=R} dT_{\text{bottom}} \\
&= \int_{y=0}^{y=H} R\, dF_{\text{circumference}} + \int_{r=0}^{r=R} r\, dF_{\text{bottom}} \\
&= \int_0^H R\mu\frac{R\Omega}{t}\left(2\pi R dy\right) + \int_0^R r\mu\frac{r\Omega}{t}\left(2\pi r dr\right) \\
&= 2\pi\mu\frac{R^3\Omega}{t}H + \frac{2\pi}{4}\mu\frac{R^4\Omega}{t}
\end{aligned}$$

$$T = 2\pi\mu\frac{\Omega R^3}{t}\left(H+\tfrac{1}{4}R\right) \tag{3}$$

Re-arrange to solve for the viscosity:

$$\boxed{\mu = \frac{T}{2\pi\frac{\Omega R^3}{t}\left(H+\frac{1}{4}R\right)}} \tag{4}$$