

Consider an ice cube with initial volume  $V_{ice,0}$  floating in a cup of water of initial volume  $V_{water,0}$ . The specific gravity of ice is  $SG_{ice}$ . Show mathematically that, as the ice cube melts, the water level in the cup remains unchanged.



SOLUTION:

If a mass of ice,  $\Delta m_{\text{ice}}$ , melts ( $\Delta m_{\text{ice}} < 0$ ), it will correspond to an equal increase in water,  $\Delta m_{\text{water}}$ , i.e.,

$$\Delta m_{\text{water}} = -\Delta m_{\text{ice}} . \quad (1)$$

Expressed in terms of volumes,

$$\rho_{\text{water}} \Delta V_{\text{water}} = -\rho_{\text{ice}} \Delta V_{\text{ice}} = -SG_{\text{ice}} \rho_{\text{water}} \Delta V_{\text{ice}} , \quad (2)$$

$$\Delta V_{\text{water}} = -SG_{\text{ice}} \Delta V_{\text{ice}} . \quad (3)$$

The volume of water displaced by the ice is found by equating the weight of the displaced water to the weight of the ice (Archimedes Law),

$$\rho_{\text{water}} V_{\text{water,disp}} g = \rho_{\text{ice}} V_{\text{ice}} g = SG_{\text{ice}} \rho_{\text{water}} V_{\text{ice}} g , \quad (4)$$

$$V_{\text{water,disp}} = SG_{\text{ice}} V_{\text{ice}} . \quad (5)$$

Thus, if a volume of ice melts,  $\Delta V_{\text{ice}}$ , then the amount of water displaced, in order to balance the new ice weight, is,

$$\Delta V_{\text{water,disp}} = SG_{\text{ice}} \Delta V_{\text{ice}} . \quad (6)$$

Note that if the ice melts ( $\Delta V_{\text{ice}} < 0$ ), less water needs to be displaced to support the (smaller) ice weight ( $\Delta V_{\text{water,disp}} < 0$ ).

Thus, the sum of the volume of water added due to melting and the change in displaced water volume due to a change in the weight of the ice is,

$$\Delta V_{\text{water}} + \Delta V_{\text{water,disp}} = -SG_{\text{ice}} \Delta V_{\text{ice}} + SG_{\text{ice}} \Delta V_{\text{ice}} = 0 . \quad (7)$$

The increase in water volume is exactly balanced by a decrease in the displaced water volume, which means that the water level height won't change!

This fact has important implications regarding the rise in sea level due to melting ice. Melting free-floating ice, e.g., icebergs, won't result in an increase in sea level. However, ice that was originally supported by land, e.g., glaciers, will contribute to an increase in sea levels.