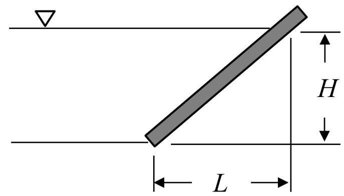
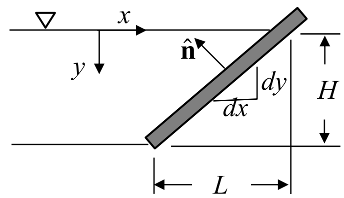
Calculate the net horizontal and vertical forces acting on the planar surface shown below. The surface has a width *w* into the page.



SOLUTION:

One approach to finding the net force on the wall is to integrate the pressure force along the wall,

$$\mathbf{F}_p = \int_A -p\,d\mathbf{A}\,, \qquad (1)$$



where,

$$p = \rho g y\,, \qquad (2)$$

and,

$$d\mathbf{A} = -wdy\hat{\mathbf{e}}_x - wdx\hat{\mathbf{e}}_y\,. \qquad (3)$$

Note that since we'll be integrating in the y direction (since the pressure varies in that direction), we should express dx in terms of dy,

$$\frac{dy}{dx} = \frac{H}{L} \Rightarrow dx = \left(\frac{L}{H}\right)dy\,. \qquad (4)$$

Substituting and integrating as y goes from zero to H,

$$\mathbf{F}_p = \int_{y=0}^{y=H} -(\rho g y)\left[-wdy\hat{\mathbf{e}}_x - w\left(\frac{L}{H}\right)dy\hat{\mathbf{e}}_y\right] = \rho g w\left[\hat{\mathbf{e}}_x \int_{y=0}^{y=H} y\,dy + \hat{\mathbf{e}}_y\left(\frac{L}{H}\right)\int_{y=0}^{y=H} y\,dy\right], \qquad (5)$$

$$\mathbf{F}_p = \rho g w\left[\tfrac{1}{2}H^2\hat{\mathbf{e}}_x + \tfrac{1}{2}\left(\frac{L}{H}\right)H^2\hat{\mathbf{e}}_y\right], \qquad (6)$$

$$\boxed{\mathbf{F}_p = \tfrac{1}{2}\rho g w H^2\hat{\mathbf{e}}_x + \tfrac{1}{2}\rho g w L H\hat{\mathbf{e}}_y}\,. \qquad (7)$$

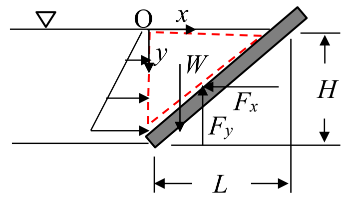
We could have also solved the integral by splitting it into two parts,

$$\mathbf{F}_p = \int_{y=0}^{y=H} -(\rho g y)(-wdy\hat{\mathbf{e}}_x) + \int_{x=0}^{x=L} -(\rho g y)(-wdx\hat{\mathbf{e}}_y) = \tfrac{1}{2}\rho g w H^2\hat{\mathbf{e}}_x + \int_{x=0}^{x=L} -\left[\rho g\left(\frac{H}{L}\right)x\right](-wdx\hat{\mathbf{e}}_y)\,, \qquad (8)$$

$$\mathbf{F}_p = \tfrac{1}{2}\rho g w H^2\hat{\mathbf{e}}_x + \tfrac{1}{2}\rho g w\left(\frac{H}{L}\right)L^2\hat{\mathbf{e}}_y = \tfrac{1}{2}\rho g w H^2\hat{\mathbf{e}}_x + \tfrac{1}{2}\rho g w H L\hat{\mathbf{e}}_y \quad \text{Same answer as before!} \qquad (9)$$

Note that in the 2nd integral in Eq. (8), the y dependence on x needed to be made explicit in order to integrate properly with respect to x. An approach similar to what was used to derive Eq. (4) was utilized.

An alternate approach to solving this problem is to balance forces on the dashed volume of fluid shown below.



$$\sum F_x = 0 = \int_{y=0}^{y=H} (\rho g y)(w dy) - F_x \Rightarrow F_x = \tfrac{1}{2}\rho g w H^2 \quad \text{The same answer as before!} \tag{10}$$

$$\sum F_y = 0 = W - F_y = \rho \tfrac{1}{2} LHwg - F_y \Rightarrow F_y = \tfrac{1}{2}\rho g L H w \quad \text{The same answer as before!} \tag{11}$$

Note that from Newton's 3rd Law, the force the wall exerts on the fluid is equal and opposite to the force the fluid exerts on the wall.

To find the center of pressure, balance moments about the origin,

$$\sum \boldsymbol{M}_O = \boldsymbol{0} = \boldsymbol{r}_{CP} \times \boldsymbol{F}_R + \boldsymbol{r}_{CM} \times \boldsymbol{W} + \int_{y=0}^{y=H} y\hat{\boldsymbol{j}} \times dF_p \hat{\boldsymbol{\imath}}, \tag{12}$$

where,

$$\boldsymbol{r}_{CP} = x_{CP}\hat{\boldsymbol{\imath}} + y_{CP}\hat{\boldsymbol{j}}, \tag{13}$$

$$\boldsymbol{F}_R = -F_x\hat{\boldsymbol{\imath}} - F_y\hat{\boldsymbol{j}} = -\tfrac{1}{2}\rho g w H^2\hat{\boldsymbol{\imath}} - \tfrac{1}{2}\rho g L H\hat{\boldsymbol{j}}, \tag{14}$$

$$\boldsymbol{W} = \tfrac{1}{2}\rho g w L H\hat{\boldsymbol{j}}, \tag{15}$$

$$\boldsymbol{r}_{CM} = \tfrac{1}{3}L\hat{\boldsymbol{\imath}} + \tfrac{1}{3}H\hat{\boldsymbol{j}}, \tag{16}$$

$$\int_{y=0}^{y=H} y\hat{\boldsymbol{j}} \times dF_p\hat{\boldsymbol{\imath}} = \int_0^H y\hat{\boldsymbol{j}} \times \rho g y w dy\hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}\rho g w \int_0^H y^2 dy = -\tfrac{1}{3}\rho g w H^3\hat{\boldsymbol{k}} \tag{17}$$

Substitute and simplify,

$$\boldsymbol{0} = (x_{CP}\hat{\boldsymbol{\imath}} + y_{CP}\hat{\boldsymbol{j}}) \times \left(-\tfrac{1}{2}\rho g w H^2\hat{\boldsymbol{\imath}} - \tfrac{1}{2}\rho g L H\hat{\boldsymbol{j}}\right) + \left(\tfrac{1}{3}L\hat{\boldsymbol{\imath}} + \tfrac{1}{3}H\hat{\boldsymbol{j}}\right) \times \left(\tfrac{1}{2}\rho g w L H\hat{\boldsymbol{j}}\right) - \tfrac{1}{3}\rho g w H^3\hat{\boldsymbol{k}}, \tag{18}$$

$$\boldsymbol{0} = y_{CP}\tfrac{1}{2}\rho g w H^2\hat{\boldsymbol{k}} - x_{CP}\tfrac{1}{2}\rho g L H\hat{\boldsymbol{k}} + \tfrac{1}{6}\rho g w L^2 H\hat{\boldsymbol{k}} - \tfrac{1}{3}\rho g w H^3\hat{\boldsymbol{k}}, \tag{19}$$

$$y_{CP}H^2 = x_{CP}LH - \tfrac{1}{3}L^2H + \tfrac{2}{3}H^3, \tag{20}$$

$$y_{CP} = x_{CP}\tfrac{L}{H} - \tfrac{1}{3}L\tfrac{L}{H} + \tfrac{2}{3}H, \tag{21}$$

Note that x_{CP} and y_{CP} are related since they are located somewhere along the wall,

$$y_{CP} = -\tfrac{H}{L}x_{CP} + H. \tag{22}$$

Substituting Eq. (22) into Eq. (21) and continuing to simplify,

$$-\tfrac{H}{L}x_{CP} + H = x_{CP}\tfrac{L}{H} - \tfrac{1}{3}L\tfrac{L}{H} + \tfrac{2}{3}H, \tag{23}$$

$$\left(-\tfrac{H}{L} - \tfrac{L}{H}\right)x_{CP} = -\tfrac{1}{3}L\tfrac{L}{H} - \tfrac{1}{3}H, \tag{24}$$

$$\left(\tfrac{H^2+L^2}{HL}\right)x_{CP} = \tfrac{1}{3}L\tfrac{L}{H} + \tfrac{1}{3}H, \tag{25}$$

$$x_{CP} = \tfrac{1}{3}\left(\tfrac{HL}{H^2+L^2}\right)\left(\tfrac{L^2+H^2}{H}\right). \tag{26}$$

Thus,

$$x_{CP} = \tfrac{1}{3}L, \tag{27}$$

and from Eq. (22),

$$y_{CP} = \tfrac{2}{3}H. \tag{27}$$