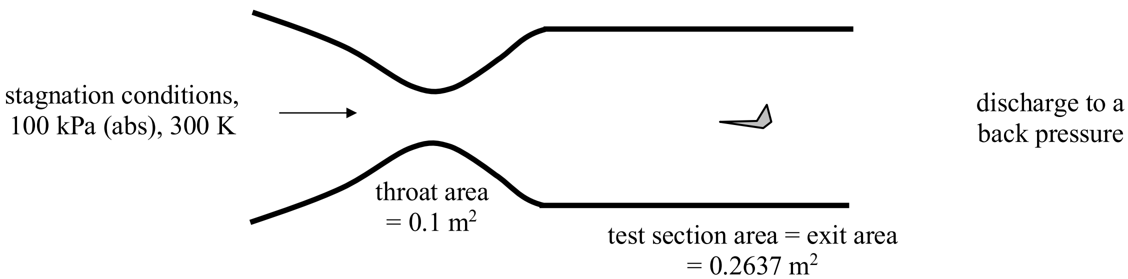
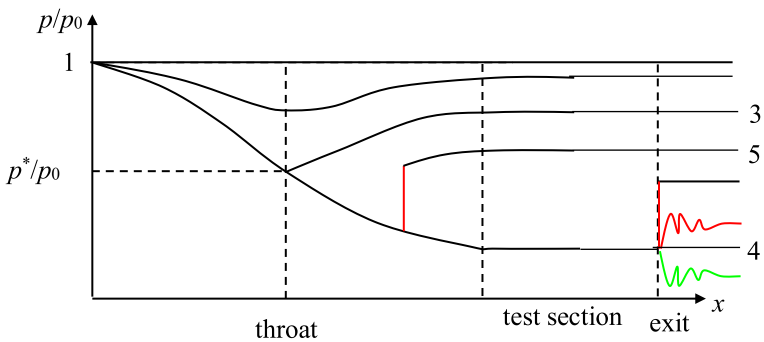
Consider the supersonic wind tunnel shown in the following schematic. Air is the working fluid and the test section area is constant.





a. What is the design Mach number of the test section?
b. What is the mass flow rate through the wind tunnel at design conditions?
c. What is the maximum back pressure at which the throat will reach sonic conditions?
d. Assume a shock wave stands in the diverging section where the area is 0.1688 m^2. What is the back pressure at these conditions?

SOLUTION:

The test section design Mach number may be found using the isentropic sonic area ratio and choosing the supersonic test section Mach number (case 4 in the diagram above),

$$\frac{A_{TS}}{A_T} = \frac{0.2637 \text{ m}^2}{0.1 \text{ m}^2} = 2.637 = \frac{A_{TS}}{A^*} = \frac{1}{\text{Ma}_{TS}}\left(\frac{1+\frac{k-1}{2}\text{Ma}_{TS}^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \boxed{\text{Ma}_{TS} = 2.50}. \quad (1)$$

Note that at design conditions, the throat Mach number is one.

The flow through the wind tunnel will be choked at design conditions, with a mass flow rate of,

$$\dot{m}_{\text{choked}} = \left(1+\frac{k-1}{2}\right)^{\frac{k+1}{2(1-k)}} p_0\sqrt{\frac{k}{RT_0}}A^* \Rightarrow \boxed{\dot{m} = 23.3 \text{ kg/s}}, \quad (2)$$

where $A^* = A_T$.

When the throat just reaches sonic conditions (case 3 in the diagram above), the throat area will equal the sonic area ($A^* = A_T$) and the exit Mach number may be found using the isentropic sonic area ratio since the flow through the entire converging-diverging nozzle will be subsonic (no shock waves),

$$\frac{A_E}{A_T} = \frac{0.2637 \text{ m}^2}{0.1 \text{ m}^2} = 2.637 = \frac{A_E}{A^*} = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{k-1}{2}\text{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \text{Ma}_E = 0.2263. \quad (3)$$

The exit pressure may be found from this Mach number using the isentropic stagnation pressure ratio,

$$\frac{p_E}{p_0} = \left(1+\frac{k-1}{2}\text{Ma}_E^2\right)^{\frac{k}{1-k}} \Rightarrow p_E/p_0 = 0.9650 \Rightarrow p_E = 96.5 \text{ kPa (abs)}, \quad (4)$$

using p_0 = 100 kPa (abs). Since the exit Mach number is subsonic, the exit and back pressures are equal. Hence,

$$\boxed{p_B = p_E = 96.5 \text{ kPa (abs)}}. \quad (5)$$

The Mach number just upstream of the shock wave may be found using the isentropic sonic area ratio since the flow leading up to the shock wave is isentropic and the throat area is at sonic conditions (since shock waves only form in supersonic flows, case 5 in the diagram shown above),

$$\frac{A_1}{A_T} = \frac{0.1688 \text{ m}^2}{0.1 \text{ m}^2} = 1.688 = \frac{A_1}{A^*} = \frac{1}{\text{Ma}_1}\left(\frac{1+\frac{k-1}{2}\text{Ma}_1^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \text{Ma}_1 = 2.00. \quad (6)$$

The stagnation pressure ratio and sonic area ratio across the shock are,

$$\frac{p_{02}}{p_{01}} = \frac{A_1^*}{A_2^*} = \left[\frac{(k+1)\text{Ma}_1^2}{2+(k-1)\text{Ma}_1^2}\right]^{k/(k-1)}\left[\frac{k+1}{2k\text{Ma}_1^2-(k-1)}\right]^{1/(k-1)} \Rightarrow p_{02}/p_{01} = A^*_1/A^*_2 = 0.7209. \quad (7)$$

The flow downstream of the shock wave is isentropic and subsonic. Thus, the pressure at the exit may be found

$$\frac{A_E}{A_2^*} = \frac{A_E}{A_1^*}\frac{A_1^*}{A_2^*} = \frac{A_E}{A_T}\frac{A_1^*}{A_2^*} = \left(\frac{0.2637 \text{ m}^2}{0.1 \text{ m}^2}\right)(0.7209) = 1.901 = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{k-1}{2}\text{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \text{Ma}_E = 0.3240. \quad (8)$$

The exit pressure may be found from the isentropic stagnation pressure ratio downstream of the shock and the exit Mach number,

$$\frac{p_{TS}}{p_{02}} = \left(1+\frac{k-1}{2}\text{Ma}_{TS}^2\right)^{\frac{k}{1-k}} \Rightarrow p_{TS}/p_{02} = 0.9299. \quad (9)$$

Accounting for the change in stagnation pressure ratio across the shock,

$$p_{TS} = \left(\frac{p_{TS}}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right)p_{01} = (0.9299)(0.7209)(100 \text{ kPa}) \Rightarrow p_{TS} = 67.03 \text{ kPa (abs)} \quad (10)$$

Since the exit is at a subsonic Mach number, the exit and back pressures are equal,

$$\boxed{p_B = p_E = 67.0 \text{ kPa (abs)}}. \quad (11)$$