A one-dimensional, unsteady velocity field is given by:

$$\mathbf{u} = U \sin\left[\omega\left(t - y/V\right)\right]\hat{\mathbf{e}}_x + V\hat{\mathbf{e}}_y$$

where U, V, and ω are positive constants. Find the equations of the streamline, streakline, and pathline that pass through the point (0, 0) at time $t = 0$.

SOLUTION:

The slope of the streamline is tangent to the velocity vector.

$$\frac{dy}{dx} = \frac{u_y}{u_x} \tag{1}$$

where

$$u_x = U \sin\left[\omega\left(t - y/V\right)\right] \tag{2}$$

$$u_y = V \tag{3}$$

Substitute Eqns. (2) and (3) into Eqn. (1) and solve the resulting differential equation.

$$\frac{dy}{dx} = \frac{V}{U \sin\left[\omega\left(t - y/V\right)\right]}$$

$$\int_{y_0}^{y} \sin\left[\omega\left(t - y/V\right)\right] dy = \frac{V}{U}\int_{x_0}^{x} dx$$

$$\frac{V}{\omega}\left\{\cos\left[\omega\left(t - y/V\right)\right] - \cos\left[\omega\left(t - y_0/V\right)\right]\right\} = \frac{V}{U}\left(x - x_0\right)$$

The streamline passing through the point $(x_0, y_0) = (0, 0)$ at time $t = 0$ is given by:

$$\boxed{x = \frac{U}{\omega}\left[\cos\left(\frac{\omega y}{V}\right) - 1\right]} \tag{4}$$

A streakline is a line that connects all of the fluid particles that pass through the same point in space. The equation for the streakline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = U \sin\left[\omega\left(t - y/V\right)\right] \tag{5}$$

$$u_y = \frac{dy}{dt} = V \tag{6}$$

Solve the second differential equation first (Eqn. (6)) since Eqn. (5) has a y term which is currently an unknown function of t.

$$\int_{y_0}^{y} dy = V\int_{t_0}^{t} dt \qquad \Rightarrow \qquad y - y_0 = V\left(t - t_0\right) \tag{7}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the streakline. Now that we know how y varies with t, substitute Eqn. (7) into Eqn. (5) and solve the resulting differential equation.

$$\frac{dx}{dt} = U \sin\left[\omega\left(t - \frac{y_0 + V\left(t - t_0\right)}{V}\right)\right] = U \sin\left[\omega\left(t - \frac{y_0}{V} - t + t_0\right)\right] = U \sin\left[\omega\left(t_0 - \frac{y_0}{V}\right)\right]$$

$$\int_{x_0}^{x} dx = U \sin\left[\omega\left(t_0 - \frac{y_0}{V}\right)\right]\int_{t_0}^{t} dt$$

$$x - x_0 = U \sin\left[\omega\left(t_0 - \frac{y_0}{V}\right)\right]\left(t - t_0\right) \tag{8}$$

Hence, the streakline passing through the point $(x_0, y_0) = (0, 0)$ at time $t = 0$ is given parametrically (in t_0) as:

$$\boxed{x = -Ut_0 \sin\left(\omega t_0\right)} \tag{9}$$

$$\boxed{y = -Vt_0} \tag{10}$$

Recall that t_0 is the time when a fluid particle passes through the point (x_0, y_0).

A pathline is a line traced out by a particular fluid particle as it moves through space. The parametric equations for a pathline will be identical to Eqns. (7) and (8) where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the pathline. Hence, the pathline for a particle passing through the point (x_0, y_0) = (0, 0) at time t_0 = 0 is given parametrically (in t) as:

$$\boxed{x = 0} \tag{11}$$

$$\boxed{y = Vt} \tag{12}$$

Note that the streamline, streakline, and pathline are all different. A plot of these lines through (0, 0) at t = 0 is shown below. Note that the pathline is a vertical line at x = 0.