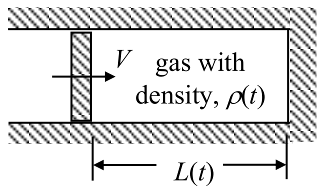
A piston compresses gas in a cylinder by moving at a constant speed, V. The gas density and the piston length are initially ρ_0 and L_0, respectively. Assume that the gas velocity varies linearly from velocity, V, at the piston face to zero velocity at the cylinder wall (at L). If the gas density varies only with time, determine $\rho(t)$.



SOLUTION:

As given in the problem statement, assume the gas velocity, u, varies linearly with distance x from the piston face with the boundary conditions: $u(x = 0) = V$ and $u(x = L(t)) = 0$.

$$\Rightarrow \quad u(x,t) = V\left(1 - \frac{x}{L(t)}\right) \tag{1}$$

However, the piston moves at a constant speed so that:

$$L(t) = L_0 - Vt \tag{2}$$

Substituting Eqn. (2) into Eqn. (1) gives:

$$u(x,t) = V\left(1 - \frac{x}{L_0 - Vt}\right) \tag{3}$$

Apply the continuity equation assuming 1D flow.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{4}$$

$$\frac{d\rho}{dt} = -\rho \frac{\partial u}{\partial x} \quad \text{(Note that } \rho = \rho(t).\text{)}$$

$$\frac{d\rho}{\rho} = V\left(\frac{1}{L_0 - Vt}\right)dt$$

$$\int_{\rho=\rho_0}^{\rho=\rho} \frac{d\rho}{\rho} = V\int_{t=0}^{t=t} \frac{dt}{L_0 - Vt}$$

$$\ln\left(\frac{\rho}{\rho_0}\right) = -\ln\left(\frac{L_0 - Vt}{L_0}\right)$$

$$\boxed{\therefore \frac{\rho}{\rho_0} = \left(1 - \frac{Vt}{L_0}\right)^{-1}} \tag{5}$$