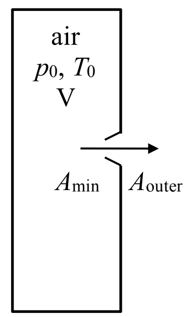
During a test docking of the Progress M-34 supply ship with the Mir space station in 1997, a collision occurred which punctures the hull of Spektr Module of Mir. Assume the puncture hole had a minimum area of 1.0 cm^2 and an outer area of 1.5 cm^2 (the size of the hole was not directly measured). The volume of the Spektr module was 61.9 m^3 and had an initial interior pressure of 100 kPa (abs) and temperature of 34 °C.

1. Determine the mass flow rate of air from the capsule when the hole initially occurred.
2. Write an equation relating how the mass of air inside the module changed with time. You may assume that the air behaved as a perfect gas throughout the entire discharge process and that the temperature remained constant inside the space station (thanks to the small discharge rate and onboard heaters).
3. Calculate the thrust acting the space station for the initial conditions.



SOLUTION:

Since the air in the space station is discharging into space, the back pressure is essentially zero and the flow will always be choked with a mass flow rate of:

$$m = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} \rho_0 \sqrt{\gamma R T_0} A^* \quad (1)$$

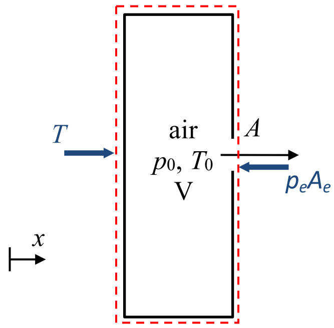
where

$$\rho_0 = \frac{p_0}{R T_0} = \frac{M}{V} \quad (2)$$

where M is the mass of air within the space station and V is the interior volume of the station. Using the given data:

γ = 1.4
R = 287 J/(kg.K)
$p_{0,t=0}$ = $100*10^3$ Pa (abs)
T_0 = 34 + 273 = 307 K
$A^* = A_{\min}$ = 1 cm^2 = $1*10^{-4}$ m^2
V = 61.9 m^3
$\Rightarrow \rho_0 = 1.135$ kg/m^3
$\Rightarrow M_{t=0} = 70.25$ kg
$\Rightarrow \boxed{m_{t=0} = 2.90*10^{-2} \text{ kg/s}}$

The mass in the space station may be found as a function of time by applying conservation of mass to a control volume surrounding the station as shown in the figure below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0 \quad (3)$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{dM}{dt} \quad (4)$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = m \quad (5)$$

Note that since the back pressure is always zero, the mass flow rate out of the space station will always be choked. Substitute and simplify.

$$\frac{dM}{dt} = -m = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\rho_0\sqrt{\gamma RT_0}A^* = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{M}{V}\right)\sqrt{\gamma RT_0}A^* \quad (6)$$

where Eqns. (1) and (2) have been used. Solve the differential equation given in Eqn. (6).

$$\int_{M=M_{t=0}}^{M=M}\frac{dM}{M} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{\sqrt{\gamma RT_0}A^*}{V}\right)\int_{t=0}^{t=t}dt \quad \text{(Note that } T_0 = \text{constant.)} \quad (7)$$

$$\ln\left(\frac{M}{M_{t=0}}\right) = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{\sqrt{\gamma RT_0}A^*}{V}\right)t \quad (8)$$

$$\boxed{M = M_{t=0}\exp\left[-\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{\sqrt{\gamma RT_0}A_{\min}}{V}\right)t\right]} \quad \text{where } A^* = A_{\min} \quad (9)$$

The thrust acting on the space station may be found by applying the linear momentum equation to the same control volume.

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \quad (10)$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(The thrust is the force required to hold Mir stationary.)} \quad (11)$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = mV_e \quad (12)$$

$$F_{B,x} = 0 \quad (13)$$

$$F_{S,x} = T - p_e A_e \quad (\text{where } A_e = A_{\text{outer}}) \quad (14)$$

Substitute and simplify.

$$T = mV_e + p_e A_e \quad (15)$$

The exit conditions may be found using isentropic relations since the flow through the hole is underexpanded.

$$\frac{A_e}{A^*} = \frac{A_{\text{outer}}}{A_{\text{min}}} = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_e = 1.8541 \quad (16)$$

$$\frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_e/p_0 = 0.1602 \Rightarrow p_e = 16.02 \text{ kPa (abs)} \quad (p_0 = 100 \text{ kPa abs}) \quad (17)$$

$$\frac{T_e}{T_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{-1} \Rightarrow T_e/T_0 = 0.5926 \Rightarrow T_e = 181.9 \text{ K} \quad (T_0 = 307 \text{ K}) \quad (18)$$

$$V_e = \text{Ma}_e\sqrt{\gamma R T_e} \Rightarrow V_e = 501.3 \text{ m/s} \quad (19)$$

Now calculate the thrust using Eqn. (15) and the mass flow rate found in the first part of this problem.

$$\boxed{T_{t=0} = 16.94 \text{ N}} \quad (20)$$

Note that this is the thrust at $t = 0$. The thrust will vary with time since the stagnation pressure, and thus exit pressure, will vary with time as mass discharges from the space station.