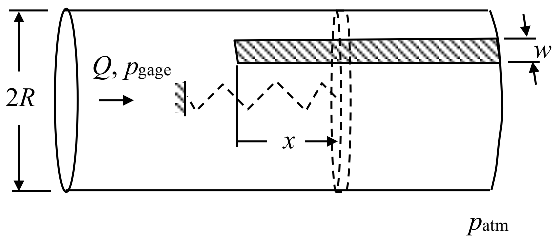
The device shown in the figure below is proposed for measuring the exhalation pressure and volume flow rate of a person (the device is known as a "peak flow meter"). A circular tube, with inside radius R, has a slit of width w running down the length of it (a cut-out in the cylinder). Inside the tube is a lightweight, freely moving piston attached to a linear spring (with spring constant k). The equilibrium position of the piston is at $x = 0$ where the slit begins.



Derive equations for:

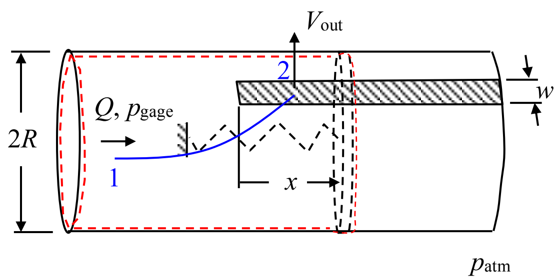
a. the volumetric flow rate, Q, and

b. the gage pressure in the tube, p_{gage},

in terms of (a subset of) the piston displacement, x, as well as the tube radius, R, slit width, w, spring constant, k, and the properties of air. Assume that the slit width, w, is so small that the outflow area is much smaller than the tube's cross-sectional area, πR^2, even at the piston's full extension.

SOLUTION:

Apply conservation of mass to the control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(at steady state)} \tag{2}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho Q + \rho V_{out} wx \tag{3}$$

Substitute and simply to get:

$$-\rho Q + \rho V_{out} wx = 0 \tag{4}$$

$$Q = V_{out} wx \tag{5}$$

where V_{out} is the speed of the air flowing out of the slit. This speed may be found by applying Bernoulli's equation from a point located within the tube (1) and a point just at the slit exit (2).

$$\left(p + \tfrac{1}{2}\rho V^2\right)_1 = \left(p + \tfrac{1}{2}\rho V^2\right)_2 \tag{6}$$

where

$$p_1 = p_{gage} \tag{7}$$

$$p_2 = 0 \; (p_{atm,gage} = 0) \tag{8}$$

$$V_1 = Q/(\pi R^2) \tag{9}$$

$$V_2 = V_{out} \tag{10}$$

Since the slit area is much smaller than the outlet area, $V_1 << V_2$, Eqn. (6) becomes

$$V_{out} = \sqrt{\frac{2p_{gage}}{\rho}} \, . \tag{11}$$

Substituting into Eqn. (5) gives:

$$Q = wx\sqrt{\frac{2p_{gage}}{\rho}} \tag{12}$$

The pressure, p_{gage}, may be found by balancing forces on the piston:

$$p_{gage}\pi R^2 - kx = 0 \tag{13}$$

$$\boxed{p_{gage} = \frac{kx}{\pi R^2}} \tag{14}$$

Note that we could have used the linear momentum equation in the x-direction on the same control volume to arrive at this expression (see below).

Combining Eqns. (12) and (14) gives:

$$\boxed{Q = wx^{3/2}\sqrt{\frac{2k}{\rho \pi R^2}}} \tag{15}$$

Thus, by measure the displacement of the piston on the simple device shown in the figure, lung functions such as pressure and volumetric flow rate can be easily determined.

Note that we could have also worked out Eq. (14) of this problem using the linear momentum equation in the x-direction applied to the same control volume.

$$\frac{d}{dt}\int_{\mathrm{CV}} u_x \rho \, dV + \int_{\mathrm{CS}} u_x \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{16}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} u_x \rho \, dV = 0 \quad \text{(steady flow)} \tag{17}$$

$$\int_{\mathrm{CS}} u_x \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = \frac{Q}{\pi R^2}\left(-\rho \frac{Q}{\pi R^2}\pi R^2\right) = -\rho \frac{Q^2}{\pi R^2} \quad \text{(no } x\text{-momentum flux out through side)} \tag{18}$$

$$F_{B,x} = 0 \tag{19}$$

$$F_{S,x} = p_{\mathrm{gage}} \pi R^2 - kx \tag{20}$$

Substitute and simplify.

$$-\rho \frac{Q^2}{\pi R^2} = p_{\mathrm{gage}} \pi R^2 - kx \tag{21}$$

Substituting from Eq. (12) ,

$$-\rho \frac{(wx)^2 \left(2p_{\mathrm{gage}} / \rho\right)}{\pi R^2} = p_{\mathrm{gage}} \pi R^2 - kx \tag{22}$$

$$-\left[2\left(\frac{wx}{\pi R^2}\right)^2 + 1\right]\pi R^2 p_{\mathrm{gage}} = -kx \tag{23}$$

But since $wx \ll \pi R^2$,

$$p_{\mathrm{gage}} = \frac{kx}{\pi R^2} \quad \text{which is the same as Eq. (14)!} \tag{24}$$