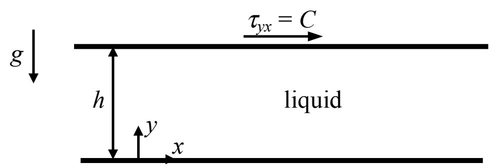
Consider a Newtonian liquid film that is driven by a constant shear stress, $\tau_{yx}(y = h) = C$, applied by a plate to the top surface. Assume that the liquid film is flat, fully developed, and has a constant pressure gradient in the x-direction such that there is zero net flow rate ($Q = 0$).



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Determine the velocity profile $u(y)$ and the pressure gradient dp/dx.

a. Use the continuity and Navier-Stokes equations to solve this problem.
b. Use a differential control volume and apply conservation of mass and the linear momentum equation to solve this problem.

SOLUTION:
Make the following assumptions about the flow:

1. The flow is planar. $\Rightarrow$ $\partial/\partial z(\cdots) = 0, u_z = \text{constant}$
2. The flow is steady. $\Rightarrow$ $\partial/\partial t(\cdots) = 0$
3. The flow is fully developed in the x-direction. $\Rightarrow$ $\partial u_x/\partial x = \partial u_y/\partial x = 0$
4. Gravity acts in the $-y$ direction. $\Rightarrow$ $g_y = -g; g_x = g_z = 0$

The continuity equation for an incompressible, planar flow is:

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \frac{\partial u_y}{\partial y} = 0 \;\Rightarrow\; \frac{\partial u_y}{\partial y} = 0. \tag{1}$$

Since the flow is also steady (#2), fully developed (#3), and planar (#1), the y-velocity can be at most a constant. Since $u_y = 0$ at the wall, then u_y everywhere is:

$$\underline{u_y = 0} \quad \text{(Call this condition \#5.)} \tag{2}$$

Now examine the x-momentum equation:

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#2)} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \underbrace{u_y}_{=0(\#5)} \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#3)} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho \underbrace{g_x}_{=0(\#4)}$$

$$\underline{0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u_x}{dy^2}} \tag{3}$$

where the partial derivatives have been replaced by ordinary derivatives since u_x is not a function of x (#3), t (#1), or z (#1).

Now consider the y-momentum equation,

$$\rho\left(\underbrace{\frac{\partial u_y}{\partial t}}_{=0(\#2)} + u_x \underbrace{\frac{\partial u_y}{\partial x}}_{=0(\#3)} + \underbrace{u_y}_{=0(\#5)} \underbrace{\frac{\partial u_y}{\partial y}}_{=0(\#5)}\right) = -\frac{\partial p}{\partial y} + \mu\left(\underbrace{\frac{\partial^2 u_y}{\partial x^2}}_{=0(\#5)} + \underbrace{\frac{\partial^2 u_y}{\partial y^2}}_{=0(\#5)}\right) + \rho \underbrace{g_y}_{=-g(\#4)}, \tag{4}$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \,, \tag{5}$$

$$\frac{\partial p}{\partial y} = -\rho g \,, \tag{6}$$

$$p(x, y) = \rho g (h - y) + f(x) \,. \tag{7}$$

Note that the pressure is not a function of the z direction since the flow is planar.

Now solve Eqn. (3) for the velocity profile.

$$\frac{d^2 u_x}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{1}{\mu}\frac{df}{dx} \,, \tag{8}$$

$$\frac{du_x}{dy} = \frac{1}{\mu}\frac{df}{dx} y + c_1 \tag{9}$$

$$\underline{u_x = \frac{1}{2\mu}\frac{df}{dx} y^2 + c_1 y + c_2} \tag{10}$$

Apply boundary conditions to determine the unknown constant c_1 and c_2.

no-slip at $y = 0$ $\Rightarrow$ $u_x(y = 0) = 0$ $\Rightarrow$ $0 = c_2$ (11)

constant stress at $y = h$ $\Rightarrow$ $\tau_{yx}(y = h) = \mu \frac{du_x}{dy}(y = h) = C$ $\Rightarrow$ $\mu\left(\frac{1}{\mu}\frac{df}{dx} h + c_1\right) = C$ (12)

$$c_1 = \frac{1}{\mu}\left(C - \frac{df}{dx} h\right) \tag{13}$$

Re-write the velocity profile,

$$u_x = \frac{1}{2\mu}\frac{df}{dx} y^2 + \frac{1}{\mu}\left(C - \frac{df}{dx} h\right) y \,, \tag{14}$$

Since the flow has zero volumetric flow rate,

$$Q = \int_{y=0}^{y=h} u_x b \, dy = 0 \,, \tag{15}$$

$$\int_{y=0}^{y=h} \left[\frac{1}{2\mu}\frac{df}{dx} y^2 + \frac{1}{\mu}\left(C - \frac{df}{dx} h\right) y \right] dy = 0 \,, \tag{16}$$

$$\frac{1}{6\mu}\frac{df}{dx} h^3 + \frac{1}{2\mu}\left(C - \frac{df}{dx} h\right) h^2 = 0 \,, \tag{17}$$

$$\frac{1}{6\mu}\frac{df}{dx} h^3 + \frac{Ch^2}{2\mu} - \frac{1}{2\mu}\frac{df}{dx} h^3 = 0 \,, \tag{18}$$

$$\frac{-1}{3\mu}\frac{df}{dx} h^3 + \frac{Ch^2}{2\mu} = 0 \,, \tag{19}$$

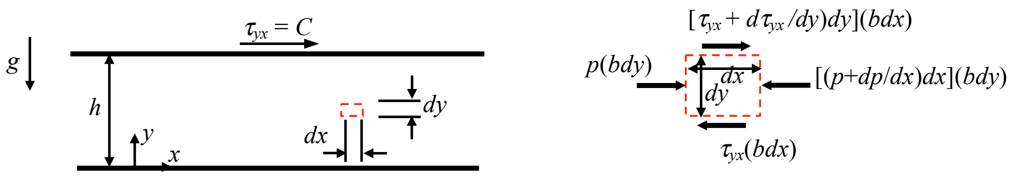
$\boxed{\frac{df}{dx} = \frac{3C}{2h}}$. Note that $\frac{\partial p}{\partial x} = \frac{df}{dx}$ (refer to Eq. (7)). (20)

Substitute back into Eq. (14) to get,

$$u_x = \frac{3}{4\mu}\frac{C}{h}y^2 + \frac{1}{\mu}\left(C - \frac{3}{2}C\right)y \,, \tag{21}$$

$$\boxed{u_x = \frac{3}{4\mu}\frac{C}{h}y^2 - \frac{1}{2\mu}Cy}. \tag{22}$$

Now solve the same problem, but using the fixed differential control volume shown in the following figure.



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Apply conservation of mass,

$$\frac{d}{dt}\int_{\text{CV}} \rho \, dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \,, \tag{23}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} \rho \, dV = 0 \text{ (steady)}, \tag{24}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho u_x\left(bdy\right) + \rho\left(u_x + \frac{\partial u_x}{\partial x}dx\right)\left(bdy\right) - \rho u_y\left(bdx\right) + \rho\left(u_y + \frac{\partial u_y}{\partial y}dy\right)\left(bdx\right), \tag{25}$$

(Note: Assuming planar flow so the *z* component isn't shown.)

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \rho \frac{\partial u_x}{\partial x}\left(bdxdy\right) + \rho \frac{\partial u_y}{\partial y}\left(bdxdy\right). \tag{26}$$

Note that the flow is fully developed in the *x* direction, planar, and steady,

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \rho \frac{du_y}{dy}\left(bdxdy\right). \tag{27}$$

Substitute and simplify,

$$\rho \frac{du_y}{dy}\left(bdxdy\right) = 0 \quad \Rightarrow \quad \frac{du_y}{dy} = 0 \,. \tag{28}$$

Using the same logic used to derive Eq. (2) gives $u_y = 0$.

Now apply the linear momentum equation in the *x* direction,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho \, dV + \int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \,, \tag{29}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho \, dV = 0 \text{ (steady)}, \tag{30}$$

$$\int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0 \text{ (fully developed in the } x\text{-direction; } u_y = 0), \tag{31}$$

$$F_{B,x} = 0 \,, \tag{32}$$

$$F_{S,x} = pbdy - \left(p + \frac{\partial p}{\partial x} dx \right) bdy - \tau_{yx} bdx + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) bdx \ . \tag{33}$$

Substitute and simplify,

$$0 = pbdy - \left(p + \frac{\partial p}{\partial x} dx \right) bdy - \tau_{yx} bdx + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) bdx \ , \tag{34}$$

$$0 = -\frac{\partial p}{\partial x} dxdy + \frac{\partial \tau_{yx}}{\partial y} dxdy \ , \tag{35}$$

$$\frac{d\tau_{yx}}{dy} = \frac{dp}{dx} \ . \tag{36}$$

Note that,

$$\tau_{yx} = \mu \left(\frac{\partial u_x}{\partial y} + \underbrace{\frac{\partial u_y}{\partial x}}_{\substack{=0 \text{ since} \\ u_y = 0}} \right) \Rightarrow \frac{\partial \tau_{yx}}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u_x}{\partial y} \right) = \mu \frac{\partial^2 u_x}{\partial y^2} \ . \tag{37}$$

Since the flow is steady, full developed in the *x* direction, and planar, Eq. (37) can be written in terms of an ordinary derivative,

$$\frac{\partial \tau_{yx}}{\partial y} = \mu \frac{d^2 u_x}{dy^2}$$

Thus, Eq. (36) becomes,

$$\mu \frac{d^2 u_x}{dy^2} = \frac{\partial p}{\partial x} \ , \tag{38}$$

$$\frac{d^2 u_x}{dy^2} = \frac{1}{\mu} \frac{\partial p}{\partial x} \ . \tag{39}$$

This equation is the same as that found previously (Eq. (8)).

The pressure gradient in the *y* direction can be found by applying the linear momentum equation in the *y* direction to the same control volume,

$$\frac{d}{dt} \int_{\text{CV}} u_y \rho \, dV + \int_{\text{CS}} u_y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} \right) = F_{B,y} + F_{S,y} \ , \tag{40}$$

where,

$$\frac{d}{dt} \int_{\text{CV}} u_y \rho \, dV = 0 \ \text{(steady; } u_y = 0\text{)}, \tag{41}$$

$$\int_{\text{CS}} u_y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} \right) = 0 \ (u_y = 0), \tag{42}$$

$$F_{B,y} = -\rho g b dx dy \ , \tag{43}$$

$$F_{S,y} = pbdx - \left(p + \frac{\partial p}{\partial y} dy \right) bdx - \tau_{xy} bdy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx \right) bdy \ . \tag{44}$$

Substitute and simplify,

$$0 = -\rho g b dx dy + pbdx - \left(p + \frac{\partial p}{\partial y} dy \right) bdx - \tau_{xy} bdy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx \right) bdy \ , \tag{45}$$

$$-\frac{\partial p}{\partial y}dxdy+\frac{\partial \tau_{xy}}{\partial x}dxdy=\rho gdxdy\ , \tag{46}$$

Note that,

$$\tau_{xy}=\mu\left(\underbrace{\frac{\partial u_y}{\partial x}}_{\substack{=0\text{ since}\\ u_y=0}}+\frac{\partial u_x}{\partial y}\right)\Rightarrow\frac{\partial \tau_{yx}}{\partial x}=\frac{\partial}{\partial x}\left(\mu\frac{\partial u_x}{\partial y}\right)=\mu\frac{\partial}{\partial y}\left(\underbrace{\frac{\partial u_x}{\partial x}}_{\substack{=0\text{ since}\\ \text{fully-developed}\\ \text{in }x\text{ direction}}}\right)=0\ , \tag{47}$$

where the order of the partial derivative have been flipped near the end of the equation.

Equation (46) now becomes,

$$-\frac{\partial p}{\partial y}dxdy=\rho gdxdy\ , \tag{48}$$

$$\frac{\partial p}{\partial y}=-\rho g\ , \tag{49}$$

which is precisely the same as Eq. (6)