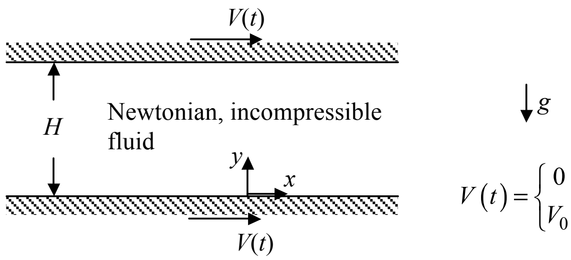
Consider two infinitely long parallel plates separated by a constant distance H as shown in the figure below. Between the plates is a Newtonian, incompressible fluid with density ρ and viscosity μ. There are no pressure gradients in the direction of the flow and gravity acts in the negative y-direction.



$$V(t) = \begin{cases} 0 & t < 0 \\ V_0 & t \geq 0 \end{cases}$$

Assume at time $t < 0$ the entire system is at rest. For $t \geq 0$, the walls are impulsively started and move at a constant speed V_0 in the $+x$-direction.

For these conditions, which of the following simplifications to the Navier-Stokes equations provides the governing equation for determining the velocity profile in the x-direction?

A. $\rho \dfrac{\partial u_x}{\partial t} = \mu \dfrac{\partial^2 u_x}{\partial y^2}$

B. $\rho \left(\dfrac{\partial u_x}{\partial t} + u_y \dfrac{\partial u_x}{\partial y} \right) = \mu \dfrac{\partial^2 u_x}{\partial y^2}$

C. $0 = \mu \dfrac{\partial^2 u_x}{\partial y^2}$

D. $0 = \mu \dfrac{\partial^2 u_x}{\partial x^2} + \rho g$

E. $\rho \dfrac{\partial u_x}{\partial t} = -\dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u_x}{\partial y^2}$

F. $\rho \left(\dfrac{\partial u_y}{\partial t} + u_y \dfrac{\partial u_y}{\partial y} \right) = \mu \dfrac{\partial^2 u_y}{\partial y^2} - \rho g$

SOLUTION:

Make the following assumption about the flow.

1. fully developed flow in the x-direction $\Rightarrow$ $\frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial x} = \frac{\partial u_z}{\partial x} = 0$ (1)
2. planar flow $\Rightarrow$ $u_z = \text{const}; \frac{\partial}{\partial z}(\ldots) = 0$ (2)
3. no pressure gradient in the x-direction $\Rightarrow$ $\frac{\partial p}{\partial x} = 0$ (3)
4. no gravity in the x-direction $\Rightarrow$ $g_x = 0$ (4)

Note that the flow velocity chanes with time. Hence, the flow is unsteady.

Consider the continuity equation for an incompressible fluid.

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0\ (\#1)} + \frac{\partial u_y}{\partial y} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0\ (\#2)} = 0 \quad \Rightarrow \quad \frac{\partial u_y}{\partial y} = 0 \tag{5}$$

Since $u_y \neq fcn(x, y, z)$ (assumption #1, Eq. (5), and assumption #2), at most we can have:

$$u_y = fcn(t) \tag{6}$$

However, at the boundaries $u_y(t) = 0$, thus, $\underline{u_y = 0}$ everywhere and for all times. Call this condition #5.

Now consider the Navier-Stokes equation for an incompressible, Newtonian fluid in the x-direction.

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0\ (\#1)} + \underbrace{u_y}_{=0\ (\#5)} \frac{\partial u_x}{\partial y} + u_z \underbrace{\frac{\partial u_x}{\partial z}}_{=0\ (\#2)}\right) = -\underbrace{\frac{\partial p}{\partial x}}_{=0\ (\#3)} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0\ (\#1)} + \frac{\partial^2 u_x}{\partial y^2} + \underbrace{\frac{\partial^2 u_x}{\partial z^2}}_{=0\ (\#2)}\right) + \rho \underbrace{g_x}_{=0\ (\#4)} \tag{7}$$

Thus,

$$\boxed{\rho \frac{\partial u_x}{\partial t} = \mu \frac{\partial^2 u_x}{\partial y^2}} \tag{8}$$

The boundary conditions for the flow are,

no-slip at the top and bottom boundaries $\Rightarrow$ $u_x(t; y = 0, H) = \begin{cases} 0 & t < 0 \\ V_0 & t \geq 0 \end{cases}$ (9)