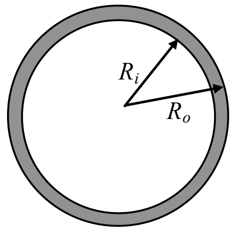
An incompressible, Newtonian liquid of density ρ and dynamic viscosity μ is sheared between concentric cylinders as shown in the sketch below. The inner cylinder radius is R_i and the outer cylinder radius is R_o.



a. Determine the velocity profile for the liquid in the gap assuming that the inner cylinder rotates with constant angular speed, ω. Do not assume that $(R_o - R_i) << R_o$.
b. Determine the torque (per unit depth into the page) acting on the outer wall of the cylinder.

SOLUTION:

The continuity and momentum equations in cylindrical coordinates for an incompressible, Newtonian fluid with constant viscosity are:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0$$

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]+\rho f_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]+\rho f_\theta$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho f_z$$

Make the following additional assumptions:

1. steady flow $\Rightarrow \frac{\partial}{\partial t}(\cdots)=0$
2. neglect gravity $\Rightarrow f_{B,z}=0, f_{B,r}=0, f_{B,\theta}=0$
3. planar flow $\Rightarrow u_z=\text{constant}, \frac{\partial}{\partial z}(\cdots)=0$
4. axi-symmetric flow $\Rightarrow \frac{\partial u_r}{\partial \theta}=\frac{\partial u_\theta}{\partial \theta}=\frac{\partial u_z}{\partial \theta}=0$
5. no pressure gradients in the θ direction $\Rightarrow \frac{\partial p}{\partial \theta}=0$ (due to the axi-symmetric flow assumption)

Simplify the continuity equation using the given assumptions:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}+\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#3)}=0 \;\Rightarrow\; \frac{\partial(ru_r)}{\partial r}=0 \;\Rightarrow\; ru_r=\text{constant}$$

Note that from assumptions #3 and #4, u_r is not a function of either z or θ. Since there is no radial flow at the inner boundary ($r = R_i$), the constant in the previous equation must be zero. Thus,

$u_r = 0$ (condition #6)

Now simplify the momentum equations using our assumptions and condition #6:

$$\rho\left(\underbrace{\frac{\partial u_r}{\partial t}}_{=0(\#1,\#6)}+\underbrace{u_r\frac{\partial u_r}{\partial r}}_{=0(\#6)}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}_{=0(\#4,\#6)}-\frac{u_\theta^2}{r}+u_z\underbrace{\frac{\partial u_r}{\partial z}}_{=0(\#3,\#6)}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_r}_{=0(\#6)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_r}{\partial \theta^2}}_{=0(\#4,\#6)}+\underbrace{\frac{\partial^2 u_r}{\partial z^2}}_{=0(\#3,\#6)}-\frac{2}{r^2}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}\right]+\rho\underbrace{f_r}_{=0(\#2)}$$

$$\Rightarrow \quad \frac{\partial p}{\partial r}=\rho\frac{u_\theta^2}{r}$$

$$\rho\left(\underbrace{\frac{\partial u_\theta}{\partial t}}_{=0(\#1)}+\underbrace{u_r\frac{\partial u_\theta}{\partial r}}_{=0(\#6)}+\frac{u_\theta}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}+\underbrace{\frac{u_r u_\theta}{r}}_{=0(\#6)}+u_z\underbrace{\frac{\partial u_\theta}{\partial z}}_{=0(\#3)}\right)=-\frac{1}{r}\underbrace{\frac{\partial p}{\partial \theta}}_{=0(\#5)}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r u_\theta\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_\theta}{\partial \theta^2}}_{=0(\#4)}+\underbrace{\frac{\partial^2 u_\theta}{\partial z^2}}_{=0(\#3,\#4)}+\frac{2}{r^2}\underbrace{\frac{\partial u_r}{\partial \theta}}_{=0(\#4,\#6)}\right]+\rho\underbrace{f_\theta}_{=0(\#2)}$$

$$\Rightarrow \quad \mu\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r u_\theta\right)\right]=0$$

Note that since u_θ is not a function of z or θ (assumptions #3 and #4), the partial derivatives with respect to r in the last equation can be written as ordinary derivatives. Integrating the second equation with respect to r twice gives:

$$u_\theta(r)=c_1 r+\frac{c_2}{r}$$

where c_1 and c_2 are constants.

The boundary conditions for the flow are:

no-slip at $r = R_i$: $\quad u_\theta(r=R_i)=\omega R_i$

no-slip at $r = R_o$: $\quad u_\theta(r=R_o)=0$

The boundary conditions are used to determine the unknown constants.

$$u_\theta(r=R_o)=0=c_1 R_o+\frac{c_2}{R_o} \quad\Rightarrow\quad c_1=-\frac{c_2}{R_0^2}$$

$$u_\theta(r=R_i)=\omega R_i=c_1 R_i+\frac{c_2}{R_i}=-\frac{c_2}{R_o^2}R_i+\frac{c_2}{R_i}=c_2\left(\frac{R_o^2-R_i^2}{R_i R_o^2}\right) \quad\Rightarrow\quad c_2=\omega\left(\frac{R_i^2 R_o^2}{R_o^2-R_i^2}\right)$$

Hence:

$$u_\theta(r)=\omega\left(\frac{R_i^2 R_o^2}{R_o^2-R_i^2}\right)\left(-\frac{r}{R_0^2}+\frac{1}{r}\right)$$

$$\boxed{\therefore u_\theta(r)=\omega r\left(\frac{R_i^2 R_o^2}{R_o^2-R_i^2}\right)\left(\frac{R_0^2-r^2}{r^2 R_0^2}\right)}$$

The torque on the outer wall (assuming unit depth) is given by:

$$T = R_o\left(2\pi R_o \sigma_{r\theta}\big|_{r=R_o}\right) = 2\pi R_o^2 \left\{ \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\underbrace{\frac{\partial u_r}{\partial \theta}}_{=0} \right]\right\}_{r=R_o}$$

$$= 2\pi R_o^2 \left\{ \mu r \frac{\partial}{\partial r}\left[\omega\left(\frac{R_i^2 R_o^2}{R_o^2 - R_i^2}\right)\left(\frac{R_0^2 - r^2}{r^2 R_0^2}\right)\right]\right\}_{r=R_o}$$

$$= 2\pi\mu\omega R_o^2 \left(\frac{R_i^2 R_o^2}{R_o^2 - R_i^2}\right)\left\{ r\frac{\partial}{\partial r}\left[\left(\frac{1}{r^2} - \frac{1}{R_0^2}\right)\right]\right\}_{r=R_o}$$

$$= 2\pi\mu\omega R_o^2 \left(\frac{R_i^2 R_o^2}{R_o^2 - R_i^2}\right)\left\{\frac{-2}{r^2}\right\}_{r=R_o}$$

$$T = -4\pi\mu\omega\left(\frac{R_i^2 R_o^2}{R_o^2 - R_i^2}\right)$$

Note that the stress $\sigma_{r\theta}$ is the stress acting on the fluid. The stress acting on the cylinder will be in the opposite direction. Thus, the torque on the cylinder is:

$$\boxed{\therefore T = 4\pi\mu\omega\left(\frac{R_i^2 R_o^2}{R_o^2 - R_i^2}\right)}$$