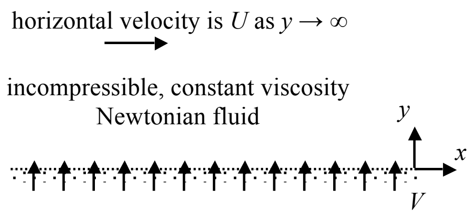
Consider steady flow at horizontal velocity U (at $y \to \infty$) past an infinitely long and wide plate. The plate is porous and there is uniform flow normal to the surface at a constant velocity, V. Assume there are no pressure gradients and that gravity is negligible.



a. Determine the y-velocity at all points in the flow field.
b. Determine the x-velocity at all points in the flow field.
c. What restriction is there on the velocity V?
d. Quantify how far into the flow the wall effects are felt. Clearly indicate what criterion you are using.

SOLUTION:

Make the following assumptions about the flow:

1. The flow is planar. $\Rightarrow \partial/\partial z(\cdots) = 0, u_z = 0$
2. The flow is steady. $\Rightarrow \partial/\partial t(\cdots) = 0$
3. The flow is fully-developed in the x-direction. $\Rightarrow \partial u_x/\partial x = \partial u_y/\partial x = 0$
4. Neglect gravity. $\Rightarrow g_x = g_y = g_z = 0$
5. No pressure gradients. $\Rightarrow \partial p/\partial x = \partial p/\partial y = \partial p/\partial z = 0$

The continuity equation for an incompressible, planar flow is:

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \frac{\partial u_y}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial u_y}{\partial y} = 0 \quad \text{(call this condition \#6)}$$

Since u_y is not a function of x (#3), z (#1), or y (#6), then u_y = constant. Since the flow at the wall has vertical velocity, V:

$$\boxed{u_y = V} \tag{1}$$

Now examine the x-momentum equation:

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#2)}+u_x\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)}+\underbrace{u_y}_{=V(\#6)}\frac{\partial u_x}{\partial y}\right)=-\underbrace{\frac{\partial p}{\partial x}}_{=0(\#5)}+\mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#3)}+\frac{\partial^2 u_x}{\partial y^2}\right)+\rho\underbrace{g_x}_{=0(\#4)}$$

$$V\frac{du_x}{dy}=\nu\frac{d^2u_x}{dy^2} \tag{2}$$

where the partial derivatives have been replaced by ordinary derivatives since u_x is not a function of x (#3) or z (#1). Not also that $\nu = \mu/\rho$.

Solving Eqn. (2):

$$\frac{V}{\nu}=\frac{d}{dy}\left(\ln\frac{du_x}{dy}\right)$$

$$\frac{V}{\nu}y+c_1=\ln\frac{du_x}{dy}$$

$$\exp\left(\frac{V}{\nu}y+c_1\right)=c_2\exp\left(\frac{V}{\nu}y\right)=\frac{du_x}{dy}$$

$$\underline{u_x=c_3\exp\left(\frac{V}{\nu}y\right)+c_4} \tag{3}$$

Apply the following boundary conditions:

no-slip at $y = 0$ $\Rightarrow$ $u_x(y=0)=0$ (4)

horz. velocity is U as $y \to \infty$ $\Rightarrow$ $u_x(y\to\infty)=U$ (5)

$$u_x(y=0)=c_3+c_4=0 \tag{6}$$

$$u_x(y\to\infty)=c_3\lim_{y\to\infty}\left[\exp\left(\frac{V}{\nu}y\right)\right]+c_4=U \tag{7}$$

Note that in order to have u_x remain finite as $y \to \infty$, we must have $\boxed{V<0}$. Hence, Eqn. (7) implies that $c_4 = U$. Substituting and simplifying gives:

$$\boxed{u_x=U\left[1-\exp\left(\frac{V}{\nu}y\right)\right]} \tag{8}$$

To quantify the distance into the flow that the wall effects are felt, use the 99% boundary layer thickness, δ, *i.e.*:

$$\frac{u_x}{U}=0.99=1-\exp\left(\frac{V}{\nu}\delta\right)$$

$$\boxed{\delta=\frac{\nu}{V}\ln 0.01} \tag{9}$$