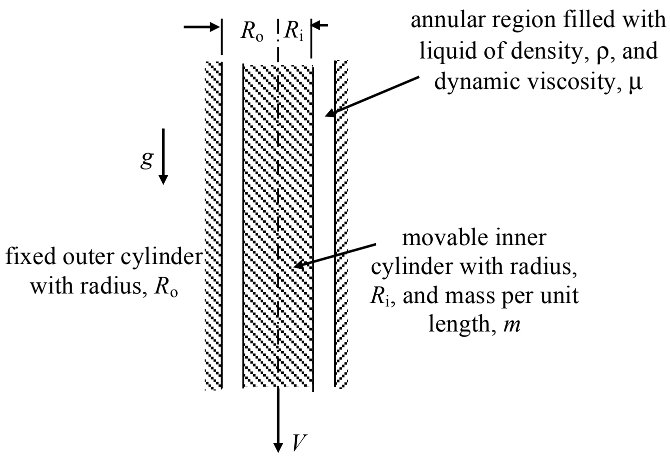
Consider two concentric cylinders with a Newtonian liquid of constant density, ρ, and constant dynamic viscosity, μ, contained between them. The outer pipe, with radius, R_o, is fixed while the inner pipe, with radius, R_i, and mass per unit length, m, falls under the action of gravity at a constant speed. There is no pressure gradient within flow and no swirl velocity component. Determine the vertical speed, V, of the inner cylinder as a function of the following (subset of) parameters: g, R_o, R_i, m, ρ, and μ.



SOLUTION:

First determine the velocity profile of the fluid within the annulus. Make the following assumptions:

1. steady flow $\Rightarrow \frac{\partial u_r}{\partial t} = \frac{\partial u_\theta}{\partial t} = \frac{\partial u_z}{\partial t} = 0$
2. gravity acts in the z-direction $\Rightarrow f_z = g, f_r = 0, f_\theta = 0$
3. fully-developed flow in the z-direction $\Rightarrow \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = \frac{\partial u_z}{\partial z} = 0$
4. the flow is axi-symmetric and there is no swirl velocity $\Rightarrow \frac{\partial u_r}{\partial \theta} = \frac{\partial u_\theta}{\partial \theta} = \frac{\partial u_z}{\partial \theta} = 0, u_\theta = 0$
5. no pressure gradients in the z direction $\Rightarrow \frac{\partial p}{\partial z} = 0$

Consider the continuity equation.

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(4)} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0(3)} = 0 \;\Rightarrow\; \frac{\partial(ru_r)}{\partial r} = 0 \;\Rightarrow\; ru_r = \text{constant} \quad (1)$$

Note that from assumptions 3 and 4, u_r is not a function of either θ or z. Since there is no radial flow at r = R_i or $r = R_o$, the constant in the previous equation must be zero. Thus,

$$u_r = 0 \text{ (condition 6)} \quad (2)$$

Now consider the Navier-Stokes equations.

$$\rho\left(\underbrace{\frac{\partial u_r}{\partial t}}_{=0(\#1,\#6)}+\underbrace{u_r\frac{\partial u_r}{\partial r}}_{=0(\#6)}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}_{=0(\#4,\#6)}-\underbrace{\frac{u_\theta^2}{r}}_{=0(\#4)}+\underbrace{u_z\frac{\partial u_r}{\partial z}}_{=0(\#3,\#6)}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_r}_{=0(\#6)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_r}{\partial \theta^2}}_{=0(\#4,\#6)}+\underbrace{\frac{\partial^2 u_r}{\partial z^2}}_{=0(\#3,\#6)}-\frac{2}{r^2}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}\right]+\rho\underbrace{f_r}_{=0(\#2)} \quad (3)$$

$$\Rightarrow \frac{\partial p}{\partial r}=0 \quad (4)$$

$$\rho\left(\underbrace{\frac{\partial u_\theta}{\partial t}}_{=0(\#1,\#4)}+\underbrace{u_r\frac{\partial u_\theta}{\partial r}}_{=0(\#6)}+\frac{u_\theta}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}+\underbrace{\frac{u_r u_\theta}{r}}_{=0(\#6)}+u_z\underbrace{\frac{\partial u_\theta}{\partial z}}_{=0(\#3)}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_\theta}_{=0(\#4)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_\theta}{\partial \theta^2}}_{=0(\#4)}+\underbrace{\frac{\partial^2 u_\theta}{\partial z^2}}_{=0(\#3,\#4)}+\frac{2}{r^2}\underbrace{\frac{\partial u_r}{\partial \theta}}_{=0(\#4,\#6)}\right]+\rho\underbrace{f_\theta}_{=0(\#2)} \quad (5)$$

$$\Rightarrow \frac{\partial p}{\partial \theta}=0 \quad (6)$$

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(\#1)}+\underbrace{u_r}_{=0(\#6)}\frac{\partial u_z}{\partial r}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}_{=0(\#4)}+u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#3)}\right)=-\underbrace{\frac{\partial p}{\partial z}}_{=0(\#5)}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_z}{\partial \theta^2}}_{=0(\#4)}+\underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(\#3)}\right]+\rho\underbrace{f_z}_{=g(\#2)} \quad (7)$$

$$\Rightarrow \underline{0=\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\rho g} \quad (8)$$

Note that since u_z is not a function of θ (assumption #4) or z (assumption #3), then $u_z = u_z(r)$ and the partial differentials in Eqn. (8) may be written as ordinary differentials. Solve the ODE given in Eqn. (8).

$$\frac{d}{dr}\left(r\frac{du_z}{dr}\right)=-\frac{\rho g}{\mu}r$$

$$r\frac{du_z}{dr}=-\frac{\rho g}{2\mu}r^2+c_1$$

$$\frac{du_z}{dr}=-\frac{\rho g}{2\mu}r+\frac{c_1}{r}$$

$$\underline{u_z=-\frac{\rho g}{4\mu}r^2+c_1\ln r+c_2} \quad (9)$$

Apply the following boundary conditions to determine the constants c_1 and c_2.

no-slip at $r = R_0$: $$u_z(r=R_o)=0 \Rightarrow 0=-\frac{\rho g}{4\mu}R_o^2+c_1\ln R_o+c_2 \quad (10)$$

no-slip at $r = R_i$: $$u_z(r=R_i)=V \Rightarrow V=-\frac{\rho g}{4\mu}R_i^2+c_1\ln R_i+c_2 \quad (11)$$

First determine c_1 by subtracting Eqn. (10) from Eqn. (11).

$$V=-\frac{\rho g}{4\mu}\left(R_i^2-R_o^2\right)+c_1\ln\frac{R_i}{R_o}$$

$$\underline{c_1=\frac{\left[V+\frac{\rho g}{4\mu}\left(R_i^2-R_o^2\right)\right]}{\ln\frac{R_i}{R_o}}} \quad (12)$$

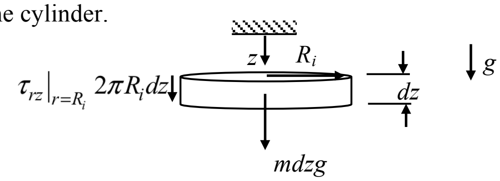
Find c_2 by applying the no-slip condition at $r = R_o$:

$$0 = -\frac{\rho g}{4\mu} R_o^2 + \frac{\left[V + \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)\right]}{\ln \frac{R_i}{R_o}} \ln R_o + c_2 \tag{13}$$

$$c_2 = \frac{\rho g}{4\mu} R_o^2 - \frac{\left[V + \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)\right]}{\ln \frac{R_i}{R_o}} \ln R_o \tag{14}$$

Perform a force balance on a small length of the cylinder.

$$\sum F_z = 0 = mdzg + \tau_{rz}\big|_{r=R_i} 2\pi R_i dz \tag{15}$$



$$\tau_{rz}\big|_{r=R_i} = -\frac{mg}{2\pi R_i} \tag{16}$$

Since the fluid is Newtonian:

$$\tau_{rz} = \mu \frac{du_z}{dr} \Rightarrow \tau_{rz}\big|_{r=R_i} = \mu\left[-\frac{\rho g}{2\mu} r + \frac{c_1}{r}\right]_{r=R_i} = \mu\left\{-\frac{\rho g}{2\mu} R_i + \frac{\left[V + \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)\right]}{R_i \ln \frac{R_i}{R_o}}\right\} \tag{17}$$

Substitute Eqn. (17) into Eqn. (16) and solve for V.

$$\mu\left\{-\frac{\rho g}{2\mu} R_i + \frac{\left[V + \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)\right]}{R_i \ln \frac{R_i}{R_o}}\right\} = -\frac{mg}{2\pi R_i} \tag{18}$$

$$\boxed{V = \left(R_i \ln \frac{R_i}{R_o}\right)\left(\frac{\rho g}{2\mu} R_i - \frac{mg}{2\pi R_i \mu}\right) - \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)} \tag{19}$$