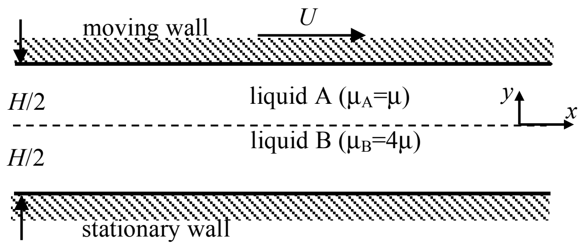
Two immiscible viscous liquids are introduced into a Couette flow device so that they form two layers of equal height as shown:



The dynamic viscosity, μ, of liquid A is one quarter that of liquid B. The upper plate is moved at a constant velocity, U, while the bottom plate remains stationary.

a. Determine the velocity of the interface between the two liquids.
b. Determine the "apparent viscosity" of the mixture as seen by an experimenter who believes that only one liquid is in the device.

SOLUTION:

Make several assumptions regarding the flow in each region:

1. planar flow $\Rightarrow$ $\partial/\partial z(\cdots)=0, u_z=0$
2. steady flow $\Rightarrow$ $\partial/\partial t(\cdots)=0$
3. fully-developed flow in the x-direction $\Rightarrow$ $\partial u_x/\partial x=\partial u_y/\partial x=0$
4. no pressure gradient in the x-direction $\Rightarrow$ $\partial p/\partial x=0$
5. no body forces $\Rightarrow$ $g_x=g_y=g_z=0$

First examine the continuity equation.

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)}+\frac{\partial u_y}{\partial y}=0 \quad\Rightarrow\quad \frac{\partial u_y}{\partial y}=0$$

This result combined with assumptions 1 and 3 indicate that u_y = constant. Since there is no flow through the walls, $u_y = 0$ everywhere (call this condition #6).

Simplify the Navier-Stokes equation in the x-direction.

$$\rho\left[\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#2)}+u_x\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)}+\underbrace{u_y}_{=0(\#6)}\frac{\partial u_x}{\partial y}\right]=-\underbrace{\frac{\partial p}{\partial x}}_{=0(\#4)}+\mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#3)}+\frac{\partial^2 u_x}{\partial y^2}\right)+\rho\underbrace{g_x}_{=0(\#5)}$$

$$\frac{d^2u_x}{dy^2}=0 \quad \text{(Note that } u_x=u_x(y) \text{ from assumptions 1, 2, and 3.)} \tag{1}$$

Integrating twice with respect to y gives:

$$\frac{du_x}{dy}=c_1 \tag{2}$$

$$u_x=c_1y+c_2 \tag{3}$$

Note that Eqns. (1) - (3) are valid for both fluids (A and B).

Now apply boundary conditions for each fluid.

Fluid B:

no-slip at $y = 0$ $\Rightarrow$ $u_{xB}\left(y=0\right)=V_i \Rightarrow c_{2B}=V_i$ (where V_i is the interface velocity)

no-slip at $y = -H/2$ $\Rightarrow$ $u_{xB}\left(y=-\frac{H}{2}\right)=0 \Rightarrow c_{1B}=\dfrac{2V_i}{H}$

$$\therefore u_{xB}=V_i\left[2\left(\frac{y}{H}\right)+1\right] \quad (4)$$

Fluid A:

no-slip at $y = 0$ $\Rightarrow$ $u_{xA}\left(y=0\right)=V_i \Rightarrow c_{2A}=V_i$ (where V_i is the interface velocity)

no-slip at $y = H/2$ $\Rightarrow$ $u_{xA}\left(y=\frac{H}{2}\right)=U \Rightarrow c_{1A}=\dfrac{2\left(U-V_i\right)}{H}$

$$\therefore u_{xA}=2\left(U-V_i\right)\left(\frac{y}{H}\right)+V_i \quad (5)$$

Also note that the shear stress is continuous at the interface.

$$\mu_A\left.\frac{du_{xA}}{dy}\right|_{y=0}=\mu_B\left.\frac{du_{xB}}{dy}\right|_{y=0}$$

$$\mu\frac{2\left(U-V_i\right)}{H}=4\mu\frac{2V_i}{H}$$

$$\boxed{\therefore V_i=\tfrac{1}{5}U} \quad (6)$$

To determine the "apparent viscosity", μ^*, note that if there was only a single fluid between the two plates with viscosity, μ^*, the shear stress exerted by the upper plate would be:

$$\tau=\mu^*\left(\frac{U}{H}\right) \quad (7)$$

With the two fluids, the shear stress exerted by the upper plate on fluid A is:

$$\tau=\mu\frac{2\left(U-V_i\right)}{H}=\mu\frac{2\left(U-\frac{1}{5}U\right)}{H}=\tfrac{8}{5}\mu\left(\frac{U}{H}\right) \quad (8)$$

Equating Eqns. (7) and (8) shows that the apparent viscosity is:

$$\boxed{\mu^*=\tfrac{8}{5}\mu} \quad (9)$$