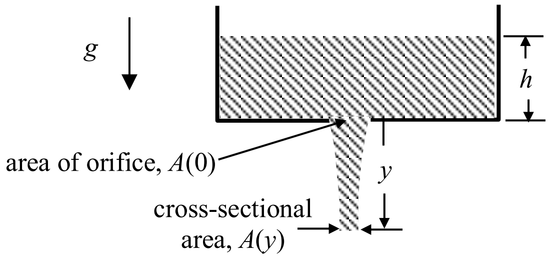
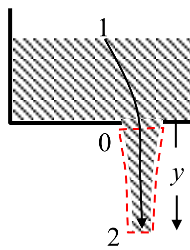
A water tank has an orifice in the bottom of the tank:



The height, h, of water in the tank is kept constant by a supply of water which is not shown. A jet of water emerges from the orifice; the cross-sectional area of the jet, $A(y)$, is a function of the vertical distance, y. Neglecting viscous effects and surface tension, find an expression for $A(y)$ in terms of $A(0)$, h, and y.

SOLUTION:

Apply conservation of mass to the following CV:



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(The flow is steady.)}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho V_0 A_0 + \rho V_2 A_2$$

Substitute and simplify:

$$V_2 = V_0 \frac{A_0}{A_2} \tag{1}$$

Now apply Bernoulli's equation from point 1 to point 0 and from point 1 to point 2:

$$\left(p + \tfrac{1}{2}\rho V^2 - \rho g y\right)_1 = \left(p + \tfrac{1}{2}\rho V^2 - \rho g y\right)_0 = \left(p + \tfrac{1}{2}\rho V^2 - \rho g y\right)_2$$

where

$p_1 = p_0 = p_2 = p_{atm}$ (These points are all at free surfaces.)

$V_1 = 0$ and V_0 and V_2 are related through Eqn. (1).

$y_1 = -h, y_0 = 0, y_2 = y$

Substitute and simplify:

$$\rho g h = \tfrac{1}{2}\rho V_0^2 = \tfrac{1}{2}\rho V_2^2 - \rho g y$$

$$\rho g h = \tfrac{1}{2}\rho V_0^2 = \tfrac{1}{2}\rho V_0^2 \left(\frac{A_0}{A_2}\right)^2 - \rho g y$$

The first two equations in the previous expression state that:

$$V_0 = \sqrt{2gh} \tag{2}$$

Eqn. (2) combined with the second two equations gives:

$$\left(\frac{A_0}{A_2}\right)^2 = 1 + \frac{\rho g y}{\tfrac{1}{2}\rho V_0^2}$$

$$\left(\frac{A_0}{A_2}\right)^2 = 1 + \frac{\rho g y}{\rho g h} = 1 + \frac{y}{h}$$

$$\boxed{\frac{A_2}{A_0} = \frac{1}{\sqrt{1 + \frac{y}{h}}}}$$