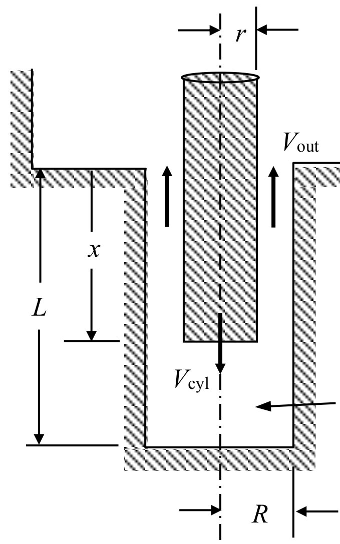
The motion of a hydraulic cylinder is cushioned at the end of its stroke by a piston that enters a hole as shown. The cavity and cylinder are filled with hydraulic fluid of uniform density, ρ.

a. Obtain an expression for the average velocity, V_{out}, at which hydraulic fluid escapes from the cylindrical hole assuming that the cylinder moves at a constant velocity, V_{cyl}.

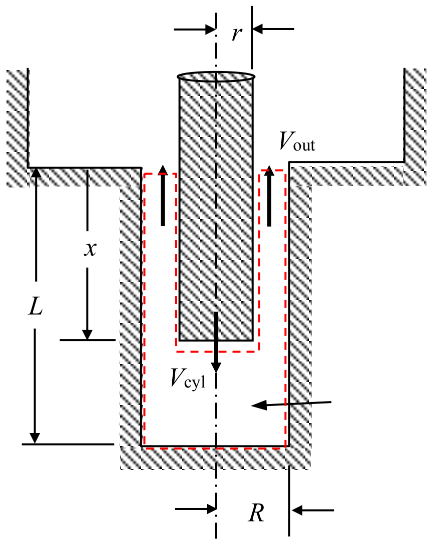


b. Determine the velocity, V_{out}, with relative uncertainty, for the following conditions.

$\rho = 900 \pm 5$ kg/m^3
$L = 100 \pm 0.1$ mm
$R = 15 \pm 0.1$ mm
$r = 10 \pm 0.1$ mm
$V_{cyl} = 100 \pm 1$ mm/s

SOLUTION:

Apply conservation of mass to a control volume that deforms to follow the piston as shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \rho\frac{d}{dt}\underbrace{\left[\pi R^2 L - \pi r^2 x\right]}_{=V_{CV}} = -\rho\pi r^2\frac{dx}{dt} = -\rho\pi r^2 V_{cyl}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = \rho V_{out}\pi\left(R^2 - r^2\right)$$

Substitute and solve for V_{out}.

$$-\rho\pi r^2 V_{cyl} + \rho V_{out}\pi\left(R^2 - r^2\right) = 0$$

$$V_{out} = V_{cyl}\frac{r^2}{R^2 - r^2}$$

$$\boxed{\therefore V_{out} = V_{cyl}\frac{1}{\left(R/r\right)^2 - 1}}$$

The total relative uncertainty in V_{out} is given by:

$$u_{V_{\text{out}}} = \left[u_{V_{\text{out}},V_{\text{cyl}}}^2 + u_{V_{\text{out}},R}^2 + u_{V_{\text{out}},r}^2 \right]^{1/2}$$

where

$$u_{V_{\text{out}},V_{\text{cyl}}} = \frac{1}{V_{\text{out}}} \frac{\partial V_{\text{out}}}{\partial V_{\text{cyl}}} \delta V_{\text{cyl}} = \frac{\left(R/r\right)^2 - 1}{V_{\text{cyl}}} \frac{1}{\left(R/r\right)^2 - 1} \delta V_{\text{cyl}} = \frac{\delta V_{\text{cyl}}}{V_{\text{cyl}}} = u_{V_{\text{cyl}}}$$

$$u_{V_{\text{out}},R} = \frac{1}{V_{\text{out}}} \frac{\partial V_{\text{out}}}{\partial R} \delta R = \frac{\left(R/r\right)^2 - 1}{V_{\text{cyl}}} \left\{ -\frac{2R/r^2 \, V_{\text{cyl}}}{\left[\left(R/r\right)^2 - 1\right]^2} \right\} \delta R = -\frac{2R/r^2}{\left(R/r\right)^2 - 1} \delta R = \frac{-2}{1-\left(r/R\right)^2} \frac{\delta R}{R} = \frac{-2u_R}{1-\left(r/R\right)^2}$$

$$u_{V_{\text{out}},r} = \frac{1}{V_{\text{out}}} \frac{\partial V_{\text{out}}}{\partial r} \delta r = \frac{\left(R/r\right)^2 - 1}{V_{\text{cyl}}} \left\{ -\frac{-2R^2/r^3 \, V_{\text{cyl}}}{\left[\left(R/r\right)^2 - 1\right]^2} \right\} \delta r = \frac{2R^2/r^3}{\left(R/r\right)^2 - 1} \delta r = \frac{2}{1-\left(r/R\right)^2} \frac{\delta r}{r} = \frac{2u_r}{1-\left(r/R\right)^2}$$

Substitute and simplify.

$$u_{V_{\text{out}}} = \left[u_{V_{\text{cyl}}}^2 + \frac{4u_R^2}{\left[1-\left(r/R\right)^2\right]^2} + \frac{4u_r^2}{\left[1-\left(r/R\right)^2\right]^2} \right]^{1/2}$$

Using the given data:

$$V_{\text{out}} = 80 \text{ mm/s}$$

$$u_{V_{\text{cyl}}} = 1 \text{ mm}/100 \text{ mm} = 1.0*10^{-2}$$

$$u_R = 0.1 \text{ mm}/15 \text{ mm} = 6.7*10^{-3}$$

$$u_r = 0.1 \text{ mm}/10 \text{ mm} = 1.0*10^{-2}$$

$$r/R = 10 \text{ mm}/15 \text{ mm} = 6.7*10^{-1}$$

$$\therefore u_{V_{\text{out}}} = 4.5*10^{-2} \Rightarrow \delta V_{\text{out}} = 3.4 \text{ mm/s}$$

$$\boxed{\therefore V_{\text{out}} = 80.0 \pm 3.6 \text{ mm/s}}$$