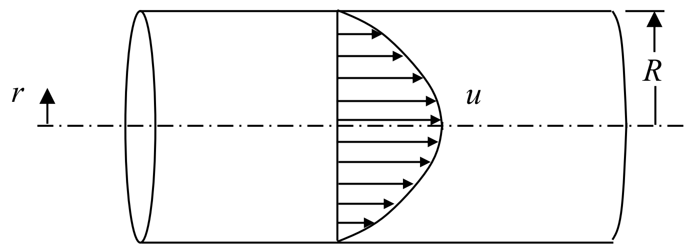
An incompressible flow in a pipe has a velocity profile given by:

$$u(r) = u_c\left(1 - \frac{r^2}{R^2}\right)$$

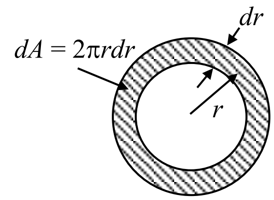
where u_c is the centerline velocity and R is the pipe radius. Determine the average velocity in the pipe.



SOLUTION:

The volumetric flow rate using the average velocity profile must give the same volumetric flow rate using the real velocity profile.

$$Q_{\text{real}} = \int_A \mathbf{u}\cdot d\mathbf{A} = \int_A dQ = \underbrace{\int_{y=-H}^{y=+H} u_c\left(1-r^2/R^2\right)\overbrace{\left(2\pi r dr\right)}^{dA}}_{=dQ} = \tfrac{1}{2}\pi u_c R^2 \qquad (1)$$



The velocity, $u(r)$, is nearly constant over the small annulus with radius dr so we can write the volumetric flow rate over this small area as $dQ = u(r)dA = u(r)(2\pi r dr)$.

$$Q_{\text{average}} = \int_A \mathbf{u}\cdot d\mathbf{A} = \bar{u}\left(\pi R^2\right) \qquad (2)$$

(There is no need to integrate since the velocity is uniform over r.)

$$Q_{\text{real}} = Q_{\text{average}} \quad \Rightarrow \quad \tfrac{1}{2}\pi u_c R^2 = \bar{u}\left(\pi R^2\right) \qquad (3)$$

$$\boxed{\therefore \bar{u} = \tfrac{1}{2}u_c} \qquad (4)$$