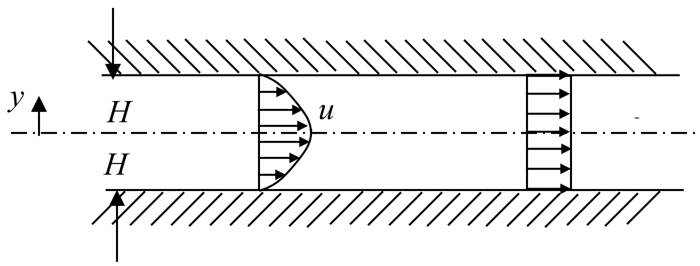
Consider the flow of an incompressible fluid between two parallel plates separated by a distance $2H$. If the velocity profile is given by:

$$u = u_c\left(1 - \frac{y^2}{H^2}\right)$$

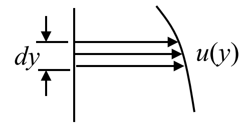
where u_c is the centerline velocity, determine the average velocity of the flow, $\bar{u}$. Assume the depth into the page is w.



SOLUTION:

The volumetric flow rate using the average velocity profile must give the same volumetric flow rate using the real velocity profile.

$$Q_{\text{real}} = \int_A \mathbf{u} \cdot d\mathbf{A} = \int_A dQ = \underbrace{\int_{y=-H}^{y=+H} u_c \left(1 - \frac{y^2}{H^2}\right) \overbrace{dyw}^{=dA}}_{=dQ} = \tfrac{4}{3} u_c w H \qquad (1)$$



The velocity, $u(y)$, is nearly constant over the small distance dy so we can write the volumetric flowrate over this small area as $dQ = u(y)dA = u(y)(dyw)$.

$$Q_{\text{average}} = \int_A \mathbf{u} \cdot d\mathbf{A} = \bar{u}\left(2Hw\right) \qquad (2)$$

(There is no need to integrate since the velocity is uniform over y.)

$$Q_{\text{real}} = Q_{\text{average}} \implies \tfrac{4}{3} u_c w H = \bar{u}\left(2Hw\right) \qquad (3)$$

$$\boxed{\therefore \bar{u} = \tfrac{2}{3} u_c} \qquad (4)$$