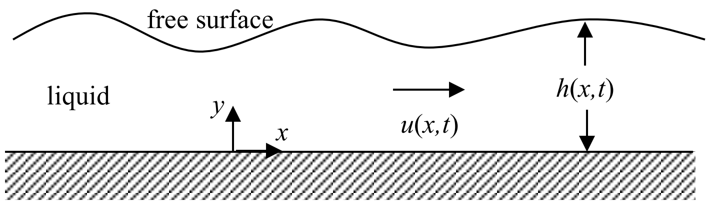
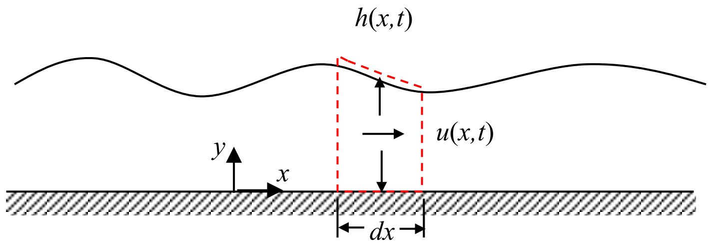
Construct from first principles an equation for the conservation of mass governing the planar flow (in the *xy* plane) of a compressible liquid lying on a flat horizontal plane. The depth, $h(x,t)$, is a function of position, x, and time, t. Assume that the velocity of the fluid in the positive x-direction, $u(x,t)$, is independent of y. Also assume that the wavelength of the wave is much greater than the wave amplitude so that the horizontal velocities are much greater than the vertical velocities.



SOLUTION:

Apply conservation of mass to the fixed control volume shown below. Assume a unit depth into the page.



Note that the mass flow rate at the center of the control volume is ρuh.

$$\frac{d}{dt}\int_{\mathrm{CV}}\rho dV + \int_{\mathrm{CS}}\rho\mathbf{u}_{\mathrm{rel}}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}}\rho dV = \frac{\partial}{\partial t}\big(\underbrace{\rho h dx}_{=M_{\mathrm{CV}}}\big) = \frac{\partial}{\partial t}(\rho h)dx$$

$$\int_{\mathrm{CS}}\rho\mathbf{u}_{\mathrm{rel}}\cdot d\mathbf{A} = -\Big[\underbrace{(\rho uh)+\frac{\partial}{\partial x}(\rho uh)\left(-\tfrac{1}{2}dx\right)}_{=m_{\mathrm{left}}}\Big] + \Big[\underbrace{(\rho uh)+\frac{\partial}{\partial x}(\rho uh)\left(\tfrac{1}{2}dx\right)}_{=m_{\mathrm{right}}}\Big]$$

$$= \frac{\partial}{\partial x}(\rho uh)dx$$

Substitute and simplify.

$$\frac{\partial}{\partial t}(\rho h)dx + \frac{\partial}{\partial x}(\rho uh)dx = 0$$

$$\boxed{\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho uh) = 0} \tag{2}$$

If the fluid is incompressible, then Eqn. (2) simplifies to:

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0} \tag{3}$$