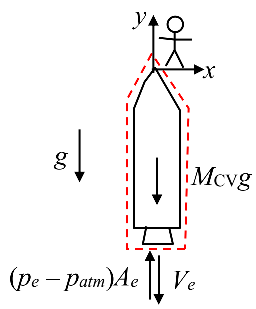
A model solid propellant rocket has a mass of 69.6 gm, of which 12.5 gm is fuel. The rocket produces 1.3 lb_f of thrust for a duration of 1.7 sec. For these conditions, calculate the maximum speed and height attainable in the absence of air resistance. Plot the rocket speed and the distance traveled as functions of time.

SOLUTION:

Assume that the mass flow rate from the rocket is constant. Also assume that the thrust remains constant over the burn duration.

Apply the linear momentum equation in the y-direction to the CV shown using a frame of reference attached to the rocket.



$$\frac{d}{dt}\int_{CV} u_y \rho dV + \int_{CS} u_y \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,y} + F_{S,y} - \int_{CV} a_{y/Y} \rho dV$$

where

$$\frac{d}{dt}\int_{CV} u_y \rho dV \approx 0 \quad \text{(Most of the fluid has zero velocity in this frame of reference.)}$$

$$\int_{CS} u_y \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = -V_e\left(\rho_e V_e A_e\right) = -\rho_e V_e^2 A_e$$

$$F_{B,y} = -M_{CV} g \quad \text{(weight)}$$

$$F_{S,y} = \left(p_e - p_{atm}\right) A_e \quad \text{(The exit pressure may be different from atmospheric pressure.)}$$

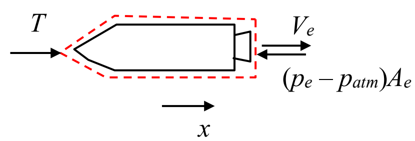
$$\int_{CV} a_{y/Y} \rho dV = a M_{CV} \quad \text{(We're using an accelerating frame of reference.)}$$

Substituting and simplifying:

$$-\rho_e V_e^2 A_e = -M_{CV} g + \left(p_e - p_{atm}\right) A_e - M_{CV} a$$

$$a = -g + \frac{\rho_e V_e^2 A_e + \left(p_e - p_{atm}\right) A_e}{M_{CV}} \tag{1}$$

Note that the thrust, T, is the force required to hold the rocket stationary (neglecting gravity).



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV \approx 0 \quad \text{(Most of the fluid has zero } x\text{-velocity.)}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = V_e\left(\rho_e V_e A_e\right) = \rho_e V_e^2 A_e$$

$$F_{B,x} = 0$$

$$F_{S,x} = -\left(p_e - p_{atm}\right)A_e + T$$

Substituting and simplifying:

$$\rho_e V_e^2 A_e = -\left(p_e - p_{atm}\right)A_e + T$$

$$T = \rho_e V_e^2 A_e + \left(p_e - p_{atm}\right)A_e \tag{2}$$

Substitute Eqn. (2) into Eqn. (1):

$$a = -g + \frac{T}{M_{CV}} \tag{3}$$

Apply COM to the same CV:

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = 0$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = \frac{dM_{CV}}{dt}$$

$$\int_{CS} \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = \rho_e V_e A_e = m$$

Substituting and simplifying:

$$\frac{dM_{CV}}{dt} + m = 0 \tag{4}$$

Assuming the mass flow rate is a constant, solve Eqn. (4) subject to initial conditions:

$$\int_{M_0}^{M_{CV}} dM_{CV} = -m\int_0^t dt$$

$$M_{CV} = M_0 - mt \tag{5}$$

where M_0 is the initial mass of the CV.

Substitute Eqn. (5) into Eqn. (3) and solve the differential equation for the velocity:

$$a = \frac{dU}{dt} = -g + \frac{T}{M_0 - mt}$$

$$\int_0^U dU = \int_0^t -g\,dt + \int_0^t \frac{T\,dt}{M_0 - mt}$$

$$U = -gt - \frac{T}{m}\ln\left(\frac{M_0 - mt}{M_0}\right)$$

$$U = -gt - \frac{T}{m}\ln\left(1 - \frac{mt}{M_0}\right) \qquad (6)$$

Solve the differential equation given in Eqn. (6) for the height of the rocket.

$$U = \frac{dh}{dt} = -gt - \frac{T}{m}\ln\left(1 - \frac{mt}{M_0}\right)$$

$$\int_0^h dh = \int_0^t -gt\,dt - \int_0^t \frac{T}{m}\ln\left(1 - \frac{mt}{M_0}\right)dt$$

$$h = -\tfrac{1}{2}gt^2 + \frac{T}{m}\left[\frac{M_0}{m}\ln\left(1 - \frac{mt}{M_0}\right) - t\ln\left(1 - \frac{mt}{M_0}\right) + t\right] \qquad (7)$$

Note that Eqns. (3), (5), (6), and (7) are written specifically for when the fuel is burning. When the fuel has been expended, the rocket equations of motion are:

$$a = -g \qquad (8)$$

$$U = U_{t=t'} - g\left(t - t'\right) \qquad (9)$$

$$h = -\tfrac{1}{2}g\left(t - t'\right)^2 + U_{t=t'}\left(t - t'\right) + h_{t=t'} \qquad (10)$$

where *t'* is the time at which the fuel has been expended.

For the given problem we're told:

M_0 = 69.6 g
M_{fuel} = 12.5 g
T = 1.3 lb_f = 5.79 N
t' = 1.7 sec

giving a mass flow rate of:

$$m = \frac{M_{\text{fuel}}}{t'} = 7.35 \text{ g/sec} = 7.35 \times 10^{-3} \text{ kg/sec}$$

The maximum velocity will occur at the moment the fuel has been expended (neglecting the velocities as the rocket falls back to the ground). The maximum height will occur when the velocity is zero.

$\underline{U_{\text{max}}}$ = $U(t = t' = 1.7$ sec) = $\underline{139.2 \text{ m/s}}$ ($h(t = t')$ = 114 m)
$\underline{h_{\text{max}}}$ = $h(t = t_m = 15.9$ sec) = $\underline{1100 \text{ m}}$

The maximum height occurs when:

$$U = U_{t=t'} - g\left(t_m - t'\right) = 0$$

$$t_m = t' + \frac{U_{t=t'}}{g}$$

The rocket speed and height are plotted below: