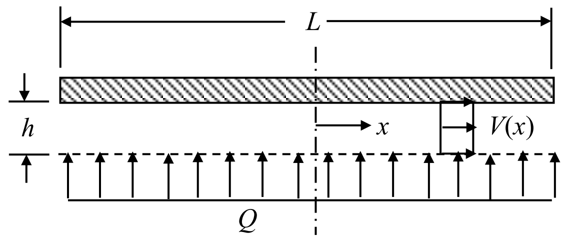
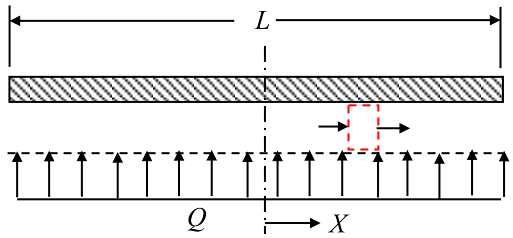
Incompressible fluid of negligible viscosity is pumped, at total volume flow rate Q, through a porous surface into the small gap between closely spaced parallel plates as shown. The fluid has only horizontal motion in the gap. Assume uniform flow across any vertical section. Obtain an expression for the pressure variation as a function of x.



Assume a depth w into the page.

SOLUTION:

Apply conservation of mass to the following differential control volume.



$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV + \int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = -\left[\rho V h(w) + \frac{d}{dx}(\rho V h w)\left(-\tfrac{1}{2}dx\right)\right] + \left[\rho V h w + \frac{d}{dx}(\rho V h w)\left(\tfrac{1}{2}dx\right)\right] - \rho \frac{Q}{Lw} w dx$$

$$= \frac{d}{dx}(\rho V h w) dx - \rho \frac{Q}{Lw} w dx = \rho h w \frac{dV}{dx} dx - \rho \frac{Q}{Lw} w dx$$

Substituting and simplifying gives:

$$\rho h w \frac{dV}{dx} dx = \rho \frac{Q}{Lw} w dx$$

$$\frac{dV}{dx} = \frac{Q}{Lhw} \tag{1}$$

$$\int_{V=0}^{V=V} dV = \int_{x=0}^{x=x} \frac{Q}{Lhw} dx$$

$$\frac{Vhw}{Q} = \frac{x}{L} \quad \text{or} \quad V = \left(\frac{Q}{hw}\right)\left(\frac{x}{L}\right) \tag{2}$$

Now apply linear momentum equation in the *X*-direction to the same control volume.

$$\frac{d}{dt}\int_{\mathrm{CV}} u_X \rho dV + \int_{\mathrm{CS}} u_X \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = F_{BX} + F_{SX}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} u_X \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\mathrm{CS}} u_X \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = -\left[\rho V^2 h w + \frac{d}{dx}\left(\rho V^2 h w\right)\left(-\tfrac{1}{2}dx\right)\right] + \left[\rho V^2 h w + \frac{d}{dx}\left(\rho V^2 h w\right)\left(\tfrac{1}{2}dx\right)\right]$$

$$= \frac{d}{dx}\left(\rho V^2 h w\right) dx = 2\rho V \frac{dV}{dx} h w dx$$

(Assume unit depth into the page. Note that the flux of mass from the porous surface has no *X*-momentum.)

$$F_{BX} = 0$$

$$F_{SX}=\left[phw+\frac{d}{dx}(phw)\left(-\tfrac{1}{2}dx\right)\right]-\left[phw+\frac{d}{dx}(phw)\left(\tfrac{1}{2}dx\right)\right]$$

$$=-\frac{d}{dx}(phw)dx=-\frac{dp}{dx}hwdx$$

Substituting and simplifying gives:

$$2\rho V\frac{dV}{dx}hwdx=-\frac{dp}{dx}hwdx$$

$$2\rho V\frac{dV}{dx}=-\frac{dp}{dx}$$

Substituting Eqns. (1) and (2) gives:

$$2\rho\left(\frac{Q}{Lhw}\right)^2x=-\frac{dp}{dx}$$

$$\int_{p=p}^{p=p_{\text{atm}}}dp=-2\rho\left(\frac{Q}{Lhw}\right)^2\int_{x=x}^{x=\frac{1}{2}L}xdx$$

$$p_{\text{atm}}-p=-\rho\left(\frac{Q}{Lhw}\right)^2\left(\tfrac{1}{4}L^2-x^2\right)$$

$$p-p_{\text{atm}}=\rho\left(\frac{Q}{hw}\right)^2\left(\tfrac{1}{4}-\left(\frac{x}{L}\right)^2\right)$$

$$\boxed{\frac{p-p_{\text{atm}}}{\rho\left(Q/hw\right)^2}=\frac{1}{4}-\left(\frac{x}{L}\right)^2} \qquad (3)$$