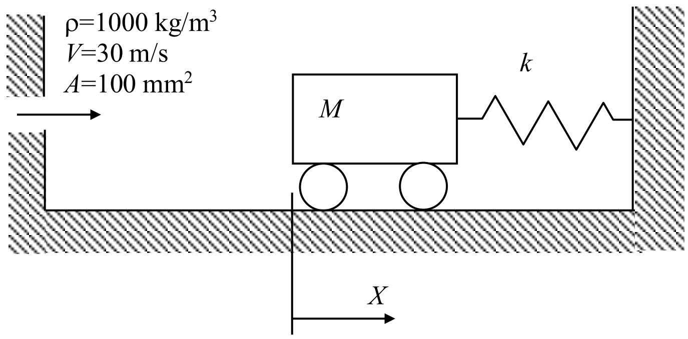
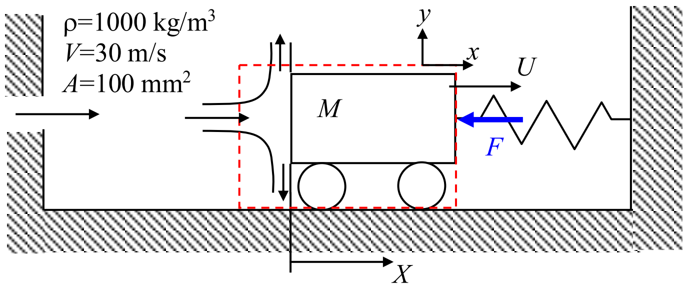
A block of mass, M=10 kg, with rectangular cross-section is arranged to slide with negligible friction along a horizontal plane. As shown in the sketch, the block is fastened to a spring that has stiffness such that $F=kx$ where k=500 N/m. The block is initially stationary. At time, t=0, a liquid jet begins to impinge on the block (the jet properties are also shown in the sketch). For t>0, the block moves laterally with speed, $U(t)$.

a. Obtain a differential equation valid for t>0 that could be solved for $U(t)$ and $X(t)$. Do not solve.
b. State appropriate boundary conditions for the differential equation of part (a).
c. Evaluate the final displacement of the block.



SOLUTION:

Apply the linear momentum equation in the *x*-direction to a control volume surrounding the block. Use a frame of reference that is fixed to the control volume (non-inertial).



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{CV} a_{x/X}\rho dV$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV \approx 0$$

(Although the fluid mass in the CV will change its velocity with time (the block mass using the given FOR is always zero), this time rate of change of momentum within the CV will be very small compared to the other terms in COLM and can be reasonably neglected.)

$$\int_{CS} u_x\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = (V-U)\left[-\rho(V-U)A\right] = -\rho(V-U)^2 A$$

$$F_{B,x} = 0$$

$$F_{S,x} = -kX$$

$$\int_{CV} a_{x/X}\rho dV = \frac{dU}{dt}M_{CV} \approx M\frac{dU}{dt}$$

(Assume the block mass is much greater than the water mass in the CV.)

Substitute and simplify.

$$-\rho\left(V-U\right)^2 A=-kX-M\frac{dU}{dt} \tag{1}$$

Note that:

$$U=\frac{dX}{dt} \text{ and } \frac{dU}{dt}=\frac{d^2X}{dt^2}$$

so that Eqn. (1) becomes:

$$-\rho\left(V-\frac{dX}{dt}\right)^2 A=-kX-M\frac{d^2X}{dt^2}$$

$$\boxed{\frac{d^2X}{dt^2}-\frac{\rho A}{M}\left(V-\frac{dX}{dt}\right)^2+\frac{k}{M}X=0} \tag{2}$$

Note that this is a non-linear 2nd order ODE.

The initial conditions for Eqn. (2) are:

$$\boxed{X\left(t=0\right)=0} \tag{3}$$

$$\boxed{\frac{dX}{dt}\left(t=0\right)=0} \tag{4}$$

The final position of the block occurs when the acceleration and velocity of the block are zero. From Eqn. (2) we have:

$$-\frac{\rho A}{M}V^2+\frac{k}{M}X_f=0$$

$$\boxed{X_f=\frac{\rho A}{k}V^2} \tag{5}$$

Note that we could have also worked this problem using an inertial frame of reference. Choose one that is fixed to the ground. Linear momentum in the X-direction using this new frame of reference gives:

$$\frac{d}{dt}\int_{CV} u_X \rho dV+\int_{CS} u_X\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=F_{B,X}+F_{S,X}$$

where

$$\frac{d}{dt}\int_{CV} u_X \rho dV \approx M\frac{dU}{dt} \quad \text{(Assume the block mass is much greater than the water mass in the CV.)}$$

$$\begin{aligned}\int_{CS} u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)&=\left(V\right)\left[-\rho\left(V-U\right)A\right]+\left(U\right)\underbrace{\left[\rho\left(V-U\right)A\right]}_{=m_{\text{sides}}}\\&=-V\rho\left(V-U\right)A+U\rho\left(V-U\right)A\\&=-\rho\left(V-U\right)^2 A\end{aligned}$$

$$F_{B,x}=0$$

$$F_{S,x}=-kX$$

Substitute and simplify.

$$M\frac{dU}{dt}-\rho\left(V-U\right)^2 A=-kX$$

$$\frac{d^2X}{dt^2}-\frac{\rho A}{M}\left(V-\frac{dX}{dt}\right)^2+\frac{k}{M}X=0 \quad \text{(This is the same as Eqn. (2)!)}$$