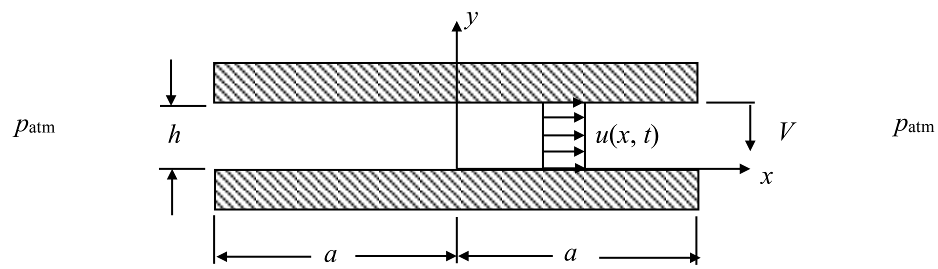
Two parallel plates of width, $2a$, (and unit depth) are separated by a gap of height, h, which changes with time. The upper plate approaches the lower plate at a constant speed, V. The space between the plates is filled with a frictionless, incompressible gas of density, ρ. Assume that the velocity is uniform across the gap width (y direction) so that $u=u(x, t)$.

Obtain algebraic expressions for:

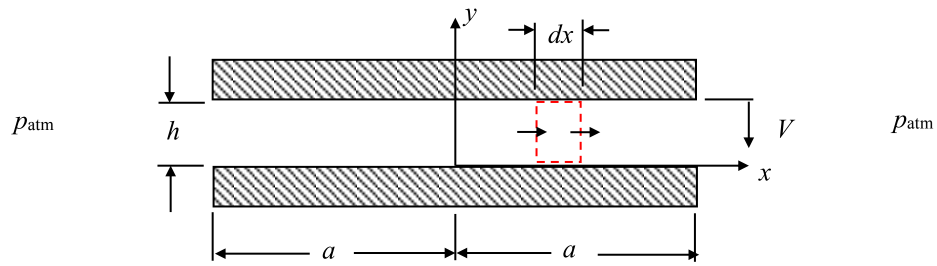
a. the velocity distribution, $u(x, t)$.

b. the pressure distribution in the gap, $p(x, t)$. The pressure outside of the gap is atmospheric pressure. Note: You do not need to use Bernoulli's equation to solve this problem.



SOLUTION:

Apply conservation of mass to the control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{d}{dt}\left(\rho h dx\right) = \rho\frac{dh}{dt}dx = -\rho V dx \tag{1}$$

$$\begin{aligned}\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} &= -\left[\left(\rho uh\right)+\frac{\partial}{\partial x}\left(\rho uh\right)\left(-\tfrac{1}{2}dx\right)\right]+\left[\left(\rho uh\right)+\frac{\partial}{\partial x}\left(\rho uh\right)\left(\tfrac{1}{2}dx\right)\right] \\ &= \frac{\partial}{\partial x}\left(\rho uh\right)dx = \rho\frac{\partial u}{\partial x}h dx\end{aligned} \tag{2}$$

Substitute and simplify.

$$-\rho V dx + \rho\frac{\partial u}{\partial x}h dx = 0$$

$$\frac{\partial u}{\partial x} = \frac{V}{h}$$

$u = V\dfrac{x}{h} + f\left(t\right)$ where $f(t)$ is an unknown function of time (Note: $u = u(x, t)$.)

Since the velocity at the center line of the plate is always zero, *i.e.* $u(x = 0, t) = 0$, then $f(t) = 0$.

$$\boxed{\therefore u = V\frac{x}{h}} \text{ (Note: } h = h(t) \Rightarrow u = u(x, t).\text{)} \tag{3}$$

Now apply the linear momentum equation in the *x*-direction to the same control volume using the given fixed frame of reference.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = \frac{d}{dt}\left(u\rho h dx\right) = \rho dx\left(u\frac{dh}{dt} + h\frac{\partial u}{\partial t}\right) = \rho dx\left(-uV + h\frac{\partial u}{\partial t}\right) \tag{4}$$

$$\begin{aligned}\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) &= -\left[\left(u\rho uh\right) + \frac{\partial}{\partial x}\left(u\rho uh\right)\left(-\tfrac{1}{2}dx\right)\right] + \left[\left(u\rho uh\right) + \frac{\partial}{\partial x}\left(u\rho uh\right)\left(\tfrac{1}{2}dx\right)\right] \\ &= \frac{\partial}{\partial x}\left(u\rho uh\right)dx = 2\rho u\frac{\partial u}{\partial x}hdx\end{aligned} \tag{5}$$

$$F_{B,x} = 0 \tag{6}$$

$$\begin{aligned}F_{S,x} &= \left[\left(ph\right) + \frac{\partial}{\partial x}\left(ph\right)\left(-\tfrac{1}{2}dx\right)\right] - \left[\left(ph\right) + \frac{\partial}{\partial x}\left(ph\right)\left(\tfrac{1}{2}dx\right)\right] \\ &= -\frac{\partial}{\partial x}\left(ph\right)dx = -\frac{\partial p}{\partial x}hdx\end{aligned} \tag{7}$$

Substitute and simplify.

$$\rho dx\left(-uV + h\frac{\partial u}{\partial t}\right) + 2\rho u\frac{\partial u}{\partial x}hdx = -\frac{\partial p}{\partial x}hdx \tag{8}$$

Substitute for *u* using the expression derived from conservation of mass.

$$\left[-\left(V\frac{x}{h}\right)V + h\left(V^2\frac{x}{h^2}\right)\right] + 2\left(V\frac{x}{h}\right)\left(V\frac{1}{h}\right)h = -\frac{1}{\rho}\frac{\partial p}{\partial x}h \tag{9}$$

$$2V^2\frac{x}{h} = -\frac{1}{\rho}\frac{\partial p}{\partial x}h$$

$$\frac{\partial p}{\partial x} = -2\rho V^2\frac{x}{h^2}$$

$$p = -\rho V^2\frac{x^2}{h^2} + f\left(t\right) \tag{10}$$

The pressure at $x = a$ is p_{atm} for all times, *i.e.* $p(x = a, t) = p_{\text{atm}}$:

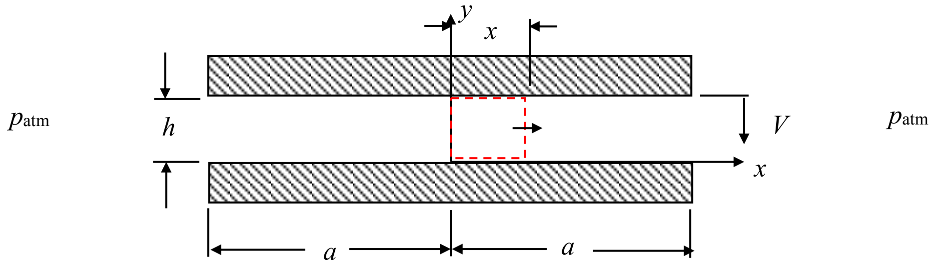
$$p_{\text{atm}} = -\rho V^2\frac{a^2}{h^2} + f\left(t\right) \Rightarrow f\left(t\right) = p_{\text{atm}} + \rho V^2\frac{a^2}{h^2} \tag{11}$$

Substituting and simplifying gives:

$$p = -\rho V^2\frac{x^2}{h^2} + p_{\text{atm}} + \rho V^2\frac{a^2}{h^2}$$

$$\boxed{\frac{p - p_{\text{atm}}}{\frac{1}{2}\rho V^2} = 2\left[\left(\frac{a}{h}\right)^2 - \left(\frac{x}{h}\right)^2\right]} \text{ (Note: } h = h(t) \Rightarrow p = p(x, t).\text{)} \tag{12}$$

Now let's work the problem using the control volume shown below.



Conservation of Mass:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{d}{dt}\left[\rho h x\right] = \rho \frac{dh}{dt} x = -\rho V x \quad (13)$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \rho u h \quad \text{(Mass flux only through right side due to symmetry.)} \quad (14)$$

Substitute and simplify.

$$-\rho V x + \rho u h = 0$$

$$\boxed{u = V\left(\frac{x}{h}\right)} \quad \text{This is the same result as before!} \quad (15)$$

Linear Momentum Equation in the x-direction:

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = \frac{d}{dt}\int_{x=0}^{x=x} \rho u h dx = \frac{d}{dt}\int_{x=0}^{x=x} \rho\left(V\frac{x}{h}\right) h dx = \rho V \frac{d}{dt}\left(\int_{x=0}^{x=x} x dx\right) = 0 \quad (16)$$

(The result from conservation of mass has been used in simplifying the previous expression.)

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \rho u^2 h = \rho V^2 \frac{x^2}{h} \quad \text{(Momentum flux only through right side due to symmetry.)} \quad (17)$$

(The result from conservation of mass has been used in simplifying the previous expression.)

$$F_{B,x} = 0 \quad (18)$$

$$F_{S,x} = p_{x=0} h - p_{x=x} h \quad (19)$$

Substitute and simplify.

$$\rho V^2 \frac{x^2}{h} = p_{x=0} h - p_{x=x} h$$

$$p_{x=x} = p_{x=0} - \rho V^2 \frac{x^2}{h^2} \quad (20)$$

Since the pressure at $x = a$ is p_{atm}, *i.e.* $p(x = a, t) = p_{\text{atm}}$:

$$p_{\text{atm}} = p_{x=0} - \rho V^2 \frac{a^2}{h^2} \Rightarrow p_{x=0} = p_{\text{atm}} + \rho V^2 \frac{a^2}{h^2} \quad (21)$$

$$p_{x=x} = p_{\text{atm}} + \rho V^2 \frac{a^2}{h^2} - \rho V^2 \frac{x^2}{h^2}$$

$$\boxed{\therefore \frac{p - p_{\text{atm}}}{\frac{1}{2}\rho V^2} = 2\left[\left(\frac{a}{h}\right)^2 - \left(\frac{x}{h}\right)^2\right]} \quad \text{This is the same result as before!} \quad (22)$$