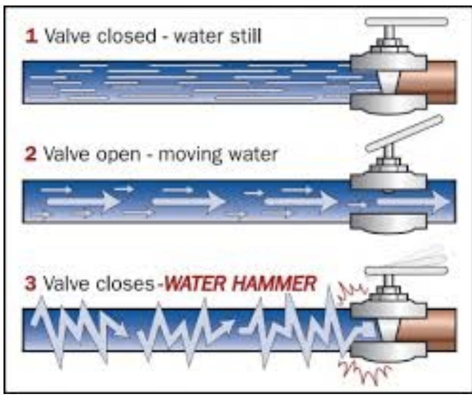
The pressure waves created by a rapid change of flow in a water line are referred to as water-hammers. To analyze the behavior of this phenomenon, consider a fluid flowing at speed U in a *rigid* pipe. The flow is stopped by a sudden closure of a valve. The pressure and the density of the fluid near the valve are suddenly increased by an amount Δp and $\Delta\rho$, respectively, and a pressure wave propagates upstream of the valve with speed, a.

a. Show that the increase in pressure, Δp, and the wave speed, a, are related by:

$$\Delta p = \rho U \left(U + a \right)$$

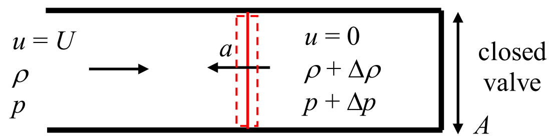
$$a\left(U + a \right) = \frac{\Delta p}{\Delta \rho}$$

b. The bulk modulus $K = \rho\,(dp/d\rho)$ is $43\text{x}10^6$ lb_f/ft^2 for water. Compute the wave speed a in a rigid pipe and Δp due to a sudden stoppage of water flowing with a speed of 1ft/s. You may assume that the pressure change across the wave is sufficiently weak to be considered an acoustic wave for the given conditions.

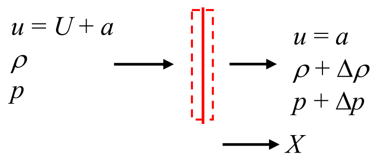


SOLUTION:

Apply conservation of mass and the linear momentum equation to a control volume surrounding the pressure wave.



Change the frame of reference so that wave appears stationary.



Apply conservation of mass to the control volume.

$$\frac{d}{dt}\int_{CV}\rho\, dV + \int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho\, dV = 0 \quad \text{(steady in the given frame of reference)} \tag{2}$$

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho(U+a)A + (\rho+\Delta\rho)aA \tag{3}$$

Combine and simplify.

$$-\rho(U+a)A + (\rho+\Delta\rho)aA = 0 \tag{4}$$

$$\rho(U+a) = (\rho+\Delta\rho)a \tag{5}$$

Apply the linear momentum in the x-direction using an inertial frame of reference.

$$\frac{d}{dt}\int_{CV}u_X\rho\, dV + \int_{CS}u_X\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X} \tag{6}$$

where

$$\frac{d}{dt}\int_{CV}u_X\rho\, dV = 0 \quad \text{(steady in the given frame of reference)} \tag{7}$$

$$\int_{CS}u_X\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = -\rho(U+a)^2A + (\rho+\Delta\rho)a^2A \tag{8}$$

$$F_{B,X} = 0 \tag{9}$$

$$F_{S,X} = pA - (p+\Delta p)A \tag{10}$$

Combine and simplify.

$$-\rho(U+a)^2A + (\rho+\Delta\rho)a^2A = pA - (p+\Delta p)A \tag{11}$$

$$-\rho(U+a)^2 + (\rho+\Delta\rho)a^2 = -\Delta p \tag{12}$$

$$-\rho(U+a)^2 + \rho(U+a)a = -\Delta p \quad \text{(making use of Eq. (5))} \tag{13}$$

$$\rho(U+a)\left[(U+a) - a\right] = \Delta p \tag{14}$$

$$\boxed{\Delta p = \rho U(U+a)} \tag{15}$$

Note that if $U \ll a$, which is typically the case, then Eq. (15) becomes,

$$\Delta p = \rho U a \tag{16}$$

Re-arranging Eq. (15) to solve for ρ gives,

$$\rho = \frac{\Delta p}{\rho U(U+a)} \tag{17}$$

Substitute this relation into Eq. (5) and simplify.

$$(U+a) = \left(1+\frac{\Delta\rho}{\rho}\right)a \tag{18}$$

$$(U+a) = \left[1+\frac{U(U+a)\Delta\rho}{\Delta p}\right]a \tag{19}$$

$$\frac{(U+a)}{a} = 1 + U(U+a)\frac{\Delta\rho}{\Delta p} \tag{20}$$

$$\frac{\Delta\rho}{\Delta p} = \frac{1}{U(U+a)}\left[\frac{(U+a)}{a}-1\right] \tag{21}$$

$$\frac{\Delta p}{\Delta\rho} = U(U+a)\left[\frac{a}{(U+a)-a}\right] \tag{22}$$

$$\boxed{\frac{\Delta p}{\Delta\rho} = a(U+a)} \tag{23}$$

Again, if $U \ll a$, then this relation becomes,

$$\frac{\Delta p}{\Delta\rho} = a^2 \tag{24}$$

In addition, if the wave is weak, meaning that the change in pressure and density across the wave are infinitesimally small, i.e., a sound wave, then Eq. (24) becomes,

$$\frac{dp}{d\rho} = a^2 \tag{25}$$

The bulk modulus is defined as,

$$K \equiv \rho\frac{dp}{d\rho}, \tag{26}$$

Since the wave is assumed to be an acoustic wave for the given conditions (refer to Eq. (25)),

$$a^2 = \frac{dp}{d\rho} \quad \Rightarrow \quad a = \sqrt{\frac{K}{\rho}} \tag{27}$$

The pressure change across the wave is found from Eq. (15). Using the given data,

K = 43*10^6 lb_f/ft^2
ρ = 1.94 slug/ft^3
U = 1 ft/s
$\Rightarrow$ $\boxed{a = 4710 \text{ ft/s and } \Delta p = 9.14*10^3 \text{ psf} = 63.4 \text{ psi}}$

Note that $U \ll a$ and $d\rho/\rho \ll 1$, consistent with the assumption of an acoustic wave.