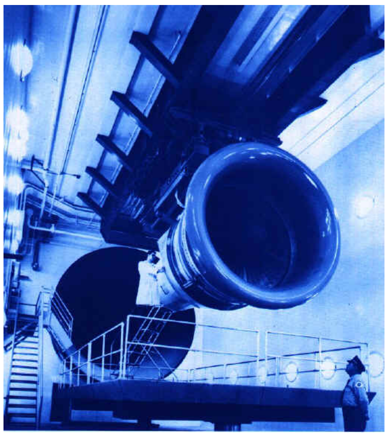
A turbojet engine in a wind tunnel receives air at a velocity of $U_i = 100$ m/s and a density of $\rho_i = 1$ kg/m^3. The velocity is uniform and the cross-sectional area of the approaching stream which enters the engine is $A_i = 0.1$ m^2. The velocity of the exhaust jet from the engine, however, is not uniform but has a velocity which varies over the cross-section according to:

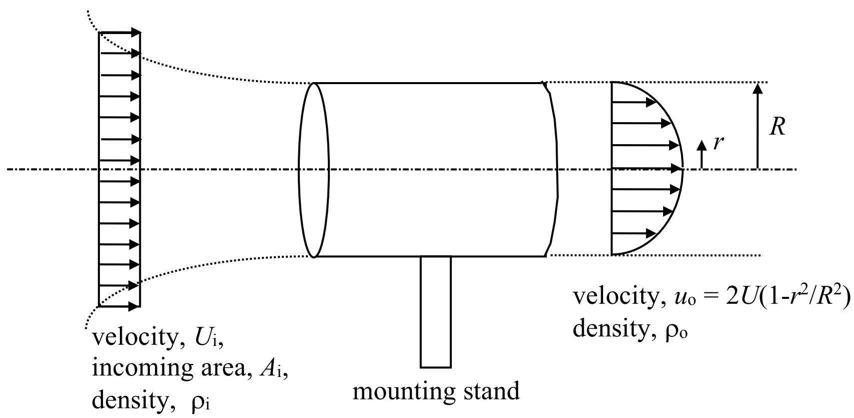
$$u_o = 2U\,(1 - r^2/R^2)$$

where the constant $U = 600$ m/s and R is the radius of the jet cross-section. Radial position with the axi-symmetric jet is denoted by r. The density of the exhaust jet is uniformly $\rho_0 = 0.5$ kg/m^3.

a. Determine the average velocity of the exhaust jet.
b. Find the thrust of the turbojet engine.
c. Find what the thrust would be if the exhaust jet had a uniform velocity, U.

Assume the pressures in both the inlet and exhaust jets are the same as the surrounding air and that mass is conserved in the flow through the engine (roughly true in practice).





BRIEF SOLUTION:

1. Determine the average exhaust velocity by equating the volumetric flow rate using the average velocity to the volumetric flow rate using the actual velocity profile.
2. Apply the linear momentum equation to a control volume crossing perpendicular to the inlet and outlet flows, follows the flow streamlines from the inlet cross-section to the engine, and follows along the exterior of the engine cutting through the mounting stand. Be sure to include the density differences, the non-uniform velocity exhaust velocity profile, and the force that the stand exerts on the control volume in the analysis. Note that the exhaust area is not necessarily the same as the inlet area.
3. Apply conservation of mass to the same control volume to determine the relation between the exhaust area and the other variables.

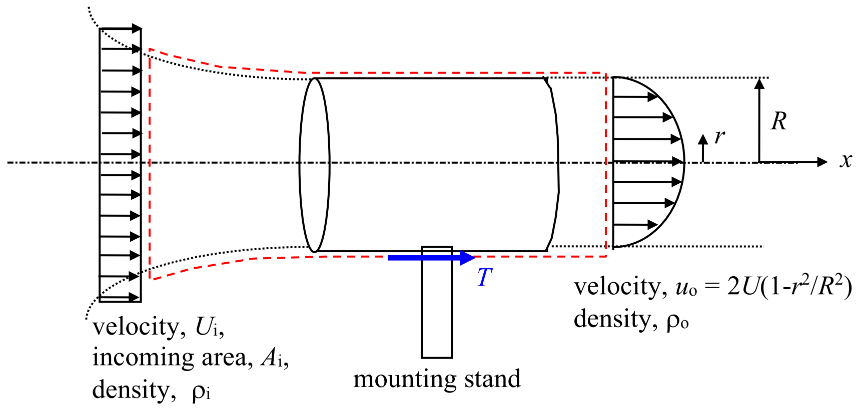
DETAILED SOLUTION:

Determine the average exhaust jet velocity using the fact that the volumetric flow rate using the average jet velocity will give the same volumetric velocity as the actual jet velocity.

$$Q = \bar{u}_o \pi R^2 = \int_{r=0}^{r=R} 2U\left(1 - r^2/R^2\right)\underbrace{\left(2\pi r dr\right)}_{=dA} = 4\pi U\left(\tfrac{1}{2}R^2 - \tfrac{1}{4}R^2\right)$$

$$\boxed{\therefore \bar{u}_o = U} \qquad (1)$$

Now apply the linear momentum equation in the x-direction to the fixed control volume shown below.



$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(assuming steady flow)}$$

$$\int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\rho_i U_i^2 A_i + \int_{r=0}^{r=R} \rho_o 4U^2\left(1 - r^2/R^2\right)^2\left(2\pi r dr\right) = -\rho_i U_i^2 A_i + \rho_o 8\pi U^2 \int_{r=0}^{r=R}\left(r - 2 r^3/R^2 + r^5/R^4\right)dr$$

$$= -\rho_i U_i^2 A_i + \rho_o 8\pi U^2\left(\tfrac{1}{2}R^2 - \tfrac{1}{2}R^2 + \tfrac{1}{6}R^2\right)$$

$$= -\rho_i U_i^2 A_i + \rho_o \tfrac{4}{3}\pi U^2 R^2$$

(The engine radius, R, is currently an unknown quantity.)

$$F_{B,x} = 0$$

$$F_{S,x} = T$$

(The thrust, T, is equal to the force the stand exerts on the engine, but in the opposite direction. Note that the pressure forces are the assumed to be the same everywhere.)

Substitute and solve for T.

$$-\rho_i U_i^2 A_i + \tfrac{4}{3}\pi \rho_o U^2 R^2 = T$$

$$T = \tfrac{4}{3}\pi \rho_o U^2 R^2 - \rho_i U_i^2 A_i \qquad (2)$$

Note that R is an unknown quantity at this point.

To determine R, apply conservation of mass to the same control volume.

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV + \int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV = 0$$

$$\int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = -\rho_i U_i A_i + \int_{r=0}^{r=R} \rho_o 2U\left(1 - r^2/R^2\right)(2\pi r dr) = -\rho_i U_i A_i + \rho_o 4\pi U \int_{r=0}^{r=R}\left(r - r^3/R^2\right)dr$$

$$= -\rho_i U_i A_i + \rho_o 4\pi U\left(\tfrac{1}{2}R^2 - \tfrac{1}{4}R^2\right)$$

$$= -\rho_i U_i A_i + \pi\rho_o U R^2$$

Substitute into conservation of mass and solve for R.

$$-\rho_i U_i A_i + \pi\rho_o U R^2 = 0$$

$$R^2 = \frac{\rho_i U_i A_i}{\pi\rho_o U} \tag{3}$$

Substitute Eqn. (3) into Eqn. (2).

$$T = \tfrac{4}{3}\pi\rho_o U^2\left(\frac{\rho_i U_i A_i}{\pi\rho_o U}\right) - \rho_i U_i^2 A_i = \tfrac{4}{3}U\rho_i U_i A_i - \rho_i U_i^2 A_i$$

$$\boxed{\therefore T = \rho_i U_i^2 A_i\left(\tfrac{4}{3}U/U_i - 1\right)} \tag{4}$$

Using the given data:

ρ_i = 1 kg/m^3
U_i = 100 m/s
A_i = 0.1 m^2
U = 600 m/s
$\Rightarrow \boxed{T = 7000\ \mathrm{N}}$

If the average velocity is used at the outlet, the momentum flux term becomes:

$$\int_{\mathrm{CS}} u_x\left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = -\rho_i U_i^2 A_i + \rho_o \bar{u}_o^2 \pi R^2 = -\rho_i U_i^2 A_i + \rho_o U^2 \pi R^2$$

And the thrust, T, becomes:

$$T = \rho_o U^2 \pi\left(\frac{\rho_i U_i A_i}{\pi\rho_o U}\right) - \rho_i U_i^2 A_i = U\rho_i U_i A_i - \rho_i U_i^2 A_i$$

$$\boxed{T = \rho_i U_i^2 A_i\left(U/U_i - 1\right)} \tag{5}$$

Using the parameters given in the problem:

$\Rightarrow \boxed{T = 5000\ \mathrm{N}}$

Using the average velocity to calculate the thrust will give a value that is incorrect. This result occurs because the velocity term is squared in the momentum flux term.