

Consider laminar flow over a flat plate ( $U=\text{constant}$ ). Approximate the boundary layer velocity distribution using a parabolic profile:

$$\frac{u}{U} = \begin{cases} 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 & \text{for } 0 \leq \frac{y}{\delta} \leq 1 \\ 1 & \frac{y}{\delta} > 1 \end{cases}$$

Determine an expression for the (99%) boundary layer thickness using the Momentum Integral Equation.

**SOLUTION:**

Evaluate the momentum thickness,  $\delta_M$ :

$$\delta_M = \delta \int_0^1 \frac{u}{U} \left(1 - \frac{u}{U}\right) d\eta = \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2) d\eta$$

$$\therefore \delta_M = \frac{2}{15} \delta$$

where in the above equation the substitution  $\eta=y/\delta$  has been used. Now substitute this momentum thickness into the Momentum Integral Equation. Note that  $dU/dx=0$ .

$$\frac{\tau_w}{\rho} = \frac{d}{dx} \left[ U^2 \delta_M \right] + \delta_D U \underbrace{\frac{dU}{dx}}_{=0}$$

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} = \frac{2}{15} \rho U^2 \frac{d\delta}{dx}$$

This should be the same shear stress as,

$$\tau_w = \mu \left. \frac{du}{dy} \right|_{y=0} = \mu \frac{U}{\delta} \left[ \frac{d}{d\eta} (2\eta - \eta^2) \right]_{\eta=0} = 2\mu \frac{U}{\delta}$$

so that by equating the two we have:

$$2\mu \frac{U}{\delta} = \frac{2}{15} \rho U^2 \frac{d\delta}{dx}$$

$$\Rightarrow \int_0^{\delta} \delta d\delta = \int_0^x \frac{15\nu}{U} dx$$

$$\Rightarrow \frac{1}{2} \delta^2 = \frac{15\nu}{U} x$$

$$\Rightarrow \frac{\delta}{x} = \sqrt{\frac{30\nu}{Ux}} \approx 5.5 \sqrt{\frac{\nu}{Ux}}$$

$$\therefore \frac{\delta}{x} \approx \frac{5.5}{\text{Re}_x^{1/2}}$$

Recall that the exact (Blasius) solution for flow over a flat plate gave:

$$\frac{\delta}{x} = \frac{5.0}{\text{Re}_x^{1/2}}$$

so that the approximate solution is off by only 10%!