Using the momentum integral theorem, determine the friction coefficient, c_f, dimensionless boundary layer momentum thickness, δ_M/x, and the dimensionless boundary layer displacement thickness, δ_D/x, for laminar flat plate flow with no pressure gradient assuming a sinusoidal velocity profile:

$$\frac{u}{U} \approx \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right),$$

where δ is the 99% boundary layer thickness, y is the distance from the plate surface, and U is the outer flow speed. Compare your answers with the Blasius' exact laminar boundary layer solution.

SOLUTION:

Use the Kármán Momentum Integral Equation (KMIE),

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(\delta_M U^2\right) + \delta_D U \frac{dU}{dx} \tag{1}$$

Assuming a flat plate flow with no pressure gradient,

$$U = \text{constant} \Rightarrow \frac{dU}{dx} = 0 \quad \text{(from Bernoulli's equation applied outside the boundary layer)} \tag{2}$$

Simplifying Eqn. (1) gives,

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} \tag{3}$$

The momentum thickness is given by,

$$\delta_M = \int_{y=0}^{y=\delta} \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \delta \int_{y/\delta=0}^{y/\delta=1} \frac{u}{U}\left(1-\frac{u}{U}\right)d\left(y/\delta\right)$$

$$= \delta \int_0^1 \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\left[1-\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]d\left(y/\delta\right)$$

$$= \delta\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\Bigg|_{y/\delta=0}^{y/\delta=1} - \frac{1}{2}\frac{y}{\delta}\Bigg|_{y/\delta=0}^{y/\delta=1} + \frac{1}{2\pi}\sin\left(\pi\frac{y}{\delta}\right)\Bigg|_{y/\delta=0}^{y/\delta=1}\right]$$

$$\therefore \delta_M = \delta\left(\frac{2}{\pi}-\frac{1}{2}\right) \approx 0.1367\delta \tag{4}$$

Substitute Eq. (4) into Eq. (3),

$$\tau_w = 0.1367 \rho U^2 \frac{d\delta}{dx} \tag{5}$$

For a laminar flow, the shear stress can also be expressed as,

$$\tau_w = \mu \frac{du}{dy}\bigg|_{y=0}$$

$$\tau_w = \frac{\pi}{2}\frac{\mu U}{\delta} \tag{6}$$

Equate Eqs. (5) and (6) and solve for δ,

$$0.1367 \rho U^2 \frac{d\delta}{dx} = \frac{\pi}{2}\frac{\mu U}{\delta}$$

$$\int_{\delta=0}^{\delta=\delta} \delta\, d\delta = 11.4908 \frac{\mu}{\rho U}\int_{x=0}^{x=x} dx$$

$$\tfrac{1}{2}\delta^2 = 11.4908 \frac{\mu}{\rho U} x$$

$$\boxed{\therefore \frac{\delta}{x} = 4.7939\sqrt{\frac{\mu}{\rho U x}} = \frac{4.7939}{\mathrm{Re}_x^{1/2}}} \tag{7}$$

Equation (7) is only 4% different from the exact Blasius solution of $\delta/x = 5.0/\mathrm{Re}_x^{1/2}$.

From Eq. (4) the momentum thickness is,

$$\boxed{\frac{\delta_M}{x} = \frac{0.6553}{\mathrm{Re}_x^{1/2}}} \tag{8}$$

This result is 1% different from the Blasius solution of $\delta_M / x = 0.664/\mathrm{Re}_x^{1/2}$.

The displacement thickness is given by,

$$\delta_D = \int_{y=0}^{y=\delta}\left(1-\frac{u}{U}\right)dy = \delta\int_{y/\delta=0}^{y/\delta=1}\left(1-\frac{u}{U}\right)d\left(y/\delta\right)$$

$$= \delta\int_0^1\left[1-\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]d\left(y/\delta\right)$$

$$= \delta\left[\left.\frac{y}{\delta}\right|_{y/\delta=0}^{y/\delta=1} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\Bigg|_{y/\delta=0}^{y/\delta=1}\right]$$

$$\therefore \delta_D = \delta\left(1-\frac{2}{\pi}\right) \approx 0.3634\delta \tag{9}$$

so that, when combined with Eq. (7),

$$\boxed{\frac{\delta_D}{x} = \frac{1.7420}{\mathrm{Re}_x^{1/2}}} \tag{10}$$

This result is 1% different from the Blasius solution of $\delta_D / x = 1.72/\mathrm{Re}_x^{1/2}$.

The friction coefficient can be found using Eq. (6),

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\frac{\pi}{2}\frac{\mu U}{\delta}}{\frac{1}{2}\rho U^2} = \pi\frac{\mu}{\rho U\delta} = \pi\frac{\mu}{\rho U x}\frac{x}{\delta}$$

$$\boxed{C_f = \frac{0.6553}{\mathrm{Re}_x^{1/2}}} \tag{11}$$

This result is 1% different form the Blasius solution of $C_f = 0.664/\mathrm{Re}_x^{1/2}$.