Notes on Thermodynamics, Fluid Mechanics, and Gas Dynamics

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For Sarah

Contents

A few comments to get things started...

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(2) Your corrections, comments, and compliments on the notes are also appreciated.

(3) I strongly encourage you to use these notes as supplemental material as opposed to serving as your primary resource. Coupling these notes with a textbook or two will provide you with a better learning experience. Seeing a topic from different points of view definitely helps to improve understanding of that topic.

(4) I give a large number of example problems in these notes. I recommend covering up the solutions and trying to work the problems on a blank piece of paper. Refer to the solutions only when you get stuck and have put in a reasonable amount of effort. Reviewing the solutions and saying to yourself, "This makes sense." is a very different experience than having to work out a problem on a blank page. Most engineering positions and university tests involve solving a problem without a solution available, so it makes sense to practice in the same way.

CHAPTER 1

The Basics

1.1. Symbolic vs. Numeric Analysis

Consider the following example. You need to determine the trajectory of a projectile fired from a cannon. The projectile has a mass of 10 kg and the cannon is tilted at an angle of 30° from the horizontal. The initial velocity of the projectile from the cannon is $100\,\mathrm{m\,s^{-1}}$. Determine:

(1) the distance the projectile will travel and
(2) how long the projectile is in flight.

We can approach this problem a couple of different ways. The first is to start with the given numbers and immediately begin the calculations. The second approach is to solve the problem symbolically and then substitute the numbers at the end.

1.1.1. Numerical Approach

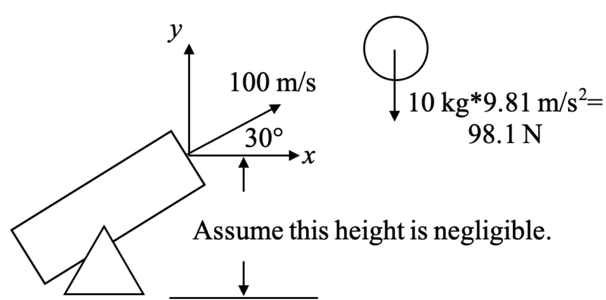


Figure 1.1. The free body diagram for the projectile example using numerical values.

Draw a free body diagram (FBD) of the projectile, as shown in Figure 1.1. Use Newton's Second Law to determine the acceleration of the projectile,

$$\sum F_x = m\ddot{x} \implies 0\,\mathrm{N} = (10\,\mathrm{kg})\ddot{x} \implies \ddot{x} = 0, \tag{1.1}$$

$$\sum F_y = m\ddot{y} \implies (10\,\mathrm{kg})(-9.81\,\mathrm{m/s^2}) = (10\,\mathrm{kg})\ddot{y} \implies \ddot{y} = -9.81\,\mathrm{m/s^2}. \tag{1.2}$$

Integrate with respect to time to determine the projectile's velocity and position given the projectile's initial x and y velocities and positions,

$$\dot{x} = \dot{x}_0 = (100\,\mathrm{m\,s^{-1}})(\cos 30°) = 86.6\,\mathrm{m\,s^{-1}}, \tag{1.3}$$

$$\dot{y} = (-9.81\,\mathrm{m/s^2})t + \dot{y}_0 = (-9.81\,\mathrm{m/s^2})t + (100\,\mathrm{m\,s^{-1}})(\sin 30°) = (-9.81\,\mathrm{m/s^2})t + 50\,\mathrm{m\,s^{-1}}, \tag{1.4}$$

$$\boxed{x = (86.6\,\mathrm{m\,s^{-1}})t}, \tag{1.5}$$

$$\boxed{y = (-4.91\,\mathrm{m/s^2})t^2 + (50\,\mathrm{m\,s^{-1}})t}. \tag{1.6}$$

The projectile will hit the ground when $y = 0$ so that by rearranging Eq. (1.6) we find that the time aloft is,

$$\boxed{t = \frac{50.0\,\mathrm{m\,s^{-1}}}{4.91\,\mathrm{m/s^2}} = 10.2\,\mathrm{s.}} \tag{1.7}$$

Substituting into Eq. (1.5) gives the distance traveled as,

$$\boxed{x = (86.6\,\mathrm{m\,s^{-1}})(10.2\,\mathrm{s}) = 883\,\mathrm{m}}. \tag{1.8}$$

As you can see, we've made a number of calculations along the way to finding the answers. Now let's address some additional questions based on these answers. How does the maximum time aloft depend on the mass of the projectile? If the initial speed from the cannon doubles, how is the range affected? What angle maximizes the distance the projectile travels? The answers to these questions are not obvious from Eqs. (1.5) – (1.8). We would need to perform additional calculations. Also, consider how many calculations would need to be made if we had to determine the range and time aloft for a variety of cannon angles, initial velocities, and cannon ball masses.

1.1.2. Symbolic Approach

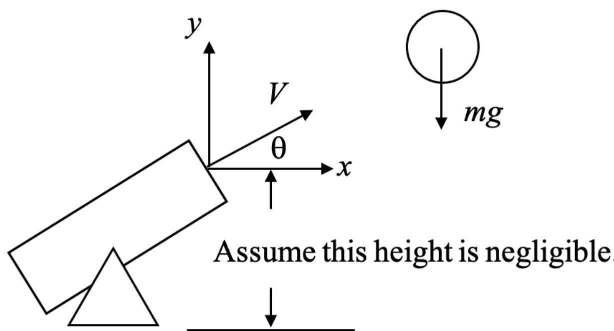


Figure 1.2. The free body diagram for the projectile example using symbols.

Now let's try working the same problem using symbols rather than numbers. We'll plug in the numbers at the very end of the problem. Draw the FBD as before (Figure 1.2). Follow the same approach as before,

$$\sum F_x = m\ddot{x} \implies 0 = m\ddot{x} \implies \ddot{x} = 0, \tag{1.9}$$

$$\sum F_y = m\ddot{y} \implies -mg = m\ddot{y} \implies \ddot{y} = -g, \tag{1.10}$$

$$\dot{x} = \dot{x}_0 = V\cos\theta, \tag{1.11}$$

$$\dot{y} = -gt + \dot{y}_0 = -gt + V\sin\theta, \tag{1.12}$$

$$\boxed{x = (V\cos\theta)t} \quad (x_0 = 0), \tag{1.13}$$

$$\boxed{y = -\frac{1}{2}gt^2 + (V\sin\theta)t} \quad (y_0 = 0). \tag{1.14}$$

The time aloft is found by setting $y = 0$,

$$\boxed{t = \frac{2V\sin\theta}{g}}, \tag{1.15}$$

and the distance traveled is,

$$\boxed{x = \frac{2V^2\cos\theta\sin\theta}{g} = \frac{V^2\sin(2\theta)}{g}}. \tag{1.16}$$

We can now plug in the given numbers to get our numerical answers,

$$t = \frac{2(100\,\mathrm{m\,s^{-1}})\sin(30°)}{9.81\,\mathrm{m/s^2}} = 10.2\,\mathrm{s}, \tag{1.17}$$

$$x = \frac{(100\,\mathrm{m\,s^{-1}})^2\sin(60°)}{9.81\,\mathrm{m/s^2}} = 883\,\mathrm{m}, \tag{1.18}$$

$$\tag{1.19}$$

which are the same answers found previously.

Using these results for t and x we can easily calculate the time aloft and distance traveled for a variety of values of θ, V, and m. Note that nowhere in Eqs. (1.13) - (1.16) does the mass appear so we conclude that the mass of the cannon ball is unimportant to our calculations. We also observe that if we double the initial velocity, the time aloft will double and the distance traveled will quadruple. This information is easily lost in our calculations where numbers were used right away (refer to Eqs. (1.7) and (1.8).

Lastly, if we wanted to determine the angle that will maximum the distance traveled for a given velocity, we observe from Eq. (1.16) that we want $\sin(2\theta)$ to be as large as possible. Thus, we should tilt our cannon at an angle of $\theta = 45°$. Substituting this result back into Eqs. (1.15) and (1.16) gives,

$$t_{\max} = \frac{\sqrt{2}V}{g}, \tag{1.20}$$

$$x_{\max} = \frac{V^2}{g}. \tag{1.21}$$

We can also easily double-check the dimensions of the equations and verify that they are dimensionally homogeneous,

$$[t] = \frac{L/T}{L/T^2} = T \quad \text{Ok!} \tag{1.22}$$

$$[x] = \frac{(L/T)^2}{L/T^2} = L \quad \text{Ok!}, \tag{1.23}$$

where L and T represent length and time, respectively. We can conclude from this exercise the following:

(1) More information is contained in our solutions when using the symbolic approach than when using the numeric approach.
(2) If several calculations must be made using different values of the parameters, solving the problem first symbolically rather than starting the problem immediately with the numbers can save considerably on the number of computations required. Furthermore, it's much easier to correct numerical mistakes at the end of the problem rather than at the beginning or in the middle of the problem.

You're almost always better off working out a problem using symbols rather than numbers!

Be Sure To:

(1) Work out problems symbolically and wait to substitute numerical values until the final relation has been derived.
(2) Try to physically interpret your equations.
(3) Make sure any relations you derive and the numbers you calculate are physically reasonable.
(4) Double check that the dimensions (or units) of your answers are correct.

1.1.3. A Note on the Use of the Ballistic Equation

From your introductory physics course, you likely recall the ballistic equation for describing the position of an object, x, as a function of time, t, subject to an acceleration a, initial velocity, $\dot{x}_0$, and initial position, x_0,

$$x = \frac{1}{2}at^2 + \dot{x}_0 t + x_0. \tag{1.24}$$

This equation was derived in the following manner. Assume an object is subject to a constant acceleration a so we can write,

$$\ddot{x} = \frac{d\dot{x}}{dt} = a, \tag{1.25}$$

where the overdots represent differentiation with respect to time. Integrate this equation twice with respect to time making use of the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ to get,

$$\frac{d\dot{x}}{dt} = a \implies \int_{\dot{x}=\dot{x}_0}^{\dot{x}=\dot{x}} d\dot{x} = \int_{t=0}^{t=t} a dt \implies \dot{x} - \dot{x}_0 = a \int_{t=0}^{t=t} dt = at \implies \dot{x} = at + \dot{x}_0, \tag{1.26}$$

$$\dot{x} = \frac{dx}{dt} = at + \dot{x}_0 \implies \int_{x=x_0}^{x=x} dx = \int_{t=0}^{t=t} (at + \dot{x}_0)dt \implies x - x_0 = \frac{1}{2}at^2 + \dot{x}_0 t \implies x = \frac{1}{2}at^2 + \dot{x}_0 t + x_0. \tag{1.27}$$

A key step in the derivation to this equation is the assumption that $a =$ constant. When a is a constant, it may be pulled out of the integrals in Eqs. (1.26) and (1.27). Thus, the ballistic equation is only valid when $a =$ constant. It is not valid when a varies with time. If a is a function of time, then it must be evaluated within the integral. For example, if we have,

$$a = ct, \tag{1.28}$$

where c is a constant, then, using the same initial conditions as before, we have,

$$\frac{d\dot{x}}{dt} = a \implies \int_{\dot{x}=\dot{x}_0}^{\dot{x}=\dot{x}} d\dot{x} = \int_{t=0}^{t=t} a dt \implies \dot{x} - \dot{x}_0 = \int_{t=0}^{t=t} (ct)dt = \frac{1}{2}ct^2 \implies \dot{x} = \frac{1}{2}ct^2 + \dot{x}_0, \tag{1.29}$$

$$\dot{x} = \frac{dx}{dt} = \frac{1}{2}ct^2 + \dot{x}_0 \implies \int_{x=x_0}^{x=x} dx = \int_{t=0}^{t=t} (\frac{1}{2}ct^2 + \dot{x}_0)dt \implies x - x_0 = \frac{1}{6}ct^3 + \dot{x}_0 t \implies x = \frac{1}{6}ct^3 + \dot{x}_0 t + x_0. \tag{1.30}$$

Thus, we see that the position in Eq. (1.30) is different than the result given by the ballistic equation.

1.2. Dimensions and Units

A dimension is a qualitative description of the physical nature of some quantity.

Notes:

(1) A basic or primary dimension is one that is not formed from a combination of other dimensions. It is an independent quantity.
(2) A secondary dimension is one that is formed by combining primary dimensions.
(3) Common dimensions include:

$$\begin{aligned} M &= \text{mass} \\ L &= \text{length} \\ T &= \text{time} \\ \theta &= \text{temperature} \\ F &= \text{force} \end{aligned}$$

(4) If M,L, and T are primary dimensions, then $F = ML/T^2$ is a secondary dimension. If F, L, and T are primary dimensions, then $M = FT^2/L$ is a secondary dimension.

A unit is a quantitative description of a dimension. A unit gives "size" to a dimension. Common systems of units in engineering are given in Table 1.1.

Notes:

(1) The mole is the amount of substance that contains the same number of elementary entities as there are atoms in 12 g of carbon 12 ($= 6.022 \times 10^{23}$, known as Avogadro's constant). The elementary entities must be specified, e.g., atoms, molecules, particles, etc. The unit kmol (aka kgmol) is also frequently used, with 1 kmol = 1000 mol = 6.022×10^{26} entities. The unit lbmol is used in the English system of units. Since 1 lbm = 0.453 kg, 1 lbmol = 453.592 mol. The number of kmols of a substance, n, is found by dividing the mass of the substance, m (kg) by the molecular weight of the substance, M (in kg/kmol): $n = m/M$. For example, the atomic weight of carbon 12 is 12

Table 1.1. Common units used in engineering.

primary dimension	SI (Systéme International d' Unités)	BG (British Gravitational	EE (English Engineering)
L, length	meter (m)	foot (ft)	foot (ft)
T, time	second (s)	second (s)	second (s)
θ, temperature	Kelvin (K)	degree Rankine (°R)	degree Rankine (°R)
M, mass	kilogram (kg)	-not primary-	pound mass (lbm or lb)
N, amount of a substance	mole (mol)	mole (mol)	pound mole (lbmol)
F, force	-not primary-	pound force (lbf)	pound force (lbf)

kmol/kg, hence, 1 kg of carbon 12 contains: (12 kg)/(12 kg/kmol) = 1 kmol = 1000 mol (or 12 g of C12 contains 1 mol).

(2) In the SI system, force is a secondary dimension and is given by: $F = ML/T^2$ where $1\,\mathrm{N} = 1\,\mathrm{kg m/s^2}$.
(3) In the EE system, both force and mass are primary dimensions. The two are related via Newton's second law, $g_c F = ma$ where $g_c = 32.2\,\mathrm{lb_m ft/(lb_f s^2)}$. It's easiest just to remember that: $1\,\mathrm{lb_f} = 32.2\,\mathrm{lb_m ft/s^2}$.
(4) The slug is the unit of mass in the British Engineering system of units. To convert between a slug and $\mathrm{lb_f}$ is: $1\,\mathrm{lb_f} = 1\,\mathrm{slug ft/s^2}$.
(5) The kilogram-force (kgf) is (unfortunately) a not uncommon quantity. The conversion between kgf and Newtons is: 1 kgf = 9.81 N.
(6) It's a good policy to carry units through your calculations. Remember that, unless it's dimensionless, every number has a unit attached to it. For example, if $m = 1\,\mathrm{lb_m}$ and $g = 32.2\,\mathrm{ft/s^2}$,

$$\text{Poor Practice:} \quad mg = (1)(32.2) = 1\,\mathrm{lb_f},$$

$$\text{Good Practice:} \quad mg = (1\,\mathrm{lb_m})(32.2\,\mathrm{ft/s^2}) = \left(32.2\,\mathrm{lb_m ft/s^2}\right)\left(\frac{1\,\mathrm{lb_f}}{32.2\,\mathrm{lb_m ft/s^2}}\right) = 1\,\mathrm{lb_f}$$

Carrying through your units makes it less likely that you'll make unit conversion errors, plus it makes it easier for you and others to follow your work.

Example: What is 70 mph in furlongs per fortnight?
Solution:

$$\left(70\frac{\text{miles}}{\text{hour}}\right)\left(\frac{5280\,\text{feet}}{\text{miles}}\right)\left(\frac{\text{rod}}{16.5\,\text{feet}}\right)\left(\frac{\text{furlong}}{40\,\text{rods}}\right)\left(\frac{24\,\text{hours}}{\text{day}}\right)\left(\frac{7\,\text{days}}{\text{week}}\right)\left(\frac{2\,\text{weeks}}{\text{fortnight}}\right) = 1.88\times10^5\,\frac{\text{furlongs}}{\text{fortnight}} \tag{1.31}$$

Example: What is weight of $1\,\mathrm{lb_m}$ of water in $\mathrm{lb_f}$?
Solution: Weight is mass multiplied by gravitational acceleration,

$$W = mg = (1\,\mathrm{lb_m})\left(32.2\,\mathrm{ft/s^2}\right)\left(\frac{1\,\mathrm{lb_f}}{32.2\,\mathrm{lb_m ft/s^2}}\right) = 1\,\mathrm{lb_f}. \tag{1.32}$$

Dimensional homogeneity is the concept whereby only quantities with similar dimensions can be added (or subtracted). It is essentially the concept of "You can't add apples and oranges." For example, consider the following equation,

$$10\,\mathrm{kg} + 16\,\mathrm{K} = 26\,\mathrm{m\,s^{-1}}. \tag{1.33}$$

This equation doesn't make sense since it is not dimensionally homogeneous. How can one add mass to temperature and get velocity?!?

Note that dimensional homogeneity is a necessary, but not sufficient, condition for an equation to be correct. In other words, an equation must be dimensionally homogeneous to be correct, but a dimensionally homogeneous equation isn't always correct. For example,

$$10\,\mathrm{kg} + 10\,\mathrm{kg} = 25\,\mathrm{kg}. \tag{1.34}$$

The equation has the right dimensions, but the wrong magnitudes!

Be sure to:

(1) Verify that equations are dimensionally homogeneous.
(2) Carefully evaluate unit conversions. A unit conversion error caused the loss of the $125M Mars Climate Observer spacecraft in 1999!
(3) Always include units with numerical values. An Air Canada Flight in 1983 (now referred to as the "Gimli Glider") ran out of fuel and had to make an emergency landing due, in part, because the fuel load was assumed to be in pounds when in fact it was reported in kilograms.

The Ideal Gas Law is used to find the volume as given in the following formula,

$V = \frac{mRT}{p}$,

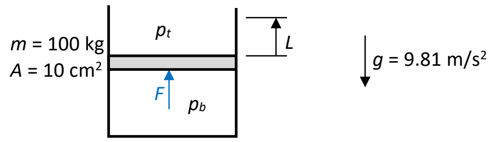
where m = 2 kg, R = 0.189 kJ/(kg.K), T = 300 K, and p = 1 bar (abs). Calculate the volume in m^3. Show all of your calculations and unit conversions.

SOLUTION:

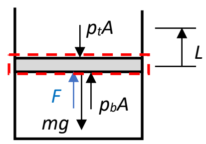
$$V = \frac{(2\ \text{kg})(0.189\ \text{kJ/(kg.K)})(300\ \text{K})}{(1\ \text{bar})} = \left(\frac{2\ \text{kg}}{1}\right)\left(\frac{0.189\ \text{kJ}}{\text{kg.K}}\right)\left(\frac{300\ \text{K}}{1}\right)\left(\frac{1}{1\ \text{bar}}\right)\left(\frac{1000\ \text{J}}{1\ \text{kJ}}\right)\left(\frac{1\ \text{bar}}{10^5\ \text{Pa}}\right)\left(\frac{1\ \text{Pa}}{1\ \text{N/m}^2}\right)\left(\frac{1\ \text{N.m}}{1\ \text{J}}\right) \tag{1}$$

$\boxed{V = 1.13\ \text{m}^3}$.

A piston, with a mass of m = 100 kg and cross-sectional area of A = 10 cm^2, is located within a cylinder as shown in the figure. The pressure on the top surface of the piston is a uniform p_t = 200 kPa (abs). The pressure on the bottom of the piston is a uniform p_b = 1 bar (abs). If a force is applied to the piston to move it slowly upwards, i.e., in a quasi-equilibrium process, a distance of L = 1 cm, determine the work done on the piston by the force, in kJ. Show all of your unit conversions.



SOLUTION:



First, determine the force by performing a vertical force balance on the piston, keeping in mind that the process is in quasi-equilibrium so that there is no acceleration of the piston,

$$\sum F = 0 = F + p_b A - mg - p_t A \;\; => \;\; F = mg + (p_t - p_b)A. \tag{1}$$

The work due to the force is,

$$W = \int_{s=0}^{s=L} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^L [mg + (p_t - p_b)A]ds = [mg + (p_t - p_b)A]L, \tag{2}$$

$$\boxed{W = mgL + (p_t - p_b)AL}. \tag{3}$$

Note that the first term on the right-hand side is the change in potential energy while the second term is the pressure difference multiplied by the volume traced out by the piston (= AL).

Using the given values, evaluate the terms in Eq. (3). Start with the mgL term,

$$mgL = \underbrace{(100 \text{ kg})}_{=m}\underbrace{\left(9.81 \frac{\text{m}}{\text{s}^2}\right)}_{=g}\underbrace{(1 \text{ cm})}_{=L}\left(\frac{1 \text{ N}}{1 \frac{\text{kg.m}}{\text{s}^2}}\right)\left(\frac{1 \text{ m}}{100 \text{ cm}}\right)\left(\frac{1 \text{ J}}{1 \text{ N.m}}\right)\left(\frac{1 \text{ kJ}}{1000 \text{ J}}\right) = 9.81 * 10^{-3} \text{ kJ}. \tag{4}$$

Now evaluate the $(p_t - p_b)AL$ term,

$$(p_t - p_b)AL = \left[\underbrace{(200 \text{ kPa})}_{=p_t}\left(\frac{1000 \text{ Pa}}{1 \text{ kPa}}\right) - \underbrace{(1 \text{ bar})}_{=p_b}\left(\frac{10^5 \text{ Pa}}{1 \text{ bar}}\right)\right]\underbrace{(10 \text{ cm}^2)}_{=A}\underbrace{(1 \text{ cm})}_{=L}\left(\frac{1 \text{ N/m}^2}{1 \text{ Pa}}\right)\left(\frac{1 \text{ m}}{100 \text{ cm}}\right)^3\left(\frac{1 \text{ J}}{1 \text{ N.m}}\right)\left(\frac{1 \text{ kJ}}{1000 \text{ J}}\right) = 1 * 10^{-3} \text{ kJ}. \tag{5}$$

Combining the numerical values in Eqs. (4) and (5),

$$\boxed{W = 1.08 * 10^{-2} \text{ kJ}}.$$

For the flow of gas in a nozzle,

$$h_2 = h_1 + \frac{1}{2}(V_1^2 - V_2^2),$$

where h_1 and h_2 are the gas's specific enthalpies at the inlet and outlet of the nozzle, respectively, and V_1 and V_2 are the gas speeds at the inlet and outlet, respectively. For the current case, h_1 = 300 kJ/kg, V_1 = 100 m/s, and V_2 = 200 m/s.

Using the given formula, calculate the value for h_2 in kJ/kg.

SOLUTION:
Substitute the given parameter into the equation to solve for h_2, including appropriate unit conversions,

$$h_2 = \left(300\,\frac{\text{kJ}}{\text{kg}}\right) + \frac{1}{2}\left[\left(100\,\frac{\text{m}}{\text{s}}\right)^2 - \left(200\,\frac{\text{m}}{\text{s}}\right)^2\right]\left(\frac{1\text{ kJ}}{1000\text{ N.m}}\right)\left(\frac{1\text{ N.m}}{1\text{ kg.m}^2/\text{s}^2}\right), \tag{1}$$

$$h_2 = \left(300\,\frac{\text{kJ}}{\text{kg}}\right) + \frac{1}{2}\left(-30000\text{ m}^2/\text{s}^2\right)\left(\frac{1\text{ kJ/kg}}{1000\text{ m}^2/\text{s}^2}\right), \tag{2}$$

$$h_2 = \left(300\,\frac{\text{kJ}}{\text{kg}}\right) - \left(15\,\frac{\text{kJ}}{\text{kg}}\right), \tag{3}$$

$$\boxed{h_2 = 285\,\frac{\text{kJ}}{\text{kg}}}. \tag{4}$$

1.3. Taylor Series Expansion Approximation

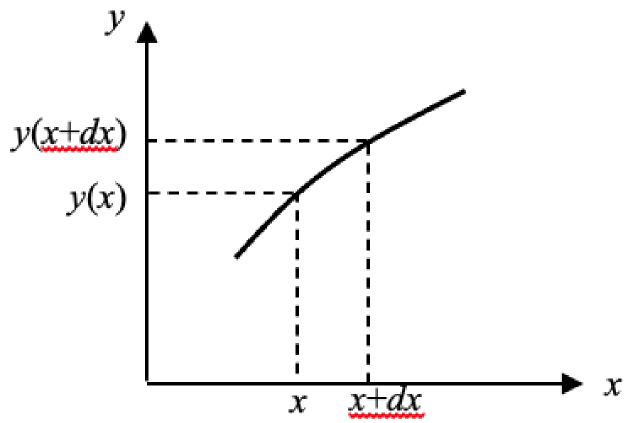


Figure 1.3. A plot used to motivate the use of Taylor Series approximations. If we know the value of $y(x)$, how can we estimate the value of $y(x+dx)$?

If we know the value of some quantity, y, at some location, x, then how can we determine the value of y at a nearby location $x+dx$ (Figure 1.3)? We can use a Taylor series expansion for y about location x,

$$y(x+\delta x) = y(x) + \left.\frac{dy}{dx}\right|_x (\delta x) + \left.\frac{d^2y}{dx^2}\right|_x \frac{(\delta x)^2}{2!} + \ldots \tag{1.35}$$

As δx becomes very small, $\delta x \to dx$, and the higher order terms become negligibly small: $\delta x \gg (\delta x)^2 \gg (\delta x)^3$,

$$\boxed{y(x+dx) \approx y(x) + \left.\frac{dy}{dx}\right|_x (dx).} \tag{1.36}$$

Note that Eq. (1.36) is simply the equation of a line. We can see what's happening more clearly in Figure 1.4. We'll use this approximation often, especially when examining how quantities vary over small distances.

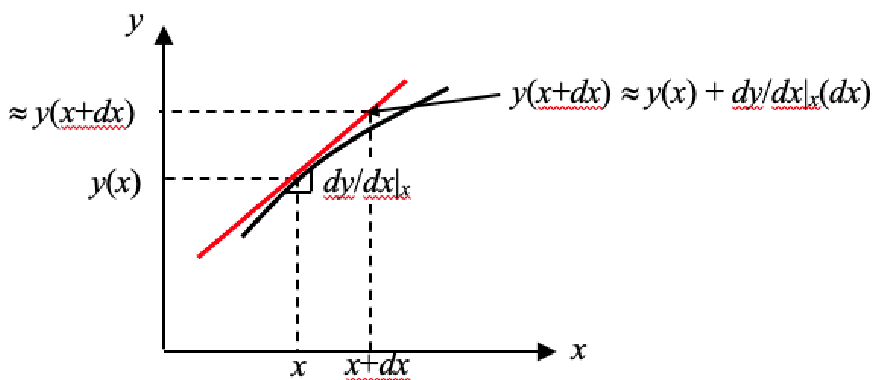


Figure 1.4. A sketch used to illustrate how a truncated Taylor Series can be used to estimate values. The black line is the original function and the red line is the truncated Taylor Series approximation.

Be sure to:

(1) Make sure you understand how this procedure works. It will be used frequently in the remainder of these notes.

1.4. Significant Digits

Significant digits (aka significant figures) are the number of digits in a value that contribute to the accuracy of the value. A large number of significant digits implies that the value is known to high accuracy. For example, let's say we report the length of a mobile phone as 14.681230924 cm. Having such a large number of digits (in this case, 11 significant digits) implies that we know the phone length very accurately, in this case down to the nanometer scale. Obviously this isn't typically the case. A more reasonable length to report would be 14.7 cm (three significant digits), implying we know the length down to the millimeter. Thus, one should use an appropriate number significant digits when reporting values. Clearly the reporting of significant digits is closely related to the uncertainty in a reported value.

Following are the rules for identifying the number of significant digits in a value.

(1) Leading zeros are not counted as significant digits. For example, 0.0023 has only two significant digits, the 2 and the 3. The three leading zeros do not count as significant digits and only serve as placeholders (the ones, tenths, and hundredths spots).
(2) Trailing zeros that serve merely as placeholders are not counted as significant digits. For example, 23000 has only two significant digits, the 2 and the 3. The three zeros are not significant and only serve to fill in the ones, tens, and thousands spots.
(3) Zeros between non-zero values are considered significant. For example, 23001 has five significant digits: 2, 3, 0, 0, and 1.
(4) Trailing zeros immediately before and immediately after a decimal point are considered significant digits because they indicate that a value is known with some accuracy. For example, 23000. has five significant digits. Similarly, 2.3000 has five significant digits.
(5) A line over a zero can indicate that that the numbers to the left of that value are significant. For example, $230\bar{0}0$ has four significant digits: the 2, the 3, the 0 in the thousands spot, and the zero in the tens spot.
(6) A handy way of determining the number of significant digits is to write the value in scientific notation then count the digits in the mantissa, i.e., the number being multiplied by 10 raised to some power. For example, 0.0023 can be written as $2.3 * 10^{-3}$. The mantissa is 2.3, which has two significant digits. The number 23000 can be written as $2.3 * 10^4$, which as two significant digits. The numbers 23000. and 2.3000 can be written as $2.3000 * 10^4$ and $2.3000 * 10^1$, respectively, and have five significant digits each.
(7) An exact number has an infinite number of significant digits. For example, if the number of apples you have is five, then the number 5 has an infinite number of significant digits since it could be written as 5.0000... with an infinite number of zeros.

There are rules for performing calculations with significant figures so that calculated values are not reported more precisely than the numbers used in the calculations.

(1) For addition and subtraction, the rightmost significant digit in the calculated value is determined by the least accurate added/subtracted value. For example, the sum $1.2 + 0.444$ should be reported as 1.6 since the accuracy of the sum is set by the tenths position of the first number. Similarly, $1.234 - 4.5$ should be reported as -3.3, where the significant digit in the tenths position has been rounded.
(2) For multiplication and division, the number of significant digits in the product/quotient should have the same number of significant digits as the lesser of the two multiplied/divided values. For example, $1.23 * 0.4$ should be written as 0.5. The product has one significant digit and has been rounded. The operation 1.2/0.456 should be written as 2.6.
(3) There are also rules for logarithms, exponential, and trig functions, but these are not covered in these notes and are instead available in introductory physics textbooks or online.
(4) To prevent a loss of accuracy, when performing intermediate calculations retain all of the digits as is practical rather than rounding the intermediate values. Only round the final result. For example, $(1.23 * 0.4) + 5.67$ is calculated as $0.\bar{4}92 + 5.67$ which should then be reported as 6.2.

Despite all of these rules, many engineering analyses (including most of the examples I present in these notes) only follow these rules loosely. A best practice would be to follow the rules exactly. The author's only excuse for not following the rules is that it would be a considerable effort to correct all of the examples in these notes, which isn't a great excuse. An engineering rule of thumb is to present final values to within three significant digits, i.e., $1.23 * 10^n$ (where n is some number) since most engineering calculations don't involve highly accurate values. That being said, if one handles numbers and calculations requiring high accuracy, then following the significant digit rules carefully would be wise.

1.5. Experimental Uncertainty

In any experimental (or even computational) study, attention must be paid to the uncertainties involved in making measurements. Including the uncertainty allows one to judge the validity or accuracy of the measurements. Uncertainty analysis can also be useful when designing an experiment so that the propagation of uncertainties can be minimized. Consider a measurement of a flow rate through a pipe. Let's say that one measures a flow rate of 1.6 kg/s. Now consider a theoretical calculation that predicts a flow rate of 1.82 kg/s. Are the theory and measurement inconsistent? The answer depends upon the uncertainty in the measurement. If the experimental uncertainty is ± 0.3 kg/s, then the true measured value could very well be equal to the theoretical value. However, if the experimental uncertainty is ± 0.1 kg/s, then the two results are likely to be inconsistent.

There are two parts to uncertainty analysis. These include:

(1) estimating the uncertainty associated with a measurement and
(2) analyzing the propagation of uncertainty in subsequent analyses.

Both of these parts will be reviewed in the following sections. There are many texts (such as Holman, J.P., *Experimental Methods for Engineers*, McGraw-Hill) that can be referred to for additional information concerning experimental uncertainty.

1.5.1. Estimation of Uncertainty

There are three common types of error. These include "blunders," systematic (or fixed) errors, and random errors.

(1) "Blunders" are errors caused by mistakes occurring due to inattention or an incorrectly configured experimental apparatus. Examples include:
   - Blatant blunder: An experimenter looks at the wrong gauge or misreads a scale and, as a result, records the wrong quantity.
   - Less blatant blunder: A measurement device has the wrong resolution (spatial or temporal) to measure the parameter of interest. For example, an experimenter who uses a manometer to measure the pressure fluctuations occurring in an automobile piston cylinder will not be able to capture the rapid changes in pressure due to the manometer's slow response time.
   - Subtle blunder: A measurement might affect the phenomenon that is being measured. For example, an experimenter using an ordinary thermometer to make a very precise measurement of a hot cavity's temperature might inadvertently affect the measurement by conducting heat out of the cavity through the thermometer's stem.

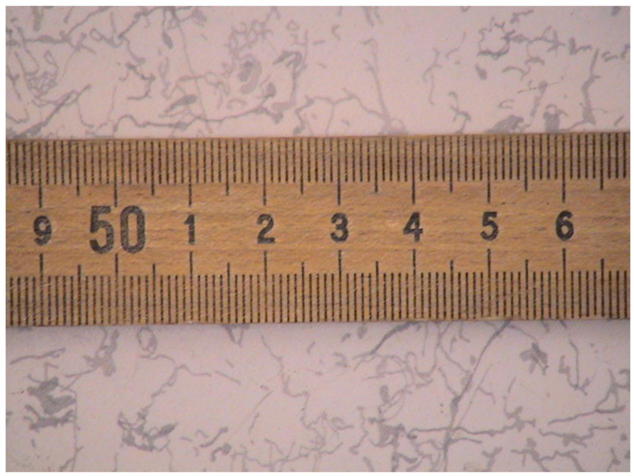
(2) Systematic (or fixed) errors occur when repeated measurements are in error by the same amount. These errors can be removed via calibration or correction. For example, the error in length caused by a blunt ruler. This error can be corrected by calibrating the ruler against a known length.
(3) Random errors occur due to unknown factors. These errors are not correctable, in general. Blunders and systematic errors can be avoided or corrected. It is the random errors that we must account for in uncertainty analyses. How we quantify random errors depends on whether we conduct a single experiment or multiple experiments. Each case is examined in the following sub-sections.

1.5.2. Single Sample Experiments (aka Type B Uncertainty)

A single sample experiment is one in which a measurement is made only once. This approach is common when the cost or duration of an experiment makes it prohibitive to perform multiple experiments.

The measure of uncertainty in a single sample experiment is $\pm\frac{1}{2}$ the smallest scale division (or least count) of the measurement device. For example, given a thermometer where the smallest discernible scale division is $1\,^\circ$C, the uncertainty in a temperature measurement will be $\pm 0.5\,^\circ$C. If your eyesight is poor and you can only see $5\,^\circ$C divisions, then the uncertainty will be $\pm 2.5\,^\circ$C. One should use an uncertainty within which they are 95% certain that the result lies.

Example: What is the least count for the ruler in the following figure?



Solution: The least count for the ruler is 1 mm. Hence, the uncertainty in the length measurement will be ± 0.5 mm.

Example: You use a manual electronic stop watch to measure the speed of a person running the 100 m dash. The stop watch gives the elapsed time to 1/1000th of a second. What is the least count for the measurement?

Solution: Although the stop watch has a precision of 1/1000th of a second, you cannot respond quickly enough to make this the limiting uncertainty. Most people have a reaction time of 1/10th of a second. (Test yourself by having a friend drop a ruler between your fingers. You can determine your reaction time by where you catch the ruler.) Hence, to be 95% certain of your time measurement, you should use an uncertainty of $\pm\frac{1}{2}(0.1\,\text{s}) = \pm 0.05\,\text{s}$.

Be sure to:

(1) Always indicate the uncertainty of any experimental measurement.
(2) Carefully design your experiments to minimize sources of error.
(3) Carefully evaluate your least count. The least count is not always $\pm\frac{1}{2}$ the smallest scale division.

1.5.3. Multiple Sample Experiments (aka Type A Uncertainty)

A multiple sample experiment is one in which many different trials are conducted in which the same measurement is made. For example, imagine taking temperature measurements in many "identical" hot cavities (Figure 1.5) or making temperature measurements in the same cavity many different times.

We can use statistics to estimate the random error associated with a multiple sample experiment. For truly random errors, the distribution of errors will approximately follow a Gaussian (aka normal) distribution

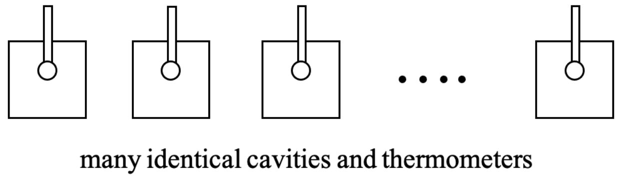


Figure 1.5. A multiple sample experiment in which temperatures are measured in many identical systems.

(Figure 1.6), which has the following probability distribution,

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \tag{1.37}$$

$$\int_{-\infty}^{+\infty} p(x)dx = 1, \tag{1.38}$$

where $p(x)$ is the probability of obtaining the value x, μ is the true mean of the distribution, and σ^2 is the true variance of the distribution. The mean is the center of the distribution and the variance is a measure of the distribution's spread about the mean.

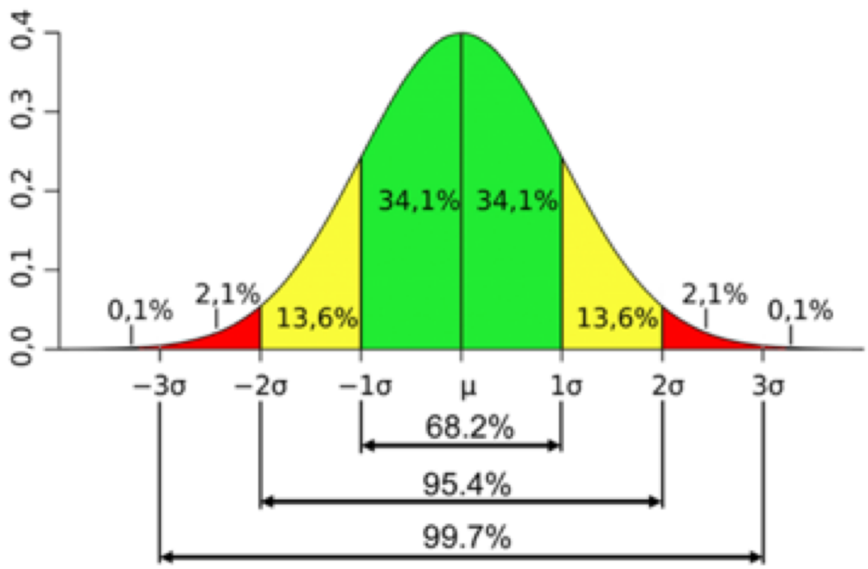


Figure 1.6. A Gaussian (aka normal) probability distribution. The parameter μ is the true mean of the distribution and σ is the true standard deviation. For a normal distribution, 68.2% of the values lie within $\pm 1\sigma$ of the mean, 95.4% lie within $\pm 2\sigma$ of the mean, and 99.7% lie within 99.7% of the mean. The area under any probability distribution curve is equal to one.

Notes:

(1) It is not possible to comprehensively discuss statistical analyses of data within the scope of these notes. The reader is encouraged to look through an introductory text on statistics for additional information (see, for example, Vardeman, S.B., *Statistics for Engineering Problem Solving*, PWS Publishing, Boston).

(2) The coefficient of variation, CoV or CV (also rsd = relative standard deviation), is defined as the ratio of the standard deviation to the mean, i.e., $CoV := \sigma/\mu$. A small CoV means that the scatter in the measurements is small compared to the mean.
(3) For random data (a Gaussian/normal distribution) and a very large number of measurements,

$$\left.\begin{matrix} 68.2\% \\ 95.4\% \\ 99.7\% \end{matrix}\right\} \text{of the measurements fall between} \left\{\begin{matrix} \mu \pm 1\sigma \\ \mu \pm 2\sigma \\ \mu \pm 3\sigma \end{matrix}\right. \tag{1.39}$$

The true mean and true variance of the experimental data aren't typically known in practice since determining those quantities would require an infinite number of measurements. Instead, we have a finite number of measurements (call this number N) and we calculate the sample mean and sample variance of the measurements,

$$\bar{x} := \frac{1}{N}\sum_{n=1}^{N} x_n \qquad \text{sample mean,} \tag{1.40}$$

$$s^2 := \frac{1}{N-1}\sum_{n=1}^{N}(x_n - \bar{x})^2 \qquad \text{sample variance } (s \text{ is the sample standard deviation}). \tag{1.41}$$

Example: The following seven measurements are randomly chosen from a normal distribution with a true mean of $\mu = 100$ and a true variance of $\sigma^2 = 400$ ($\sigma = 20$). Calculate the sample mean and sample variance (and standard deviation) of the measurements.

#	$\boldsymbol{x_i}$
1	99.36
2	121.02
3	131.73
4	119.56
5	94.31
6	114.74
7	78.33

Solution: Using Eqs. (1.40) and (1.41), the sample mean and sample variance are $\bar{x} = 108.4$ and $s^2 = 342.05$ ($s = 18.49$). Notice that the sample mean and sample variance are different from the true mean ($\mu = 100$) and true variance ($\sigma^2 = 400$). The reason for the difference is that we're making a mean and variance calculation using a small number of samples ($N = 7$) from the real distribution. The larger our number of samples, the closer the sample mean and sample variance will be to the true mean and true variance.

Now imagine we collect seven new measurements and calculate the sample mean and sample variance for that set of data. Call this Trial 2. Do this multiple times to obtain a table of sample means and variances for many trials (Table 1.2). Notice the sample means and sample variances are different for each trial. Plotting the sample means from a large number of trials produces the frequency distribution shown in Figure 1.7. The vertical axis is the fraction of the total number of trials with sample means in the given range on the horizontal axis, divided by the size of the range. Defined in this manner, the total area under the columns is equal to one. There are a large number of trials with sample means close to the true mean, and a handful with sample means far from the true mean.

The standard deviation of the distribution of sample means is known as the standard error, $s_{\bar{x}}$. The standard error can be approximated (proof not given here) from a single trial's measurements using,

$$s_{\bar{x}} \approx \frac{s}{\sqrt{N}}. \tag{1.42}$$

For the current example using Trial 1 data, $s_{\bar{x}} \approx 6.99$. A normal distribution using the mean of the sample means and standard deviation equal to the standard error is superimposed on the previous plot as an orange curve. Clearly the data from the trials is approximated well by this normal distribution. The true mean of

TABLE 1.2. A table of values sampled from a normal distribution with a true mean and true variance of $(\mu, \sigma^2) = (100, 400)$. In each trial, seven samples are collected and the sample mean $\bar{x}$ and sample variance s^2 are calculated for that trial.

	Trial 1	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**	**Trial 6**
#	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$
1	99.36	120.20	80.92	72.20	130.41	86.54
2	121.02	76.56	88.64	95.92	116.38	100.09
3	131.73	93.18	93.32	100.04	113.11	112.06
4	119.56	65.21	98.33	82.72	103.82	128.79
5	94.31	105.08	143.42	72.67	102.18	88.30
6	114.74	102.21	116.85	147.12	101.71	99.12
7	78.33	76.47	98.88	104.78	96.41	79.68
$\bar{x} =$	**108.44**	**91.27**	**102.91**	**96.49**	**109.15**	**99.22**
$s^2 =$	**342.05**	**377.60**	**442.04**	**665.40**	**135.73**	**283.68**

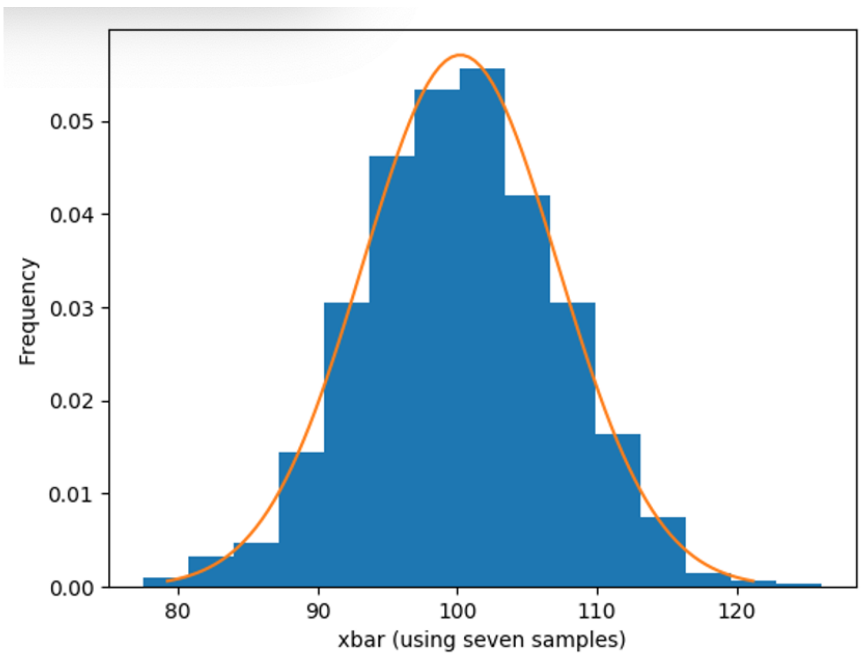


FIGURE 1.7. A frequency distribution of the sample means calculated from the trials in Table 1.2. Note that this plot includes many more trials than the six shown in Table 1.2. The orange curve is a normal distribution centered on the mean of the sample means with a standard deviation equal to the standard error.

the distribution lies somewhere within this distribution. Since we don't know exactly what the true mean value is without an exceedingly large number of measurements, at best we can estimate its value from the sample mean measurement and the standard error. Using the properties of a normal distribution discussed in a previous note, we can state, for example, that for a large number of measurements N that the true mean will lie within the range,

$$\bar{x} - 2s_{\bar{x}} < \mu < \bar{x} + 2s_{\bar{x}} \quad \text{or} \quad \mu = \bar{x} \pm 2s_{\bar{x}}, \tag{1.43}$$

95.4% of the time.

Notes:

Table 1.3. A table of t values from the Student t-distribution.

N	2	3	4	5	6	7	8	9	10	15	20	30	∞
$t_{95\%}$	12.71	4.30	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.14	2.09	2.04	1.96

(1) When we use two standard errors to bound the true mean, i.e., $\pm 2s_{\bar{x}}$, we call this a 95.4% confidence interval (CI). In engineering, the preferred confidence interval is 95%, which corresponds to $\pm 1.96s_{\bar{x}}$, at least for a large number of measurements.

(2) If the number of measurements is not very large ($N < 30$, for example), it is more accurate to use the Student t-distribution for estimating the uncertainty rather than a normal distribution (refer to an introductory text on statistics such as Vardeman, S.B., *Statistics for Engineering Problem Solving*, PWS Publishing, Boston),

$$\mu = \bar{x} \pm t s_{\bar{x}}, \tag{1.44}$$

where t is a factor related to the degree of confidence desired (again, a 95% uncertainty is typically desired in engineering applications), $s_{\bar{x}}$ is the standard error, and N is the number of measurements made. Table 1.3 gives the value of t for various values of N and a 95% confidence level. Note that as $N \to \infty$ the t factor approaches the large sample size value of 1.96. For the previous example,

$$N = 7, \bar{x} = 108.44, s = 18.49 \implies s_{\bar{x}} = \frac{s}{\sqrt{N}} = 6.99,\ t_{95\%} = 2.45, \tag{1.45}$$

$$\implies \mu = 108.44 \pm 17.12\,(95\%CI) \quad \text{or} \quad 91.32 < \mu < 125.56\,(95\%CI). \tag{1.46}$$

This range is shown in Figure 1.8. Recall that the true mean is $\mu = 100$.

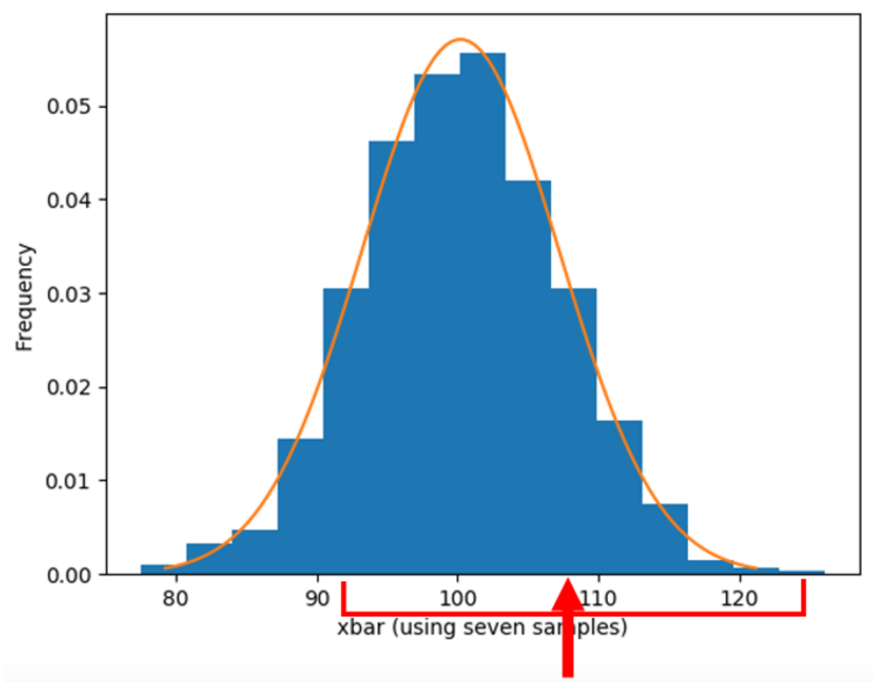


Figure 1.8. The same frequency distribution shown in Figure 1.7, but this one also shows the range within which the true mean lies using a 95% confidence interval.

(3) It's possible that the true mean could lie outside of our stated range. For a confidence interval of 95%, it's unlikely, but possible.

(4) To improve the precision of the true mean estimate, one should increase the number of measurements N, which decreases the standard error and the t factor. Decreasing the sample standard deviation would also improve the precision (by decreasing the standard error), but this may not be possible depending on what is generating the variability. If it's environmental noise, then it may not be possible to decrease the standard deviation of the measurements. If it's equipment noise, then improvements in equipment design would help.

Be sure to:

(1) Report the uncertainty in an individual measurement as well as the sample mean and 95% confidence interval for multiple sample experiments.

An engineer makes five "identical" pressure measurements in an experiment. The computer display on which the pressure measurement is displayed has a least count of 0.01 psi; however, the pressure values fluctuate over a wider range of values as indicated in the following table containing the pressure measurement readings.

Measurement	1	2	3	4	5
Reading [psi]	16.77	16.29	16.66	16.33	16.76

What is the true pressure that the engineer should report?

SOLUTION:

Even though the transducer's least count is 0.01 psi, the uncertainty per measurement is much larger than this based on the range over which the pressures fluctuate.

The sample mean for the $N = 5$ measurements is $\bar{x}$ =16.56 psi and the sample standard deviation is $s = 0.23$ psi. Since the number of measurements is small, a Student's t-distribution should be used to give the 95% confidence level in the measurement. With $N = 5$, $t_{0.95} = 2.78$ (found from a t distribution table). The standard error of the sample means is,

$$s_{\bar{x}} = \frac{s}{\sqrt{N}} = \frac{(0.23\text{ psi})}{\sqrt{5}} = 0.10\text{ psi}.$$

Hence, the measurement with uncertainty is,

$$\bar{x} \pm ts_{\bar{x}} = 16.56 \pm (2.78)(0.10)\text{ psi},$$

$$\boxed{\bar{x} \pm ts_{\bar{x}} = 16.56 \pm 0.29\text{ psi}}.$$

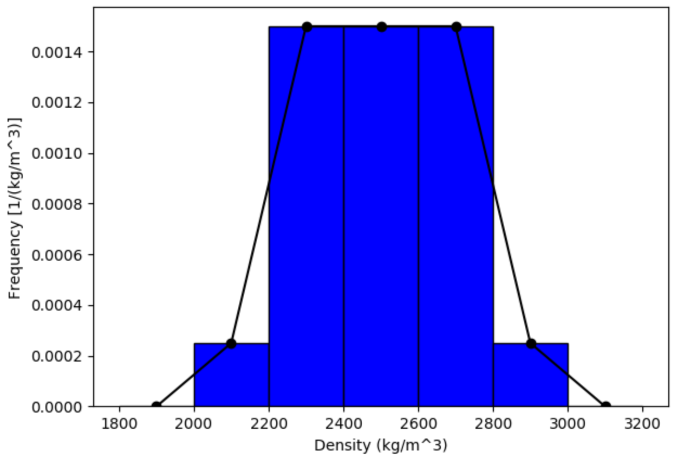
The following table lists repeated measurements of the density of glass particles.

a. Plot a frequency distribution of the density values in a plot with the x-axis ranging from [1800, 3200] kg/m^3 with seven total bins (each bin size is 200 kg/m^3).
b. Determine the sample mean of the distribution.
c. Determine the true mean of the particle density within a confidence interval of 95%.
d. What fraction of the density measurements lie within the range [2200, 2800] kg/m^3?

Measurement #	Density [kg/m^3]
1	2694
2	2516
3	2628
4	2831
5	2342
6	2505
7	2612
8	2531
9	2452
10	2380
11	2657
12	2335
13	2668
14	2516
15	2701
16	2222
17	2003
18	2565
19	2222
20	2316

SOLUTION:

Following is a frequency distribution plot of the data. Refer to the python code at the end of this document for how it was generated.



Note that the area under the frequency distribution curve is equal to one.

The sample mean of the measurements, m, is,

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{i=N} x_i, \tag{1}$$

where $N = 20$ and x_i is measurement number i. Using the given data,

$$\boxed{\bar{x} = 2484.8\ \text{kg/m}^3}.$$

The sample standard deviation of the measurements, s, is,

$$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{i=N}(x_i - \bar{x})^2}. \tag{2}$$

Using the given data,

$$\boxed{s = 201.9\ \text{kg/m}^3}.$$

The standard error of the sample means, SEM, is,

$$SEM = \frac{s}{\sqrt{N}}. \tag{3}$$

Using the given data,

$$\boxed{SEM = 45.2\ \text{kg/m}^3}.$$

The true mean, μ, will lie within the range,

$$\mu = \bar{x} \pm t_{95\%} SEM, \tag{3}$$

where the value for $t_{95\%}$ is found from a Student's t distribution at a 95% confidence interval to be 2.093 for $N = 20$ ($N - 1 = 19$ degrees of freedom). Thus,

$$\boxed{\mu = 2484.8 \pm 94.5\ \text{kg/m}^3\ (95\%\ \text{CI})}. \tag{4}$$

The fraction of density measurements in the range [2200, 2800] is,

$$\text{fraction}(x_i, x_f) = \sum_{x_i}^{x_f - \Delta x_{f-1}} f(x_i, x_i + \Delta x_i)\Delta x_i \tag{5}$$

$$\text{fraction}(2200, 2800) = f(x_{2200}, x_{2400})(200\ \text{kg/m}^3) + f(x_{2400}, x_{2600})(200\ \text{kg/m}^3) + f(x_{2600}, x_{2800})(200\ \text{kg/m}^3) \tag{6}$$

$$\text{fraction}(2200, 2800) = [f(x_{2200}, x_{2400}) + f(x_{2400}, x_{2600}) + f(x_{2600}, x_{2800})](200\ \text{kg/m}^3) \tag{7}$$

$$\text{fraction}(2200, 2800) = \left[0.001500\,\frac{\text{m}^3}{\text{kg}} + 0.001500\,\frac{\text{m}^3}{\text{kg}} + 0.001500\,\frac{\text{m}^3}{\text{kg}}\right](200\ \text{kg/m}^3) \tag{8}$$

$$\text{fraction}(2200, 2800) = 0.9. \tag{9}$$

$\boxed{\text{Thus, 90\% of the measurements lie in the range [2200, 2800] kg/m}^3}$.

```
# uncertainty_10.py

import scipy.stats as stats
import numpy as np
import pylab as plt

# Put the data into an array.  Normally we would read this data from
# an input file.
my_data = np.array([2694, 2516, 2628, 2831, 2342, 2505, 2612, 2531, 2452, 2380, 2657, 2335, 2668, 2516, 2701,
2222, 2003, 2565, 2222, 2316])

# Report some statistics about the data.
N = len(my_data)  # number of samples
sample_mean = np.mean(my_data)  # sample mean
sample_stdev = np.std(my_data, ddof=1)      # sample standard deviation;
                                            # divisor is N-1 since we
                                            # don't know the entire
                                            # population
sem = stats.sem(my_data) # std error of the sample mean
CI = 0.05 # 95% confidence interval (alpha = 0.05)
t = stats.t.ppf(1-CI/2, N-1)  # compute t-factor for the specified confidence interval

# Print the data
print("# of data entries =", N)
print("sample mean (kg/m^3) = %.1f" % sample_mean)
print("sample standard deviation (kg/m^3) = %.1f" % sample_stdev)
print("standard error of the sample means (kg/m^3) = %.1f" % sem)
print("t_95 for %d" % N, "samples = %.3f" % t)
print("true mean (kg/m^3) = %.1f" % sample_mean, " +/- %.1f" % (t*sem))

# Generate the frequency distribution data.  Set the bin edges.
bin_list = np.linspace(1800, 3200, num=8)
#Nbins = 6  # number of bins to use in the frequency plot
counts, bin_edges = np.histogram(my_data, bins=bin_list, density=True)

# Determine the bin centers.
bin_centers = np.empty([len(bin_edges)-1])
for i in range(len(bin_edges)-1):
    bin_centers[i] = (bin_edges[i]+bin_edges[i+1])/2

# Print the bin edges, the bin centers, and the counts.
print("[lower_bin_value, upper_bin_value)\tbin_center\tfrequency [1/(kg/m^3)]")
for i in range(len(bin_edges)-1):
    print("[%.1f," % bin_edges[i], "%.1f)" % bin_edges[i+1], "\t%.1f" % bin_centers[i], "\t%3f" % counts[i])

# Plot the frequency distribution.  Plotting it two ways: once showing
# the bin sizes with a bar chart and once showing the center of the
# bins with a scatter plot.
plt.figure(1)
plt.hist(my_data, bins=bin_list, density=True, color="blue", edgecolor="black")
plt.plot(bin_centers, counts, color='black', marker='o', linestyle='solid')
plt.ylabel('Frequency [1/(kg/m^3)]')
plt.xlabel('Density (kg/m^3)')
plt.show()
```

Running the program gives the following output:

# of data entries = 20
sample mean (kg/m^3) = 2484.8
sample standard deviation (kg/m^3) = 201.9
standard error of the sample means (kg/m^3) = 45.2
t_95 for 20 samples = 2.093
true mean (kg/m^3) = 2484.8 +/- 94.5

[lower_bin_value, upper_bin_value)	bin_center	frequency [1/(kg/m^3)]
[1800.0, 2000.0)	1900.0	0.000000
[2000.0, 2200.0)	2100.0	0.000250
[2200.0, 2400.0)	2300.0	0.001500
[2400.0, 2600.0)	2500.0	0.001500
[2600.0, 2800.0)	2700.0	0.001500
[2800.0, 3000.0)	2900.0	0.000250
[3000.0, 3200.0)	3100.0	0.000000

1.5.4. Propagation of Uncertainty

Let R be a result that depends on several measurements $(x_1, \ldots, x_N)$ or, in mathematical terms,

$$R = R(x_1, \ldots, x_N). \tag{1.47}$$

For example, the volume of a cylinder is,

$$V = \pi r^2 h \implies V = V(r, h). \tag{1.48}$$

How do we determine the uncertainty in the result R due to the uncertainties in the measurements $(x_1, \ldots, x_N)$? In the example above, what is the uncertainty in the volume V given the uncertainties in the radius, r, and height, h?

To address this issue, consider how a small variation in parameter, x_n, call it δx_n, causes a variation in R, call this variation δR_{x_n},

$$\delta R_{x_n} = R(x_1, \ldots, x_n + \delta x_n, \ldots, x_N) - R(x_1, \ldots, x_n, \ldots, x_N), \tag{1.49}$$

$$\delta R_{x_n} = \frac{R(x_1, \ldots, x_n + \delta x_n, \ldots, x_N) - R(x_1, \ldots, x_n, \ldots, x_N)}{\delta x_n} \delta x_n, \tag{1.50}$$

$$\boxed{\underbrace{\delta R_{x_n}}_{\substack{\text{uncertainty}\\ \text{in } R \text{ due to}\\ \text{uncertainty in } x_n}} \approx \underbrace{\frac{\partial R}{\partial x_n}}_{\substack{\text{partial derivative}\\ \text{of } R \text{ wrt } x_n}} \underbrace{\delta x_n}_{\substack{\text{uncertainty in}\\ \text{measurement } x_n}}} \tag{1.51}$$

Note that an "=" is only strictly true as $\delta x_n \to dx_n$.

The total uncertainty in R, δR, due to uncertainties in all measurements $x_1, \ldots, x_N$, assuming that the x_n are independent so that the variations in one parameter do not affect the variations in the others, is estimated as,

$$\boxed{\delta R = \left[\sum_{n=1}^{N} (\delta R_{x_n})^2\right]^{1/2} = \left[\sum_{n=1}^{N} \left(\frac{\partial R}{\partial x_n} \delta x_n\right)^2\right]^{1/2}}. \tag{1.52}$$

The <u>relative uncertainty</u> in R (u_R) is given by,

$$\boxed{u_R = \frac{\delta R}{R}}. \tag{1.53}$$

For example, the uncertainty in the cylinder volume, $V = \pi r^2 h$, due to uncertainties in the radius, r, and height, h, is,

$$\delta V = \left[\left(\frac{\partial V}{\partial r} \delta r\right)^2 + \left(\frac{\partial V}{\partial h} \delta h\right)^2\right]^{1/2}, \tag{1.54}$$

$$= \left[(2\pi r h \delta r)^2 + (\pi r^2 \delta h)^2\right]^{1/2}, \tag{1.55}$$

and the relative uncertainty is,

$$u_V = \frac{\delta V}{V} = \frac{1}{\pi r^2 h} \left[(2\pi r h \delta r)^2 + (\pi r^2 \delta h)^2\right]^{1/2}, \tag{1.56}$$

$$= \left[\left(2\frac{\delta r}{r}\right)^2 + \left(\frac{\delta h}{h}\right)^2\right]^{1/2}, \tag{1.57}$$

$$= \left[(2u_r)^2 + (u_h)^2\right]^{1/2}, \tag{1.58}$$

where $u_r = \delta r / r$ and $u_h = \delta h / h$ are the relative uncertainties in r and h, respectively.

Notes:

(1) Use absolute quantities when calculating the uncertainty. For example, use or °R or K as opposed to °F or °C for temperature, and use absolute pressures rather than gage pressures.
(2) In an uncertainty analysis the uncertainty of some quantities may be so small compared to the uncertainties in the remaining quantities that they can be considered "exactly" known. This is generally the case for well-characterized constants and material parameters, e.g., the acceleration due to gravity.

1.5.5. Significant Figures

A topic closely related to uncertainty is "significant figures".

Notes:

(1) The zeros between the decimal point and the first non-zero number are not counted as significant digits. For example, 0.001 23 kg has three significant digits, i.e., the "123". The leading zeros aren't necessary to report the value. For example, we could have also reported the number as 1.23×10^{-2} kg, which doesn't include the leading zeros.
(2) Trailing zeros are also not counted as significant digits if they're only used as placeholders. For example, 12 300 kg has three significant digits, i.e., the "123". For example, we could have written 1.23×10^{4} kg, which doesn't include the trailing zeros.
(3) In typical engineering calculations, if uncertainty is included in the parameter values, then reporting results to three significant figures is typical.

Be sure to:

(1) Use absolute quantities when evaluating uncertainties, e.g., absolute temperature and pressure.
(2) Review your uncertainty analyses to determine which measurements result in the greatest error in a derived quantity. Design your experiments to reduce these uncertainties.

Using the ruler in the photograph shown below, determine the diameter of the tennis ball including uncertainty. Note that the finest divisions on the ruler are in 1 mm increments.

SOLUTION:

Even though the ruler's divisions are in 1 mm increments, the photograph's resolution is too poor to clearly make out the divisions. A much more reasonable measurement least count is 5 mm since these increments are more easily seen. Using this least count, the left side of the tennis ball, l_L, is located at 50.2±0.25 cm and the right side, l_R, is located at 56.7±0.25 cm. The diameter, D, is:

$$D = l_R - l_L = 56.7 - 50.2 \text{ cm} = 6.5 \text{ cm} \quad (1)$$

The absolute uncertainty in the diameter is:

$$\delta D = \sqrt{\left(\delta D_{l_R}\right)^2 + \left(\delta D_{l_L}\right)^2} = \sqrt{\left(\frac{\partial D}{\partial l_R}\delta l_R\right)^2 + \left(\frac{\partial D}{\partial l_L}\delta l_L\right)^2} \quad (2)$$

where

$$\frac{\partial D}{\partial l_R} = 1 \text{ and } \frac{\partial D}{\partial l_L} = -1 \quad (3)$$

Thus,

$$\delta D = \sqrt{\left(\delta l_R\right)^2 + \left(\delta l_L\right)^2} = \sqrt{2\left(0.25 \text{ cm}\right)^2} = 0.35 \text{ cm} \quad (4)$$

Thus, the tennis ball diameter, with uncertainty, is:

$$\boxed{D = 6.5 \pm 0.35 \text{ cm}} \quad (5)$$

Note that the International Tennis Federation (the United States Tennis Association is a member of this organization) indicates that a tennis ball should have a diameter between 6.541 and 6.858 cm for Type 1 (fast speed) and Type 2 (medium speed) balls (Type 3 (slow speed) balls are bigger). The measurement given above is within the upper limit, but could potentially be smaller than the allowable size.

Reference

International Tennis Federation, *The Rules of Tennis*, available at:
http://dps.altdc3.va.twimm.net/usta_master/usta/doc/content/doc_13_4198.pdf
(2005 Dec 15).

The estimated dimensions of a soda can are $D \approx 66.0$ mm and $H \approx 110$ mm. Determine the accuracy with which the diameter and height must be measured to estimate the volume of the can within an uncertainty of ±0.5%.

SOLUTION:

The volume of a cylinder (*e.g.* the soda can) is:

$$V = \tfrac{\pi}{4} D^2 H \tag{1}$$

The relative uncertainty in V is:

$$u_V = \left[u_{V,D}^2 + u_{V,H}^2 \right]^{1/2} \tag{2}$$

where

$$u_{V,D} = \frac{1}{V}\frac{\partial V}{\partial D}\delta D = \frac{4}{\pi D^2 H}\left(\frac{2\pi DH}{4}\right)\delta D = 2\frac{\delta D}{D} = 2u_D \tag{3}$$

$$u_{V,H} = \frac{1}{V}\frac{\partial V}{\partial H}\delta H = \frac{4}{\pi D^2 H}\left(\frac{\pi D^2}{4}\right)\delta H = \frac{\delta H}{H} = u_H \tag{4}$$

Substitute into Eqn. (2).

$$u_V = \left[4u_D^2 + u_H^2 \right]^{1/2} \tag{5}$$

Express the right-hand side of the previous equation in terms of absolute uncertainties and re-arrange to solve for the absolute uncertainty in the diameter and height measurements.

$$u_V^2 = 4\left(\frac{\delta x}{D}\right)^2 + \left(\frac{\delta x}{H}\right)^2 \tag{6}$$

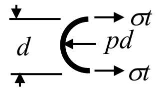
$$u_V^2 = \left(\frac{4}{D^2} + \frac{1}{H^2}\right)(\delta x)^2 \tag{7}$$

$$\boxed{\therefore \delta x = u_V \left(\frac{4}{D^2} + \frac{1}{H^2}\right)^{-1/2}} \tag{8}$$

Since we wish to measure the volume to within a relative uncertainty of $u_V = 0.005$, and $D = 66.0$ mm and $H = 110$ mm, we must have a length measurement precision of $\boxed{\delta x = 0.158 \text{ mm}}$.

The hoop stress, σ, in a thin-walled cylindrical pressure vessel may be estimated using:

$$\sigma = \frac{pd}{2t}$$



where p is the cylinder's interior gage pressure, d is the cylinder diameter, and t is the vessel wall thickness. The pressure in the vessel is measured to be 30 ± 2 psig, the tank diameter is 2.45 ± 0.03 in., and the wall thickness is 0.0050 ± 0.0002 in.

a. Determine the hoop stress including its uncertainty.
b. Which measurement should be improved first in order to reduce the uncertainty in the hoop stress?

SOLUTION:

The relative uncertainty in σ is:

$$u_\sigma = \left[u_{\sigma,p}^2 + u_{\sigma,d}^2 + u_{\sigma,t}^2\right]^{1/2} \tag{1}$$

where

$$u_{\sigma,p} = \frac{1}{\sigma}\frac{\partial\sigma}{\partial p}\delta p = \frac{2t}{pd}\left(\frac{d}{2t}\right)\delta p = \frac{\delta p}{p} = u_p \tag{2}$$

$$u_{\sigma,d} = \frac{1}{\sigma}\frac{\partial\sigma}{\partial d}\delta d = \frac{2t}{pd}\left(\frac{p}{2t}\right)\delta d = \frac{\delta d}{d} = u_d \tag{3}$$

$$u_{\sigma,t} = \frac{1}{\sigma}\frac{\partial\sigma}{\partial t}\delta t = \frac{2t}{pd}\left(-\frac{pd}{2t^2}\right)\delta t = -\frac{\delta t}{t} = -u_t \tag{4}$$

Substitute into Eqn. (1).

$$u_\sigma = \left[u_p^2 + u_d^2 + u_t^2\right]^{1/2} \tag{5}$$

The relative uncertainties in the pressure, diameter, and thickness are:

$$u_p = \frac{\delta p}{p} = \frac{2\text{ psi}}{30\text{ psi}} = 6.7\% \tag{6}$$

$$u_d = \frac{\delta d}{d} = \frac{0.03\text{ in.}}{2.45\text{ in.}} = 1.2\% \tag{7}$$

$$u_t = \frac{\delta t}{t} = \frac{0.0002\text{ in.}}{0.005\text{ in.}} = 4.0\% \tag{8}$$

$$\Rightarrow\ u_\sigma = 7.9\% \tag{9}$$

$$\boxed{\therefore \sigma = 7350 \pm 580\text{ psi}}$$

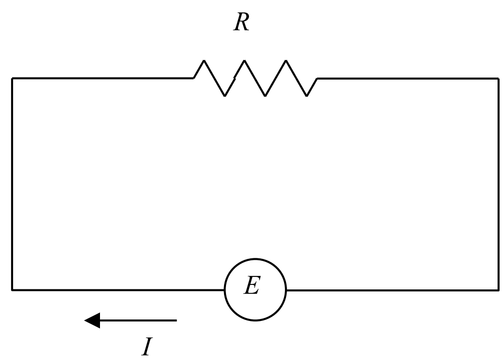
Since the relative uncertainty in the pressure measurement is the greatest, an attempt should be made to improve the accuracy of this measurement first.

A resistor has a nominal stated value of 10±0.1 Ω. A voltage difference occurs across the resister and the power dissipation is to be calculated in two different ways:

a. from $P=E^2/R$
b. from $P=EI$

In (a) only a voltage measurement will be made while both current and voltage will be measured in (b). Calculate the uncertainty in the power for each case when the measured values of E and I are:

E = 100±1 V (for both cases)
I = 10±0.1 A



SOLUTION:

Perform an uncertainty analysis using the first formula for power.

$$P = E^2/R \tag{1}$$

The relative uncertainty in P is:

$$u_P = \left[u_{P,E}^2 + u_{P,R}^2\right]^{1/2} \tag{2}$$

where

$$u_{P,E} = \frac{1}{P}\frac{\partial P}{\partial E}\delta E = \frac{R}{E^2}\left(\frac{2E}{R}\right)\delta E = 2\frac{\delta E}{E} = 2u_E \tag{3}$$

$$u_{P,R} = \frac{1}{P}\frac{\partial P}{\partial R}\delta R = \frac{R}{E^2}\left(\frac{-E^2}{R^2}\right)\delta R = -\frac{\delta R}{R} = -u_R \tag{4}$$

Substitute into Eqn. (2).

$$u_P = \left[4u_E^2 + u_R^2\right]^{1/2} \tag{5}$$

The relative uncertainties in the voltage and resistance are:

$$u_E = \frac{\delta E}{E} = \frac{1\text{ V}}{100\text{ V}} = 1\% \tag{6}$$

$$u_R = \frac{\delta R}{R} = \frac{0.1\ \Omega}{10\ \Omega} = 1\% \tag{7}$$

$$\Rightarrow \boxed{u_P = 2.24\%}$$

Now perform an uncertainty analysis using the second relation for power.

$$P = EI \tag{8}$$

The relative uncertainty in P is:

$$u_P = \left[u_{P,E}^2 + u_{P,I}^2\right]^{1/2}$$

where

$$u_{P,E} = \frac{1}{P}\frac{\partial P}{\partial E}\delta E = \frac{1}{EI}(I)\delta E = \frac{\delta E}{E} = u_E \tag{9}$$

$$u_{P,I} = \frac{1}{P}\frac{\partial P}{\partial R}\delta R = \frac{1}{EI}(E)\delta I = \frac{\delta I}{I} = u_I \tag{10}$$

Substitute into Eqn. (2).

$$u_P = \left[u_E^2 + u_I^2\right]^{1/2} \tag{11}$$

The relative uncertainties in the voltage and resistance are:

$$u_E = \frac{\delta E}{E} = \frac{1\text{ V}}{100\text{ V}} = 1\% \tag{12}$$

$$u_I = \frac{\delta I}{I} = \frac{0.1\text{ A}}{10\text{ A}} = 1\% \tag{13}$$

$$\Rightarrow \boxed{u_P = 1.41\%}$$

We observe that using the second relation ($P = EI$) gives a smaller uncertainty for the given values.

A certain obstruction-type flowmeter is used to measure the flow of air at low velocities. The relation describing the flow rate is:

$$\dot{m} = CA\left[\frac{2p_1}{RT_1}(p_1 - p_2)\right]^{1/2}$$

where C is an empirical discharge coefficient, A is the flow area, p_1 and p_2 are the upstream and downstream pressures, T_1 is the upstream temperature, and R is the gas constant for air.

Calculate the relative uncertainty in the mass flow rate for the following conditions:

C = 0.92±0.005 (from calibration data)
p_1 = 25±0.5 psia
T_1 = 530±2 °R
$\Delta p = p_1 - p_2$ = 1.4±0.005 psia
A = 1.0±0.001 in^2

What factors contribute the most to the uncertainty in the mass flow rate?

SOLUTION:

The relative uncertainty in the mass flow rate is given by:

$$u_{\dot{m}} = \left[u_{\dot{m},C}^2 + u_{\dot{m},A}^2 + u_{\dot{m},p_1}^2 + u_{\dot{m},T_1}^2 + u_{\dot{m},\Delta p}^2\right]^{1/2} \tag{1}$$

where

$$u_{\dot{m},C} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial C}\delta C = \frac{\delta C}{C} = u_C \tag{2}$$

$$u_{\dot{m},A} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial A}\delta A = \frac{\delta A}{A} = u_A \tag{3}$$

$$u_{\dot{m},p_1} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial p_1}\delta p_1 = \frac{1}{2}\frac{\delta p_1}{p_1} = \tfrac{1}{2}u_{p_1} \tag{4}$$

$$u_{\dot{m},T_1} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial T_1}\delta T_1 = -\frac{1}{2}\frac{\delta T_1}{T_1} = -\tfrac{1}{2}u_{T_1} \tag{5}$$

$$u_{\dot{m},\Delta p} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta p}\delta(\Delta p) = \frac{1}{2}\frac{\delta(\Delta p)}{\Delta p} = \tfrac{1}{2}u_{\Delta p} \tag{6}$$

Note that the there is negligible uncertainty in the gas constant R since it is presumed to be known to a high degree of accuracy.

Substitute into Eqn. (1).

$$u_{\dot{m}} = \left[u_C^2 + u_A^2 + \tfrac{1}{4}u_{p_1}^2 + \tfrac{1}{4}u_{T_1}^2 + \tfrac{1}{4}u_{\Delta p}^2\right]^{1/2} \tag{7}$$

where the relative uncertainties are:

$$u_C = \frac{\delta C}{C} = \frac{0.005}{0.92} = 0.54\% \tag{8}$$

$$u_A = \frac{\delta A}{A} = \frac{0.001 \text{ in}^2}{1.0 \text{ in}^2} = 0.10\% \tag{9}$$

$$u_{p_1} = \frac{\delta p_1}{p_1} = \frac{0.5 \text{ psia}}{25 \text{ psia}} = 2.0\% \tag{10}$$

$$u_{T_1} = \frac{\delta T_1}{T_1} = \frac{2\ ^\circ\text{R}}{530\ ^\circ\text{R}} = 0.38\% \tag{11}$$

$$u_{\Delta p} = \frac{\delta(\Delta p)}{\Delta p} = \frac{0.005 \text{ psia}}{1.4 \text{ psia}} = 0.36\% \tag{12}$$

$$\Rightarrow \boxed{u_{\dot{m}} = 1.2\%}$$

Examine the contributions of each term on the right hand side of Eqn. (7) to determine which uncertainty has the greatest influence on the uncertainty in $\dot{m}$.

$$u_C^2 = \left(5.4*10^{-3}\right)^2 = 2.9*10^{-5}$$

$$u_A^2 = \left(1.0*10^{-3}\right)^2 = 1.0*10^{-6}$$

$$\tfrac{1}{4}u_{p_1}^2 = \tfrac{1}{4}\left(2.0*10^{-2}\right)^2 = 1.0*10^{-4}$$

$$\tfrac{1}{4}u_{T_1}^2 = \tfrac{1}{4}\left(3.8*10^{-3}\right)^2 = 3.6*10^{-6}$$

$$\tfrac{1}{4}u_{\Delta p}^2 = \tfrac{1}{4}\left(3.6*10^{-3}\right)^2 = 3.2*10^{-6}$$

The uncertainty in the p_1 measurement contributes the most to the uncertainty in $\dot{m}$.

In pneumatic conveying, solid particles such as flour or coal are carried through a duct by a moving air stream. Solids density at any duct location can be measured by passing a laser beam of known intensity, I_0, through the duct and measuring the light intensity transmitted to the other side, I. A transmission factor is found using:

$$T = \frac{I}{I_0} = \exp(-KEW) \quad \text{where } 0 \le T \le 1$$

Here W is the width of the duct, K is the solids density, and E is a factor taken as 2.0±0.4 kg/m^2 for spheroidal particles. Determine how the relative uncertainty in K is related to the relative uncertainties of the other variables. If the transmission factor and duct width can be measured to within ±1%, can the solids density be measured to within 5%? 10%? Discuss your answer remembering that T varies from 0 to 1.

SOLUTION:

Solve for the solids density using the definition of the transmission factor.

$$T = \exp(-KEW)$$

$$K = \frac{-1}{EW}\ln T \tag{1}$$

The relative uncertainty in the solids density is given by:

$$u_K = \left[u_{K,E}^2 + u_{K,W}^2 + u_{K,T}^2\right]^{1/2} \tag{2}$$

where

$$u_{K,E} = \frac{1}{K}\frac{\partial K}{\partial E}\delta E = \left(\frac{-EW}{\ln T}\right)\left(\frac{\ln T}{E^2 W}\right)\delta E = -\frac{\delta E}{E} = -u_E \tag{3}$$

$$u_{K,W} = \frac{1}{K}\frac{\partial K}{\partial W}\delta W = \left(-\frac{EW}{\ln T}\right)\left(\frac{\ln T}{EW^2}\right)\delta W = -\frac{\delta W}{W} = -u_W \tag{4}$$

$$u_{K,T} = \frac{1}{K}\frac{\partial K}{\partial T}\delta T = \left(-\frac{EW}{\ln T}\right)\left(\frac{-1}{EWT}\right)\delta T = \frac{\delta T}{T\ln T} = \frac{u_T}{\ln T} \tag{5}$$

Substitute into Eqn. (2).

$$u_K = \left[u_E^2 + u_W^2 + \left(\frac{u_T}{\ln T}\right)^2\right]^{1/2} \tag{6}$$

where the relative uncertainties are:

$$u_E = \frac{\delta E}{E} = \frac{0.4}{2.0} = 20\% \tag{7}$$

$$u_W = 1\% \tag{8}$$

$$u_T = 1\% \tag{9}$$

Recall that $0 \le T \le 1$ so that:

$$T = 0: \quad \lim_{T\to 0}(\ln T) = -\infty \quad \Rightarrow \quad \lim_{T\to 0}\left(\frac{u_T}{\ln T}\right) = 0 \quad \Rightarrow \quad \lim_{T\to 0}(u_K) = 20\% \tag{10}$$

$$T = 1: \quad \ln T\big|_{T=1} = 0 \quad \Rightarrow \quad \lim_{T\to 1}\left(\frac{u_T}{\ln T}\right) = \infty \quad \Rightarrow \quad \lim_{T\to 1}(u_K) = \infty \tag{11}$$

$$\Rightarrow \quad 20\% \le u_K \le \infty \tag{12}$$

Hence, it is not possible to measure K to within either 5% or 10%. In fact, it is not possible to measure K to better than 20% relative uncertainty.

Two ME309 students wish to measure the height of the Mechanical Engineering building. The first student suggests dropping a ball bearing from the top of the building and measuring the time it takes for the ball to hit the ground using a digital stopwatch. (Air drag may be neglected. Legal Disclaimer: I do not recommend dropping anything off the building!) The second student recommends using a tape measure to measure a horizontal distance from the building, a protractor to measure the angle to the top of the building, and then using trigonometry to determine the height. The time for the ball to fall to the ground is measured at 2.2 s while the angle to the roofline measured from a distance of 20.0 m is 44.4 deg. The uncertainty in the ball-dropping method is ±0.2 sec and the uncertainty in the length and angle measurements, respectively, are ±0.5 m and ±1 deg.

a. What is the height of the ME building?
b. Which measurement method is most accurate?
c. Is there a particular angle for which the uncertainty in the angle method is minimized?

SOLUTION:

First consider the ball-dropping method. The distance the ball travels in time T is:

$$H = \tfrac{1}{2} g T^2 \Rightarrow H = 23.7 \text{ m} \tag{1}$$

Determine the relative uncertainty in H given a relative uncertainty in T. Note that the acceleration due to gravity, g, is an accurately known constant and thus the uncertainty in this quantity is considered negligible.

$$u_H = \sqrt{u_{H,T}^2} \tag{2}$$

where

$$u_{H,T} = \frac{1}{H}\frac{\partial H}{\partial T}\delta T = \frac{1}{\frac{1}{2} g T^2}(gT)\delta T = 2\frac{\delta T}{T} = 2u_T \tag{3}$$

Thus,

$$u_H = |2u_T| \tag{4}$$

For the given values of $\delta T = 0.2$ s and $T = 2.2$ s, $u_T = 0.091 \Rightarrow \underline{u_H = 0.182}$. Thus,

$\boxed{H = 23.7 \pm 4.3 \text{ m}}$ using the ball dropping method. (5)

Now consider the relative uncertainty using method 2 (angle method).

$$H = L\tan\theta \Rightarrow H = 19.6 \text{ m} \tag{6}$$

$$u_H = \sqrt{u_{H,\theta}^2 + u_{H,L}^2} \tag{7}$$

where

$$u_{H,\theta} = \frac{1}{H}\frac{\partial H}{\partial \theta}\delta\theta = \frac{1}{L\tan\theta}\left(L\sec^2\theta\right)\delta\theta = \frac{\theta}{\sin\theta\cos\theta}\frac{\delta\theta}{\theta} = \frac{\theta}{\sin\theta\cos\theta}u_\theta \tag{8}$$

$$u_{H,L} = \frac{1}{H}\frac{\partial H}{\partial L}\delta L = \frac{1}{L\tan\theta}(\tan\theta)\delta L = \frac{\delta L}{L} = u_L \tag{9}$$

Substituting,

$$u_H = \sqrt{\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 u_\theta^2 + u_L^2} \tag{10}$$

For the given values of $\delta\theta = 1$ deg (= 0.0175 rad), $\theta = 44.4$ deg, $\delta L = 0.5$ m, and $L = 20.0$ m, $u_\theta = 0.022$, $u_L = 0.025$, and $\underline{u_H = 0.043}$. Note that the angle θ should be evaluated in terms of radians, not degrees.
Thus,

$\boxed{H = 19.6 \pm 0.8 \text{ m}}$ using the angle method.

$\boxed{\text{The angle method is more accurate than the ball dropping method.}}$

To determine the angle that minimizes the height uncertainty measurement, minimize Eq. (10) with respect to θ,

$$\frac{\partial u_H}{\partial\theta} = 0 = \frac{\partial}{\partial\theta}\sqrt{\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 u_\theta^2 + u_L^2}. \tag{11}$$

For simplicity, take the derivative of u_H^2 instead of u_H. We'll get the same result, but the derivative will be easier to evaluate,

$$\frac{\partial (u_H)^2}{\partial\theta} = 0 = \frac{\partial}{\partial\theta}\left[\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 u_\theta^2 + u_L^2\right]. \tag{12}$$

Expand the relative uncertainty in θ, u_θ, since u_θ is a function of θ,

$$\frac{\partial}{\partial\theta}\left[\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2\left(\frac{\delta\theta}{\theta}\right)^2 + u_L^2\right] = 0, \tag{13}$$

$$\frac{\partial}{\partial\theta}\left[\left(\frac{1}{\sin\theta\cos\theta}\right)^2(\delta\theta)^2 + u_L^2\right] = 0, \tag{14}$$

$$(\delta\theta)^2\frac{\partial}{\partial\theta}\left[\left(\frac{1}{\sin\theta\cos\theta}\right)^2\right] = 0, \quad (\delta\theta \text{ is a constant and } u_L \text{ isn't a function of } \theta) \tag{15}$$

$$\frac{\partial}{\partial\theta}\left[\left(\frac{1}{\frac{1}{2}\sin(2\theta)}\right)^2\right] = 0, \quad \text{(using a trigonometric identity)} \tag{16}$$

$$\frac{\partial}{\partial\theta}\left[\sin^{-2}(2\theta)\right] = 0, \tag{17}$$

$$-2\sin^{-3}(2\theta)\cos(2\theta)\cdot 2 = 0, \quad \text{(using the chain rule)} \tag{18}$$

$$\frac{\cos(2\theta)}{[\sin(2\theta)]^3} = 0. \tag{19}$$

For the previous expression to hold true, $\theta = 45°$. $\boxed{\text{Thus, an angle of } \theta = 45° \text{ minimizes the uncertainty}}$.
The given value of $\theta = 44.4°$ is close to this optimal angle.

An engineer wishes to determine the efficiency with which paint is applied to a sample surface using a particular spray nozzle. The mass deposition efficiency, *MDE*, is defined as:

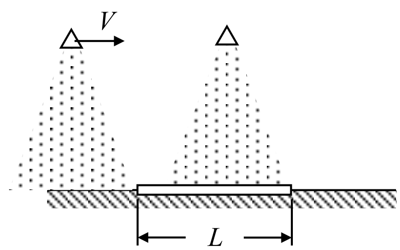
$$MDE \equiv \frac{m_f - m_i}{m_a}$$

where m_f is the mass of the surface after painting and drying, m_i is the initial mass of the surface (no paint applied), and m_a is the mass of paint that came out of the spray nozzle in a specified period of time. The mass of paint from the spray nozzle, m_a, may be calculated using:

$$m_a = \dot{m}Ts$$

where $\dot{m}$ is the mass flow rate through the nozzle, T is the duration that the spray is applied to the surface, and s is the percentage of (paint) solids present in the spray. The paint is applied by traversing the nozzle over the surface, with a traverse distance, L, at a constant speed, V, as shown in the figure below. Hence, the duration T may be found from:

$$T = \frac{L}{V}$$



Given the following uncertainties:

$\delta\dot{m}$ = ±0.025 kg/min
δs = ±2%
δL = ±1.5 mm
δV = ±0.5 mm/sec
δm_f = ±0.0001 kg
δm_i = ±0.0001 kg
δm_f = ±0.0001 kg

determine the mass deposition efficiencies, *MDE*s, with uncertainties for the following cases.

$\dot{m}$ [kg/min]	V [m/s]	L [m]	m_f [kg]	m_i [kg]	s [%]
0.92	0.127	0.304	0.0498	0.0341	51.2
1.79	0.254	0.306	0.0502	0.0339	51.0
1.66	0.254	0.302	0.0523	0.0368	50.9

SOLUTION:

First determine the relative uncertainty in the duration, T.

$$u_T = \frac{\delta T}{T} = \sqrt{u_{T,L}^2 + u_{T,V}^2} \tag{1}$$

where

$$u_{T,L} = \frac{1}{T}\frac{\partial T}{\partial L}\delta L = \left(\frac{V}{L}\right)\left(\frac{1}{V}\right)(\delta L) = \frac{\delta L}{L} = u_L \tag{2}$$

$$u_{T,V} = \frac{1}{T}\frac{\partial T}{\partial V}\delta V = \left(\frac{V}{L}\right)\left(-\frac{L}{V^2}\right)(\delta V) = -\frac{\delta V}{V} = -u_V \tag{3}$$

$$\therefore u_T = \sqrt{u_L^2 + u_V^2} \tag{4}$$

Now determine the relative uncertainty in the applied mass, m_a:

$$u_{m_a} = \frac{\delta m_a}{m_a} = \sqrt{u_{m_a,\dot{m}}^2 + u_{m_a,T}^2 + u_{m_a,s}^2} \tag{5}$$

where

$$u_{m_a,\dot{m}} = \frac{1}{m_a}\frac{\partial m_a}{\partial \dot{m}}\delta\dot{m} = \left(\frac{1}{\dot{m}Ts}\right)(Ts)(\delta\dot{m}) = \frac{\delta\dot{m}}{\dot{m}} = u_{\dot{m}} \tag{6}$$

$$u_{m_a,T} = \frac{1}{m_a}\frac{\partial m_a}{\partial T}\delta T = \left(\frac{1}{\dot{m}Ts}\right)(\dot{m}s)(\delta T) = \frac{\delta T}{T} = u_T \tag{7}$$

$$u_{m_a,s} = \frac{1}{m_a}\frac{\partial m_a}{\partial s}\delta s = \left(\frac{1}{\dot{m}Ts}\right)(\dot{m}T)(\delta s) = \frac{\delta s}{s} = u_s \tag{8}$$

$$\therefore u_{m_a} = \sqrt{u_{\dot{m}}^2 + u_T^2 + u_s^2} \tag{9}$$

Lastly, determine the relative uncertainty in the mass deposition efficiency, *MDE*:

$$u_{MDE} = \frac{\delta(MDE)}{MDE} = \sqrt{u_{MDE,m_f}^2 + u_{MDE,m_i}^2 + u_{MDE,m_a}^2} \tag{10}$$

where

$$u_{MDE,m_f} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_f}\delta m_f = \left(\frac{m_a}{m_f - m_i}\right)\left(\frac{1}{m_a}\right)(\delta m_f) = \frac{\delta m_f}{m_f - m_i} = \frac{\delta m_f / m_f}{1 - m_i / m_f} = \frac{u_{m_f}}{1 - m_i / m_f} \tag{11}$$

$$u_{MDE,m_i} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_i}\delta m_i = \left(\frac{m_a}{m_f - m_i}\right)\left(\frac{-1}{m_a}\right)(\delta m_i) = \frac{-\delta m_i}{m_f - m_i} = \frac{-\delta m_i / m_i}{m_f / m_i - 1} = \frac{u_{m_i}}{1 - m_f / m_i} \tag{12}$$

$$u_{MDE,m_a} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_a}\delta m_a = \left(\frac{m_a}{m_f - m_i}\right)\left[\frac{-(m_f - m_i)}{m_a^2}\right](\delta m_a) = \frac{-\delta m_a}{m_a} = -u_{m_a} \tag{13}$$

$$\therefore u_{MDE} = \sqrt{\frac{u_{m_f}^2}{\left(1 - m_i / m_f\right)^2} + \frac{u_{m_i}^2}{\left(1 - m_f / m_i\right)^2} + u_{m_a}^2} \tag{14}$$

Create a spreadsheet to perform the calculations.

mdot [kg/min]	u_{mdot}	V [m/s]	u_V	L [m]	u_L	m_f [kg]	u_{mf}	m_i [kg]	u_{mi}	s	u_s	T [s]	u_T	m_a [kg]	u_{ma}	MDE	u_{MDE}
0.92	2.7%	0.127	0.4%	0.304	0.5%	0.0498	0.2%	0.0341	0.3%	51.2%	3.9%	2.39	0.6%	0.0188	4.8%	83.5%	4.9%
1.79	1.4%	0.254	0.2%	0.306	0.5%	0.0502	0.2%	0.0339	0.3%	51.0%	3.9%	1.20	0.5%	0.0183	4.2%	88.9%	4.3%
1.66	1.5%	0.254	0.2%	0.302	0.5%	0.0523	0.2%	0.0368	0.3%	50.9%	3.9%	1.19	0.5%	0.0167	4.2%	92.6%	4.3%

Another approach to determining the uncertainties is to substitute the supporting formulas directly into the expression for the mass deposition efficiency.

$$MDE = \frac{m_f - m_i}{\dot{m}\left(L/V\right)s} \tag{15}$$

$$u_{MDE} = \frac{\delta(MDE)}{MDE} = \sqrt{u_{MDE,m_f}^2 + u_{MDE,m_i}^2 + u_{MDE,\dot{m}}^2 + u_{MDE,L}^2 + u_{MDE,V}^2 + u_{MDE,s}^2} \tag{16}$$

where

$$u_{MDE,m_f} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_f}\delta m_f = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left(\frac{1}{\dot{m}\left(L/V\right)s}\right)(\delta m_f) = \frac{\delta m_f}{m_f - m_i} = \frac{\delta m_f / m_f}{1 - m_i/m_f} = \frac{u_{m_f}}{1 - m_i/m_f} \tag{17}$$

$$u_{MDE,m_i} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_i}\delta m_i = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left(\frac{-1}{\dot{m}\left(L/V\right)s}\right)(\delta m_i) = \frac{-\delta m_i}{m_f - m_i} = \frac{-\delta m_i / m_i}{m_f/m_i - 1} = \frac{u_{m_i}}{1 - m_f/m_i} \tag{18}$$

$$u_{MDE,\dot{m}} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial \dot{m}}\delta \dot{m} = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left[\frac{-(m_f - m_i)}{\dot{m}^2\left(L/V\right)s}\right](\delta \dot{m}) = \frac{-\delta \dot{m}}{\dot{m}} = -u_{\dot{m}} \tag{19}$$

$$u_{MDE,L} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial L}\delta L = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left[\frac{-(m_f - m_i)}{\dot{m}\left(L^2/V\right)s}\right](\delta L) = \frac{-\delta L}{L} = -u_L$$

$$u_{MDE,V} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial V}\delta V = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left[\frac{(m_f - m_i)}{\dot{m}Ls}\right](\delta V) = \frac{\delta V}{V} = u_V$$

$$u_{MDE,s} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial s}\delta s = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left[\frac{-(m_f - m_i)}{\dot{m}\left(L/V\right)s^2}\right](\delta s) = \frac{-\delta s}{s} = -u_s$$

$$\therefore u_{MDE} = \sqrt{\frac{u_{m_f}^2}{\left(1 - m_i/m_f\right)^2} + \frac{u_{m_i}^2}{\left(1 - m_f/m_i\right)^2} + u_{\dot{m}}^2 + u_L^2 + u_V^2 + u_s^2} \tag{20}$$

The uncertainties calculated using Eqn. (20) are exactly the same as those found using Eqn. (14) (this can be proven by simply substituting in for the relative uncertainty expressions in Eqn. (14)).

Two colleagues are tasked with measuring the mass of five nearly identical pennies using a mass balance. One colleague recommends measuring the mass of each of the five pennies and obtain an average value from the five measurements. The other colleague recommends measuring the mass of the five pennies simultaneously then dividing by five. Which measurement will have the least uncertainty? Support your answer.

SOLUTION:
First consider the case where each penny mass is measured separately. Each of these measurements will have the same uncertainty, δm, since the same mass balance is used. Thus, we will have the following five measurements:

$m_1 \pm \delta m$, $m_2 \pm \delta m$, $m_3 \pm \delta m$, $m_4 \pm \delta m$, $m_5 \pm \delta m$

The average penny mass is,

$$\bar{m} = \frac{1}{5}(m_1 + m_2 + m_3 + m_4 + m_5), \tag{1}$$

and the uncertainty is,

$$\delta\bar{m} = \sqrt{\left(\delta\bar{m}_{m_1}\right)^2 + \left(\delta\bar{m}_{m_2}\right)^2 + \left(\delta\bar{m}_{m_3}\right)^2 + \left(\delta\bar{m}_{m_4}\right)^2 + \left(\delta\bar{m}_{m_5}\right)^2}, \tag{2}$$

where,

$$\delta\bar{m}_{m_i} = \frac{\partial\bar{m}}{\partial m_i}\delta m_i = \frac{1}{5}\delta m \tag{3}$$

where δm is the uncertainty in an individual penny mass measurement. Thus, Eq. (2) becomes,

$$\delta\bar{m} = \sqrt{5\left(\frac{1}{5}\delta m\right)^2} = \frac{1}{\sqrt{5}}\delta m. \tag{4}$$

Now consider the case where all five pennies are measured simultaneously. For this case we have a single measurement,

$$\bar{m} = \frac{1}{5}M, \tag{6}$$

where,

$$M = m_1 + m_2 + m_3 + m_4 + m_5. \tag{7}$$

The uncertainty for this case is,

$$\delta\bar{m} = \sqrt{(\delta\bar{m}_M)^2} = \delta\bar{m}_M = \frac{\partial\bar{m}}{\partial M}\delta M = \frac{1}{5}\delta M. \tag{8}$$

The uncertainty in this single measurement is δm, i.e., $\delta M = \delta m$, since the same mass balance is used. Thus,

$$\delta\bar{m} = \frac{1}{5}\delta m. \tag{9}$$

Thus, we observe that the uncertainty is smaller using the latter method (measuring the mass of the five pennies simultaneously). This technique is known as "stacking" and can be used to reduce measurement uncertainty.

1.6. Statistical vs. Continuum Approach

Most substances consist of a collection of molecules or atoms. In studying how substances behave, we can either explicitly account for the molecular behavior, referred to as the statistical approach, or instead treat the substance as being continuous, referred to as the continuum approach.

1.6.1. Statistical Approach

In the statistical approach we treat the substance of interest as a collection of individual objects (e.g., molecules). The interactions between molecules are modeled (e.g., using Newton's laws) and the macroscopic behavior of the substance is determined by utilizing probability and statistics. This approach is useful in understanding the behavior of material properties (e.g., fluid viscosity), but it is not very practical for modeling typical engineering applications. Statistical Mechanics and Kinetic Theory utilize the statistical approach.

1.6.2. Continuum Approach

The continuum approach ignores individual molecules and instead treats the substance of interest as being continuously distributed in space. The macroscopic behavior of the substance is modeled using basic conservation laws (e.g., mass, momentum, energy). The continuum method requires that the smallest length scale of interest, referred to as the macroscopic length scale (e.g., the length scale over which significant changes in properties occur), be much larger than the microscopic length scale, typically the mean free path of a molecule (for gases). The mean free path, λ, of a molecule is the average distance a molecule travels before colliding with another molecule. By requiring that the (macroscopic length scale) $\gg$ (microscopic length scale), there will be enough molecules at a "point" (the smallest macroscopic length scale of interest) so that meaningful averages of the molecular behavior can be made at that point.

Consider the following experiment. Let's measure the density, ρ, of a fluid in a small cube of length, L (Figure 1.9). The local density of the fluid is defined as the total mass of molecules in the cube, $\sum m_i$, divided by the cube's volume, L^3, as the volume approaches the smallest macroscopic length scale of interest, ϵ,

$$\rho := \lim_{L\to\epsilon} \frac{\sum m_i}{L^3}. \tag{1.59}$$

At large length scales, L, the density will not have very good spatial resolution and, hence, may not be a very

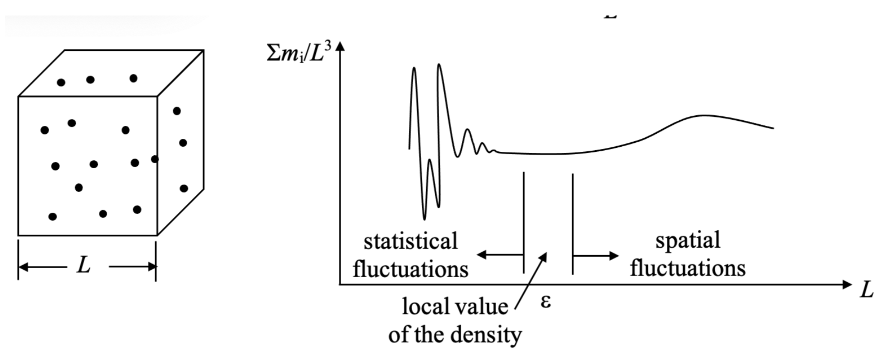


Figure 1.9. An illustration showing how the density varies with the size of the averaging cube.

useful quantity. At very small length scales the number of molecules within the box will vary significantly with time since molecules continuously enter and exit the box. Since the small box can contain only a few molecules to begin with, the density fluctuations will be very large. There is an intermediate region between the extremes that will have few fluctuations but good spatial resolution.

Notes:

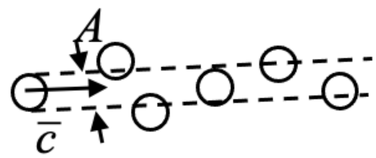


Figure 1.10. An illustration used in deriving the mean free path.

(1) In most engineering applications the length scale condition for a substance to be considered a continuum is easily satisfied. The mean free path of a gas can be estimated using the following simple analysis. Consider gas molecules with an effective cross-sectional area of A and a mean (molecular) speed of $\bar{c}$. The volume swept out by a molecule per unit time is (Figure 1.10),

$$\dot{V} = \bar{c}A. \tag{1.60}$$

The expected number of collisions with other molecules per unit time is,

$$\dot{n} = \nu\dot{V}, \tag{1.61}$$

where ν is the number of molecules per unit volume. The average time between collisions, t', is the inverse of the collision rate,

$$t' = \frac{1}{\dot{n}} = \frac{1}{\nu\dot{V}} = \frac{1}{\nu\bar{c}A}. \tag{1.62}$$

Thus, the typical distance between collisions, aka the mean free path λ, is,

$$\lambda = \bar{c}t' = \frac{1}{\nu A}. \tag{1.63}$$

For air at standard conditions,

$$\nu \approx 2.7 \times 10^{19}\,\text{cm}^{-3} \quad \text{and} \quad A \approx 1 \times 10^{-15}\,\text{cm}^2 \implies \lambda \approx 3.7 \times 10^{-5}\,\text{cm}. \tag{1.64}$$

This mean free path is certainly much smaller than any length scale in which we are typically interested for most engineering applications. Examples of where the continuum assumption may not be valid include,

- flow in the upper atmosphere where the mean free path is large (e.g., at an altitude of 100 miles, the mean free path is approximately 80 m!),
- flow within a shock wave where the macroscopic length scale is very small (the width of a shock wave is on the order of 1 µm, and
- granular flows (e.g., flowing sand) where the microscopic length scale ≈ the macroscopic length scale.

(2) The Knudsen number, Kn, is a dimensionless parameter that indicates when the continuum assumption is valid,

$$\text{Kn} := \frac{\text{microscopic length scale}}{\text{macroscopic length scale}} \ll 1 \implies \text{the continuum assumption is valid.} \tag{1.65}$$

Four flow regimes are typically defined based on the Knudsen number. These are (according to Zucrow, M.J. and Hoffman, J.D., *Gas Dynamics Vol. 1*, Wiley):

- $\text{Kn} < 0.01$. Continuum flow. This is the flow regime that occurs in most engineering applications.
- $0.01 < \text{Kn} < 0.1$. Slip flow. In this regime, the fluid may still be treated as a continuous substance, but the no-slip condition at boundaries does not hold. Instead, fluid may slip at boundaries.
- $0.1 < \text{Kn} < 3.0$. Transitional flow. The flow in this regime is very difficult to analyze since the fluid cannot be considered a continuum and molecules still interact to a considerable degree.
- $3.0 < \text{Kn}$. Free molecular flow. In this regime molecules are spaced so far apart that they rarely interact. The flow may be modeled as a collection of non-interacting molecular impacts.

(3) The continuum assumption is valid in the vast majority of engineering applications. For example, the Knudsen number for a typical engineering flow in which the mean free path is $\sim 1 \times 10^{-5}$ cm (refer to the calculation in an earlier note) and the macroscopic length of interest is 1 mm, is $\mathrm{Kn} \sim 0.0001$.

(4) The Knudsen number can be related to the Reynolds and Mach numbers using some results from kinetic theory. From kinetic theory, the kinematic viscosity of a gas, ν, is related to the mean free path, λ, and mean molecular speed, $\bar{c}$, by,

$$\nu = \frac{1}{2}\lambda\bar{c}. \tag{1.66}$$

The mean molecular speed is related (via kinetic theory) to the speed of sound, c, by,

$$c = \bar{c}\sqrt{\frac{\pi\gamma}{8}}, \tag{1.67}$$

where γ is the specific heat ratio ($= c_p/c_v$). Substituting Eq. (1.66) into Eq. (1.67) and simplifying gives,

$$c = \frac{2\nu}{\lambda}\sqrt{\frac{\pi\gamma}{8}}, \tag{1.68}$$

$$\lambda = \sqrt{\frac{\pi\gamma}{2}}\frac{\nu}{c}. \tag{1.69}$$

From the definitions of the Knudsen, Reynolds, and Mach numbers,

$$\mathrm{Kn}_L = \frac{\lambda}{L} = \sqrt{\frac{\pi\gamma}{2}}\frac{\nu}{cL} = \sqrt{\frac{\pi\gamma}{2}}\underbrace{\frac{\nu}{VL}}_{=\frac{1}{\mathrm{Re}_L}}\underbrace{\frac{V}{c}}_{=\mathrm{Ma}}, \tag{1.70}$$

where L is the macroscopic length scale of interest. Thus,

$$\boxed{\mathrm{Kn}_L = \sqrt{\frac{\pi\gamma}{2}}\frac{\mathrm{Ma}}{\mathrm{Re}_L}}. \tag{1.71}$$

Hence, flows occurring at large Mach numbers and small Reynolds numbers tend toward non-continuum flows.

Note that for large Reynolds numbers over an object, the significant length scale that is often used in the defining the Reynolds number is the boundary layer thickness, δ. Chapter 9 shows that for laminar boundary layers, the boundary layer thickness is related to the macroscopic length scale, L, by,

$$\frac{\delta}{L} \propto \frac{1}{\sqrt{\mathrm{Re}_L}} \implies \mathrm{Re}_\delta \propto \sqrt{\mathrm{Re}_L} \quad (\text{where } \mathrm{Re}_L = VL/\nu \text{ and } \mathrm{Re}_\delta = V\delta/\nu). \tag{1.72}$$

Hence, Eq. (1.71) for boundary flows (occurring at large Reynolds numbers) is,

$$\mathrm{Kn}_\delta = \sqrt{\frac{\pi\gamma}{2}}\frac{\mathrm{Ma}}{\mathrm{Re}_\delta} \propto \frac{\mathrm{Ma}}{\sqrt{\mathrm{Re}_L}}. \tag{1.73}$$

(5) To learn more about non-continuum flows, refer to the following texts:
- Schaaf, S.A. and Chambré, P.L., *Flow of Rarefied Gases*, Princeton University Press.
- Nguyen, N-T. and Wereley, S.T., *Fundamentals and Applications of Microfluidics*, Artech House.

Below what characteristic time interval would you expect the continuum assumption to be invalid?

SOLUTION:

In order to treat a substance as a continuum, a perturbation applied to a few molecules must be "smeared" out to the neighboring molecules. This will occur after a sufficient number of collisions between molecules. The typical collision time between molecules (in a gas) is the mean free path divided by the average molecular speed:

$$t_{\substack{\text{between}\\ \text{collisions}}} = \frac{\lambda}{\bar{c}} \tag{1}$$

where λ is the mean free path and $\bar{c}$ is the mean velocity of a molecule. Hence, in order to be treated as a continuum, one should have:

$$\boxed{t_{\text{macro}} \gg t_{\substack{\text{between}\\ \text{collisions}}} = \frac{\lambda}{\bar{c}}} \tag{2}$$

where t_{macro} is the macroscopic time scale of interest.

For air at standard conditions:

$\lambda \approx 4*10^{-7}$ m

$\bar{c} \approx 400$ m/s

$\Rightarrow \boxed{t_{\text{between collisions}} \approx 1*10^{-9} \text{ sec}}$

1.7. Fluid Properties

Some good fluid property references include:

- Avallone, E.A. and Baumeister III, T., *Marks' Standard Handbook for Mechanical Engineers*, McGraw-Hill.
- Kestin, J., and Wakeham, W.A., *Transport Properties of Fluids*, CINDAS Data Series on Material Properties, C.Y. Ho, ed., Hemisphere Publishing.

1.7.1. Density, ρ

- The density of a substance is a measure of how much mass there is of the substance per unit volume.
- The dimensions of density are M/L^3. Typical units are $\mathrm{kg/m^3}$, $\mathrm{slug/ft^3}$, and $\mathrm{lb_m/ft^3}$.
- The density of water at 4 °C is $\rho_{\mathrm{H2O,4\,^\circ C}} = 1000\,\mathrm{kg/m^3} = 1.94\,\mathrm{slug/ft^3} = 62.4\,\mathrm{lb_m/ft^3}$.
- The density of air at Standard Temperature (= 15 °C) and Pressure (= 101.3 kPa (abs)) (STP) is $\rho_{\mathrm{air,\ STP}} = 1.23\,\mathrm{kg/m^3} = 2.38 \times 10^{-3}\,\mathrm{slug/ft^3} = 7.68 \times 10^{-2}\,\mathrm{lb_m/ft^3}$.
- Density does not vary greatly with temperature for liquids, in general.
- Density does change considerably with temperature and pressure for gases.
- A substance where the density remains constant for all conditions is considered incompressible.
  - A good engineering rule of thumb is that if there are no significant temperature changes and for fluid velocities less than approximately 1/3 the speed of sound in the fluid, the fluid can be approximated as incompressible (the proof of this is examined in Chapter 13).
  - In air, the speed of sound at standard conditions ($p = 1\,\mathrm{atm\,(abs)}, T = 59\,^\circ\mathrm{F} = 15\,^\circ\mathrm{C}$), is approximately $1100\,\mathrm{ft\,s^{-1}}$ ($340\,\mathrm{m\,s^{-1}}$).
  - The speed of sound in water is approximately $4800\,\mathrm{ft\,s^{-1}}$ ($1500\,\mathrm{m\,s^{-1}}$).
  - The speed of sound in steel is approximately $16\,400\,\mathrm{ft\,s^{-1}}$ ($5000\,\mathrm{m\,s^{-1}}$).
  - In most instances, liquids and gases flowing at low speeds can be approximated as incompressible.
- specific gravity, SG
  - Specific gravity is dimensionless. It has no units.
  - The specific gravity of a liquid is the ratio of the liquid's density to the density of water at some specified condition, typically at 4 °C,

$$SG_{\mathrm{liquid}} \coloneqq \frac{\rho_{\mathrm{liquid}}}{\rho_{\mathrm{H2O,4\,^\circ C}}}. \tag{1.74}$$

    For example, the density of mercury (Hg) at 20 °C is $13.6 \times 10^3\,\mathrm{kg/m^3}$. Hence, $SG_{\mathrm{Hg}} = 13.6$.
- Specific weight, γ
  - The specific weight of a substance is the weight of the substance per unit volume,
  - The dimensions of specific weight are F/L^3 and common units are $\mathrm{N/m^3}$ and $\mathrm{lb_f/ft^3}$.

$$\gamma \coloneqq \frac{W}{V} = \frac{mg}{V} = \frac{\rho V g}{V} = \rho g. \tag{1.75}$$

    For example, the specific weight of water at 4 °C is $9.81 \times 10^3\,\mathrm{N/m^3} = 62.4\,\mathrm{lb_f/ft^3}$.
- Specific volume, v
  - The specific volume of a substance is how much volume the substance occupies per unit mass.
  - The dimensions of specific volume are L^3/M. Common units are $\mathrm{m^3/kg}$, $\mathrm{ft^3/slug}$, and $\mathrm{ft^3/lb_m}$.
  - The specific volume is simply the inverse of the density,

$$v = \frac{1}{\rho}. \tag{1.76}$$

  - In thermodynamics, the specific volume is commonly used in place of density.

Example:
What is the specific weight of water if its density is $62.4\,\mathrm{lb_m/ft^3}$?

Solution:

$$\gamma = \rho g = \left(62.4\,\mathrm{lb_m/ft^3}\right)\left(32.2\,\mathrm{ft/s^2}\right)\left(\frac{1\,\mathrm{lb_f}}{32.2\,\mathrm{lb_m ft/s^2}}\right) = 62.4\,\mathrm{lb_f/ft^2} \tag{1.77}$$

Be sure to:

(1) Be careful when using the incompressible flow assumption. Make sure that the assumption is reasonable for your flow situation. For liquids, the assumption is usually reasonable. For gases, you need to check the flow velocities and temperatures.

1.7.2. Pressure, p

- The dimensions of pressure are F/L^2. Common units are Pa = N/m^2, psf = lb$_f$/ft^2, psi = lb$_f$/in^2, torr, atm, bar, and mmHg and inHg.
- Pressure is a scalar quantity. It has a magnitude, but no direction.
- The differentially small pressure force, $d\boldsymbol{F}_p$, on a differentially small area $d\boldsymbol{A}$ is,

$$d\boldsymbol{F}_p = p(-d\boldsymbol{A}), \tag{1.78}$$

  where p is the pressure acting on the surface. Notice that the area is a vector. Areas have both a magnitude and a direction. The area's direction is specified by the unit normal vector $\hat{\boldsymbol{n}}$, as shown in Figure 1.11. The pressure force acts inward on the area because the pressure force is the result of the momentum impulse of molecular impacts with the surface.

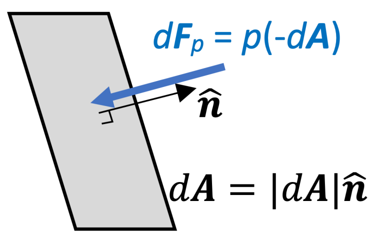


FIGURE 1.11. Illustration of the differentially small pressure force $d\boldsymbol{F}_p$ acting inward on the differentially small area $d\boldsymbol{A}$.

- Absolute pressure is referenced to zero pressure. For example, a perfect vacuum has $p_{\text{vacuum}} = 0\,(\text{abs})$. To indicate an absolute pressure, one should use either the text "(abs)" after the reported pressure or, if using EE or BG units, specify the absolute pressure with the units "psia" or "psfa".
  - There are no molecules in a perfect vacuum and, thus, there is no pressure.
  - Atmospheric pressure at standard conditions is $p_{\text{atm}} = 101.33\,\text{kPa}\,(\text{abs}) = 1\,\text{atm}\,(\text{abs}) = 1.0133\,\text{bar}\,(\text{abs}) = 14.696\,\text{psia} = 760\,\text{torr}\,(\text{abs})$.
- Gage pressure is referenced to atmospheric pressure,

$$p_{\text{gage}} = p_{\text{abs}} - p_{\text{atm,abs}}. \tag{1.79}$$

  - To indicate a gage pressure, one should use either the text "(gage)" after the reported pressure or, if using EE or BG units, specify the absolute pressure with the units "psig" or "psfg".
  - Atmospheric pressure at standard conditions is $p_{\text{atm}} = 0\,(\text{gage})$.
  - A perfect vacuum has $p_{\text{vacuum}} = -101.33\,\text{kPa}\,(\text{gage}) = -14.7\,\text{psig}$ (referencing to standard atmospheric conditions).
- An important equation of state for an ideal gas is the Ideal Gas Law,

$$p = \rho R T, \tag{1.80}$$

  where p and T are absolute quantities, e.g., $[p]$ = psia or Pa (abs) and $[T]$ = °R or K.
- The vapor pressure, p_v, of a liquid is the pressure at which the liquid is in equilibrium with its own vapor (Figure 1.12).

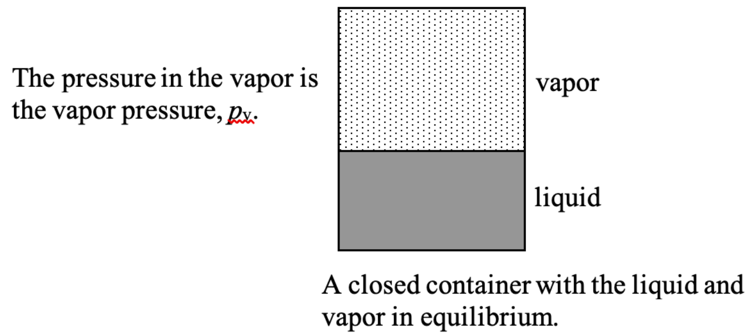


Figure 1.12. Illustration of liquid and vapor in equilibrium in a closed container. The pressure in the vapor is p_v.

- The vapor pressure for a liquid increases with increasing temperature.
- If the pressure in the liquid falls below the vapor pressure, then the liquid will turn to vapor. This change can occur via boiling or cavitation.
  - Boiling occurs when the temperature of the liquid increases so that the vapor pressure equals the surrounding atmospheric pressure. For example, the vapor pressure of water at 20 °C is 0.023 atm (abs) while the vapor pressure at 100 °C is 1 atm (abs). Hence, one method of turning liquid water to vapor is to bring the water temperature to 100 °C while holding the surrounding pressure at 1 atm (abs). This path is known as boiling and is shown schematically in the phase plot shown on the left-hand side of Figure 1.13.
  - Cavitation occurs when the surrounding pressure drops below the vapor pressure. Using the previous example, we can also turn liquid water to water vapor by dropping the surrounding pressure to 0.023 atm (abs) at 20 °C. This path is shown schematically in the phase plot shown on the right-hand side of Figure 1.13.

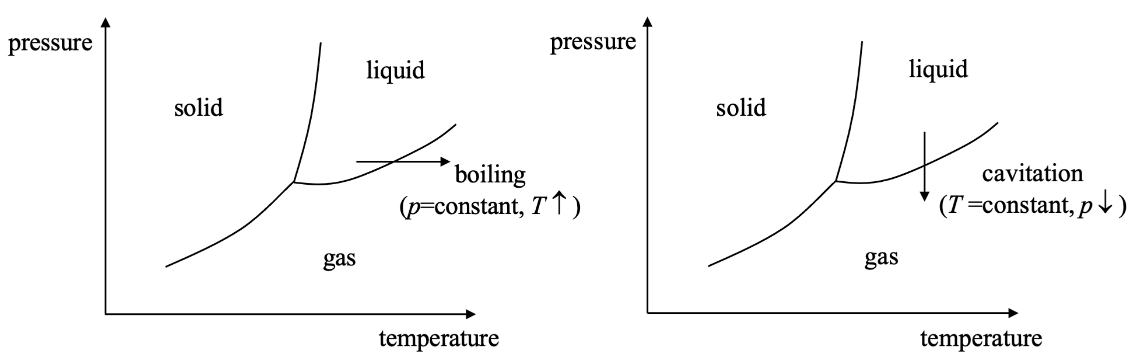


Figure 1.13. Phase diagrams (pressure vs. temperature) showing the processes of boiling (left) and cavitation (right).

  - Cavitation can cause considerable damage to surfaces. When cavitation occurs in a liquid, pockets of vapor form (either as bubbles or large "voids"). When the vapor pockets travel into a region where the surrounding pressure is greater than the vapor pressure, the vapor region rapidly collapses. This collapse can be so rapid that shock waves and high speed water jets propagate from the collapsing region and impact on nearby surfaces causing small bits of the surface to erode away. Hence, cavitation is typically avoided when designing pumps, pipe bends, and underwater propellers.

Example:
A tire pressure gage measures the pressure inside a tire as 40 psig for a car at Standard Temperature and Pressure. What is the corresponding absolute pressure in the tire?
Solution:
The corresponding absolute pressure is,

$$p_{\text{tire,abs}} = p_{\text{tire,gage}} + p_{\text{atm,abs}} = 40\,\text{psi} + 14.7\,\text{psi} = 54.7\,\text{psia}. \tag{1.81}$$

Be sure to:

- Always indicate whether a pressure is an absolute or gage pressure in order to avoid ambiguity.
- Use an absolute pressure, and not a gage pressure, in the Ideal Gas Law.
- Be careful not to mix gage and absolute pressures when evaluating pressure forces.
- Use the correct area when calculating a pressure force.
- Integrate to find a pressure force when the pressure is not uniform over the area over which the pressure acts. This topic is covered in detail in Chapter 2.
- Check for cavitation in low pressure flows of liquids.

1.7.3. Temperature

- The dimensions of temperature are simply temperature θ. Typical units are K, °R, °C, and °F.
- An object's temperature is a quantitative way of describing how "hot" the object is (temperature is a measure of the agitation or random kinetic energy of the molecules). We typically measure an object's temperature using a device called a "thermometer."
- Experience tells us that when two objects are placed in contact with each other and have different temperatures, the hotter object (i.e., the one with a larger temperature) will become cooler while the cooler object becomes hotter. When the two objects have the same temperature, they no longer change temperature. The objects are then said to be in thermal equilibrium.
- A simple but fundamental concept concerning thermal equilibrium is the Zeroth Law of Thermodynamics: If two bodies are in thermal equilibrium with a third body, then the two bodies will also be in thermal equilibrium. This concept is key when comparing the temperatures of two objects not in contact using a thermometer since the thermometer acts as the third body.
- To use the concept of temperature, we must first define some scale on which we'll measure temperatures. Perhaps the easiest scale to define is one where we reference all temperatures to some reproducible and unique physical phenomena such as the freezing or boiling points of water. Two point scales use two phenomena to define the temperature scale. For example, let's define the ice point of water (when ice and water are in equilibrium at a pressure of 1 atm (abs)) as our 0° temperature and the steam point of water (when water and water vapor are in equilibrium at a pressure of 1 atm (abs)) as our 100° temperature. We can now measure all temperatures relative to this scale. Examples of two-point scales include the Celsius and Fahrenheit scales:

	ice point of H_2O	steam point of H_2O
Celsius scale	0 °C	100 °C
Fahrenheit scale	32 °F	212 °F

  Long ago, researchers noticed something curious when measuring the temperature of various gases at different pressures (and constant volume). They found that the temperature of a gas at a constant volume and at low pressures is proportional to its pressure, (Figure 1.14),

$$T = a + bp, \tag{1.82}$$

  where a and b are constants. By extrapolating the temperature of the gases at zero pressure, we find that the lowest possible temperature, or the absolute zero temperature, is $-273.15\,°\text{C}$ on the Celsius scale and $-459.67\,°\text{F}$ on the Fahrenheit scale (i.e., $a = -273.15\,°\text{C} = -459.67\,°\text{F}$). This temperature scale, based on the behavior of ideal gases, is referred to as the ideal gas temperature scale.

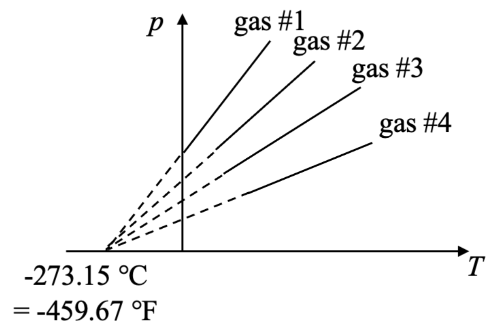


Figure 1.14. Gas pressure plotted as a function of temperature for different gases using a two-point temperature scale.

To make things a bit easier, let's redefine our scale so that absolute zero is the zero point of our temperature scale (Figure 1.15), i.e.,

$$T = bp \quad (a = 0). \tag{1.83}$$

This new scale is called an absolute temperature scale since the lowest temperature is zero.

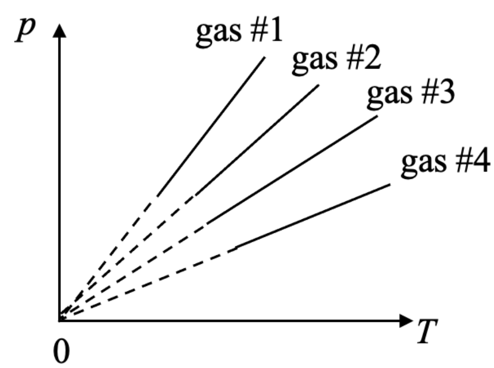


Figure 1.15. Gas pressure plotted as a function of temperature for different gases using an absolute temperature scale.

Note that so far our temperature scales have been based on the behavior of a particular substance (e.g., water) or a particular class of substances (e.g., gases). A better way to define a temperature scale is to make the scale independent of substances. Such a scale is called a thermodynamic temperature scale. In order to define such a temperature scale, we would need to first learn about the Second Law of Thermodynamics, a topic covered in Chapter 3. Suffice it to say here that the scale using this method gives the same result as the one using the ideal gas temperature scale.
To summarize, the lowest possible temperature is $0\,\text{K} = 0\,^\circ\text{R}$ and $\Delta(1\,\text{K}) = \Delta(1\,^\circ\text{C})$ and $\Delta(1\,^\circ\text{R}) = \Delta(1\,^\circ\text{F})$. Some additional helpful conversions are (in the equations below, θ refers to temperature):

$$\theta(\text{K}) = 1.8\theta(^\circ\text{R}) \quad (1.8 = 9/5), \tag{1.84}$$

$$\theta(^\circ\text{C}) = [\theta(^\circ\text{F}) - 32]/1.8, \tag{1.85}$$

$$\theta(^\circ\text{C}) = \theta(\text{K}) - 273.15, \tag{1.86}$$

$$\theta(^\circ\text{F}) = \theta(^\circ\text{R}) - 459.67. \tag{1.87}$$

Another convenient conversion formula is,

$$10\,^\circ\text{C} = 50\,^\circ\text{F} \quad \text{and for every } 5\,^\circ\text{C increase, add } 9\,^\circ\text{F}. \tag{1.88}$$

Another very approximate conversion, and one that should only be used for everyday convenience and not in engineering calculations is,

$$\theta(^\circ\mathrm{F}) \approx 2\theta(^\circ\mathrm{C}) + 30 \quad \text{(a few degrees error over the range of typical weather temps)}, \tag{1.89}$$

$$\theta(^\circ\mathrm{C}) \approx [\theta(^\circ\mathrm{F}) - 30]/2. \tag{1.90}$$

Example:
A pressure gage measures the pressure inside a car tire as 240 kPa (gage) at the National Institute of Standards and Technology (NIST) Standard Temperature and Pressure. To get the tire to a volume of 10 L, what mass of air needs to be added to the tire?

Solution:
The corresponding absolute pressure inside the tire is, $p_{\text{tire,abs}} = 240\,\text{kPa} + 101.325\,\text{kPa} = 341.325\,\text{kPa (abs)}$. The absolute temperature inside the tire is $T_{\text{tire,abs}} = 293.15\,\text{K}$. From the Ideal Gas Law,

$$M = \frac{p_{\text{abs}}V}{R_{\text{air}}T_{\text{abs}}} = \frac{(341.325 \times 10^3\,\text{Pa})(10\,\text{L})\left(\frac{1\,\text{m}^3}{1000\,\text{L}}\right)}{(287.058\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1})(293.15\,\text{K})} = 4.06 \times 10^{-2}\,\text{kg}, \tag{1.91}$$

where R_{air} is the gas constant for air.

Be sure to:

(1) Use an absolute temperature when using the Ideal Gas Law.

1.7.4. (Dynamic) Viscosity, μ

- Viscosity is the "internal friction" within a fluid. It's a measure of how easily a fluid flows.
- The dimensions of dynamic viscosity are FT/L^2. Typical units are Pa s and $\text{lb}_\text{f}\text{s}/\text{ft}^2$. Another common unit is the Poise (P) (pronounced "'pwäz"): $10\,\text{P} = 1\,\text{Pa s}$ and $1\,\text{cP} = 0.01\,\text{P} = 1 \times 10^{-3}\,\text{Pa s}$.
- The viscous stresses in a fluid will be related to the deformation rate of a small element of fluid. Recall that for solids, the force is related to the deformation, e.g., Hooke's Law for springs. For fluids, however, the forces are related to deformation rates.

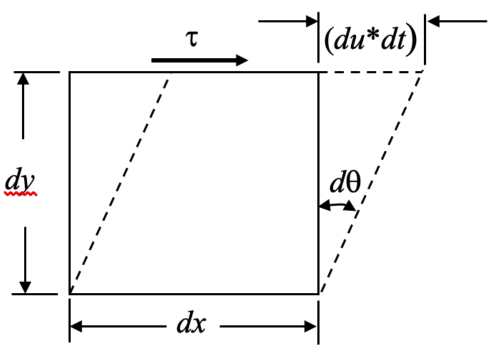


Figure 1.16. A small element of fluid subject to a small degree of shear deformation.

Consider the deformation of a small piece of fluid with area $(dxdy)$ as shown in Figure 1.16. The top of the fluid element is subject to a shear stress, τ, over a short time, dt. During this time, the top of the fluid element moves with a small velocity, du, with respect to the bottom of the element. The total distance the top moves relative to the bottom will be $dudt$. The angular deformation of the fluid element can be measured by the angle the vertical sides of the element have deformed. The small angle $d\theta$ is found from simple trigonometry,

$$\tan(d\theta) = \frac{dudt}{dy}. \tag{1.92}$$

Since the angle is very small, $\tan(d\theta) \approx d\theta$. The *rate* at which the element deforms is found by dividing both sides by dt,

$$\boxed{\frac{d\theta}{dt} = \frac{du}{dy}}. \tag{1.93}$$

This quantity is known as the shear or angular deformation rate. The rate of angular deformation on the element, $d\theta/dt$ is equal to the velocity gradient, du/dy, in the fluid. Furthermore, since the shear stress acting on the fluid element, τ, is a function of the deformation rate, we have,

$$\boxed{\tau = f\left(\frac{d\theta}{dt}\right) = f\left(\frac{du}{dy}\right)}, \tag{1.94}$$

where the function is unknown at this point.

- A Newtonian fluid is one in which the shear stress varies proportionally with the deformation rate. The constant of proportionality is call the dynamic viscosity, μ,

$$\boxed{\tau = \mu\frac{du}{dy}}. \tag{1.95}$$

- Air and water are two examples of Newtonian fluids.
- A slightly more precise definition of the shear stress is,

$$\tau_{yx} = \mu\frac{du_x}{dy}. \tag{1.96}$$

  where the subscript on the stress indicates that the shear stress acts on a y-surface in the x-direction. The x-subscript on the velocity indicates that it is the x-component. We will review this sign convention in greater detail in Chapter 5.

• In a non-Newtonian fluid the shear stress is not proportional to the deformation rate, but instead varies in some other way.
  - Non-Newtonian fluids are further classified by how the shear stress varies with deformation rate. The apparent viscosity, μ_{app}, is the viscosity at the local conditions as shown in Figure 1.17. For a Newtonian fluid the apparent viscosity remains constant.

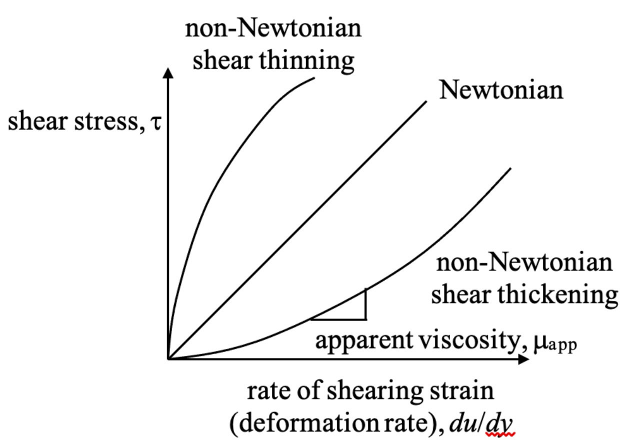


Figure 1.17. A plot of shear stress as a function of the shear strain rate for different types of fluids. The apparent viscosity is the local slope of the shear stress-deformation rate curve.

  - In shear thinning (aka psuedoplastic) fluids, the apparent viscosity decreases as the shear stress increases. Examples of shear thinning fluids include blood, latex paint, and cookie dough.

  - In shear thickening (aka dilatant) fluids, the apparent viscosity increases as the shear stress increases. An example of a shear thickening fluid is quicksand or a thick cornstarch-water mixture (aka oobleck).
- Viscosity is weakly dependent on pressure, but is sensitive to temperature. For liquids, the viscosity generally decreases as the temperature increases and increases as pressure increases. Changes in temperature and pressure can be very significant in lubrication problems. For gases, the viscosity increases as the temperature increases (in fact, from kinetic theory one can show that $\mu \propto \sqrt{T}$). Figure 1.18 shows the variation in dynamic viscosity with temperature for several fluids. Two commonly-used dynamic viscosities are,
$$\mu_{\text{H2O},20\,^\circ\text{C}} = 1.00 \times 10^{-3}\,\text{Pa}\,\text{s} = 1.00\,\text{cP},$$
$$\mu_{\text{air},20\,^\circ\text{C}} = 1.81 \times 10^{-5}\,\text{Pa}\,\text{s} = 1.8 \times 10^{-2}\,\text{cP}.$$
- The kinematic viscosity, ν, is a quantity that often appears in fluid mechanics. It is defined as,
$$\boxed{\nu := \frac{\mu}{\rho}}. \tag{1.97}$$
The dimensions of kinematic viscosity are L^2/T and common units are m^2/s, ft^2/s, and 1 Stoke = St = $1\,\text{cm}^2/\text{s}$. Kinematic viscosity has the dimensions of a diffusion coefficient. The kinematic viscosity is a measure of how rapidly momentum diffuses into a flow. This point will be discussed again in Chapter 8. The kinematic viscosities of water and air at 20 °C are:
$$\nu_{\text{H2O},20\,^\circ\text{C}} = 1.00 \times 10^{-6}\,\text{m}^2/\text{s} = 1.00\,\text{cSt},$$
$$\nu_{\text{air},20\,^\circ\text{C}} = 1.50 \times 10^{-5}\,\text{m}^2/\text{s} = 15\,\text{cSt}.$$
- Viscometers are devices used to measure the viscosity of fluids. A good reference on experimental viscometry is: Dinsdale, A. and Moore, F., *Viscosity and its Measurement*, Chapman and Hall.
- Let's examine the common flow situation shown in Figure 1.19. A fluid is contained between two, infinitely long parallel plates separated by a distance H. The bottom plate is fixed while the top plate moves at a constant speed V. There are no pressure gradients in the fluid. This type of flow situation is called a planar Couette flow. One of the first things we would notice when conducting this experiment is that the fluid sticks to the solid boundaries, i.e., there is zero relative velocity between the fluid and the boundary. This behavior is referred to as the no-slip condition. The no-slip condition occurs for all real, viscous fluids under normal conditions. (The no-slip condition may be violated in rarefied flows where the motion of individual molecules becomes significant, i.e., when the Knudsen number is Kn > 0.1.)
The second thing we would notice is that the fluid velocity profile is linear,
$$u_x = V\left(\frac{y}{H}\right). \tag{1.98}$$
Note that the velocity at the bottom plate is zero and at the top plate the velocity is V. Thus, the no-slip condition is satisfied. We will derive this velocity profile in Chapter 8.
If the fluid is Newtonian, then the shear stress acting on the fluid is,
$$\tau_{yx} = \mu\frac{du_x}{dy} = \mu\left(\frac{V}{H}\right). \tag{1.99}$$
The shear stress is a constant everywhere in the fluid, i.e., there is no y dependence. Additionally, the shear stress acts to resist the motion of the top plate and tries to carry the bottom plate along with the fluid (Figure 1.20).
- An inviscid flow is one in which the viscous stresses are negligible ($\tau \approx 0$). There are two ways to have negligible viscous stresses. First, the viscosity could be negligibly small, but there are no common fluids that have $\mu \approx 0$ (although super-fluid helium does have $\mu = 0$, but it's not a common fluid!). The second method is to have a small velocity gradient, e.g., $du_x/dy \approx 0$. This condition is quite common. For example, a plug flow where the velocity profile is constant (u_x = constant) is

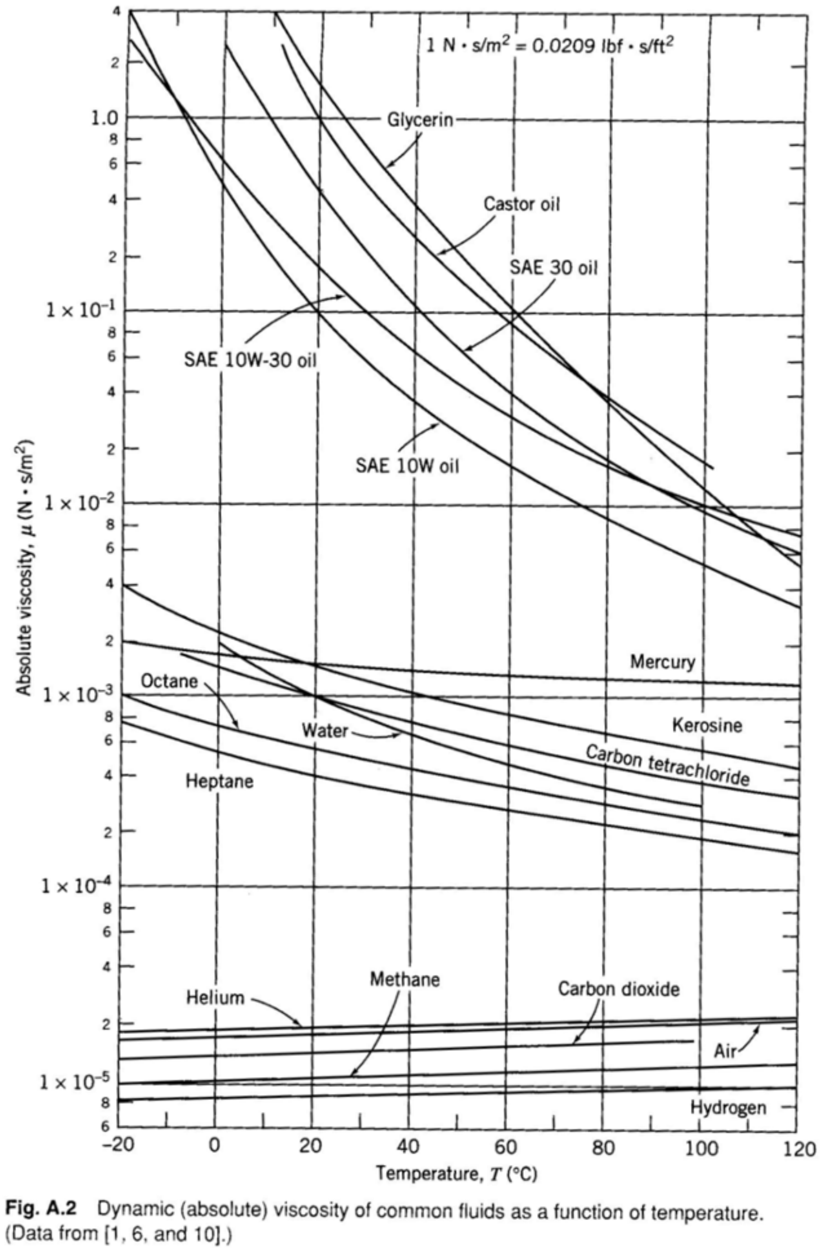


Figure 1.18. The dynamic viscosity for several liquids plotted as a function of temperature. This plot is from Fox, R.W. and McDonald, A.T., *Introduction to Fluid Mechanics*, 5th ed., Wiley.

truly an inviscid flow since $du_x/dy = 0 \implies \tau = 0$. There are many cases where assuming that the flow is inviscid is a reasonable approximation. In addition, many analyses of fluid flow rely on an inviscid flow assumption in order to make the mathematics of the analyses tractable without the use of computers. Note that the no-slip condition does not hold when the flow is inviscid.

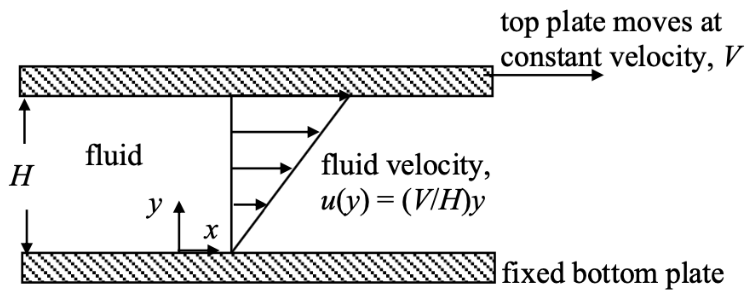


Figure 1.19. A schematic of the Couette flow geometry.

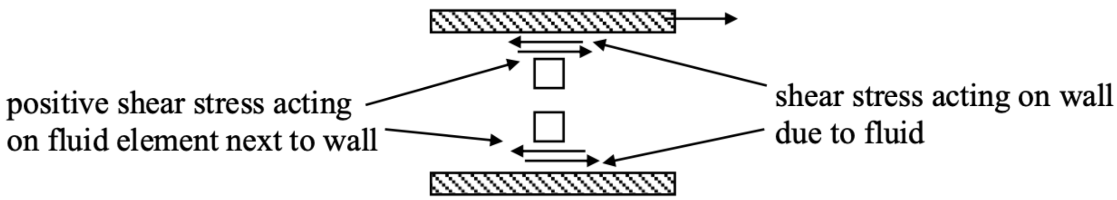


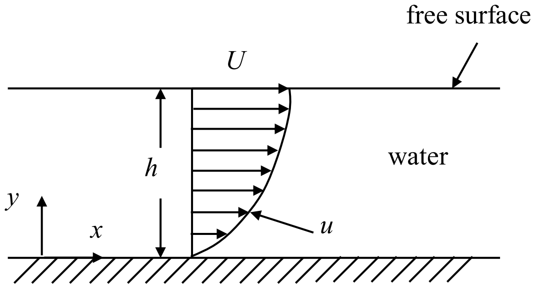
Figure 1.20. Positive shear stresses acting on fluid elements near the wall boundaries.

- An ideal flow is one that is both incompressible and inviscid. The ideal flow approximation is often reasonable and is commonly used in fluid mechanics analyses. We'll consider ideal fluid flow in Chapter 6.

Be sure to:

(1) Get your shear stress directions correct. Equation (1.95) is the shear stress acting on the fluid element.
(2) Use the correct area when evaluating shear forces.
(3) Integrate to determine a shear force on an area where the shear stress is not uniform over the area.
(4) Evaluate the velocity gradient in Eq. (1.95) at the location where you're interested in determining the shear stress.

Determine the magnitude and direction of the shear stress that the water applies:
a. to the base
b. to the free surface



$$\frac{u}{U} = 2\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2$$

SOLUTION:

The shear stress, τ_{yx}, acting on a Newtonian fluid is given by:

$$\tau_{yx} = \mu \frac{du}{dy} \quad (1)$$

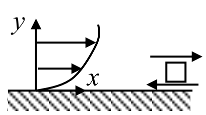
where

$$\frac{du}{dy} = U\left(2\frac{1}{h} - \frac{2y}{h^2}\right) \quad (2)$$

Evaluating the shear stress at the base and free surface gives:

base ($y = 0$): $\boxed{\tau_{yx}\big|_{y=0} = \frac{2\mu U}{h}}$ (3)

This is the stress the wall exerts on the fluid. The fluid will exert an equal but opposite stress on the wall.



fluid element adjacent to the wall with positive τ_{yx} shown

free surface ($y = h$): $\boxed{\tau_{yx}\big|_{y=h} = 0}$ (4)

The air at the free surface does not exert a stress on the water. Although in reality the air will exert a small shear stress on the water, assuming that the shear stress is negligible is reasonable in most engineering applications.

The viscosity of blood is to be determined from measurements of shear stress and shear rate obtained from a small blood sample tested in a suitable viscometer. Based on the data given in the table below, determine if the blood is a Newtonian or a non-Newtonian fluid. Explain how you arrived at your answer.

data set	1	2	3	4	5	6	7	8
shear rate [s^{-1}]	2.25	4.50	11.25	22.5	45.0	90.0	225	450
shear stress [N/m^2]	0.04	0.06	0.12	0.18	0.30	0.52	1.12	2.10

SOLUTION:

Plot the ratio of the shear stress to the shear rate to give the apparent dynamic viscosity:

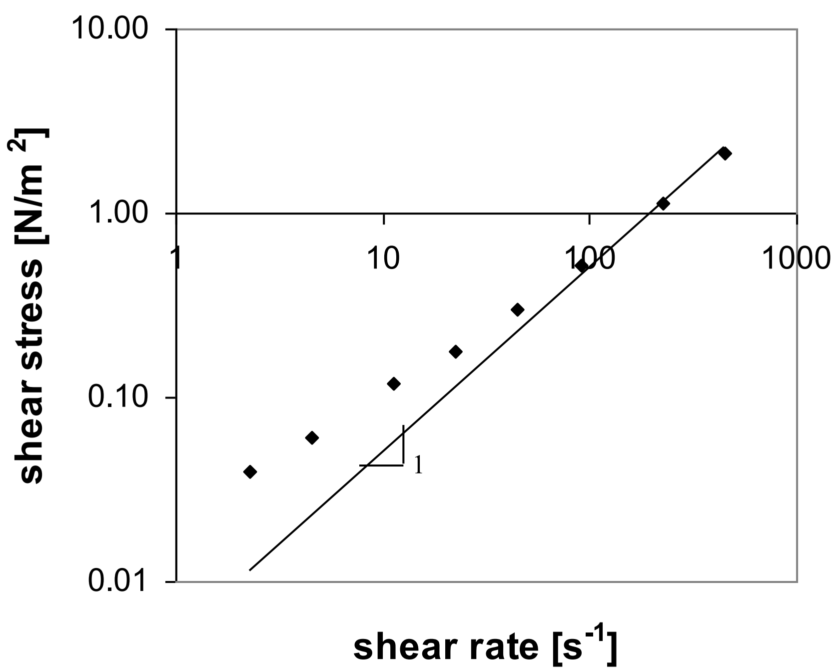
$$\mu_{app} = \frac{\tau}{\left(du/dy\right)}$$

data set	1	2	3	4	5	6	7	8
apparent viscosity, μ_{app} [kg/(m·s)]	0.0178	0.0133	0.0107	0.0080	0.0067	0.0058	0.0050	0.0047

Since the apparent viscosity is decreasing with increasing shear rate (increasing data set number), blood is not Newtonian, but is instead shear thinning.

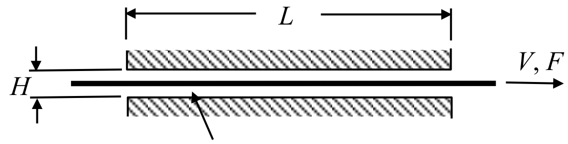
Another way to look at the problem:

Plot the data on a log-log scale as shown below. Note that if $y = x^n$ (*i.e.* a power law function), then $\ln(y) = n\ln(x)$ (*i.e.* the function is a straight line with slope n on a log-log scale). Hence, if blood is Newtonian, then the shear rate-shear stress data plotted on a log-log scale will have a slope of one since $\tau \propto du/dy$ for a Newtonian fluid.



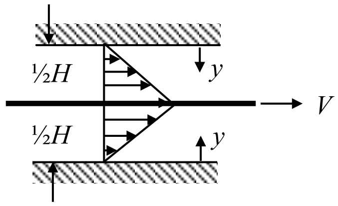
The slope of the blood data is not equal to one indicating that blood is non-Newtonian. In fact, since the slope is less than one over most of the range of shear rate, blood is shear thinning.

During a coating process, a thin, flat tape of width w is pulled through a channel of length L containing a Newtonian fluid of density ρ and dynamic viscosity μ. The fluid is in contact with both sides of the tape. Estimate the force required to pull the tape through the channel if the tape has velocity V and the channel has height H. You may assume that the tape is much thinner than H.



SOLUTION:

Assume that the gap between the tape and the channel walls is sufficiently small so that a laminar Couette flow can be assumed in the gaps. Hence, the velocity profile in each gap is:



$$u = V\left(\frac{y}{\frac{1}{2}H}\right) \tag{1}$$

The shear stress acting on the tape is:

$$\tau\big|_{y=\frac{1}{2}H} = \mu \frac{du}{dy}\bigg|_{y=\frac{1}{2}H} = \frac{2\mu V}{H} \tag{2}$$

The total shear force acting on the tape is then:

$$F_{\text{shear}} = \underbrace{2}_{\substack{\text{since there}\\ \text{are two sides}\\ \text{to the tape}}} \underbrace{\left(\tau\big|_{y=\frac{1}{2}H}\right)(Lw)}_{\substack{\text{shear force acting on}\\ \text{one side of the tape;}\\ Lw \text{ is the area over which}\\ \text{the shear stress acts}}} \tag{3}$$

$$\boxed{\therefore F_{\text{shear}} = \frac{4\mu V L w}{H}} \tag{4}$$

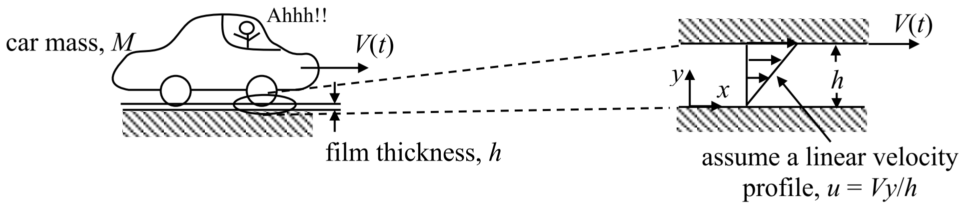
Video solution: https://www.youtube.com/watch?v=2KgQtt8kRuY

When a vehicle such as an automobile slams on its brakes (locking the wheels) on a very wet road it can "hydro-plane." In these circumstances a film of water is created between the tires and the road. Theoretically, a vehicle could slide a very long way under these conditions though in practice the film is destroyed before such distances are achieved (indeed, tire treads are designed to prevent the persistence of such films).

To analyze this situation, consider a vehicle of mass, M, sliding over a horizontal plane covered with a film of liquid of viscosity, μ. Let the area of the film under all four tires be A (the area under each tire is ¼A) and the film thickness (assumed uniform) be h.

a. If the velocity of the vehicle at some instant is V, find the force slowing the vehicle down in terms of A, V, h, and μ.
b. Find the distance, L, that the vehicle would slide before coming to rest assuming that A and h remain constant (this is not, of course, very realistic).
c. What is this distance, L, for a 1000 kg vehicle if $A = 0.1\ \text{m}^2$, $h = 0.1$ mm, $V = 10$ m/s, and the water viscosity is $\mu = 0.001$ kg/(m·s)?

SOLUTION:



The shear stress, τ_{yx}, the tires exert on the fluid is, for a Newtonian fluid:

$$\left.\tau_{yx}\right|_{y=h} = \mu \frac{du}{dy} \tag{1}$$

Assuming a linear velocity profile in the fluid film, the shear stress the tires exert on the fluid is:

$$\left.\tau_{yx}\right|_{y=h} = \mu \frac{V}{h} \tag{2}$$

The fluid will exert an equal but opposite shear stress on the tires. The total shear force acting on the four tires (with a combined area of A) is:

$$F = -\left.\tau_{yx}\right|_{y=h} A$$

$$\boxed{\therefore F = -\mu \frac{V}{h} A} \quad \text{(The negative sign implies that the force is acting to slow down the car.)} \tag{3}$$

The distance the vehicle slides before coming to rest can be determined using Newton's 2nd Law applied to the vehicle.

$$F = M\frac{dV}{dt} = -\mu \frac{V}{h} A \tag{4}$$

Solve the differential equation for the velocity as a function of time.

$$\int_{V_0}^{V} \frac{dV}{V} = -\frac{\mu A}{Mh}\int_0^t dt$$

$$\ln\left(\frac{V}{V_0}\right) = -\frac{\mu A}{Mh} t$$

$$\therefore V = V_0 \exp\left(-\frac{\mu A}{Mh} t\right) \tag{5}$$

Note that as $t \to \infty$, $V \to 0$ (the car comes to rest). Integrate Eqn. (5) with respect to time once more to determine the travel distance, x, as a function of time.

$$V = \frac{dx}{dt} = V_0 \exp\left(-\frac{\mu A}{Mh}t\right)$$

$$\int_0^x dx = V_0 \int_0^t \exp\left(-\frac{\mu A}{Mh}t\right) dt$$

$$x = \frac{MhV_0}{\mu A}\left[1 - \exp\left(-\frac{\mu A}{Mh}t\right)\right] \tag{6}$$

The car comes to rest as $t \to \infty$ so the total distance the car travels, L, will be:

$$\boxed{L = \frac{MhV_0}{\mu A}} \tag{7}$$

For the following parameters:

M = 1000 kg
h = $0.1*10^{-3}$ m
V_0 = 10 m/s
μ = 0.001 kg/(m·s)
A = 0.1 m^2
$\Rightarrow$ $\boxed{L = 10{,}000 \text{ m } (\approx 6.2 \text{ miles})}$

Obviously this isn't very realistic. Our assumptions of constant tire area and film thickness aren't very good ones.

Another approach to solving this problem is to set the small amount of work performed by the viscous force over a small displacement dx equal to the small change in the kinetic energy of the vehicle,

$$dW = d(KE) \Rightarrow Fdx = d\left(\tfrac{1}{2}MV^2\right) = MVdV\ , \tag{8}$$

where F is given by Eq. (3),

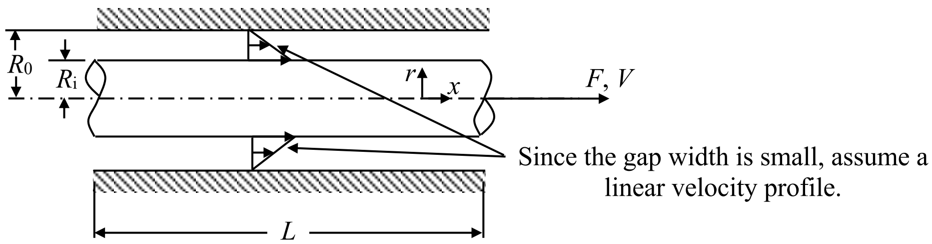
$$-\mu\frac{V}{h}Adx = MVdV \Rightarrow -\frac{\mu A}{Mh}dx = dV \Rightarrow -\frac{\mu A}{Mh}\int_{x=0}^{x=L} dx = \int_{V=V_0}^{V=0} dV\ , \text{ (since the velocity is zero at } x = L\text{)} \tag{9}$$

$$-\frac{\mu AL}{Mh} = -V_0 \Rightarrow L = \frac{MhV_0}{\mu A} \quad \text{which is the same answer as given in Eq. (7)!} \tag{10}$$

Magnet wire is to be coated with varnish for insulation by drawing it through a circular die of 0.9 mm diameter. The wire diameter is 0.8 mm and it is centered in the die. The varnish, with a dynamic viscosity of 20 centipoise, completely fills the space between the wire and the die for a length of 20 mm. The wire is drawn through the die at a speed of 50 m/s. Determine the force required to pull the wire.

SOLUTION:

Apply Newton's 2nd Law in the x-direction to the wire shown in the diagram below.



$$\sum F_x = F + \tau_{rx}\left(2\pi R_i L\right) = 0 \quad \text{(Note that the wire is not accelerating.)} \tag{1}$$

where τ_{rx} is the shear stress the fluid exerts on the wire and ($2\pi R_i L$) is the area over which the shear stress acts. The shear stress the wire exerts on the fluid (assumed to be Newtonian) is:

$$\tau_{rx} = \mu \frac{du}{dr} \tag{2}$$

where

$$\frac{du}{dr} = \frac{0 - V}{R_o - R_i} = \frac{-V}{R_o - R_i} \quad \text{(Note that } u(r = R_o) = 0 \text{ and } u(r = R_i) = V\text{.)} \tag{3}$$

Recall that the shear stress the fluid exerts on the wire will be equal to, but opposite, the value given by Eqn. (2). Substitute Eqns. (2) and (3) into Eqn. (1) and simplify.

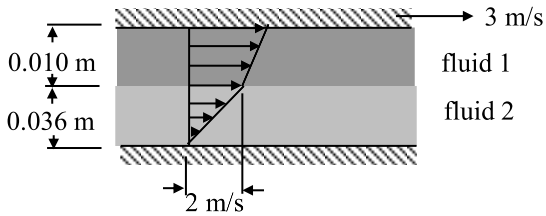
$$F - \left(\mu \frac{V}{R_o - R_i}\right)\left(2\pi R_i L\right) = 0$$

$$\boxed{\therefore F = \frac{2\pi R_i L \mu V}{R_o - R_i}} \tag{4}$$

Use the numerical values given in the problem statement.

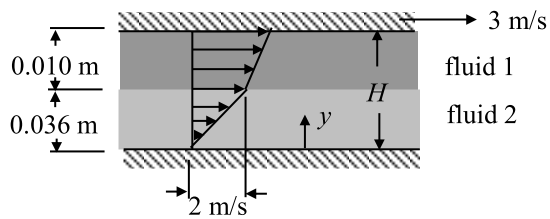
R_i = 4.0×10^{-4} m
R_o = 4.5×10^{-4} m
L = 2.0×10^{-2} m
μ = 20 cP = 0.02 kg/(m·s) (Note: 100 cP = 0.1 kg/(m·s).)
V = 50 m/s
$\Rightarrow \boxed{F = 1.0 \text{ N}}$

The no-slip condition states that fluid "sticks" to solid surfaces. Two immiscible layers of Newtonian fluid are dragged along by the motion of an upper plate as shown in the figure. The bottom plate is stationary and the velocity profiles for each fluid are linear. The top fluid (fluid 1), with a specific gravity of 0.8 and kinematic viscosity of 1.0 cSt, puts a shear stress on the upper plate, and the lower fluid (fluid 2), with a specific gravity of 1.1 and kinematic viscosity of 1.3 cSt, puts a shear stress on the bottom plate. Determine the ratio of the shear stress on the top plate to the shear stress on the bottom plate.



SOLUTION:

The shear stress acting on either plate may be found using the expression relating the shear stress to the velocity gradient for a Newtonian fluid:



$$\tau_{yx}{}_{\text{on fluid}} = \mu \frac{\partial u_x}{\partial y} \tag{1}$$

For the bottom plate:

$$\tau_{yx}{}_{\text{on bottom plate}} = -\tau_{yx}{}_{\text{on fluid}}\Big|_{y=0} = -\mu_2 \frac{\Delta u_2}{\Delta y_2} \tag{2}$$

where the derivative in Eqn. (1) may be replace with $\Delta u/\Delta y$ since the velocity profile is linear, μ_2 is the dynamic viscosity of fluid 2, and Δu_2 and Δy_2 are the change in the velocity over the change in y-position, respectively, in fluid 2. Following a similar approach for the top plate gives:

$$\tau_{yx}{}_{\text{on top plate}} = -\tau_{yx}{}_{\text{on fluid}}\Big|_{y=H} = -\mu_1 \frac{\Delta u_1}{\Delta y_1} \tag{3}$$

Taking the ratio of Eqn. (3) to Eqn. (2) gives:

$$\frac{\tau_{yx}{}_{\text{on top plate}}}{\tau_{yx}{}_{\text{on bottom plate}}} = \frac{\mu_1}{\mu_2}\frac{\Delta u_1}{\Delta u_2}\frac{\Delta y_2}{\Delta y_1} \tag{4}$$

Note that the dynamic viscosity is related to the specific gravity, SG, and kinematic viscosity, ν, by:

$$\mu = \rho\nu = SG\rho_{H_2O}\nu \tag{5}$$

where ρ is the fluid density and ρ_{H2O} is the density of water. Substituting Eqn. (5) into Eqn. (4) gives:

$$\frac{\tau_{yx}{}_{\text{on top plate}}}{\tau_{yx}{}_{\text{on bottom plate}}} = \frac{SG_1}{SG_2}\frac{\nu_1}{\nu_2}\frac{\Delta u_1}{\Delta u_2}\frac{\Delta y_2}{\Delta y_1} \tag{6}$$

Using the given values:

$SG_1 = 0.8$ $\quad SG_2 = 1.1$

$\nu_1 = 1.0$ cSt $\quad \nu_2 = 1.3$ cSt

$\Delta u_1 = (3 - 2)$ m/s = 1 m/s

$\Delta u_2 = (2 - 0)$ m/s = 2 m/s

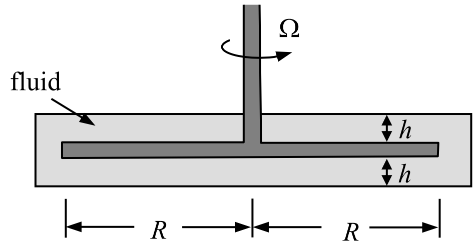
$\Delta y_1 = 0.010$ m

$\Delta y_2 = 0.036$ m

$$\Rightarrow \boxed{\frac{\tau_{yx}{}_{\text{on top plate}}}{\tau_{yx}{}_{\text{on bottom plate}}} = 1.0}$$

Note that the stress across a fluid interface is continuous and since the shear stress is constant in the fluid, the shear stresses on the plates should be identical.

A rotating disk viscometer has a radius, R = 50 mm, and a clearance, h = 1 mm, as shown in the figure.



a. If the torque required to rotate the disk at Ω = 900 rpm is 0.537 N·m, determine the dynamic viscosity of the fluid. You may neglect the viscous forces acting on the rim of the disk and on the vertical shaft.
b. If the uncertainty in each parameter is ±1%, determine the uncertainty in the viscosity.

SOLUTION:

The torque acting on the rotating section is due to the fluid shear stresses acting on the rotating surface. Consider only the stresses acting on the upper and lower surfaces since the edge stresses act over a negligibly small area. Determine the force acting on a small area of the rotating disk.

$$dF = 2\tau_{y\theta}\big|_{y=0}\, dA = 2\tau_{y\theta}\big|_{y=0}\left(2\pi r dr\right) \tag{1}$$

(Note that the force acts on both the upper and lower surfaces.)

The shear stress acting on the upper and lower surfaces, assuming a Newtonian fluid and a linear velocity profile in the gap (a reasonable assumption if the gap width is narrow), is:

$$\tau_{y\theta} = \mu\frac{du}{dy} = \mu\frac{0-r\Omega}{0-h} = \mu\frac{r\Omega}{h} \quad \text{(Note } u(y=0)=0 \text{ and } u(y=h)=r\Omega\text{.)} \tag{2}$$

Substitute into Eqn. (1).

$$dF = 2\mu\frac{r\Omega}{h}\left(2\pi r dr\right) \tag{3}$$

The torque due to this force contribution is:

$$dT = r dF = r2\mu\frac{r\Omega}{h}\left(2\pi r dr\right) \tag{4}$$

The total torque is found by integrating over the disk radius.

$$T = \int_{r=0}^{r=R} dT = 4\pi\mu\frac{\Omega}{h}\int_{r=0}^{r=R} r^3 dr$$

$$\therefore T = \pi\mu\frac{\Omega R^4}{h} \tag{5}$$

Re-arrange to solve for the viscosity.

$$\boxed{\mu = \frac{Th}{\pi\Omega R^4}} \tag{6}$$

Using the given values:

T = 0.537 N·m
h = $1.0*10^{-3}$ m
Ω = 900 rpm = 94.2 rad/s
R = $50.0*10^{-3}$ m
⇒ $\boxed{\mu = 0.29 \text{ kg/(m·s)} \quad \text{or } 0.29 \text{ Pa·s or } 290 \text{ cP}}$

Perform an uncertainty analysis on Eqn. (6).

$$u_\mu = \left[u_{\mu,T}^2 + u_{\mu,h}^2 + u_{\mu,\Omega}^2 + u_{\mu,R}^2\right]^{1/2} \tag{7}$$

where

$$u_{\mu,T} = \frac{1}{\mu}\frac{\partial \mu}{\partial T}\delta T = \left(\frac{Th}{\pi\Omega R^4}\right)^{-1}\frac{h}{\pi\Omega R^4}\delta T = \frac{\delta T}{T} = u_T$$

$$u_{\mu,h} = \frac{1}{\mu}\frac{\partial \mu}{\partial h}\delta h = \left(\frac{Th}{\pi\Omega R^4}\right)^{-1}\frac{T}{\pi\Omega R^4}\delta h = \frac{\delta h}{h} = u_h$$

$$u_{\mu,\Omega} = \frac{1}{\mu}\frac{\partial \mu}{\partial \Omega}\delta\Omega = \left(\frac{Th}{\pi\Omega R^4}\right)^{-1}\frac{-Th}{\pi\Omega^2 R^4}\delta\Omega = -\frac{\delta\Omega}{\Omega} = -u_\Omega$$

$$u_{\mu,R} = \frac{1}{\mu}\frac{\partial \mu}{\partial R}\delta R = \left(\frac{Th}{\pi\Omega R^4}\right)^{-1}\frac{-4Th}{\pi\Omega R^5}\delta R = -4\frac{\delta R}{R} = -4u_R$$

Substitute into Eqn. (7).

$$\boxed{u_\mu = \left[u_T^2 + u_h^2 + u_\Omega^2 + 16u_R^2\right]^{1/2}} \tag{8}$$

Note that the relative uncertainty in R contributes the most to the uncertainty in the viscosity.

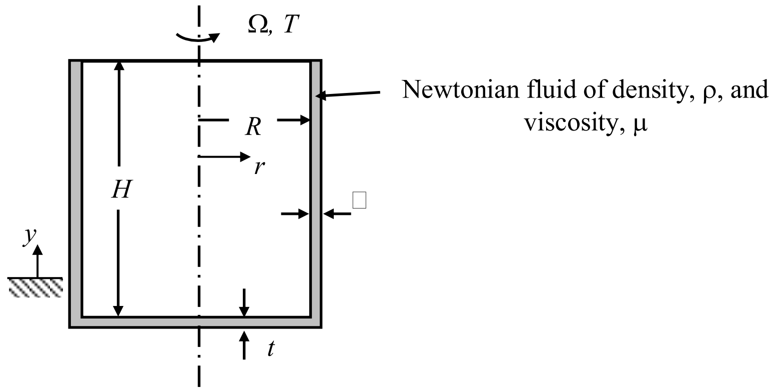
From the problem statement, the relative uncertainty in each of the measurements is ±1% so that:

$$\boxed{u_\mu = \pm 4.4\%} \tag{9}$$

$$\Rightarrow \boxed{\mu = 0.29 \pm 0.01 \text{ Pa}\cdot\text{s}}$$

Video solution: https://www.youtube.com/watch?v=B2IK8SuDWv0

A rotating cylinder viscometer is shown in the figure below. The inner cylinder has radius, R, and height, H. An incompressible, viscous, Newtonian fluid of density, ρ, and viscosity, μ, is contained between the cylinders. The narrow gap between the cylinders has width, t (<< R and H). A torque, T, is required to rotate the inner cylinder at constant speed Ω. Determine the fluid viscosity, μ, in terms of the other system parameters.



SOLUTION:

Assume that the velocity profiles in the narrow gaps are linear since the gap size is small and curvature may be neglected (t/R << 1) . In the circumferential gap the fluid velocity gradient will be:

$$\left.\frac{du_\theta}{dr}\right|_{\text{circumference}} = \frac{0 - R\Omega}{(R+t) - R} = -\frac{R\Omega}{t} \tag{1}$$

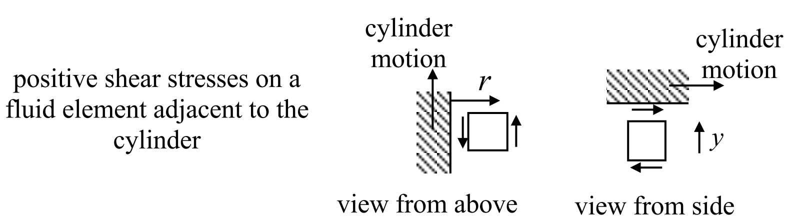
In the gap at the bottom of the cylinder the fluid velocity gradient will be:

$$\left.\frac{du_\theta}{dy}\right|_{\text{bottom}} = \frac{r\Omega - 0}{t - 0} = \frac{r\Omega}{t} \tag{2}$$

The shear forces acting on the fluid at the inner cylinder wall and base will be:

$$dF_{\text{circumference}} = \tau|_{\text{circumference}}(2\pi R dy) = \mu\left.\frac{du_\theta}{dr}\right|_{\text{circumference}}(2\pi R dy) = -\mu\frac{R\Omega}{t}(2\pi R dy)$$

$$dF_{\text{bottom}} = \tau|_{\text{bottom}}(2\pi r dr) = \mu\left.\frac{du_\theta}{dr}\right|_{\text{bottom}}(2\pi r dr) = \mu\frac{r\Omega}{t}(2\pi r dr)$$



Note that the force acting on the cylinder will be equal in magnitude but in the opposite direction to the force acting on the fluid (Newton's 3rd Law).

The total torque acting on the cylinder (neglected the contribution from the corner regions) is:

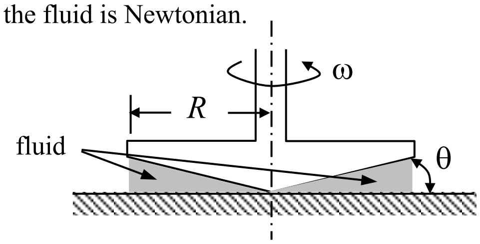
$$\begin{aligned}
T &= \int_{y=0}^{y=H} dT_{\text{circumference}} + \int_{r=0}^{r=R} dT_{\text{bottom}} \\
&= \int_{y=0}^{y=H} R\,dF_{\text{circumference}} + \int_{r=0}^{r=R} r\,dF_{\text{bottom}} \\
&= \int_{0}^{H} R\mu \frac{R\Omega}{t}\left(2\pi R dy\right) + \int_{0}^{R} r\mu \frac{r\Omega}{t}\left(2\pi r dr\right) \\
&= 2\pi\mu \frac{R^3\Omega}{t} H + \frac{2\pi}{4}\mu \frac{R^4\Omega}{t} \\
T &= 2\pi\mu \frac{\Omega R^3}{t}\left(H + \tfrac{1}{4}R\right)
\end{aligned} \tag{3}$$

Re-arrange to solve for the viscosity:

$$\boxed{\mu = \frac{T}{2\pi \frac{\Omega R^3}{t}\left(H + \tfrac{1}{4}R\right)}} \tag{4}$$

The cone and plate viscometer shown in the figure is an instrument used frequently to characterize non-Newtonian fluids. It consists of a flat plate and a rotating cone with a very obtuse angle (typically θ is less than 0.5 degrees). The apex of the cone just touches the plate surface and the liquid to be tested fills the narrow gap formed by the cone and plate.

a. Derive an expression for the shear rate in the liquid that fills the gap in terms of the geometry of the system and the operating conditions.
b. Evaluate the torque on the driven cone in terms of the shear stress and geometry of the system.
c. Evaluate the torque on the driven cone in terms of the geometry, operating conditions, and fluid properties if the fluid is Newtonian.



SOLUTION:

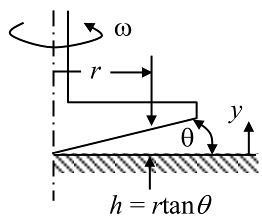
The shear rate in the fluid is equal to the fluid's velocity gradient. Since the gap width is small, assume that the velocity profile in the gap is linear in the vertical (i.e., y-direction) direction.

$$\text{shear rate } = \frac{du}{dy} = \frac{r\omega - 0}{h-0} = \frac{r\omega}{h}$$ (Note: $u(y = h) = rw$ and $u(y = 0) = 0$.)

where

$$h = r\tan\theta$$

$$\boxed{\therefore \text{ shear rate } = \frac{du}{dy} = \frac{\omega}{\tan\theta}} \tag{1}$$



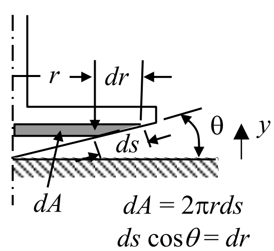
The torque acting on the cone is due to the shear stresses exerted by the fluid,

$$dF = \tau_{y\theta}\big|_{y=h} \underbrace{dA\cos\theta}_{\text{use horz. proj. area}} = \tau_{y\theta}\big|_{y=h}(2\pi r ds\cos\theta) = \tau_{y\theta}\big|_{y=h}\left(2\pi r\frac{dr}{\cos\theta}\cos\theta\right) = \tau_{y\theta}\big|_{y=h}(2\pi r dr), \tag{2}$$

$$dT = rdF = r\tau_{y\theta}\big|_{y=h}(2\pi r dr), \tag{3}$$

$$T = \int_{r=0}^{r=R} dT = \int_0^R r\tau_{y\theta}\big|_{y=h}(2\pi r dr), \tag{4}$$

$$\boxed{T = 2\pi\int_0^R \tau_{y\theta}\big|_{y=h} r^2 dr}. \tag{5}$$



Note that in Eq. (2) the horizontal projected area of the area is used since it's the velocity gradient in the y direction that's causing the shear stress on the cone (it's the gap in the vertical direction that the cause for the shear stress).

For a Newtonian fluid,

$$\tau_{y\theta} = \mu\frac{du}{dy}. \tag{2}$$

Use the shear rate given in Eq. (1) and substitute into Eq. (5),

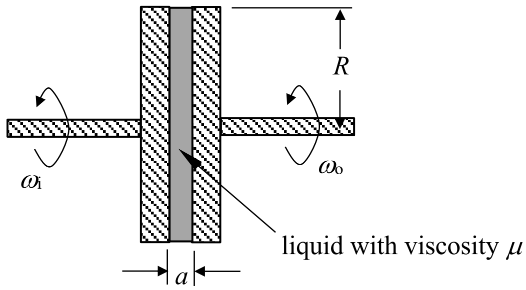
$$T = 2\pi\int_0^R \left(\mu\frac{\omega}{\tan\theta}\right) r^2 dr. \tag{7}$$

$$\boxed{T = \frac{2\pi}{3}\frac{\mu\omega R^3}{\tan\theta}}. \tag{8}$$

Note that since the angle is small, $\tan\theta \approx \theta$.

Video solution: https://www.youtube.com/watch?v=s0iiZi0dfMo

A viscous clutch is made from a pair of closely spaced parallel, circular disks enclosing a thin layer of viscous liquid.

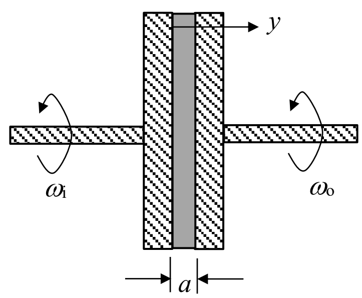


Develop an expression for the torque, T, transmitted by the disk pair, in terms of the liquid dynamic viscosity, μ, the disk radius, R, the disk spacing, a, and the angular speeds of the input disk, ω_i, and output disk, ω_o.

SOLUTION:

Since the disks are closely spaced, assume that the velocity profile in the liquid is linear, with the velocity gradient at a radius r being,

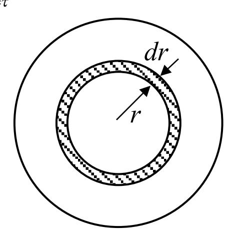
$$\frac{du}{dy} \overset{\text{since linear}}{=} \frac{\Delta u}{\Delta y} = \frac{\omega_o r - \omega_i r}{a} = \frac{(\omega_o - \omega_i) r}{a}. \tag{1}$$



The torque acting on the output disk due to the shear force exerted by the liquid is,

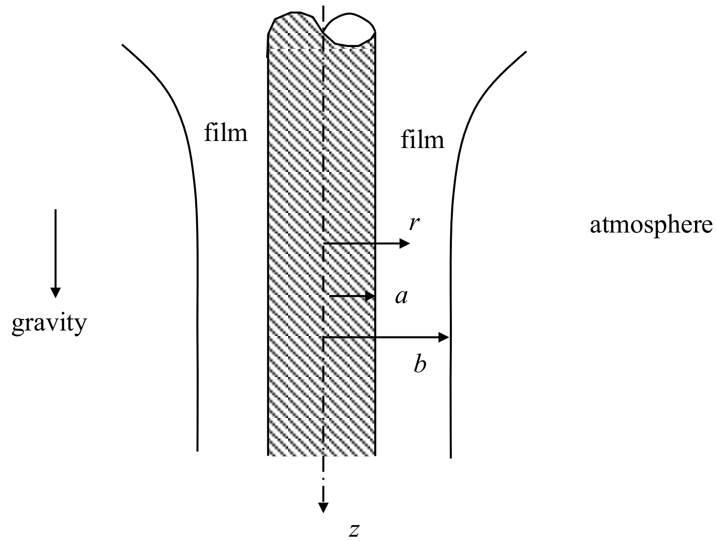
$$T = \int_{r=0}^{r=R} r\,dF = \int_{r=0}^{r=R} r \underbrace{\tau\big|_{y=a} dA}_{=dF} = \int_{r=0}^{r=R} r \underbrace{\left[\mu \frac{(\omega_o - \omega_i) r}{a}\right]}_{=\tau} \underbrace{(2\pi r dr)}_{=dA} = 2\pi\mu \frac{(\omega_o - \omega_i)}{a} \int_{r=0}^{r=R} r^3 dr, \tag{2}$$

$$\boxed{T = \frac{\pi}{2} \mu \frac{(\omega_o - \omega_i)}{a} R^4}. \tag{3}$$



Video solution: https://www.youtube.com/watch?v=m0ZuyCPzN64

A viscous, Newtonian liquid film falls under the action of gravity down the surface of a rod as shown in the figure below.



The velocity of the fluid is given by:

$$u_z = -\frac{\rho g}{4\mu}\left(r^2 - a^2\right) + \frac{\rho g b^2}{2\mu}\ln\left(\frac{r}{a}\right)$$

where ρ is the liquid density, g is the acceleration due to gravity, μ is the liquid's dynamic viscosity, a is the radius of the rod, b is the radius to the free surface of the film, and r is the radius measured from the centerline of the rod.

Determine:

a. What is the shear stress, τ_{rz}, at the free surface of the liquid?

b. What force acts on the rod per unit length of the rod due to the viscous liquid?

c. Set up, but do not solve, the integral for determining the volumetric flow rate of liquid flowing down the rod.

SOLUTION:

The shear stress acting on a fluid element is given by:

$$\tau_{rz} = \mu \frac{\partial u_z}{\partial r} \tag{1}$$

Using the given velocity profile gives:

$$\tau_{rz} = \mu\left[-\frac{\rho g}{2\mu} r + \frac{\rho g b^2}{2\mu}\frac{1}{r}\right] \tag{2}$$

At the liquid's free surface, $r = b$ so:

$$\tau_{rz}\big|_{r=b} = \mu\left[-\frac{\rho g}{2\mu} b + \frac{\rho g b^2}{2\mu}\frac{1}{b}\right] \tag{3}$$

$$\boxed{\therefore \tau_{rz}\big|_{r=b} = 0} \tag{4}$$

Note that typically the surrounding atmosphere exerts a negligible shear stress on the surface of a flowing liquid so we often assume that the shear stress there is zero.

Video solution: https://www.youtube.com/watch?v=m0ZuyCPzN64

The viscous force acting on the rod per unit length is:

$$\frac{F}{L} = 2\pi a \, \tau_{rz}\big|_{r=a} \tag{5}$$

Use Eqn. (2) to determine the shear stress acting at the rod's surface.

$$\frac{F}{L} = 2\pi a \mu\left[-\frac{\rho g}{2\mu}a + \frac{\rho g b^2}{2\mu}\frac{1}{a}\right] \tag{6}$$

$$\boxed{\therefore \frac{F}{L} = \rho g \pi a^2\left[\left(\frac{b}{a}\right)^2 - 1\right]} \tag{7}$$

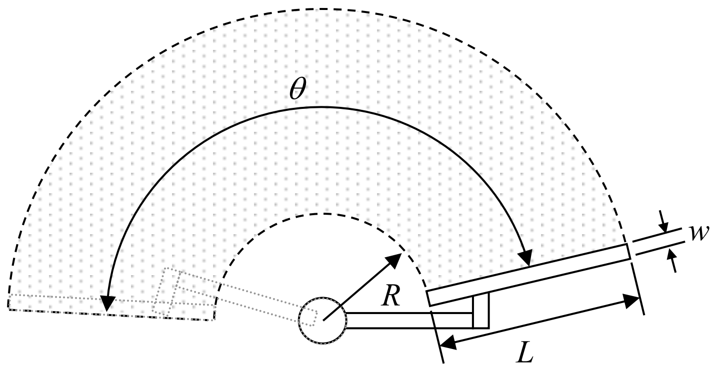
Note that the viscous force acting on the rod points in the downstream direction.

The volumetric flow rate of the liquid is given by:

$$\boxed{Q = \int_{r=a}^{r=b} u_z \underbrace{\left(2\pi r dr\right)}_{=dA}} \text{ where } u_z \text{ is given in the problem statement.} \tag{8}$$

Video solution: https://www.youtube.com/watch?v=1NEMiNyeSX4

Derive an expression estimating the torque required to rotate a windshield wiper blade over the surface of a wet windshield in terms of the parameters given in the figure below,

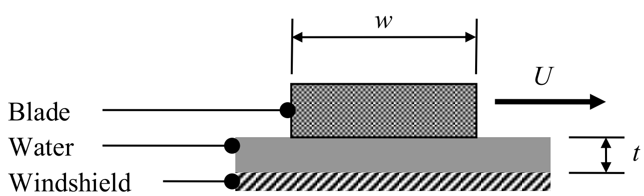


Here, R is the radius of the inner most point swept by the wiper blade, L is the length of the wiper blade, θ is the angle swept by the blade in a time T, w is the width of the part of the blade in contact with the windshield, t is the thickness of the water layer, and μ is the viscosity of water.

For a 1996 Toyota Rav 4, R = 6", L = 11", θ = 110°, T = 2 sec, w = ¼", and t = $^1/_{16}$". Evaluate your expression to estimate the torque.

SOLUTION:

Consider a differential element of the wiper along the radial direction. The cross-section for this element at radius r can be imagined as shown in the figure below.



If the wiper blade sweeps out an angle θ in a time T, the average speed U of the element over the windshield is,

$$U = \frac{\theta r}{T}. \tag{1}$$

Assuming Couette flow between the wiper blade and the windshield, the shear stress applied to the differential element is,

$$\tau = \mu \frac{du}{dy} = \mu \frac{U}{t} = \mu \frac{\theta r}{Tt}, \tag{2}$$

where t is the thickness of the water layer between the wiper and the windshield. The force on this differential element is the shear stress acting on this element multiplied by the area of the element,

$$dF = \tau dA = \mu \frac{\theta r}{Tt}(wdr). \tag{3}$$

The small moment on the wiper blade pivot caused by this small force is,

$$dM = rdF = r\mu \frac{\theta r}{Tt}(wdr). \tag{4}$$

Integrating over the length of the blade gives the total moment,

$$M = \int_{r=R}^{r=R+L} dM = \int_{r=R}^{r=R+L} \mu \frac{r^2\theta}{Tt} w\,dr\,, \tag{5}$$

$$\boxed{\therefore M = \frac{1}{3}\mu\frac{\theta}{T}R^3\left[\left(1+\frac{L}{R}\right)^3 - 1\right]\frac{w}{t}}. \tag{6}$$

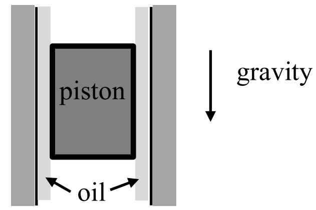
The numerical solution for the given parameters is,

$$\begin{aligned} M &= \frac{1}{3}\left(2*10^{-5}\ \tfrac{\text{lb}_\text{f}\cdot\text{s}}{\text{ft}^2}\right)\left(\frac{110^\circ\cdot\frac{\pi}{180^\circ}}{2\text{ s}}\right)\left(\frac{6\text{ in}}{12\,\frac{\text{in}}{\text{ft}}}\right)^3\left[\left(1+\frac{11\text{ in}}{6\text{ in}}\right)^3 - 1^3\right]\left(\frac{\frac{1}{4}\text{ in}}{\frac{1}{16}\text{ in}}\right) \\ &\approx 7*10^{-5}\text{ ft-lb}_\text{f} \end{aligned} \tag{7}$$

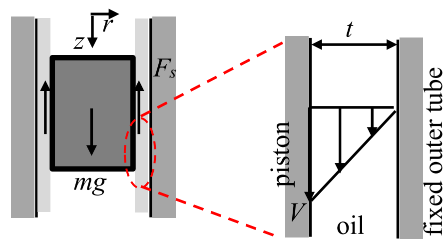
This torque is quite small. An important factor contributing to this small value is that the actual wiper geometry is not parallel to window as given in this simple example, but instead has an inclined surface. This inclined surface will result in a pressure force acting on the blade surface in addition to a shear force. This pressure force will be much larger than the shear force and, hence, a larger torque will result. Wiper blade dynamics are actually quite complex and have been the study of a number of research projects.

Note: Thanks to Dr. Ben Freireich for helping to prepare this problem.

A 73 mm diameter aluminum (specific gravity of 2.64) piston of 100 mm length is centered in a stationary 75 mm inner diameter steel tube lined with SAE 10W-30 oil at 25 °C. The piston is set into motion by cutting a support cord. What is the terminal velocity of the piston? You may assume a linear velocity profile within the oil.



SOLUTION:
Draw a free body diagram of the piston and, since we're interested in the terminal speed of the piston, sum the forces in the z direction and set them equal to zero.



$$\sum F_z = 0 = mg - F_s, \tag{1}$$

where F_s is the shear force due to the viscous stress the fluid exerts on the piston. The mass of the piston is,

$$m = SG_{Al}\rho_{H20}\pi R^2 L, \tag{2}$$

where R is the radius of the piston.

The shear force acting on the surface of the piston is due to the viscous stress applied by the oil. To find this viscous stress, first assume a linear profile in the oil,

$$u_z = V\left(\frac{r-R}{t}\right), \tag{3}$$

where V is the speed of the piston. Hence, the gradient of the z velocity component in the r direction in the oil is,

$$\frac{du_z}{dr} = \frac{V}{t}. \tag{4}$$

The shear stress acting on the surface of the piston due to the oil, assuming Newtonian fluid behavior, is,

$$\tau_{rz}\Big|_{r=R} = \mu \frac{du_z}{dr}\Big|_{r=R} = \mu\frac{V}{t}. \tag{5}$$

Note that this shear stress is constant. The corresponding shear force due to this shear stress is determined by multiplying the shear stress by the area over which this stress acts,

$$F_s = \tau_{rz}\Big|_{r=R}(2\pi RL) = \left(\mu\frac{V}{t}\right)(2\pi RL). \tag{6}$$

Substitute Eqs. (6) and (2) into Eq. (1) and solve for V,

$$0 = \left(SG_{Al}\rho_{H20}\pi R^2 L\right)g - \left(\mu\frac{V}{t}\right)(2\pi RL), \tag{7}$$

$$V = \frac{SG_{Al}\rho_{H20}\pi R^2 Lgt}{2\pi RL\mu}, \tag{8}$$

$$\boxed{V = \frac{SG_{Al}\rho_{H20}Rgt}{2\mu}}. \tag{9}$$

Note that the piston length isn't a factor.

Substitute the given parameters,

$SG_{Al} = 2.64$

$\rho_{H20} = 1000$ kg/m^3,

$R = 0.5*73*10^{-3}$ m = $36.5*10^{-3}$ m,

$g = 9.81$ m/s^2,

$t = 0.5*(75 - 73)*10^{-3}$ m = $1.0*10^{-3}$ m

$\mu = 0.13$ Pa.s (from Fig. A.2 of Pritchard et al.),

=> $\boxed{V = 3.64 \text{ m/s}}$.

1.7.5. Bulk Modulus, E_ν

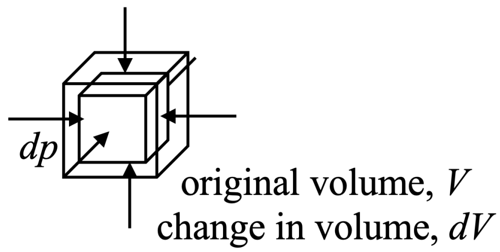


Figure 1.21. Illustration of how an applied differential pressure dp results in a differential change in volume dV.

- The bulk modulus is a measure of a substance's compressibility. The larger the bulk modulus, the less compressible the substance. An incompressible substance has an infinite bulk modulus.
- The dimensions of bulk modulus are F/L^2 and common units are Pa, bar, and psi.
- The bulk modulus is defined as the ratio of the differential pressure, dp, required to cause a relative decrease in the differential volume, $-dV/V$ (Figure 1.21),

$$\boxed{E_\nu := -\frac{dp}{dV/V}}. \tag{1.100}$$

- If $dp > 0$ (i.e., an increase in pressure) and $dV < 0$ (i.e., a decrease in volume), then $E_\nu > 0$.
- The bulk modulus can be written in terms of density rather than volume,

$$E_\nu = \frac{dp}{d\rho/\rho}, \tag{1.101}$$

since $m = \rho V$ and $dm = 0 = V d\rho + \rho dV \implies d\rho/\rho = -dV/V$.
- Water at 20 °C has $E_\nu = 2.21\,\text{GPa}$. This large bulk modulus indicates that water is nearly incompressible. Indeed, liquids are generally modeled as being incompressible.
- For an ideal gas under isothermal conditions,

$$p = \rho RT \implies \left.\frac{dp}{d\rho}\right|_T = RT \implies E_\nu|_T = \rho RT = p. \tag{1.102}$$

- For an ideal gas under isentropic conditions,

$$p = c\rho^\gamma \implies \left.\frac{dp}{d\rho}\right|_s = c\gamma\rho^{\gamma-1} \implies E_\nu|_s = \gamma p = \gamma\rho RT. \tag{1.103}$$

where c is a constant and $\gamma = c_p/c_v$ is the specific heat ratio.
- The speed of sound in a substance is related to how compressible the substance is. It can be shown (in Chapter 13) that the speed of sound, c, is given by,

$$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s} \quad \text{The subscript "s" indicates the process occurs isentropically.} \tag{1.104}$$

Since the bulk modulus can be written as,

$$E_\nu = \frac{dp}{d\rho/\rho} = \rho\left(\frac{dp}{d\rho}\right), \tag{1.105}$$

we have,

$$\boxed{c = \sqrt{\frac{E_\nu|_s}{\rho}}}. \tag{1.106}$$

Hence, we observe that the speed of sound in a substance is related to the ratio of the substance's compressibility to its density.

- Recall that for an isentropic process involving an ideal gas: $E_\nu = \gamma p = \gamma \rho RT$, so that,

$$c = \sqrt{\gamma RT} \quad \text{(speed of sound for an ideal gas).} \tag{1.107}$$

Example:
Does air become more or less compressible, or does the compressibility remain unchanged, as the air temperature increases for isentropic conditions?

Solution:
From Eq. (1.103), $E_\nu|_s = \gamma \rho RT$. Thus, $E_\nu|_s \uparrow$ and $T \uparrow$, which means that the air becomes less compressible as the temperature increases.

What is the fractional change in density of water in the ocean over a depth of 1000 m (3281 ft)? The change in pressure over this depth, Δp, is $9.81 * 10^6$ Pa.

SOLUTION:

Assume that the bulk modulus for water remains constant at $E_v = 2.2 * 10^9$ Pa (found from a fluid properties table). Recall that the bulk modulus is defined as:

$$E_v = \frac{dp}{d\rho / \rho} \tag{1}$$

$$\int_{\rho_0}^{\rho_f} \frac{d\rho}{\rho} = \frac{1}{E_v} \int_{p_0}^{p_f} dp$$

$$\ln\left(\frac{\rho_f}{\rho_0}\right) = \frac{\left(p_f - p_0\right)}{E_v} = \frac{\Delta p}{E_v}$$

$$\frac{\rho_f}{\rho_0} = \exp\left(\frac{\Delta p}{E_v}\right) \tag{2}$$

Note that $\rho_f = \rho_0 + \Delta\rho$ so that:

$$\frac{\rho_f}{\rho_0} = \frac{\rho_0 + \Delta\rho}{\rho_0} = 1 + \frac{\Delta\rho}{\rho_0} \tag{3}$$

Substitute Eqn. (3) into Eqn. (2):

$$1 + \frac{\Delta\rho}{\rho_0} = \exp\left(\frac{\Delta p}{E_v}\right)$$

$$\boxed{\frac{\Delta\rho}{\rho_0} = \exp\left(\frac{\Delta p}{E_v}\right) - 1} \tag{4}$$

In the given problem:

$\Delta p = 9.81 * 10^6$ Pa

$E_v = 2.2 * 10^9$ Pa

$\Rightarrow \boxed{\Delta\rho / \rho_0 = 4.5 * 10^{-3} = 0.45\%}$ Water remains essentially incompressible over this depth!

Note that in most engineering problems (where pressure variations are not enormous), it is reasonable to assume that water is incompressible.

1.7.6. Surface tension, σ

- Surface tension is caused by unbalanced molecular forces occurring at the interface between a liquid and a solid surface, two dissimilar liquids, or between a liquid and a gas. For example, water "beads" up on a waxed surface because the inter-molecular forces between the water molecules is greater than the inter-molecular forces between the water and the wax.
- The dimensions of surface tension are F/L and common units are $\mathrm{N\,m^{-1}}$, $\mathrm{lb_f\,ft^{-1}}$, and dyne/cm where 1 dyne $= 1 \times 10^{-5}$ N.
- Examples where we observe surface tension include:
  - bubbles,
  - bugs walking on the surface of water, and
  - soda in a straw.
- Surface tension effects become more significant as length scales (L) decrease since,

$$F_{\text{surface tension}} \sim L \quad \text{(proportional to interface length)}, \tag{1.108}$$

$$F_{\text{inertia or weight}} \sim L^3 \quad \text{(proportional to volume)}. \tag{1.109}$$

  Taking the ratio of the two,

$$\frac{F_{\text{surface tension}}}{F_{\text{inertia or weight}}} \sim \frac{L}{L^3} = \frac{1}{L^2}. \tag{1.110}$$

  Thus, as L becomes smaller, surface tension forces begin to dominate over weight.
- The contact angle between a liquid and a surface is the angle the liquid surface makes at the contact point with the surface as shown in Figure 1.22. The contact angle is measured through the liquid.

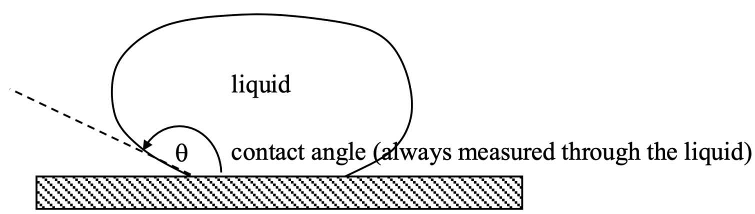


Figure 1.22. The geometry for evaluating the contact angle.

  - If $\theta > 90°$, then the surface is considered non-wetting. When water is the liquid, then a non-wetting surface is considered hydrophobic (Figure 1.23).
  - If $\theta < 90°$, then the surface is considered wetting. When water is the liquid, then a wetting surface is considered hydrophilic (Figure 1.24). Precise measurements of contact angle are difficult to make due to the sensitivity of the contact angle to surface chemical variations.

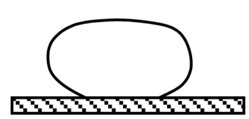


Figure 1.23. A non-wetting or hydrophobic contact angle.

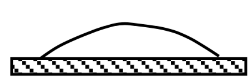


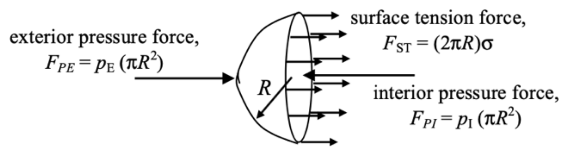
Figure 1.24. A wetting or hydrophilic contact angle.

Be sure to:

(1) Consider the effects of surface tension at fluid interfaces, especially at small length scales.

Determine the relationship between the surface tension in a soap bubble and the pressure difference between the inside and outside of the bubble.

SOLUTION:
Analyze the problem by cutting the spherical bubble with radius, *R*, in half. The free body diagram of the half-bubble is shown in the figure below. The forces acting on the bubble include the surface tension force holding the two bubble halves together, and the pressure force acting to push the two bubble halves apart.



$$\sum F = 0 = p_E(\pi R^2) + (2\pi R)\sigma - p_I(\pi R^2), \tag{1}$$

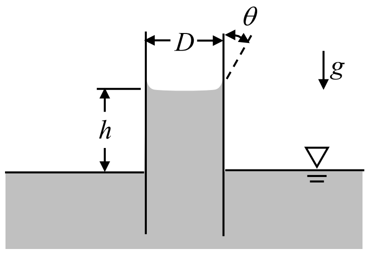
$$\boxed{p_I - p_E = \frac{2\sigma}{R}}. \tag{2}$$

The larger the surface tension or the smaller the bubble the radius, the larger the pressure difference between the bubble interior and the exterior.

A capillary tube of internal diameter 1 mm is placed vertically in a bucket of water. How high will the level in the capillary rise above the level in the bucket if the contact angle at the inner walls of the tube is 15º and the surface tension is 0.07 N/m? Consider a smaller capillary with the same contact angle and surface tension. If the water will vaporize below a pressure of 0.017 bar, what is the maximum capillary height which can, in principle, be achieved and what size of capillary is necessary to achieve this elevation?

SOLUTION:

Balance the vertical forces on the fluid in the tube shown below.



$$\sum F = 0 = F_S - F_W \tag{1}$$

where F_S is the surface tension force and F_W is the weight of the fluid in the tube. The surface tension force is:

$$F_S = \sigma \pi D \cos\theta \tag{2}$$

while the weight of the fluid in the tube is:

$$F_W = \rho \tfrac{\pi}{4} D^2 h g \tag{3}$$

Substitute and simplify.

$$0 = \sigma \pi D \cos\theta - \rho \tfrac{\pi}{4} D^2 h g \tag{4}$$

$$\boxed{\therefore h = \frac{4\sigma\cos\theta}{\rho g D}} \tag{5}$$

Using the given data:

σ = 0.07 N/m
θ = 15°
ρ = 1000 kg/m^3
g = 9.81 m/s^2
D = 10^{-3} m
$\boxed{\therefore h = \text{2.76 cm}}$

The height giving the minimum pressure may be found via manometry.

$$p_{\text{min}} = p_{\text{atm}} - \rho g h \tag{6}$$

$$\boxed{h_{\text{max}} = \frac{p_{\text{min}} - p_{\text{atm}}}{\rho g}} \tag{7}$$

Using the given data:

p_{min} = 0.017 bar = 0.017*10^5 Pa
p_{atm} = 1.01*10^5 Pa
ρ = 1000 kg/m^3
g = 9.81 m/s^2
$\boxed{\therefore h_{\text{max}} = \text{10.2 m}}$

Re-arrange Eqn. (5) to find the capillary tube diameter.

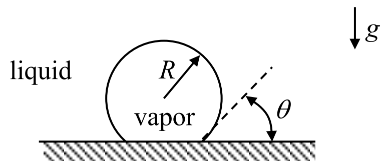
$$\boxed{D = \frac{4\sigma\cos\theta}{\rho g h}} \tag{8}$$

Using the given data:

σ = 0.07 N/m
θ = 15°
ρ = 1000 kg/m^3
g = 9.81 m/s^2
h = 10.2 m

$\boxed{\therefore D = 2.7\ \mu\text{m}}$

Vapor bubbles that form in the bottom of a pan of boiling water remain attached until they are of sufficient size so that the upward buoyant force acting on the bubble just overcomes the surface tension force at the contact line of their attachment to the bottom.



Find an expression for the radius, R, of a bubble rising through the liquid assuming that the attached bubble shape is a truncated sphere and that the line of attachment is a circle. Express your result in terms of the surface tension, σ, the contact angle, θ, the density of the liquid, ρ, and the acceleration due to gravity, g.

SOLUTION:

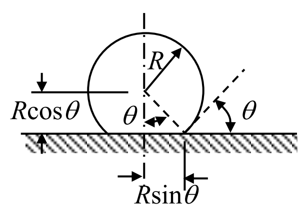
Balance forces acting on the bubble in the vertical direction:

$$\sum F = 0 = F_B - F_S \tag{1}$$

where the F_B and F_S are the buoyant and surface tension forces, respectively. The weight of the bubble is neglected since the density of the vapor is much smaller than the liquid density.

The buoyant force, F_B, is:

$$F_B = \rho V g \tag{2}$$



where the (truncated sphere) bubble volume is:

$$\begin{aligned} V &= \int_{z=-R\cos\theta}^{z=R} \pi r^2 dz = \int_{z=-R\cos\theta}^{z=R} \pi\left(R^2 - z^2\right) dz \\ &= \pi\left[R^3\left(1+\cos\theta\right) - \tfrac{1}{3}R^3\left(1+\cos^3\theta\right)\right] \\ &= \pi R^3\left[\tfrac{2}{3} + \cos\theta - \tfrac{1}{3}\cos^3\theta\right] \end{aligned} \tag{3}$$

and the surface tension force, F_S, is:

$$F_S = \sigma\left(2\pi R\sin\theta\right)\sin\theta = 2\pi\sigma R\sin^2\theta \tag{4}$$

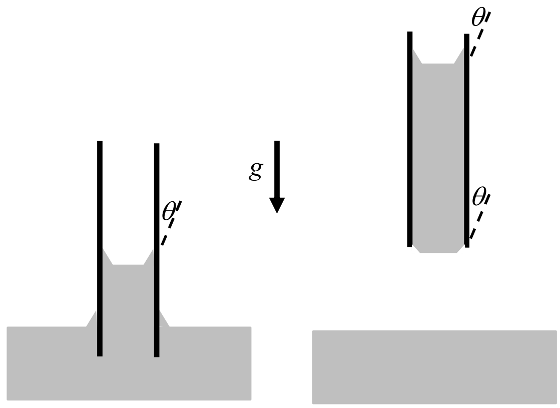
Substitute Eqns. (2) – (4) into Eqn. (1) and solve for R.

$$0 = \rho\pi R^3\left[\tfrac{2}{3} + \cos\theta - \tfrac{1}{3}\cos^3\theta\right]g - 2\pi R\sigma\sin^2\theta \tag{5}$$

$$\boxed{\therefore R = \sqrt{\frac{2\sigma\sin^2\theta}{\rho\left[\frac{2}{3} + \cos\theta - \frac{1}{3}\cos^3\theta\right]g}}} \tag{6}$$

Show that a tube containing liquid that is not submerged can support twice the fluid that a tube that is submerged can.

SOLUTION:



Balance forces on the liquid in each tube (Let y be the vertical direction):

$$\sum F_y = 0 = \underbrace{-\rho g \frac{\pi D^2}{4} H_1}_{\text{weight}} + \underbrace{\sigma \pi D \cos\theta}_{\substack{\text{surface tension}\\ \text{at top}}} \Rightarrow H_1 = \frac{4\sigma\cos\theta}{\rho g D}, \tag{1}$$

$$\sum F_y = 0 = \underbrace{-\rho g \frac{\pi D^2}{4} H_2}_{\text{weight}} + \underbrace{\sigma \pi D \cos\theta}_{\substack{\text{surface tension}\\ \text{at top}}} + \underbrace{\sigma \pi D \cos\theta}_{\substack{\text{surface tension}\\ \text{at bottom}}} \Rightarrow H_2 = \frac{8\sigma\cos\theta}{\rho g D} = 2H_1\,. \tag{2}$$

1.8. Flow Kinematics

A good reference on the experimental aspects of this topic is: Merzkirch, W., *Flow Visualization*, Academic Press.

In 1986, the Chernobyl Nuclear Power Plant in the Soviet Union released radioactive fallout into the atmosphere as a result of an explosion in one of the reactors (Figure 1.25). The radioactive plume covered regions of the western Soviet Union, Europe, and even parts of eastern North America. If such an accident were to happen again, how would you know which communities would be affected by the drifting radioactive cloud? If one had predictions of wind velocity measurements as a function of location and time, i.e., $\boldsymbol{u} = \boldsymbol{u}(x, t)$, could you figure out what areas would be covered by the cloud? In this section, we'll present three forms of flow kinematics: streamlines, pathlines, and streaklines. Each of these lines provides different information on the movement of fluid. For the toxic cloud release, one would be most interested in determining the streakline passing through the location of the damaged reactor.

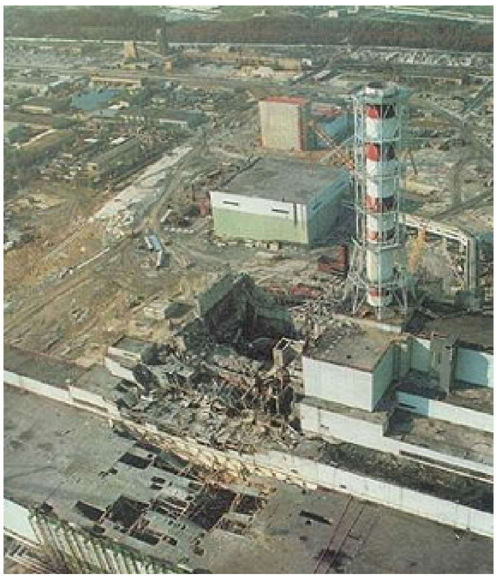


FIGURE 1.25. Photo of the damaged Chernobyl reactor. Photo from http://en.wikipedia.org/wiki/Chernobyl_disaster.

1.8.1. Streamlines

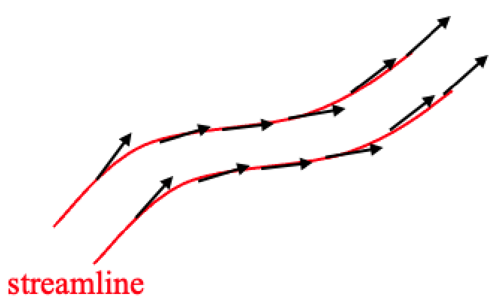


FIGURE 1.26. Illustration of streamlines. Streamlines are everywhere tangent to the velocity vectors.

A streamline is a line that is everywhere tangent to the velocity field vectors (Figure 1.26). Experimentally, one can visualize streamlines using the Particle Image Velocimetry (PIV) technique. In PIV, fluid particles are "tagged", usually by mixing in very small, neutrally buoyant bits of "paint," and taking two photographs in rapid succession. Velocity vectors can then be produced by "connecting the dots" (actually this method is technically Particle Tracking Velocimetry (PTV), but PIV operates in a similar manner, but matching up regions of particles rather than individual particles) (Figure 1.27).

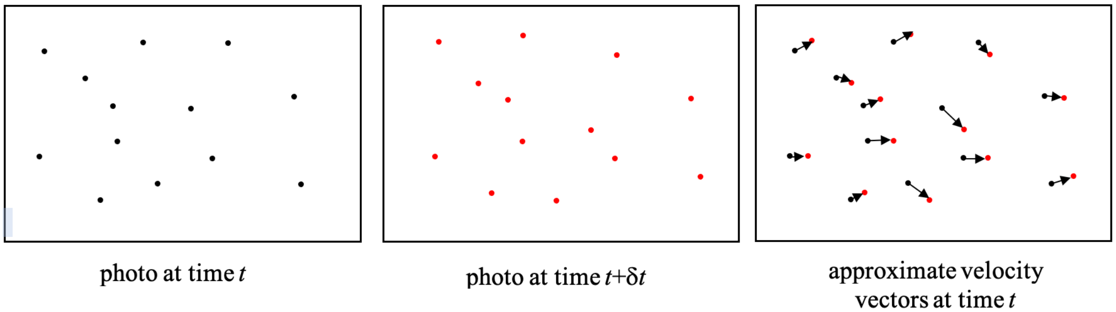


Figure 1.27. An illustration of the Particle Tracking Velocimetry technique for obtaining a velocity vector field. Particle Image Velocimetry works in a similar manner, but tracks regions of particles rather than individual particles.

Note that the approximate velocity vectors can be found using,

$$\dot{\boldsymbol{x}}(t) \approx \frac{\boldsymbol{x}(t+\delta t)-\boldsymbol{x}(t)}{\delta t}. \tag{1.111}$$

We can determine the equation of a streamline given a velocity field by simply using the definition of a streamline. Since the streamline is tangent to the velocity vector, the slope of the streamline will be equal to the slope of the velocity vector,

$$\underbrace{\frac{dy}{dx}}_{\substack{\text{slope of}\\ \text{streamline}}} = \underbrace{\frac{u_y}{u_x}}_{\substack{\text{slope of}\\ \text{velocity vector}}} \tag{1.112}$$

Similarly, in the x-z and y-z planes we have,

$$\frac{dz}{dx}=\frac{u_z}{u_x} \quad \frac{dz}{dy}=\frac{u_z}{u_y}. \tag{1.113}$$

We can combine these equations into a more compact form,

$$\boxed{\frac{dx}{u_x}=\frac{dy}{u_y}=\frac{dz}{u_z}}. \tag{1.114}$$

Notes:

(1) There is no flow across a streamline since the velocity component normal to the streamline is zero.
(2) A stream tube is a tube made by all the streamlines passing through a closed curve (Figure 1.28). There is no flow through a stream tube wall.
(3) A stream filament is a stream tube with infinitesimally small cross-sectional area.

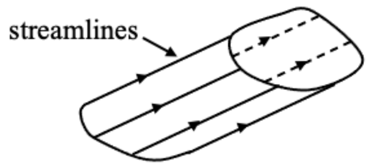


Figure 1.28. An illustration of a stream tube.

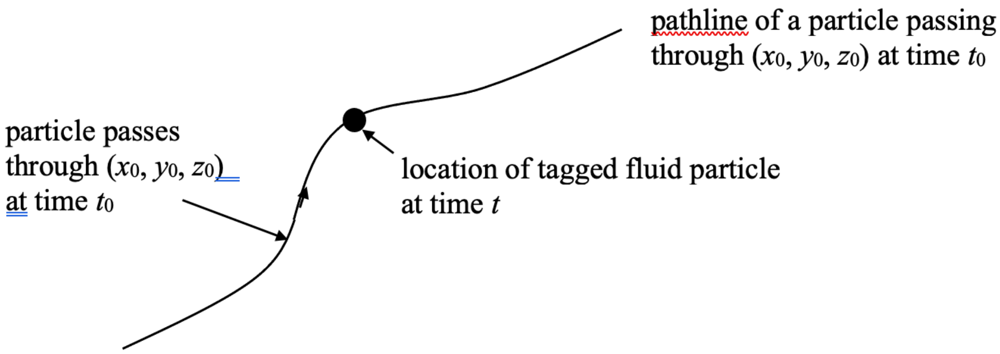


Figure 1.29. An illustration of a pathline.

1.8.2. Pathlines

A pathline is the line traced out by a particular particle as it moves from one point to another (Figure 1.29). It is the actual path a particle takes. Experimentally, a pathline can be visualized by "tagging" a particular fluid particle and taking a long-exposure photograph of the particle's motion.

To determine the equation of a pathline at time, t, for a particle passing through the point (x_0, y_0, z_0) at some previous time t_0, we solve the differential equation describing the particle's position,

$$\boldsymbol{u} = \frac{d\boldsymbol{x}}{dt}, \tag{1.115}$$

where $\boldsymbol{u}$ is the velocity field subject to the initial condition that the particle passes through the point $\boldsymbol{x}_0 = (x_0, y_0, z_0)$ at time, t_0,

$$\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0. \tag{1.116}$$

For a pathline t_0 will be a particular value. The solution of Eq. (1.115) subject to the initial condition in Eq. (1.116) will consist of a set of parametric equations in t (by varying t, we can trace out the location of the particle at various times).

1.8.3. Streaklines

A streakline is a line that connects all fluid particles that have passed through the same point in space at a previous (or later) time. Experimentally a streakline can be visualized by injecting dye into a fluid flow at a particular point (Figure 1.30).

To determine the equation of a streakline at time, t, passing through the point (x_0, y_0, z_0), we solve the differential equation describing a particle's position as a function of time,

$$\boldsymbol{u} = \frac{d\boldsymbol{x}}{dt}, \tag{1.117}$$

where $\boldsymbol{u}$ is the velocity field subject to the initial condition that a particle pass through the point $\boldsymbol{x}_0 = (x_0, y_0, z_0)$ at some previous (or later) time, t_0,

$$\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0. \tag{1.118}$$

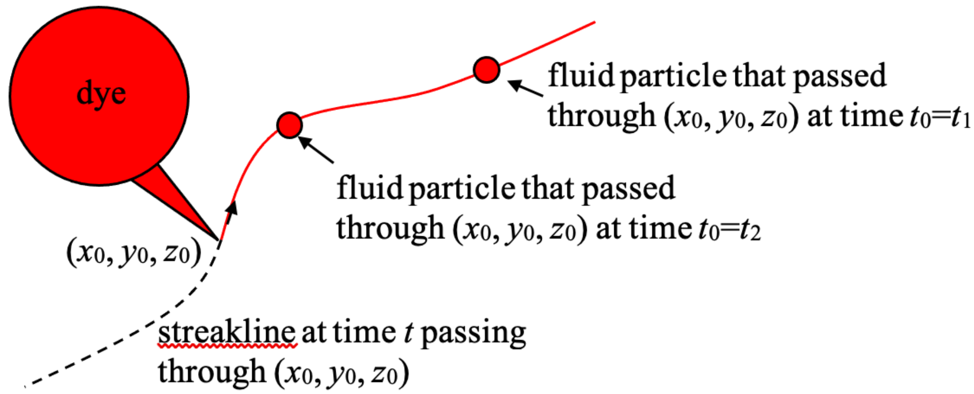


Figure 1.30. An illustration of a streakline.

Note that t_0 will be different for each particle, i.e., it varies, unlike a pathline where t_0 is fixed. The solution of Eq. (1.117) subject to the initial condition in Eq. (1.118) will consist of a set of parametric equations in t_0 (note that t is a known value since we want to know the streakline at a particular time, t).

Notes:

(1) The streamline, streakline, and pathline passing through a particular location can be different in an unsteady flow, but will be identical in a steady flow.
(2) The quantity t_0 is the time when a fluid particle passes through the point $\boldsymbol{x}_0$. Hence, for a pathline t_0 is fixed since there is only one fluid particle. However, for a streakline t_0 varies since there are many fluid particles passing through the point $\boldsymbol{x}_0$, each at a different t_0.
(3) Why don't we use a Lagrangian derivative (covered in Chapter 5) when solving Eq. (1.115) for a particle's pathline (since the pathline is a Lagrangian concept)? It turns out that the Lagrangian derivative of a particle's position is equal to its Eulerian derivative. Consider, for example, the change in the x position of the particle as we follow it. Note that the position, $\boldsymbol{x}$, is an Eulerian quantity,
$$\frac{Dx}{Dt} = \frac{\partial x}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})x = \underbrace{\frac{\partial x}{\partial t}}_{=0} + u_x\underbrace{\frac{\partial x}{\partial x}}_{=1} + u_y\underbrace{\frac{\partial x}{\partial y}}_{=0} + u_z\underbrace{\frac{\partial x}{\partial z}}_{=0} = u_x. \tag{1.119}$$

Be sure to:

(1) Understand the definitions for streamlines, streaklines, and pathlines.
(2) Understand what initial conditions to use when evaluating streaklines and pathlines.
(3) Draw the direction of flow on the streamlines, streaklines, and pathlines.
(4) It's perfectly correct to represent the position of a fluid particle parametrically, i.e., $x = x(t)$ and $y = y(t)$.

One technique for visualizing fluid flow over a surface is to attach short, lightweight pieces of thread or "tufts" to the surface. A photograph of the tufts on the surface of an automobile is shown in the figure below. Do tufts trace out the streamlines, streaklines, pathlines, or some other type of flow line? What if the flow is unsteady? Explain your answers.

SOLUTION:

The tufts show the local direction of the fluid velocity and, hence, are indicators of the local streamline slope.

If the flow is steady, then the streamlines, streaklines, and pathlines are all the same. If the flow is unsteady, then the tufts will, in general, only indicate the local slope of the streamlines.

Video solution: https://www.youtube.com/watch?v=2PMolyW39vI

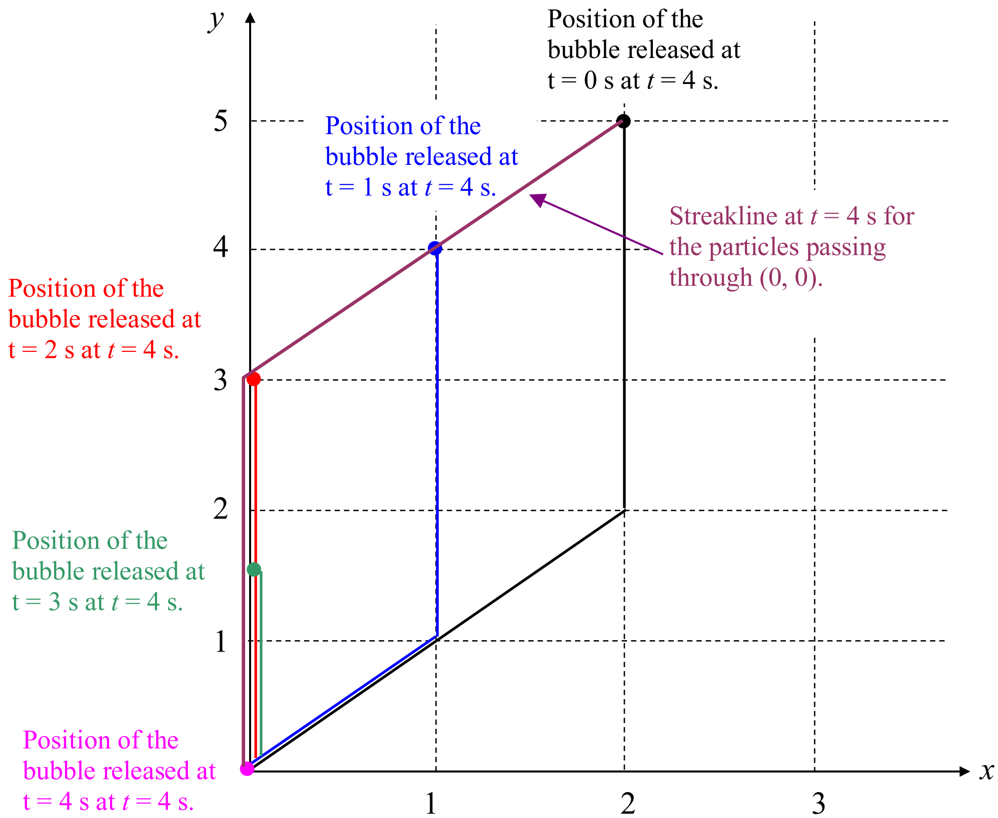
Tiny hydrogen bubbles are being used as tracers to visualize a flow. All the bubbles are generated at the origin ($x = 0$, $y = 0$). The velocity field is unsteady and obeys the equations:

$$u = 1 \text{ m/s} \qquad v = 1 \text{ m/s} \qquad 0 \le t < 2 \text{ s}$$
$$u = 0 \qquad v = 1.5 \text{ m/s} \qquad 2 \le t \le 4 \text{ s}$$

Plot the pathlines of bubbles that leave the origin at t = 0, 1, 2, 3, and 4 s. Mark the locations of these five bubbles at t = 4 s. Use a dashed line to indicate the position of the streakline passing through (0, 0) at t = 4 s. What does the streamline passing through (0, 0) look like at t = 4 s?

SOLUTION:

One could solve the differential equations describing the particle pathlines and streakline using the velocities given above, or, more easily, simply plot the positions of the fluid particles at different times. The plot below shows the particle positions, pathlines, and streakline.



The streamline passing through (0, 0) at t = 4 s (or any other point for that matter) will be a vertical line since the velocity at t = 4 s is purely vertical.

The two-dimensional velocity field for an unsteady flow is given by,

$$\boldsymbol{u} = \begin{cases} \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} & 0 \le t < 1\text{ s} \\ \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} & 1\text{ s} \le t < 2\text{ s} \\ \hat{\boldsymbol{\imath}} & 2\text{ s} \le t < 3\text{ s} \end{cases}$$

a. Write an equation for the streamline passing through the point (x, y) = (1, 1) for $0 \le t \le 3$ s.
b. Sketch the pathline for a fluid particle released from the origin at t = 0 s for $0 \le t \le 3$ s.
c. Sketch the streakline through the point (x, y) = (1, 1) at t = 3 s.

SOLUTION:
The slope of a streamline at a point is tangent to the velocity vector at that same point,

$$\frac{dy}{dx} = \frac{u_y}{u_x} = \begin{cases} 1 & 0 \le t < 1\text{ s} \\ -1 & 1\text{ s} \le t < 2\text{ s}, \\ 0 & 2\text{ s} \le t < 3\text{ s} \end{cases} \tag{1}$$

$$\begin{aligned} \int_{y_0}^{y} dy &= \int_{x_0}^{x} dx & 0 \le t < 1\text{ s} \\ \int_{y_0}^{y} dy &= -\int_{x_0}^{x} dx & 1\text{ s} \le t < 2\text{ s}, \\ \int_{y_0}^{y} dy &= 0 & 2\text{ s} \le t < 3\text{ s} \end{aligned} \tag{2}$$

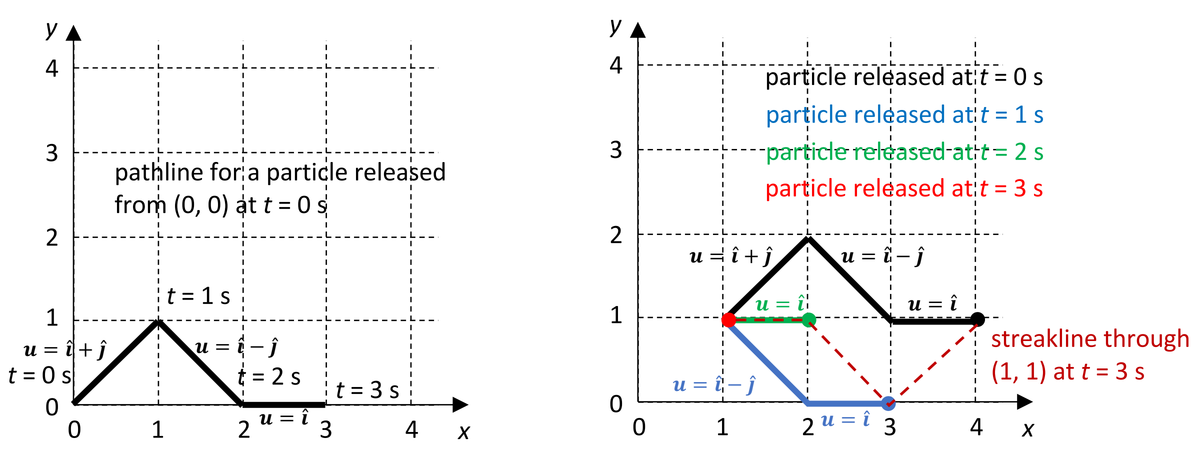
$$\begin{aligned} y - y_0 &= x - x_0 & 0 \le t < 1\text{ s} \\ y - y_0 &= x_0 - x & 1\text{ s} \le t < 2\text{ s}, \\ y - y_0 &= 0 & 2\text{ s} \le t < 3\text{ s} \end{aligned} \tag{3}$$

Using (x_0, y_0) = (1, 1),

$$\begin{aligned} y - 1 &= x - 1 & 0 \le t < 1\text{ s} \\ y - 1 &= 1 - x & 1\text{ s} \le t < 2\text{ s}, \\ y - 1 &= 0 & 2\text{ s} \le t < 3\text{ s} \end{aligned} \tag{4}$$

$$\boxed{\begin{aligned} y &= x & 0 \le t < 1\text{ s} \\ y &= 2 - x & 1\text{ s} \le t < 2\text{ s} \\ y &= 1 & 2\text{ s} \le t < 3\text{ s} \end{aligned}}, \tag{5}$$

Sketches are shown below for (a) the pathline of a fluid particle released from the origin (x_0, y_0) = (0, 0) at t = 0 s, and (b) the streakline through the point (x_0, y_0) = (1, 1) at t = 3 s.



Show that for a steady flow, streamlines, streaklines, and pathlines are identical.

SOLUTION:

Streamlines are defined as lines that are everywhere tangent to the instantaneous velocity vectors. (The rest of the problem will be worked out in Cartesian coordinates for convenience.)

$$\begin{aligned}\frac{dy}{dx} &= \frac{u_y}{u_x} \Rightarrow \frac{dy}{u_y} = \frac{dx}{u_x} \\ \frac{dz}{dx} &= \frac{u_z}{u_x} \Rightarrow \frac{dz}{u_z} = \frac{dx}{u_x} \\ \frac{dz}{dy} &= \frac{u_z}{u_y} \Rightarrow \frac{dz}{u_z} = \frac{dy}{u_y}\end{aligned} \tag{1}$$

Re-writing:

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z} \tag{2}$$

where $\mathbf{u}$ is not a function of time since the flow is assumed steady but is, in general, a function of position, *i.e.* $\mathbf{u} = \mathbf{u}(\mathbf{x})$.

Streaklines are lines connecting all fluid particles that pass through the same point in space.

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} \quad \text{where } \mathbf{x}\left(t = t_0\right) = \mathbf{x}_0 \tag{3}$$

where t_0 is the time at which a fluid particle on the streamline passes through the point $\mathbf{x}_0$ on the streakline. Note that t_0 will be different for each fluid particle on a given streakline.

Pathlines trace the motion of individual fluid particles over time.

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} \quad \text{where } \mathbf{x}\left(t = t_0\right) = \mathbf{x}_0 \tag{4}$$

where t_0 is the time at which a fluid particle passes through the point $\mathbf{x}_0$ on the pathline. Note that t_0 is a fixed quantity for a given pathline.

We can re-write the differential equations for the streakline and pathline as:

$$\begin{aligned}u_x &= \frac{dx}{dt} \Rightarrow \frac{dx}{u_x} = dt \\ u_y &= \frac{dy}{dt} \Rightarrow \frac{dy}{u_y} = dt \\ u_z &= \frac{dz}{dt} \Rightarrow \frac{dz}{u_z} = dt\end{aligned} \tag{5}$$

Note that $\mathbf{u}$ is not a function of t (steady flow $\Rightarrow \mathbf{u} = \mathbf{u}(\mathbf{x})$) so that we needn't worry about how the slope of the lines change with time. Thus, we can write:

$$\therefore \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z} \tag{6}$$

Since Eqns. (6) and (2) are identical, we can conclude that streamlines, streaklines, and pathlines are identical for a steady flow.

A velocity field is given by:

$$\mathbf{u} = \frac{-V_0 y}{(x^2+y^2)^{1/2}}\hat{\mathbf{i}} + \frac{V_0 x}{(x^2+y^2)^{1/2}}\hat{\mathbf{j}}$$

where V_0 is a positive constant, *i.e.* $V_0 > 0$. Determine:

a. where in the flow the speed is V_0
b. the equation and sketch of the streamlines
c. the equations for the streaklines and pathlines

SOLUTION:

The speed is given by:

$$|\mathbf{u}| = \sqrt{u_x^2 + u_y^2} \tag{1}$$

where

$$u_x = \frac{-V_0 y}{(x^2+y^2)^{1/2}} \tag{2}$$

$$u_y = \frac{V_0 x}{(x^2+y^2)^{1/2}} \tag{3}$$

Substituting into Eqn. (1) gives:

$$|\mathbf{u}| = \sqrt{\frac{V_0^2 y^2}{(x^2+y^2)} + \frac{V_0^2 x^2}{(x^2+y^2)}}$$

$$\boxed{\therefore |\mathbf{u}| = V_0} \tag{4}$$

The flow speed is everywhere equal to V_0.

The slope of the streamline is tangent to the slope of the velocity vector:

$$\frac{dy}{dx} = \frac{u_y}{u_x} \tag{5}$$

Substitute Eqns. (2) and (3) and solving the resulting differential equation.

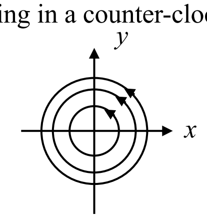
$$\frac{dy}{dx} = \frac{\dfrac{V_0 x}{(x^2+y^2)^{1/2}}}{\dfrac{-V_0 y}{(x^2+y^2)^{1/2}}} = \frac{x}{-y}$$

$$-\int_{y_0}^{y} y\,dy = \int_{x_0}^{x} x\,dx \quad \text{(where } (x_0, y_0) \text{ is a point located on the streamline)}$$

$$-\tfrac{1}{2}\left(y^2 - y_0^2\right) = \tfrac{1}{2}\left(x^2 - x_0^2\right)$$

$$\boxed{x^2 + y^2 = x_0^2 + y_0^2 = \text{constant}} \tag{6}$$

The streamlines are circles! Note that when $x > 0$ and $y > 0$, Eqns. (2) and (3) indicate that $u_x < 0$ and $u_y > 0$ (note that $V_0 > 0$) so that the flow is moving in a counter-clockwise direction.



Since the flow is steady, the streaklines and pathlines will be identical to the streamlines.

Consider a 2D flow with a velocity field given by:

$$\mathbf{u} = x(1+2t)\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Determine the equations for the streamline, streakline, and pathline passing through the point $(x,y)=(1,1)$ at time $t=0$.

SOLUTION:

The slope of a streamline is tangent to the velocity vector.

$$\frac{dy}{dx} = \frac{u_y}{u_x} \tag{1}$$

where

$$u_x = x(1+2t) \tag{2}$$

$$u_y = y \tag{3}$$

Substitute Eqns. (2) and (3) into Eqn. (1) and solve the resulting differential equation.

$$\frac{dy}{dx} = \frac{y}{x(1+2t)}$$

$$(1+2t)\int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} \frac{dx}{x} \quad \text{(where } (x_0, y_0) \text{ is a point passing through the streamline)}$$

$$(1+2t)\ln\left(\frac{y}{y_0}\right) = \ln\left(\frac{x}{x_0}\right)$$

$$\left(\frac{y}{y_0}\right)^{(1+2t)} = \left(\frac{x}{x_0}\right) \tag{4}$$

For the streamline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$:

$$\boxed{y = x} \tag{5}$$

A streakline is a line that connects all of the fluid particles that pass through the same point in space. The equation for the streakline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = x(1+2t) \tag{6}$$

$$u_y = \frac{dy}{dt} = y \tag{7}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} (1+2t)dt \quad \Rightarrow \quad \ln\left(\frac{x}{x_0}\right) = t + t^2 - t_0 - t_0^2 \tag{8}$$

$$\int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} dt \quad \Rightarrow \quad \ln\left(\frac{y}{y_0}\right) = t - t_0 \tag{9}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the streakline. Hence, the streakline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$ is given parametrically (in t_0) as:

$$\ln(x) = -t_0 - t_0^2 \quad \Rightarrow \quad \boxed{x = \exp\left(-t_0 - t_0^2\right)} \tag{10}$$

$$\ln(y) = -t_0 \quad \Rightarrow \quad \boxed{y = \exp\left(-t_0\right)} \tag{11}$$

Recall that t_0 is the time when a fluid particle passes through the point (x_0, y_0).

Video solution: https://www.youtube.com/watch?v=LnclBvW5wvc

A pathline is a line traced out by a particular fluid particle as it moves through space. The equation for the pathline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = x(1+2t) \tag{12}$$

$$u_y = \frac{dy}{dt} = y \tag{13}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} (1+2t)dt \quad \Rightarrow \quad \ln\left(\frac{x}{x_0}\right) = t + t^2 - t_0 - t_0^2 \tag{14}$$

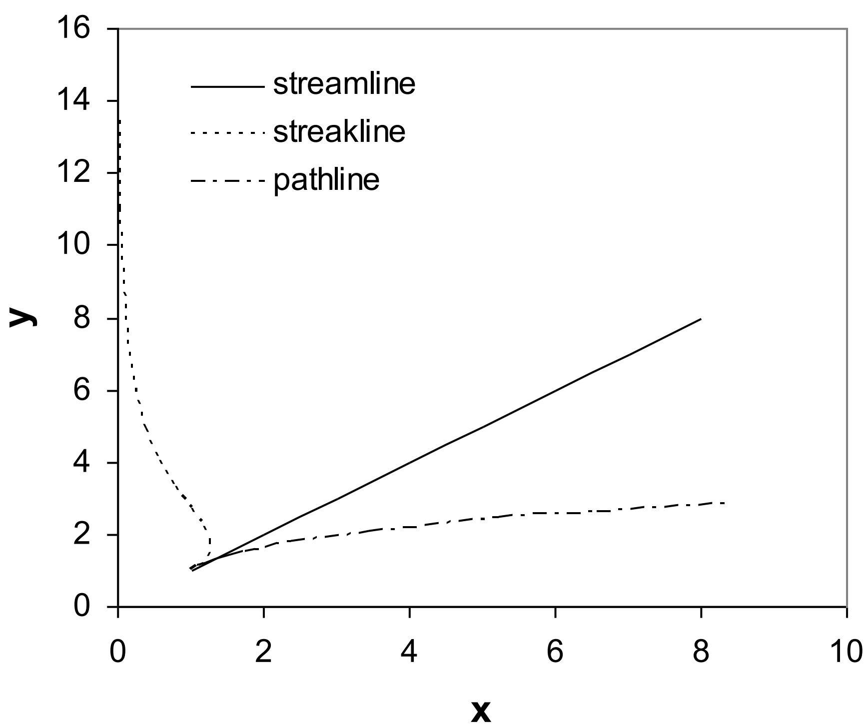
$$\int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} dt \quad \Rightarrow \quad \ln\left(\frac{y}{y_0}\right) = t - t_0 \tag{15}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the pathline. Hence, the pathline for a particle passing through the point (x_0, y_0) = (1, 1) at time t_0 = 0 is given parametrically (in t) as:

$$\ln(x) = t + t^2 \quad \Rightarrow \quad \boxed{x = \exp\left(t + t^2\right)} \tag{16}$$

$$\ln(y) = t \quad \Rightarrow \quad \boxed{y = \exp(t)} \tag{17}$$

Note that the streamline, streakline, and pathline are all different. A plot of these lines through (1, 1) at t = 0 is shown below.



A tornado can be represented in polar coordinates by the velocity field,

$$\mathbf{u} = -\frac{a}{r}\hat{\mathbf{e}}_r + \frac{b}{r}\hat{\mathbf{e}}_\theta$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are unit vectors pointing in the radial (r) and tangential (θ) directions, respectively, and a and b are constants. Show that the streamlines for this flow form logarithmic spirals, *i.e.*

$$r = c\exp\left(-\frac{a}{b}\theta\right)$$

where c is a constant.

SOLUTION:

The slope of a streamline is tangent to the velocity vector. In polar coordinates, the streamline slope is given by:

$$\frac{\text{small displacement in } r\text{-direction}}{\text{small displacement in } \theta\text{-direction}} = \frac{dr}{rd\theta} \tag{1}$$

so that the relation describing the streamline slope is:

$$\frac{dr}{rd\theta} = \frac{u_r}{u_\theta} \tag{2}$$

where

$$u_r = -\frac{a}{r} \tag{3}$$

$$u_\theta = \frac{b}{r} \tag{4}$$

Substitute Eqns. (3) and (4) into Eqn. (2) and solve the resulting differential equation.

$$\frac{dr}{rd\theta} = \frac{-a/r}{b/r} = -\frac{a}{b}$$

$$\int_{r_0}^{r}\frac{dr}{r} = -\frac{a}{b}\int_{\theta_0}^{\theta}d\theta$$

$$\ln\left(\frac{r}{r_0}\right) = -\frac{a}{b}(\theta-\theta_0)$$

$$\frac{r}{r_0} = \exp\left[-\frac{a}{b}(\theta-\theta_0)\right]$$

$$\boxed{\therefore r = c\exp\left[-\frac{a}{b}\theta\right]} \tag{5}$$

where the constants r_0 and θ_0 have been incorporated into the constant c.

Consider the 2D flow field defined by the following velocity:

$$\mathbf{u} = \left(\frac{1}{1+t}\right)\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

For this flow field, find the equation of:

a. the streamline through the point (1,1) at $t = 0$,
b. the pathline for a particle released at the point (1,1) at $t = 0$, and
c. the streakline at $t = 0$ which passes through the point (1,1).

SOLUTION:

The slope of a streamline is tangent to the velocity vector.

$$\frac{dy}{dx} = \frac{u_y}{u_x} \tag{1}$$

where

$$u_x = \frac{1}{1+t} \tag{2}$$

$$u_y = 1 \tag{3}$$

Substitute Eqns. (2) and (3) into Eqn. (1) and solve the resulting differential equation.

$$\frac{dy}{dx} = \frac{1}{1/(1+t)} = 1+t$$

$$\int_{y_0}^{y} dy = (1+t)\int_{x_0}^{x} dx \quad \text{(where } (x_0, y_0) \text{ is a point passing through the streamline)}$$

$$y - y_0 = (1+t)(x - x_0) \tag{4}$$

For the streamline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$:

$$\boxed{y = x} \tag{5}$$

A streakline is a line that connects all of the fluid particles that pass through the same point in space. The equation for the streakline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = \frac{1}{1+t} \tag{6}$$

$$u_y = \frac{dy}{dt} = 1 \tag{7}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} \frac{dt}{1+t} \quad \Rightarrow \quad x - x_0 = \ln\left(\frac{1+t}{1+t_0}\right) \tag{8}$$

$$\int_{y_0}^{y} dy = \int_{t_0}^{t} dt \quad \Rightarrow \quad y - y_0 = t - t_0 \tag{9}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the streakline. Hence, the streakline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$ is given parametrically (in t_0) as:

$$x - 1 = \ln\left(\frac{1}{1+t_0}\right) \quad \Rightarrow \quad \boxed{x = \ln\left(\frac{1}{1+t_0}\right) + 1} \tag{10}$$

$$y - 1 = -t_0 \quad \Rightarrow \quad \boxed{y = 1 - t_0} \tag{11}$$

Recall that t_0 is the time when a fluid particle passes through the point (x_0, y_0).

A pathline is a line traced out by a particular fluid particle as it moves through space. The equation for the pathline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = \frac{1}{1+t} \tag{12}$$

$$u_y = \frac{dy}{dt} = 1 \tag{13}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} \frac{dt}{1+t} \quad \Rightarrow \quad x - x_0 = \ln\left(\frac{1+t}{1+t_0}\right) \tag{14}$$

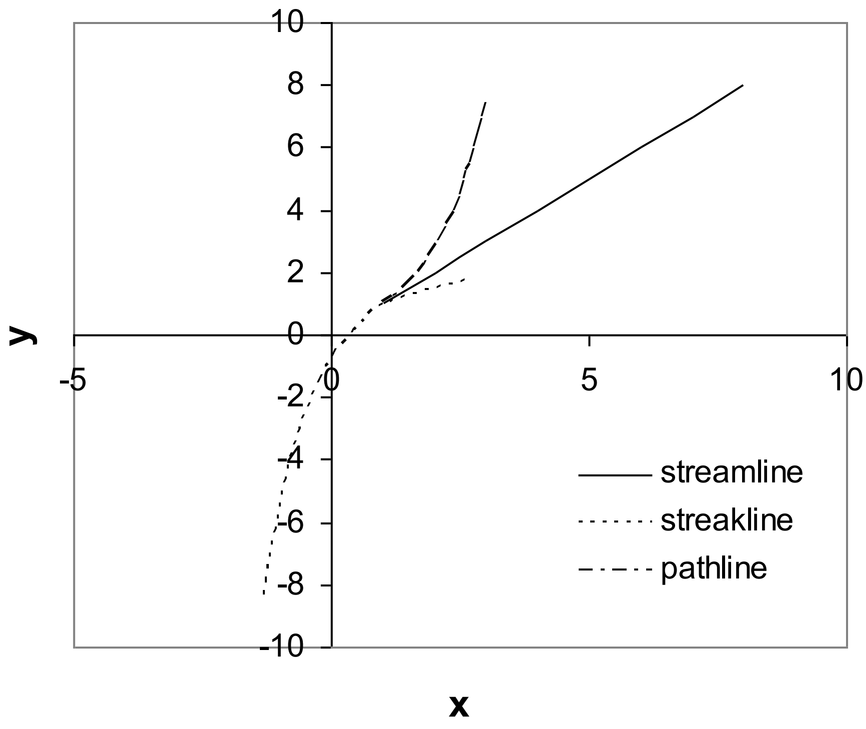
$$\int_{y_0}^{y} dy = \int_{t_0}^{t} dt \quad \Rightarrow \quad y - y_0 = t - t_0 \tag{15}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the pathline. Hence, the pathline for a particle passing through the point $(x_0, y_0) = (1, 1)$ at time $t_0 = 0$ is given parametrically (in t) as:

$$x - 1 = \ln(1+t) \quad \Rightarrow \quad \boxed{x = \ln(1+t) + 1} \tag{16}$$

$$y - 1 = t \quad \Rightarrow \quad \boxed{y = 1 + t} \tag{17}$$

Note that the streamline, streakline, and pathline are all different. A plot of these lines through (1, 1) at t = 0 is shown below.



1.9. Some Basic Definitions and Concepts

Before we begin to study the behavior of fluids, we need to first define a few commonly used terms.

- A fluid is a substance that deforms continuously when subject to non-isotropic stresses.

  Notes:
  (1) An isotropic stress state is one in which the stresses are the same in all directions. A non-isotropic stress state means that the stresses are unbalanced.
  (2) Liquids and gases are considered fluids. Their properties are different (e.g., the density of air at standard conditions is 1.23 kg/m^3 while the density of water at standard conditions is 1000 kg/m^3), but their bulk motion is similar.
  (3) Solids will not deform continuously when subject to unbalanced forces. They will deform up until a point and then resist further deformation (e.g., behavior similar to that of a spring).
  (4) Some substances are difficult to classify, e.g., slurries and granular materials. The study of material behavior (deformation and flow) is known as rheology. For additional information, refer to Steffe, J.F., *Rheological Methods in Food Process Engineering*, Freeman Press and Bird, R.B., Armstrong, R.C., and Hassager, O., *Dynamics of Polymeric Liquids*, Wiley.
- A system is a particular quantity of matter chosen for study. The surroundings include everything that is not the system.
- A control volume (CV) is a particular volume or region in space. A control surface (CS) is the surface enclosing the control volume. The orientation of the CS at a particular location is given by the direction of its outward-pointing unit normal vector, $\hat{\boldsymbol{n}}$, at that location. The outward-pointing unit normal vector has a magnitude of one, is perpendicular to the control surface, and always points out of the CV (Figure 1.31).

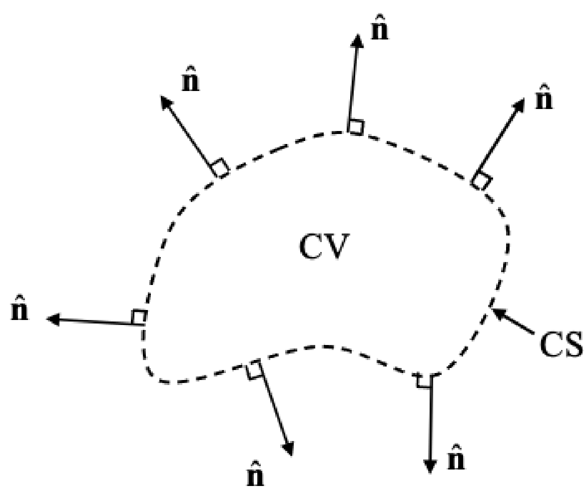


Figure 1.31. An illustration showing the unit outward-pointing normal vectors on a control surface at different locations.

- Properties are macroscopic characteristics of a system. An extensive property is one that depends on the amount of mass in the system. An intensive property is one that is independent of the mass in the system. A specific property is an extensive property per unit mass (a specific property is also an intensive property). Some examples: mass, m, is an extensive property, pressure, p, is an intensive property, and specific volume, $v := V/m$, is a specific property. An easy way to determine whether a property is extensive or intensive is to divide the system into two parts and see how the property is affected. If the property value changes, then it's an extensive property. If the property value remains the same, then it's an intensive property.

- The state of a system is the system's condition or configuration as described by its properties in sufficient detail so that it is distinguishable from other states. Often a state can be described by a subset of the system's properties since the properties themselves may be related.
- A field representation of a quantity gives the value of that quantity at all locations and times, i.e., the field representation of the density, ρ, is,

$$\rho = \rho(x, y, z, t) \tag{1.120}$$

A scalar is a quantity that has only a magnitude, e.g., temperature, T. A vector is a quantity that has both magnitude and direction, e.g., velocity, $\boldsymbol{u}$.
- Steady-state conditions are conditions for which properties at each location do not change with time, i.e.,

$$\frac{\partial}{\partial t}(\dots) = 0. \tag{1.121}$$

Unsteady conditions are conditions for which properties do change with time. A steady flow is a flow with properties that do not change with time.
- The dimension of a flow is equal to the number of spatial coordinates required to describe the flow. For example:
  - 0D flow: $u = u(t)$ or $u =$ constant
  - 1D flow: $u = u(t; x)$ or $u = u(t; x)$
  - 2D flow: $u = u(t; r, \theta)$ or $u = u(t; r, \theta)$
  - 3D flow: $u = u(t; x, y, z)$ or $u = u(x, y, z)$

A flow is uniform if it does not vary in a spatial direction. A flow is non-uniform if it does vary in that spatial direction. For example, the velocity profile in the figure below at left is non-uniform in the y-direction. The figure at the right is uniform in the y-direction.

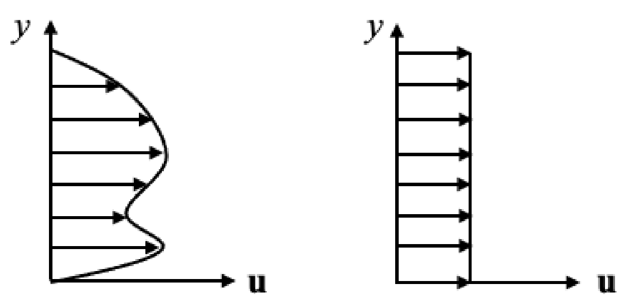


Figure 1.32. An illustration showing a non-uniform velocity field (left) and a uniform velocity field (right).

Example:
The velocity field for a flow is $\boldsymbol{u} = xy\hat{\boldsymbol{\imath}} - \frac{1}{2}y^2\hat{\boldsymbol{\jmath}} + xt\hat{\boldsymbol{k}}$. How many dimensions is this flow? Is this flow steady or unsteady?
Solution:
The flow is 2D since two spatial dimensions (x and y) are required to describe the velocity field. The flow is unsteady since it is a function of time.

- A fully-developed flow is a flow in which the velocity no longer varies in a particular direction. For example, a fully-developed, laminar flow of a Newtonian fluid through a circular pipe (Figure 1.33) has the velocity profile,

$$u_z(r) = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1 - \left(\frac{r}{R}\right)^2\right]. \tag{1.122}$$

This velocity profile is fully-developed in the z direction since the velocity profile doesn't change as we move in the z direction.

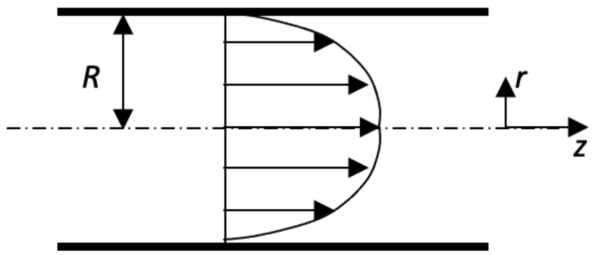


Figure 1.33. The velocity profile for laminar, fully-developed flow in a circular pipe. The flow is fully-developed in the z direction since the velocity profile doesn't vary in the z direction.

- A laminar flow is one in which the fluid flows in layers (or lamina). There is little macroscopic mixing between fluid layers. A turbulent flow is one in which the fluid velocity varies in time and space in a chaotic manner. There is significant macroscopic mixing in a turbulent flow. A photograph of laminar and turbulent flow is shown in Figure 1.34. A transitional flow has characteristics of both laminar and turbulent flow. The flow may be laminar for a period of time, but then randomly transition to turbulence for some time, then transition back to laminar flow.

Figure 1.34. A photograph of the plume from a candle. The flow leaving the candle is initially laminar, but transitions to turbulence some distance above the candle. Image from https://en.wikipedia.org/wiki/File:Laminar-turbulent_transition.jpg.

Be sure to:

(1) Understand the previous definitions since they are used frequently in the discussion of fluid mechanics.

1.10. Review Questions

(1) What is the conversion between lb_m and slug? What is the conversion between lb_f and lb_m? Estimate, without a calculator, what 20 °C is in °F.
(2) What are the dimensions (in M, L, T) of power?
(3) Do you understand the physical concept behind the Taylor series approximation given below?

$$y(x+dx) = y(x) + \left.\frac{dy}{dx}\right|_x (dx) \tag{1.123}$$

Can you draw a sketch corresponding to your explanation?
(4) What is meant by "least count"? Give an example of a situation where the least count for a measurement may be different for two experimenters using the same equipment.
(5) Set up and work out an example concerning propagation of experimental uncertainty.
(6) What is the difference between relative and absolute uncertainty?
(7) Why should uncertainty be included in experimental measurements?
(8) Under what conditions can a fluid be considered a continuum?
(9) Give an example of where the continuum approximation is not appropriate.
(10) What is the no-slip condition?
(11) Discuss the definition and meaning of the Knudsen number.
(12) What is meant (in words) by a "Newtonian fluid"? Write the shear stress-shear strain rate relationship for a Newtonian fluid. Is this the shear stress that acts on the fluid or the stress the fluid exerts on its neighbors?
(13) Give an example of a shear thickening fluid.
(14) How does viscosity vary with temperature for liquids? with pressure?
(15) What is the difference between "dynamic" and "kinematic" viscosities?
(16) What is meant by an "inviscid" fluid?
(17) What is meant by an "incompressible" fluid? Under what conditions can a flow be considered incompressible?
(18) What is meant by an "ideal" fluid?
(19) What are the pressure, temperature, and density at standard conditions?
(20) What is the density of air at standard conditions (in kg/m^3)? What is the approximate density of water under ordinary conditions (in kg/m^3)? What is the specific gravity of mercury?
(21) What is the difference between absolute and gage pressures?
(22) An engineer wants to minimize the resistance on a block sliding over a thin film of liquid. Would it be better to increase the thickness of the film, decrease the thickness, or does it matter? Explain your reasoning in words and provide analytical support for your argument (i.e., supporting equations).
(23) Estimate the absolute and kinematic viscosities of water under ordinary conditions.
(24) What is meant by a Couette flow? Under what geometric conditions would it be reasonable to apply the Couette flow assumption to flow between two rotating concentric cylinders?
(25) What is cavitation? Can cavitation be a problem in a pipeline containing steam? Give some examples of where cavitation occurs. Will a submarine propeller be more likely to cavitate near the ocean surface or deep in the ocean? Why?
(26) How do you expect the speed of sound in water containing small air bubbles to compare with the speed of sound in water and the speed of sound in air? (Hint: Consider the mixture's bulk modulus and density to the pure substance values.)
(27) What is the approximate speed of sound in air at standard conditions?
(28) Which bubble has greater internal pressure: one with a smaller radius or one with a larger radius?
(29) What are the dimensions of surface tension?
(30) Draw a picture defining the contact angle measurement.
(31) Describe, in words and equations, what is meant by a streamline, streakline, and pathline.
(32) Under what conditions will a streamline, streakline, and pathline be identical, in general?
(33) Experimentally, how would one produce a streamline, streakline, and pathline?
(34) Describe analytically the difference between a streakline and pathline.

(35) What is the definition of a fluid? What is the fundamental difference between fluids and solids? Give an example of a substance that has similarities with both fluids and solids.
(36) What is meant by a "system"? What is meant by a "control volume"?
(37) What is meant by "uniform"? What is meant by "steady"? Give examples of velocity fields that are uniform/non-uniform and steady/unsteady.
(38) What is meant by an "extensive" property? Give an example of an extensive property. What is meant by an "intensive" property? Give an example of an intensive property.
(39) How is the "dimension" of a flow determined? Give an example of a 2D, unsteady velocity field. Is it possible to have a 2D velocity field but a 3D temperature field?
(40) What is meant by a "uniform" flow? Give an example of a velocity profile that is uniform in the x-direction but is unsteady.

CHAPTER 2

Fluid Statics

2.1. Hydrostatic Pressure Variation

For a flow in which there are no velocity gradients, e.g., $du/dy = 0$, such as a static fluid, the shear stresses are zero. Draw a free body diagram of a (differentially) small piece of fluid, with width, height, and depth of dx, dy, and dz, respectively, under static conditions. Note that only pressure forces and weight will act on the fluid element. If we assume that the pressure, density, and gravitational acceleration at the center of the element are p, ρ, and g, respectively, then the free body diagram looks as shown in Figure 2.1 (making use of the Taylor Series approximation discussed in Chapter 1).

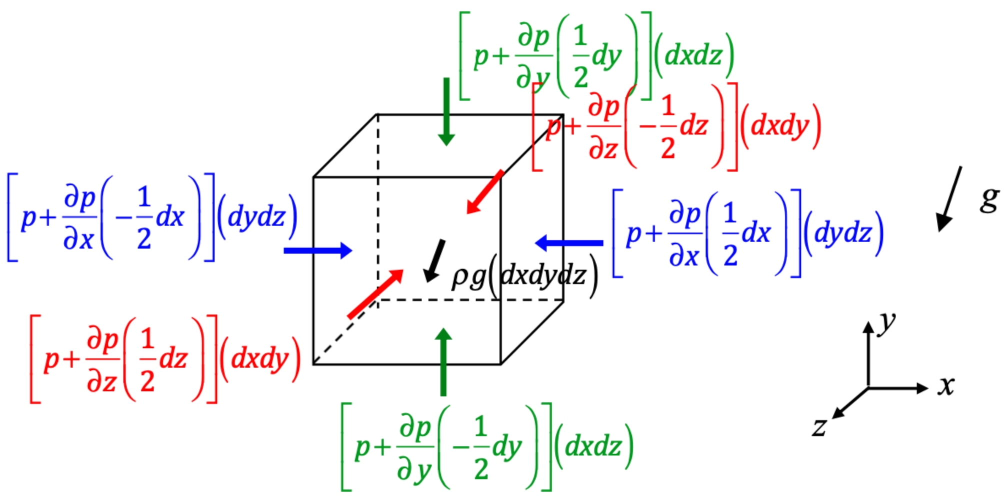


Figure 2.1. A free body diagram of a differentially-small fluid element with no shear stresses.

Summing forces in each of the three directions and noting that the fluid element is static,

$$\sum F_x = 0 = \left[p + \frac{\partial p}{\partial x}\left(-\frac{1}{2}dx\right)\right](dydz) - \left[p + \frac{\partial p}{\partial x}\left(\frac{1}{2}dx\right)\right](dydz) + \underbrace{\rho dxdydz}_{=dm} g_x, \tag{2.1}$$

$$0 = -\frac{\partial p}{\partial x}dxdydz + \rho dxdydz g_x, \tag{2.2}$$

$$0 = -\frac{\partial p}{\partial x} + \rho g_x. \tag{2.3}$$

A similar approach can be taken in the y and z directions,

$$0 = -\frac{\partial p}{\partial y} + \rho g_y, \tag{2.4}$$

$$0 = -\frac{\partial p}{\partial z} + \rho g_z. \tag{2.5}$$

The last three equations may be written more compactly in vector form as,

$$\boxed{\nabla p = \rho \boldsymbol{g}}. \quad \text{Force balance for a static fluid particle.} \tag{2.6}$$

What this equation tells us is that for a static piece of fluid, a difference in pressure is required to balance the weight of the fluid particle.

Now consider the case where the gravitational acceleration points in the positive y direction, $g = g\hat{\boldsymbol{\jmath}}$, so that the y-component of Eq. (2.6) is,

$$\frac{dp}{dy} = \rho g. \tag{2.7}$$

Note that the x and z components indicate that there is no change in pressure in those directions since there is no component of weight to balance, i.e., the pressure only changes in the y direction (hence the use of an ordinary derivative in Eq. (2.7) as opposed to a partial derivative since $p = p(y)$).

2.1.1. Hydrostatic Pressure Variation in an Incompressible Fluid

To determine how the pressure varies in the y direction, we must solve the differential equation in Eq. (2.7),

$$dp = \rho g dy \implies \int_{p=p_0}^{p=p} dp = \int_{y=0}^{y=y} \rho g dy \implies p - p_0 = \int_0^y \rho g dy. \tag{2.8}$$

In order to solve the integral on the right-hand side of the previous equation, we must know how the density and gravitational acceleration vary with y. It's reasonable in most applications to assume that the gravitational acceleration is constant, so it can be pulled outside the integral. If we further assume that we're dealing with an incompressible fluid, then the density can also be pulled outside the integral and we're left with,

$$p - p_0 = \rho g \int_0^y dy = \rho g y, \tag{2.9}$$

$$\boxed{p = p_0 + \rho g y} \quad \text{or, alternately,} \quad \Delta p = \rho g \Delta y. \tag{2.10}$$

The previous equation is the hydrostatic pressure variation in an incompressible fluid in which gravity points in the positive y direction.

Notes:

(1) The pressure in Eq. (2.10) only changes when there are variations in elevation in the direction of gravity (the y direction). Moving perpendicular to the direction of gravity does not change the pressure.
(2) The pressure increases linearly in the direction of the gravitational acceleration. A plot of this variation is shown in Figure 2.2.

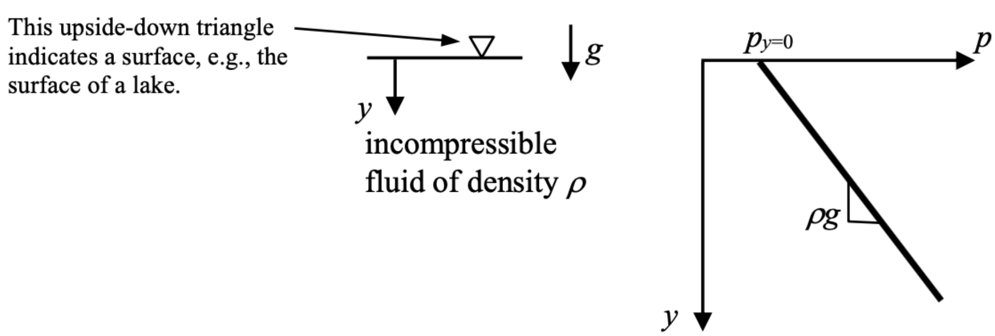


Figure 2.2. The hydrostatic pressure plotted against depth for an incompressible fluid in a constant gravity field.

(3) As mentioned previously, the reason the pressure increases with depth is because the pressure must balance the weight of all the fluid sitting above it.

(4) Since changes in pressure correspond to changes in elevation (refer to Eq. (2.10)), pressure differences are often expressed in terms of lengths, or depths of fluid. For example, the standard atmospheric pressure of 101 kPa (abs) corresponds to 760 mmHg,

$$\underbrace{101\,\text{kPa}}_{\Delta p} = \underbrace{(13\,600\,\text{kg/m}^3)}_{=\rho_{\text{Hg}}}\underbrace{(9.81\,\text{m/s}^2)}_{=g}\underbrace{(760\times10^{-3}\,\text{m})}_{=\Delta y}. \tag{2.11}$$

(5) When measuring pressure differences in Eq. (2.10), either both pressures must be absolute pressures or both must be gage. Don't mix gage and absolute pressures.

2.1.2. Hydrostatic Pressure Variation in a Compressible Fluid

Now consider the pressure variation in a compressible fluid. This case would be of particular interest for airplanes, rockets, and mountain climbers where large changes of elevation in the atmosphere are common. Recall from Eq. (2.7),

$$\frac{dp}{dy} = \rho g \quad (y \text{ and } g \text{ point in the same direction}). \tag{2.12}$$

For convenience, since we typically deal with elevation or altitude instead of depth when considering the hydrostatic pressure variation in compressible fluids like air, let's change our coordinate system so that y points in the direction *opposite* to gravity. Thus,

$$\frac{dp}{dy} = -\rho g \quad (y \text{ and } g \text{ point in opposite directions}). \tag{2.13}$$

If we're dealing with an ideal gas (like air), then the pressure and density are related via the Ideal Gas Law,

$$p = \rho RT \implies \rho = \frac{p}{RT}, \tag{2.14}$$

where R is the gas constant. Substituting Eq. (2.14) into Eq. (2.13) and re-arranging gives,

$$\frac{dp}{p} = -\frac{g}{RT}dy. \tag{2.15}$$

Numerous measurements have been made of the average atmospheric temperature as a function of altitude, i.e., $T = T(y)$, and can be substituted into Eq. (2.15). For example, the temperature variation with altitude from the U.S. Standard Atmosphere (standardized in 1976) is shown in Figure 2.3. In each region of the atmosphere, the temperature varies linearly and can be expressed as,

$$T = T_a - \beta y, \tag{2.16}$$

where T_a and β (known as the temperature lapse rate) are constants and y is the altitude measured from sea level. Table 2.1 lists these constants for the different parts of the atmosphere. For example, from sea level to an altitude of 11000 m, the temperature decreases by 6.5 K/km.

Table 2.1. Pressure (p_a), temperature (T_a), and temperature lapse rate (β), starting at different altitudes from the (1976) U.S. Standard Atmosphere.

y (m)	p_a (Pa (abs))	T_a (K)	β (K/m)
0	101325	288.15	0.0065
11000	22632.1	216.65	0.0
20000	5474.89	216.65	-0.001
32000	868.019	228.65	-0.0028
47000	110.906	270.65	0.0
51000	66.9389	270.65	0.0028
71000	3.95642	214.65	0.002

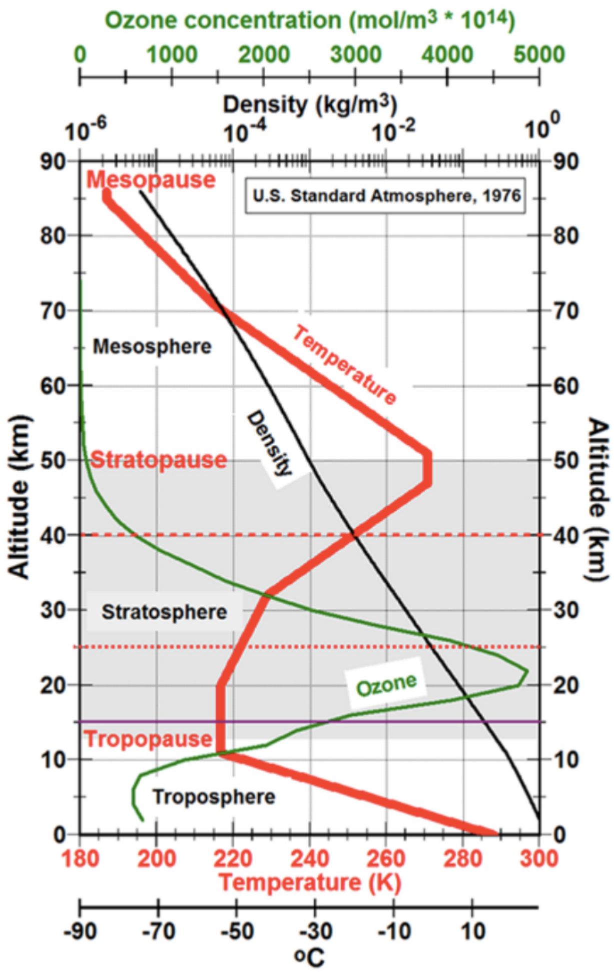


Figure 2.3. The (1976) U.S. Standard Atmosphere.

If we substitute Eq. (2.16) into Eq. (2.15) and solve the differential equation, we get,

$$\frac{dp}{p} = -\frac{g}{R(T_a - \beta y)}dy \implies \int_{p_0}^{p} \frac{dp}{p} = -\frac{g}{R}\int_0^y \frac{dy}{T_a - \beta y} \implies \ln\left(\frac{p}{p_a}\right) = \frac{g}{\beta R}\ln\left(\frac{T_a - \beta y}{T_a}\right), \tag{2.17}$$

$$\boxed{\frac{p}{p_a} = \left(1 - \frac{\beta y}{T_a}\right)^{\frac{g}{\beta R}}}. \tag{2.18}$$

Notes:

(1) The temperature lapse rate in the troposphere is $\beta = 6.5\,\mathrm{K\,km^{-1}}$ ($\approx 3.57\,^\circ\mathrm{F}/1000\,\mathrm{ft}$). Thus, a handy rule-of-thumb when hiking in the mountains (or deep canyons) is that the temperature will decrease $6.5\,^\circ\mathrm{C}$ for every 1 km of elevation gain or approximately $3.5\,^\circ\mathrm{F}$ for every 1000 ft of elevation gain.

(2) Large changes in elevation are required to make appreciable changes in pressure in a gas, such as air. For example, estimate the altitude change required to drop the pressure by 1% using the U.S. Standard Atmosphere in the troposphere,

$$\frac{p}{p_a} = 0.99 = \left(1 - \frac{\beta y}{T_a}\right)^{\frac{g}{\beta R}} \implies (0.99)^{\frac{\beta R}{g}} = 1 - \frac{\beta y}{g} \implies y = \frac{T_a}{\beta}\left[1 - (0.99)^{\frac{\beta R}{g}}\right], \tag{2.19}$$

which gives $y = 85\,\mathrm{m}$ using the troposphere values for the constants. Thus, it's reasonable to assume that unless large elevation differences occur, the pressure does not vary with elevation in the atmosphere, or any gas for that matter. However, even small elevation differences in liquids do result in appreciable pressure changes since the density of liquids is much larger than the density of gases.

What is the pressure at the bottom of the Marianas Trench (11,000 m = 36,201 ft = 6.9 mi)?

SOLUTION:

The pressure at the bottom of the Marianas Trench, assuming salt water to be incompressible, is:

$$p_{\text{bottom}} = p_{\text{top}} + \rho_{\text{saltH20}} g h$$

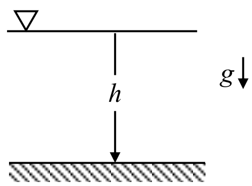
where

p_{top} = 101 kPa (abs)

ρ_{saltH20} = 1025 kg/m^3

g = 9.81 m/s^2

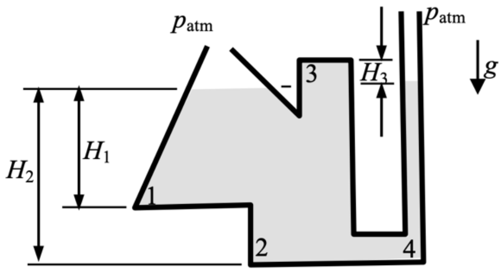
h = 11000 m



Hence, the pressure at the bottom is:

$\boxed{p_{\text{bottom}} = 110 \text{ MPa} = 1100 \text{ atm!}}$

Determine the pressure at points 1, 2, 3, and 4.



SOLUTION:
Recall that the shape of the container doesn't matter when calculating hydrostatic pressure. It's only the depth of the fluid that matters.

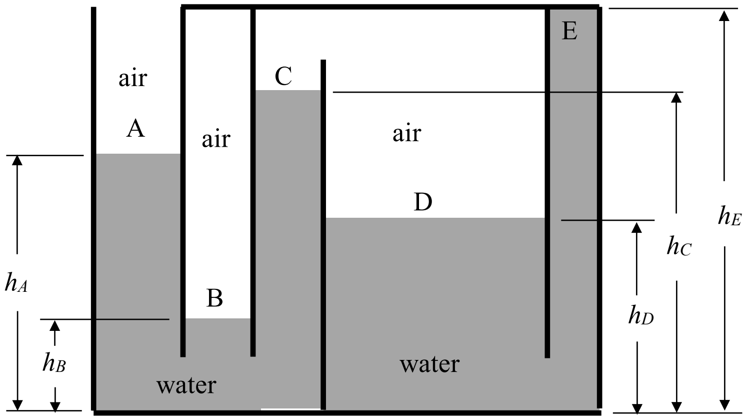
$p_1 = p_{atm} + \rho g H_1,$ (1)

$p_2 = p_{atm} + \rho g H_2,$ (2)

$p_3 = p_{atm} - \rho g H_3,$ (3)

$p_4 = p_{atm} + \rho g H_2$ (Point 4 is at the same depth as point 2.) (4)

Determine the gage pressure at points B, C, D, and E in the system shown below.



h_A = 6 m
h_B = 2 m
h_C = 7 m
h_D = 5 m
h_E = 10 m

SOLUTION:

First determine the pressure at point B,

$$\boxed{p_B = p_A + \rho g\left(h_A - h_B\right)}. \tag{1}$$

Note that the pressure at A is $p_A = p_{\text{atm}}$.

Now determine the gage pressure at C using the known pressure at B,

$$\boxed{p_C = p_B - \rho g\left(h_C - h_B\right)} \tag{2}$$

The pressure at point D will be the same as the pressure at point C since both contact the same air and we're assuming the variations in air pressure over the small elevations in this problem are negligible,

$$\boxed{p_D = p_C.} \tag{3}$$

The pressure at point E is,

$$\boxed{p_E = p_D - \rho g\left(h_E - h_D\right)}. \tag{4}$$

Using the given data,

p_A = p_{atm} = 0 (gage)
ρ = 1000 kg/m^3
g = 9.81 m/s^2
h_A = 6 m
h_B = 2 m
h_C = 7 m
h_D = 5 m
h_E = 10 m

$$\Rightarrow \boxed{\begin{aligned} p_B &= 39.2 \text{ kPa (gage)} \\ p_C &= -9.8 \text{ kPa (gage)} \\ p_D &= -9.8 \text{ kPa (gage)} \\ p_E &= -58.9 \text{ kPa (gage)} \end{aligned}}$$

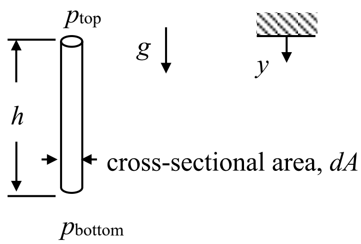
Assuming that air is incompressible, determine the height of a column of air required to give a pressure difference of 0.1 psi. Assume that the density of air is 2.38×10^{-3} slug/ft^3.

SOLUTION:

Assuming air as being incompressible:

$$p_{\text{bottom}} = p_{\text{top}} + \rho_{\text{air}} g h$$

$$\boxed{h = \frac{p_{\text{bottom}} - p_{\text{top}}}{\rho_{\text{air}} g}}$$



For:

$$p_{\text{bottom}} - p_{\text{top}} = 0.1 \text{ psi} = 14.4 \text{ lb}_f/\text{ft}^2$$
$$\rho_{\text{air}} = 2.38 \times 10^{-3} \text{ slug/ft}^3$$
$$g = 32.2 \text{ ft/s}^2$$

gives:

$$\boxed{h = 188 \text{ ft}}$$

Hence, very large elevation differences must occur to give appreciable differences in pressure when dealing with atmospheric air (or gases in general).

Another way to determine the height, h, is to perform a vertical force balance on the column.

$$\sum F_y = 0 = -p_{\text{bottom}} dA + p_{\text{top}} dA + \rho_{\text{air}} g h dA$$

$$h = \frac{p_{\text{bottom}} - p_{\text{top}}}{\rho_{\text{air}} g} \quad \text{(Same answer as above!)}$$

What is the air pressure at the top of the Burj Khalifa, which has a height of 828 m (2717 ft)? If there was a pipe containing water that extended from the top of the Burj Khalifa to the ground, what would be the gage pressure in the water at the bottom of the pipe?

SOLUTION:

Assuming constant air density,

$$p_{y=H} = p_{y=0} - \rho_{\text{air}} g H \tag{1}$$

where

$\rho = 1.225\ \text{kg/m}^3$
$g = 9.81\ \text{m/s}^2$
$H = 828\ \text{m}$
$p_{y=0} = 101.33$ kPa (abs) ($= p_{\text{atm}}$)

Thus, $\boxed{p_{y=H} = 91.4 \text{ kPa (abs) or } p_{y=H}/p_{y=0} = 0.902.}$

If we treat air as a compressible, ideal gas and assume the air temperature varies according to the U.S. Standard Atmosphere,

$$p_y = p_{y=0}\left(1 - \frac{\beta y}{T_{y=0}}\right)^{\frac{g}{R\beta}} \tag{2}$$

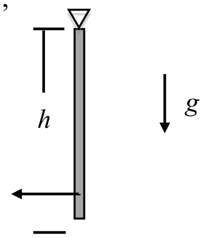
where the previous values for g, $p_{y=0}$, and H are assumed, and,

$\beta = 0.00650$ K/m
$T_{y=0} = 288$ K (= 15 degC)
$R = 286.9$ J/(kg.K)
$\Rightarrow \boxed{p/p_{y=0} = 0.906}$, which is nearly identical to the previous calculation.

Video solution: https://www.youtube.com/watch?v=roYusHpCWPA

The gage pressure in a water column with a depth of 828 m is given by,

$$p_{\text{gage}} = \rho g h\,, \tag{3}$$



where

ρ = 1000 kg/m^3
g = 9.81 m/s^2
h = 828 m
$\Rightarrow$ $\boxed{p_{\text{gage}} = 8.12*10^6\text{ Pa}}$ = 80.2 atm!

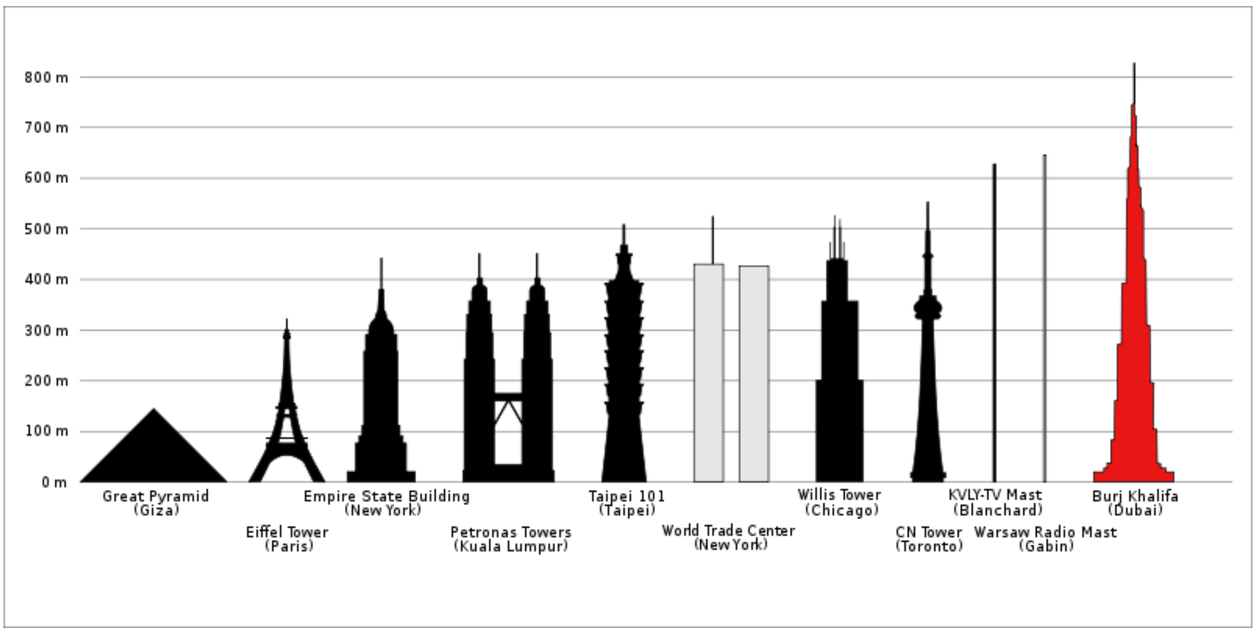
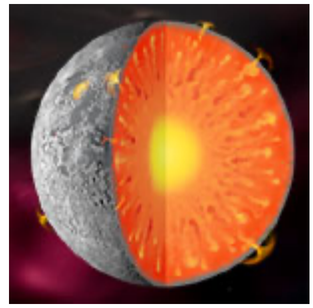


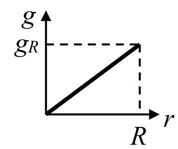
Image from Wikipedia (2012 Jan 10; http://en.wikipedia.org/wiki/File:BurjKhalifaHeight.svg)

It is often conjectured that the Earth was, at one time, comprised of molten material. If the acceleration due to gravity within this fluid sphere (with a radius of 6440 km) varied linearly with distance, r, from the Earth's center, the acceleration due to gravity at r = 6440 km was 9.81 m/s^2, and the density of the fluid was uniformly 5600 kg/m^3, determine the gage pressure at the center of this fluid Earth.



SOLUTION:

Since the acceleration due to gravity, g, varies linearly with r:



$$g = cr \tag{1}$$

where c is a constant. Since $g(r = R = 6440 \text{ km}) = g_R = 9.81 \text{ m/s}^2$:

$$c = \frac{g_R}{R} = \frac{9.81 \text{ m/s}^2}{6440*10^3 \text{ m}} = 1.523*10^{-6} \text{ s}^{-2} \tag{2}$$

From the hydrostatic pressure distribution (neglecting the curvature of the Earth):

$$\frac{dp}{dr} = -\rho g \tag{3}$$

Substitute Eqn. (1) and solve the resulting differential equation.

$$\frac{dp}{dr} = -\rho c r \Rightarrow \int_{p=p_0}^{p=0} dp = -\rho c \int_{r=0}^{r=R} r dr \tag{4}$$

$$\boxed{\therefore p_0 = \tfrac{1}{2}\rho c R^2} \tag{5}$$

Using the given data:

ρ = 5600 kg/m^3

c = 1.523*10^{-6} s^{-2}

R = 6.440*10^6 m

$\boxed{p_0 = 1.769*10^{11} \text{ Pa} = 1.769*10^6 \text{ atm}}$

2.2. Pressure Measurements using Barometers and Manometers

As noted in the previous section, differences in elevation can be used to measure differences in pressure. This is the principle by which barometers and manometers operate.

2.2.1. Barometers

Let's first consider a barometer, which is most often used to measure atmospheric pressure. A sketch of a barometer is shown in Figure 2.4. A barometer consists of a tube that is open on one end. The tube is filled with a working liquid, often mercury or water, which is then immersed in a large bath of the liquid and turned upside down and lifted out of the bath to the configuration shown in the figure. Using this method, the weight of the liquid in the tube is balanced by the pressure difference between the external pressure (normally atmospheric pressure, p_{atm}) and the pressure at the top of the column of liquid column, which is the vapor pressure of the liquid (p_v).

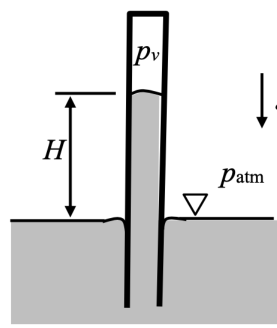


Figure 2.4. A sketch of a simple barometer.

Using Eq. (2.10),

$$p_v = p_{\text{atm}} - \rho g H \implies p_{\text{atm}} = p_v + \rho g H. \tag{2.20}$$

Thus, atmospheric pressure can be measured by measuring the height of the column of liquid in the barometer and knowing the liquid density and vapor pressure.

Notes:

(1) Vapor pressure varies with temperature. Thus, it's important to also measure the temperature when using a barometer for obtaining accurate results. Since the vapor pressure is often much smaller than atmospheric pressure, it is sometimes neglected in Eq. (2.20), but doing so does introduce some inaccuracy into the atmospheric pressure calculation.

(2) At a standard atmospheric pressure and temperature of 101.3 kPa (abs) and 15 °C, respectively, the height of a column of mercury ($\rho = 13\,600\,\text{kg/m}^3$, $p_v = 1.11 \times 10^{-4}\,\text{kPa (abs)}$) is 760 mm, which is a reasonable height to have in a laboratory setting. Using water, ($\rho = 1000\,\text{kg/m}^3$, $p_v = 1.71\,\text{kPa (abs)}$), the height is 10.2 m, which is more challenging to accommodate. Hence, most barometers still use mercury as a working liquid even though mercury is toxic.

2.2.2. Manometers

A manometer is similar to a barometer in that the height difference in a working liquid is used to measure pressure differences. However, a manometer does not have one end of the working liquid at vapor pressure. An example of a U-tube manometer is shown in Figure 2.5.

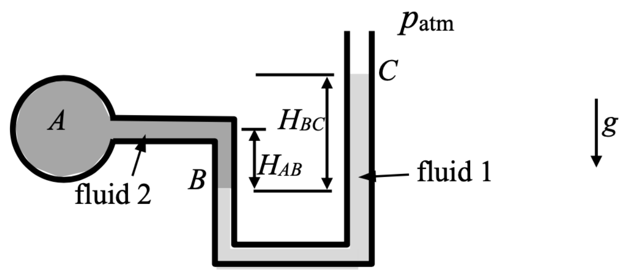


Figure 2.5. A sketch of a U-tube manometer.

In this figure, there are two incompressible fluids, fluid 1 and fluid 2 with corresponding densities ρ_1 and ρ_2. Let's determine the pressure at A starting with the pressure at C using Eq. (2.10), which depends only on elevation differences in a given fluid,

$$p_C = p_{\text{atm}}, \tag{2.21}$$
$$p_B = p_C + \rho_1 g H_{BC} \quad \text{(moving through fluid 1)}, \tag{2.22}$$
$$p_A = p_B - \rho_2 g H_{AB} \quad \text{(moving through fluid 2)}, \tag{2.23}$$
$$\implies p_A = p_{\text{atm}} + \rho_1 g H_{BC} - \rho_2 g H_{AB} \quad \text{or} \quad p_A - p_{\text{atm}} = \rho_1 g H_{BC} - \rho_2 g H_{AB}. \tag{2.24}$$

Thus, by measuring differences in height, it's possible to measure differences in pressure.

Another common type of manometer is known as an inclined tube manometer and is shown in Figure 2.6. This type of manometer is used most often when small differences in pressure are to be measured since small elevation differences correspond to large differences in length in the inclined arm, especially for small angles θ. As before, determine the pressure at A starting with the pressure at C,

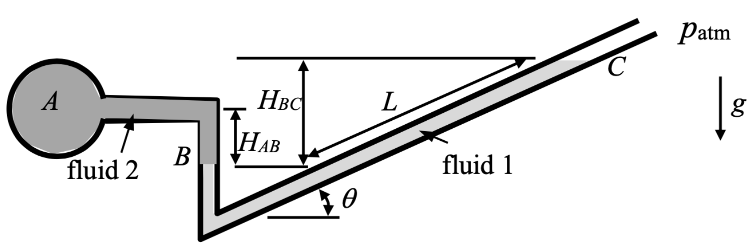


Figure 2.6. A sketch of an inclined-tube manometer.

$$p_C = p_{\text{atm}}, \tag{2.25}$$
$$p_B = p_C + \rho_1 g H_{BC} \quad \text{(moving through fluid 1)}, \tag{2.26}$$
$$p_A = p_B - \rho_2 g H_{AB} \quad \text{(moving through fluid 2)}, \tag{2.27}$$
$$\implies p_A = p_{\text{atm}} + \rho_1 g H_{BC} - \rho_2 g H_{AB} \quad \text{or} \quad p_A - p_{\text{atm}} = \rho_1 g L \sin\theta - \rho_2 g H_{AB}, \tag{2.28}$$

where,

$$H_{BC} = L \sin\theta. \tag{2.29}$$

Thus, for small θ, small variations in H_{BC} will be magnified into large variations in L.

Notes:

(1) If a gas is used as one of the fluids in the manometer, then the pressure in that gas can be reasonably assumed to remain constant with elevation.

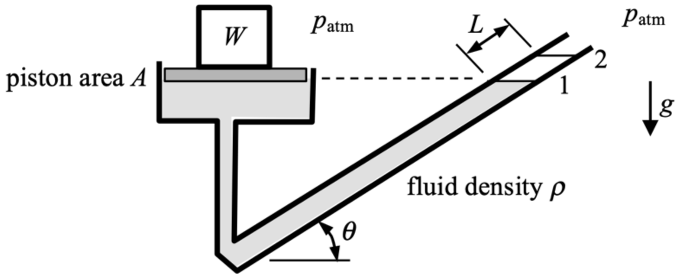
(2) One of the reasons we use gage pressures instead of absolute pressures is because if one of the ends of the manometer is open to the atmosphere, then the pressure at the other end can be treated as a gage pressure, such as in Eqs. (2.24) and (2.28).

(3) A good approach to working through manometer pressures is to start at one end of the manometer and calculate the pressure at each fluid interface until reaching the other end of the manometer, as done in the previous two examples. Moving down in the fluid adds pressure (to support the weight of the fluid above it) while moving up in the manometer subtracts pressure (less weight to support). Note that moving horizontally *in the same fluid* does not change the pressure.

(4) There are other styles of manometers, but they all operate on the same principle: pressure differences are measured using differences in fluid elevations.

(5) Nowadays, the use of electronic pressure transducers is common for measuring pressures. Pressure transducers have much faster response times than manometers and can more accurately measure small pressure differences. Nevertheless, manometers are still useful since (a) they are simple and cheap and (b) need not be calibrated.

When a weight W is placed on a piston with an area A, fluid in an inclined manometer moves from point 1 to point 2. What is W in terms of the fluid density ρ, gravitational acceleration g, the displacement L, the piston area A, and the tube arm angle θ?



SOLUTION:
Analyzing the manometer after the weight is applied,

$$p_{atm} = p_{piston} - \rho g L \sin\theta, \tag{1}$$

where the (absolute) pressure in the fluid just below the piston is,

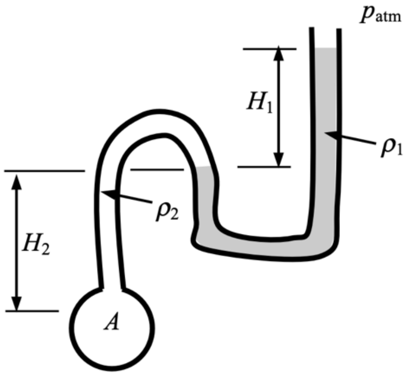
$$p_{piston} = p_{atm} + \frac{W}{A}. \tag{2}$$

Combine both equations and solve for W,

$$p_{atm} = p_{atm} + \frac{W}{A} - \rho g L \sin\theta, \tag{3}$$

$$\boxed{W = \rho g L A \sin\theta}. \tag{4}$$

Determine the gage pressure at point A.

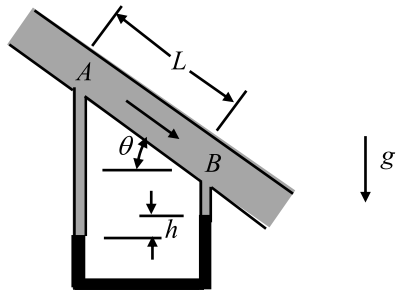


SOLUTION:

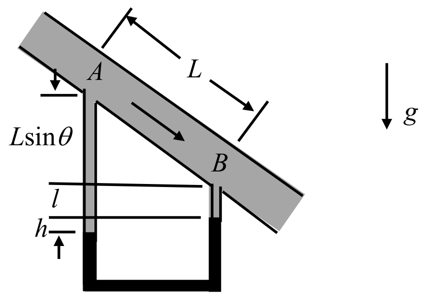
$$p_A = p_{atm} + \rho_1 g H_1 + \rho_2 g H_2, \quad (1)$$

$$\boxed{p_{A,gage} = p_A - p_{atm} = \rho_1 g H_1 + \rho_2 g H_2}. \quad (2)$$

Water flows downward through a pipe inclined at a $\theta = 45°$ to the horizon as shown in the figure. The pressure difference $p_A - p_B$ is due partly to gravity and partly to viscous dissipation. Determine the pressure difference if L = 5 m and h = 6 cm. Mercury is the working fluid in the manometer.



SOLUTION:



The pressure at B may be written in terms of the pressure at A using,

$$p_B = p_A + \rho_{H_2O} g \left(L\sin\theta + l + h \right) - \rho_{Hg} g h - \rho_{H_2O} g l \tag{1}$$

$$p_B - p_A = \rho_{H_2O} g \left(L\sin\theta + h \right) - \rho_{H_2O} SG_{Hg} g h \tag{2}$$

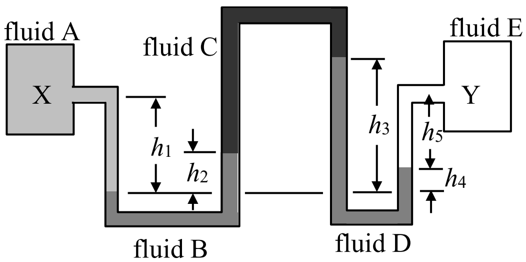
$$p_A - p_B = \rho_{H_2O} g \left[SG_{Hg} h - \left(L\sin\theta + h \right) \right] \tag{3}$$

$$p_A - p_B = \rho_{H_2O} g \left[\left(SG_{Hg} - 1 \right) h - L\sin\theta \right] \tag{4}$$

Using the given data,

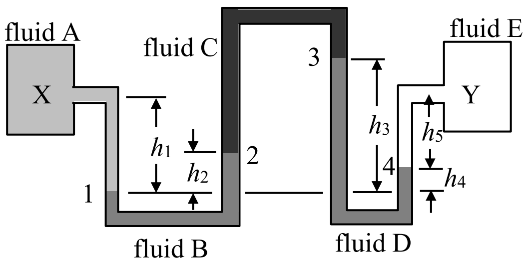
ρ_{H2O} = 1000 kg/m^3
g = 9.81 m/s^2
SG_{Hg} = 13.6
h = 0.06 m
L = 5 m
θ = 45°
$\Rightarrow$ $\boxed{p_A - p_B = \text{-27.3 kPa}}$

Determine the pressure difference between points X and Y in the system shown below.



SOLUTION:

First indicate some reference points in the manometer system as shown in the figure below.



Now determine the pressure at the various reference points.

$$p_1 = p_X + \rho_A g h_1 \tag{1}$$

$$p_2 = p_1 - \rho_B g h_2 \tag{2}$$

$$p_3 = p_2 - \rho_C g \left(h_3 - h_2\right) \tag{3}$$

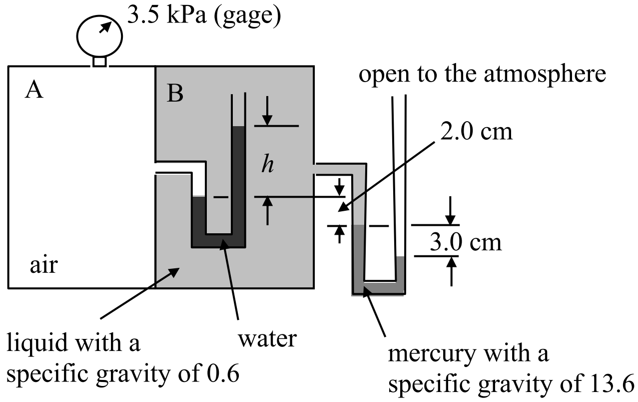
$$p_4 = p_3 + \rho_D g \left(h_3 - h_4\right) \tag{4}$$

$$p_Y = p_4 - \rho_E g h_5 \tag{5}$$

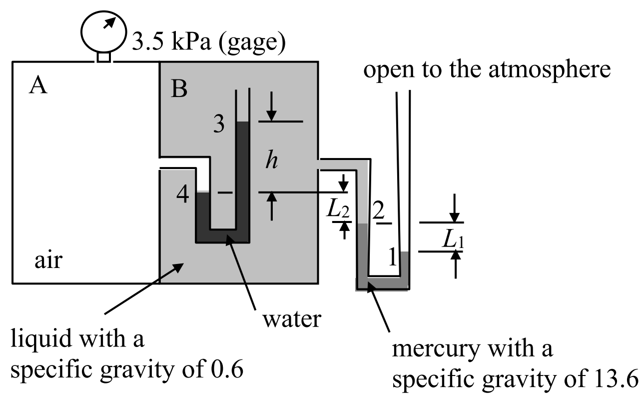
Now combine Eqns. (1) - (5).

$$\boxed{\therefore\ p_Y = p_X + \rho_A g h_1 - \rho_B g h_2 - \rho_C g \left(h_3 - h_2\right) + \rho_D g \left(h_3 - h_4\right) - \rho_E g h_5} \tag{6}$$

Compartments A and B of the tank shown in the figure below are closed and filled with air and a liquid with a specific gravity equal to 0.6. If atmospheric pressure is 101 kPa (abs) and the pressure gage reads 3.5 kPa (gage), determine the manometer reading, *h*.



SOLUTION:



First determine the pressure at 2 in terms of the pressure at 1.

$$p_2 = p_1 - \rho_{Hg} g L_1 \quad (1)$$

Now determine the pressure at 3 in terms of the pressure at 2.

$$p_3 = p_2 - \rho_{liquid} g \left(h + L_2\right) \quad (2)$$

Now determine the pressure at 4 in terms of the pressure at 3.

$$p_4 = p_3 + \rho_{H20} g h \quad (3)$$

Combine Eqns. (1)-(3).

$$p_4 = p_1 - \rho_{Hg} g L_1 - \rho_{liquid} g \left(h + L_2\right) + \rho_{H20} g h$$

$$p_4 = p_1 - \rho_{H20} SG_{Hg} g L_1 - \rho_{H20} SG_{liquid} g \left(h + L_2\right) + \rho_{H20} g h$$

$$p_4 - p_1 = -\rho_{H20} g \left[SG_{Hg} L_1 + SG_{liquid} h + SG_{liquid} L_2 - h\right]$$

$$\frac{p_1 - p_4}{\rho_{H20} g} - SG_{Hg} L_1 - SG_{liquid} L_2 = h\left(SG_{liquid} - 1\right)$$

$$\boxed{h = \frac{1}{\left(1 - SG_{liquid}\right)}\left[SG_{Hg} L_1 + SG_{liquid} L_2 + \frac{p_4 - p_1}{\rho_{H20} g}\right]} \quad (4)$$

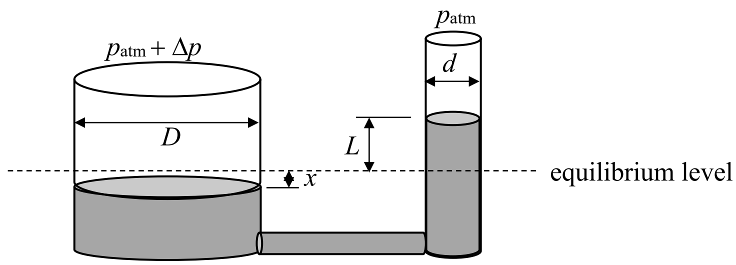
Using the given data:

p_1 = 101 kPa (abs) = 0 Pa (gage)
p_4 = 3.5 kPa (gage) = 3500 Pa (gage)
SG_{Hg} = 13.6
SG_{liquid} = 0.6
g = 9.81 m/s^2
ρ_{H20} = 1000 kg/m^3
L_1 = 3.0 cm = 3.0*10^{-2} m
L_2 = 2.0 cm = 2.0*10^{-2} m

Solving Eqn. (4) for h gives:

$\boxed{h = 1.9 \text{ m}}$

A reservoir manometer has vertical tubes of diameter D and d. When the pressure at the liquid surfaces in both tubes is the same, the liquid levels in each tube are at the same elevation. When an additional pressure Δp is applied to the left tube, the liquid layer in that tube drops a distance x while the liquid in the right tube rises a distance L. Develop an algebraic expression for the liquid deflection L in the small tube when the additional pressure Δp is applied to the large tube.



SOLUTION:
Relate the pressure at the liquid surface in the left tube to the pressure at the liquid surface in the right tube using manometry,

$$p_{atm} = \left(p_{atm} + \Delta p\right) - \rho g\left(x + L\right), \tag{1}$$

$$x + L = \frac{\Delta p}{\rho g}. \tag{2}$$

The distances x and L may be related by noting that the liquid mass remains the same in the system. Assuming that the liquid is incompressible (a good assumption), the volume displaced in the left tube will equal the volume gained in the right tube,

$$x\frac{\pi D^2}{4} = L\frac{\pi d^2}{4}, \tag{3}$$

$$x = L\left(\frac{d}{D}\right)^2. \tag{4}$$

Now substitute Eq. (4) into Eq. (2) and solve for L,

$$L\left(\frac{d}{D}\right)^2 + L = \frac{\Delta p}{\rho g}, \tag{5}$$

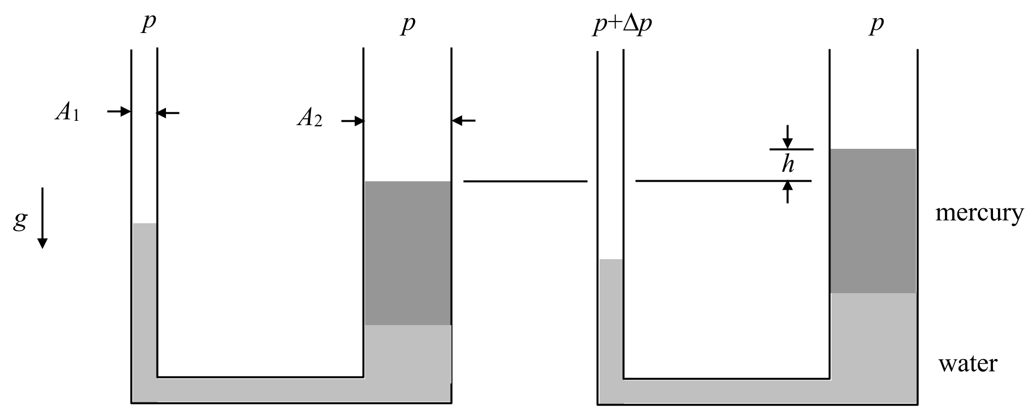
$$\boxed{L = \frac{\Delta p}{\rho g}\left[\frac{1}{1+\left(d/D\right)^2}\right]}. \tag{6}$$

Video solution: https://www.youtube.com/watch?v=pt-XhFEnmIE

Determine the deflection, h, in the manometer shown below in terms of A_1, A_2, Δp, g, and ρ_{H2O}. Determine the sensitivity of this manometer. The manometer sensitivity, s, is defined here as the change in the elevation difference, h, with respect to a change in the applied pressure, Δp:

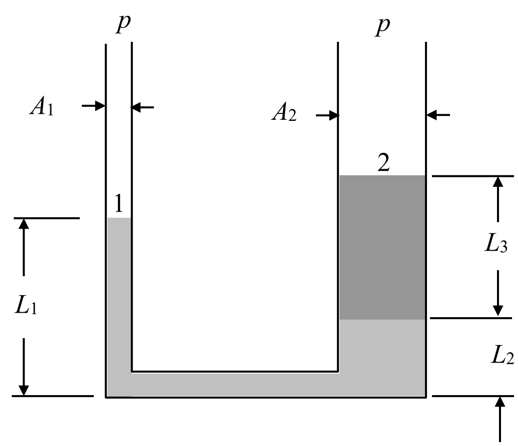
$$s \equiv \frac{dh}{d(\Delta p)}$$

Manometers with larger sensitivity will give larger changes in h for the same Δp.



SOLUTION:

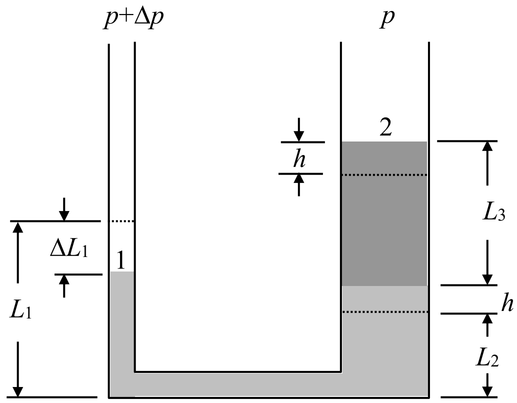
First analyze the initial system.



$$\underbrace{p_2}_{=p} = \underbrace{p_1}_{=p} + \rho_{H20} g L_1 - \rho_{H20} g L_2 - \rho_{Hg} g L_3$$

$$L_1 - L_2 = SG_{Hg} L_3 \tag{1}$$

Now analyze the displaced system.



$$\underbrace{p_2}_{=p} = \underbrace{p_1}_{=p+\Delta p} + \rho_{H20} g \left(L_1 - \Delta L_1\right) - \rho_{H20} g \left(L_2 + h\right) - \rho_{Hg} g L_3$$

$$-\frac{\Delta p}{\rho_{H20} g} = \left(L_1 - \Delta L_1 - L_2 - h\right) - SG_{Hg} L_3 \tag{2}$$

Substitute Eqn. (1) into Eqn. (2).

$$-\frac{\Delta p}{\rho_{H20} g} = \left(L_1 - \Delta L_1 - L_2 - h\right) - \left(L_1 - L_2\right)$$

$$\frac{\Delta p}{\rho_{H20} g} = \Delta L_1 + h \tag{3}$$

Note also that the displaced volume will also be conserved.

$$\Delta L_1 A_1 = h A_2$$

$$\Delta L_1 = h \frac{A_2}{A_1} \tag{4}$$

Substitute Eqn. (4) into Eqn. (3).

$$\frac{\Delta p}{\rho_{H20} g} = h \frac{A_2}{A_1} + h$$

$$\boxed{h = \frac{1}{1 + A_2/A_1} \left(\frac{\Delta p}{\rho_{H20} g}\right)} \tag{5}$$

Note that the density of the secondary fluid (i.e., mercury) does not factor into the displaced height.

The manometer sensitivity, s, is defined as the change in the elevation difference, h, with respect to a change in the applied pressure, Δp.

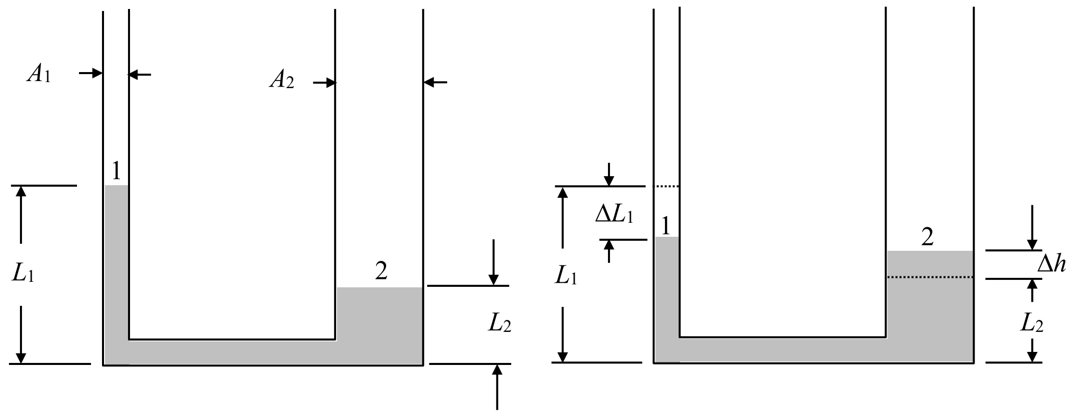
$$s \equiv \frac{dh}{d(\Delta p)} \tag{6}$$

Manometers with larger sensitivity will give larger changes in h for the same Δp. Using Eqn. (5), the sensitivity of this manometer is:

$$\boxed{s = \frac{1}{1 + {A_2}/{A_1}}\left(\frac{1}{\rho_{H20} g}\right)} \tag{7}$$

To increase the manometer's sensitivity, one should decrease the area ratio, A_2/A_1, and use a lower density fluid than water.

Why doesn't Eqn. (5) involve the properties of mercury? In fact, the properties of the secondary fluid (*i.e.* the mercury) do influence the system. Consider the change in potential energy of the water during the displacement as shown in the plots below.



$$\begin{aligned}
\Delta PE_{\text{left,H20}} &= \underbrace{\rho_{H20} A_1 (L_1 - \Delta L_1)}_{=m_{\text{after}}} g \underbrace{\tfrac{1}{2}(L_1 - \Delta L_1)}_{=L_{\text{CM,after}}} - \underbrace{\rho_{H20} A_1 L_1}_{=m_{\text{before}}} g \underbrace{\tfrac{1}{2} L_1}_{=L_{\text{CM,before}}} \\
&= \tfrac{1}{2}\rho_{H20} g A_1 (L_1 - \Delta L_1)^2 - \tfrac{1}{2}\rho_{H20} g A_1 L_1^2 \\
&= \tfrac{1}{2}\rho_{H20} g A_1 \left(-2L_1 \Delta L_1 + \Delta L_1^2\right)
\end{aligned}$$

$$\begin{aligned}
\Delta PE_{\text{right,H20}} &= \underbrace{\rho_{H20} A_2 (L_2 + h)}_{=m_{\text{after}}} g \underbrace{\tfrac{1}{2}(L_2 + h)}_{=L_{\text{CM,after}}} - \underbrace{\rho_{H20} A_2 L_2}_{=m_{\text{before}}} g \underbrace{\tfrac{1}{2} L_2}_{=L_{\text{CM,before}}} \\
&= \tfrac{1}{2}\rho_{H20} g A_2 (L_2 + h)^2 - \tfrac{1}{2}\rho_{H20} g A_2 L_2^2 \\
&= \tfrac{1}{2}\rho_{H20} g A_2 \left(2L_2 h + h^2\right)
\end{aligned}$$

$$\begin{aligned}
\Delta PE_{\text{total,H2O}} &= \tfrac{1}{2}\rho_{H20} g A_1 \left(-2L_1 \Delta L_1 + \Delta L_1^2\right) + \tfrac{1}{2}\rho_{H20} g A_2 \left(2L_2 h + h^2\right) \\
&= \tfrac{1}{2}\rho_{H20} g \left(-2L_1 \Delta L_1 A_1 + \Delta L_1^2 A_1 + 2L_2 h A_2 + h^2 A_2\right)
\end{aligned}$$

Substitute Eqn. (4).

$$\Delta PE_{\text{total,H2O}} = \tfrac{1}{2}\rho_{H20} g \left[-2L_1 h \frac{A_2}{A_1} A_1 + h^2 \left(\frac{A_2}{A_1}\right)^2 A_1 + 2L_2 h A_2 + h^2 A_2\right]$$

$$\Delta PE_{\text{total,H2O}} = \tfrac{1}{2}\rho_{H20} g A_2 h \left[\left(1 + {A_2}/{A_1}\right) h + 2(L_2 - L_1)\right] \tag{8}$$

The change in potential energy of the water will depend not only on h, but also on the initial state of the water, L_2-L_1. From Eqn. (1) we see that L_1-L_2 is related to the specific gravity of the secondary fluid.

Another way to solve the problem is to apply the 1st Law of Thermodynamics to the system (consisting of the fluids within the manometer):

$$\Delta E_{\text{system}} = Q_{\text{into system}} + W_{\text{on system}} \tag{9}$$

where $Q_{\text{into system}} = 0$ (assuming adiabatic conditions – a reasonable assumption) and the only work on the system is the pressure work causing the displacement:

$$W_{\substack{\text{pressure}\\\text{on system}}} = \left(p + \Delta p\right) A_1 \Delta L_1 - p A_2 h \tag{10}$$

Note that using Eqn. (4), Eqn. (10) becomes:

$$W_{\substack{\text{pressure}\\\text{on system}}} = \Delta p A_1 \Delta L_1 \tag{11}$$

The total change in the system's energy (which is the potential energy) is:

$$\begin{aligned}\Delta PE_{\text{left}} &= \underbrace{\rho_{\text{H20}} A_1 \left(L_1 - \Delta L_1\right)}_{=m_{\text{after}}} g \underbrace{\tfrac{1}{2}\left(L_1 - \Delta L_1\right)}_{=L_{\text{CM,after}}} - \underbrace{\rho_{\text{H20}} A_1 L_1}_{=m_{\text{before}}} g \underbrace{\tfrac{1}{2} L_1}_{=L_{\text{CM,before}}} \\ &= \tfrac{1}{2}\rho_{\text{H20}} A_1 g \left(L_1^2 - 2L_1\Delta L_1 + \Delta L_1^2 - L_1^2\right) \\ &= -\tfrac{1}{2}\rho_{\text{H20}} A_1 g \left(2L_1\Delta L_1 - \Delta L_1^2\right)\end{aligned} \tag{12}$$

$$\begin{aligned}\Delta PE_{\text{right}} &= \underbrace{\rho_{\text{H20}} A_2 \left(L_2 + h\right) g \tfrac{1}{2}\left(L_2 + h\right) - \rho_{\text{H20}} A_2 L_2 g \tfrac{1}{2} L_2}_{=\Delta PE_{\text{H20}}} + \underbrace{\rho_{\text{Hg}} A_2 L_3 g h}_{\Delta PE_{\text{Hg}}} \\ &= \tfrac{1}{2}\rho_{\text{H20}} A_2 g \left(L_2^2 + 2L_2 h + h^2 - L_2^2\right) + \rho_{\text{Hg}} A_2 L_3 g h \\ &= \tfrac{1}{2}\rho_{\text{H20}} A_2 g \left(2L_2 h + h^2\right) + \rho_{\text{Hg}} A_2 L_3 g h\end{aligned}$$

$$\begin{aligned}\Delta PE_{\text{system}} &= \Delta PE_{\text{left}} + \Delta PE_{\text{right}} \\ &= -\tfrac{1}{2}\rho_{\text{H20}} A_1 g \left(2L_1\Delta L_1 - \Delta L_1^2\right) + \tfrac{1}{2}\rho_{\text{H20}} A_2 g \left(2L_2 h + h^2\right) + \rho_{\text{Hg}} A_2 L_3 g h\end{aligned} \tag{13}$$

Substitute Eqns. (11) and (13) into Eqn. (9) gives:

$$-\tfrac{1}{2}\rho_{\text{H20}} A_1 g \left(2L_1\Delta L_1 - \Delta L_1^2\right) + \tfrac{1}{2}\rho_{\text{H20}} A_2 g \left(2L_2 h + h^2\right) + \rho_{\text{Hg}} A_2 L_3 g h = \Delta p A_1 \Delta L_1 \tag{14}$$

$$\frac{\Delta p}{\rho_{\text{H20}} g} = -L_1 + \tfrac{1}{2}\Delta L_1 + \frac{A_2}{A_1}\frac{1}{\Delta L_1}\left(L_2 h + \tfrac{1}{2}h^2 + SG_{\text{Hg}} L_3 h\right) \tag{15}$$

Substitute Eqn. (1).

$$\begin{aligned}\frac{\Delta p}{\rho_{\text{H20}} g} &= -L_1 + \tfrac{1}{2}\Delta L_1 + \frac{A_2}{A_1}\frac{1}{\Delta L_1}\left(L_2 h + \tfrac{1}{2}h^2 + L_1 h - L_2 h\right) \\ &= -L_1 + \tfrac{1}{2}\Delta L_1 + \frac{1}{2}\frac{A_2}{A_1}\frac{h^2}{\Delta L_1} + \frac{A_2}{A_1}\frac{L_1 h}{\Delta L_1}\end{aligned} \tag{16}$$

Substitute Eqn. (4) and simplify:

$$\frac{\Delta p}{\rho_{\text{H20}} g} = -L_1 + \tfrac{1}{2} h \frac{A_2}{A_1} + \frac{1}{2}\frac{A_2}{A_1}\frac{h^2}{h\frac{A_2}{A_1}} + \frac{A_2}{A_1}\frac{L_1 h}{h\frac{A_2}{A_1}} = -L_1 + \tfrac{1}{2} h \frac{A_2}{A_1} + \tfrac{1}{2} h + L_1 = \tfrac{1}{2}\left(1 + {}^{A_2}\!/\!_{A_1}\right) h \tag{17}$$

$$\therefore h = \frac{2}{1 + {}^{A_2}\!/\!_{A_1}}\left(\frac{\Delta p}{\rho_{\text{H20}} g}\right) \quad \text{(This is the same as Eqn. (5)!)}$$

2.3. Pressure Forces on Submerged Surfaces and Center of Pressure

2.3.1. Flat Surfaces

Recall from Chapter 1 that the small pressure force $d\boldsymbol{F}_p$ acting on a surface with a small area $d\boldsymbol{A}$ is,

$$d\boldsymbol{F}_p = p(-d\boldsymbol{A}). \tag{2.30}$$

This force relationship was written specifically for a differentially small area since it's possible that over a large area, the pressure and the direction of the area could vary over the area (Figure 2.7). Thus, to find the total pressure force on the whole area, the (small) force on a small area, where the area direction and pressure are well defined, is calculated first and then these are added, or integrated, over the whole area, i.e.,

$$\boldsymbol{F}_P = \int_A d\boldsymbol{F}_p = \int_A p(-d\boldsymbol{A}). \tag{2.31}$$

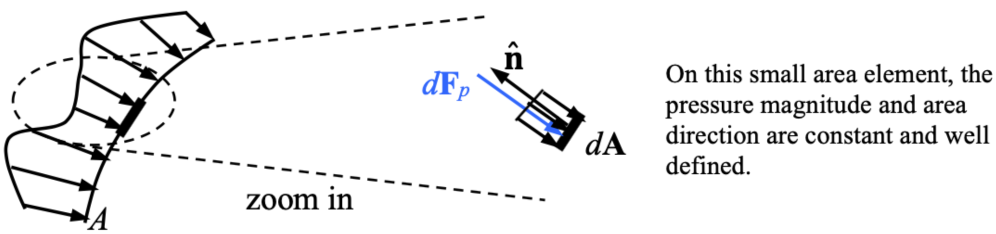


Figure 2.7. A sketch showing how the pressure magnitude and area orientation may change over a large area. However, over a differentially-small area, both the pressure and surface orientation are well defined.

Let's consider the example of a fish tank completely filled with water, as shown in Figure 2.8. We wish to determine the net pressure force acting on bottom and right tank walls. Start first with the pressure force

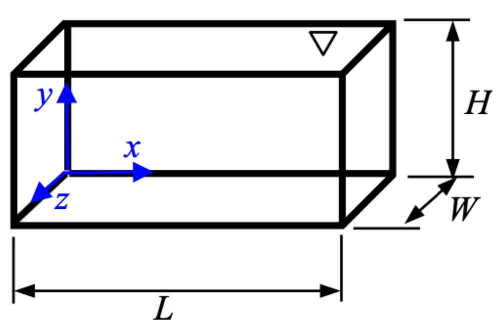


Figure 2.8. A completely-filled fish tank used in the example.

on the tank bottom, (Figure 2.9),

$$\boldsymbol{F}_{p,\text{bottom}} = \int_{z=0}^{z=W} \int_{x=0}^{x=L} p \underbrace{\left[-dxdz\hat{\boldsymbol{j}}\right]}_{=-d\boldsymbol{A}} = -\hat{\boldsymbol{j}} \int_{z=0}^{z=W} \int_{x=0}^{x=L} \underbrace{\rho g H}_{=p_{\text{gage}}} dxdz = -\hat{\boldsymbol{j}} \rho g H W L, \tag{2.32}$$

where, at the bottom of the tank, the gage pressure remains constant at,

$$p_{\substack{\text{bottom,}\\ \text{gage}}} = \rho g H. \tag{2.33}$$

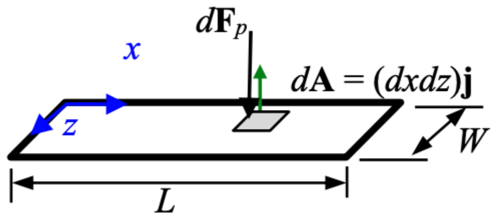


Figure 2.9. The bottom surface of the fish tank.

Notes:

(1) The magnitude of the pressure force on the bottom is equal to the weight of the water in the tank. This makes sense because if there are no shear stresses at the side walls, then the pressure force at the bottom of the tank must support all of the weight of the liquid sitting above it.

(2) A gage pressure is used in Eq. (2.32) to simplify the pressure force calculation. Since there is atmosphere on the other side of the tank bottom, then the gage pressure due to the atmosphere is zero ($p_{\text{atm,gage}} = 0$) and the corresponding pressure force is zero. We get the same result as Eq. (2.32) if absolute pressures are used everywhere instead,

$$\boldsymbol{F}_{p,\text{bottom}} = \underbrace{\int_{z=0}^{z=W}\int_{x=0}^{x=L}(p_{\text{atm}} + \rho g H)(-dxdz\hat{\boldsymbol{j}})}_{\substack{\text{pressure force due to water}\\ \text{using an absolute pressure}}} + \underbrace{\int_{z=0}^{z=W}\int_{x=0}^{x=L}(p_{\text{atm}})(dxdz\hat{\boldsymbol{j}})}_{\substack{\text{pressure force due to}\\ \text{atmosphere using}\\ \text{an absolute pressure}}} = -\hat{\boldsymbol{j}}\rho g H W L. \tag{2.34}$$

Note that the unit normal vector for the atmospheric side (bottom side, second integral) is in the opposite direction of the unit normal vector for the water side (first integral) since we're on opposite sides of the wall.

(3) Since the pressure and the area orientation don't vary over the bottom surface, we could have also found the pressure force on the bottom of the tank using,

$$\boldsymbol{F}_{p,\text{bottom}} = p(-\boldsymbol{A}) = \rho g H(-WL\hat{\boldsymbol{j}}). \tag{2.35}$$

It's important to emphasize that we can only avoid integration if both the pressure and area orientation are constant on the macroscopic area.

Now let's calculate the pressure force acting on the right side wall (Figure 2.10),

$$\boldsymbol{F}_{p,\text{right}} = \int_{z=0}^{z=W}\int_{y=0}^{y=H} p\underbrace{\left[dydz\hat{\boldsymbol{i}}\right]}_{=-d\boldsymbol{A}} = \hat{\boldsymbol{i}}\int_{z=0}^{z=W}\int_{y=0}^{y=H}\underbrace{\rho g(H-y)}_{=p_{\text{gage}}}dydz = \hat{\boldsymbol{i}}\frac{1}{2}\rho g H^2 W. \tag{2.36}$$

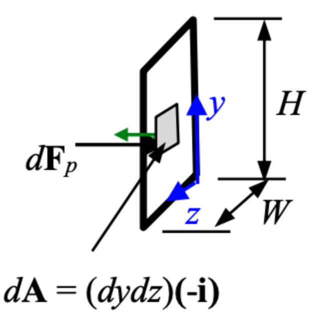


Figure 2.10. The right surface of the fish tank.

Notes:

(1) Recall from the diagram that the coordinate system is located at the bottom of the tank. Thus, the (gage) pressure varies as,

$$p_{\text{gage}} = \rho g(H - y). \tag{2.37}$$

This pressure still varies linearly with depth, as shown in Figure 2.11.

Figure 2.11. The pressure variation with depth in the fish tank example.

(2) The small area element $d\boldsymbol{A} = -dydz\hat{\boldsymbol{\imath}}$ (Figure 2.10) is used since the pressure has a well-defined value on this area. Since the pressure only varies in the y direction, we could have also used the area element $d\boldsymbol{A} = -Wdy\hat{\boldsymbol{\imath}}$ (Figure 2.12). The pressure is well defined on this "strip" of area too,

$$\boldsymbol{F}_{p,\text{right}} = \int_{y=0}^{y=H} p(-Wdy\hat{\boldsymbol{\imath}}) = \hat{\boldsymbol{\imath}}W\int_{y=0}^{y=H} \rho g(H-y)dy = \hat{\boldsymbol{\imath}}\rho g\frac{1}{2}H^2W. \tag{2.38}$$

A vertical strip of area, i.e., $d\boldsymbol{A} = -Hdz\hat{\boldsymbol{\imath}}$, can't be used to determine the pressure force since the pressure isn't well defined on this surface. The pressure varies in the y direction so over this vertical strip, the pressure doesn't remain constant.

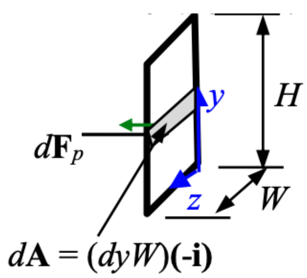


Figure 2.12. An alternate, and easier differential area for integrating the pressure force on the right side wall.

(3) The pressure force is equal in magnitude to the area under the pressure curve shown in Note #1,

$$|d\boldsymbol{F}_p| = \frac{1}{2}\underbrace{(\rho g H)}_{\text{base}}\underbrace{(H)}_{\text{height}}\underbrace{(W)}_{\text{depth}} = \frac{1}{2}\rho g H^2 W. \tag{2.39}$$

This same behavior is true for the pressure force on the base.

Now that we've determined the resultant pressure force on the right surface, let's determine where this resultant force acts (Figure 2.13). This location is known as the center of pressure (CP). The center of pressure is found by ensuring that the moments generated by the resultant pressure force equal the moments

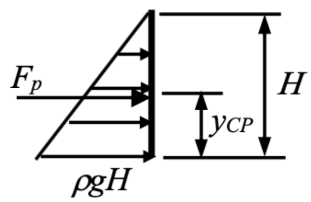


Figure 2.13. A sketch showing the distributed pressure forces, the resultant pressure force, and the location of the center of pressure.

generated by the actual, distributed pressure forces. Consider the right side of the tank and Figure 2.10. Balancing moments about the origin,

$$\underbrace{\boldsymbol{x}_{CP} \times \boldsymbol{F}_{p,\text{right}}}_{\substack{\text{moment due to}\\ \text{resultant force}\\ \text{acting at the CP}}} = \int_{y=0}^{y=H} \int_{z=0}^{z=W} \underbrace{(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})}_{=\text{moment arm}} \times \underbrace{p(-d\boldsymbol{A})}_{=d\boldsymbol{F}_p} \tag{2.40}$$

$$(x_{CP}\hat{\boldsymbol{i}} + y_{CP}\hat{\boldsymbol{j}} + z_{CP}\hat{\boldsymbol{k}}) \times \left(\frac{1}{2}\rho g H^2 W \hat{\boldsymbol{i}}\right) = \int_0^H \int_0^W (x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}) \times \left[\rho g (H-y) dz dy (\hat{\boldsymbol{i}})\right], \tag{2.41}$$

$$y_{CP}\frac{1}{2}\rho g H^2 W(-\hat{\boldsymbol{k}}) + z_{CP}\frac{1}{2}\rho g H^2 W(\hat{\boldsymbol{j}}) = \int_0^H y\rho g(H-y) W dy(-\hat{\boldsymbol{k}}) + \int_0^H \frac{1}{2}W^2 \rho g (H-y) dy(\hat{\boldsymbol{j}}), \tag{2.42}$$

$$y_{CP}\frac{1}{2}\rho g H^2 W(-\hat{\boldsymbol{k}}) + z_{CP}\frac{1}{2}\rho g H^2 W(\hat{\boldsymbol{j}}) = \frac{1}{6}\rho g H^3 W(-\hat{\boldsymbol{k}}) + \frac{1}{4}W^2 \rho g H^2(\hat{\boldsymbol{j}}), \tag{2.43}$$

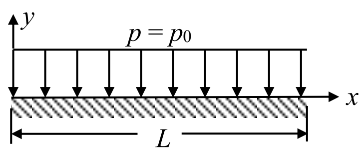
$$\therefore y_{CP} = \frac{1}{3}H \text{ and } z_{CP} = \frac{1}{2}W. \tag{2.44}$$

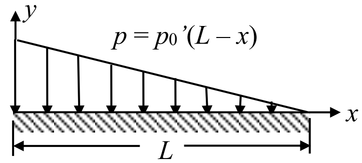
The center of pressure in the x direction is undefined since the resultant and distributed pressure forces act in the x direction and, thus, there is no moment generated by the forces about the x axis.

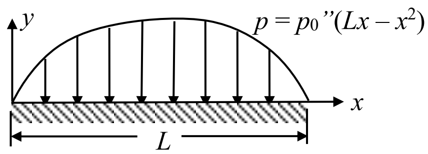
Notes:

(1) The center of pressure is also equal to the center of area under the pressure distribution curve.
(2) We can take moments about any location and get the same result.
(3) The center of pressure for the right wall in the z direction may also be determined from symmetry.

For each of the following pressure profiles,
a. Determine the magnitude of the total pressure force acting on the horizontal plate.
b. Determine the location of the center of pressure.
Assume the plate has unit depth in the z direction. Show all of your work.

1. 

2. 

3. 

SOLUTION:

The total pressure force may be found via integration of the differential pressure force.

$$\left|F_p\right| = \int_{x=0}^{x=L} p\,\underbrace{dx(1)}_{=dA} \qquad \text{(Note: The differential area is } dA = dx(1) \text{ since the plate has unit depth.)} \tag{1}$$

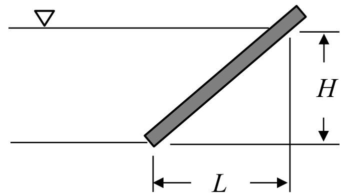
1. $$\left|F_p\right| = \int_{x=0}^{x=L} p_0 dx \Rightarrow \boxed{\left|F_p\right| = p_0 L} \tag{2}$$
2. $$\left|F_p\right| = \int_{x=0}^{x=L} p_0'(L-x)dx = p_0'\left(L^2 - \tfrac{1}{2}L^2\right) \Rightarrow \boxed{\left|F_p\right| = \tfrac{1}{2}p_0'L^2} \tag{3}$$
3. $$\left|F_p\right| = \int_{x=0}^{x=L} p_0''\left(Lx - x^2\right)dx = p_0''\left(\tfrac{1}{2}L^3 - \tfrac{1}{3}L^3\right) \Rightarrow \boxed{\left|F_p\right| = \tfrac{1}{6}p_0''L^3} \tag{4}$$

The center of pressure may be found by equating the moment resulting from the pressure distribution to the moment caused by the total pressure force acting at the center of pressure.

$$\int_{x=0}^{x=L} xp\,\underbrace{dx(1)}_{=dA} = x_{CP}F_p \Rightarrow x_{CP} = \frac{1}{F_p}\int_{x=0}^{x=L} xp\,\underbrace{dx(1)}_{=dA} \tag{5}$$

1. $$x_{CP} = \frac{1}{p_0 L}\int_{x=0}^{x=L} xp_0 dx = \frac{1}{p_0 L}\left(\tfrac{1}{2}p_0 L^2\right) \Rightarrow \boxed{x_{CP} = \tfrac{1}{2}L} \tag{6}$$
2. $$x_{CP} = \frac{1}{\tfrac{1}{2}p_0'L^2}\int_{x=0}^{x=L} xp_0'(L-x)dx = \frac{1}{\tfrac{1}{2}p_0'L^2}\left[p_0'\left(\tfrac{1}{2}L^3 - \tfrac{1}{3}L^3\right)\right] \Rightarrow \boxed{x_{CP} = \tfrac{1}{3}L} \tag{7}$$
3. $$x_{CP} = \frac{1}{\tfrac{1}{6}p_0''L^3}\int_{x=0}^{x=L} xp_0''\left(Lx - x^2\right)dx = \frac{1}{\tfrac{1}{6}p_0''L^3}\left[p_0''\left(\tfrac{1}{3}L^4 - \tfrac{1}{4}L^4\right)\right] \Rightarrow \boxed{x_{CP} = \tfrac{1}{2}L} \tag{8}$$

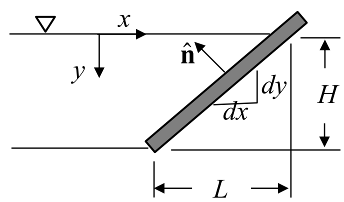
Calculate the net horizontal and vertical forces acting on the planar surface shown below. The surface has a width w into the page.



SOLUTION:

One approach to finding the net force on the wall is to integrate the pressure force along the wall,

$$\mathbf{F}_p = \int_A -p\,d\mathbf{A}\,, \tag{1}$$



where,

$$p = \rho g y\,, \tag{2}$$

and,

$$d\mathbf{A} = -wdy\hat{\mathbf{e}}_x - wdx\hat{\mathbf{e}}_y\,. \tag{3}$$

Note that since we'll be integrating in the y direction (since the pressure varies in that direction), we should express dx in terms of dy,

$$\frac{dy}{dx} = \frac{H}{L} \Rightarrow dx = \left(\frac{L}{H}\right)dy\ . \tag{4}$$

Substituting and integrating as y goes from zero to H,

$$\mathbf{F}_p = \int_{y=0}^{y=H} -(\rho g y)\left[-wdy\hat{\mathbf{e}}_x - w\left(\frac{L}{H}\right)dy\hat{\mathbf{e}}_y\right] = \rho g w\left[\hat{\mathbf{e}}_x \int_{y=0}^{y=H} y\,dy + \hat{\mathbf{e}}_y\left(\frac{L}{H}\right)\int_{y=0}^{y=H} y\,dy\right], \tag{5}$$

$$\mathbf{F}_p = \rho g w\left[\tfrac{1}{2}H^2\hat{\mathbf{e}}_x + \tfrac{1}{2}\left(\frac{L}{H}\right)H^2\hat{\mathbf{e}}_y\right], \tag{6}$$

$$\boxed{\mathbf{F}_p = \tfrac{1}{2}\rho g w H^2\hat{\mathbf{e}}_x + \tfrac{1}{2}\rho g w L H\hat{\mathbf{e}}_y}. \tag{7}$$

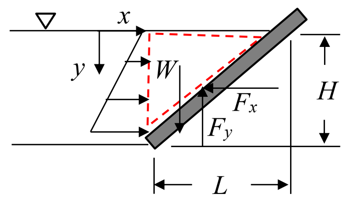
We could have also solved the integral by splitting it into two parts,

$$\mathbf{F}_p = \int_{y=0}^{y=H} -(\rho g y)(-wdy\hat{\mathbf{e}}_x) + \int_{x=0}^{x=L} -(\rho g y)(-wdx\hat{\mathbf{e}}_y) = \tfrac{1}{2}\rho g w H^2\hat{\mathbf{e}}_x + \int_{x=0}^{x=L} -\left[\rho g\left(\frac{H}{L}\right)x\right](-wdx\hat{\mathbf{e}}_y)\ , \tag{8}$$

$$\mathbf{F}_p = \tfrac{1}{2}\rho g w H^2\hat{\mathbf{e}}_x + \tfrac{1}{2}\rho g w\left(\frac{H}{L}\right)L^2\hat{\mathbf{e}}_y = \tfrac{1}{2}\rho g w H^2\hat{\mathbf{e}}_x + \tfrac{1}{2}\rho g w H L\hat{\mathbf{e}}_y \quad \text{Same answer as before!} \tag{9}$$

Note that in the 2nd integral in Eq. (8), the y dependence on x needed to be made explicit in order to integrate properly with respect to x. An approach similar to what was used to derive Eq. (4) was utilized.

An alternate approach to solving this problem is to balance forces on the dashed volume of fluid shown below.



$$\sum F_x = 0 = \int_{y=0}^{y=H} (\rho g y)(w\,dy) - F_x \Rightarrow F_x = \tfrac{1}{2}\rho g w H^2 \quad \text{The same answer as before!} \tag{10}$$

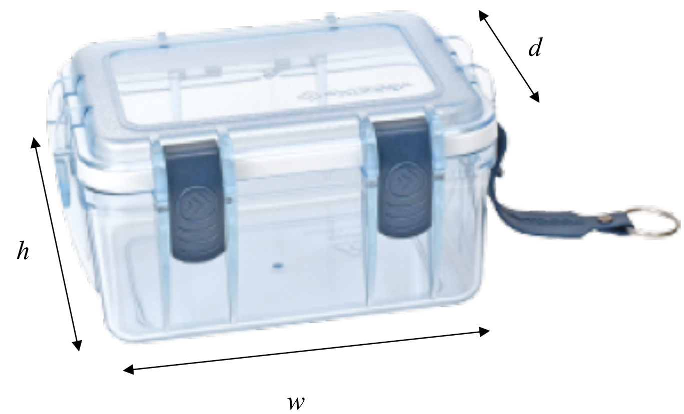
$$\sum F_y = 0 = W - F_y = \rho \tfrac{1}{2} L H w g - F_y \Rightarrow F_y = \tfrac{1}{2}\rho g L H w \quad \text{The same answer as before!} \tag{11}$$

Note that from Newton's 3rd Law, the force the wall exerts on the fluid is equal and opposite to the force the fluid exerts on the wall.

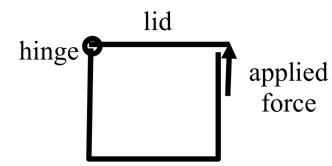
Your professor purchased a watertight box to hold his camera while traveling to Ft. Myers Beach, FL during winter break. The box's dimensions are shown in the photograph. During the flight, he opened the box and then re-sealed it. Upon reaching his destination, he found that he had significant difficulty trying to open the box.

a. Why was opening the box such a challenge?

b. Estimate the force required to open the box if the force is applied at the front of the box. Note that the box is hinged at the back.



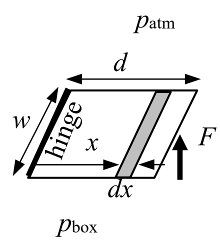
$w = 6.46$ in
$d = 5.11$ in
$h = 3.25$ in



SOLUTION:

The box was difficult to open because the air in the interior of the box was at the cabin pressure of the aircraft (required to be pressurized to a maximum altitude of 8000 ft altitude[1]) and the air outside the box was at the local atmospheric pressure (Ft. Myers Beach, FL which is at sea level). This pressure difference resulted in a net pressure force acting to hold the lid shut.

Sum moments about the lid's hinge,



$$\sum M_{\text{hinge}} = 0 = Fd - \int_{x=0}^{x=d} \underbrace{x}_{\text{moment arm}} \underbrace{\left(p_{\text{atm}} - p_{\text{box}}\right)}_{\text{pressure difference}} \underbrace{w\,dx}_{=dA}, \tag{1}$$

$$Fd = \left(p_{\text{atm}} - p_{\text{box}}\right)\frac{1}{2}wd^2, \tag{2}$$

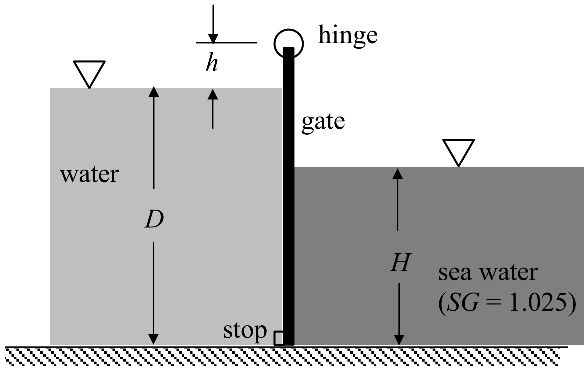
$$\boxed{F = \left(p_{\text{atm}} - p_{\text{box}}\right)\frac{1}{2}wd}. \tag{3}$$

where

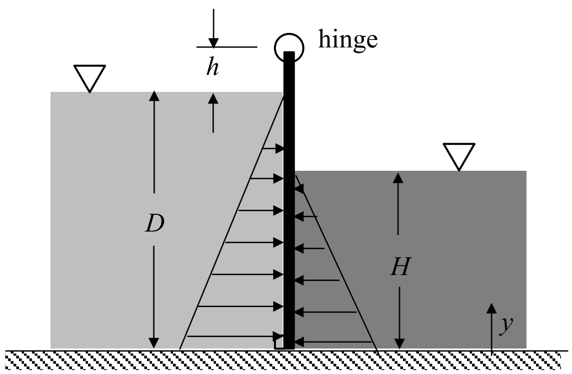
$p_{\text{FMB}} = p_{\text{sea level}} = 14.7$ psia (using a U.S. Standard Atmosphere)
$p_{\text{cabin}} = p_{\text{8000 ft altitude}} = 10.9$ psia (using a U.S. Standard Atmosphere)
$A_{\text{lid}} = wd = (6.46\text{ in})\,(5.11\text{ in}) = 33.0\text{ in}^2$
$\Rightarrow$ $\boxed{F_{\text{lid}} = 62.5\text{ lb}_{\text{f}}!}$

[1] https://en.wikipedia.org/wiki/Cabin_pressurization

The gate shown below has a width of $w = 8$ ft and opens to let fresh water out when the ocean tide drops. The hinge is a height $h = 2$ ft above the freshwater level. At what ocean level H will the gate first open? You may neglect the weight of the gate.



SOLUTION:



Balance moments about the hinge,

$$\sum M_{\text{hinge}} = 0 = \int_{y=0}^{y=D} \underbrace{(D+h-y)}_{\text{moment arm length}} \underbrace{\rho_{\text{fresh}} g(D-y)}_{\text{pressure}} \underbrace{(w\,dy)}_{\text{area}} - \int_{y=0}^{y=H} \underbrace{(D+h-y)}_{\text{moment arm length}} \underbrace{\rho_{\text{sea}} g(H-y)}_{\text{pressure}} \underbrace{(w\,dy)}_{\text{area}}, \tag{1}$$

$$\int_{y=0}^{y=D} (D+h-y)\rho_{\text{fresh}} g(D-y)(w\,dy) = \int_{y=0}^{y=H} (D+h-y)\rho_{\text{sea}} g(H-y)(w\,dy), \tag{2}$$

$$\rho_{\text{fresh}} \int_{y=0}^{y=D} (D+h-y)(D-y)\,dy = \rho_{\text{sea}} \int_{y=0}^{y=H} (D+h-y)(H-y)\,dy, \tag{3}$$

$$\rho_{\text{fresh}} \int_{y=0}^{y=D} \left(D^2 + Dh - 2Dy - hy + y^2\right) dy = \rho_{\text{sea}} \int_{y=0}^{y=H} \left(DH + Hh - Hy - Dy - hy + y^2\right) dy, \tag{4}$$

$$\rho_{\text{fresh}} \left[\left(D^2 + Dh\right) y - \tfrac{1}{2}(2D+h) y^2 + \tfrac{1}{3} y^3 \right]_{y=0}^{y=D} = \rho_{\text{sea}} \left[(DH + Hh) y - \tfrac{1}{2}(H+h+D) y^2 + \tfrac{1}{3} y^3 \right]_{y=0}^{y=H}, \tag{5}$$

$$\rho_{\text{fresh}} \left[\left(D^2 + Dh\right) D - \tfrac{1}{2}(2D+h) D^2 + \tfrac{1}{3} D^3 \right] = \rho_{\text{sea}} \left[(DH + Hh) H - \tfrac{1}{2}(H+h+D) H^2 + \tfrac{1}{3} H^3 \right], \tag{6}$$

$$D^3 + D^2 h - D^3 - \tfrac{1}{2} D^2 h + \tfrac{1}{3} D^3 = \frac{\rho_{\text{sea}}}{\rho_{\text{fresh}}} \left(DH^2 + H^2 h - \tfrac{1}{2} H^3 - \tfrac{1}{2} H^2 h - \tfrac{1}{2} DH^2 + \tfrac{1}{3} H^3 \right), \tag{7}$$

$$\tfrac{1}{2}D^2h+\tfrac{1}{3}D^3 = SG_{\text{sea}}\left(-\tfrac{1}{6}H^3+\tfrac{1}{2}H^2h+\tfrac{1}{2}DH^2\right), \tag{8}$$

$$\tfrac{1}{6}SG_{\text{sea}}H^3-\tfrac{1}{2}SG_{\text{sea}}\left(D+h\right)H^2+\tfrac{1}{2}D^2h+\tfrac{1}{3}D^3=0\ , \tag{9}$$

$$H^3-3\left(D+h\right)H^2+\frac{\left(3h+2D\right)D^2}{SG_{\text{sea}}}=0\ . \tag{10}$$

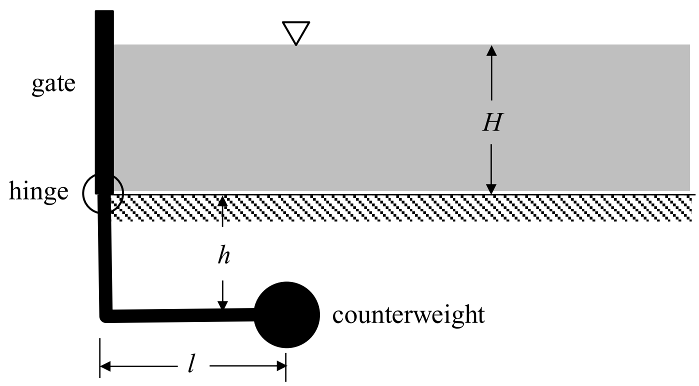
Using the given data,

SG_{sea} = 1.025
h = 2 ft
D = 10 ft

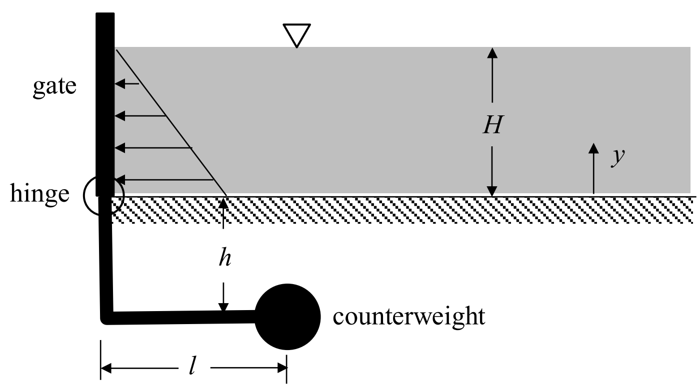
$$\text{Eq. (10)} \Rightarrow\ H^3-\left(36\text{ ft}\right)H^2+\left(2536.6\text{ ft}^3\right)=0 \tag{11}$$

Solving this equation numerically gives $\boxed{H = 9.85\text{ ft}}$
For sea levels less than this critical value, the gate will open.

The w = 4 ft wide gate shown in the figure pivots about a hinge. The gate is held in place by a counterweight with a weight of W = 2000 lb$_f$, which is located a distance h = 5 ft below the base of the water and a distance l = 3 ft from the gate. Determine the depth of the water, H, for which the gate remains in the equilibrium position shown. You may assume the gate mass is small compared to the counterweight mass, and that the hinge friction is negligible.



SOLUTION:



Balance moments about the hinge,

$$\sum M_{\text{hinge}} = 0 = \int_{y=0}^{y=H} \underbrace{y}_{\text{moment arm length}} \underbrace{\rho g(H-y)}_{\text{pressure}} \underbrace{(w dy)}_{\text{area}} - \underbrace{lW}_{\text{moment due to counterweight}}, \tag{1}$$

$$\rho g w \int_{y=0}^{y=H} y(H-y)dy = lW, \tag{2}$$

$$\rho g w \left(\tfrac{1}{2}Hy^2 - \tfrac{1}{3}y^3\right)_{y=0}^{y=H} = lW, \tag{3}$$

$$\tfrac{1}{6}H^3 = \frac{lW}{\rho g w}, \tag{4}$$

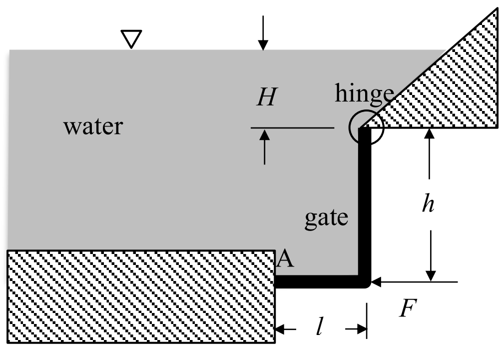
$$\boxed{H = \left(\frac{6lW}{\rho g w}\right)^{1/3}}. \tag{5}$$

Using the given data,

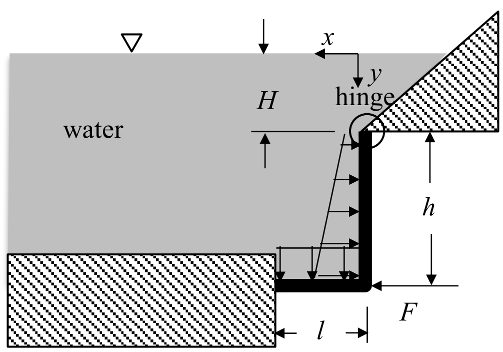
ρg = 62.4 $\text{lb}_\text{f}/\text{ft}^3$
W = 2000 lb_f
l = 3 ft
w = 4 ft

$\Rightarrow$ $\boxed{H = 5.2 \text{ ft}}$

The rigid, L-shaped gate shown in the figure can rotate about the hinge and rests against the rigid support at point A. What is the minimum horizontal force, F required to hold the gate closed if its width is $w = 3$ m and the lengths are $h = 4$ m and $l = 2$ m? The height of the free surface above the hinge is $H = 3$ m. You may neglect the weight of the gate and the friction in the hinge. Note that the back of the gate is exposed to the atmosphere.



SOLUTION:



Balance moments about the hinge,

$$\sum M_{\text{hinge}} = 0 = \int_{y=H}^{y=H+h} \underbrace{(y-H)}_{\text{moment arm length}} \underbrace{\rho g y}_{\text{pressure}} \underbrace{(w dy)}_{\text{area}} + \int_{x=0}^{x=l} \underbrace{x}_{\text{moment arm length}} \underbrace{\rho g (H+h)}_{\text{pressure}} \underbrace{(w dx)}_{\text{area}} - \underbrace{hF}_{\text{moment due to applied force}}, \tag{1}$$

$$\rho g w \int_{y=H}^{y=H+h} (y-H) y \, dy + \rho g (H+h) w \int_{x=0}^{x=l} x \, dx - hF = 0, \tag{2}$$

$$hF = \rho g w \left(\tfrac{1}{3} y^3 - \tfrac{1}{2} H y^2 \right)_{y=H}^{y=H+h} + \tfrac{1}{2} \rho g (H+h) w l^2, \tag{3}$$

$$hF = \rho g w \left\{ \left(\tfrac{1}{3} \left[(H+h)^3 - H^3 \right] - \tfrac{1}{2} H \left[(H+h)^2 - H^2 \right] \right) \right\} + \tfrac{1}{2} \rho g (H+h) w l^2, \tag{4}$$

$$F = \rho g w \left[\tfrac{1}{2} H h + \tfrac{1}{3} h^2 + \tfrac{1}{2} \left(H/h + 1 \right) l^2 \right]. \tag{5}$$

Using the given data,

ρ = 1000 kg/m^3
g = 9.81 m/s^2
w = 3 m
H = 3 m
h = 4 m
l = 2 m

$\Rightarrow$ $\boxed{F = 437\text{ kN}}$

A rectangular block of concrete (SG=2.5) is used as a retaining wall or dam for a reservoir of water:

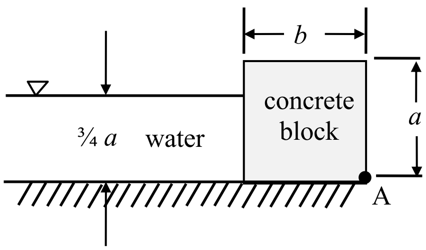


Figure (a)

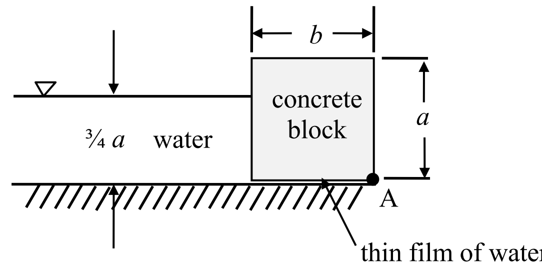


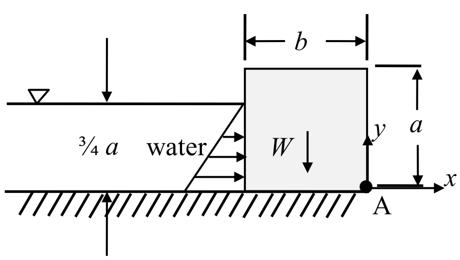
Figure (b)

The block has a height, *a*, a breadth, *b*, and unit depth into the page. The depth of the water is 3*a*/4.

a. Determine the critical ratio, *b*/*a*, below which the block will be overturned by the water (figure a). Assume the block does not slide on the base but can rotate about the point A. For figure (a), there is no fluid underneath the block.
b. What is the critical ratio, *b*/*a*, if there is seepage and a thin film of water forms under the block (figure b)? Assume that a seal at point A prevents water from flowing out from underneath the block.

SOLUTION:

Draw a free body diagram of the block. Note that when the block is on the verge of tipping over, the vertical force the ground exerts on the block is zero.



Sum moments about point A.

$$\sum M_A = 0 = \left(\tfrac{1}{2}b\right)W - \int_{y=0}^{y=\frac{3}{4}a} y \underbrace{p\underbrace{\left(dy\cdot 1\right)}_{=dA}}_{=dF} \tag{1}$$

where

$$W = \rho_{\text{block}}\left(b\cdot a\cdot 1\right)g \tag{2}$$

$$p = \rho_{H20}\,g\left(\tfrac{3}{4}a - y\right) \quad \text{(note that this is a gage pressure)} \tag{3}$$

Substitute and simplify.

$$\left(\tfrac{1}{2}b\right)\left(\rho_{\text{block}}bag\right) - \int_{y=0}^{y=\frac{3}{4}a} y\rho_{H20}\,g\left(\tfrac{3}{4}a - y\right)dy = 0 \tag{4}$$

$$\tfrac{1}{2}\rho_{\text{block}}b^2a-\rho_{H20}\int_{y=0}^{y=\frac{3}{4}a}\left(\tfrac{3}{4}ay-y^2\right)dy=0 \tag{5}$$

$$\tfrac{1}{2}\rho_{\text{block}}b^2a-\rho_{H20}\left(\tfrac{3}{8}ay^2-\tfrac{1}{3}y^3\right)\Big|_{y=0}^{y=\frac{3}{4}a}=0 \tag{6}$$

$$\tfrac{1}{2}\rho_{\text{block}}b^2a-\rho_{H20}\left(\tfrac{3}{8}\tfrac{9}{16}a^3-\tfrac{1}{3}\tfrac{27}{64}a^3\right)=0 \tag{7}$$

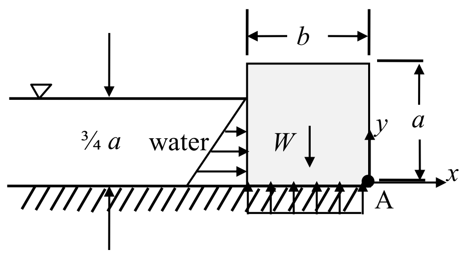
$$\tfrac{1}{2}SG_{\text{block}}b^2a-\tfrac{9}{128}a^3=0 \tag{8}$$

$$\left(\frac{b}{a}\right)^2=\frac{9}{64}\frac{1}{SG_{\text{block}}} \tag{9}$$

$$\therefore\frac{b}{a}=\frac{3}{8}\frac{1}{\sqrt{SG_{\text{block}}}}\quad\text{when the block is just about to tip over} \tag{10}$$

Thus, the block will tip over for $SG_{\text{block}} = 2.5$ if $\boxed{b/a < 0.237}$.

Now draw the free body diagram for a block with a thin liquid layer underneath it.



Sum moments about point A.

$$\sum M_A=0=\left(\tfrac{1}{2}b\right)W-\int_{y=0}^{y=\frac{3}{4}a}yp\left(dy\cdot1\right)-\left(\tfrac{1}{2}b\right)\underbrace{\underbrace{\rho_{H20}g\tfrac{3}{4}a}_{=p}\underbrace{\left(b\cdot1\right)}_{=A}}_{=F} \tag{11}$$

where the weight and pressure on the side are given in Eqs. (2) and (3). The last term in the previous equation is the (gage) pressure that the liquid layer on the bottom exerts on the block.

Substitute and simplify.

$$\tfrac{1}{2}SG_{\text{block}}b^2a-\tfrac{9}{128}a^3-\tfrac{3}{8}ab^2=0 \tag{12}$$

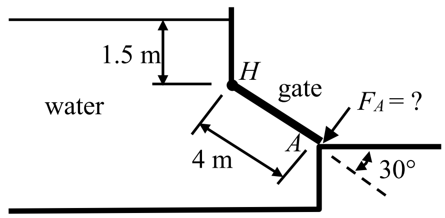
$$SG_{\text{block}}b^2a-\tfrac{9}{64}a^3-\tfrac{3}{4}ab^2=0 \tag{13}$$

$$SG_{\text{block}}\left(\frac{b}{a}\right)^2-\frac{3}{4}\left(\frac{b}{a}\right)^2=\frac{9}{64} \tag{14}$$

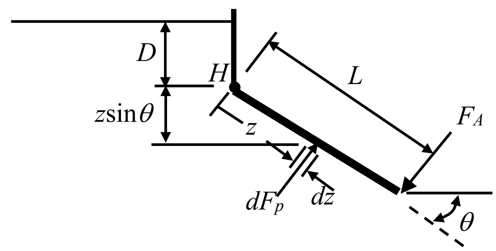
$$\therefore\frac{b}{a}=\frac{3}{8}\left(SG_{\text{block}}-\frac{3}{4}\right)^{-\frac{1}{2}}\quad\text{when the block is just about to tip over} \tag{15}$$

Thus, the block will tip over for $SG_{\text{block}} = 2.5$ if $\boxed{b/a < 0.283}$.

The 3 m wide (into the page) gate shown in the figure is hinged at point H. Calculate the force required at point A to hold the gate closed



SOLUTION:
Draw a free body diagram of the gate, just as the gate is about to open.



Sum moments about the hinge H and set them equal to zero since the gate isn't accelerating,

$$\sum M_H = 0 = \int_{z=0}^{z=L} z \underbrace{\underbrace{\rho g\left(D + z\sin\theta\right)}_{=p}\underbrace{\left(Tdz\right)}_{=dA}}_{=dF_p} - LF_A\,, \tag{1}$$

where T is the thickness of the gate into the page. The first z in the integral is the moment arm out to the differential hydrostatic pressure force dF_P acting on area $dA = Tdz$. Note that the pressure is a function of the depth from the free surface, $D + z\sin\theta$.

Simplify Eq. (1) and solve for F_A,

$$LF_A = \rho g T \int_{z=0}^{z=L} \left(Dzdz + \sin\theta z^2 dz\right), \tag{2}$$

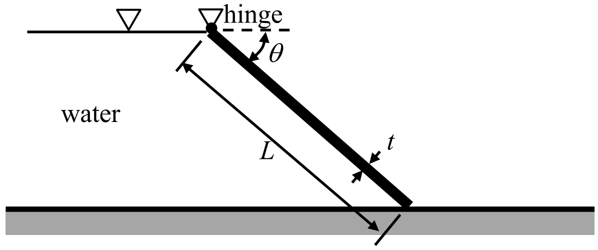
$$LF_A = \rho g T\left(\tfrac{1}{2}DL^2 + \tfrac{1}{3}L^3\sin\theta\right), \tag{3}$$

$$\boxed{F_A = \rho g T L\left(\tfrac{1}{2}D + \tfrac{1}{3}L\sin\theta\right)}. \tag{4}$$

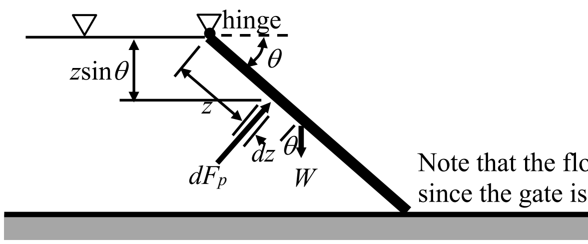
Using the given data,

ρ = 1000 kg/m^3,
g = 9.81 m/s^2,
T = 3 m,
L = 4 m,
D = 1.5 m,
θ = 30°,
=> $\boxed{\square_{\square} = 167\ \text{kN}}$

A plane gate of uniform thickness *t* and width into the page *w* holds back a depth of water as shown. Find the minimum weight of the gate needed to keep the gate closed.



SOLUTION:
Draw a free body diagram of the gate.



Note that the floor exerts no force on the gate since the gate is just about to open.

Sum moments about the gate's hinge, noting that the gate is in equilibrium and just about to open,

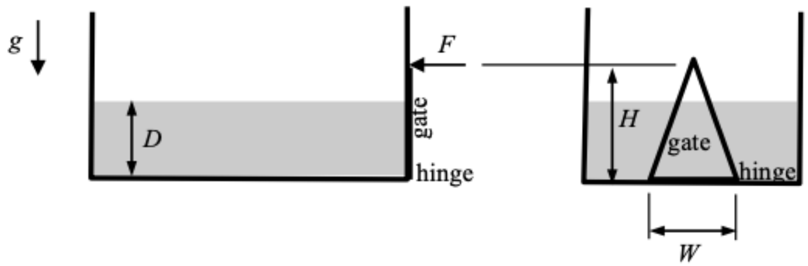
$$\sum M_{\text{hinge}} = 0 = -\left(\frac{L}{2}\right)\left(W\cos\theta\right) + \int_{z=0}^{z=L} z\underbrace{\underbrace{\left(\rho g z\sin\theta\right)}_{=p}\underbrace{\left(w\,dz\right)}_{=dA}}_{=dF_p}, \tag{1}$$

$$\left(\frac{L}{2}\right)\left(W\cos\theta\right) = \rho g w\sin\theta\int_{z=0}^{z=L} z^2\,dz, \tag{2}$$

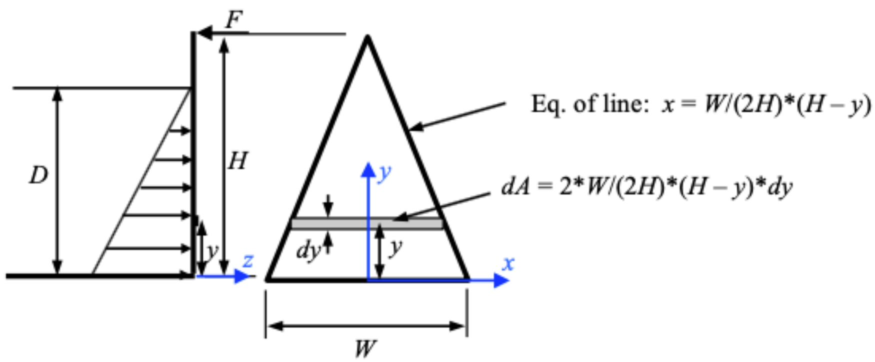
$$\left(\frac{L}{2}\right)\left(W\cos\theta\right) = \frac{1}{3}\rho g w L^3\sin\theta, \tag{3}$$

$$\boxed{W = \tfrac{2}{3}\rho g w L^2\tan\theta}. \tag{4}$$

The tank shown below is partially filled with a liquid of density ρ and is open to the atmosphere. A triangular gate is hinged at the bottom and held closed by a force applied at the top. Determine the force F in terms of the liquid density ρ, the acceleration due to gravity g, the liquid depth D, the gate height H, and the gate width W.



SOLUTION:



Balance moments on the gate. Since the pressure varies over the surface of the gate and because the gate width changes with depth, we'll need to integrate the pressure force over the gate surface. To do this, divide the gate into small areas over which the pressure remains constant.

$$\sum \boldsymbol{M}_{hinge} = \boldsymbol{0} = \left(H\hat{\boldsymbol{j}} \times -F\hat{\boldsymbol{k}}\right) + \int_A \underbrace{y\hat{\boldsymbol{j}}}_{\substack{\text{moment}\\\text{arm}}} \times \underbrace{d\boldsymbol{F}_p}_{\substack{\text{pressure}\\\text{force}}}, \tag{1}$$

where,

$$d\boldsymbol{F}_p = -pd\boldsymbol{A}, \tag{2}$$

$$d\boldsymbol{A} = 2\left(\frac{W}{2H}\right)(H-y)dy\left(-\hat{\boldsymbol{k}}\right), \tag{3}$$

$$p_{gage} = \rho g(D-y). \tag{4}$$

Substitute and solve for F,

$$\boldsymbol{0} = HF\hat{\boldsymbol{\imath}} + \int_{y=0}^{y=D} y\rho g(D-y)2\left(\frac{W}{2H}\right)(H-y)dy(-\hat{\boldsymbol{\imath}}), \tag{5}$$

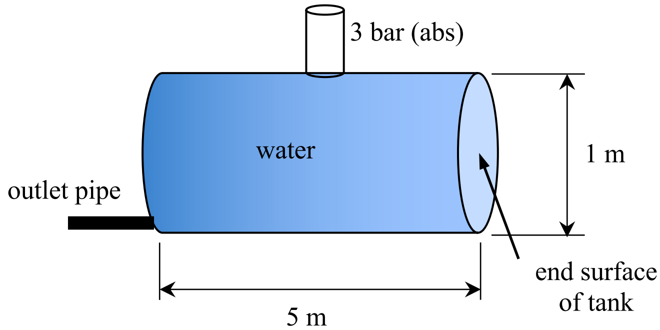
$$HF = \rho g\left(\frac{W}{H}\right)\int_0^D y(D-y)(H-y)dy = \rho g\left(\frac{W}{H}\right)\int_0^D [DHy-(D+H)y^2+y^3]dy, \tag{6}$$

$$F = \rho g\left(\frac{W}{H^2}\right)\left[\frac{1}{2}D^3H - \frac{1}{3}(D+H)D^3 + \frac{1}{4}D^4\right], \tag{7}$$

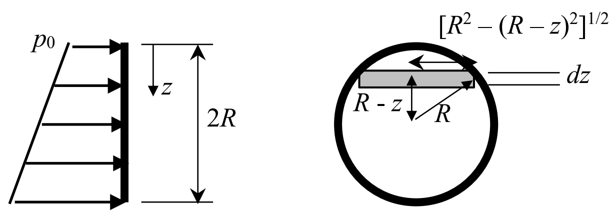
$$F = \rho g\left(\frac{W}{H^2}\right)\left[\frac{1}{2}D^3H - \frac{1}{3}D^4 - \frac{1}{3}HD^3 + \frac{1}{4}D^4\right], \tag{8}$$

$$\boxed{F = \frac{1}{6}\rho g\left(\frac{WD^3}{H^2}\right)\left(H - \frac{1}{2}D\right)}. \tag{9}$$

A cylindrical tank if filled with water. In order to control the flow rate from the tank, a pressure can be applied to the water surface by a compressor. For an applied absolute pressure of 3 bar, calculate the hydrostatic force exerted by the water on the end surface of the tank.



SOLUTION:
Draw the pressure distribution acting on the tank end surface due to the water in the tank.



The hydrostatic pressure force on the tank surface due to the water is,

$$F = \int_{z=0}^{z=2R} p\,dA = \int_{z=0}^{z=2R} \underbrace{\left(p_0 + \rho g z\right)}_{=p} \underbrace{\left[2\sqrt{R^2 - \left(R - z\right)^2}\,dz\right]}_{=dA}, \tag{1}$$

$$F = 2\int_{z=0}^{z=2R} \left(p_0 + \rho g z\right)\sqrt{2Rz - z^2}\,dz = 2\left[p_0 \int_{z=0}^{z=2R} \sqrt{2Rz - z^2}\,dz + \rho g \int_{z=0}^{z=2R} z\sqrt{2Rz - z^2}\,dz\right], \tag{2}$$

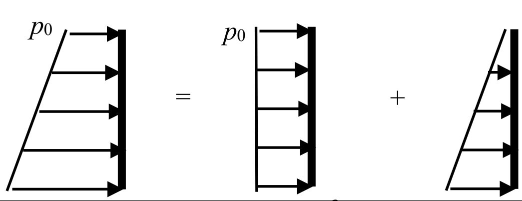
$$F = 2\left[p_0 \frac{\pi R^2}{2} + \rho g \frac{\pi R^3}{2}\right], \tag{3}$$

$$\boxed{F = \pi R^2\left(p_0 + \rho g R\right)}. \tag{4}$$

Using the following parameters:

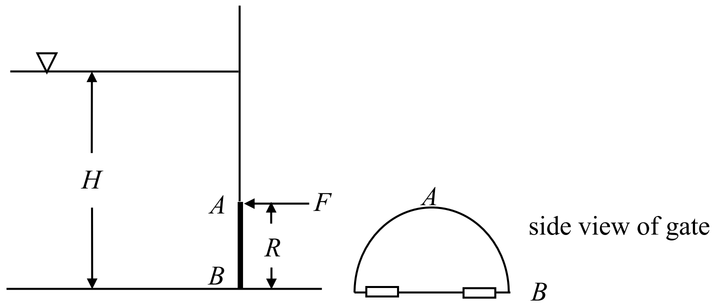
$R = 0.5$ m
$p_0 = 3$ bar (abs) = 300 kPa (abs)
$\rho = 1000$ kg/m^3
$g = 9.81$ m/s^2
=> $\boxed{F = 23.6 \text{ MN}}$.

Note an alternate approach to solving the problem is to break the applied pressure into a constant part at pressure p_0 and the linearly increasing part, as shown in the figures below.

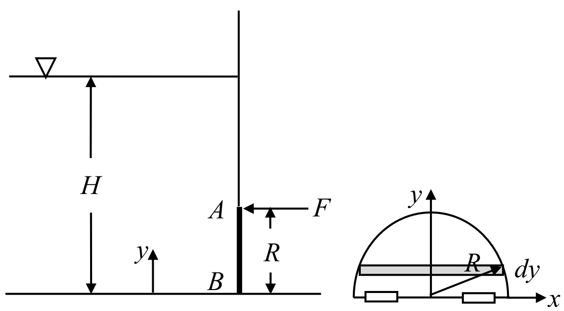


$$F_1 = p_0 \pi R^2 \qquad F_2 = 2\rho g \int_{z=0}^{z=2R} z\sqrt{2Rz - z^2}\,dz$$

A semi-circular plane gate is hinged along B and held by horizontal force *F* applied at point *A*. The liquid in the reservoir is water. Calculate the minimum force required to hold the gate closed. Hint: An integral table or symbolic algebra software will be helpful in solving the integrals that appear in the derivation.



SOLUTION:



Sum moments about point B.

$$\sum M_B = 0 = RF - \int_{y=0}^{y=R} y p dA \tag{1}$$

$$RF = \int_{y=0}^{y=R} y \underbrace{\rho g (H-y)}_{=p_{\text{gage}}} \underbrace{2\sqrt{R^2-y^2}dy}_{=dA} \tag{2}$$

$$F = \frac{2\rho g}{R} \int_{y=0}^{y=R} y(H-y)\sqrt{R^2-y^2}dy \tag{3}$$

$$F = \frac{2\rho g}{R}\left[H\int_{y=0}^{y=R} y\sqrt{R^2-y^2}dy - \int_{y=0}^{y=R} y^2\sqrt{R^2-y^2}dy\right] \tag{4}$$

Evaluate the integrals using an integral table or symbolic algebra software (*e.g.*, Mathematica).

$$F = \frac{2\rho g}{R}\left[-\frac{1}{3}H\left(R^2-y^2\right)^{3/2}\Big|_{y=0}^{y=R} - \frac{1}{8}\left(y\sqrt{R^2-y^2}\left(2y^2-R^2\right)+R^4\tan^{-1}\left(\frac{y}{\sqrt{R^2-y^2}}\right)\right)\Bigg|_{y=0}^{y=R}\right] \tag{5}$$

$$F = \frac{2\rho g}{R}\left(\frac{1}{3}HR^3 - \frac{1}{8}R^4\frac{\pi}{2}\right) \tag{6}$$

$$\boxed{\therefore F = 2\rho g R^2\left(\frac{1}{3}H - \frac{\pi}{16}R\right)} \tag{7}$$

2.3.2. Curved Surfaces

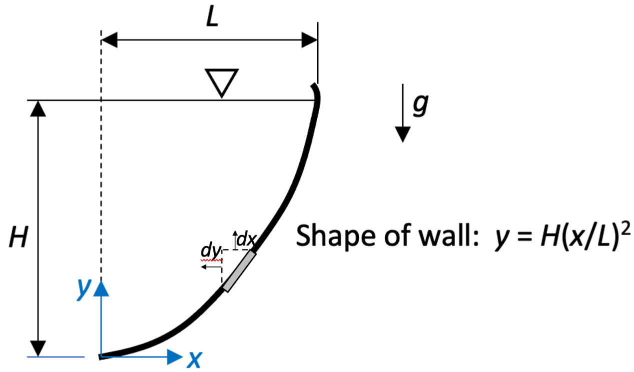


Figure 2.14. The parabolically-shaped wall used in the example.

The resultant pressure force and center of pressure location for curved surfaces may be found in the same way as for flat surfaces. The only significant difference is that the unit normal vectors for the differentially-small area elements may change with position. For example, let's determine the net pressure force and center of pressure on the parabolically-shaped wall shown in Figure 2.14. Assume the wall is planar and has a depth W into the page.

$$\boldsymbol{F}_p = \int_A p(-d\boldsymbol{A}) = \int_A \underbrace{\rho g(H-y)}_{=p_{\text{gage}}} \underbrace{\left[-(W dx\hat{\boldsymbol{j}} - W dy\hat{\boldsymbol{i}})\right]}_{-d\boldsymbol{A}} = -\rho g W \int_A (H-y)(dx\hat{\boldsymbol{j}} - dy\hat{\boldsymbol{i}}). \tag{2.45}$$

Before setting the limits on the integral, note that y is a function of x on the wall surface, which also means that a small displacement in the y direction is related to a small displacement in the x direction,

$$y = H\left(\frac{x}{L}\right)^2 \implies dy = \frac{2H}{L^2} x dx. \tag{2.46}$$

We can use this information to express the integral in terms of a single variable (we'll use x, but we could use y too). Substituting Eq. (2.46) into Eq. (2.45) gives,

$$\boldsymbol{F}_p = -\rho g W \int_{x=0}^{x=L} \left[H - H\left(\frac{x}{L}\right)^2\right]\left(dx\hat{\boldsymbol{j}} - \frac{2H}{L^2} x dx\hat{\boldsymbol{i}}\right), \tag{2.47}$$

$$= -\rho g W H \left[\hat{\boldsymbol{j}} \int_{x=0}^{x=L} \left(1 - \frac{x^2}{L^2}\right) dx - \hat{\boldsymbol{i}} \frac{2H}{L^2} \int_{x=0}^{x=L} \left(x - \frac{x^3}{L^2}\right) dx\right], \tag{2.48}$$

$$= -\rho g W H \left[\hat{\boldsymbol{j}} \left(L - \frac{1}{3} L^3 L^2\right) - \hat{\boldsymbol{i}} \frac{2H}{L^2}\left(\frac{1}{2} L^2 - \frac{1}{4} L^4 L^2\right)\right], \tag{2.49}$$

$$= \rho g W H \left(\frac{1}{2} H\hat{\boldsymbol{i}} - \frac{2}{3} L\hat{\boldsymbol{j}}\right), \tag{2.50}$$

$$\boldsymbol{F}_p = \frac{1}{2}\rho g W H^2 \hat{\boldsymbol{i}} - \frac{2}{3}\rho g W H L\hat{\boldsymbol{j}}. \tag{2.51}$$

This result is the pressure force the fluid exerts on the wall.

The center of pressure is found by balancing moments, identical to what was used for planar surfaces. Balance moments about the origin,

$$\boldsymbol{x}_{CP} \times \boldsymbol{F}_p = \int_A (x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}) \times [\rho g(H-y)] \left[-(W dx\hat{\boldsymbol{j}} - W dy\hat{\boldsymbol{i}})\right] = -\rho g W \int_A (H-y)(xdx + ydy)\hat{\boldsymbol{k}}, \tag{2.52}$$

$$(x_{CP}\hat{\boldsymbol{i}} + y_{CP}\hat{\boldsymbol{j}}) \times \left(\frac{1}{2}\rho g W H^2 \hat{\boldsymbol{i}} - \frac{2}{3}\rho g W H L \hat{\boldsymbol{j}}\right) \tag{2.53}$$

$$= -\rho g W \hat{\boldsymbol{k}} \int_{x=0}^{x=L} \left[H\left(1 - \frac{x^2}{L^2}\right) x dx + H\left(1 - \frac{x^2}{L^2}\right) H \left(\frac{x^2}{L^2}\right) \frac{2H}{L^2} x dx\right], \tag{2.54}$$

$$-\rho g W H \left(x_{CP}\frac{2}{3}L + y_{CP}\frac{1}{2}H\right)\hat{\boldsymbol{k}} = -\rho g W H \hat{\boldsymbol{k}} \int_{x=0}^{x=L} \left[\left(x - \frac{x^3}{L^2}\right) + \frac{2H^2}{L^4}\left(x^3 - \frac{x^5}{L^2}\right)\right] dx, \tag{2.55}$$

$$x_{CP}\frac{2}{3}L + y_{CP}\frac{1}{2}H = \frac{1}{2}L^2 - \frac{1}{4}\frac{L^4}{L^2} + \frac{2H^2}{L^4}\left(\frac{1}{4}L^4 - \frac{1}{6}\frac{L^6}{L^2}\right) = \frac{1}{4}L^2 + \frac{1}{6}H^2, \tag{2.56}$$

$$\therefore y_{CP} = \left(-\frac{4}{3}\frac{L}{H}\right) x_{CP} + \left(\frac{1}{2}\frac{L^2}{H} + \frac{1}{3}H\right). \tag{2.57}$$

The previous equation, which is the equation of a line, is known as the line of action. It is the line along which the resultant force acts. This line of action is shown graphically in Figure 2.15. Now find the intersection of

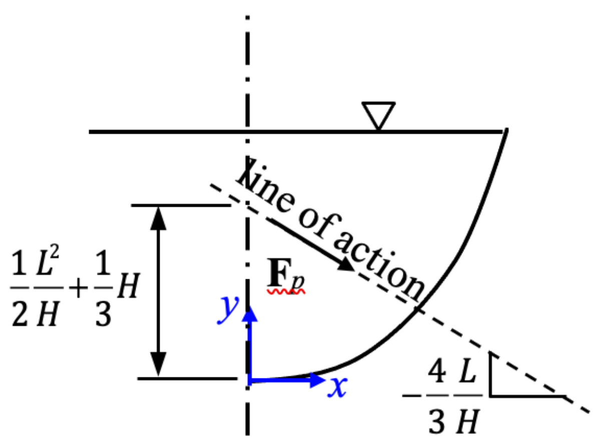


FIGURE 2.15. A sketch showing the line of action for the parabolic wall example.

the line of action and the wall by substituting Eq. (2.46) into Eq. (2.57),

$$H\left(\frac{x_{CP}}{L}\right)^2 = \left(-\frac{4}{3}\frac{L}{H}\right) x_{CP} + \left(\frac{1}{2}\frac{L^2}{H} + \frac{1}{3}H\right), \tag{2.58}$$

$$\left(\frac{x_{CP}}{L}\right)^2 + \frac{4}{3}\left(\frac{L}{H}\right)^2 \left(\frac{x_{CP}}{L}\right) - \left[\frac{1}{2}\left(\frac{L}{H}\right)^2 + \frac{1}{3}\right] = 0. \tag{2.59}$$

Solving this (unfortunately messy) equation gives,

$$\frac{x_{CP}}{L} = -\frac{2}{3}\left(\frac{L}{H}\right)^2 + \sqrt{\frac{4}{9}\left(\frac{L}{H}\right)^4 + \frac{1}{2}\left(\frac{L}{H}\right)^2 + \frac{1}{3}}. \tag{2.60}$$

Note that only the positive root of the previous equation makes physical sense. Now that we have x_{CP}, the value for y_{CP} can then be found by substituting this value into Eq. (2.46).

Notes:

(1) The horizontal component of the resultant pressure force in Eq. (2.51) is equal to the resultant force acting on the vertical projected area HW, i.e., $F_{P,x} = \frac{1}{2}\rho g H^2 W$.

An alternate method for determining the resultant force and center of pressure is to balance forces on a region of fluid bordered by the wall. For example, balance forces on the region of fluid identified by the dotted line in Figure 2.16.

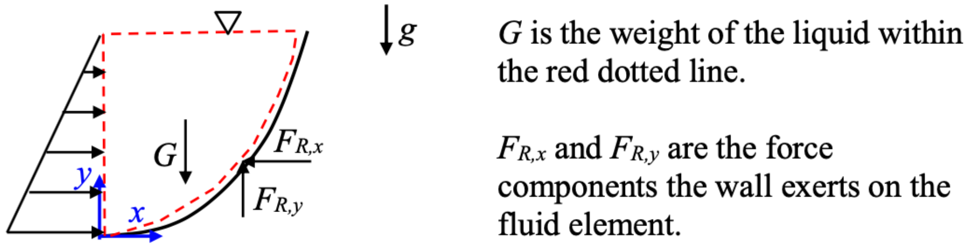


FIGURE 2.16. Free body diagram for the region of fluid enclosed by the red dashed line.

$$\sum F_x = 0 = \frac{1}{2}\rho g H^2 W - F_{R,x} \implies F_{R,x} = \frac{1}{2}\rho g H^2 W, \tag{2.61}$$

$$\sum F_y = 0 = -G + F_{R,y} \implies F_{R,y} = G, \tag{2.62}$$

where,

$$G = \int_{x=0}^{x=L} \rho g \underbrace{(H-y)dxW}_{=dV} = \int_{x=0}^{x=L} \rho g \left[H - H\left(\frac{x}{L}\right)^2\right] dxW, \tag{2.63}$$

$$= \rho g H W \int_{x=0}^{x=L} \left(1 - \frac{x^2}{L^2}\right) dx = \rho g H W \left(L - \frac{1}{3}\frac{L^3}{L^2}\right), \tag{2.64}$$

$$G = \frac{2}{3}\rho g H W L, \tag{2.65}$$

so that,

$$F_{R,y} = \frac{2}{3}\rho g H W L. \tag{2.66}$$

These magnitudes for $F_{R,x}$ and $F_{R,y}$ are exactly the same as what was found in Eq. (2.51). Note that here $F_{R,x}$ and $F_{R,y}$ are the force components the wall exerts on the fluid so, from Newton's Third Law, the fluid exerts equal and opposite force components on the wall.

The center of pressure about the z axis is found by balancing moments about the origin, the same as what was done for planar walls,

$$\boldsymbol{x}_{CP} \times \boldsymbol{F}_p = \left(\underbrace{\frac{1}{3}H\hat{\boldsymbol{j}}}_{\substack{\text{CP on}\\ \text{left side}}} \times \underbrace{\frac{1}{2}\rho g H^2 W \hat{\boldsymbol{i}}}_{\substack{\text{resultant force}\\ \text{on left side}}}\right) + \left[\underbrace{(x_{CM}\hat{\boldsymbol{i}} + y_{CM}\hat{\boldsymbol{j}})}_{\text{center of mass}} \times \underbrace{-\frac{2}{3}\rho g H W L \hat{\boldsymbol{j}}}_{\substack{\text{weight of}\\ \text{fluid region}}}\right]. \tag{2.67}$$

Since the weight has no x component, we need not worry about calculating y_{CM}. However, we do need the x component of the center of mass, which we can find via integration (refer to the figure),

$$x_{CM}G = \int_{x=0}^{x=L} x\rho g \underbrace{(H-y)W\,dx}_{=dV}, \tag{2.68}$$

$$= \int_{x=0}^{x=L} x\rho g \left[H - H\left(\frac{x}{L}\right)^2\right] W\,dx, \tag{2.69}$$

$$= \rho g H W \int_{x=0}^{x=L} \left(x - \frac{x^3}{L^2}\right) dx, \tag{2.70}$$

$$x_{CM}\frac{2}{3}\rho g H W L = \rho g H W \left(\frac{1}{2}L^2 - \frac{1}{4}\frac{L^4}{L^2}\right), \tag{2.71}$$

$$= \frac{1}{4}\rho g H W L^2, \tag{2.72}$$

$$x_{CM} = \frac{3}{8}L. \tag{2.73}$$

Substituting this value back into the right-hand side of Eq. (2.67) and making use of the resultant pressure force on the left-hand side,

$$(x_{CP}\hat{\boldsymbol{i}} + y_{CP}\hat{\boldsymbol{j}}) \times \left(\frac{1}{2}\rho g H^2 W\hat{\boldsymbol{i}} - \frac{2}{3}\rho g H W L\hat{\boldsymbol{j}}\right) \tag{2.74}$$

$$= \left(\frac{1}{3}H\hat{\boldsymbol{j}} \times \frac{1}{2}\rho g H^2 W\hat{\boldsymbol{i}}\right) + \left(\frac{3}{8}L\hat{\boldsymbol{i}} \times -\frac{2}{3}\rho g H W L\hat{\boldsymbol{j}}\right), \tag{2.75}$$

$$-x_{CP}\frac{2}{3}\rho g H W L\hat{\boldsymbol{k}} - y_{CP}\frac{1}{2}\rho g H^2 W\hat{\boldsymbol{k}} = -\frac{1}{6}\rho g H^3 W\hat{\boldsymbol{k}} - \frac{1}{4}\rho g H W L^2\hat{\boldsymbol{k}}, \tag{2.76}$$

$$y_{CP} = \left(-\frac{4}{3}\frac{L}{H}\right)x_{CP} + \left(\frac{1}{3}H + \frac{1}{2}\frac{L^2}{H}\right), \tag{2.77}$$

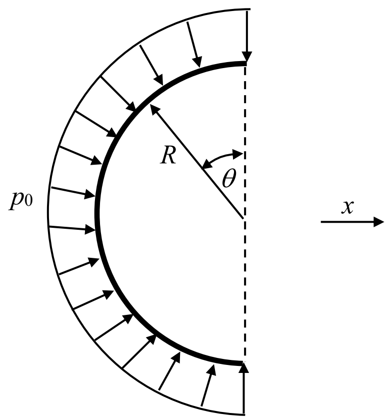
which is the same line of action found previously.

Notes:

(1) Either approach to finding the resultant force and center of pressure (integration or balancing forces on a wisely-chosen region of fluid) is fine. One method may be easier than the other, depending on the geometry of the problem.

(2) Yet another method to finding the resultant pressure force and center of pressure relies on calculating the center of area of the wall surface and calculating moments of inertia. This approach isn't described in these notes since it's a more "formulaic" approach and is less connected to the actual physics of the problem. Moreover, this moment-of-inertia approach often requires access to moment of inertia tables, which may be inconvenient. A number of texts that discuss fluid statics present this "moments-of-inertia" approach, but it's not this author's preferred method.

Calculate the net horizontal pressure force acting on the half cylinder shown below. The half cylinder has radius R unit depth into the page, and the gage pressure acting on it is p_0.



SOLUTION:

We can determine the net horizontal pressure force in two ways. The first method directly integrates the horizontal pressure force components over the entire surface and the second method uses the surface's projected area.

Method 1: Integrate the horizontal pressure force components over the entire surface area.

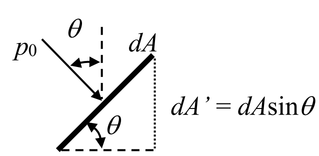
$$dF_{p,x} = \underbrace{p_0 \underbrace{Rd\theta}_{=dA}}_{=dF_p} \sin\theta \tag{1}$$

$$F_{p,x} = \int_{\theta=0}^{\theta=\pi} dF_{p,x} = \int_{\theta=0}^{\theta=\pi} p_0 R d\theta \sin\theta = p_0 R \int_{\theta=0}^{\theta=\pi} \sin\theta d\theta = -p_0 R \cos\theta\big|_0^{\pi} = -p_0 R(-1-1) \tag{2}$$

$$\boxed{\therefore F_{p,x} = p_0(2R)} \tag{3}$$

Method 2: Multiply the pressure with the surface's area projected in the x-direction.

The small amount of horizontal pressure force $dF_{p,x}$ due to the pressure p_0 acting on a small area dA inclined at an angle θ as shown in the figure to the right is,



$$dF_{p,x} = \underbrace{p_0 dA}_{=dF_p} \sin\theta \tag{4}$$

By grouping terms, we see that horizontal pressure force is equivalent to multiplying the pressure by the area projected in the horizontal direction, dA', *i.e.*, the area of the surface viewed from the x-direction.

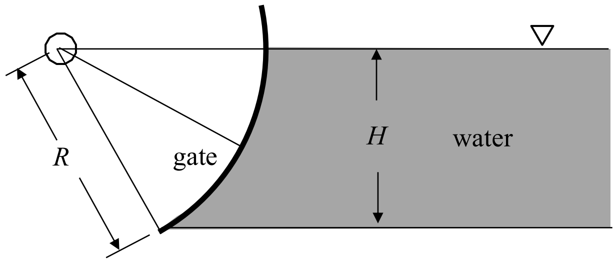
$$dF_{p,x} = p_0 \underbrace{dA\sin\theta}_{=dA'} \tag{5}$$

Thus, the horizontal pressure force acting on the half-cylinder is simply the pressure multiplied by the cylinder's horizontal projected area, $2R$,

$$\boxed{\therefore F_{p,x} = p_0(2R)} \text{ (This is the same result as before!)} \tag{6}$$

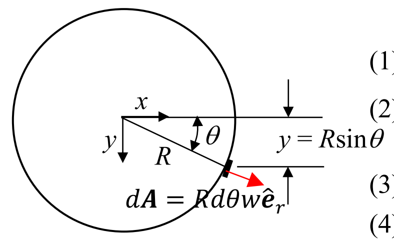
Video solution: https://www.youtube.com/watch?v=EHbeLGXqSn4

The figure shows a Tainter gate used to control water flow from a dam. The gate radius is $R = 20$ m, the gate width is $w = 35$ m, and the water depth is $H = 10$ m. Determine the force components, magnitude, and line of action of the force that the water exerts on the gate.



SOLUTION:

First determine the force components acting on the gate,



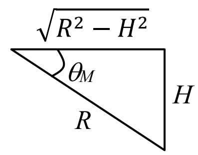
$$\boldsymbol{F} = \int_{y=0}^{y=H} p(-d\boldsymbol{A}) = \int_{y=0}^{y=H} (\rho g y)[-(Rd\theta w \hat{\boldsymbol{e}}_r)], \quad (1)$$

$$\boldsymbol{F} = \int_{\theta=0}^{\theta=\theta_M} (\rho g R \sin\theta)(-Rd\theta w \hat{\boldsymbol{e}}_r), \quad (2)$$

where,

$$\sin\theta_M = \frac{H}{R} \Rightarrow \theta_M = \sin^{-1}\left(\frac{H}{R}\right), \quad (3)$$

$$\hat{\boldsymbol{e}}_r = \cos\theta\, \hat{\boldsymbol{\imath}} + \sin\theta\, \hat{\boldsymbol{\jmath}}. \quad (4)$$



Substitute and simplify,

$$\boldsymbol{F} = \int_0^{\theta_M} (\rho g R \sin\theta)[-Rd\theta w(\cos\theta\, \hat{\boldsymbol{\imath}} + \sin\theta\, \hat{\boldsymbol{\jmath}})], \quad (5)$$

$$\boldsymbol{F} = -\rho g R^2 w \int_0^{\theta_M} (\sin\theta\cos\theta\, d\theta \hat{\boldsymbol{\imath}} + \sin^2\theta\, d\theta \hat{\boldsymbol{\jmath}}), \quad (6)$$

$$\boldsymbol{F} = -\rho g R^2 w \left\{\left(\frac{1}{2}\sin^2\theta_M\right)\hat{\boldsymbol{\imath}} + \left[\frac{1}{2}\theta_M - \frac{1}{4}\sin(2\theta_M)\right]\hat{\boldsymbol{\jmath}}\right\}, \quad (7)$$

$$F_x = -\frac{1}{2}\rho g R^2 w \sin^2\theta_M, \quad (8)$$

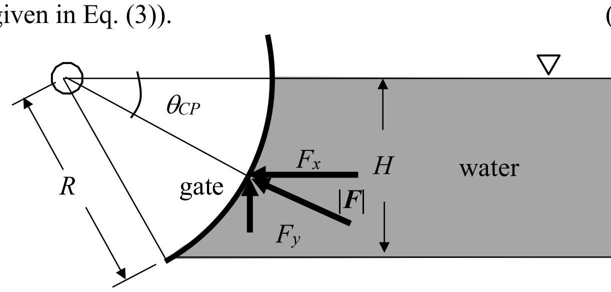
$$F_y = -\frac{1}{2}\rho g R^2 w \left[\theta_M - \frac{1}{2}\sin(2\theta_M)\right], \quad (9)$$

$$F_x = -\frac{1}{2}\rho g R^2 w \left(\frac{H}{R}\right)^2 \Rightarrow F_x = -\frac{1}{2}\rho g H^2 w. \quad (10)$$

$$F_y = -\frac{1}{2}\rho g R^2 w \left[\theta_M - \frac{1}{2}\sin(2\theta_M)\right] \text{ (where } \theta_M \text{ is given in Eq. (3)).} \quad (11)$$

Using the given data,

ρ = 1000 kg/m^3,
g = 9.81 m/s^2,
w = 35 m,
H = 10 m,
R = 20 m,
=> $\boxed{F_x = -17.2 \text{ MN and } F_y = -6.22 \text{ MN}}$



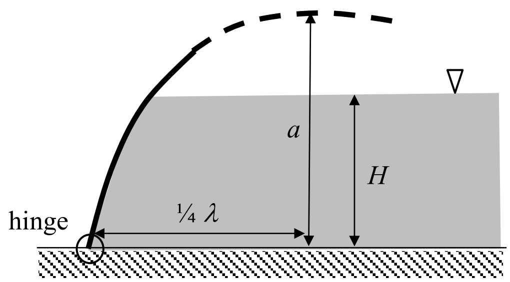
and the force magnitude is |**F**| = 18.3 MN. The angle from the horizontal is,

$$\tan\theta_{CP} = \frac{F_y}{F_x}, \text{ (refer to the figure to the right)} \quad (12)$$

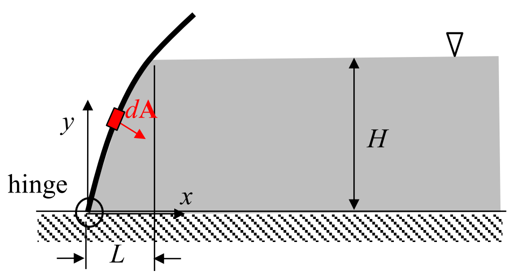
$\boxed{\theta_{CP} = 19.9°}$

Note that the resultant force will pass through the center of the circle (the hinge) since the pressure force acts normal to the surface.

A spring-loaded hinge is designed to hold closed the sinusoidally-shaped gate shown in the figure (assume unit depth into the page). The wavelength of the gate shape is λ and its amplitude is a. The water depth is $H < a$. liquid in the figure is water. Determine the horizontal and vertical components of the force acting in the hinge due to the gate, as well as the moment the hinge must supply to keep the gate in the configuration shown. You may neglect the weight of the gate in your calculations.



SOLUTION:



First determine the water force acting on the gate.

$$\mathbf{F} = -\int_A p\,d\mathbf{A}\,, \tag{1}$$

where

$$p = \rho g(H - y)\,, \tag{2}$$

$$d\mathbf{A} = dy(1)\hat{\mathbf{i}} - dx(1)\hat{\mathbf{j}}\,. \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1) gives,

$$\mathbf{F} = -\int_A \rho g(H - y)\left[dy(1)\hat{\mathbf{i}} - dx(1)\hat{\mathbf{j}}\right], \tag{4}$$

Split the integral into two parts: one concerning the vertical force component and one concerning the horizontal force component. The integral limits for the horizontal force component are simply $y = 0$ to $y = H$. The integral limits for the vertical force component are $x = 0$ to $x = L$, where L may be found by noting that the gate is sinusoidal in shape,

$$y = a\sin\left(2\pi\frac{x}{\lambda}\right) \;\Rightarrow\; L = \frac{\lambda}{2\pi}\sin^{-1}\left(\frac{H}{a}\right). \tag{5}$$

Thus, Eq. (4) may be written as,

$$\mathbf{F} = -\hat{\mathbf{i}}\rho g\int_{y=0}^{y=H}(H - y)dy + \hat{\mathbf{j}}\rho g\int_{x=0}^{x=L}(H - y)dx\,, \tag{6}$$

$$\mathbf{F} = -\hat{\mathbf{i}}\rho g\int_{y=0}^{y=H}(H - y)dy + \hat{\mathbf{j}}\rho g\int_{x=0}^{x=L}\left[H - a\sin\left(2\pi\frac{x}{\lambda}\right)\right]dx\,, \tag{7}$$

where Eq. (5) has been used to substitute in for y. Evaluating the integrals in Eq. (7) gives,

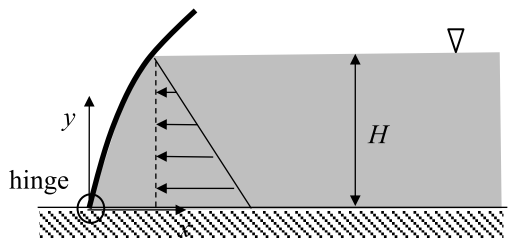
$$\mathbf{F} = -\hat{\mathbf{i}}\rho g\left(Hy - \tfrac{1}{2}y^2\right)_{y=0}^{y=H} + \hat{\mathbf{j}}\rho g\left[Hx + \frac{a\lambda}{2\pi}\cos\left(2\pi\frac{x}{\lambda}\right)\right]_{x=0}^{x=L}, \tag{8}$$

$$\mathbf{F} = -\hat{\mathbf{i}}\tfrac{1}{2}\rho gH^2 + \hat{\mathbf{j}}\rho g\left\{HL - \frac{a\lambda}{2\pi}\left[1-\cos\left(2\pi\frac{L}{\lambda}\right)\right]\right\}, \tag{9}$$

where L is given in Eq. (5). Note that these are the forces acting on the gate due to the water. The forces acting on the hinge would have the same magnitude, but opposite sign,

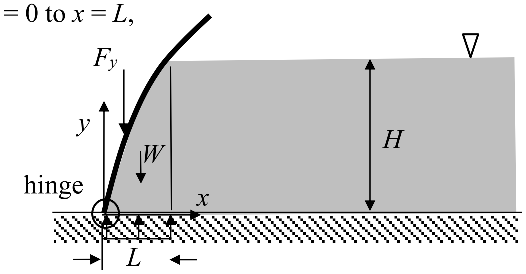
$$\boxed{\mathbf{F}_{\text{hinge}} = \hat{\mathbf{i}}\tfrac{1}{2}\rho gH^2 - \hat{\mathbf{j}}\rho g\left\{HL - \frac{a\lambda}{2\pi}\left[1-\cos\left(2\pi\frac{L}{\lambda}\right)\right]\right\}}. \tag{10}$$

The horizontal pressure force is the same pressure force that's exerted on the horizontal, projected area of the gate,



$$F_x = -\int_{y=0}^{y=H} \rho g(H-y)\,dy(1) = -\tfrac{1}{2}\rho gH^2. \tag{11}$$

The vertical pressure force could have also been found by balancing forces on the fluid contained within the span from $x = 0$ to $x = L$,



$$\sum F_y = 0 = -F_y - W + \rho gHL(1), \tag{12}$$

where the $\rho gHL(1)$ term is the (uniform) pressure force acting on the bottom of the section of fluid under consideration. The weight of the fluid in the section is given by,

$$W = \rho g\int_{x=0}^{x=L} y\,dx(1) = \rho g\int_{x=0}^{x=L} a\sin\left(\frac{2\pi}{\lambda}x\right)dx(1) = \rho g\frac{a\lambda}{2\pi}\left[1-\cos\left(\frac{2\pi}{\lambda}L\right)\right]. \tag{13}$$

Combining Eqs. (12) and (13) and solving for F_y gives,

$$F_y = \rho gHL - \rho g\frac{a\lambda}{2\pi}\left[1-\cos\left(\frac{2\pi}{\lambda}L\right)\right] = \rho g\left\{HL - \frac{a\lambda}{2\pi}\left[1-\cos\left(\frac{2\pi}{\lambda}L\right)\right]\right\}, \tag{14}$$

which is the same as the expression found previously.

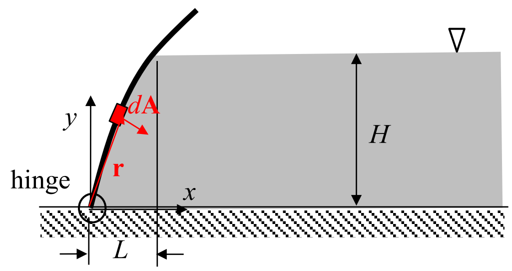
The moment exerted by the hinge may be found by summing moments at the hinge,

$$\sum \mathbf{M}_{\text{hinge}} = 0 = \mathbf{M}_{\text{hinge}} + \int_A \mathbf{r} \times d\mathbf{F}_p \,, \tag{15}$$

where

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\,, \tag{16}$$

$$d\mathbf{F}_p = -pd\mathbf{A} = -\rho g(H-y)\left[dy(1)\hat{\mathbf{i}} - dx(1)\hat{\mathbf{j}}\right]. \tag{17}$$



Substituting these relations into Eq. (15) and simplifying gives,

$$\mathbf{M}_{\text{hinge}} = -\int_A \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right) \times -\rho g(H-y)\left[dy(1)\hat{\mathbf{i}} - dx(1)\hat{\mathbf{j}}\right] = \rho g\hat{\mathbf{k}}\int_A (H-y)(-xdx - ydy)\,, \tag{18}$$

$$\mathbf{M}_{\text{hinge}} = -\rho g\hat{\mathbf{k}}\int_A (H-y)(xdx + ydy)\,, \tag{19}$$

$$\mathbf{M}_{\text{hinge}} = -\rho g\hat{\mathbf{k}}\left\{\int_{x=0}^{x=L}\left[H - a\sin\left(\frac{2\pi}{\lambda}x\right)\right](xdx) + \int_{y=0}^{y=H}(H-y)(ydy)\right\}, \tag{20}$$

where Eq. (5) has been substituted in for y in the first integral. Solving the integrals in the previous equation gives,

$$\mathbf{M}_{\text{hinge}} = -\rho g\hat{\mathbf{k}}\left\{\tfrac{1}{2}HL^2 - \int_{x=0}^{x=L} a\sin\left(\frac{2\pi}{\lambda}x\right)x\,dx + \tfrac{1}{6}H^3\right\}, \tag{21}$$

$$\mathbf{M}_{\text{hinge}} = -\rho g\hat{\mathbf{k}}\left\{\tfrac{1}{2}HL^2 - \left[\left(\frac{\lambda}{2\pi}\right)^2\sin\left(\frac{2\pi}{\lambda}x\right) - \left(\frac{\lambda}{2\pi}\right)x\cos\left(\frac{2\pi}{\lambda}x\right)\right]_{x=0}^{x=L} + \tfrac{1}{6}H^3\right\}, \tag{22}$$

$$\boxed{\mathbf{M}_{\text{hinge}} = -\rho g\hat{\mathbf{k}}\left\{\tfrac{1}{2}HL^2 - \left(\frac{\lambda}{2\pi}\right)^2\sin\left(\frac{2\pi}{\lambda}L\right) + \left(\frac{\lambda L}{2\pi}\right)\cos\left(\frac{2\pi}{\lambda}L\right) + \tfrac{1}{6}H^3\right\}}. \tag{23}$$

2.4. Buoyant Force and Center of Buoyancy

When an object is submerged in a fluid, the pressure acting on the object deeper in the fluid (i.e., in the direction of gravity) will be larger than the pressure acting on the object shallower in the fluid. As a result, there will be a net pressure force acting on the object. This net pressure force is known as the buoyant force.

To derive the value of the buoyant force, consider the vertical pressure forces acting on a narrow cylinder with cross-sectional area dA within a fully-submerged object as shown in Figure 2.17.

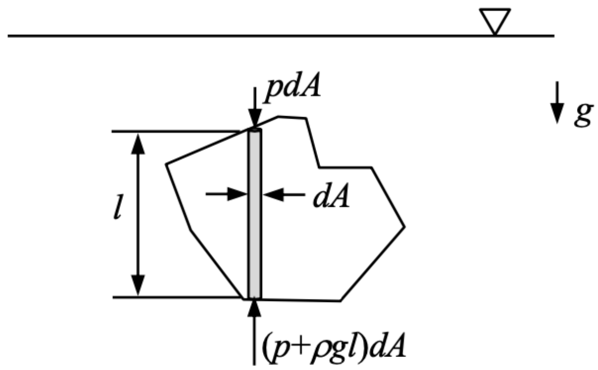


Figure 2.17. Pressure forces on a thin cylinder of cross-sectional area dA and height l from within a fully-submerged object.

The net pressure force in the vertical direction on the narrow cylinder, assuming an incompressible fluid, is,

$$dF_{p,\text{net}} = (p + \rho_{\text{fluid}} g l)dA - pdA = \rho_{\text{fluid}} g l dA \quad \text{(acting opposite to gravity).} \tag{2.78}$$

The total net pressure force acting on the object is found by integrating these small bits of pressure force over the entire cross-sectional area of the object,

$$F_{p,\text{net}} = \int_A dF_{p,\text{net}} = \int_A \rho_{\text{fluid}} g l dA \quad \text{(acting opposite to gravity).} \tag{2.79}$$

Since the density and gravity are assumed constant here, they may be pulled outside the integral,

$$F_{p,\text{net}} = \rho_{\text{fluid}} g \int_A l dA \quad \text{(acting opposite to gravity),} \tag{2.80}$$

$$= \rho_{\text{fluid}} g \int_V dV \quad \text{(acting opposite to gravity),} \tag{2.81}$$

$$\boxed{F_{p,\text{net}} = F_B = \rho_{\text{fluid}} g V_{\substack{\text{submerged}\\ \text{object}}}} \quad \text{(acting opposite to gravity),} \tag{2.82}$$

where the integral is simply the volume of the submerged object. This net pressure force acting on the object is referred to as the buoyant force.

Notes:

(1) Equation (2.82) states that the buoyant force is equal to the weight of the fluid that's been displaced by the submerged object. This relationship is also known as Archimede's Principle.

(2) The same analysis can be used for partially submerged objects. In that case, the pressure acting on the top of the object is atmospheric pressure while the pressure at the bottom is $p_{\text{atm}} + \rho_{\text{fluid}} g l'$, where l' is the length of the narrow cylinder that's submerged in the fluid (Figure 2.18). After integrating over the objects cross-sectional area (similar to Eq. (2.80), we would arrive at exactly the same relation as in Eq. (2.82) except that the $V_{\text{submerged object}}$ refers to just that volume that is submerged in the fluid.

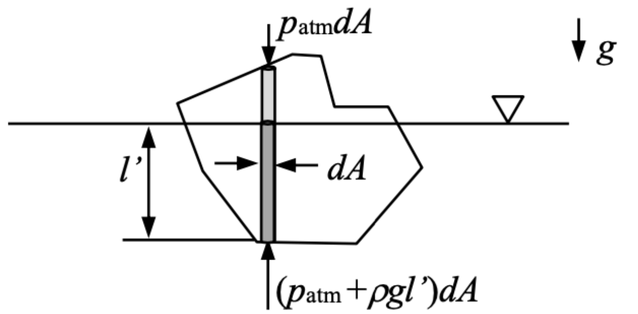


Figure 2.18. Pressure forces on a thin cylinder of cross-sectional area dA. The depth of the cylinder below the free surface is l'.

(3) There is no net pressure force on the object in the directions perpendicular to gravity since the pressure only varies parallel to the gravitational vector.

The resultant buoyant force acts at the center of buoyancy. The center of buoyancy is found by equating the moment caused by the resultant buoyant force acting at the center of buoyancy to the distributed moment caused by the distributed pressure forces. Consider moments about the z axis in Figure 2.19.

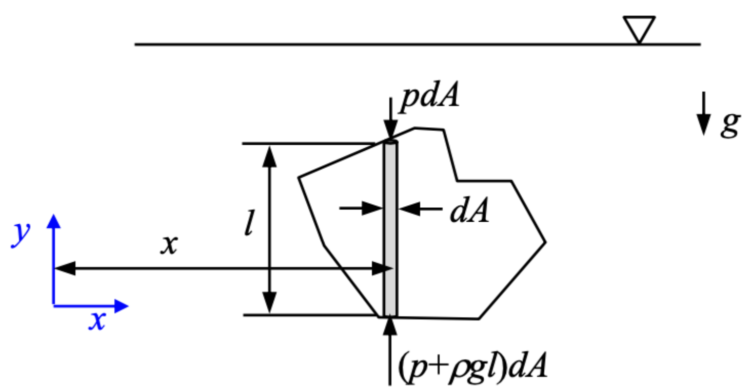


Figure 2.19. Moments about the z axis due to the pressure forces acting on a thin cylinder of cross-sectional area dA and height l from within a fully-submerged object.

$$x_{CB}\hat{\boldsymbol{i}} \times \underbrace{\rho g V \hat{\boldsymbol{j}}}_{\substack{\text{buoyant}\\ \text{force}}} = \int_A x\hat{\boldsymbol{i}} \times \underbrace{\rho g l dA \hat{\boldsymbol{j}}}_{\substack{\text{net pressure force}\\ \text{on cylinder}}}, \tag{2.83}$$

$$x_{CB}\rho g V \hat{\boldsymbol{k}} = \rho g \hat{\boldsymbol{k}} \int_A x dV \quad (dV = l dA), \tag{2.84}$$

$$x_{CB} = \frac{1}{V}\int_V x dV, \tag{2.85}$$

which is the center of displaced volume. Performing similar analyses about the x and y axes produces similar results. Thus, *the center of buoyancy is located at the center of the displaced volume.* This is true for both fully submerged and partially submerged objects.

2.5. Stable Orientation of a Submerged Object

Submerged objects will be in an equilibrium orientation when the forces acting on the object are such that there is no net moment on the object. Considering only the object weight and a buoyant force, an equilibrium orientation will only occur when the two forces are co-linear, as shown in Figure 2.20. Neither object experiences a net moment.

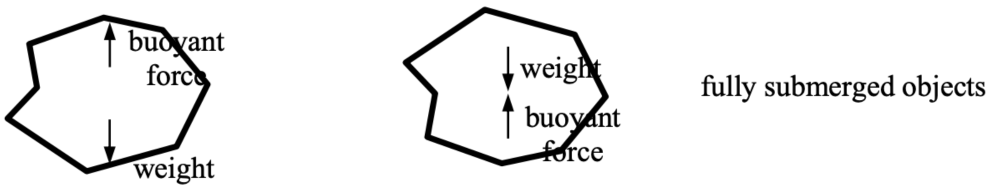


Figure 2.20. The buoyant force, acting at the center of buoyancy, and weight, acting at the center of gravity, for two fully-submerged objects in equilibrium. The object on the left is stable, but the object on the right is unstable.

The object on the left is in a stable equilibrium while the object on the right is in an unstable equilibrium. The reason for the difference is that if each object is rotated slightly, the object on the left will experience a moment that restores it back to its original configuration. However, a small perturbation to the right-hand object will result in a moment that will cause the object to move away from its initial configuration.

The stability of partially submerged objects is a particularly important topic when considering the design of ships. The Swedish ship *Vasa* is a famous example of a ship that was unstable and "turtled" shortly after setting sail for the first time. Unfortunately, stability analysis of partially submerged objects can be complicated since the submerged volume and center of buoyancy changes as the orientation of the object rotates. For example, consider the stability of the simple shape shown in Figure 2.21 (CG is the center of gravity and CB is the center of buoyancy). The initial configuration of the object (on the left) appears to be

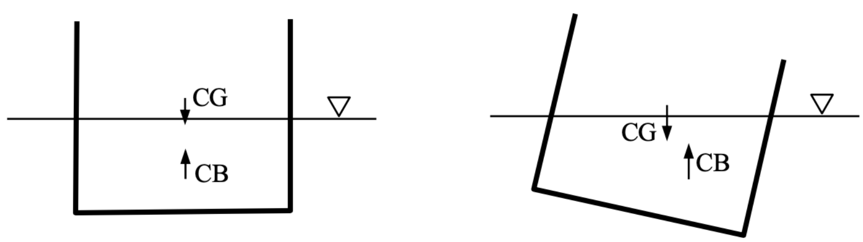


Figure 2.21. The center of buoyancy and center of gravity for a partially-submerged object. The center of buoyancy changes location as the submerged volume changes.

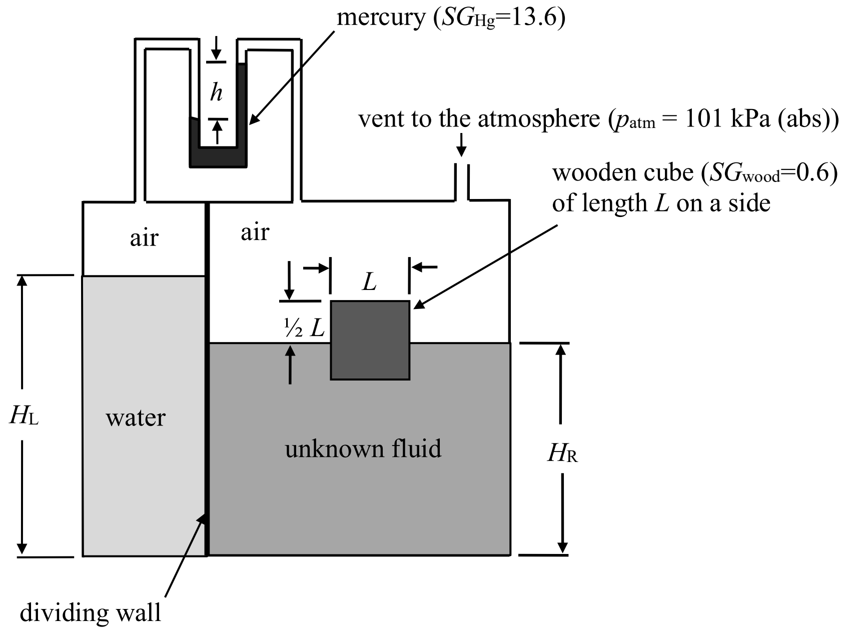
in unstable equilibrium with the center of gravity above the center of buoyancy. However, when the object is tilted (on the right), the center of buoyancy shifts to one side such that it acts to restore the object to its initial configuration. Hence, the object is actually initially in stable equilibrium.

A tank is divided by a wall into two independent chambers. The left chamber is filled to a depth of H_L=6m with water (ρ_{H20}=1000 kg/m^3) and the right side if filled to a depth of H_R=5m with an unknown fluid. A wooden cube (SG_{wood}=0.6) with a length of L=0.20m on each side floats half submerged in the unknown fluid. Air (ρ_{air}=1.2 kg/m^3) fills the remainder of the container above each fluid. The right container has a pipe that is vented to the atmosphere while the left container is sealed from the atmosphere. A manometer using mercury as the gage fluid (SG_{Hg}=13.6) connects the two chambers and indicates that h=0.150 m.

a. Determine the density of the unknown fluid.

b. Determine the magnitude of the force (per unit depth into the page) acting on the dividing wall due to the unknown fluid.

c. Determine the magnitude of the force (per unit depth into the page) acting on the dividing wall due to the water.



SOLUTION:

Balance forces on the wooden cube.

$$\sum F_y = 0 = \rho_{\text{fluid}} g\left(\tfrac{1}{2}L\right)L^2 - \rho_{\text{wood}} g L^3 \quad (1)$$

$$\boxed{\therefore \rho_{\text{fluid}} = 2\rho_{\text{wood}} = 2SG_{\text{wood}}\rho_{\text{H}_2\text{O}}} \quad (2)$$

Using the given data:

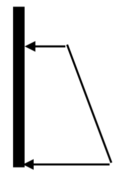
SG_{wood} = 0.6

ρ_{H2O} = 1000 kg/m^3

$\Rightarrow$ $\boxed{\rho_{fluid} = 1200 \text{ kg/m}^3}$

Now determine the force acting on the wall due to the unknown fluid.

$$F_{p,R} = \int_{y=0}^{y=H_R} \underbrace{\left(p_{\text{atm}} + \rho_{\text{fluid}} g y\right)}_{=p\ (\text{abs})} \underbrace{dy(1)}_{=dA} \tag{3}$$



$$\boxed{\therefore F_{p,R} = p_{\text{atm}} H_R + \tfrac{1}{2}\rho_{\text{fluid}} g H_R^2} \tag{4}$$

Using the given data:

p_{atm} = 101 kPa (abs)
H_R = 5 m
ρ_{fluid} = 1200 kg/m^3
g = 9.81 m/s^2
$\Rightarrow$ $\boxed{F_{p,R} = 506 \text{ kN/m}}$

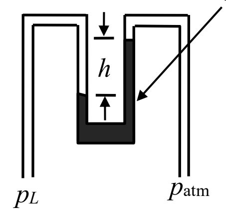
Now find the pressure force due to the water.

$$F_{p,L} = \int_{y=0}^{y=H_L} \underbrace{\left(p_L + \rho_{H_2O} g y\right)}_{=p\ (\text{abs})} \underbrace{dy(1)}_{=dA} \tag{5}$$

$$\therefore F_{p,L} = p_L H_L + \tfrac{1}{2}\rho_{H_2O} g H_L^2 \tag{6}$$

where p_L is the (absolute) pressure acting on the free surface of the water. This pressure may be found using the manometer.

$$p_L = p_{\text{atm}} + \rho_{\text{Hg}} g h = p_{\text{atm}} + SG_{\text{Hg}} \rho_{H_2O} g h \tag{7}$$



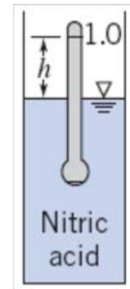
Substitute Eqn. (7) into Eqn. (6).

$$\boxed{\therefore F_{p,L} = \left(p_{\text{atm}} + SG_{\text{Hg}} \rho_{H_2O} g h\right) H_L + \tfrac{1}{2}\rho_{H_2O} g H_L^2} \tag{8}$$

Using the given data:

p_{atm} = 101 kPa (abs)
SG_{Hg} = 13.6
ρ_{H2O} = 1000 kg/m^3
g = 9.81 m/s^2
h = 0.150 m
H_L = 6 m
$\Rightarrow$ $\boxed{F_{p,L} = 903 \text{ kN/m}}$

A hydrometer is a specific gravity indicator, the value being indicate by the level at which the free surface intersects the stem when floating in a liquid. The 1.0 mark is the level when in distilled water. For the unit shown, the immersed volume in distilled water is 15 cm^3. The stem is 6 mm in diameter. Find the distance, h, from the 1.0 mark to the surface when the hydrometer is placed in a nitric acid solution of specific gravity 1.5.



SOLUTION:
Since the hydrometer is in equilibrium, its weight and the buoyant force should equal each other. When submerged in distilled water,

$$W = \rho_{H2O} g V_{disp,H2O} = \rho_{H2O} g h \frac{\pi}{4} d^2 \;\Rightarrow\; h = \frac{W}{\rho_{H2O} g \frac{\pi}{4} d^2}, \tag{1}$$

where A is the hydrometer's cross-sectional area. The height h is the location where the mark is made for distilled water.

When submerged in nitric acid,

$$W = \rho_{HNO_3} g \frac{\pi}{4} d^2 (h + \Delta h) \;\Rightarrow\; h + \Delta h = \frac{W}{\rho_{HNO_3} g \frac{\pi}{4} d^2} = \frac{W}{SG_{HNO_3} \rho_{H2O} g \frac{\pi}{4} d^2}. \tag{2}$$

Combining Eqs. (1) and (2),

$$\frac{W}{\rho_{H2O} g \frac{\pi}{4} d^2} + \Delta h = \frac{W}{SG_{HNO_3} \rho_{H2O} g \frac{\pi}{4} d^2}, \tag{3}$$

$$\Delta h = \frac{W}{SG_{HNO_3} \rho_{H2O} g \frac{\pi}{4} d^2} - \frac{W}{\rho_{H2O} g \frac{\pi}{4} d^2}, \tag{4}$$

$$\Delta h = \frac{W}{\rho_{H2O} g \frac{\pi}{4} d^2} \left(\frac{1}{SG_{HNO_3}} - 1 \right), \tag{5}$$

$$\Delta h = \frac{\rho_{H2O} g V_{disp,H2O}}{\rho_{H2O} g \frac{\pi}{4} d^2} \left(\frac{1}{SG_{HNO_3}} - 1 \right), \tag{6}$$

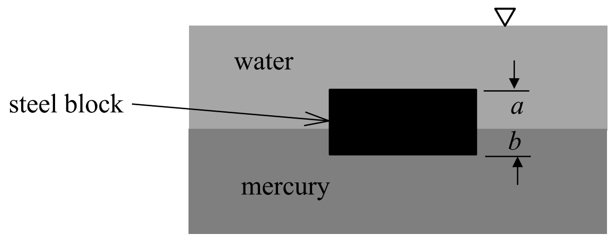
$$\Delta h = \frac{V_{disp,H2O}}{\frac{\pi}{4} d^2} \left(\frac{1}{SG_{HNO_3}} - 1 \right). \tag{7}$$

Using the given data,

$V_{disp,H2O} = 15$ cm^3,
$d = 0.6$ cm,
$SG_{HNO3} = 1.5$,
=> $\boxed{\Delta h = -17.7 \text{ cm}}$.

The hydrometer moves upward a distance of 17.7 cm from where the distilled water mark is located.

A uniform block of steel (with a specific gravity of 7.85) will "float" at a mercury-water interface as shown in the figure. What is the ratio of the distances *a* and *b*?



SOLUTION:

Balance forces in the vertical direction,

$$\sum F_V = 0 = -W_{\text{block}} + F_{B,Hg} + F_{B,H_2O} = -\rho_{\text{block}} V_{\text{block}} g + \rho_{Hg} V_{\substack{\text{block,}\\ \text{in } Hg}} g + \rho_{H_2O} V_{\substack{\text{block,}\\ \text{in } H_2O}} g \,, \tag{1}$$

where the buoyant forces are equal to the weights of the displaced fluids.

Re-writing in terms of the lengths *a* and *b* and the block's cross-sectional area A_{block},

$$-\rho_{\text{block}} A_{\text{block}} \left(a+b\right) + \rho_{Hg} A_{\text{block}} b + \rho_{H_2O} A_{\text{block}} a = 0 \,, \tag{2}$$

$$-\rho_{\text{steel}} \left(a+b\right) + \rho_{Hg} b + \rho_{H_2O} a = 0 \,, \tag{3}$$

$$-\rho_{H_2O} SG_{\text{steel}} b \left(\frac{a}{b} + 1\right) + \rho_{H_2O} SG_{Hg} + \rho_{H_2O} b \frac{a}{b} = 0 \,, \tag{4}$$

$$-SG_{\text{steel}} \left(\frac{a}{b} + 1\right) + SG_{Hg} + \frac{a}{b} = 0 \,, \tag{5}$$

$$\boxed{\frac{a}{b} = \frac{SG_{Hg} - SG_{\text{steel}}}{SG_{\text{steel}} - 1}} . \tag{6}$$

Using the given data,

SG_{Hg} = 13.6
SG_{steel} = 7.85
$\Rightarrow$ $\boxed{a/b = 0.83}$

Note that we could also solve this problem by balancing the block's weight with the pressure forces acting on the top and bottom block surfaces.

$$\sum F_V = 0 = -W_{\text{block}} + F_{p,H_2O} + F_{p,Hg} = -\rho_{\text{block}} A_{\text{block}} \left(a+b\right) g - \rho_{H_2O} g \left(H-a\right) A_{\text{block}} + \left(\rho_{H_2O} g H + \rho_{Hg} g b\right) A_{\text{block}} \,, \tag{7}$$

where *H* is the depth of the water-mercury interface. Simplifying this equation gives,

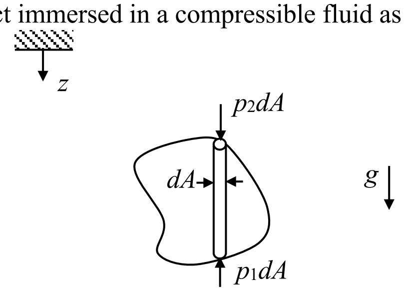
$$-\rho_{\text{block}} \left(a+b\right) - \rho_{H_2O} \left(H-a\right) + \rho_{H_2O} H + \rho_{Hg} b = 0 \,, \tag{8}$$

$$-\rho_{\text{block}} \left(a+b\right) + \rho_{H_2O} a + \rho_{Hg} b = 0 \,, \tag{9}$$

which is exactly the same as Eq. (3).

Archimedes principle states that the buoyant force acting on a submerged object is equal to the weight of the fluid displaced by that object. Is this true for *compressible* fluids?

SOLUTION:
Consider an arbitrary object immersed in a compressible fluid as shown in the figure below.



Determine the net pressure force acting on a parallelpiped of the material with a differential cross-sectional area,

$$dF_P = \left(p_1 - p_2\right) dA, \tag{1}$$

where,

$$p_1 = p_{z=0} + \int_{z=0}^{z=z_1} \rho g \, dz , \tag{2}$$

and,

$$p_2 = p_{z=0} + \int_{z=0}^{z=z_2} \rho g \, dz , \tag{3}$$

where ρ is the density of the fluid (not the object).

Equation (1) becomes,

$$dF_P = \left(p_{z=0} + \int_{z=0}^{z=z_1} \rho g \, dz - p_{z=0} - \int_{z=0}^{z=z_2} \rho g \, dz \right) dA , \tag{4}$$

$$dF_P = dA \int_{z=z_2}^{z=z_1} \rho g dz . \tag{5}$$

The net pressure force acting over the entire object, i.e., the buoyant force, is,

$$F_P = \int_A dF_P = \int_A \int_{z=z_2}^{z=z_1} \rho g dz dA . \tag{6}$$

Assuming that the gravitational acceleration is constant (usually a good assumption),

$$F_P = g \int_A \int_{z=z_2}^{z=z_1} \rho \, dz dA , \tag{7}$$

Note that the integrals in the previous equation give the mass of the fluid displaced by the object, i.e.,

$$M_{\substack{\text{fluid displaced} \\ \text{by object}}} = \int_A \int_{z=z_2}^{z=z_1} \rho \, dz dA . \tag{8}$$

Thus, just as with the incompressible case, <u>the buoyant force in a compressible fluid is equal to the weight of the fluid displaced by the object</u>,

$$F_p = M_{\substack{\text{fluid displaced} \\ \text{by object}}} g . \tag{9}$$

Consider an ice cube with initial volume $V_{ice,0}$ floating in a cup of water of initial volume $V_{water,0}$. The specific gravity of ice is SG_{ice}. Show mathematically that, as the ice cube melts, the water level in the cup remains unchanged.

SOLUTION:
If a mass of ice, Δm_{ice}, melts ($\Delta m_{ice} < 0$), it will correspond to an equal increase in water, Δm_{water}, i.e.,

$$\Delta m_{water} = -\Delta m_{ice} . \tag{1}$$

Expressed in terms of volumes,

$$\rho_{water}\Delta V_{water} = -\rho_{ice}\Delta V_{ice} = -SG_{ice}\rho_{water}\Delta V_{ice} , \tag{2}$$

$$\Delta V_{water} = -SG_{ice}\Delta V_{ice} . \tag{3}$$

The volume of water displaced by the ice is found by equating the weight of the displaced water to the weight of the ice (Archimedes Law),

$$\rho_{water}V_{water,disp}g = \rho_{ice}V_{ice}g = SG_{ice}\rho_{water}V_{ice}g , \tag{4}$$

$$V_{water,disp} = SG_{ice}V_{ice} . \tag{5}$$

Thus, if a volume of ice melts, ΔV_{ice}, then the amount of water displaced, in order to balance the new ice weight, is,

$$\Delta V_{water,disp} = SG_{ice}\Delta V_{ice} . \tag{6}$$

Note that if the ice melts ($\Delta V_{ice} < 0$) , less water needs to be displaced to support the (smaller) ice weight ($\Delta V_{water,disp} < 0$).

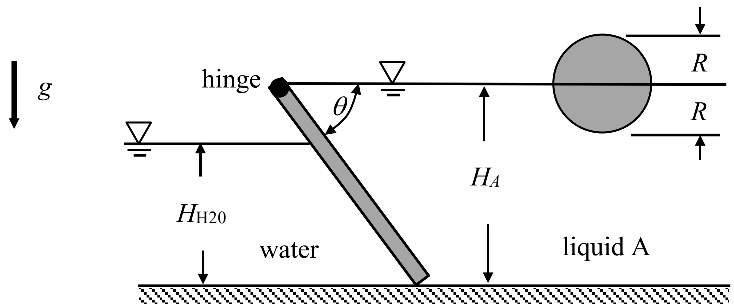
Thus, the sum of the volume of water added due to melting and the change in displaced water volume due to a change in the weight of the ice is,

$$\Delta V_{water} + \Delta V_{water,disp} = -SG_{ice}\Delta V_{ice} + SG_{ice}\Delta V_{ice} = 0 . \tag{7}$$

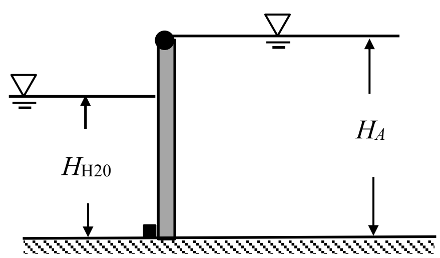
The increase in water volume is exactly balanced by a decrease in the displaced water volume, which means that the water level height won't change!

This fact has important implications regarding the rise in sea level due to melting ice. Melting free-floating ice, e.g., icebergs, won't result in an increase in sea level. However, ice that was originally supported by land, e.g., glaciers, will contribute to an increase in sea levels.

Consider the system shown below. A wooden sphere of radius R and specific gravity SG_{wood} is half submerged in an unknown liquid, referred to as liquid A. Liquid A, which has a depth H_A, is separated from a pool of water, which has a depth H_{H2O}, by a hinged gate tilted at an angle θ with respect to the horizontal. The gate has a width b into the page.



a. What is the density of liquid A, ρ_A, in terms of the specific gravity of the wooden sphere (SG_{wood}) and the density of water (ρ_{H20})?
b. What is the pressure force liquid A exerts on the inclined gate in terms of (a subset of) ρ_A, H_A, g, b, and θ? Write this force as a vector.
c. Assuming the gate has negligible mass and the angle θ is 90° so the gate is vertical (figure shown below), at what height H_{H20} will the gate just start to rotate about its hinge? Write this height in terms of (a subset of) ρ_A, ρ_{H20}, H_A, g, and b.



SOLUTION:

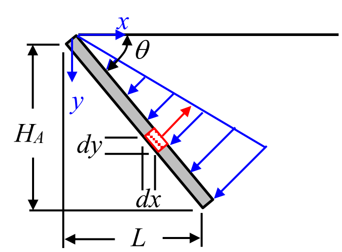
The density of liquid A may be found by balancing the weight of the wooden sphere with the buoyant force acting on it,

$$F_W = F_B \Rightarrow \rho_{\text{wood}} \tfrac{4}{3}\pi R^3 g = \rho_A \underbrace{\tfrac{1}{2}\tfrac{4}{3}\pi R^3}_{\substack{\text{half of the} \\ \text{sphere is} \\ \text{submerged}}} g \Rightarrow \rho_A = 2\rho_{\text{wood}}\,, \tag{1}$$

$$\boxed{\rho_A = 2SG_{\text{wood}}\rho_{\text{H20}}}. \tag{2}$$

The force that liquid A exerts on the gate may be found by integrating pressure forces along the length of the gate,

$$\mathbf{F}_{A\text{ on gate}} = \int_A -p\,d\mathbf{A}\,, \tag{3}$$



where,

$$p = \rho_A g y \text{ (gage pressure)}, \tag{4}$$

$$d\mathbf{A} = bdy\hat{\mathbf{i}} - bdx\hat{\mathbf{j}}\,, \tag{5}$$

so that,

$$\mathbf{F}_{A\text{ on gate}} = \int_A -(\rho_A g y)\left(bdy\hat{\mathbf{i}} - bdx\hat{\mathbf{j}}\right) = \rho_A g b\left(-\hat{\mathbf{i}}\int_{y=0}^{y=H_A} y\,dy + \hat{\mathbf{j}}\int_{x=0}^{x=L} y\,dx\right). \tag{6}$$

Note that,

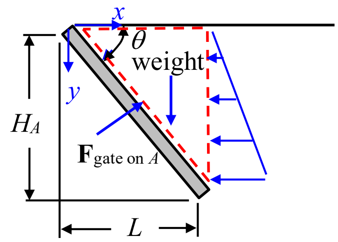
$$y = x\tan\theta \quad \text{and} \quad H_A = L\tan\theta\,, \tag{7}$$

so that Eq. (6) becomes,

$$\mathbf{F}_{A\text{ on gate}} = \rho_A g b\left(-\hat{\mathbf{i}}\int_{y=0}^{y=H_A} y\,dy + \hat{\mathbf{j}}\int_{x=0}^{x=H_A/\tan\theta} x\tan\theta\,dx\right), \tag{8}$$

$$\boxed{\mathbf{F}_{A\text{ on gate}} = -\tfrac{1}{2}\rho_A g b H_A^2\hat{\mathbf{i}} + \tfrac{1}{2}\rho_A g b\frac{H_A^2}{\tan\theta}\hat{\mathbf{j}}}. \tag{9}$$

Another approach to calculating the force on the gate is to balance forces on the triangular block of liquid shown in the figure below.

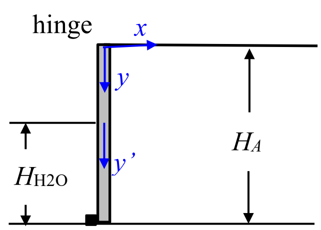


$$\sum\mathbf{F} = \mathbf{0} = \underbrace{\int_{y=0}^{y=H_A} -(\rho_A g y)\left(dyb\hat{\mathbf{i}}\right)}_{\text{pressure force on side of fluid block}} \underbrace{+\rho_A g\tfrac{1}{2}LH_A b\hat{\mathbf{j}}}_{\text{weight of fluid block}} + \underbrace{\mathbf{F}_{\text{gate on }A}}_{\substack{\text{force gate}\\\text{exerts on block}}}\,, \tag{10}$$

$$\mathbf{F}_{\text{gate on }A} = \tfrac{1}{2}\rho_A g b H_A^2\hat{\mathbf{i}} - \tfrac{1}{2}\rho_A g\frac{H_A^2}{\tan\theta}b\hat{\mathbf{j}}\,, \tag{11}$$

where Eq. (7) has been used. Note that since $\mathbf{F}_{A\text{ on gate}} = -\mathbf{F}_{\text{gate on }A}$, the final result is the same as what was found in Eq. (9)!

For the specific case when $\theta = 90°$ (figure shown below), the moments about the hinge are,



$$\sum M_{\text{hinge},z} = 0 = -\int_{y'=0}^{y'=H_{H20}}\left[y' + \left(H_A - H_{H2O}\right)\right]\left(\rho_{H20}gy'\right)\left(bdy'\right) + \int_{y=0}^{y=H_A} y\left(\rho_A g y\right)\left(bdy\right), \tag{12}$$

$$0 = gb\left(-\rho_{H20}\int_{y'=0}^{y=H_{H2O}}\left[y'^2 + \left(H_A - H_{H2O}\right)y'\right]dy' + \rho_A\int_{y=0}^{y=H_A} y^2\,dy\right), \tag{13}$$

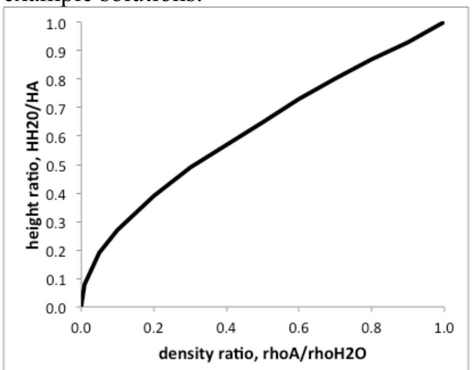
$$\rho_{H20}\left[\tfrac{1}{3}H_{H2O}^3 + \tfrac{1}{2}\left(H_A - H_{H20}\right)H_{H2O}^2\right] = \tfrac{1}{3}\rho_A H_A^3, \tag{14}$$

$$\left(\frac{H_{\mathrm{H20}}}{H_A}\right)^3+\frac{3}{2}\left[1-\left(\frac{H_{\mathrm{H20}}}{H_A}\right)\right]\left(\frac{H_{\mathrm{H20}}}{H_A}\right)^2=\frac{\rho_A}{\rho_{\mathrm{H20}}}, \tag{15}$$

$$\left(\frac{H_{\mathrm{H20}}}{H_A}\right)^3+\frac{3}{2}\left(\frac{H_{\mathrm{H20}}}{H_A}\right)^2-\frac{3}{2}\left(\frac{H_{\mathrm{H20}}}{H_A}\right)^3=\frac{\rho_A}{\rho_{\mathrm{H20}}}, \tag{16}$$

$$\boxed{\left(\frac{H_{\mathrm{H20}}}{H_A}\right)^3-3\left(\frac{H_{\mathrm{H20}}}{H_A}\right)^2+2\left(\frac{\rho_A}{\rho_{\mathrm{H20}}}\right)=0}. \tag{17}$$

This equation could be solved numerically for H_{H20}/H_A given a value for $\rho_A/\rho_{\mathrm{H20}}$. The following plot shows example solutions.



An alternate approach for deriving Eq. (17) is to sum moments about the hinge, but make note of the fact that the center of pressure on each wall is one-third of the liquid depth from the bottom of the wall,

$$\sum M_{\mathrm{hinge},z}=0=-\left[\left(H_A-H_{\mathrm{H2O}}\right)+\tfrac{2}{3}H_{\mathrm{H2O}}\right]\left(\tfrac{1}{2}\rho_{\mathrm{H2O}}gbH_{\mathrm{H2O}}^2\right)+\left(\tfrac{2}{3}H_A\right)\left(\tfrac{1}{2}\rho_A gbH_A^2\right), \tag{18}$$

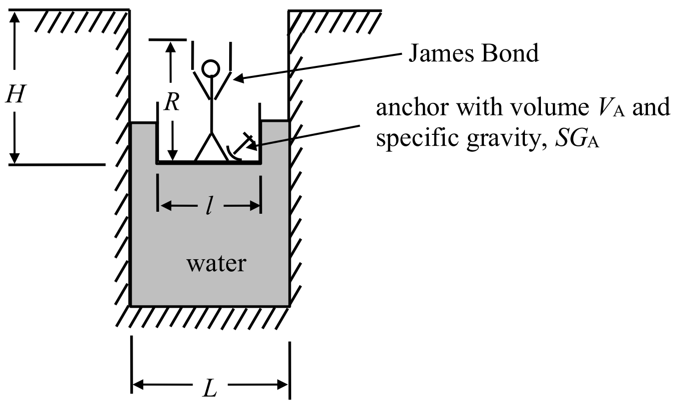
$$\tfrac{1}{2}\rho_{\mathrm{H2O}}H_{\mathrm{H2O}}^2H_A-\tfrac{1}{6}\rho_{\mathrm{H2O}}H_{\mathrm{H2O}}^3=\tfrac{1}{3}\rho_A H_A^3, \tag{19}$$

$$3\left(\frac{H_{\mathrm{H2O}}}{H_A}\right)^2-\left(\frac{H_{\mathrm{H2O}}}{H_A}\right)^3=2\left(\frac{\rho_A}{\rho_{\mathrm{H2O}}}\right), \tag{20}$$

$$\left(\frac{H_{\mathrm{H2O}}}{H_A}\right)^3-3\left(\frac{H_{\mathrm{H2O}}}{H_A}\right)^2+2\left(\frac{\rho_A}{\rho_{\mathrm{H2O}}}\right)=0, \tag{21}$$

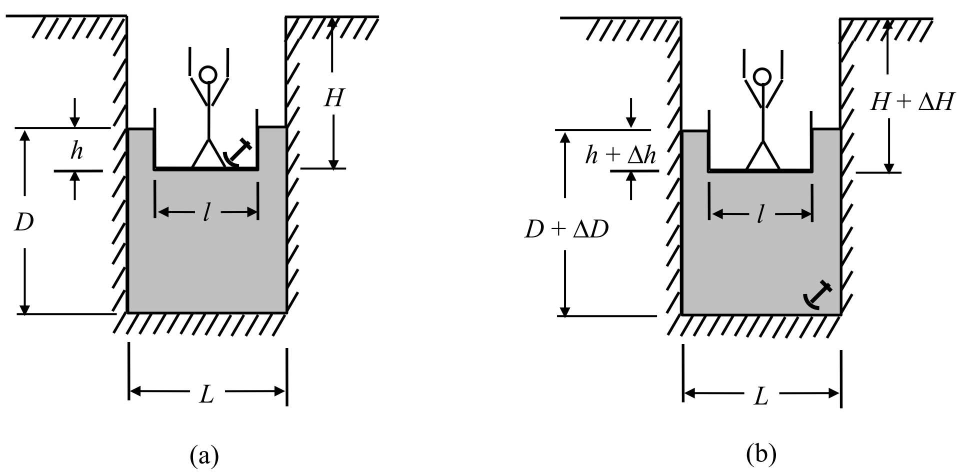
which is the same as Eq. (17).

James Bond is trapped on a small raft in a steep walled pit filled with water as shown in the figure. Both the raft and pit have square cross-sections with a side length of l=3 ft for the raft and L=4 ft for the pit. In the raft there is a steel anchor (SG_A=7.85) with a volume of V_A=1 ft^3. In the current configuration, the distance from the floor of the raft to the top of the pit is H=7.5 ft. Unfortunately, Bond can only reach a distance of R=7 ft from the floor of the raft. In order for Bond to escape, would it be helpful for him to toss the anchor overboard? Justify your answer with calculations. (Hint: The mass of water is conserved in this problem.)



SOLUTION:

Consider the cases when the anchor is in the raft and out of the raft as shown in the figures below.



First consider the change in the position of the raft floor relative to the free surface of the water.

Case (a): $$\underbrace{\left(m_{\text{raft+Bond}} + m_{\text{anchor}}\right)g}_{\text{weight of raft \& contents}} = \underbrace{\rho_{H_2O} g l^2 h}_{\text{weight of displaced water}} \tag{1}$$

Case (b): $$\underbrace{\left(m_{\text{raft+Bond}}\right)g}_{\text{weight of raft \& contents}} = \underbrace{\rho_{H_2O} g l^2 \left(h + \Delta h\right)}_{\text{weight of displaced water}} \tag{2}$$

Subtract Eqn. (2) from Eqn. (1) and simplify.

$$\left(m_{\text{raft+Bond}} + m_{\text{anchor}}\right) g - \left(m_{\text{raft+Bond}}\right) g = \rho_{H_2O} g l^2 h - \rho_{H_2O} g l^2 \left(h + \Delta h\right) \tag{3}$$

$$m_{\text{anchor}} = -\rho_{H_2O} l^2 \Delta h \tag{4}$$

$$\Delta h = -\frac{m_{\text{anchor}}}{\rho_{H_2O} l^2} \tag{5}$$

$$\therefore \Delta h = -\frac{SG_{\text{anchor}} V_{\text{anchor}}}{l^2} \tag{6}$$

Note that since $V_{\text{anchor}} > 0$, $\Delta h < 0$ and thus the raft moves up relative to the free surface. However, the free surface will also move so we still don't yet know whether Bond moves up or down relative to the surface of the pit.

We must now consider the movement of the free surface of the water.

Case (a): $$V_{H_2O} = \underbrace{L^2 D}_{\text{volume of } H_2O \text{ in pit}} - \underbrace{l^2 h}_{\text{volume of raft in } H_2O} \tag{7}$$

Case (b): $$V_{H_2O} = \underbrace{L^2\left(D + \Delta D\right)}_{\text{volume of } H_2O \text{ in pit}} - \underbrace{l^2\left(h + \Delta h\right)}_{\text{volume of raft in } H_2O} - V_{\text{anchor}} \tag{8}$$

Since the volume of water is conserved, Eqns. (7) and (8) must be equal.

$$L^2\left(D + \Delta D\right) - l^2\left(h + \Delta h\right) - V_{\text{anchor}} = L^2 D - l^2 h \tag{9}$$

$$L^2 \Delta D - l^2 \Delta h - V_{\text{anchor}} = 0$$

$$\therefore \Delta D = \frac{l^2 \Delta h + V_{\text{anchor}}}{L^2} \tag{10}$$

$$\therefore \Delta D = \frac{\left(1 - SG_{\text{anchor}}\right) V_{\text{anchor}}}{L^2} \quad \text{(where Eqn. (6) has been utilized)} \tag{11}$$

Note that since $SG_{\text{anchor}} > 1$, $\Delta D < 0$, i.e. the free surface moves downward.

Combine the expressions for Δh and ΔD to determine the movement of the raft bottom relative to the pit walls.

$$D + H - h = \left(D + \Delta D\right) + \left(H + \Delta H\right) - \left(h + \Delta h\right) \tag{12}$$

$$\Delta H = -\Delta D + \Delta h \tag{13}$$

$$\Delta H = -\frac{\left(1 - SG_{\text{anchor}}\right) V_{\text{anchor}}}{L^2} - \frac{SG_{\text{anchor}} V_{\text{anchor}}}{l^2} \tag{14}$$

$$\boxed{\therefore \Delta H = \frac{V_{\text{anchor}}}{L^2}\left[SG_{\text{anchor}}\left(1 - \frac{L^2}{l^2}\right) - 1\right]} \tag{15}$$

Use the given data to determine ΔH.

$V_{\text{anchor}} = 1 \text{ ft}^3$
$L = 4$ ft
$SG_{\text{anchor}} = 7.85$
$l = 3$ ft
$\Rightarrow$ $\boxed{\Delta H = -0.44 \text{ ft (The raft moves closer to the top of the pit.)}}$

Recall that $H = 7.5$ ft and Bond can only reach $R = 7$ ft. After tossing the anchor overboard, the bottom of the raft is $H + \Delta H = 7.06 \text{ ft} > R = 7$ ft. $\boxed{\text{Hence, Bond still can't reach the top of the pit.}}$

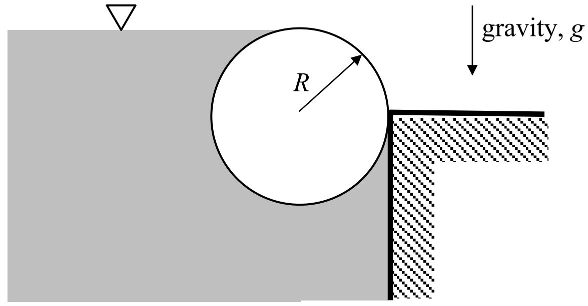
Goodbye, Mr. Bond.

Video solution: https://www.youtube.com/watch?v=w5pVYc30GQA

A cylindrical log of radius R and length L rests against the top of a dam. The water is level with the top of the log and the center of the log is level with the top of the dam. You may assume that the contact point with the dam is frictionless. Obtain expressions for

a. the mass of the log, and
b. the contact force between the log and dam.

Express your answers in terms of (a subset of) ρ_{H2O}, g, L, and R.

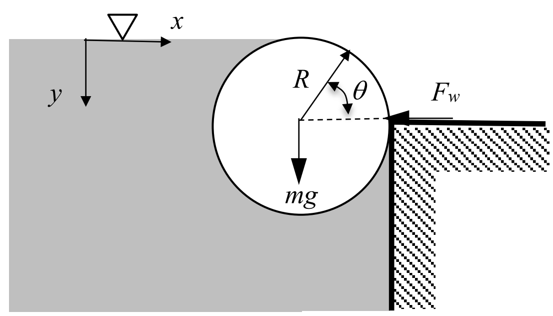


SOLUTION:

The mass of the log, m, may be found by performing a force balance in the vertical direction,

$$\sum F_y = 0 = mg + F_{P,y}\ , \tag{1}$$

where g is the acceleration due to gravity. Note that the point of contact with the dam is assumed to be frictionless.



The net vertical pressure force, $F_{P,y}$, is found by integrating the vertical component of the pressure force around the log,

$$F_{P,y} = \int_{\theta=\pi/2}^{\theta=2\pi} p\sin\theta\, dA = \int_{\theta=\pi/2}^{\theta=2\pi} \underbrace{\rho g y}_{=p}\sin\theta \underbrace{R d\theta (L)}_{=dA}, \tag{2}$$

$$F_{P,y} = \int_{\theta=\pi/2}^{\theta=2\pi} \rho g\underbrace{(R - R\sin\theta)}_{=y}\sin\theta R\, d\theta (L) = \rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} (1-\sin\theta)\sin\theta\, d\theta\ , \tag{3}$$

$$F_{P,y} = \rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} \left(\sin\theta - \sin^2\theta\right) d\theta\ , \tag{4}$$

where ρ is the density of the water. Evaluating the integral in Eq. (4) gives,

$$F_{P,y} = \rho g R^2 L\left\{ -\cos\theta\Big|_{\theta=\pi/2}^{\theta=2\pi} - \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta)\right]_{\theta=\pi/2}^{\theta=2\pi} \right\} = \rho g R^2 L\left[-1 - \tfrac{1}{2}\left(2\pi - \tfrac{\pi}{2}\right)\right], \tag{5}$$

$$F_{P,y} = -\left(1 + \tfrac{3\pi}{4}\right)\rho g R^2 L\ . \tag{6}$$

Substituting into Eq. (1) and solving for m gives,

$$\boxed{m=\left(1+\frac{3\pi}{4}\right)\rho R^2 L}. \tag{7}$$

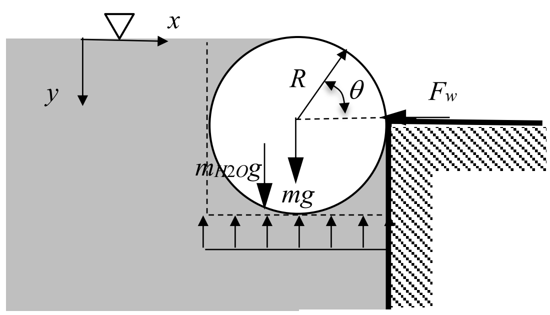
An alternate, easier method for determining the vertical pressure force acting on the log is to note that the vertical surface forces acting along a horizontal plane at the bottom of the log is,

$$\sum F_y = 0 = \underbrace{\rho g(2R)(2RL)}_{\text{pressure force at bottom}} \underbrace{-mg}_{\text{log weight}} \underbrace{-\rho g L \tfrac{3}{4}\left(4R^2-\pi R^2\right)}_{\text{weight of water}} = 4\rho g L R^2 - mg - \rho g L \tfrac{3}{4}\left(4R^2-\pi R^2\right), \tag{8}$$

$$mg = \rho g L R^2\left[4-\tfrac{3}{4}(4-\pi)\right] = \rho g L R^2\left[4-3+\tfrac{3}{4}\pi\right], \tag{9}$$

$$m=\rho R^2 L\left(1+\frac{3\pi}{4}\right), \tag{10}$$

which is the same result found in Eq. (7).



An even easier method is to use a buoyant force, although one must recognize the appropriate volume to use to determine the displaced volume. A vertical force balance for the log gives,

$$\sum F_y = 0 = -mg + F_B \;\Rightarrow\; m=\frac{F_B}{g}, \tag{11}$$

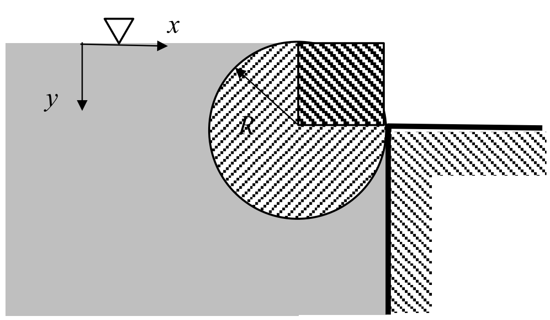
where F_B is the buoyant force, which is the weight of the displaced fluid. Note that in this case, the displaced volume of fluid is the volume of the log, plus the volume above the right, upper quadrant of the log as shown in the figure below,

$$F_B = \rho g V_{\text{displaced}} = \rho g\left(\tfrac{3}{4}\pi R^2 + R^2\right)L = \rho g R^2 L\left(\tfrac{3\pi}{4}+1\right), \tag{12}$$

Combining Eqs. (11) and (12) gives the mass of the log,

$$m=\rho R^2 L\left(\tfrac{3\pi}{4}+1\right), \tag{13}$$

which is exactly the same result as found in the previous two methods.



Now consider a horizontal force balance for the log.

$$\sum F_x = 0 = -F_w + F_{P,x}\,, \tag{14}$$

where F_w is the horizontal force exerted by the wall on the wall and $F_{P,x}$ is the horizontal component of the net pressure force acting on the log due to the water. The net horizontal pressure force is given by,

$$F_{P,x} = \int_{\theta=\pi/2}^{\theta=2\pi} -p\cos\theta\, dA = \int_{\theta=\pi/2}^{\theta=2\pi} \underbrace{(-\rho g y)}_{=p}\cos\theta \underbrace{(Rd\theta L)}_{=dA} = \int_{\theta=\pi/2}^{\theta=2\pi}\left[-\rho g\underbrace{(R-R\sin\theta)}_{=y}\right]\cos\theta (Rd\theta L)\,, \tag{15}$$

$$F_{P,x} == -\rho g R^2 L\int_{\theta=\pi/2}^{\theta=2\pi}(1-\sin\theta)\cos\theta\, d\theta = -\rho g R^2 L\int_{\theta=\pi/2}^{\theta=2\pi}(\cos\theta - \sin\theta\cos\theta)\, d\theta\,. \tag{16}$$

Evaluate the integrals in Eq. (16),

$$F_{P,x} = -\rho g R^2 L\left[\sin\theta\Big|_{\theta=\pi/2}^{\theta=2\pi} - \tfrac{1}{2}\sin^2\theta\Big|_{\theta=\pi/2}^{\theta=2\pi}\right] = -\rho g R^2 L\left[\sin\theta\Big|_{\theta=\pi/2}^{\theta=2\pi} - \tfrac{1}{2}\sin^2\theta\Big|_{\theta=\pi/2}^{\theta=2\pi}\right], \tag{17}$$

$$F_{P,x} = -\rho g R^2 L\left[-1+\tfrac{1}{2}\right] = \tfrac{1}{2}\rho g R^2 L\,. \tag{18}$$

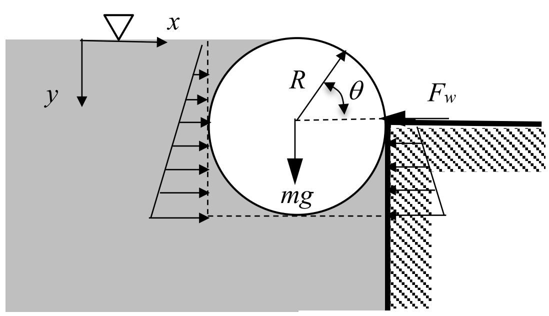
Substitute into Eq. (14) and solve for the wall force.

$$\boxed{F_w = \tfrac{1}{2}\rho g R^2 L}. \tag{19}$$

Another, much simpler method for finding the wall force is to note that the horizontal pressure force acting on the log will simply be the pressure force acting on the horizontally projected area.

$$F_{P,x} = \int_{y=0}^{y=R} p\, dA = \int_{y=0}^{y=R}\underbrace{(\rho g y)}_{=p}\underbrace{dyL}_{=dA} = \rho g L\int_{y=0}^{y=R} y\, dy = \tfrac{1}{2}\rho g R^2 L\,, \tag{20}$$

which is precisely the same result found in Eq. (18). Note that the horizontal pressure force is only evaluated from $y = 0$ to $y = R$ since on the bottom half of the log, the pressure forces from either side of the log cancel each other out.



2.6. Pressure Distribution Due to Rigid Body Motion

Recall that for a static fluid with no shear stresses, a balance of forces on a small fluid element gives (Eq. (2.6)),

$$-\boldsymbol{\nabla} p + \rho \boldsymbol{g} = \boldsymbol{0}. \tag{2.86}$$

If the fluid element is now subject to an acceleration $\boldsymbol{a}$, then from Newton's Second Law, Eq. (2.86) becomes,

$$-\boldsymbol{\nabla} p + \rho \boldsymbol{g} = \rho \boldsymbol{a}. \tag{2.87}$$

One method for producing an acceleration without shear stresses is through rigid body translation, e.g., where the whole body of fluid moves together in a straight line. For this case, we can easily combine the acceleration due to gravity with the rigid body translational acceleration,

$$\boldsymbol{\nabla} p = \rho(\boldsymbol{g} - \boldsymbol{a}). \tag{2.88}$$

For example, consider the case in which a tank of liquid is accelerated upwards and to the right as shown in Figure 2.22 (assume a two-dimensional system for simplicity).

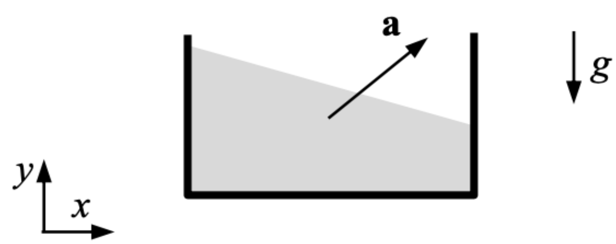


FIGURE 2.22. A two-dimensional container of liquid accelerated in rigid-body translation in the positive x and y directions.

From Eq. (2.88), the pressure distribution in the liquid is,

$$\frac{\partial p}{\partial x} = -\rho a_x, \tag{2.89}$$

$$\frac{\partial p}{\partial y} = -\rho (g + a_y), \tag{2.90}$$

where a_x and a_y are the components of the applied acceleration. Note that gravity points in the -y direction. The pressure varies in both the x and y directions. We can determine the pressure anywhere in the fluid by integrating and combining the previous two equations (and noting that they involve *partial* derivatives),

$$\frac{\partial p}{\partial x} = -\rho a_x \implies p(x, y) = -\rho a_x x + f(y), \tag{2.91}$$

$$(f(y) \text{ is an unknown function of } y \text{ and can include a constant}), \tag{2.92}$$

$$\frac{\partial p}{\partial y} = -\rho(g + a_y) \implies p(x, y) = -\rho(g + a_y)y + g(x), \tag{2.93}$$

$$(g(x) \text{ is an unknown function of } x \text{ and can include a constant}), \tag{2.94}$$

$$\therefore p(x, y) = -\rho a_x x - \rho(g + a_y)y + \text{constant}. \tag{2.95}$$

Thus, the pressure varies linearly in both the x and y directions. The constant can be found by applying the boundary condition on the (inclined) free surface where the pressure is atmospheric.

The isobars in the fluid, i.e., the surfaces on which the pressure is constant, are found by noting,

$$p = p(x, y) \implies dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \overset{dp=0}{\implies} \frac{dy}{dx} = -\frac{\partial p / \partial x}{\partial p / \partial y}, \tag{2.96}$$

where, along an isobar, $dp = 0$. Making use of Eqs. (2.91) and (2.93), for this particular case,

$$\frac{dy}{dx} = -\frac{-\rho a_x}{-\rho(g + a_y)} = -\frac{a_x}{g + a_y}. \tag{2.97}$$

Thus, the isobars are lines of constant slope, as shown in Figure 2.23 as dotted lines. Note that the free

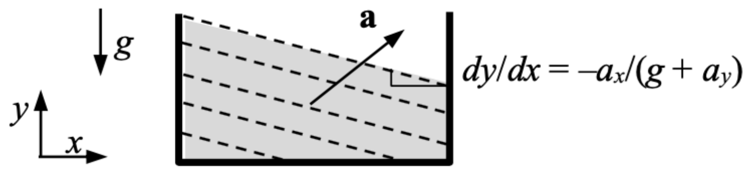
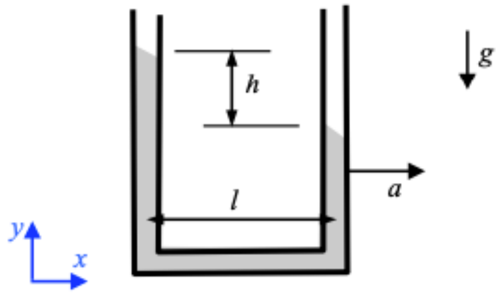


Figure 2.23. Isobars in a 2D container of liquid accelerating in translation.

surface is also an isobar. This simple example demonstrates that it's possible to measure accelerations by measuring the surface slope.

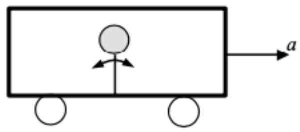
A U-tube manometer is accelerated in the positive x direction with an acceleration a. Determine the height difference h in the manometer arms in terms of the acceleration a and the distance between the arms l.



SOLUTION:
The manometer is undergoing rigid body translation in the x direction,
$$-\nabla p - \rho g \hat{\boldsymbol{j}} = \rho a \hat{\boldsymbol{\imath}}, \quad (1)$$
which, when expanded is,
$$-\frac{\partial p}{\partial x} = \rho a, \quad (2)$$
$$-\frac{\partial p}{\partial y} - \rho g = 0. \quad (3)$$
Since the pressure is a function of both x and y, i.e., $p = p(x, y)$,
$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy. \quad (4)$$
Substituting Eqs. (2) and (3) into Eq. (4) and solving for the pressure,
$$\int dp = \int (-\rho a) dx + \int (-\rho g) dy, \quad (5)$$
$$p = -\rho a x - \rho g y + \text{constant}. \quad (6)$$

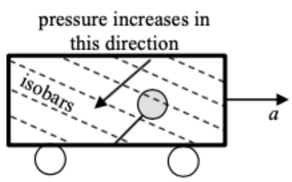
Taking the difference between two arbitrary points,
$$\Delta p = -\rho a \Delta x - \rho g \Delta y. \quad (7)$$
Since on the free surface of the two arms the pressure is a constant, $\Delta p = 0$, from Eq. (7),
$$0 = -\rho a l - \rho g h, \quad (8)$$
$$\boxed{h = \left(\frac{a}{g}\right) l}. \quad (9)$$

A person is driving with a floating helium balloon in a car. When the car accelerates, in what direction does the balloon move, i.e., forward, backward, stay in the same location, something else?



See, for example: https://www.youtube.com/watch?v=y8mzDvpKzfY

SOLUTION:
When the car accelerates, the pressure gradient in the air increases as one moves from the front of the car to the rear.



Thus, the balloon will move forward as the car accelerates (perpendicular to the isobars) due to the buoyant force acting on the balloon!

Another type of rigid body motion is rigid body rotation, where the fluid rotates at a constant speed about a central axis (Figure 2.24). As with rectilinear acceleration, in rigid body rotation there are no shear stresses acting on small elements of fluid.

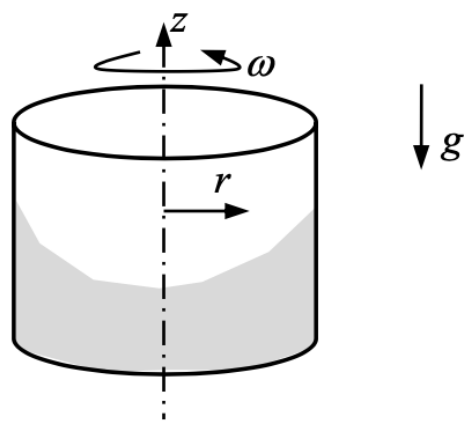


Figure 2.24. A cylindrical container of liquid in rigid body rotation.

Equation (2.87) still holds for this system, but rather than using Cartesian coordinates, the system is better suited to cylindrical coordinates. In cylindrical coordinates,

$$\nabla p = \frac{\partial p}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial p}{\partial z}\hat{\mathbf{e}}_z. \tag{2.98}$$

Thus, Eq (2.87) becomes,

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \quad \text{(refer to Figure 2.25)}, \tag{2.99}$$

$$\frac{\partial p}{\partial \theta} = 0, \tag{2.100}$$

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.101}$$

Since $p = p(r, z)$, the slope of the isobars for this case is,

$$p = p(r, z) \implies dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz \overset{dp=0}{\implies} \frac{dz}{dr} = -\frac{\partial p/\partial r}{\partial p/\partial z}. \tag{2.102}$$

Substituting Eqs. (2.99) and (2.101) gives,

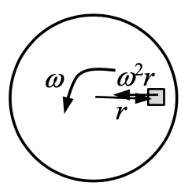


Figure 2.25. Acceleration in the radial direction for a small element of fluid in rigid body rotation.

$$\frac{dz}{dr} = -\frac{\rho\omega^2 r}{-\rho g} = \frac{\omega^2 r}{g}. \tag{2.103}$$

Solving this differential equation subject to the boundary condition that $z(r=0)=0$, i.e., setting the origin on the fluid free surface at the centerline,

$$\int_{z=0}^{z=z} dz = \frac{\omega}{g}\int_{r=0}^{r=r} r dr, \tag{2.104}$$

$$\therefore z = \frac{1}{2}\frac{\omega^2 r^2}{g}. \tag{2.105}$$

Thus, the isobars for this case are parabolas (actually a paraboloids since they're rotated about the z-axis)! The pressure anywhere in the fluid is found by integrating Eqs. (2.99) and (2.101) and combining (note that the pressure derivatives are partial derivatives),

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \implies p(r,z) = \frac{1}{2}\rho\omega^2 r^2 + f(z), \tag{2.106}$$

$$(f(z) \text{ is an unknown function of } z \text{ and can include a constant}), \tag{2.107}$$

$$\frac{\partial p}{\partial z} = -\rho g \implies p(r,z) = -\rho g z + g(r), \tag{2.108}$$

$$(g(r) \text{ is an unknown function of } r \text{ and can include a constant}), \tag{2.109}$$

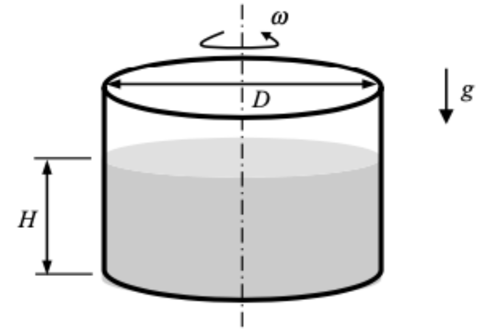
$$\therefore p(r,z) = \frac{1}{2}\rho\omega^2 r^2 - \rho g z + \text{constant}. \tag{2.110}$$

The constant can be determined from the boundary condition at the (parabolic) free surface where the pressure is atmospheric. Note that at a constant elevation z, the pressure increases with the square of the (increasing) radius while at a constant radius, the pressure varies linearly with elevation.

Notes:

(1) Rigid body rotation of liquids is often used in the manufacture of large mirrors for telescopes since the surface shape is parabolic.

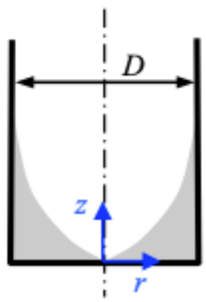
For the cylindrical tank containing a liquid of density ρ as shown below, at what rotational speed ω will the bottom of the tank first be exposed?



SOLUTION:
Recall from the notes that the free surface of the liquid forms a parabolic shape when in rigid body rotation,

$$z = \frac{1}{2}\frac{\omega^2}{g}r^2, \tag{1}$$

where z is measured from the minimum height location (located along the centerline) and r is the radius from the centerline. When the bottom of the tank is first exposed, a side view of the tank looks like the figure shown below.

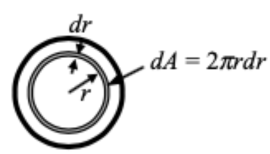


Note that the liquid volume remains constant in this problem. The initial volume of liquid is simply,

$$V = \frac{\pi}{4}D^2H. \tag{2}$$

The volume when the tank bottom is first exposed is,

$$V = \int_{r=0}^{r=D/2} z\underbrace{(2\pi r dr)}_{=dA} = \int_0^{D/2}\frac{1}{2}\frac{\omega^2}{g}r^2(2\pi r dr) = \pi\frac{\omega^2}{g}\int_0^{D/2}r^3 dr = \frac{\pi}{4}\frac{\omega^2}{g}\left(\frac{D}{2}\right)^4, \tag{3}$$



Set Eqs. (2) and (3) equal,

$$\frac{\pi}{4}D^2H = \frac{\pi}{4}\frac{\omega^2}{g}\left(\frac{D}{2}\right)^4, \tag{4}$$

$$\boxed{\omega = 4\sqrt{\frac{gH}{D^2}}}. \tag{5}$$

CHAPTER 3

Introductory Thermodynamics

3.1. Basics

3.1.1. What is thermodynamics?

Thermodynamics is the study of energy, work, and heat, and the transformation between these quantities.

3.1.2. Where is thermodynamics used?

- power generation, e.g., fossil fuel power plants, internal combustion engines, gas turbine engines, solar thermal power plants
- high speed flows of gases, i.e., compressible flows, e.g., jet engines, rockets, high speed aircraft
- heating, ventilation, and air conditioning
- refrigeration
- combustion
- phase changes (evaporation, condensation, sublimation)

Thermodynamics serves as a foundation for many other topics, including fluid mechanics and heat transfer.

3.1.3. Definitions

- A closed system (aka system, aka control mass) is a particular quantity of matter chosen for study. The system may change shape and location, but it is always the same matter.
- The surroundings consist of everything that is not the system.
- The boundary of a system is the surface separating the system and surroundings.
- An isolated system is a closed system that does not interact with its surroundings. For example, if the system consists of air in a sealed, rigid, insulated container, then the air may be consider an isolated system since it has no mass, work, or heat transfer with the surroundings.
- A control volume (CV) (aka open system) is a particular volume chosen for study. Unlike a system, matter may change within a control volume. Note that the control volume does not need to remain fixed in size or location; it may move or change size and shape.
- A control surface (CS) is the surface enclosing a control volume. The orientation of the CS at a particular location is given by the direction of its outward-pointing unit normal vector, $\hat{\mathbf{n}}$, at that location. The outward-pointing unit normal vector has a magnitude of one, is perpendicular to the control surface, and always points out of the CV (Figure 3.1).
- Properties are macroscopic characteristics of a system. Example properties include mass, volume, energy, pressure, and temperature. A quantity is a property if and only if its change in value between two states is independent of the process between these states. For example, pressure is a property since its value only depends on the current state, but the work done on a system is not since the work depends on the process taken to reach a given state.
- An extensive property is one that depends on the mass in the system. For example, kinetic energy and mass are extensive properties since their values are proportional to the mass in the system.
- An intensive property is one that is independent of the mass in the system. For example, temperature and pressure are intensive properties since their values are independent of how much mass is in the system.

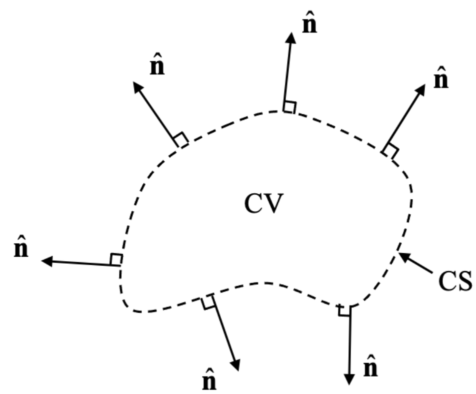


Figure 3.1. Illustration of control volume (CV), control surface (CS), and outward-pointing, unit normal vectors.

- A specific property is an extensive property per unit mass. A specific property is also an intensive property. An example of a specific property is specific volume $v = V/m$ where V is the system property and m is the system mass.
- An easy way to determine whether a property is extensive or intensive is to divide the system into two parts and see how the property is affected.
- The state of a system is the system's condition or configuration as described by its properties in sufficient detail so that it is distinguishable from other states. Often only a subset of properties is needed to define a state since some properties may be related.
- A process is the transformation of a system from one state to another. A few common processes include:
  - isothermal process: A process that occurs at constant temperature.
  - isobaric process: A process that occurs at constant pressure.
  - isochoric or isometric process: A process that occurs at constant volume.
  - adiabatic process: A process in which there is no heat transfer between the system and surroundings.
  - isentropic process: A process that occurs at constant entropy.
- A process is in steady state if the system's state does not change with time.
- A system is in a state of equilibrium if there are no potentials driving the system to another state. Examples of driving potentials include unbalanced forces, unbalanced temperatures, an electric potential (aka voltage).
- A process path is the series of states that a system passes through during some process.
- A quasi-equilibrium process is one where the process proceeds in such a manner that the system remains infinitesimally close to an equilibrium state at all times. One can interpret a quasi-equilibrium process as occurring slowly enough so that the system has time to adjust internally such that properties in part of the system do not change any faster than those properties in other parts of the system.
- A reversible process is one in which the system is in a state of equilibrium at all points in its path. In a reversible process, the system and the surroundings can be restored exactly to their initial states.
- An irreversible process is one where the system is not in a state of equilibrium at all points in its path. The system and surroundings cannot be returned to their exact initial states in an irreversible process. Note that all natural processes are irreversible. Several effects causing irreversibility include viscosity, heat conduction, and mass diffusion.

- A cycle is a sequence of processes that begins and ends at the same state. At the conclusion of a cycle, all properties have the same values they had at the beginning of the cycle. Thus, there is no change in the system's state at the end of a cycle.
- An equation of state is a relationship between properties of a particular substance or class of substances. Equations of state cannot be obtained from thermodynamics but are obtained either from experimental measurements or from some molecular model. Note that there can be various types of equations of state, e.g., two equations of state for an ideal gas include a thermal equation of state, which is the ideal gas law, $p = \rho RT$, and a caloric equation of state, which describes the relationship between the internal energy and temperature, $du = c_v dT$.

3.2. Energy, Work, and Heat

Now let's move our discussion to the three basic thermodynamic concepts of energy, work, and heat.

3.2.1. Energy

The energy associated with some phenomenon is not a physical quantity but is, in fact, just a number resulting from a formula containing physically measurable quantities related to that phenomenon. For example, the energy associated with the macroscopic motion of a system of mass, m, moving with a speed, V, is equal to $\frac{1}{2}mV^2$. By itself, the energy associated with a phenomenon is not a very useful quantity. However, experiments examining the total energy of a system, i.e., the sum of all the various energies, have resulted in a very remarkable observation. When the system does not interact with its surroundings, the total energy of the system remains constant. The energy associated with a particular phenomenon may change; however, it can only change at the expense of the energy associated with some other phenomenon. We'll examine this observation in greater detail a little later, but for now we will define the various types of energy that are most commonly encountered in engineering thermodynamics.

3.2.1.1. Kinetic Energy, KE

The energy associated with the macroscopic motion of a system relative to a coordinate system xyz is known as the kinetic energy, KE,

$$KE = \frac{1}{2}mV^2, \tag{3.1}$$

where m is the mass of the system and V_{xyz} is the speed of the system in the coordinate system xyz.

3.2.1.2. Potential Energy, PE

The energy associated with a system's ability to do work in an external force field, such as a gravity field, is known as the potential energy, PE. For example, the gravitational potential energy for a mass, m, located in a gravitational field with gravitational acceleration, g, pointing in the $-z$ direction is,

$$PE = mgz, \tag{3.2}$$

where z is the height of the mass above some reference plane.

3.2.1.3. Internal Energy, U

The internal energy of a system is comprised of a number of sub-classes of energy which include:

(1) sensible energy. This is the energy associated with the internal molecular translational, rotational, and vibrational motion. Temperature is a measure of this type of internal energy. The larger the temperature of a system, the greater its sensible energy.
(2) latent energy. This is the energy associated with the attraction between molecules. We concern ourselves with latent energy most often when examining processes that involve a change of phase, such as going from a solid to a liquid or from a liquid to a gas (or vice versa).
(3) chemical energy. This is the energy associated with the attraction between atoms.

(4) nuclear energy. This is the energy associated with the attraction between particles within an atom, such as the attraction between protons and neutrons. There are other forms of internal energy (e.g., the energy associated with electric and magnetic dipole moments) but we rarely encounter these in typical engineering applications. In these notes we'll only concern ourselves with sensible energy.

The total energy of a system, E, is the sum of these various types of energy,

$$E = U + KE + PE. \tag{3.3}$$

Note that the total, internal, kinetic, and potential energies are extensive properties, i.e., the magnitude of these energies depends on the system mass. In terms of specific quantities (extensive properties per unit mass) we have,

$$e = u + \frac{1}{2}V^2 + G, \tag{3.4}$$

where e is the specific total energy, u is the specific internal energy, V is the velocity magnitude (i.e., speed), and G is a (conservative) potential energy function. Note that the force per unit mass resulting from a conservative potential energy function is found by taking the negative of its gradient, i.e., if $G = gz$ where g is the gravitational acceleration and z is the height of the system above some reference plane, then $\mathbf{f}_{\text{gravity}} = -\boldsymbol{\nabla} G = -g\hat{\mathbf{k}}$.

The values of the specific internal energy, u, at different states for various substances are tabulated in thermodynamic property tables. Most introductory thermodynamics books have such tables for steam, refrigerants, and a variety of gases.

3.2.2. Work, W

Work is an energy interaction (a way to transfer energy) occurring at the boundary between a system and its surroundings. Thus, work is not a property of a system but rather is associated with a process that the system is undergoing. The work done on the system by its surroundings depends on the path of the process. A quantity that is also commonly encountered when discussing work is the power, $\dot{W}$, defined as the work done per unit time,

$$\dot{W} = \frac{\delta W}{dt}. \tag{3.5}$$

The small amount of work done on a system, $\delta W_{\text{on sys}}$, is equal to the dot product of the force acting on the system, $\mathbf{F}_{\text{on sys}}$, and the distance over which the force acts, $d\mathbf{s}$,

$$\underbrace{\delta W_{\text{on sys}}}_{\substack{\text{small amount of work}\\ \text{on the system}}} = \underbrace{\mathbf{F}_{\text{on sys}}}_{\substack{\text{force acting}\\ \text{on the system}}} \cdot \underbrace{d\mathbf{s}}_{\substack{\text{small distance over}\\ \text{which the force acts}}}. \tag{3.6}$$

The corresponding <u>power</u> is,

$$\dot{W} = \frac{\delta W}{dt} = \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} = \mathbf{F} \cdot \mathbf{V}, \tag{3.7}$$

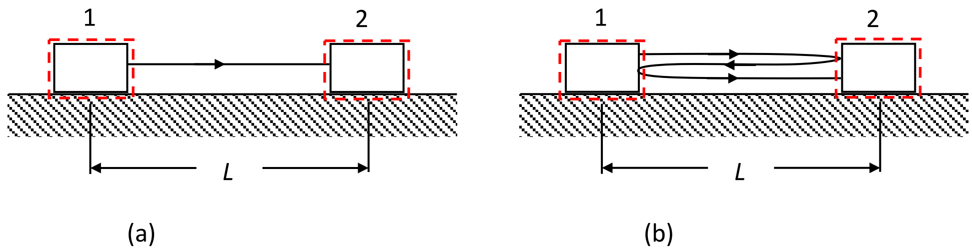
where $\mathbf{V}$ is the velocity.

The total work required in going from state 1 (indicated by s_1) to state 2 (s_2), W_{12}, may be found by integrating Eq. (3.6) between the two states,

$$W_{12} = \int_{s_1}^{s_2} \delta W = \int_{s_1}^{s_2} \mathbf{F} \cdot d\mathbf{s}. \tag{3.8}$$

Note that the <u>work depends on the path</u> taken from s_1 to s_2, so in addition to the integral, the path in going from s_1 to s_2 must be known.

A block with weight, w, is pushed on a frictional surface. The friction coefficient between the block and the surface is μ. Determine the amount of work done by the friction force on the block when moving the block from state 1 to state 2 using the paths shown.



SOLUTION:
The magnitude of the friction force acting on the block is $F = \mu w$, which acts over the small horizontal displacement dx. Since the friction force always acts in the direction opposite to the displacment,

$$\delta W_{\text{on block}} = \mathbf{F} \cdot d\mathbf{s} = -\mu w dx \ . \tag{1}$$

For case (a) the total displacement is just L. Hence, the total work done on the block is,

$$W_{\text{on block},12} = \int_1^2 \delta W = -\mu w L \ . \tag{2}$$

The total displacement for case (b) is $3L$. Hence, the total work for this case is,

$$W_{\text{on block},12} = \int_1^2 \delta W = -3\mu w L \ . \tag{3}$$

Thus, even though the block starts and ends at the same location, the work done on the block by the friction during the process is different since the paths are different.

Now let's consider a few different types of work that can be done by or on a system. The types of work we'll present here include work due to gravity, acceleration, pressure, electricity, springs, and rotating shafts. In the following drawings, the system is enclosed by a dashed line.

3.2.2.1. Gravitational Work (aka Potential Energy)

Consider the minimum amount of work required to move an object with mass, m, to a higher elevation in a gravity field, assuming a quasi-static process so that accelerations can be neglected (Figure 3.2),

$$W_{\text{on sys},12} = \int_0^{\Delta h} \mathbf{F}_{\text{on sys}} \cdot d\mathbf{s}, \tag{3.9}$$

$$= \int_0^{\Delta h} (mg\hat{\mathbf{e}}_z) \cdot (dz\hat{\mathbf{e}}_z), \tag{3.10}$$

$$\boxed{W_{\text{on sys},12} = mg\Delta h}. \tag{3.11}$$

The work is equal to the change in the potential energy of the system! Note that the work on the surroundings

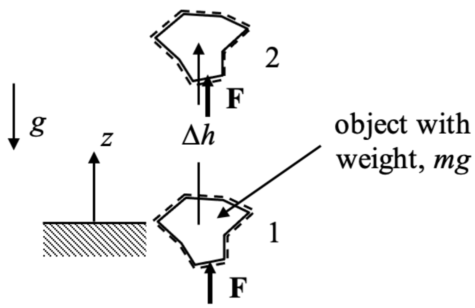


Figure 3.2. Illustration of the work due to a change in elevation.

is equal to, but has the opposite sign, of the work done on the system.

3.2.2.2. Acceleration Work (aka Kinetic Energy)

Consider the minimum work required to accelerate an object with mass, m, from speed, V_1, to speed, V_2 (Figure 3.3),

$$W_{\text{on sys},12} = \int_1^2 \mathbf{F}_{\text{on sys}} \cdot d\mathbf{s}, \tag{3.12}$$

$$= \int_{V_1}^{V_2} \left(\underbrace{m\frac{dV}{dt}}_{\text{Newton's 2nd Law}} \hat{\mathbf{e}}_x \right) \cdot \left(\underbrace{V\,dt}_{=dx} \hat{\mathbf{e}}_x \right), \tag{3.13}$$

$$= m\int_{V_1}^{V_2} V\,dV, \tag{3.14}$$

$$\boxed{W_{\text{on sys},12} = \frac{1}{2}m\left(V_2^2 - V_1^2\right)}. \tag{3.15}$$

The work is equal to the change in the kinetic energy of the system!

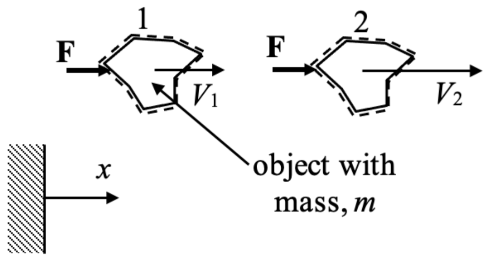


Figure 3.3. Illustration of the work due to acceleration.

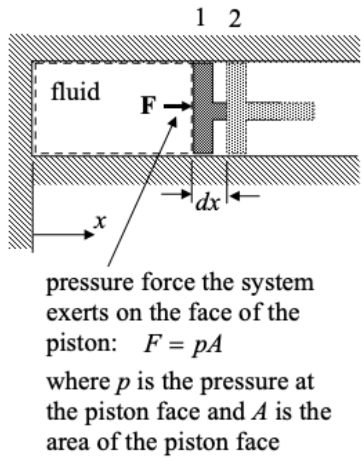


Figure 3.4. Illustration of the work due to a pressure force.

3.2.2.3. Pressure Work

Consider the work done by the expansion of a fluid (a gas or liquid) in a piston (Figure 3.4),

$$W_{\text{on surr},12} = \int_1^2 \mathbf{F}_{\text{on surr}} \cdot d\mathbf{s}, \tag{3.16}$$

$$= \int_{x_1}^{x_2} (pA\hat{\mathbf{e}}_x) \cdot (dx\hat{\mathbf{e}}_x), \tag{3.17}$$

$$= \int_{x_1}^{x_2} pAdx, \tag{3.18}$$

$$\boxed{W_{\text{on surr},12} = \int_{V_1}^{V_2} pdV}. \tag{3.19}$$

Note that $dV = Adx$. Note also that in this example, the work on the surroundings has been calculated instead of the work acting on the system. To get the work done on the system, we simply have, $W_{\text{on sys}} = -W_{\text{on surr}}$.

If we plot how the pressure changes with volume, we get a p-V diagram, as shown in Figure 3.5. Note that different paths from state 1 to state 2 will give different work values, as shown in Figure 3.6. One example of a particular pressure-volume relationship is known as a polytropic process in which the pressure and volume are related by,

$$pV^n = c, \tag{3.20}$$

where n and c are constants.

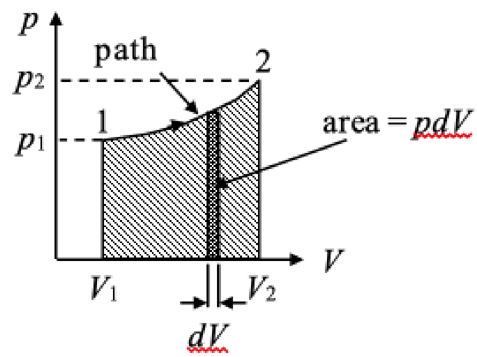
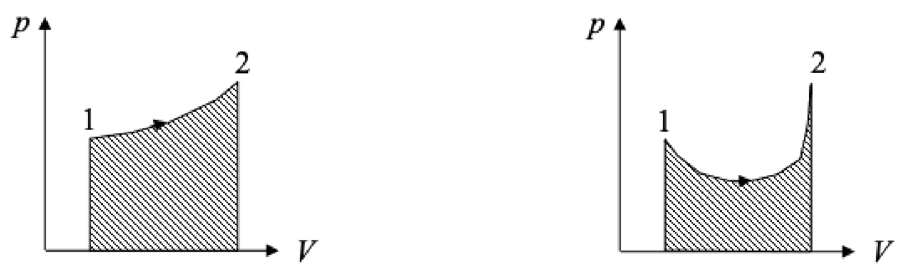


Figure 3.5. Illustration of an example path on a $p - V$ plot. The area under the curve is equal to the work done by the fluid on the surroundings in going from volume 1 to volume 2.



(a) (b)

Figure 3.6. Different paths result in different values for the work. Here, $W_{12,A} > W_{12,B}$.

3.2.2.4. Electric Work

Electrons moving across a system boundary can do work on a system since, in an electric field a force acts on an electron. When N Coulombs of electrons pass through a potential difference, V (the voltage), the electric work done on the system is,

$$W_{onsys} = NV. \tag{3.21}$$

The corresponding power is,

$$\boxed{\dot{W}_{\text{on sys}} = V\dot{N} = VI = I^2R = \frac{V^2}{R}}, \tag{3.22}$$

where I is the current and R is the resistance of the system. Note that Ohm's Law ($V = IR$) has been used in deriving the last two expressions on the right hand side of Eq. (3.22).

3.2.2.5. Spring Work

Now let's examine the work required to compress a spring with stiffness, k (Figure 3.7),

$$W_{\text{on sys},12} = \int_1^2 \mathbf{F}_{\text{on sys}} \cdot d\mathbf{s}, \tag{3.23}$$

$$= \int_{x_1}^{x_2} \left(-kx\hat{\mathbf{e}}_x\right) \cdot \left(dx\hat{\mathbf{e}}_x\right), \tag{3.24}$$

$$\boxed{W_{\text{on sys},12} = \frac{1}{2}k\left(x_1^2 - x_2^2\right)}. \tag{3.25}$$

Note that k is assumed constant in this equation.

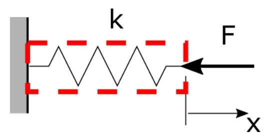


Figure 3.7. An illustration of spring work.

3.2.2.6. Shaft Work

Another method of transferring energy between a system and the surroundings is through shaft work (Figure 3.8). Shaft work is most often associated with rotating fluid machines such as compressors, pumps, turbines, fans, propellers, and windmills. The power acting on a system due to a rotating shaft is given by,

$$\boxed{\dot{W}_{\text{on sys}} = \mathbf{T}_{\text{on sys}} \cdot \boldsymbol{\omega}}, \tag{3.26}$$

where $\mathbf{T}_{\text{on sys}}$ is the torque acting on the system (assumed constant here) and $\boldsymbol{\omega}$ is the angular velocity of the shaft.

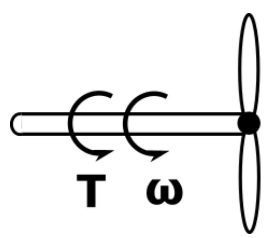
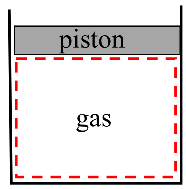


Figure 3.8. An illustration of shaft work.

A gas in a piston assembly undergoes a polytropic expansion from an initial volume, V_i=0.1 m^3, and initial pressure, p_i = 2 bar (abs) (1 bar = 1*10^5 Pa), to a final volume of V_f = 0.5 m^3. Determine the work the gas does on the piston for n = 1.5 and n = 1 (where pV^n = constant).

SOLUTION:



The work the gas performs on the piston is given by:

$$W_{i\to f} = \int_{V=0.1\text{ m}^3}^{V=0.5\text{ m}^3} p dV \tag{1}$$

where, for a polytropic expansion,

$$pV^n = \text{constant} = c \tag{2}$$

where n is a constant. Substitute Eq. (2) into Eq. (1).

$$W_{i\to f} = \int_{V=0.1\text{ m}^3}^{V=0.5\text{ m}^3} cV^{-n} dV = \begin{cases} \frac{c}{1-n} V^{1-n}\Big|_{0.1\text{ m}^3}^{0.5\text{ m}^3} & n \neq 1 \\ c \ln V\Big|_{0.1\text{ m}^3}^{0.5\text{ m}^3} & n = 1 \end{cases} \tag{3}$$

When n = 1.5, the constant is

$$c = \Big(\underbrace{2 * 10^5\text{ Pa}}_{=p_i}\Big)\Big(\underbrace{0.1\text{ m}^3}_{=V_i}\Big)^{1.5} = 6.32 * 10^3\text{ N}\cdot\text{m}^{2.5} \tag{4}$$

and the work performed by the gas, using Eq. (3), is:

$$W_{i\to f} = \frac{6.32*10^3\text{ N}\cdot\text{m}^{2.5}}{-0.5}[(0.5\text{ m}^3)^{-0.5} - (0.1\text{ m}^3)^{-0.5}], \tag{5}$$

$$\boxed{W_{i\to f} = 2.2 * 10^4\text{ N}\cdot\text{m}}. \tag{6}$$

When n = 1, the constant is:

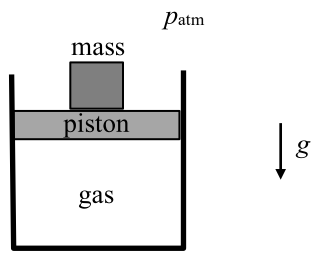
$$c = \Big(\underbrace{2 * 10^5\text{ Pa}}_{=p_i}\Big)\Big(\underbrace{0.1\text{ m}^3}_{=V_i}\Big) = 2 * 10^4\text{ N}\cdot\text{m} \tag{7}$$

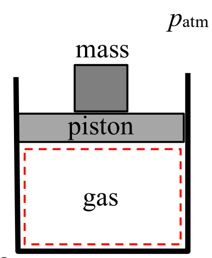
and the work performed by the gas, using Eq. (3), is:

$$W_{i\to f} = (2 * 10^4\text{ N}\cdot\text{m}) \ln\left(\frac{0.5\text{ m}^3}{0.1\text{ m}^3}\right), \tag{8}$$

$$\boxed{W_{i\to f} = 3.2 * 10^4\text{ N}\cdot\text{m}}. \tag{9}$$

Determine the work done by the gas on the piston shown below as it expands quasi-statically from a volume of 0.02 m^3 to 0.04 m^3 given that the piston area is 0.01 m^2 and the mass resting on the piston is 100 kg (neglect the weight of the piston). Assume that atmospheric pressure is 101 kPa (abs).





SOLUTION:
The work done by the gas on the surroundings is,

$$W_{\text{by gas}} = \int_{V_1}^{V_2} p\,dV \,, \tag{1}$$

where,

$p = p_{atm} + mg/A = 101\text{ kPa (abs)} + (100\text{ kg})(9.81\text{ m/s}^2)/(0.01\text{ m}^2) = 1.99*10^5\text{ Pa}$ (2)

The pressure in the gas balances the atmospheric pressure plus the weight of the mass divided by the piston area. Note that this pressure is a constant throughout the process since we're always balancing the same mass and atmospheric pressure.

$V_1 = 0.02\text{ m}^3$
$V_2 = 0.04\text{ m}^3$

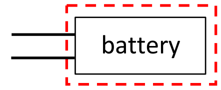
Since the pressure remains constant throughout the process, Eq. (1) may be written as,

$$W_{\text{by gas}} = \int_{V_1}^{V_2} p\,dV = p\int_{V_1}^{V_2} dV = p\left(V_2 - V_1\right). \tag{3}$$

Substituting the numbers given above,

$\boxed{W_{\text{by gas}} = 3.9\text{ kJ}}$.

A 12 V automotive battery is charged with a constant current of 1.5 A for 3 hrs. Determine the work done on the battery.



SOLUTION:
The work done on the battery is,

$$W_{\text{on battery}} = \int_{t=0}^{t=T} \dot{W}\,dt = \int_{t=0}^{t=T} VI\,dt = VIT = \left(12\text{ V}\right)\left(1.5\text{ A}\right)\left(3\text{ hr}\cdot 3600\text{ s}/\text{hr}\right),$$

$$\boxed{\therefore W_{\text{on battery}} = 0.2\text{ kJ}}.$$

3.2.3. Heat

Heat is another form of boundary energy interaction occurring between a system and its surroundings. The difference between heat and work is that heat transfer occurs due to differences in temperature and work occurs through mechanical or electrical means. Heat moves from regions of high temperature to regions of low temperature. Like work, heat is not a property of a system but rather is associated with a process. The amount of heat transferred during a process depends on the path taken during the process. To signify its path dependence, the small amount of heat transferred in a process is signified using the inexact differential, δQ.

Heat can be transferred between the system and surroundings via three methods: conduction, convection, and radiation.

3.2.3.1. Conduction

Conduction is the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a result of interactions between the particles. Conduction can occur in any substance: solids, liquids, and gases. In gases and liquids, conduction occurs due to collisions between molecules during their random motion. In solids, conduction occurs as a result of molecular vibrations and electron transfer. As the more energetic particles collide or contact the less energetic particles, there is a transfer of energy causing an increase in the energy of the less energetic particles and a decrease in energy of the more energetic particles.

The temperature of a region containing many molecules is a measure of the energy due to random translational, rotational, and vibrational motions of the molecules. The larger the temperature, the more random energy the molecules have. Thus, conduction or the transfer of energy due to molecular interactions, will occur from regions of high temperature to regions of low temperature.

The rate of heat transfer, $\dot{\mathbf{Q}}$, (this is a vector quantity since the heat energy travels in a particular direction) due to conduction through an area, A, of a substance is given by Fourier's Law of Heat Conduction,

$$\dot{\mathbf{Q}} = -kA\boldsymbol{\nabla}T, \tag{3.27}$$

where k is a material property of the substance known as the thermal conductivity, and $\boldsymbol{\nabla}T$ is the temperature gradient in the substance. Note that the negative sign in the equation is required so that heat moves from regions of higher temperature to regions of lower temperature.

The thermal conductivity is a measure of how well a material can conduct heat energy. Materials that conduct heat energy well have large k, e.g., $k_{\text{diamond}} = 2300\,\text{W}\,\text{m}^{-1}\,\text{K}$, and those that conduct heat energy poorly have small k, e.g., $k_{\text{air}} = 0.026\,\text{W}\,\text{m}^{-1}\,\text{K}$.

3.2.3.2. Convection

Convection is the mode of energy transfer between and a solid surface and an adjacent fluid that is in motion; it involves the combined effects of conduction and relative fluid motion (also known as advection).

Convection can be further classified as forced convection or free (or natural) convection. In forced convection, the fluid motion is produced via external means, e.g., by a fan or pump. In free convection, the fluid motion is induced by buoyant forces arising from density difference in the fluid caused by temperature variations.

The rate of heat transfer, $\dot{Q}$, leaving a surface with area, A_s, and entering the fluid due to convection is given by Newton's Law of Cooling,

$$\dot{Q} = hA_s\left(T_s - T_f\right), \tag{3.28}$$

where h is the heat transfer coefficient for the system and T_s is the temperature of the surface that is in contact with the fluid with temperature, T_f. The heat transfer coefficient depends on the surface and fluid properties as well as the flow characteristics. It is generally an experimentally determined property for all but the simplest flow situations. Typical ranges for the free convection heat transfer coefficient are

$2 - 25\,\mathrm{W\,m^{-1}\,K^{-1}}$ and $50 - 1000\,\mathrm{W\,m^{-1}\,K^{-1}}$ for gases and liquids, respectively. For forced convection, the range is $25 - 250\,\mathrm{W\,m^{-1}\,K^{-1}}$ and $50 - 20000\,\mathrm{W\,m^{-1}\,K^{-1}}$ for gases and liquids, respectively.

3.2.3.3. Radiation

Radiation is the energy emitted by matter in the form of electromagnetic waves as a result of changes in the electronic configurations of the atoms or molecules. Unlike conduction and convection, radiation does not require an intervening medium for transferring heat.

The rate at which heat is emitted from a surface with area, A_s, depends on the *absolute* temperature of the surface, T_s, as indicated by the modified Stefan-Boltzmann Law,

$$\dot{Q}_{\text{emitted}} = \epsilon \sigma A_s T_s^4, \tag{3.29}$$

where $\dot{Q}_{\text{emitted}}$ is the rate at which heat is emitted from the surface, ϵ is the emissivity of the surface ($0 \leq \epsilon \leq 1$), and σ is the Stefan-Boltzmann constant ($\sigma = 5.67 \times 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}) = 0.1714 \times 10^{-8}\,\mathrm{Btuh^{-1}ft^{-2\circ}R^{-4}}$)).

A blackbody is an object with an emissivity of one, $\sigma_{\text{blackbody}} = 1$, i.e., a blackbody is a perfect emitter of radiation.

Surfaces can also absorb radiation. The heat flux absorbed by a surface via radiation is given by,

$$\dot{Q}_{\text{absorbed}} = \alpha \dot{Q}_{\text{incident}}, \tag{3.30}$$

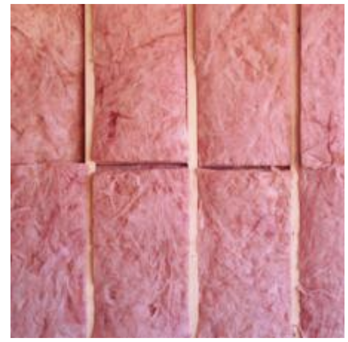
where α is the absorptivity of the surface ($0 \leq \alpha \leq 1$). Note that a blackbody is defined as having $\alpha = 1$ making it is both a perfect emitter and perfect absorber of radiation.

Actual determination of the rate at which radiation is emitted and absorbed by a surface can be complicated since the rate depends on factors such as surface orientation, the effects of the intervening medium, and the surface spectral characteristics.

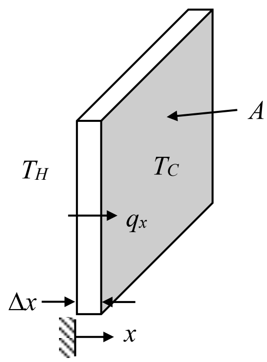
For the special case in which a small surface interacts with a much larger surface, the intervening fluid has no affect on the radiation transfer, and $\alpha = \sigma$ (termed a grey body), the rate of heat transfer from the surface to the surroundings via radiation is,

$$\dot{Q}_{\text{emitted}} = \epsilon \sigma A_s \left(T_s^4 - T_{\text{surr}}^4\right). \tag{3.31}$$

An insulated frame wall of a house has an average thermal conductivity of 0.0318 Btu/(hr·ft·°R). The thickness of the wall is 6 in. At steady state, the rate of energy transfer by conduction through an area of 160 ft^2 is 400 Btu/hr, and the temperature decreases linearly from the inner surface to the outer surface. If the outside surface temperature of the wall is 30 °F, what is the inner surface temperature in °F?



SOLUTION:



k = 0.0318 Btu/(hr·ft·°R)
Δx = 6 in = 0.5 ft
A = 160 ft^2
$\dot{Q}$ = 400 Btu/hr
T_C = 30 °F = 490 °R

From Fourier's Law, the heat transfer through the wall is:

$$\dot{Q}_x = -kA\frac{dT}{dx} \approx -kA\frac{T_C - T_H}{\Delta x} \tag{1}$$

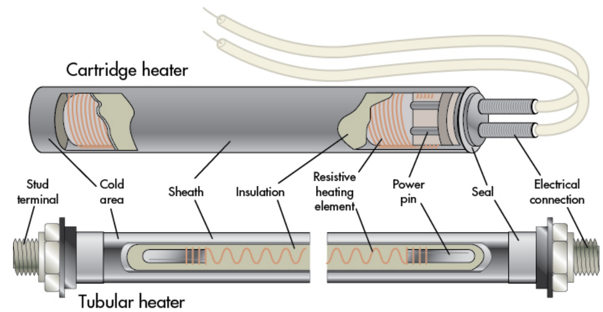
Re-arrange to solve for T_H.

$$\boxed{T_H = T_C + \frac{\dot{Q}}{kA}\Delta x} \tag{2}$$

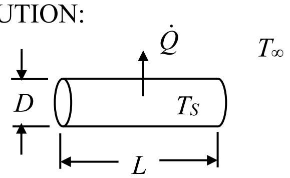
Using the given parameters:

k = 0.0318 Btu/(hr·ft·°R)
Δx = 6 in = 0.5 ft
A = 160 ft^2
$\dot{Q}$ = 400 Btu/hr
T_C = 30 °F = 490 °R
$\Rightarrow$ $\boxed{T_H = 529\ °\text{R} = 69\ °\text{F}}$

A cartridge electrical heater is shaped as a cylinder of length 200 mm and outer diameter of 20 mm. Under normal operating conditions the heater dissipates 2 kW while submerged in a water flow which is at 20 °C and provides a convection heat transfer coefficient of 5000 W/(m^2·K). Neglecting heat transfer from the ends of the heater, determine the heater's surface temperature. If the water flow is inadvertently terminated while the heater continues to operate, the heater surface is exposed to air which is also at 20 °C but for which the heat transfer coefficient is 50 W/(m^2·K). What is the corresponding surface temperature? What are the consequences of such an event?



SOLUTION:



Determine the surface temperature using Newton's Law of Cooling.

$$\dot{Q} = hA\left(T_S - T_\infty\right) \tag{1}$$

where $A = \pi DL$. Re-arranging gives:

$$\boxed{T_S = T_\infty + \frac{\dot{Q}}{hA}} \tag{2}$$

Using the given parameters:

T_∞ = 20 °C = 293 K

$\dot{Q}$ = 2000 W

h = 5000 W/(m^2·K)

D = 20*10^{-3} m

L = 200*10^{-3} m

⇒ A = 1.3*10^{-2} m^2

$\boxed{T_S = 325\text{ K} = 52\text{ °C}}$

If instead, h = 50 W/(m^2·K), then:

$\boxed{T_S = 3500\text{ K} = 3200\text{ °C}}$

This temperature is probably large enough to melt the cartridge heater!

An uninsulated steam pipe passes through a room in which the air and walls are at 25 °C. The outside diameter of the pipe is 70 mm, and its surface temperature and emissivity are 200 °C and 0.8, respectively. If the coefficient associated with free convection heat transfer from the surface to the air is 15 W/(m^2·K), what is the rate of heat loss from the surface per unit length of pipe?

SOLUTION:

The convective heat transfer rate, $\dot{Q}_C$, is given by Newton's Law of Cooling:

$$\dot{Q}_C = hA_S\left(T_S - T_\infty\right) \tag{1}$$

where h is the convection heat transfer coefficient, A_S is the surface area of the pipe, T_S is the surface temperature of the pipe and T_∞ is the ambient temperature.

The radiative heat transfer rate, $\dot{Q}_R$, is given by:

$$\dot{Q}_R = \varepsilon\sigma A_S\left(T_S^4 - T_\infty^4\right) \tag{2}$$

where ε is the surface emissivity and σ is the Stefan-Boltzmann constant.

The total heat transfer rate from the pipe is:

$$\dot{Q}_T = \dot{Q}_C + \dot{Q}_R \tag{3}$$

Using the given parameters:

h = 15 W/(m^2·K)
D = 70*10^{-3} m $\Rightarrow$ $A_S = \pi DL \Rightarrow A_S/L = 0.22$ m
T_S = 200 °C = 473 K
T_∞ = 15 °C = 288 K
ε = 0.8
σ = 5.67*10^{-8} W/(m^2·K^4)

$\Rightarrow$ $\dot{Q}_C / L$ = 580 W/m
$\dot{Q}_R / L$ = 420 W/m
$\boxed{\dot{Q}_T / L = 1 \text{ kW/m}}$

3.3. The First Law of Thermodynamics

In words and in mathematical form, the First Law of Thermodynamics is: The increase in total energy of a system is equal to the energy added to the system via heat transfer plus the energy added to the system via work done on the system,

$$dE_{\text{sys}} = \delta Q_{\text{into sys}} + \delta W_{\text{on sys}}, \tag{3.32}$$

where dE_{sys} is a small increase in the total energy of the system, $\delta Q_{\text{into sys}}$ is a small amount of energy transferred into the system via heat transfer, and $\delta W_{\text{on sys}}$ is a small amount of energy added to the system via work done on the system by the surroundings (Figure 3.9). Note that work and heat are just methods of transferring energy, hence, the First Law of Thermodynamics can also be thought of as Conservation of Energy.

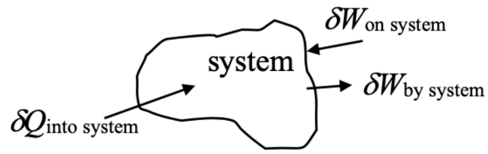


Figure 3.9. A schematic showing a system and the directions of energy transfer.

Notes:

(1) Since energy is a property of a system, an exact differential (the "d" operator in dE) is used to specify the small change in the energy. In other words, the difference in energy between two states depends only upon the endpoint states and is independent of the path between the two states. The small change in heat and work are indicated using an inexact differential (the "δ" operator in δQ and δW) to signify that both heat and work are path dependent processes.

(2) Note that different disciplines have different notations for the First Law. In particular, in thermodynamics, work is usually discussed in terms of the work done *by* the system on the surroundings so that the First Law becomes,

$$dE_{\text{sys}} = \delta Q_{\text{into sys}} - \delta W_{\text{by sys}}. \tag{3.33}$$

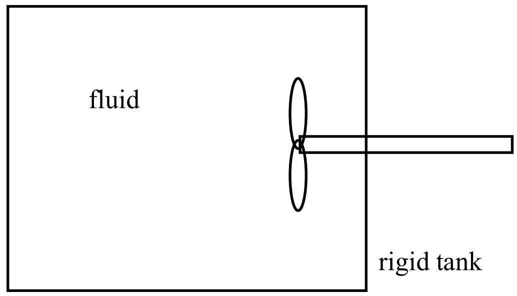
In order to avoid confusion regarding the proper sign for work, these notes will try to clearly specify whether work is being done on or by the system. Understanding that if one does work on a system, the system's energy will increase is generally sufficient to avoid most sign convention problems.

(3) We can also write the First Law in terms of time rates of changes by taking the limit of the changes in the properties over a short amount of time as the time duration approaches zero,

$$\frac{dE_{\text{sys}}}{dt} = \delta \dot{Q}_{\text{into sys}} + \delta \dot{W}_{\text{on sys}} = \delta \dot{Q}_{\text{into sys}} - \delta \dot{W}_{\text{by sys}}. \tag{3.34}$$

Now let's consider a few simple examples.

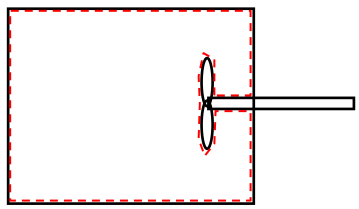
A rigid tank contains a hot fluid that is cooled while being stirred. Initially the internal energy of the fluid is 800 kJ. During the cooling process, the fluid loses 500 kJ of heat and the stirring propeller does 100 kJ of work on the fluid. What is the final internal energy of the fluid?



SOLUTION:

Apply the 1st Law to the system of fluid contained within the tank.

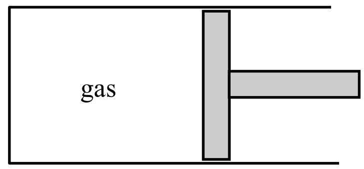
$$\Delta E = E_f - E_i = Q_{\text{into system}} + W_{\text{on system}}$$



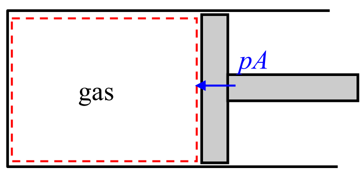
where

$E_i = U_i = 800$ kJ
$Q_{\text{into system}} = -500$ kJ
$W_{\text{on system}} = 100$ kJ
$\Rightarrow \boxed{E_f = U_f = 400 \text{ kJ}}$

Four kilograms of a certain gas is contained within a piston-cylinder assembly. The gas undergoes a polytropic process where: $pV^{1.5}$=constant. The initial pressure is 3 bars (abs), the initial volume is 0.1 m^3, and the final volume is 0.2 m^3. The change in the specific internal energy of the gas in the process is Δu = -4.5 kJ/kg. There are no significant changes in the kinetic or potential energies of the gas. What is the net heat transfer for the process?



SOLUTION:
Apply the First Law to the system of gas as shown in the figure below,



$$\Delta E_{\text{sys}} = Q_{\substack{\text{added}\\\text{to sys}}} + W_{\text{on sys}}, \tag{1}$$

where,

$$W_{on\ sys} = \int_{V=V_1}^{V=V_2} -pdV = \int_{V=V_1}^{V=V_2} -(cV^{-1.5})dV = 2c(V^{-0.5})_{V_1}^{V_2} = 2\underbrace{p_1 V_1^{1.5}}_{=c}(V_2^{-0.5} - V_1^{-0.5}), \tag{2}$$

and,

$$\Delta E_{\text{sys}} = m_{\text{sys}}\Delta e_{\text{sys}} = m_{\text{sys}}\Delta u_{\text{sys}}. \quad \text{(The kinetic and potential energy changes are negligible.)} \tag{2}$$

Re-arranging Eq. (1) and substituting Eqs. (2) and (3) gives,

$$\boxed{Q_{\substack{added\\to\ sys}} = m_{sys}\Delta u_{sys} - 2p_1 V_1^{1.5}(V_2^{-0.5} - V_1^{-0.5})}. \tag{3}$$

Using the given values:

m_{sys}	=	4 kg
Δu_{sys}	=	-4500 J/kg
p_1	=	3*10^5 Pa
V_1	=	0.1 m^3
V_2	=	0.2 m^3

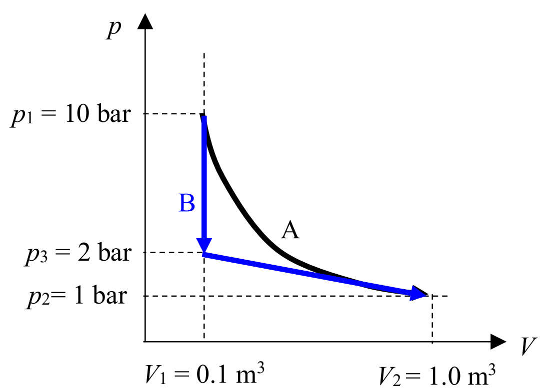
$\Rightarrow$ $\boxed{Q_{\text{added}} = -0.426\text{ kJ}}$ (heat is leaving the system)

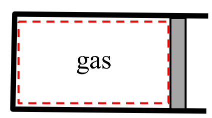
A gas contained within a piston-cylinder assembly undergoes two processes, A and B, between the same end states, 1 and 2, where at state 1 the pressure is 10 bar, the volume is 0.1 m^3, the internal energy is 400 kJ, and at state 2 the pressure is 1 bar, the volume is 1.0 m^3, and the internal energy is 200 kJ.

Process A: Process from 1 to 2 during which the pressure-volume relation is pV = constant.
Process B: Constant volume process from state 1 to a pressure of 2 bar, followed by a linear pressure-volume process to state 2.

Kinetic and potential energy effects can be ignored. For each of the processes A and B,
a. Sketch the process on a *p*-*V* diagram,
b. evaluate the work by the gas on the piston, in kJ, and
c. evaluate the heat transfer from the gas in kJ.

SOLUTION:
The processes are sketched on the plot shown below.





The work may be found by integrating the pdV work given the two processes described,

$$W_{\substack{\text{by gas}\\\text{on piston}}} = \int_1^2 p\,dV\,, \tag{1}$$

where for process A,

$$W_{\substack{\text{by gas}\\\text{on piston,}\\\text{path A}}} = c\int_{V=V_1}^{V=V_2} \frac{dV}{V} = c\ln\left(\frac{V_2}{V_1}\right), \tag{2}$$

noting that $pV = c \Rightarrow p = c/V$. The constant c may be found from the initial (or final) conditions,

$$p_1V_1 = c = p_2V_2\,, \tag{3}$$

$$W_{\substack{\text{by gas}\\\text{on piston,}\\\text{path A}}} = p_1V_1\ln\left(\frac{V_2}{V_1}\right). \tag{4}$$

Substituting the given numbers,

p_1 = 10 bar = 10*10^2 kPa
V_1 = 0.1 m^3
V_2 = 1.0 m^3

$$\Rightarrow \boxed{W_{\text{by gas on piston, path A}} = 230\text{ kJ}} \tag{5}$$

The heat transferred from the gas may be found using the 1st Law of Thermodynamics,

$$\underbrace{\Delta E_{\text{gas}}}_{=\Delta U_{\text{gas}}} = Q_{\substack{\text{into}\\ \text{gas}}} - W_{\substack{\text{by}\\ \text{gas}}} \Rightarrow Q_{\substack{\text{into}\\ \text{gas}}} = \Delta U_{\text{gas}} + W_{\substack{\text{by}\\ \text{gas}}}, \tag{6}$$

where the total change of energy in the gas is due only to changes in internal energy (U). Using the given parameters,

U_1 = 400 kJ
U_2 = 200 kJ
$W_{\text{by gas}}$ = 230 kJ (from Eq. (5))
$\Rightarrow \Delta U$ = -200 kJ $\Rightarrow$ $\boxed{Q_{\text{into gas}} = 30 \text{ kJ}}$ (7)

Thus, 30 kJ of heat is transferred into the gas (-30 kJ of heat is transferred from the gas).

For process B, there is no work done in the constant volume part of the process since the volume doesn't change. The work in the linear pressure-volume part of the process is,

$$W_{\substack{\text{by gas}\\ \text{on piston,}\\ \text{path B}}} = \int_{V=V_1}^{V=V_2} p\,dV = \int_{V=V_1}^{V=V_2} \left[\left(\frac{p_2 - p_3}{V_2 - V_1} \right) \left(V - V_1 \right) + p_3 \right] dV = \left[\left(\frac{p_2 - p_3}{V_2 - V_1} \right) \left(\tfrac{1}{2} V^2 - V_1 V \right) + p_3 V \right]_{V_1}^{V_2}, \tag{8}$$

$$W_{\substack{\text{by gas}\\ \text{on piston,}\\ \text{path B}}} = \left(\frac{p_2 - p_3}{V_2 - V_1} \right) \left(\tfrac{1}{2} V_2^2 - \tfrac{1}{2} V_1^2 - V_1 V_2 + V_1^2 \right) + p_3 \left(V_2 - V_1 \right), \tag{9}$$

$$W_{\substack{\text{by gas}\\ \text{on piston,}\\ \text{path B}}} = \tfrac{1}{2} \left(\frac{p_2 - p_3}{V_2 - V_1} \right) \left(V_2^2 - 2 V_1 V_2 + V_1^2 \right) + p_3 \left(V_2 - V_1 \right), \tag{10}$$

where the pressure varies linearly with the volume,

$$p = \left(\frac{p_2 - p_3}{V_2 - V_1} \right) \left(V - V_1 \right) + p_3, \text{ (equation of a line)} \tag{11}$$

Substituting the given data,

p_2 = 1 bar = 1*10^2 kPa
p_3 = 2 bar = 2*10^2 kPa
V_1 = 0.1 m^3
V_2 = 1.0 m^3
$\Rightarrow \boxed{W_{\substack{\text{by gas}\\ \text{on piston,}\\ \text{path B}}} = 135 \text{ kJ}}$ (12)

The heat transferred into the gas may be found using Eq. (6) with the following parameters,

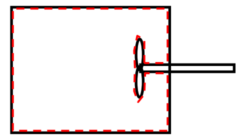
U_1 = 400 kJ (Note that the internal energies are independent of the path. They're a property!)
U_2 = 200 kJ
$W_{\text{by gas}}$ = 135 kJ (from Eq. (12))
$\Rightarrow$ $\boxed{Q_{\text{into gas}} = -65 \text{ kJ}}$ (13)

Thus, 65 kJ of heat is transferred from the gas to the surroundings.

A gas is contained in a closed rigid tank fitted with a paddle wheel. The paddle wheel stirs the gas for 20 min, with the power varying with time t according to (10 W/min)t. Heat transfer from the gas to the surroundings takes place at a constant rate of 50 W. Determine:

a. the rate of change of energy of the gas at time 10 min, in watts, and
b. the net change in energy of the gas after 20 min, in kJ.

SOLUTION:
Apply the 1st Law to the system of gas within the tank,



$$\frac{dE_{\text{sys}}}{dt} = \dot{Q}_{\substack{\text{added}\\\text{to sys}}} + \dot{W}_{\text{on sys}}, \tag{1}$$

where $\dot{Q}_{\text{into sys}} = -50\text{ W}$ and $\dot{W}_{\text{on sys}} = \left(10\,{}^{\text{W}}\!/_{\text{min}}\right)t$. Thus, at t = 10 min,

$$\frac{dE_{\text{sys}}}{dt} = -50\text{ W} + \left(10\,{}^{\text{W}}\!/_{\text{min}}\right)t, \tag{2}$$

$$\boxed{\therefore \frac{dE_{\text{sys}}}{dt} = 50\text{ W}}.$$

The net change in energy of the gas is found by integrating Eqn. (2) in time,

$$\Delta E_{\text{sys}} = \int_{t=0}^{t=20\text{ min}} \frac{dE_{\text{sys}}}{dt}dt = \int_{t=0}^{t=20\text{ min}} \left[50\text{ W} + \left(10\,{}^{\text{W}}\!/_{\text{min}}\right)t\right]dt = \left[\left(50\text{ W}\right)t + \tfrac{1}{2}\left(10\,{}^{\text{W}}\!/_{\text{min}}\right)t^2\right]_{t=0}^{t=20\text{ min}} \tag{3}$$

$$\boxed{\therefore \Delta E_{\text{sys}} = 60\text{ kJ}}$$

3.4. Thermodynamics Cycles

A cycle is a sequence of processes that begins and ends at the same state. At the conclusion of a cycle, all properties have the same values they had at the beginning of the cycle. Thus, there is no change in the system's state at the end of a cycle. Mathematically,

$$\underbrace{\Delta E_{\text{sys,cycle}}}_{=0} = Q_{\substack{\text{into sys,}\\ \text{cycle}}} + W_{\substack{\text{on sys,}\\ \text{cycle}}} = Q_{\substack{\text{into sys,}\\ \text{cycle}}} - W_{\substack{\text{by sys,}\\ \text{cycle}}}, \tag{3.35}$$

or,

$$Q_{\substack{\text{into sys,}\\ \text{cycle}}} = -W_{\substack{\text{on sys,}\\ \text{cycle}}} \quad \text{or} \quad \boxed{Q_{\substack{\text{into sys,}\\ \text{cycle}}} = W_{\substack{\text{by sys,}\\ \text{cycle}}}}, \tag{3.36}$$

where the total energy change over the cycle is zero since over a cycle we start and end at the same state. Note that $Q_{\text{into sys,cycle}}$ is the net amount of heat added to the system over the cycle and $W_{\text{by sys, cycle}}$ is the net amount of work done by the system over the cycle.

Cycles are common in many engineering applications. For example, power generation, heat pumps, and refrigeration all involve thermodynamic cycles. Let's consider two general classes of cycles: a power cycle and a refrigeration (or heat pump) cycle.

3.4.1. Power Cycle

In a power cycle, illustrated in Figure 3.10, heat moves from a hot reservoir (part of the surroundings) into the system, which then makes use of the heat to do work on the surroundings, and then ejects the remaining heat to a cold reservoir (another part of the surroundings).

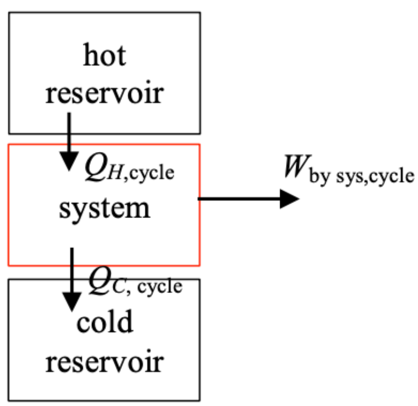


Figure 3.10. Illustration of a power cycle.

Utilizing Eq. (3.36) gives,

$$W_{\substack{\text{by sys,}\\ \text{cycle}}} = Q_{H,\text{cycle}} - Q_{C,\text{cycle}}, \tag{3.37}$$

where Q_H is the heat transfer interaction with the hot reservoir and Q_C is the heat transfer interaction with the cold reservoir.

Notes:

(1) The heat added to the system (Q_H) must be greater than the heat removed from the system (Q_C) in order for the system to do work on the surroundings.
(2) The heat into the system is generally produced by the combustion of fuel, solar radiation, or a nuclear reaction. The heat out of the system is generally discharged into the surrounding air or a body of water.

(3) The efficiency of the power cycle, η, is defined as the ratio of the amount of work produced in the cycle to the amount heat added to the system,

$$\eta \coloneqq \frac{W_{\text{by,cycle}}}{Q_{H,\text{cycle}}} = \frac{Q_{H,\text{cycle}} - Q_{C,\text{cycle}}}{Q_{H,\text{cycle}}} = 1 - \frac{Q_{H,\text{cycle}}}{Q_{C,\text{cycle}}}, \tag{3.38}$$

where Eq. (3.37) has been used. The efficiency can never be more than one since the heat out of the system will never be more than the heat into the system over a cycle. We'll discuss the limits on power cycle efficiency in greater detail when discussing the Second Law of Thermodynamics.

3.4.2. Refrigeration and Heat Pump Cycles

In refrigeration and heat pump cycles, illustrated in Figure 3.11, heat moves from a cold reservoir (part of the surroundings) into the system, work is done on the system to then eject heat from the system to a hot reservoir (another part of the surroundings). Utilizing Eq. (3.36),

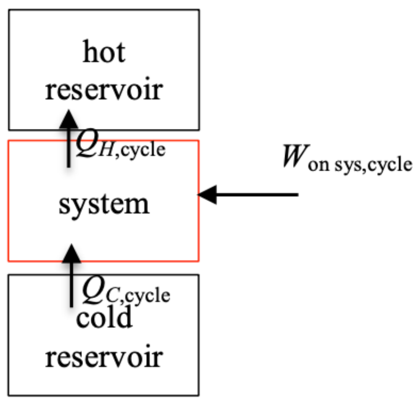


FIGURE 3.11. Illustration of refrigeration and heat pump cycles.

$$W_{\substack{\text{on sys,}\\ \text{cycle}}} = Q_{H,\text{cycle}} - Q_{C,\text{cycle}}. \tag{3.39}$$

Notes:

(1) In refrigeration and heat pump cycles, work is done on the system to make the heat out of the system larger than the heat into the system. Hence, $Q_{H,\text{cycle}} > Q_{C,\text{cycle}}$.

(2) The objective of a refrigeration cycle is to remove heat from the cold reservoir, e.g., a house or refrigerator, to a hot reservoir, e.g., the surrounding environment. A heat pump does the same thing (moves heat from a cold reservoir to a hot reservoir), but the objective is to add heat to the hot reservoir, e.g., remove heat from the environment to raise the temperature in a house. In many of these applications, the power supplied to the system is electrical power.

(3) The system in a typical refrigerator or heat pump is the fluid used within the device. It is what moves the heat (energy, actually) between the cold and hot reservoirs.

(4) Since the goal of a refrigerator is to efficiently remove heat from the cold reservoir, we can define the coefficient of performance (COP) of the refrigeration cycle, to be the ratio of the amount heat added to the system from the cold reservoir to the work done on the system over the cycle,

$$COP_{\text{ref}} \coloneqq \frac{Q_{C,\text{cycle}}}{W_{\text{on sys,cycle}}} = \frac{Q_{C,\text{cycle}}}{Q_{H,\text{cycle}} - Q_{C,\text{cycle}}} = \frac{1}{Q_{H,\text{cycle}}/Q_{C,\text{cycle}} - 1}, \tag{3.40}$$

where Eq. (3.39) has been used. Note that the heat out of the system is greater than the heat into the system (since work is done on the system). The refrigeration coefficient of performance can vary from zero to very large values. The larger the COP_{ref}, the larger the transfer of heat from the cold reservoir for a given amount of work over the cycle. The COP_{ref} for a high efficiency consumer refrigerator is between 1.6 – 3.1. Limits on the value for COP_{ref} will be discussed further after examining the Second Law of Thermodynamics.

(5) The goal of a heat pump is to efficiently move heat into a hot reservoir; hence, we can define the coefficient of performance (COP) of the heat pump cycle, as the ratio of the amount heat added to the hot reservoir to the work done on the system over the cycle,

$$COP_{\text{HP}} \coloneqq \frac{Q_{H,\text{cycle}}}{W_{\text{on sys,cycle}}} = \frac{Q_{H,\text{cycle}}}{Q_{H,\text{cycle}} - Q_{C,\text{cycle}}} = \frac{1}{1 - Q_{C,\text{cycle}}/Q_{H,\text{cycle}}}, \tag{3.41}$$

where Eq. (3.39) has been used. The heat pump coefficient of performance can never be less than one since the heat out of the system always be larger than the heat into the system (work is done on the system). The larger the COP_{HP}, the larger the transfer of heat for a given amount of work over the cycle. A typical COP_{HP} for a commercial heat pump is between 3 – 4. Limits on the value for COP_{HP} will be discussed further after examining the Second Law of Thermodynamics.

(6) In the U.S., refrigeration and heat pump COPs are often expressed as Energy Efficiency Ratios, EERs, which are simply COPs, but with an unfortunate mix of English and SI units,

$$EER \coloneqq \frac{\dot{Q} \quad \text{in Btu/h}}{\dot{W}_{\text{on sys,cycle}} \quad \text{in W}}. \tag{3.42}$$

Using appropriate unit conversions,

$$COP = \left(0.292\,\frac{\text{W}}{\text{Btu/h}}\right) EEP. \tag{3.43}$$

(7) Note that Eqs. (3.36) - (3.41) may all be written in rates of change too,

$$\dot{Q}_{\substack{\text{into sys,}\\ \text{cycle}}} = -\dot{W}_{\substack{\text{on sys,}\\ \text{cycle}}} \quad \text{or} \quad \dot{Q}_{\substack{\text{into sys,}\\ \text{cycle}}} = \dot{W}_{\substack{\text{by sys,}\\ \text{cycle}}} \tag{3.44}$$

$$\dot{W}_{\substack{\text{by sys,}\\ \text{cycle}}} = \dot{Q}_{H,\text{cycle}} - \dot{Q}_{C,\text{cycle}}, \tag{3.45}$$

$$\eta \coloneqq \frac{\dot{W}_{\text{by,cycle}}}{\dot{Q}_{H,\text{cycle}}}, \tag{3.46}$$

$$\dot{W}_{\substack{\text{on sys,}\\ \text{cycle}}} = \dot{Q}_{H,\text{cycle}} - \dot{Q}_{C,\text{cycle}}, \tag{3.47}$$

$$COP_{\text{ref}} \coloneqq \frac{\dot{Q}_{C,\text{cycle}}}{\dot{W}_{\text{on sys,cycle}}}, \tag{3.48}$$

$$COP_{\text{HP}} \coloneqq \frac{\dot{Q}_{H,\text{cycle}}}{\dot{W}_{\text{on sys,cycle}}}. \tag{3.49}$$

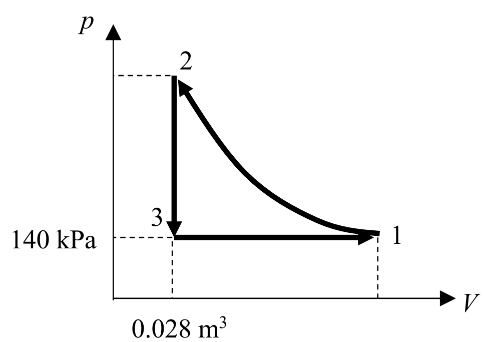
A gas within a piston-cylinder assembly undergoes a thermodynamic cycle consisting of three processes in series:

Process 1 – 2: Compression with constant internal energy (pV = constant)
Process 2 – 3: Constant volume cooling to a pressure of 140 kPa (abs) and a volume of 0.028 m^3
Process 3 – 1: Constant pressure expansion with a total work of 10.5 kJ acting on the piston

For the cycle, the net amount of work done by the gas on the piston is -8.3 kJ. There are no changes in kinetic or potential energy.

a. Sketch the processes on a p-V diagram.
b. Determine the volume at state 1, in m^3.
c. Determine the work and heat transfer for process 1 – 2, each in kJ.
d. Is this a power cycle or a refrigeration/heat pump cycle? Explain.

SOLUTION:



The volume at state 1 may be found by knowing that the work in going from state 3 to state 1 is 10.5 kJ,

$$W_{\substack{\text{by gas}\\\text{on piston,}\\3\to1}} = \int_3^1 p\,dV = p\int_{V=V_3}^{V=V_1} dV = p\left(V_1 - V_3\right), \quad \text{(since the pressure is constant from 3 to 1)} \tag{1}$$

$$V_1 = V_3 + \frac{W_{\substack{\text{by gas}\\\text{on piston,}\\3\to1}}}{p}. \tag{2}$$

Using the given parameters,

V_3 = 0.028 m^3
$W_{\substack{\text{by gas}\\\text{on piston,}\\3\to1}}$ = 10.5 kJ
p = 140 kPa (abs)

$$\Rightarrow \boxed{V_1 = 0.103\ \text{m}^3} \tag{3}$$

The work in going from state 1 to state 2 can be found by knowing that the total work done by the gas on the piston over the whole cycle is -8.3 kJ, because the volume remains constant in going from state 2 to state 3, the corresponding work is zero, and the work on the piston in going from state 3 to state 1 is 10.5 kJ,

$$\underbrace{W_{\substack{\text{by gas}\\\text{on piston,}\\\text{cycle}}}}_{=-8.5\ \text{kJ}} = W_{\substack{\text{by gas}\\\text{on piston,}\\1\to2}} + \underbrace{W_{\substack{\text{by gas}\\\text{on piston,}\\2\to3}}}_{=0} + \underbrace{W_{\substack{\text{by gas}\\\text{on piston,}\\3\to1}}}_{=10.5\ \text{kJ}}, \tag{4}$$

$$\boxed{W_{\substack{\text{by gas}\\\text{on piston,}\\1\to2}} = -18.8\ \text{kJ}}. \tag{5}$$

The heat transferred in the process from state 1 to state 2 can be found using the 1st Law of Thermodynamics and noting that the energy remains unchanged in going from 1 to 2,

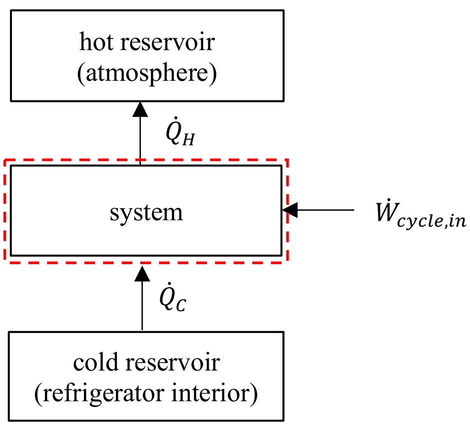
$$\underbrace{\Delta E_{\substack{\text{gas,}\\ 1\to 2}}}_{=0} = Q_{\substack{\text{into gas,}\\ 1\to 2}} - \underbrace{W_{\substack{\text{by gas,}\\ 1\to 2}}}_{=-18.8\text{ kJ}} \quad \Rightarrow \quad \boxed{Q_{\substack{\text{into gas,}\\ 1\to 2}} = -18.8\text{ kJ}} \tag{6}$$

Since $W_{\text{by gas,cycle}}$ = -8.3 kJ < 0, <u>this is a refrigeration (or heat pump) cycle</u>.

A refrigerator steadily receives a power input of 0.15 kW while rejecting energy by heat transfer to the surroundings at a rate of 0.6 kW.

a. Determine the rate at which energy is removed by heat transfer from the refrigerated space.
b. Determine the refrigerator's coefficient of performance.

SOLUTION:



Apply the 1st Law to the system to determine the rate at which heat is transferred from the refrigerator interior into the system,

$$\dot{W}_{cycle,in} = \dot{Q}_H - \dot{Q}_C, \tag{1}$$

$$\dot{Q}_C = \dot{Q}_H - \dot{W}_{cycle,in}. \tag{2}$$

Using the given data,

$\dot{Q}_H = 0.6$ kW,
$\dot{W}_{cycle,in} = 0.15$ kW,
⇨ $\underline{\dot{Q}_C = 0.45 \text{ kW}}$.

The coefficient of performance for a refrigeration cycle is,

$$COP_{ref} = \frac{\dot{Q}_C}{\dot{W}_{cycle,in}}. \tag{3}$$

Using the given data,

$\boxed{COP_{ref} = 3.0}$.

3.5. Properties

In order to analyze real-world systems, we must be able to relate describe the properties and the relation between the properties of systems. In this section we examine how properties are presented and how they're related for a few important classes of materials. First, however, we must define a few terms:

- The phase of a substance has homogeneous chemical composition and physical structure. The three phases of matter are solid, liquid, and vapor.
- A pure substance is one that is uniform and invariable in chemical composition. A pure substance can exist in more than one phase, but all of the phases must have identical chemical compositions.
- A simple, compressible system is one in which electrical, magnetic, surface tension, gravitational, and motion effects are negligible. Systems consisting of pure water or uniform mixtures of non-reacting gases are examples of simple, compressible systems. The state principle states that any two independent intensive thermodynamic properties will uniquely define the system's state. If additional effects are significant, e.g., gravitational forces and accelerations, then additional properties are required, e.g., elevation and velocity.

p-v-T Diagrams
For simple, compressible systems, we can show the relationship between pressure, specific volume, and temperature in (3D) p-v-T diagrams (Figures 3.12 and 3.13). It's most convenient, however, to show (2D) projections of these diagrams onto the p-T, p-v, and T-v planes. Please refer to the following two figures in the following discussion of the important features of these diagrams.

- Three single-phase regions can be identified on the plots: solid, liquid, and vapor.
  - The state in these regions is fixed by two independent properties, e.g., (p, v), (p, T), or (v, T).
  - Above a critical pressure, temperature, and specific volume, known as the critical point, the difference between liquid and vapor is no longer discernible. The properties at the critical point are referred to as critical properties. Values for the critical pressure and temperature may often be found in the back of textbooks (e.g., Table A-1 in Moran et al., 8th ed.) or online.
- Three two-phase regions in which two phases exist in equilibrium can also be identified in the plots. These regions correspond to solid-vapor, solid-liquid, and liquid-vapor.
  - These regions correspond to situations involving melting or freezing (transition from solid to liquid or liquid to solid, respectively); vaporization (boiling – turning to vapor by increasing the temperature while holding pressure constant, or cavitation – turning to vapor by decreasing the pressure while holding the temperature constant) or condensation (transition from vapor to a liquid); and sublimation (transition from a solid to a vapor).
  - In these two-phase regions, the pressure and temperature are not independent. Hence, to define a state we need the specific volume, v, and either the pressure or the temperature, (p or T).
- A single three-phase line, along which solid, liquid, and vapor exist in equilibrium. This line is referred to as the triple line.
- The state at which a phase change begins is known as a saturation state.
  - The two-phase (liquid-vapor) dome-shaped region is known as the vapor dome.
  - The lines bordering the vapor dome are known as the saturated liquid and saturated vapor lines.
  - The point at the top of the dome is known as the critical point, which is at the critical temperature, T_c, and critical pressure, p_c, and critical specific volume, v_c. The critical temperature is the maximum temperature at which liquid and vapor phases can co-exist in equilibrium.
- The projection of the p-v-T diagram onto the p-T plane is known as a phase diagram.
  - The two-phase regions are projected as lines in this view.
  - The saturation temperature is the temperature at which a phase change takes place at a given pressure. The corresponding pressure is known as the saturation pressure. For each saturation pressure there is a unique corresponding saturation temperature, hence, T and p are not independent during a change of phase.
  - The triple line projects to the triple point in a phase diagram.

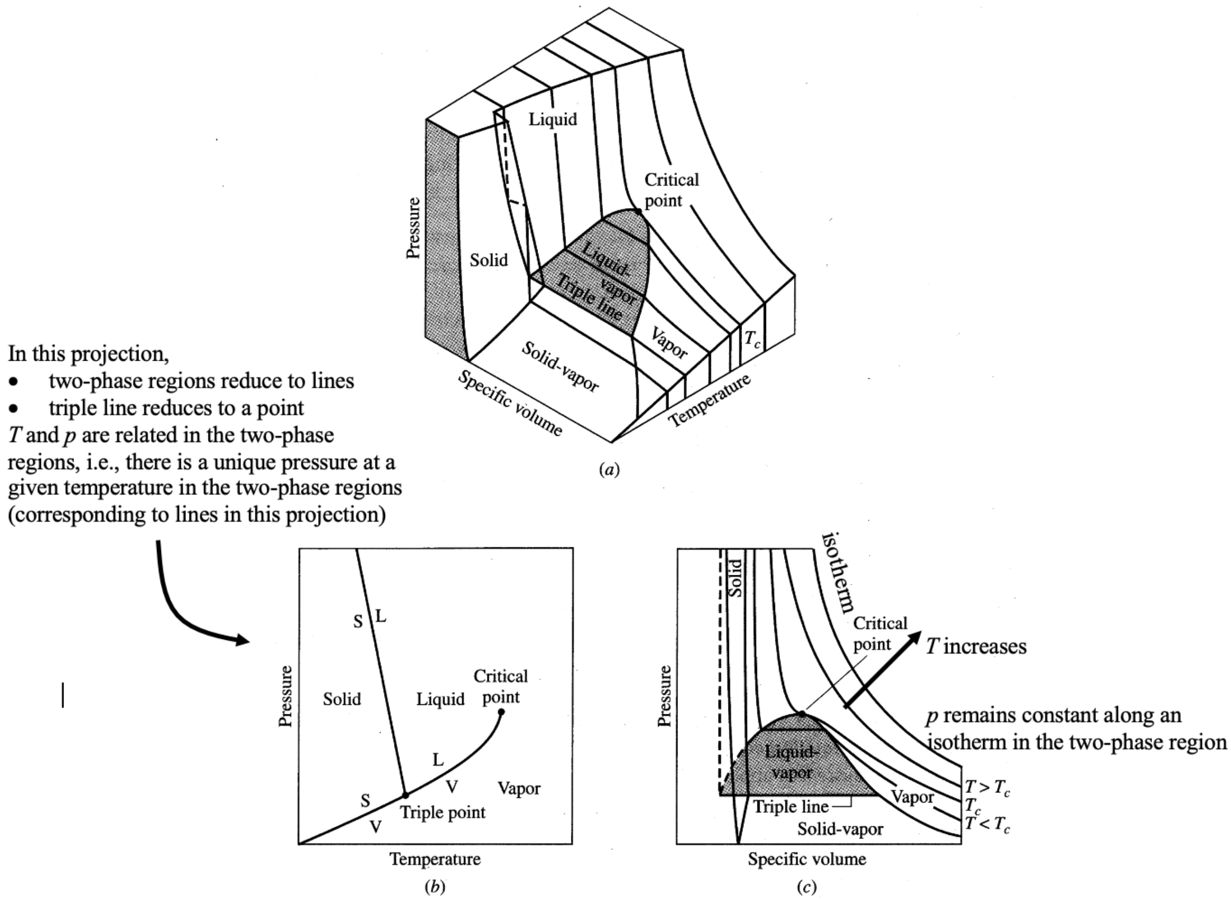


FIGURE 3.12. p-v-T surface and projections for a substance that expands on freezing. (a) 3D view, (b) phase diagram, (c) p-v diagram. Figure 3.2 from Moran et al., 7th ed.

  - The triple point of water occurs at 273.16 K and 0.6113 kPa (abs).
  - For a substance that expands on freezing, e.g., water, the solid-liquid phase line tilts toward the left. For a substance that contracts upon freezing, the solid-liquid saturation line tilts toward the right.
- The projection of the p-v-T diagram onto the p-v plane is also useful (Figure 3.14).
  - Lines of constant temperature are referred to as isotherms.
  - For $T < T_c$, the pressure remains constant in the two-phase regions along an isotherm. In the single-phase regions along an isotherm, the pressure decreases with increasing v.
  - The isotherm is at an inflection point when passing through the critical point.
- The projection of the p-v-T diagram onto the T-v plane is frequently used (Figure 3.15).
  - Lines of constant pressure are known as isobars.
  - The temperature remains constant with pressure along an isobar in the two-phase region.
  - In the single-phase regions, the temperature increases with increasing specific volume along an isobar.
  - For pressures greater than the critical pressure, the temperature increases continuously with increasing specific volume along an isobar.

The p-v-T sketches shown in the previous figures are distorted from what they actual plots look like. Figure 3.16 shows an example of a p-v plot for water drawn to scale. Note the use of logarithmic axes.

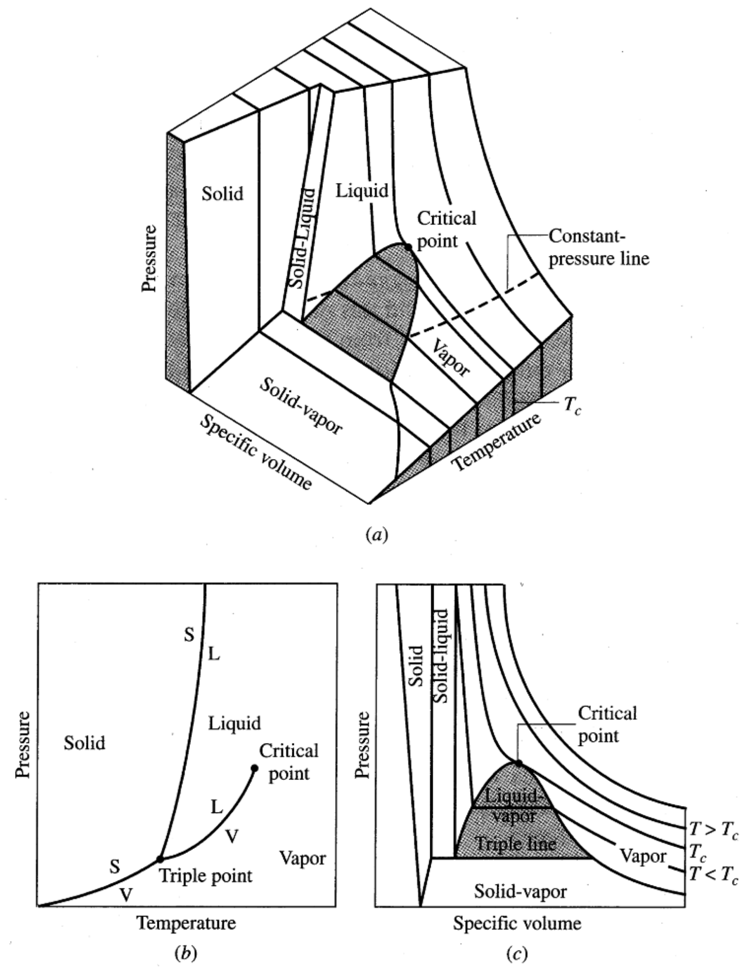


FIGURE 3.13. p-v-T surface and projections for a substance that contracts on freezing. (a) 3D view, (b) phase diagram, (c) p-v diagram. Figure 3.2 from Moran et al., 7th ed.

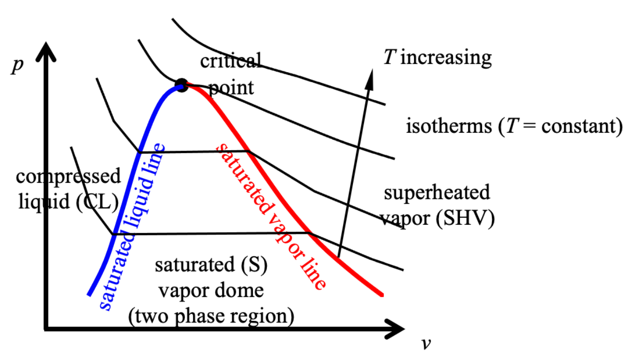


FIGURE 3.14. Sketch of a p-v plot.

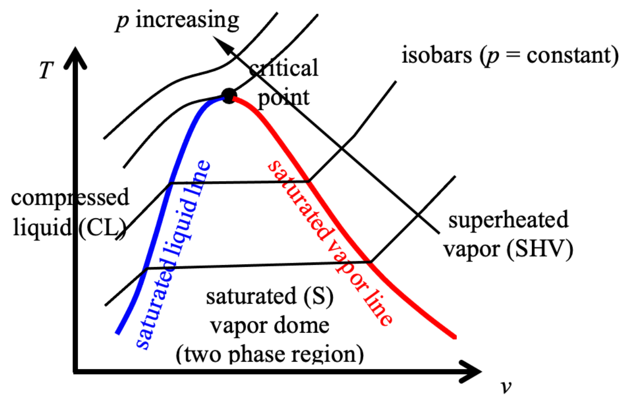


Figure 3.15. Sketch of a T - v plot.

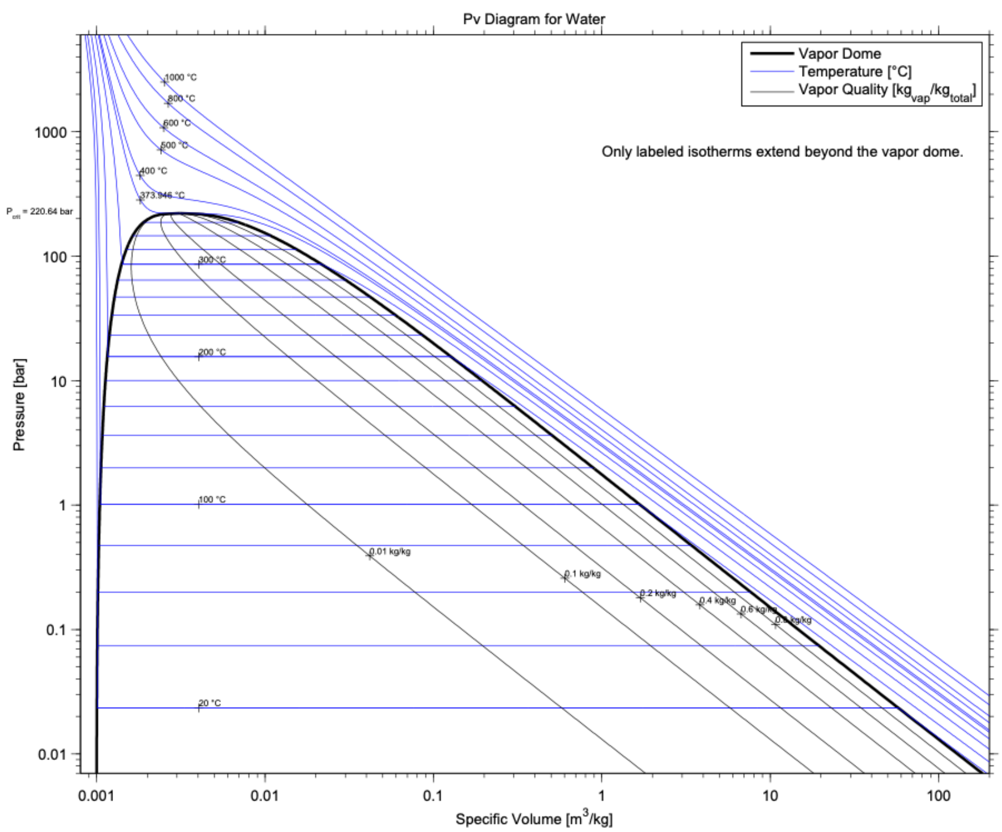


Figure 3.16. A p - v plot for water. Note the logarithmic scales.

Now consider the T - v projection more closely and, specifically, the region near the vapor dome as shown in Figure 3.17.

- The phase of a substance to the left of the vapor dome is known as a liquid, subcooled liquid, or compressed liquid (CL).
  - Point "l" in the figure is in this liquid region.
  - The term "subcooled" refers to the fact that along an isobar, the temperature is too low for the substance to be a vapor.

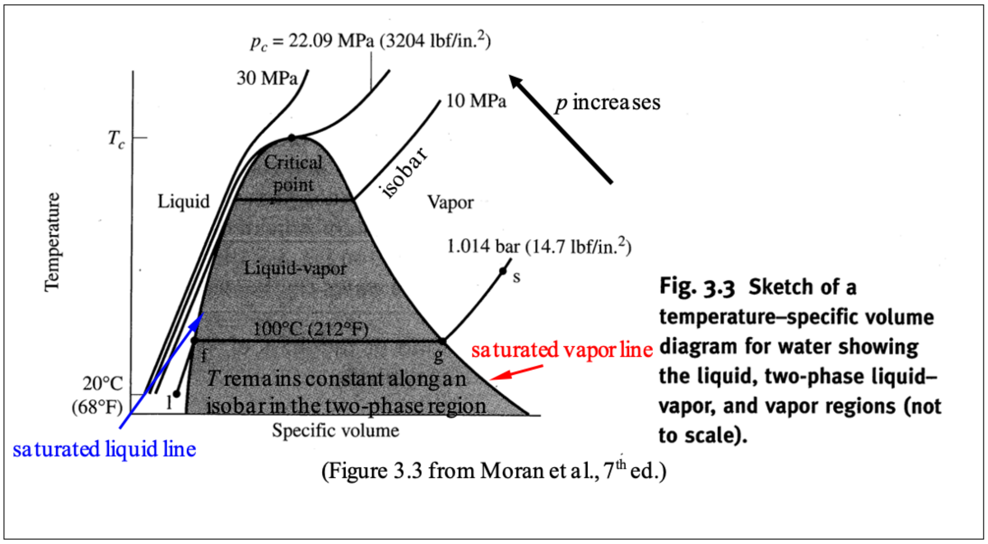


Figure 3.17. Sketch of a T - v diagram. Figure 3.3 from Moran et al., 7th ed.

  - The term "compressed" refers to the fact that at a given temperature, the pressure is larger than the pressure required to reach the saturation state. The pressure increases moving upward and toward the left across the isobars.
- The phase of a substance to the right of the vapor dome is known as a superheated vapor (SHV).
  - The term "superheated" refers the fact that the temperature is larger than what would be required to reach a liquid-vapor saturation state along a given isobar.
  - Point "s" is in the superheated vapor region.
- Within the vapor dome (the two-phase, or saturated (S) region), both liquid and vapor can exist in equilibrium. In order to specify how much of the substance is in liquid form versus vapor form, we define the quality of the mixture, x, which is the mass fraction of vapor at a given state, i.e., how much of the total mass is vapor. We'll come back to this term a little later. For now, it's sufficient to know that, from the definition of quality, a quality of zero corresponds to a saturated liquid (no vapor) while a quality of one corresponds to a saturated vapor (all vapor). Similar parameters can be defined for two-phase regions consisting of solid-vapor and solid-liquid.
- There are three similar sounding terms used frequently in the two-phase region, but each of these terms represents a different thing:
  - saturated liquid: In this phase, the state is on the saturated liquid line, i.e., at the left edge of the saturated phase, which means it's 100% liquid, i.e., it has a quality of zero.
  - saturated vapor: In this phase, the state is on the saturated vapor line, i.e., at the right edge of the saturated phase, which means it's 100% vapor, i.e., it has a quality of one.
  - saturated: In this phase, the substance contains both liquid and vapor. The state is within the vapor dome. The quality for a saturated substance is between zero and one.

Although the property plots are helpful for qualitatively understanding the relationship between properties and phases of a substance, they're not particularly useful for quantitative analysis. Fortunately, tables (and computer databases – see for example `http://webbook.nist.gov/chemistry/fluid/` have been prepared that provide quantitative values for the relationship between properties.

Figure 3.18 highlights the saturation (liquid-vapor) region in a T - v diagram. The properties corresponding to this region for water are given in Figure 3.19 as two tables. Recall that the pressure and temperature are related in this two-phase region. Thus, there is a unique temperature, known as the saturation temperature, at each pressure, known as the saturation pressure. The difference between the two tables is that one is ordered according to convenient temperature increments (top table) while the other is ordered by convenient pressure increments (bottom table).

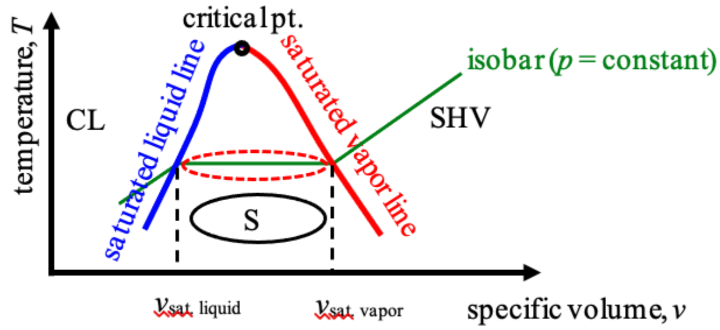


FIGURE 3.18. A sketch of a T-v diagram highlighting the region under the vapor dome corresponding to the saturation properties.

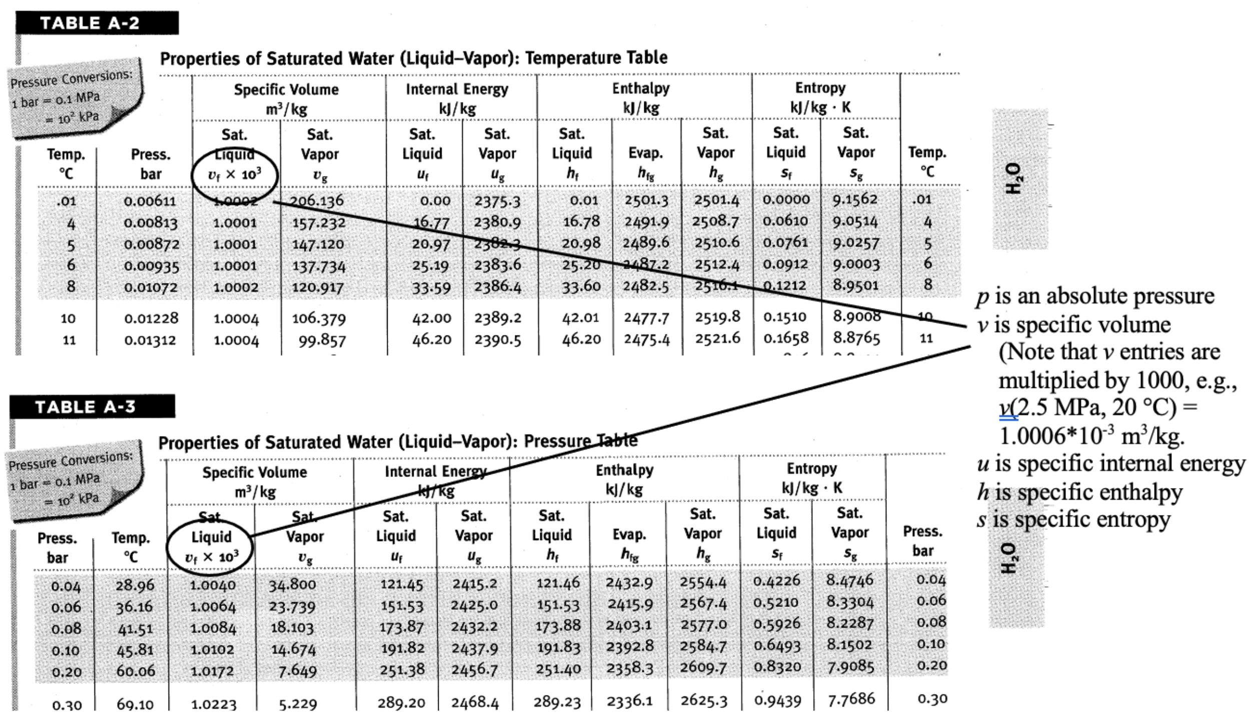


TABLE A-2

Pressure Conversions: 1 bar = 0.1 MPa = 10² kPa

Properties of Saturated Water (Liquid–Vapor): Temperature Table

		Specific Volume m³/kg		Internal Energy kJ/kg		Enthalpy kJ/kg			Entropy kJ/kg · K		
Temp. °C	Press. bar	Sat. Liquid $v_f \times 10^3$	Sat. Vapor v_g	Sat. Liquid u_f	Sat. Vapor u_g	Sat. Liquid h_f	Evap. h_{fg}	Sat. Vapor h_g	Sat. Liquid s_f	Sat. Vapor s_g	Temp. °C
.01	0.00611	1.0002	206.136	0.00	2375.3	0.01	2501.3	2501.4	0.0000	9.1562	.01
4	0.00813	1.0001	157.232	16.77	2380.9	16.78	2491.9	2508.7	0.0610	9.0514	4
5	0.00872	1.0001	147.120	20.97	2382.3	20.98	2489.6	2510.6	0.0761	9.0257	5
6	0.00935	1.0001	137.734	25.19	2383.6	25.20	2487.2	2512.4	0.0912	9.0003	6
8	0.01072	1.0002	120.917	33.59	2386.4	33.60	2482.5	2516.1	0.1212	8.9501	8
10	0.01228	1.0004	106.379	42.00	2389.2	42.01	2477.7	2519.8	0.1510	8.9008	10
11	0.01312	1.0004	99.857	46.20	2390.5	46.20	2475.4	2521.6	0.1658	8.8765	11

H_2O

TABLE A-3

Pressure Conversions: 1 bar = 0.1 MPa = 10² kPa

Properties of Saturated Water (Liquid–Vapor): Pressure Table

		Specific Volume m³/kg		Internal Energy kJ/kg		Enthalpy kJ/kg			Entropy kJ/kg · K		
Press. bar	Temp. °C	Sat. Liquid $v_f \times 10^3$	Sat. Vapor v_g	Sat. Liquid u_f	Sat. Vapor u_g	Sat. Liquid h_f	Evap. h_{fg}	Sat. Vapor h_g	Sat. Liquid s_f	Sat. Vapor s_g	Press. bar
0.04	28.96	1.0040	34.800	121.45	2415.2	121.46	2432.9	2554.4	0.4226	8.4746	0.04
0.06	36.16	1.0064	23.739	151.53	2425.0	151.53	2415.9	2567.4	0.5210	8.3304	0.06
0.08	41.51	1.0084	18.103	173.87	2432.2	173.88	2403.1	2577.0	0.5926	8.2287	0.08
0.10	45.81	1.0102	14.674	191.82	2437.9	191.83	2392.8	2584.7	0.6493	8.1502	0.10
0.20	60.06	1.0172	7.649	251.38	2456.7	251.40	2358.3	2609.7	0.8320	7.9085	0.20
0.30	69.10	1.0223	5.229	289.20	2468.4	289.23	2336.1	2625.3	0.9439	7.7686	0.30

H_2O

p is an absolute pressure
v is specific volume
(Note that v entries are multiplied by 1000, e.g., v(2.5 MPa, 20 °C) = 1.0006×10^{-3} m³/kg.
u is specific internal energy
h is specific enthalpy
s is specific entropy

FIGURE 3.19. Example table for the saturation properties for water organized by temperature (top) and pressure (bottom). These tables are from Moran et al., 7th ed.

At each saturation temperature and pressure, the tables provide the specific volume and other properties (specific internal energy, specific enthalpy, and specific entropy) at saturated liquid and saturated vapor conditions. The actual properties of the water will lie somewhere between or equal to the saturated liquid and saturated vapor conditions when the water is in a saturated state, e.g., $v_{\text{sat. liquid}} \leq v \leq v_{\text{sat. vapor}}$.

What the actual property values are depends on the quality of the water (the mass fraction of vapor in the two-phase mixture), a topic to be discussed later.

Figure 3.20 presents super-heated vapor phase region in a T - v plot. Example properties for water in this region are given in Figure 3.21. Recall that the pressure and temperature are independent in this single-phase region and, thus, the table entries are organized based on a given pressure and range of temperatures. For each pressure, the saturation temperature (the temperature at which the vapor touches the vapor dome along the saturated vapor line) is also reported.

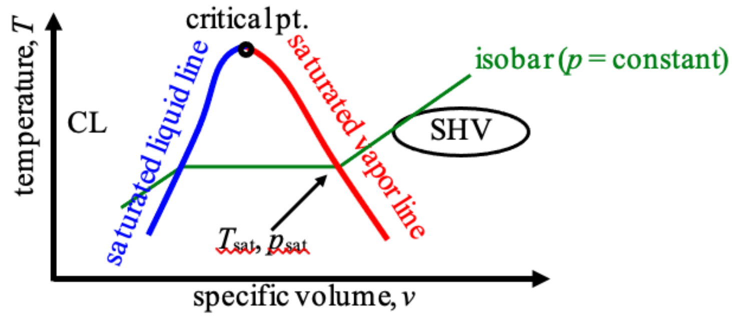


Figure 3.20. A sketch of a T - v diagram highlighting the super-heated vapor region.

TABLE A-4

Properties of Superheated Water Vapor

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
	p = 0.06 bar = 0.006 MPa (T_{sat} = 36.16°C)				p = 0.35 bar = 0.035 MPa (T_{sat} = 72.69°C)			
Sat.	23.739	2425.0	2567.4	8.3304	4.526	2473.0	2631.4	7.7158
80	27.132	2487.3	2650.1	8.5804	4.625	2483.7	2645.6	7.7564
120	30.219	2544.7	2726.0	8.7840	5.163	2542.4	2723.1	7.9644
160	33.302	2602.7	2802.5	8.9693	5.696	2601.2	2800.6	8.1519
200	36.383	2661.4	2879.7	9.1398	6.228	2660.4	2878.4	8.3237
240	39.462	2721.0	2957.8	9.2982	6.758	2720.3	2956.8	8.4828
280	42.540	2781.5	3036.8	9.4464	7.287	2780.9	3036.0	8.6314
320	45.618	2843.0	3116.7	9.5859	7.815	2842.5	3116.1	8.7712
360	48.696	2905.5	3197.7	9.7180	8.344	2905.1	3197.1	8.9034
400	51.774	2969.0	3279.6	9.8435	8.872	2968.6	3279.2	9.0291
440	54.851	3033.5	3362.6	9.9633	9.400	3033.2	3362.2	9.1490
500	59.467	3132.3	3489.1	10.1336	10.192	3132.1	3488.8	9.3194

Pressure Conversions:
1 bar = 0.1 MPa
= 10^2 kPa

H_2O

p is an absolute pressure
v is specific volume
u is specific internal energy
h is specific enthalpy
s is specific entropy

Figure 3.21. Example table for the superheated vapor (SHV) properties for water. This table is from Moran et al., 7th ed.

Figure 3.22 presents the compressed liquid phase region in a T - v plot. Example properties for water in this region are given in Figure 3.23. Recall that the pressure and temperature are independent in this single-phase region and, thus, the table entries are organized based on a given pressure and range of temperatures. For each pressure, the saturation temperature (the temperature at which the liquid touches the vapor dome along the saturated liquid line) is also reported.

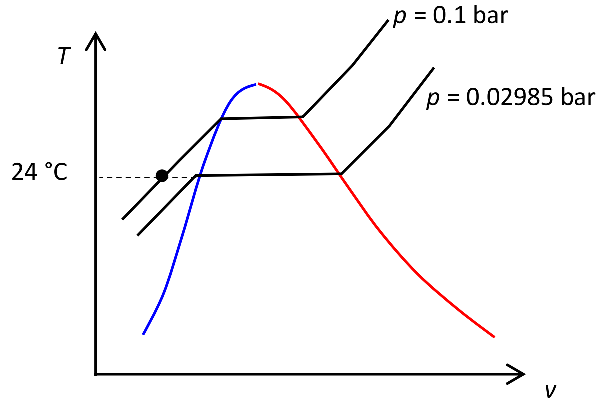
In what phase (liquid, saturated, vapor, etc.) is water for the following conditions?

a. T = 24 °C, p = 0.1 bar (abs)
b. p = 4 bar (abs), T = 180 °C
c. p = 20 bar (abs), v = 0.01 m^3/kg
d. T = 30 °C, p = 0.04246 bar (abs)
e. T = 30 °C, v = 1.0043*10^{-3} m^3/kg
f. p = 25 bar (abs), v = 0.07998 m^3/kg

SOLUTION:

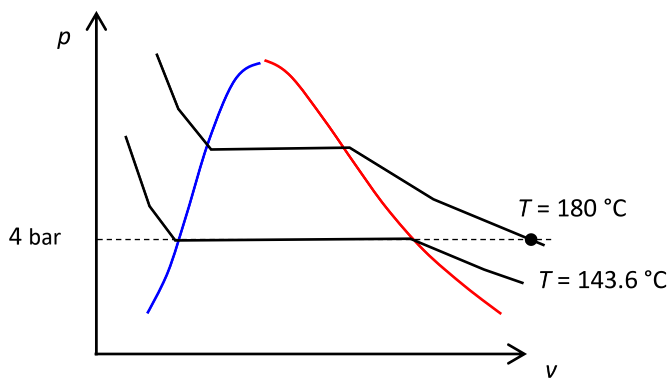
a. T = 24 °C, p = 0.1 bar (abs)

Using Table A.2 (attached to the end of this example), at T = 24 °C, p_{sat} = 0.02985 bar (abs). Since $p > p_{sat}$, the water will be in a compressed liquid phase.



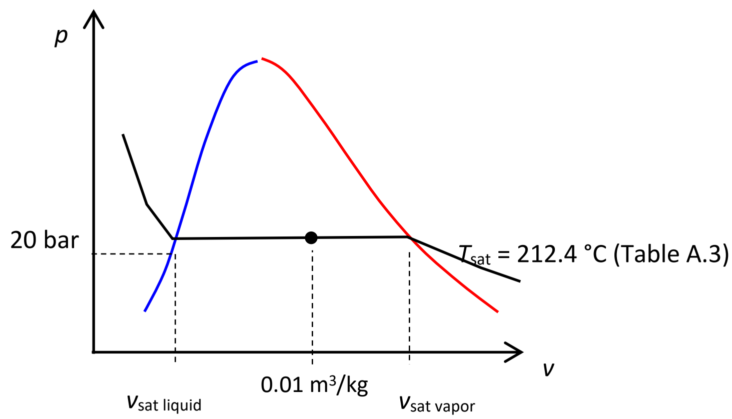
b. p = 4 bar (abs), T = 180 °C

Using Table A.3 (attached to the end of this example), at p = 4 bar (abs), T_{sat} = 143.6 °C. Since $T > T_{sat}$, the water will be in a superheated vapor phase.



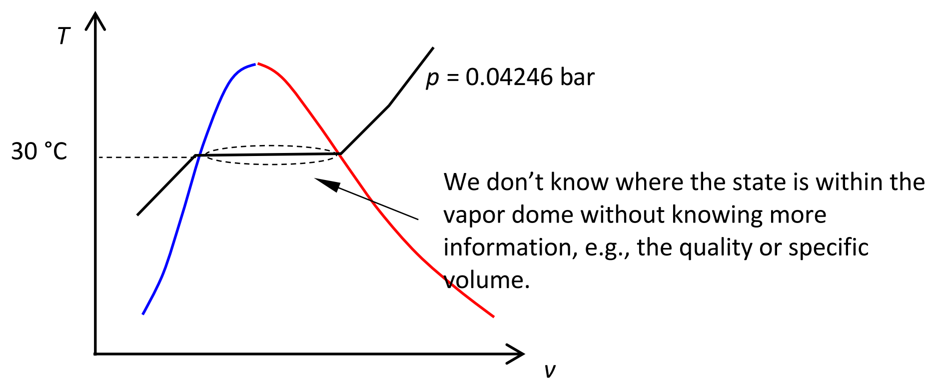
c. p = 20 bar (abs), v = 0.01 m^3/kg

Using Table A.3 (attached to the end of this example), at p = 20 bar (abs), $v_{\text{sat liquid}} = 1.1767 \times 10^{-3}$ m^3/kg and $v_{\text{sat vapor}}$ = 0.09963 m^3/kg. Since $v_{\text{sat liquid}} < v < v_{\text{sat vapor}}$, the water will be in a <u>saturated</u> (i.e., two phase) phase.



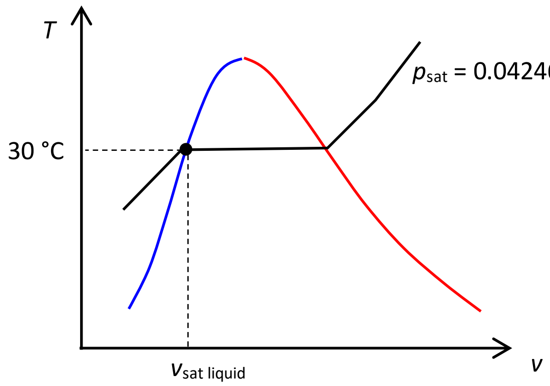
d. T = 30 °C, p = 0.04246 bar (abs)

Using Table A.2 (attached to the end of this example), at T = 30 °C, p_{sat} = 0.04246 bar (abs). Since $p = p_{sat}$, the water will be in a <u>saturated</u> (two-phase) phase.



e. T = 30 °C, v = 1.0043*10^{-3} m^3/kg

Using Table A.2 (attached to the end of this example), at T = 30 °C, $v_{\text{sat liquid}}$ = 1.0043*10^{-3} m^3/kg. Since v = $v_{\text{sat liquid}}$, the water will be in saturated liquid phase.



f. p = 25 bar (abs), v = 0.07998 m^3/kg

Using Table A.3 (attached to the end of this example), at p = 25 bar (abs), $v_{\text{sat vapor}}$ = 0.07998 m^3/kg. Since v = $v_{\text{sat vapor}}$, the water will be in a saturated vapor phase.

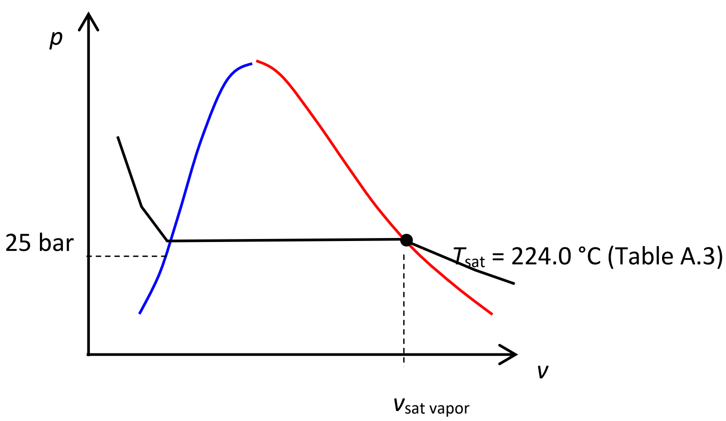


TABLE A-2

Pressure Conversions: 1 bar = 0.1 MPa = 10^2 kPa

Properties of Saturated Water (Liquid–Vapor): Temperature Table

H_2O

		Specific Volume m^3/kg		Internal Energy kJ/kg		Enthalpy kJ/kg			Entropy kJ/kg · K		
Temp. °C	Press. bar	Sat. Liquid $v_f \times 10^3$	Sat. Vapor v_g	Sat. Liquid u_f	Sat. Vapor u_g	Sat. Liquid h_f	Evap. h_{fg}	Sat. Vapor h_g	Sat. Liquid s_f	Sat. Vapor s_g	Temp. °C
.01	0.00611	1.0002	206.136	0.00	2375.3	0.01	2501.3	2501.4	0.0000	9.1562	.01
4	0.00813	1.0001	157.232	16.77	2380.9	16.78	2491.9	2508.7	0.0610	9.0514	4
5	0.00872	1.0001	147.120	20.97	2382.3	20.98	2489.6	2510.6	0.0761	9.0257	5
6	0.00935	1.0001	137.734	25.19	2383.6	25.20	2487.2	2512.4	0.0912	9.0003	6
8	0.01072	1.0002	120.917	33.59	2386.4	33.60	2482.5	2516.1	0.1212	8.9501	8
10	0.01228	1.0004	106.379	42.00	2389.2	42.01	2477.7	2519.8	0.1510	8.9008	10
11	0.01312	1.0004	99.857	46.20	2390.5	46.20	2475.4	2521.6	0.1658	8.8765	11
12	0.01402	1.0005	93.784	50.41	2391.9	50.41	2473.0	2523.4	0.1806	8.8524	12
13	0.01497	1.0007	88.124	54.60	2393.3	54.60	2470.7	2525.3	0.1953	8.8285	13
14	0.01598	1.0008	82.848	58.79	2394.7	58.80	2468.3	2527.1	0.2099	8.8048	14
15	0.01705	1.0009	77.926	62.99	2396.1	62.99	2465.9	2528.9	0.2245	8.7814	15
16	0.01818	1.0011	73.333	67.18	2397.4	67.19	2463.6	2530.8	0.2390	8.7582	16
17	0.01938	1.0012	69.044	71.38	2398.8	71.38	2461.2	2532.6	0.2535	8.7351	17
18	0.02064	1.0014	65.038	75.57	2400.2	75.58	2458.8	2534.4	0.2679	8.7123	18
19	0.02198	1.0016	61.293	79.76	2401.6	79.77	2456.5	2536.2	0.2823	8.6897	19
20	0.02339	1.0018	57.791	83.95	2402.9	83.96	2454.1	2538.1	0.2966	8.6672	20
21	0.02487	1.0020	54.514	88.14	2404.3	88.14	2451.8	2539.9	0.3109	8.6450	21
22	0.02645	1.0022	51.447	92.32	2405.7	92.33	2449.4	2541.7	0.3251	8.6229	22
23	0.02810	1.0024	48.574	96.51	2407.0	96.52	2447.0	2543.5	0.3393	8.6011	23
24	0.02985	1.0027	45.883	100.70	2408.4	100.70	2444.7	2545.4	0.3534	8.5794	24
25	0.03169	1.0029	43.360	104.88	2409.8	104.89	2442.3	2547.2	0.3674	8.5580	25
26	0.03363	1.0032	40.994	109.06	2411.1	109.07	2439.9	2549.0	0.3814	8.5367	26
27	0.03567	1.0035	38.774	113.25	2412.5	113.25	2437.6	2550.8	0.3954	8.5156	27
28	0.03782	1.0037	36.690	117.42	2413.9	117.43	2435.2	2552.6	0.4093	8.4946	28
29	0.04008	1.0040	34.733	121.60	2415.2	121.61	2432.8	2554.5	0.4231	8.4739	29
30	0.04246	1.0043	32.894	125.78	2416.6	125.79	2430.5	2556.3	0.4369	8.4533	30
31	0.04496	1.0046	31.165	129.96	2418.0	129.97	2428.1	2558.1	0.4507	8.4329	31
32	0.04759	1.0050	29.540	134.14	2419.3	134.15	2425.7	2559.9	0.4644	8.4127	32
33	0.05034	1.0053	28.011	138.32	2420.7	138.33	2423.4	2561.7	0.4781	8.3927	33
34	0.05324	1.0056	26.571	142.50	2422.0	142.50	2421.0	2563.5	0.4917	8.3728	34
35	0.05628	1.0060	25.216	146.67	2423.4	146.68	2418.6	2565.3	0.5053	8.3531	35
36	0.05947	1.0063	23.940	150.85	2424.7	150.86	2416.2	2567.1	0.5188	8.3336	36
38	0.06632	1.0071	21.602	159.20	2427.4	159.21	2411.5	2570.7	0.5458	8.2950	38
40	0.07384	1.0078	19.523	167.56	2430.1	167.57	2406.7	2574.3	0.5725	8.2570	40
45	0.09593	1.0099	15.258	188.44	2436.8	188.45	2394.8	2583.2	0.6387	8.1648	45

TABLE A-3

Pressure Conversions:
1 bar = 0.1 MPa
= 10^2 kPa

Properties of Saturated Water (Liquid–Vapor): Pressure Table

H_2O

		Specific Volume m^3/kg		Internal Energy kJ/kg		Enthalpy kJ/kg			Entropy kJ/kg · K		
Press. bar	Temp. °C	Sat. Liquid $v_f \times 10^3$	Sat. Vapor v_g	Sat. Liquid u_f	Sat. Vapor u_g	Sat. Liquid h_f	Evap. h_{fg}	Sat. Vapor h_g	Sat. Liquid s_f	Sat. Vapor s_g	Press. bar
0.04	28.96	1.0040	34.800	121.45	2415.2	121.46	2432.9	2554.4	0.4226	8.4746	0.04
0.06	36.16	1.0064	23.739	151.53	2425.0	151.53	2415.9	2567.4	0.5210	8.3304	0.06
0.08	41.51	1.0084	18.103	173.87	2432.2	173.88	2403.1	2577.0	0.5926	8.2287	0.08
0.10	45.81	1.0102	14.674	191.82	2437.9	191.83	2392.8	2584.7	0.6493	8.1502	0.10
0.20	60.06	1.0172	7.649	251.38	2456.7	251.40	2358.3	2609.7	0.8320	7.9085	0.20
0.30	69.10	1.0223	5.229	289.20	2468.4	289.23	2336.1	2625.3	0.9439	7.7686	0.30
0.40	75.87	1.0265	3.993	317.53	2477.0	317.58	2319.2	2636.8	1.0259	7.6700	0.40
0.50	81.33	1.0300	3.240	340.44	2483.9	340.49	2305.4	2645.9	1.0910	7.5939	0.50
0.60	85.94	1.0331	2.732	359.79	2489.6	359.86	2293.6	2653.5	1.1453	7.5320	0.60
0.70	89.95	1.0360	2.365	376.63	2494.5	376.70	2283.3	2660.0	1.1919	7.4797	0.70
0.80	93.50	1.0380	2.087	391.58	2498.8	391.66	2274.1	2665.8	1.2329	7.4346	0.80
0.90	96.71	1.0410	1.869	405.06	2502.6	405.15	2265.7	2670.9	1.2695	7.3949	0.90
1.00	99.63	1.0432	1.694	417.36	2506.1	417.46	2258.0	2675.5	1.3026	7.3594	1.00
1.50	111.4	1.0528	1.159	466.94	2519.7	467.11	2226.5	2693.6	1.4336	7.2233	1.50
2.00	120.2	1.0605	0.8857	504.49	2529.5	504.70	2201.9	2706.7	1.5301	7.1271	2.00
2.50	127.4	1.0672	0.7187	535.10	2537.2	535.37	2181.5	2716.9	1.6072	7.0527	2.50
3.00	133.6	1.0732	0.6058	561.15	2543.6	561.47	2163.8	2725.3	1.6718	6.9919	3.00
3.50	138.9	1.0786	0.5243	583.95	2546.9	584.33	2148.1	2732.4	1.7275	6.9405	3.50
4.00	143.6	1.0836	0.4625	604.31	2553.6	604.74	2133.8	2738.6	1.7766	6.8959	4.00
4.50	147.9	1.0882	0.4140	622.25	2557.6	623.25	2120.7	2743.9	1.8207	6.8565	4.50
5.00	151.9	1.0926	0.3749	639.68	2561.2	640.23	2108.5	2748.7	1.8607	6.8212	5.00
6.00	158.9	1.1006	0.3157	669.90	2567.4	670.56	2086.3	2756.8	1.9312	6.7600	6.00
7.00	165.0	1.1080	0.2729	696.44	2572.5	697.22	2066.3	2763.5	1.9922	6.7080	7.00
8.00	170.4	1.1148	0.2404	720.22	2576.8	721.11	2048.0	2769.1	2.0462	6.6628	8.00
9.00	175.4	1.1212	0.2150	741.83	2580.5	742.83	2031.1	2773.9	2.0946	6.6226	9.00
10.0	179.9	1.1273	0.1944	761.68	2583.6	762.81	2015.3	2778.1	2.1387	6.5863	10.0
15.0	198.3	1.1539	0.1318	843.16	2594.5	844.84	1947.3	2792.2	2.3150	6.4448	15.0
20.0	212.4	1.1767	0.09963	906.44	2600.3	908.79	1890.7	2799.5	2.4474	6.3409	20.0
25.0	224.0	1.1973	0.07998	959.11	2603.1	962.11	1841.0	2803.1	2.5547	6.2575	25.0
30.0	233.9	1.2165	0.06668	1004.8	2604.1	1008.4	1795.7	2804.2	2.6457	6.1869	30.0
35.0	242.6	1.2347	0.05707	1045.4	2603.7	1049.8	1753.7	2803.4	2.7253	6.1253	35.0
40.0	250.4	1.2522	0.04978	1082.3	2602.3	1087.3	1714.1	2801.4	2.7964	6.0701	40.0
45.0	257.5	1.2692	0.04406	1116.2	2600.1	1121.9	1676.4	2798.3	2.8610	6.0199	45.0
50.0	264.0	1.2859	0.03944	1147.8	2597.1	1154.2	1640.1	2794.3	2.9202	5.9734	50.0
60.0	275.6	1.3187	0.03244	1205.4	2589.7	1213.4	1571.0	2784.3	3.0267	5.8892	60.0
70.0	285.9	1.3513	0.02737	1257.6	2580.5	1267.0	1505.1	2772.1	3.1211	5.8133	70.0
80.0	295.1	1.3842	0.02352	1305.6	2569.8	1316.6	1441.3	2758.0	3.2068	5.7432	80.0
90.0	303.4	1.4178	0.02048	1350.5	2557.8	1363.3	1378.9	2742.1	3.2858	5.6772	90.0
100.	311.1	1.4524	0.01803	1393.0	2544.4	1407.6	1317.1	2724.7	3.3596	5.6141	100.
110.	318.2	1.4886	0.01599	1433.7	2529.8	1450.1	1255.5	2705.6	3.4295	5.5527	110.

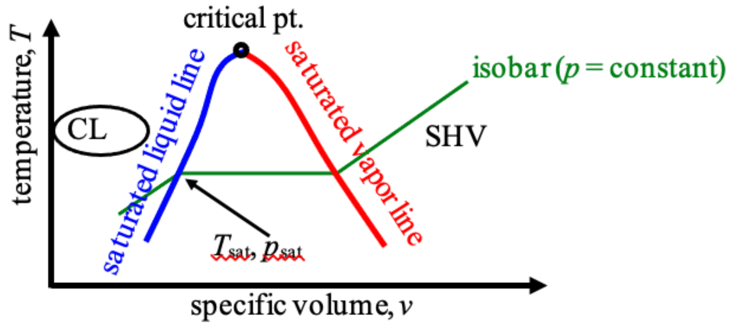


Figure 3.22. A sketch of a T - v diagram highlighting the compressed liquid region.

TABLE A-5

Properties of Compressed Liquid Water

T °C	$v \times 10^3$ m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K	$v \times 10^3$ m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
	p = 25 bar = 2.5 MPa (T_{sat} = 223.99°C)				p = 50 bar = 5.0 MPa (T_{sat} = 263.99°C)			
20	1.0006	83.80	86.30	.2961	.9995	83.65	88.65	.2956
40	1.0067	167.25	169.77	.5715	1.0056	166.95	171.97	.5705
80	1.0280	334.29	336.86	1.0737	1.0268	333.72	338.85	1.0720
100	1.0423	418.24	420.85	1.3050	1.0410	417.52	422.72	1.3030
140	1.0784	587.82	590.52	1.7369	1.0768	586.76	592.15	1.7343
180	1.1261	761.16	763.97	2.1375	1.1240	759.63	765.25	2.1341
200	1.1555	849.9	852.8	2.3294	1.1530	848.1	853.9	2.3255
220	1.1898	940.7	943.7	2.5174	1.1866	938.4	944.4	2.5128
Sat.	1.1973	959.1	962.1	2.5546	1.2859	1147.8	1154.2	2.9202

Pressure Conversions:
1 bar = 0.1 MPa
= 10^2 kPa

H_2O

p is an absolute pressure
v is specific volume
(Note that *v* entries are multiplied by 1000, e.g., v(2.5 MPa, 20 °C) = 1.0006×10^{-3} m³/kg.
u is specific internal energy
h is specific enthalpy
s is specific entropy

(Table from Moran et al., 7th ed.)

Figure 3.23. Example table for the compressed liquid (CL) properties for water. This table is from Moran et al., 7th ed.

3.5.1. Linear Interpolation

In order to save space, the properties in the tables are listed in coarse increments, e.g., increments of 2.5 MPa and 20 or 40 °C. To approximate property values between the ones stated in the tables, we can use linear interpolation. Linear interpolation is the process of estimating the values of a property assuming a linear relationship between neighboring data points. Hence, to estimate the value of a property P at a value V given the property's values at neighboring points V_S and V_L, corresponding to points in the table just smaller than and larger than the value V of interest, we can use the equation of a line (Figure 3.24),

$$P - P_S = \left(\frac{P_L - P_S}{V_L - V_S}\right)(V - V_S) \tag{3.50}$$

where P_S and P_L are the property values at V_S and V_L, respectively.

What is the specific volume of compressed liquid water at 5.0 MPa (abs) and 60 °C?

SOLUTION:
Since there is no specific volume data at 60 °C at 5.0 MPa in Table A-5 shown below, we can approximate the specific volume at 60 °C using linear interpolation,

$$v_{60\ ^\circ\mathrm{C}} - v_{40\ ^\circ\mathrm{C}} = \left(\frac{v_{80\ ^\circ\mathrm{C}} - v_{40\ ^\circ\mathrm{C}}}{T_{80\ ^\circ\mathrm{C}} - v_{40\ ^\circ\mathrm{C}}}\right)\left(T_{60\ ^\circ\mathrm{C}} - T_{40\ ^\circ\mathrm{C}}\right), \tag{1}$$

where

$v_{40\ ^\circ\mathrm{C}}$ = 1.0056*10^-3 m^3/kg
$v_{40\ ^\circ\mathrm{C}}$ = 1.0268*10^-3 m^3/kg
$T_{40\ ^\circ\mathrm{C}}$ = 40 °C
$T_{60\ ^\circ\mathrm{C}}$ = 60 °C
$T_{80\ ^\circ\mathrm{C}}$ = 80 °C
$\Rightarrow \boxed{v_{60\ ^\circ\mathrm{C}} = 1.0162 \times 10^{-3}\ \mathrm{m^3/kg}}$

TABLE A-5

Properties of Compressed Liquid Water

T °C	$v \times 10^3$ m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K	$v \times 10^3$ m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
	p = 25 bar = 2.5 MPa (T_{sat} = 223.99°C)				p = 50 bar = 5.0 MPa (T_{sat} = 263.99°C)			
20	1.0006	83.80	86.30	.2961	.9995	83.65	88.65	.2956
40	1.0067	167.25	169.77	.5715	1.0056	166.95	171.97	.5705
80	1.0280	334.29	336.86	1.0737	1.0268	333.72	338.85	1.0720
100	1.0423	418.24	420.85	1.3050	1.0410	417.52	422.72	1.3030
140	1.0784	587.82	590.52	1.7369	1.0768	586.76	592.15	1.7343
180	1.1261	761.16	763.97	2.1375	1.1240	759.63	765.25	2.1341
200	1.1555	849.9	852.8	2.3294	1.1530	848.1	853.9	2.3255
220	1.1898	940.7	943.7	2.5174	1.1866	938.4	944.4	2.5128
Sat.	1.1973	959.1	962.1	2.5546	1.2859	1147.8	1154.2	2.9202

Pressure Conversions:
1 bar = 0.1 MPa
= 10^2 kPa

H_2O

(Table from Moran et al., 7th ed.)

What is the specific internal energy of compressed liquid water at 3.0 MPa and 60 °C?

SOLUTION:
For this case, there is no table entry for either 3.0 MPa or 60 °C so we must linearly interpolate with respect to both temperature and pressure (called <u>bilinear interpolation</u>),

$$u_{2.5\text{ MPa},60\text{ °C}} - u_{2.5\text{ MPa},40\text{ °C}} = \left(\frac{u_{2.5\text{ MPa},80\text{ °C}} - u_{2.5\text{ MPa},40\text{ °C}}}{T_{2.5\text{ MPa},80\text{ °C}} - T_{2.5\text{ MPa},40\text{ °C}}}\right)\left(T_{2.5\text{ MPa},60\text{ °C}} - T_{2.5\text{ MPa},40\text{ °C}}\right), \tag{1}$$

$$u_{5.0\text{ MPa},60\text{ °C}} - u_{5.0\text{ MPa},40\text{ °C}} = \left(\frac{u_{5.0\text{ MPa},80\text{ °C}} - u_{5.0\text{ MPa},40\text{ °C}}}{T_{5.0\text{ MPa},80\text{ °C}} - T_{5.0\text{ MPa},40\text{ °C}}}\right)\left(T_{5.0\text{ MPa},60\text{ °C}} - T_{5.0\text{ MPa},40\text{ °C}}\right),$$

$$u_{3.0\text{ MPa},60\text{ °C}} - u_{2.5\text{ MPa},60\text{ °C}} = \left(\frac{u_{5.0\text{ MPa},60\text{ °C}} - u_{2.5\text{ MPa},60\text{ °C}}}{p_{5.0\text{ MPa},60\text{ °C}} - p_{2.5\text{ MPa},60\text{ °C}}}\right)\left(p_{3.0\text{ MPa},60\text{ °C}} - p_{2.5\text{ MPa},60\text{ °C}}\right),$$

where,

$u_{2.5\text{ MPa, }40\text{ °C}}$ = 167.25 kJ/kg
$u_{2.5\text{ MPa, }80\text{ °C}}$ = 334.29 kJ/kg
$u_{5.0\text{ MPa, }40\text{ °C}}$ = 166.95 kJ/kg
$u_{5.0\text{ MPa, }80\text{ °C}}$ = 333.72 kJ/kg
$T_{2.5\text{ MPa, }40\text{ °C}}$ = $T_{5.0\text{ MPa, }40\text{ °C}}$ = 40 °C
$T_{2.5\text{ MPa, }60\text{ °C}}$ = $T_{5.0\text{ MPa, }60\text{ °C}}$ = 60 °C
$T_{2.5\text{ MPa, }80\text{ °C}}$ = $T_{5.0\text{ MPa, }80\text{ °C}}$ = 80 °C
$p_{2.5\text{ MPa, }60\text{ °C}}$ = 2.5 MPa
$p_{3.0\text{ MPa, }60\text{ °C}}$ = 3.0 MPa
$p_{5.0\text{ MPa, }60\text{ °C}}$ = 5.0 MPa

$\Rightarrow u_{2.5\text{ MPa, }60\text{ °C}} = 250.77\text{ kJ/kg}$, $u_{5.0\text{ MPa, }60\text{ °C}} = 250.34\text{ kJ/kg} \Rightarrow \boxed{\upsilon_{3.0\text{ MPa, }60\text{ °C}} = 250.68\text{ kJ/kg}}$

Note that the same result would be achieved if we interpolated first with respect to pressure and then with respect to temperature.

TABLE A-5

Properties of Compressed Liquid Water

T °C	$v \times 10^3$ m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K	$v \times 10^3$ m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
	p = 25 bar = 2.5 MPa (T_{sat} = 223.99°C)				p = 50 bar = 5.0 MPa (T_{sat} = 263.99°C)			
20	1.0006	83.80	86.30	.2961	.9995	83.65	88.65	.2956
40	1.0067	167.25	169.77	.5715	1.0056	166.95	171.97	.5705
80	1.0280	334.29	336.86	1.0737	1.0268	333.72	338.85	1.0720
100	1.0423	418.24	420.85	1.3050	1.0410	417.52	422.72	1.3030
140	1.0784	587.82	590.52	1.7369	1.0768	586.76	592.15	1.7343
180	1.1261	761.16	763.97	2.1375	1.1240	759.63	765.25	2.1341
200	1.1555	849.9	852.8	2.3294	1.1530	848.1	853.9	2.3255
220	1.1898	940.7	943.7	2.5174	1.1866	938.4	944.4	2.5128
Sat.	1.1973	959.1	962.1	2.5546	1.2859	1147.8	1154.2	2.9202

Pressure Conversions:
1 bar = 0.1 MPa
= 10^2 kPa

H_2O

(Table from Moran et al., 7th ed.)

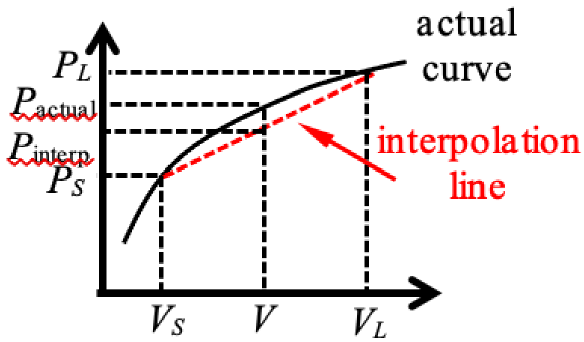


Figure 3.24. Schematic showing how to linearly interpolate a property value.

3.5.2. Quality

Recall that Figure 3.19 presented two tables for saturated water properties (aka, water in two-phase, liquid-vapor equilibrium along the saturated liquid and the saturated vapor lines). The pressure and temperature are uniquely related in this two-phase region (and along the saturated liquid and vapor lines). The top table (Table A-2) presents the property data organized according to temperature while the bottom table (Table A-3) presents the same data organized according to pressure. The subscripts "f" and "g" in the table refer to "fluid" and "gas", which is a historical notation. It is better to refer to the properties as being either at the saturated liquid state (subscript "f" in the table) or in the saturated vapor state (subscript "g" in the table). Similar tables exist for two-phase solid-liquid and solid-vapor.

Within the two-phase liquid-vapor region (i.e, the vapor dome), the fraction of mass that is vapor is given by the quality, x, which is defined as,

$$\boxed{x := \frac{m_v}{m_l + m_v}} \tag{3.51}$$

where m_v and m_l are the masses of vapor and liquid, respectively. Note that the fraction of mass that is liquid is,

$$\frac{m_l}{m_l + m_v} = \frac{m_l + m_v - m_v}{m_l + m_v} = \frac{m_l + m_v}{m_l + m_v} - \frac{m_v}{m_l + m_v} = 1 - x \tag{3.52}$$

Hence, a quality of zero corresponds to a saturated liquid (all liquid, $m_v = 0$) while a quality of one corresponds to a saturated vapor (all vapor, $m_l = 0$). The quality can be used to determine the value of properties within the two-phase region, given the saturated liquid and saturated vapor properties. For example, the specific volume of a mixture (subscript "m") of liquid (subscript "l") and vapor (subscript "v") in equilibrium (i.e., in the vapor dome), assuming the quality x is known, is,

$$V_m = V_l + V_v \tag{3.53}$$

$$v_m = \frac{V_m}{m_m} = \frac{V_l + V_v}{m_m} = \frac{V_l}{m_m} + \frac{V_v}{m_m} \tag{3.54}$$

where V_m is the total volume of the mixture. The quantity m_m is the total mass of the mixture, i.e., $m_m = m_l + m_v$. Hence,

$$v_m = \frac{V_l}{m_l + m_v} + \frac{V_v}{m_l + m_v} = \frac{m_l v_l}{m_l + m_v} + \frac{m_v + v_v}{m_l + m_v} = \left(\frac{m_l}{m_l + m_v}\right) v_l + \left(\frac{m_v}{m_l + m_v}\right) v_v \tag{3.55}$$

where the volume is related to the specific volume via $V = mv$. Making use of Eqs. (3.51) and (3.52),

$$\boxed{v_m = (1 - x)\, v_l + x v_v} \tag{3.56}$$

Thus, the specific volume of a mixture of liquid and vapor can be thought of as the specific volume of the saturated liquid multiplied by its mass fraction ($(1 - x)v_l$) plus the specific volume of the saturated vapor multiplied by its mass fraction (xv_v). Equation (3.56) may also be re-arranged to give,

$$v_m = v_l + x\underbrace{(v_v - v_l)}_{=v_{lv}} \tag{3.57}$$

where v_{lv} is the change in the specific volume during vaporization (liquid turns to vapor). Hence, the specific volume of the liquid-vapor mixture is the specific volume of the liquid (v_l) plus the mass fraction that has turned to vapor multiplied by the change in specific volume during vaporization ($x(v_v - v_l)$).

A similar approach may be used to find other properties in the two-phase liquid-vapor region, such as specific internal energy, e.g.,

$$\boxed{u_m = (1-x)\,u_l + xu_v = u_l + x\,(u_v - u_l)} \tag{3.58}$$

What is the quality of water at a pressure of 1.00 bar (abs) and specific volume of 0.01 m^3/kg?

SOLUTION:
The specific volume of a saturated substance is,

$$v = xv_v + (1-x)v_l \,. \tag{1}$$

Re-arrange to solve for the quality,

$$x = \frac{v - v_l}{v_v - v_l} \,. \tag{2}$$

For water at 1.00 bar (abs) (using Table A.3),

v_v = 1.694 m^3/kg,

v_l = 1.0432*10^{-3} m^3/kg.

Solving Eq. (2) when v = 0.01 m^3/kg,

$\boxed{x = 0.0053.}$

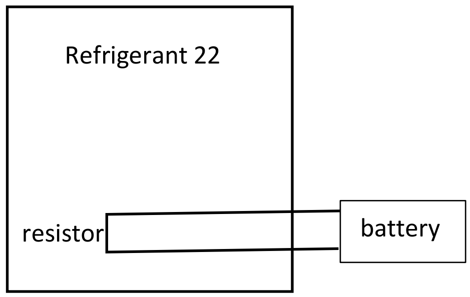
TABLE A-3

Pressure Conversions: 1 bar = 0.1 MPa = 10^2 kPa

Properties of Saturated Water (Liquid–Vapor): Pressure Table

		Specific Volume m^3/kg		Internal Energy kJ/kg		Enthalpy kJ/kg			Entropy kJ/kg · K		
Press. bar	Temp. °C	Sat. Liquid $v_f \times 10^3$	Sat. Vapor v_g	Sat. Liquid u_f	Sat. Vapor u_g	Sat. Liquid h_f	Evap. h_{fg}	Sat. Vapor h_g	Sat. Liquid s_f	Sat. Vapor s_g	Press. bar
0.04	28.96	1.0040	34.800	121.45	2415.2	121.46	2432.9	2554.4	0.4226	8.4746	0.04
0.06	36.16	1.0064	23.739	151.53	2425.0	151.53	2415.9	2567.4	0.5210	8.3304	0.06
0.08	41.51	1.0084	18.103	173.87	2432.2	173.88	2403.1	2577.0	0.5926	8.2287	0.08
0.10	45.81	1.0102	14.674	191.82	2437.9	191.83	2392.8	2584.7	0.6493	8.1502	0.10
0.20	60.06	1.0172	7.649	251.38	2456.7	251.40	2358.3	2609.7	0.8320	7.9085	0.20
0.30	69.10	1.0223	5.229	289.20	2468.4	289.23	2336.1	2625.3	0.9439	7.7686	0.30
0.40	75.87	1.0265	3.993	317.53	2477.0	317.58	2319.2	2636.8	1.0259	7.6700	0.40
0.50	81.33	1.0300	3.240	340.44	2483.9	340.49	2305.4	2645.9	1.0910	7.5939	0.50
0.60	85.94	1.0331	2.732	359.79	2489.6	359.86	2293.6	2653.5	1.1453	7.5320	0.60
0.70	89.95	1.0360	2.365	376.63	2494.5	376.70	2283.3	2660.0	1.1919	7.4797	0.70
0.80	93.50	1.0380	2.087	391.58	2498.8	391.66	2274.1	2665.8	1.2329	7.4346	0.80
0.90	96.71	1.0410	1.869	405.06	2502.6	405.15	2265.7	2670.9	1.2695	7.3949	0.90
1.00	99.63	1.0432	1.694	417.36	2506.1	417.46	2258.0	2675.5	1.3026	7.3594	1.00
1.50	111.4	1.0528	1.159	466.94	2519.7	467.11	2226.5	2693.6	1.4336	7.2233	1.50
2.00	120.2	1.0605	0.8857	504.49	2529.5	504.70	2201.9	2706.7	1.5301	7.1271	2.00

A closed, rigid tank fitted with a fine-wire electric resistor is filled with Refrigerant 22, initially at -10 °C, a quality of 80%, and a volume of 0.01 m^3. A 12 V battery provides a 5 A current to the resistor for 5 min. If the final temperature of the refrigerant is 40 °C, determine the heat transfer, in kJ, from the refrigerant.



SOLUTION:

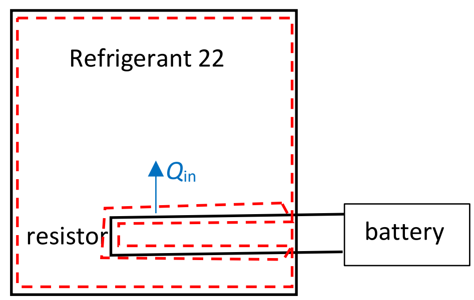
The heat transferred from the refrigerant to the surroundings may be found using the First Law applied to the refrigerant (our system),

$$\Delta E_{R22} = Q_{\substack{\text{into}\\R22}} + W_{\substack{\text{on}\\R22}} \Rightarrow Q_{\substack{\text{into}\\R22}} = \Delta E_{R22} - W_{\substack{\text{on}\\R22}}, \quad (1)$$

where,

$$\Delta E_{R22} = \Delta U_{R22} = U_2 - U_1 = m(u_2 - u_1), \quad (2)$$

assuming that other forms of energy change, e.g., kinetic and potential, are negligible. Note that since the container is closed, the initial and final refrigerant masses will be the same. Furthermore, the resistor wire is not considered to be part of the system.



The specific internal energy at state 1 is also found using the thermodynamic property tables,

$$u_1 = xu_v + (1-x)u_l, \quad (3)$$

where, at -10 °C in the saturated liquid-vapor phase,

x = 0.80,
u_v = 223.02 kJ/kg,
u_l = 33.27 kJ/kg,
$\Rightarrow u_1$ = 185.07 kJ/kg.

The specific volume at state 1 may be found in a similar manner,

$$v_1 = xv_v + (1-x)v_l, \quad (4)$$

where,

x = 0.80,
v_v = 0.0652 m^3/kg,
v_l = 0.7606*10^{-3} m^3/kg,
$\Rightarrow v_1$ = 0.0523 m^3/kg.

The mass of the refrigerant may be found from the initial state,

$$m = \frac{V}{v_1}, \text{ (The electrical wire volume is assumed negligible compared to the tank volume.)} \quad (5)$$

where,

V = 0.01 m^3,
$\Rightarrow m$ = 0.191 kg.

The specific internal energy at state 2 (after the 5 min) is found using the thermodynamic property tables for Refrigerant 22 at a temperature of 40 °C and a specific volume of,

$v_2 = v_1$ (since the container volume and refrigerant mass remain constant). (6)

Using the two-phase liquid-vapor thermodynamic table, observe that at the final temperature of T_2 = 40 °C, the saturated vapor specific volume is 0.0151 m^3/kg, which is smaller than the specific volume at state 2, v_2 = 0.0523 m^3/kg. Hence, the refrigerant must be in a superheated vapor phase. Interpolating from the superheated vapor table using T_2 and v_2,

u_2 = 250.33 kJ/kg.

Combining m, u_2, and u_1, Eq. (2) becomes,

ΔU = 12.46 kJ/kg.

There is no work acting on the refrigerant since the container volume remains constant and because the electrical work goes into the wire, which is not part of the system,

$$W_{\substack{\text{on}\\ \text{R22}}} = 0\,. \tag{7}$$

There is, however, heat that is transferred from the wire into the system. This heat may be found by applying the 1st Law to the wire. Assuming steady conditions so that the change in total energy of the wire is zero, the total heat from the wire will equal the total (electrical) work done on the wire,

$$\underbrace{\Delta E_{\text{wire}}}_{=0\text{ (steady)}} = -Q_{\substack{\text{from}\\ \text{wire}}} + W_{\substack{\text{on}\\ \text{wire}}} \Rightarrow Q_{\substack{\text{from}\\ \text{wire}}} = W_{\substack{\text{on}\\ \text{wire}}}\,, \tag{8}$$

where the total work done on the wire is,

$$W_{\substack{\text{on}\\ \text{wire}}} = VI\Delta t \text{ (assuming that neither the voltage nor current change over time } \Delta t\text{)}, \tag{9}$$

with,

V = 12 V,
I = 5 A,
Δt = 5 min = 300 s,
$\Rightarrow W_{\text{on wire}}$ = 18 kJ $\Rightarrow Q_{\text{from wire}}$ = 18 kJ.

Break the heat into the refrigerant into two heat components, one from the wire and one from the remainder of the surroundings,

$$Q_{\text{into R22}} = Q_{\substack{\text{into R22,}\\ \text{from wire}}} + Q_{\substack{\text{into R22,}\\ \text{from elsewhere}}}\,. \tag{10}$$

Substituting the expressions for heat, work, and energy into Eq. (1),

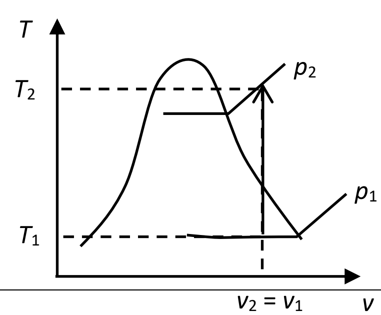
$$Q_{\substack{\text{into R22,}\\ \text{from elsewhere}}} = \Delta U - Q_{\substack{\text{into R22,}\\ \text{from wire}}}\,, \tag{11}$$

$$\Rightarrow Q_{\substack{\text{into R22,}\\ \text{from elsewhere}}} = \text{-5.54 kJ.}$$

Since we're interested in the heat from the refrigerant,

$$\boxed{Q_{\substack{\text{from R22,}\\ \text{into elsewhere}}} = -Q_{\substack{\text{into R22,}\\ \text{from elsewhere}}} = 5.54\text{ kJ.}} \tag{12}$$

The process and states are shown schematically in the following T-v plot.



SLVM Table for R22 (from Moran et al., 8th ed., Wiley).

TABLE A-7

Pressure Conversions:
1 bar = 0.1 MPa
= 10^2 kPa

Properties of Saturated Refrigerant 22 (Liquid–Vapor): Temperature Table

		Specific Volume m³/kg		Internal Energy kJ/kg		Enthalpy kJ/kg			Entropy kJ/kg · K		
Temp. °C	Press. bar	Sat. Liquid $v_f \times 10^3$	Sat. Vapor v_g	Sat. Liquid u_f	Sat. Vapor u_g	Sat. Liquid h_f	Evap. h_{fg}	Sat. Vapor h_g	Sat. Liquid s_f	Sat. Vapor s_g	Temp. °C
−60	0.3749	0.6833	0.5370	−21.57	203.67	−21.55	245.35	223.81	−0.0964	1.0547	−60
−50	0.6451	0.6966	0.3239	−10.89	207.70	−10.85	239.44	228.60	−0.0474	1.0256	−50
−45	0.8290	0.7037	0.2564	−5.50	209.70	−5.44	236.39	230.95	−0.0235	1.0126	−45
−40	1.0522	0.7109	0.2052	−0.07	211.68	0.00	233.27	233.27	0.0000	1.0005	−40
−36	1.2627	0.7169	0.1730	4.29	213.25	4.38	230.71	235.09	0.0186	0.9914	−36
−32	1.5049	0.7231	0.1468	8.68	214.80	8.79	228.10	236.89	0.0369	0.9828	−32
−30	1.6389	0.7262	0.1355	10.88	215.58	11.00	226.77	237.78	0.0460	0.9787	−30
−28	1.7819	0.7294	0.1252	13.09	216.34	13.22	225.43	238.66	0.0551	0.9746	−28
−26	1.9345	0.7327	0.1159	15.31	217.11	15.45	224.08	239.53	0.0641	0.9707	−26
−22	2.2698	0.7393	0.0997	19.76	218.62	19.92	221.32	241.24	0.0819	0.9631	−22
−20	2.4534	0.7427	0.0926	21.99	219.37	22.17	219.91	242.09	0.0908	0.9595	−20
−18	2.6482	0.7462	0.0861	24.23	220.11	24.43	218.49	242.92	0.0996	0.9559	−18
−16	2.8547	0.7497	0.0802	26.48	220.85	26.69	217.05	243.74	0.1084	0.9525	−16
−14	3.0733	0.7533	0.0748	28.73	221.58	28.97	215.59	244.56	0.1171	0.9490	−14
−12	3.3044	0.7569	0.0698	31.00	222.30	31.25	214.11	245.36	0.1258	0.9457	−12
−10	3.5485	0.7606	0.0652	33.27	223.02	33.54	212.62	246.15	0.1345	0.9424	−10
−8	3.8062	0.7644	0.0610	35.54	223.73	35.83	211.10	246.93	0.1431	0.9392	−8
−6	4.0777	0.7683	0.0571	37.83	224.43	38.14	209.56	247.70	0.1517	0.9361	−6
−4	4.3638	0.7722	0.0535	40.12	225.13	40.46	208.00	248.45	0.1602	0.9330	−4
−2	4.6647	0.7762	0.0501	42.42	225.82	42.78	206.41	249.20	0.1688	0.9300	−2
0	4.9811	0.7803	0.0470	44.73	226.50	45.12	204.81	249.92	0.1773	0.9271	0
2	5.3133	0.7844	0.0442	47.04	227.17	47.46	203.18	250.64	0.1857	0.9241	2
4	5.6619	0.7887	0.0415	49.37	227.83	49.82	201.52	251.34	0.1941	0.9213	4
6	6.0275	0.7930	0.0391	51.71	228.48	52.18	199.84	252.03	0.2025	0.9184	6
8	6.4105	0.7974	0.0368	54.05	229.13	54.56	198.14	252.70	0.2109	0.9157	8
10	6.8113	0.8020	0.0346	56.40	229.76	56.95	196.40	253.35	0.2193	0.9129	10
12	7.2307	0.8066	0.0326	58.77	230.38	59.35	194.64	253.99	0.2276	0.9102	12
16	8.1268	0.8162	0.0291	63.53	231.59	64.19	191.02	255.21	0.2442	0.9048	16
20	9.1030	0.8263	0.0259	68.33	232.76	69.09	187.28	256.37	0.2607	0.8996	20
24	10.164	0.8369	0.0232	73.19	233.87	74.04	183.40	257.44	0.2772	0.8944	24
28	11.313	0.8480	0.0208	78.09	234.92	79.05	179.37	258.43	0.2936	0.8893	28
32	12.556	0.8599	0.0186	83.06	235.91	84.14	175.18	259.32	0.3101	0.8842	32
36	13.897	0.8724	0.0168	88.08	236.83	89.29	170.82	260.11	0.3265	0.8790	36
40	15.341	0.8858	0.0151	93.18	237.66	94.53	166.25	260.79	0.3429	0.8738	40
45	17.298	0.9039	0.0132	99.65	238.59	101.21	160.24	261.46	0.3635	0.8672	45
50	19.433	0.9238	0.0116	106.26	239.34	108.06	153.84	261.90	0.3842	0.8603	50
60	24.281	0.9705	0.0089	120.00	240.24	122.35	139.61	261.96	0.4264	0.8455	60

v_f = (table value)/1000

SHV Table for R22 (from Moran et al., 8th ed., Wiley)

(Continued)

p = 2.5 bar = 0.25 MPa (T_{sat} = −19.51°C)

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
Sat.	0.09097	219.55	242.29	0.9586
−15	0.09303	222.03	245.29	0.9703
−10	0.09528	224.79	248.61	0.9831
−5	0.09751	227.55	251.93	0.9956
0	0.09971	230.33	255.26	1.0078
5	0.10189	233.12	258.59	1.0199
10	0.10405	235.92	261.93	1.0318
15	0.10619	238.74	265.29	1.0436
20	0.10831	241.58	268.66	1.0552
25	0.11043	244.44	272.04	1.0666
30	0.11253	247.31	275.44	1.0779
35	0.11461	250.21	278.86	1.0891
40	0.11669	253.13	282.30	1.1002

p = 3.0 bar = 0.30 MPa (T_{sat} = −14.66°C)

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
Sat.	0.07651	221.34	244.29	0.9502
−10	0.07833	223.96	247.46	0.9623
−5	0.08025	226.78	250.86	0.9751
0	0.08214	229.61	254.25	0.9876
5	0.08400	232.44	257.64	0.9999
10	0.08585	235.28	261.04	1.0120
15	0.08767	238.14	264.44	1.0239
20	0.08949	241.01	267.85	1.0357
25	0.09128	243.89	271.28	1.0472
30	0.09307	246.80	274.72	1.0587
35	0.09484	249.72	278.17	1.0700
40	0.09660	252.66	281.64	1.0811

p = 3.5 bar = 0.35 MPa (T_{sat} = −10.39°C)

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
Sat.	0.06605	222.88	246.00	0.9431
−10	0.06619	223.10	246.27	0.9441
−5	0.06789	225.99	249.75	0.9572
0	0.06956	228.86	253.21	0.9700
5	0.07121	231.74	256.67	0.9825
10	0.07284	234.63	260.12	0.9948
15	0.07444	237.52	263.57	1.0069
20	0.07603	240.42	267.03	1.0188
25	0.07760	243.34	270.50	1.0305
30	0.07916	246.27	273.97	1.0421
35	0.08070	249.22	227.46	1.0535
40	0.08224	252.18	280.97	1.0648
45	0.08376	255.17	284.48	1.0759

p = 4.0 bar = 0.40 MPa (T_{sat} = −6.56°C)

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
Sat.	0.05812	224.24	247.48	0.9370
−5	0.05860	225.16	248.60	0.9411
0	0.06011	228.09	252.14	0.9542
5	0.06160	231.02	225.66	0.9670
10	0.06306	233.95	259.18	0.9795
15	0.06450	236.89	262.69	0.9918
20	0.06592	239.83	266.19	1.0039
25	0.06733	242.77	269.71	1.0158
30	0.06872	245.73	273.22	1.0274
35	0.07010	248.71	276.75	1.0390
40	0.07146	251.70	280.28	1.0504
45	0.07282	254.70	283.83	1.0616

p = 4.5 bar = 0.45 MPa (T_{sat} = −3.08°C)

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
Sat.	0.05189	225.45	248.80	0.9316
0	0.05275	227.29	251.03	0.9399
5	0.05411	230.28	254.63	0.9529
10	0.05545	233.26	258.21	0.9657
15	0.05676	236.24	261.78	0.9782
20	0.05805	239.22	265.34	0.9904
25	0.05933	242.20	268.90	1.0025
30	0.06059	245.19	272.46	1.0143
35	0.06184	248.19	276.02	1.0259
40	0.06308	251.20	279.59	1.0374
45	0.06430	254.23	283.17	1.0488
50	0.06552	257.28	286.76	1.0600
55	0.06672	260.34	290.36	1.0710

p = 5.0 bar = 0.50 MPa (T_{sat} = 0.12°C)

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg · K
Sat.	0.04686	226.54	249.97	0.9269
5	0.04810	229.52	253.57	0.9399
10	0.04934	232.55	257.22	0.9530
15	0.05056	235.57	260.85	0.9657
20	0.05175	238.59	264.47	0.9781
25	0.05293	241.61	268.07	0.9903
30	0.05409	244.63	271.68	1.0023
35	0.05523	247.66	275.28	1.0141
40	0.05636	250.70	278.89	1.0257
45	0.05748	253.76	282.50	1.0371
50	0.05859	256.82	286.12	1.0484
55	0.05969	259.90	289.75	1.0595

SHV Table for R22 (from Moran et al., 8th ed., Wiley)

TABLE A-9

(Continued)

Pressure Conversions:
1 bar = 0.1 MPa
= 10^2 kPa

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg·K	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg·K
	p = 5.5 bar = 0.55 MPa (T_{sat} = 3.08°C)				p = 6.0 bar = 0.60 MPa (T_{sat} = 5.85°C)			
Sat.	0.04271	227.53	251.02	0.9226	0.03923	228.44	251.98	0.9186
5	0.04317	228.72	252.46	0.9278				
10	0.04433	231.81	256.20	0.9411	0.04015	231.05	255.14	0.9299
15	0.04547	234.89	259.90	0.9540	0.04122	234.18	258.91	0.9431
20	0.04658	237.95	263.57	0.9667	0.04227	237.29	262.65	0.9560
25	0.04768	241.01	267.23	0.9790	0.04330	240.39	266.37	0.9685
30	0.04875	244.07	270.88	0.9912	0.04431	243.49	270.07	0.9808
35	0.04982	247.13	274.53	1.0031	0.04530	246.58	273.76	0.9929
40	0.05086	250.20	278.17	1.0148	0.04628	249.68	277.45	1.0048
45	0.05190	253.27	281.82	1.0264	0.04724	252.78	281.13	1.0164
50	0.05293	256.36	285.47	1.0378	0.04820	255.90	284.82	1.0279
55	0.05394	259.46	289.13	1.0490	0.04914	259.02	288.51	1.0393
60	0.05495	262.58	292.80	1.0601	0.05008	262.15	292.20	1.0504

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg·K	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg·K
	p = 7.0 bar = 0.70 MPa (T_{sat} = 10.91°C)				p = 8.0 bar = 0.80 MPa (T_{sat} = 15.45°C)			
Sat.	0.03371	230.04	253.64	0.9117	0.02953	231.43	255.05	0.9056
15	0.03451	232.70	256.86	0.9229				
20	0.03547	235.92	260.75	0.9363	0.03033	234.47	258.74	0.9182
25	0.03639	239.12	264.59	0.9493	0.03118	237.76	262.70	0.9315
30	0.03730	242.29	268.40	0.9619	0.03202	241.04	266.66	0.9448
35	0.03819	245.46	272.19	0.9743	0.03283	244.28	270.54	0.9574
40	0.03906	248.62	275.96	0.9865	0.03363	247.52	274.42	0.9700
45	0.03992	251.78	279.72	0.9984	0.03440	250.74	278.26	0.9821
50	0.04076	254.94	283.48	1.0101	0.03517	253.96	282.10	0.9941
55	0.04160	258.11	287.23	1.0216	0.03592	257.18	285.92	1.0058
60	0.04242	261.29	290.99	1.0330	0.03667	260.40	289.74	1.0174
65	0.04324	264.48	294.75	1.0442	0.03741	263.64	293.56	1.0287
70	0.04405	267.68	298.51	1.0552	0.03814	266.87	297.38	1.0400

T °C	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg·K	v m³/kg	u kJ/kg	h kJ/kg	s kJ/kg·K
	p = 9.0 bar = 0.90 MPa (T_{sat} = 19.59°C)				p = 10.0 bar = 1.00 MPa (T_{sat} = 23.40°C)			
Sat.	0.02623	232.64	256.25	0.9001	0.02358	233.71	257.28	0.8952
20	0.02630	232.92	256.59	0.9013				
30	0.02789	239.73	264.83	0.9289	0.02457	238.34	262.91	0.9139
40	0.02939	246.37	272.82	0.9549	0.02598	245.18	271.17	0.9407
50	0.03082	252.95	280.68	0.9795	0.02732	251.90	279.22	0.9660
60	0.03219	259.49	288.46	1.0033	0.02860	258.56	287.15	0.9902
70	0.03353	266.04	296.21	1.0262	0.02984	265.19	295.03	1.0135
80	0.03483	272.62	303.96	1.0484	0.03104	271.84	302.88	1.0361
90	0.03611	279.23	311.73	1.0701	0.03221	278.52	310.74	1.0580
100	0.03736	285.90	319.53	1.0913	0.03337	285.24	318.61	1.0794
110	0.03860	292.63	327.37	1.1120	0.03450	292.02	326.52	1.1003
120	0.03982	299.42	335.26	1.1323	0.03562	298.85	334.46	1.1207
130	0.04103	306.28	343.21	1.1523	0.03672	305.74	342.46	1.1408
140	0.04223	313.21	351.22	1.1719	0.03781	312.70	350.51	1.1605
150	0.04342	320.21	359.29	1.1912	0.03889	319.74	358.63	1.1790

3.5.3. Enthalpy

The sum of the internal energy U and the pressure multiplied by the volume pV frequently appears in thermo-fluid analyses and so is given the special name, enthalpy, H,

$$\boxed{H \coloneqq U + pV}. \tag{3.59}$$

Note that enthalpy is a property since U, p, and V are also properties. On a per unit mass basis (making it an intensive property), the enthalpy becomes the specific enthalpy, h,

$$\boxed{h \coloneqq u + pv}. \tag{3.60}$$

Notes:

(1) It is important to note that the specific internal energies and specific enthalpies in the tables (along with the specific entropies, a property to be discussed later), are calculated with respect to a particular reference state. For example, the internal energy of water is defined to be zero at a saturated liquid state of 0.01 °C, i.e., $U_{\text{sat,H2O @ 0.01 °C}} \coloneqq 0$. Since most thermodynamic analyses involve calculating differences in properties, e.g., ΔU, the choice of reference state does not affect the results. For example, if a new reference state is chosen such that $U_{\text{new}} = U_{\text{table}} + c$ where U_{table} is the internal energy found in one of the tables and c is an arbitrary constant, then $\Delta U_{\text{new}} = U_{\text{new},2} - U_{\text{new},1} = (U_{\text{table},2} + c) - (U_{\text{table},1} + c) = U_{\text{table},2} - U_{\text{table},1} = \Delta U_{\text{table}}$. Thus, the choice of reference state is irrelevant.

(2) Property values for compressed liquids are often approximated using the saturated liquid property values at the corresponding temperature, i.e.,

$$v(T,p)_{CL} \approx v_l(T), \tag{3.61}$$

$$u(T,p)_{CL} \approx u_l(T). \tag{3.62}$$

These approximations may be made because in the compressed liquid region, the spacing between isobars is very small as one moves between specific volumes (and specific internal energies), as can be seen in Figure 3.25.

The specific enthalpy may be approximated as,

$$h(T,p)_{CL} \approx u_l(T) + pv_l(T). \tag{3.63}$$

Note that,

$$h_l(T) = u_l(T) + p_{\text{sat}}(T)v_l(T) \implies u_l(T) = h_l(T) - p_{\text{sat}}(T)v_l(T). \tag{3.64}$$

Substituting into Eq. (3.63) gives,

$$h(T,p)_{CL} \approx h_l(T) - p_{\text{sat}}(T)v_l(T) + pv_l(T), \tag{3.65}$$

$$h(T,p)_{CL} \approx h_l(T) + [p - p_{\text{sat}}(T)]v_l(T). \tag{3.66}$$

Determine the relative errors in calculating the specific volume, specific internal energy, and specific enthalpy for liquid water at a temperature and pressure of 100 °C and 100 bar, respectively, using thermodynamic property tables and using the saturated liquid state approximations.

SOLUTION:
From the thermodynamic property tables for liquid water at 100 °C and 100 bar (e.g., Table A-5, Moran et al., 7th ed.),

v = 1.0385*10^{-3} m^3/kg
u = 416.12 kJ/kg
h = 426.50 kJ/kg

Using the saturated liquid approximations (e.g., Table A-2, Moran et al., 7th ed.),

$v \approx 1.0435*10^{-3}$ m^3/kg at 100 °C
$u_l \approx 418.94$ kJ/kg at 100 °C
$h \approx 429.37$ kJ/kg at 100 °C, $h_l = 419.04$ kJ/kg and $p_{\text{sat},100\,°\text{C}} = 1.014$ bar

The relative error, ε, in a property, P, is,

$$\varepsilon_P = \frac{P_{\text{approx}} - P_{\text{actual}}}{P_{\text{actual}}}. \tag{1}$$

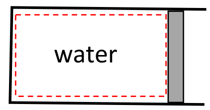
Thus, $\boxed{\varepsilon_v = 0.005,\ \varepsilon_u = 0.007, \text{ and } \varepsilon_h = 0.007}$. The error is less than 1% in all cases, implying that the approximations are good ones.

Five kg of water is contained in a piston-cylinder assembly, initially at 5 bar and 240 °C. The water is slowly heated at constant pressure to a final state. If the heat transfer into the water for the process is 2960 kJ, determine the temperature at the final state, in °C, and the work done by the water on the piston, in kJ. Kinetic and potential energy effects are negligible.

SOLUTION:

Apply the 1st Law to the water in the piston,

$$\Delta E_{\text{H2O}} = Q_{\substack{\text{into}\\ \text{H2O}}} - W_{\substack{\text{by}\\ \text{H20}}}\ , \tag{1}$$



where

$$\Delta E_{\text{H2O}} = \Delta U_{\text{H2O}} = m(u_2 - u_1). \tag{2}$$

The specific internal energy at the initial state (state 1) may be found from the thermodynamic tables for water at p_1 = 5 bar and T_1 = 240 °C. Note that saturation temperature for liquid water at 5 bar is 151.86 °C; hence, the water must be in the superheated vapor region since T_1 is greater than the saturation temperature. Using the table,

u_1 = 2707.6 kJ/kg,
v_1 = 0.4646 m^3/kg,
h_1 = 2939.9 kJ/kg.

The work done by the water on the piston is,

$$W_{\text{by H2O}} = \int_{V_1}^{V_2} p\,dV = p\left(V_2 - V_1\right) = pm\left(v_2 - v_1\right), \tag{3}$$

where the pressure is constant ($p_1 = p_2$ = 5 bar) throughout the process. Substituting into Eq.

$$m\left(u_2 - u_1\right) = Q_{\substack{\text{into}\\ \text{H2O}}} - pm\left(v_2 - v_1\right), \tag{4}$$

$$m\left(u_2 + p_2 v_2 - u_1 - p_1 v_1\right) = Q_{\substack{\text{into}\\ \text{H2O}}}\ , \tag{5}$$

$$m\left(h_2 - h_1\right) = Q_{\substack{\text{into}\\ \text{H2O}}}\ , \tag{6}$$

$$h_2 = h_1 + \frac{Q_{\substack{\text{into}\\ \text{H2O}}}}{m}\ , \tag{7}$$

where h is the specific enthalpy of the water. Substituting values,

h_1 = 2939.9 kJ/kg
Q_{into} = 2960 kJ
m = 5 kg
$\Rightarrow h_2$ = 3531.9 kJ/kg

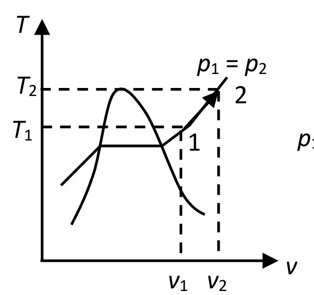
From the thermodynamic two-phase liquid-vapor table for water at p_2 = 5 bar, the saturated vapor specific enthalpy is 2748.7 kJ/kg, which is smaller than h_2. Hence, the water at state 2 will be in the superheated vapor state. Using the thermodynamic superheated vapor table with h_2 and p_2, and interpolating,

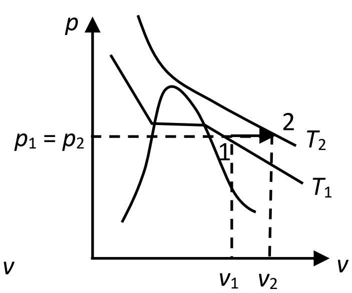
$\boxed{T_2 = 522\ ^\circ\text{C}}$
v_2 = 0.7314 m^3/kg

Using Eq. (3) and the values for v_1, v_2, m, and p,

$\boxed{W_{\text{by H2O}} = 667\ \text{kJ/kg}}$

Sketches of the process on T-v and p-v plots are shown.

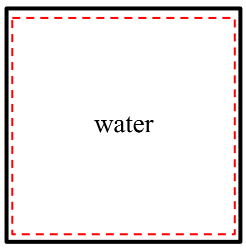




A closed, rigid tank is initially filled with 0.8 kg of water at 70 bar (abs) and a volume of 0.001 m^3 (state 1). Heat transfer occurs between the water and the surroundings until the pressure in the water is 35 bar (abs) (state 2).

a. Is the initial phase of the water a compressed liquid, saturated, or superheated vapor? Justify your answer.
b. Determine the specific internal energy at the initial state, in kJ/kg.
c. Calculate the specific volume at the final state, in m^3/kg.
d. Is the final phase of the water a compressed liquid, saturated, or superheated vapor? Justify your answer.
e. Determine the final specific internal energy of the water, in kJ/kg.
f. Determine the work done by the water during the process, in kJ.

SOLUTION:
The system is the water as shown in the following figure.



The specific volume at state 1 is,

$$v_1 = \frac{V}{m}, \tag{1}$$

where $V = 0.001$ m^3 and $m = 0.8$ kg. Hence,

$$v_1 = 1.25*10^{-3} \text{ m}^3/\text{kg}. \tag{2}$$

At a pressure of $p_1 = 70$ bar, the specific volume for a saturated liquid state is $1.3513*10^{-3}$ m^3/kg (from Table A-3 in Moran et al., 7th ed.). Since the specific volume at state 1 is smaller than this value, state 1 must be in a compressed liquid phase.

Since the water is in a compressed liquid state, we can use the saturated liquid value of the specific internal energy at the same temperature (recall that $u_{CL}(p,T) \approx u_l(T)$) to approximate the actual specific internal energy (found from Table A-3). Since the temperature isn't given we can estimate it from the specific volume. Recall that for a compressed liquid, $v_{CL}(p,T) \approx v_l(T)$. Thus, using Table A-2 in Moran et al., the temperature corresponding to $v_1 = v_{CL} = 1.25*10^{-3}$ m^3/kg is approximately $T = 250$ °C. The corresponding saturated liquid specific internal energy is,

$$\boxed{u_1 = 1080.4 \text{ kJ/kg}}. \tag{3}$$

The specific volume at state 2 will be identical to the specific volume at state 1 since the tank volume and water mass remain unchanged, i.e.,

$$\boxed{v_2 = v_1 = 1.25*10^{-3} \text{ m}^3/\text{kg}}. \tag{4}$$

At $p_2 = 35$ bar (abs), the specific volumes for the saturated liquid and saturated vapor states (Table A-3) are, respectively,

$$v_{l2} = 1.2347*10^{-3} \text{ m}^3/\text{kg}, \tag{5}$$

$$v_{v2} = 0.05707 \text{ m}^3/\text{kg}. \tag{6}$$

The specific volume for state 2 (Eq. (4)) falls between these two values. Hence, state 2 is in a saturated phase.

Since state 2 is in a saturated phase, the specific internal energy is found using the quality at state 2. The quality at state 2 can be found using the specific volume at state 2,

$$v_2 = x_2 v_{v2} + (1 - x_2) v_{l2} \Rightarrow x_2 = \frac{v_2 - v_{l2}}{v_{v2} - v_{l2}}, \tag{7}$$

$$x_2 = 2.74 * 10^{-4}. \tag{8}$$

The specific internal energy at the saturated liquid and saturated vapors states (Table A-3) is, respectively,

$$u_{l2} = 1045.4 \text{ kJ/kg}, \tag{9}$$

$$u_{v2} = 2603.7 \text{ kJ/kg}. \tag{10}$$

Hence, the specific internal energy at state 2 is,

$$u_2 = x_2 u_{v2} + (1 - x_2) u_{l2}, \tag{11}$$

$$\boxed{u_2 = 1045.8 \text{ kJ/kg}}. \tag{12}$$

Since the volume doesn't change during the process, there is no work done by the water,

$$\boxed{W_{\substack{\text{on}\\ \text{H}_2\text{O}}} = 0}. \tag{13}$$

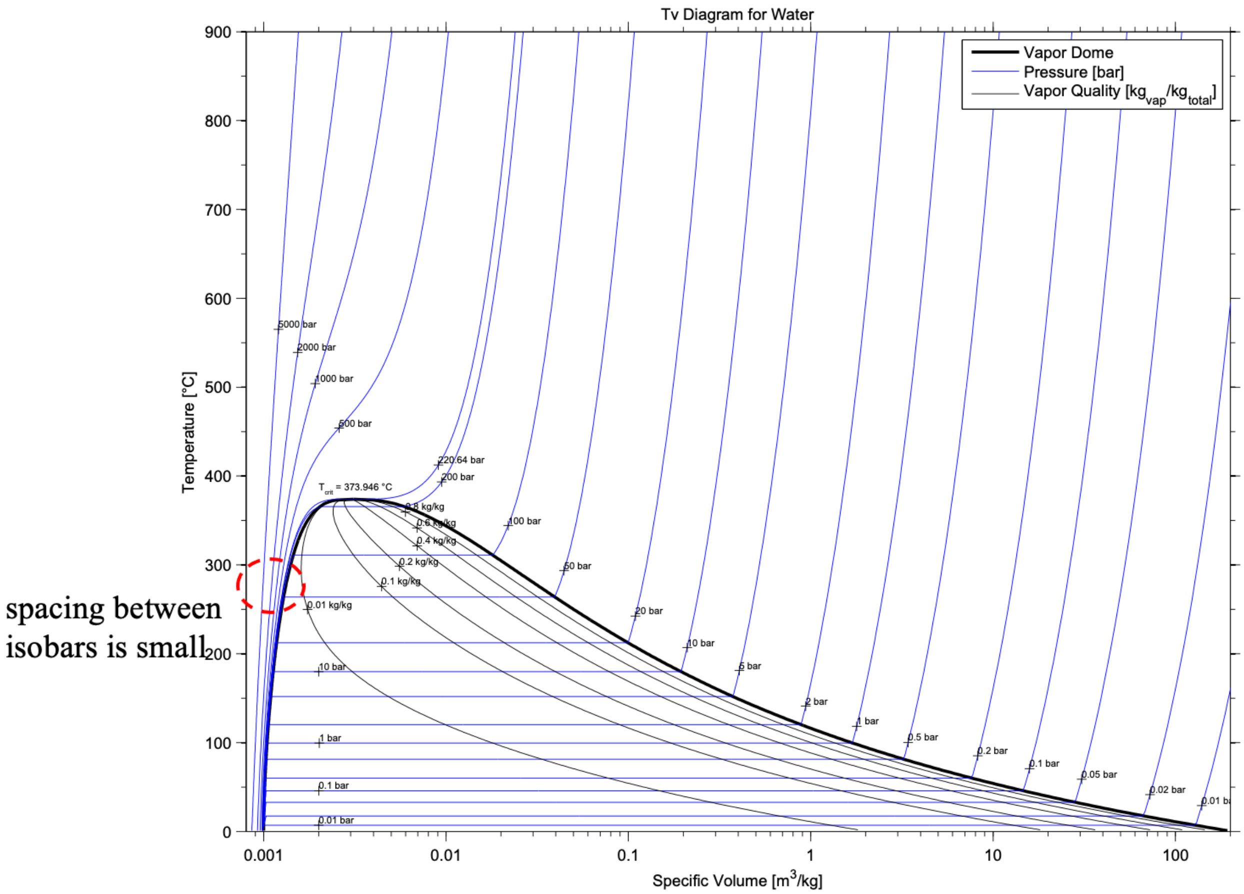


FIGURE 3.25. A T-v plot for water highlighting the close spacing of isobars in the compressed liquid region.

3.5.4. Specific Heat

Recall that internal energy, sensible energy in particular, is related to temperature. Furthermore, we know from experience that some materials heat up at different rates than others. For example, 4.5 kJ of energy added to a 1 kg mass of iron will raise the iron's temperature from 20 °C to 30 °C. To raise 1 kg of water from 20 °C to 30 °C, however, requires 41.8 kJ; about nine times the amount of energy than is required to raise the iron's temperature an equivalent amount.

The property that quantifies the energy storage capability of matter is called the specific heat (aka specific heat capacity). The specific heat of a substance is the energy required to raise the temperature of a unit mass of a substance by one degree. In general, the energy required to raise the temperature of a substance will depend on the process path. Two particular processes of interest are where the system's volume is held constant while energy is added and where the pressure in the system is held constant while energy is added.

The specific heat at constant volume, c_v, is the energy required to raise the temperature of a unit mass by one degree during a constant volume process. For a pure, simple compressible substance with $u(T, v)$, the specific heat at constant volume is defined as,

$$\boxed{c_v := \left.\frac{\partial u}{\partial T}\right|_v}, \tag{3.67}$$

where the subscript "v" indicates that the volume is held constant during differentiation.

The specific heat at constant pressure, c_p, is the energy required to raise the temperature of a unit mass by one degree during a constant pressure process. For a pure, simple compressible substance with $h(T, p)$, the specific heat at constant pressure is defined as,

$$\boxed{c_p := \left.\frac{\partial h}{\partial T}\right|_p}, \tag{3.68}$$

where the subscript "p" indicates that the pressure is held constant during differentiation.

Notes:

(1) Note that c_p is greater than c_v since at constant pressure, the system is allowed to expand and the energy for this expansion work must also be supplied to the system. The exception is for incompressible substances where the system does not expand and thus $c_p = c_v$.

(2) The ratio of the specific heats frequently appears in thermo-fluid analyses and is defined as the specific heat ratio, k,

$$\boxed{k := \frac{c_p}{c_v}}. \tag{3.69}$$

Since $c_p \geq c_v$, $k \geq 1$.

(3) In general, c_v and c_p are functions of temperature and pressure; however, for certain classes of substances, such as incompressible materials and ideal gases, the pressure dependence is negligible and only the temperature dependence is considered.

(4) The specific heat at constant volume (c_v) can be used in processes where the volume varies. Likewise, the specific heat at constant pressure (c_p) can be used in processes where the pressure varies. The terms "at constant volume" and "at constant pressure" are used only in the definitions of the specific heats. For example, consider the constant pressure process path shown in Figure 3.26. Although the process is at constant pressure, we can still evaluate c_v at different points along the path since at each point one can imagine a local constant volume process.

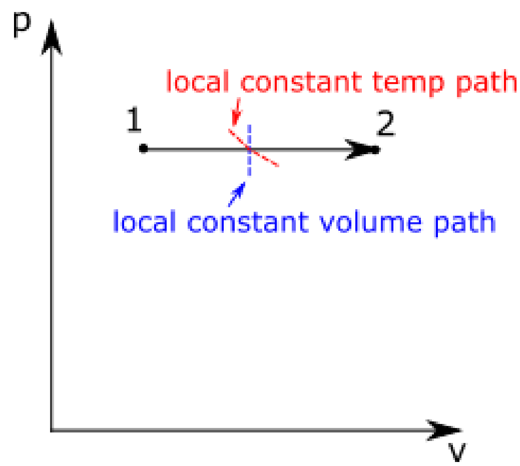


Figure 3.26. Illustration of a constant pressure process path. At each point along the path, one can locally evaluate the specific heat at constant volume, $c_v = \frac{\partial u}{\partial T}|_v$, even though the process is at constant pressure.

3.5.5. Incompressible Substances

Liquids and solids are often approximated in engineering applications as being incompressible, meaning that their specific volumes (or densities) remain constant, i.e., $v =$ constant. For example, referring to Figure 3.25, large changes in pressure produce very small changes in specific volume in the compressed liquid region. Similarly, since the isobars are nearly vertical in the compressed liquid region, large changes in temperature at a given pressure also only change the specific volume only slightly.

In addition to the specific volume assumption, it's a reasonable approximation to assume that the specific internal energy is a function only of temperature, i.e., $u(T)$, for incompressible substances. Recall that when using the compressed liquid tables we often use the approximation that $u_{CL}(T,p) \approx u_l(T)$, i.e., for compressed liquids the pressure plays little role in determining the specific internal energy. From Eq. (3.67),

$$c_v = \frac{du}{dT}, \tag{3.70}$$

where $u(T)$ and, thus, $c_v = c_v(T)$. In addition,

$$h(T,p) = u(T) + pv \implies c_p = \left.\frac{\partial h}{\partial T}\right|_p = \frac{du}{dT} = c_v. \tag{3.71}$$

Thus,

$$\boxed{c_p(T) = c_v(T) = c(T)}, \tag{3.72}$$

for an incompressible substance. Note that $c = c(T)$, in general.

Notes:

(1) Recall that the properties of liquids can be approximated using the corresponding saturated liquid properties (Eqs. (3.61), (3.62), and (3.66)). The saturated liquid specific internal energy is only a function of temperature, identical to what is noted in Eq. (3.70).

(2) The change in specific internal energy may be found by integrating Eq. (3.70),

$$c = \frac{du}{dT} \implies du = cdT, \tag{3.73}$$

$$u_2 - u_1 = \int_{T_1}^{T_2} c(T)dT. \tag{3.74}$$

The specific enthalpy may be found using its definition and the previous relation,

$$h_2 - h_1 = u_2 - u_1 + (p_2 - p_1)v, \tag{3.75}$$

$$h_2 - h_1 = \int_{T_1}^{T_2} c(T)dT + (p_2 - p_1)v, \tag{3.76}$$

where $v =$ constant. In applications where the change in temperature is small ("small" depends on the substance, but as a rough guide assume less than a few hundreds of Kelvin), the specific heat may be reasonably assumed to be constant and thus Eqs. (3.74) and (3.76) may be written as,

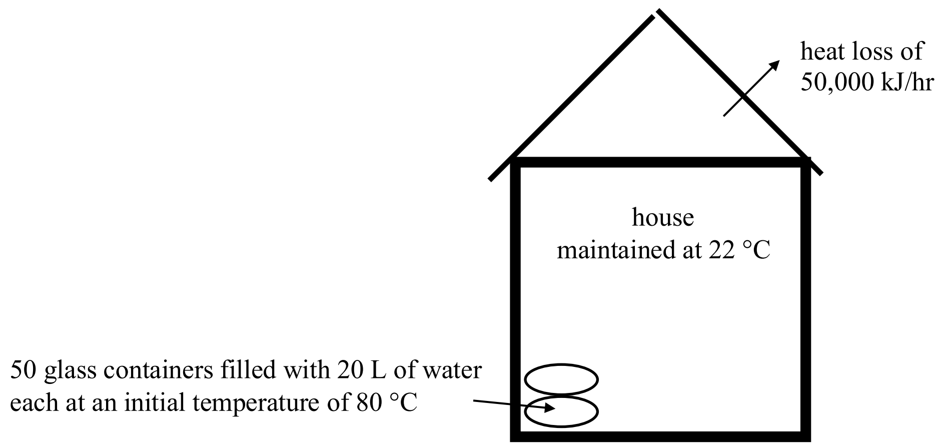
$$u_2 - u_1 = c(T_2 - T_1), \tag{3.77}$$

$$h_2 - h_1 = c(T_2 - T_1) + (p_2 - p_1)v. \tag{3.78}$$

(3) Values for the specific heat for liquids and solids are often tabulated in the back of textbooks (e.g., Table A-1 in Moran et al., 8th ed.) or can be found online.

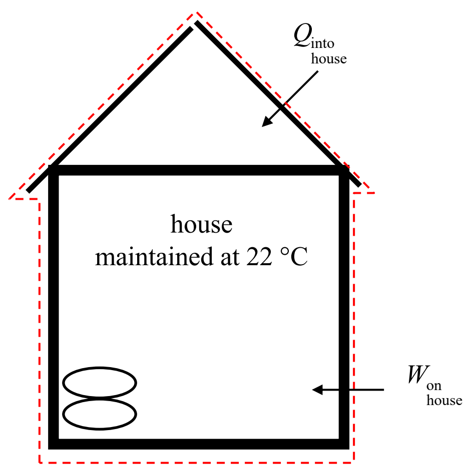
A passive solar house that is losing heat to the outdoors at an average rate of 50,000 kJ/hr is maintained at 22 °C at all times during a winter night for 10 hr. The house is to be heated by 50 glass containers each containing 20 L of water that is heated to 80 °C during the day by absorbing solar energy. A thermostat-controlled, 15 kW back-up electric resistance heater turns on whenever necessary to keep the house at 22 °C.

a. How long will the electric heating system need to run during the night?
b. How long would the electric heater run during the night if the house did not incorporate solar heating?



SOLUTION:

Apply the 1st Law to the house.



$$\Delta E_{\text{system}} = Q_{\substack{\text{into}\\\text{system}}} + W_{\substack{\text{on}\\\text{system}}} \tag{1}$$

The change in total energy of the house will consist of the change in the internal energy (potential and kinetic energy changes will be negligible). Furthermore, the total internal energy change will include the total energy change in the house structure, house air, and water tanks.

$$\Delta E_{\text{system}} = \Delta U_{\text{system}} = \Delta U_{\substack{\text{house}\\\text{structure}}} + \Delta U_{\text{air}} + \Delta U_{\text{water}} \tag{2}$$

Since the house structure and air are maintained at a constant temperature, $\Delta U_{\text{house}} = \Delta U_{\text{air}} = 0$. Hence, Eqn. (1) can be re-written as:

$$\Delta U_{\text{water}} = Q_{\substack{\text{into}\\\text{system}}} + W_{\substack{\text{on}\\\text{system}}} \tag{3}$$

The total change in the internal energy of the water (assuming an incompressible fluid) is given by:

$$\Delta U_{\text{water}} = m_{\text{water}} c_{\text{water}} \left(T_{f,\text{water}} - T_{i,\text{water}}\right) \tag{4}$$

The total heat added to the house is:

$$Q_{\substack{\text{into}\\ \text{system}}} = \left(-50,000 \text{ kJ/hr}\right)\left(10 \text{ hr}\right) = -500,000 \text{ kJ} \tag{5}$$

and the total work done on the house by the electric heater is:

$$W_{\substack{\text{on}\\ \text{system}}} = \left(15 \text{ kW}\right)\Delta t \tag{6}$$

where Δt is the time over which the heater operates.

Substitute Eqns. (4)-(6) into Eqn. (3).

$$m_{\text{water}} c_{\text{water}} \left(T_{f,\text{water}} - T_{i,\text{water}}\right) = -500,000 \text{ kJ} + \left(15 \text{ kW}\right)\Delta t \tag{7}$$

Using the given parameters in Eqn. (7).

m_{water} = 50(20 L)(0.001 m^3/L)(1000 kg/m^3) = 1000 kg
c_{water} = 4.179 kJ/(kg·K) (from a thermodynamics table)
$T_{f,\text{water}}$ = 22 °C
$T_{i,\text{water}}$ = 80 °C
$\Rightarrow$ $\boxed{\Delta t = 4.8 \text{ hrs}}$ Hence, the heater must be on for 4.8 hrs at night with the water tanks.

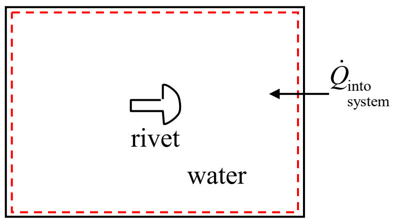
If the water containers were not present, then the left-hand side of Eqn (7) would be zero ($\Delta U_{\text{water}} = 0$) and:

$\Rightarrow$ $\boxed{\Delta t = 9.3 \text{ hrs}}$ Hence, the heater must be on for 9.3 hrs at night without the water tanks.

A steel rivet of mass 2 lb_m, initially at 1000 °F, is placed in a large tank containing 5 ft^3 of liquid water initially at 70 °F. Eventually, the rivet and water cool back to 70 °F as a result of heat transfer to the surroundings. Taking the rivet and water as the system, determine the heat transfer, in Btu, to the surroundings. The specific heat for steel is 0.11 Btu/(lb_m·°R).

SOLUTION:

Apply the 1st Law to the rivet/water system.



$$\Delta E_{\text{system}} = Q_{\substack{\text{into}\\\text{system}}} + W_{\substack{\text{on}\\\text{system}}} \tag{1}$$

Assuming that the tank is rigid, $W_{\text{on system}} = 0$. Furthermore, the change in the total energy of the system will be due solely to changes in the internal energy, *i.e.* $\Delta E_{\text{system}} = \Delta U_{\text{system}}$ where:

$$\begin{aligned}\Delta U_{\text{system}} &= \Delta U_{\text{rivet}} + \Delta U_{\text{water}} \\ &= m_{\text{rivet}} c_{\text{rivet}} \left(T_{f,\text{rivet}} - T_{i,\text{rivet}}\right) + m_{\text{water}} c_{\text{water}} \left(T_{f,\text{water}} - T_{i,\text{water}}\right)\end{aligned} \tag{2}$$

Substitute Eqn. (2) into Eqn. (1) and simplify.

$$Q_{\substack{\text{into}\\\text{system}}} = m_{\text{rivet}} c_{\text{rivet}} \left(T_{f,\text{rivet}} - T_{i,\text{rivet}}\right) + m_{\text{water}} c_{\text{water}} \left(T_{f,\text{water}} - T_{i,\text{water}}\right) \tag{3}$$

Since the final water temperature is the same as the initial water temperature, *i.e.* $T_{f,\text{water}} = T_{i,\text{water}}$, the change in the water internal energy will be zero. Hence,

$$Q_{\substack{\text{into}\\\text{system}}} = m_{\text{rivet}} c_{\text{rivet}} \left(T_{f,\text{rivet}} - T_{i,\text{rivet}}\right) \tag{4}$$

Using the given parameters.

$T_{f,\text{rivet}}$ = 70 °F
$T_{i,\text{rivet}}$ = 1000 °F
c_{rivet} = 0.11 Btu/(lb_m·°R)
m_{rivet} = 2 lb_m
⇒ $\boxed{Q_{\text{into system}} = \text{-205 Btu}}$ (205 Btu leave the system)

3.5.6. Ideal Gases

One particularly important class of substances that is the ideal gas. An ideal gas is a model describing the behavior of real gases in the limit of zero pressure and infinite temperature (i.e., zero density). It does not account for the interaction between molecules of the gas (e.g., inter-molecular forces). Nevertheless, the ideal gas model is a reasonably accurate model for gases where the system pressure is less than 0.05 times the critical pressure or the system temperature is more than twice the critical temperature (to be discussed).

3.5.6.1. Equation of State

An equation of state is a relationship between properties of a particular substance or class of substances. Equations of state cannot be obtained from thermodynamics but are obtained either from experimental measurements or from some molecular model. Note that there can be various types of equations of state, e.g., two equations of state for an ideal gas include a thermal equation of state which is the ideal gas law, $p = \rho RT$, and a caloric equation of state which describes the relationship between the internal energy and temperature, $du = c_v(T)dT$.

Thermal Equation of State
The thermal equation of state for an ideal gas is given by what is commonly referred to as the ideal gas law,

$$p = \rho RT \quad \text{or} \quad pv = RT \quad \text{or} \quad p\bar{v} = \bar{R}_U T \quad \text{or} \quad pV = mRT \quad \text{or} \quad pV = n\bar{R}_U T, \tag{3.79}$$

where p is the *absolute* pressure of the gas, ρ is the gas density ($= 1/v$, i.e., the inverse of the specific volume), R is the gas constant for the gas of interest, and T is the *absolute* temperature. The quantity $\bar{v}$ is the specific volume on a per mole basis, e.g., $[\bar{v}] = \text{m}^3/\text{mol}$, V is the volume, e.g., $[V] = \text{m}^3$, m is the mass, e.g., $[m] = \text{kg}$, and n is the amount of the substance, e.g., $[n] = \text{kmol}$. The quantity $\bar{R}_U$ is the universal gas constant (discussed in the following notes).

Notes:

(1) Absolute pressures and temperatures must be used when using the ideal gas law or anything derived from the ideal gas law.
(2) The gas constant, R, will be different for different gases. The gas constant can be determined in terms of the universal gas constant,

$$R = \frac{\bar{R}_U}{MW}, \tag{3.80}$$

where $\bar{R}_U = 8314\,\text{J/(kmol}\cdot\text{K)} = 1545.4\,(\text{ft}\cdot\text{lb}_\text{f})/(\text{lb}_\text{mol}\cdot{}^\circ\text{R}) = 1.986\,\text{Btu}/(\text{lb}_\text{mol}\cdot{}^\circ\text{R})$ and MW is the molecular weight of the gas. The gas constant for air is $R_\text{air} = 287\,\text{J/(kg}\cdot\text{K)}$ or, in English units, $R_\text{air} = 53.3\,(\text{ft}\cdot\text{lb}_\text{f})/(\text{lb}_\text{m}\cdot{}^\circ\text{R})$. Note that $MW_\text{air} = 28.98\,\text{kg}\,\text{kmol}^{-1}$.
(3) The compressibility factor, Z, is defined as,

$$Z \coloneqq \frac{pv}{RT} = \frac{p}{\rho RT}. \tag{3.81}$$

If $Z \approx 1$ for a gas, then it can be modeled well with the ideal gas model. The compressibility factor, Z, is plotted in Figure 3.27 for a variety of substances as a function of the reduced pressure, p/p_c, and reduced temperature, T/T_c, where p_c and T_c are the critical pressure and temperature for the substance. Note that the critical temperature is the temperature above which a gas cannot be liquefied no matter how large a pressure is applied. The critical pressure is the minimum pressure for liquefying a gas at the critical temperature.
   (a) For values of $p/p_c < 0.05$ or $T/T_c > 2$, $Z \approx 1$ so in this range the ideal gas model works well.
   (b) Table 3.1 lists the critical temperature and pressure values for various substances.
(4) One should be careful in treating superheated water vapor, i.e., steam, as an ideal gas. Figure 3.28 shows a T - v plot for water. The points in the plot are the relative error between a specific volume found using the thermodynamic property table, which is very accurate, and the specific volume using the Ideal Gas Law, which is approximate ($[|v_\text{table} - v_\text{ideal}|/v_\text{table}] * 100\%$). The shaded area in

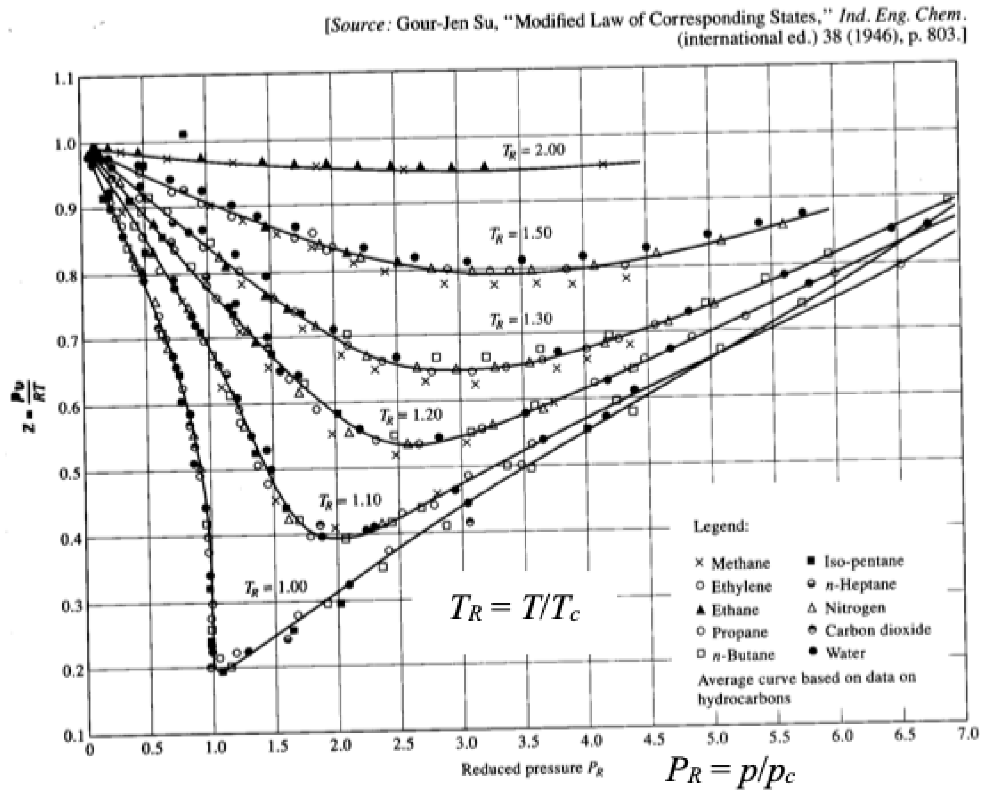


Figure 3.27. A plot of the compressibility factor Z as a function of the reduced pressure $P_R = p/p_c$ for different reduced temperatures $T_R = T/T_c$.

Table 3.1. Critical temperatures and critical pressures for various gases.

Gas	T_c [K]	p_c [atm]
air	132.41	37.25
He	5.19	2.26
H_2	33.24	12.80
N_2	126.2	33.54
O_2	154.78	50.14
CO_2	304.20	72.90
CO	132.91	34.26

the plot is the region where the relative error is less than 1%, i.e., the region where steam can be reasonably modeled using the Ideal Gas Law. As a rule of thumb, it's better to use the property tables for water in the superheated vapor region rather than assuming ideal gas behavior.

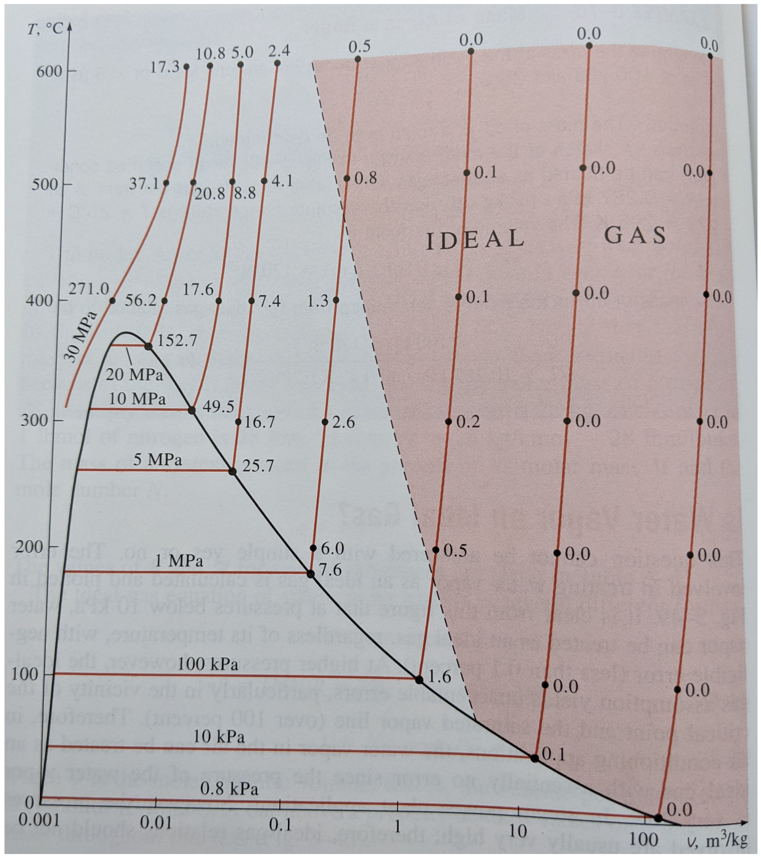


Figure 3.28. A T - v plot for water. The numbers on the isobars indicate the relative error in the specific volume found from the table vs. the Ideal Gas Law, i.e., $[|v_{\text{table}} - v_{\text{ideal}}|/v_{\text{table}}] *$ 100%. The shaded error is the region where the relative error is less than 1%, i.e., using the Ideal Gas Law in this region will give an accurate prediction of the specific volume. The figure is Figure 3.49 from Çengel, Y.A. and Boles, M.A., *Thermodynamics - An Engineering Approach*, 5th ed., McGraw-Hill.

A tank contains 0.5 m^3 of nitrogen (N_2) at -71 °C and 1356 kPa (abs). Determine the mass of nitrogen, in kg, using,

a. the ideal gas model, and
b. data from the compressibility chart.

SOLUTION:

Using the ideal gas model,

$$p\bar{v} = \bar{R}T \quad \Rightarrow \quad \bar{v} = \frac{\bar{R}T}{p}, \tag{1}$$

Using the given data,

p = 1356*10^3 Pa
T = -71 °C = 202 K
$\bar{R}$ = 8314 J/(kmol.K)
$\Rightarrow$ $\bar{v}$ = 1.23 m^3/kmol

The molecular weight of N_2 is M_{N2} = 28.01 kg/kmol. Hence,

$$v = \frac{\bar{v}}{M} \quad \Rightarrow \quad v = 0.0442 \text{ m}^3\text{/kg} \tag{2}$$

The mass may be found from the specific volume and the volume of the tank,

$$m = \frac{V}{v} \quad \Rightarrow \quad \boxed{m = 11.31 \text{ kg}} \tag{3}$$

using V = 0.5 m^3.

In order to use the compressibility chart, we must first calculate the reduced temperature and reduced pressure. Note that the critical temperature and critical pressure for N_2 are T_c = 126 K and p_c = 33.9 bar (= 33.9*10^5 Pa). Hence,

$$T_R = \frac{T}{T_c} \quad \Rightarrow \quad T_R = 1.603, \tag{4}$$

$$p_R = \frac{p}{p_c} \quad \Rightarrow \quad p_R = 0.400. \tag{5}$$

Using the compressibility chart (e.g., Figure A-1 from Moran et al., 7$^{\text{th}}$ ed.) with these reduced values,

Z = 0.98. (6)

Hence, the specific volume using the compressibility chart will be 0.98 times the specific volume using the ideal gas law (where Z = 1, Eq. (2)),

v = 0.0433 m^3/kg. (7)

Thus, from Eq. (3), the mass is,

$\boxed{m = 11.54 \text{ kg}}$ (8)

The relative error in using the ideal gas model is,

$$\varepsilon = \frac{m_{\text{ideal gas}} - m_{\text{compressibility}}}{m_{\text{compressibility}}}, \tag{9}$$

ε = 0.020. (10)

Assuming ideal gas behavior only results in a 2% error in the calculation of the mass, which is sufficiently small for most engineering calculations.

Caloric Equation of State
Now let's return to a discussion of the caloric equation of state for an ideal gas. Since an ideal gas is considered a simple, compressible system, the internal energy, u, is uniquely determined by two properties. Here we'll use the properties of temperature, T, and specific volume, $v(= 1/\rho)$,

$$u = u(T, v), \tag{3.82}$$

so that any change in the internal energy is given by,

$$du = \left.\frac{\partial u}{\partial T}\right|_v dT + \left.\frac{\partial u}{\partial v}\right|_T dv. \tag{3.83}$$

Utilizing our definition for specific heat given in Eq. (3.67) we have,

$$du = c_v dT + \left.\frac{\partial u}{\partial v}\right|_T dv. \tag{3.84}$$

The second term on the right-hand side of Eq. (3.84) is zero for an ideal gas. (This can be shown using Maxwell's relations, a topic not addressed in these notes. See for example, Moran and Shapiro, Fundamentals of Engineering Thermodynamics, 3rd ed., Wiley, Section 11.4.2.) Thus, the internal energy of an ideal gas, u, is a function only of the temperature,

$$u = u(T) \implies du = \frac{du}{dT} dT. \tag{3.85}$$

Using the definition given in Eq. (3.67) we have,

$$du = c_v(T)dT. \tag{3.86}$$

Integrating both sides and noting that the specific heat can be a function of temperature in general,

$$u - u_{\text{ref}} = \int_{T_{\text{ref}}}^{T} c_v(T)dT, \tag{3.87}$$

where the subscript "ref" indicates some reference state. Evaluating at two different states,

$$u_2 - u_1 = \int_{T_1}^{T_2} c_v(T)dT. \tag{3.88}$$

The specific enthalpy is also only a function of temperature for an ideal gas as shown below,

$$h = u + \frac{p}{\rho} = u + RT \implies h = h(T). \tag{3.89}$$

Taking the derivative of the specific enthalpy with respect to the temperature and using the definition given in Eq. (3.68) we see that,

$$dh = \frac{dh}{dt} dT = c_p(T)dT, \tag{3.90}$$

$$h - h_{\text{ref}} = \int_{T_{\text{ref}}}^{T} c_p(T)dT, \tag{3.91}$$

or,

$$h_2 - h_1 = \int_{T_1}^{T_2} c_p(T)dT, \tag{3.92}$$

where c_p is the specific heat at constant pressure which can, in general, be a function of temperature and the subscript "ref" indicates some reference state. In addition,

$$dh = du + RdT = c_v dT + RdT, \tag{3.93}$$

$$= (c_v + R)dT. \tag{3.94}$$

Comparing Eqs. (3.90) and (3.94) we see that the specific heats for an ideal gas are related in the following manner,

$$\boxed{c_p = c_v + R}. \tag{3.95}$$

The specific heat ratio, k, defined as,

$$\boxed{k := \frac{c_p}{c_v}}, \tag{3.96}$$

appears frequently in thermo-fluid systems. Other helpful relations that can be derived by combining Eqs. (3.95) and (3.96) include,

$$c_p = \frac{kR}{k-1}, \tag{3.97}$$

$$c_v = \frac{R}{k-1}. \tag{3.98}$$

Notes:

(1) When the temperature change for a process is sufficiently small (small depends on the substance, but a few hundred Kelvin is a good rule of thumb), the dependence of c_v and c_p on temperature can be reasonably neglected, i.e., $c_v = \text{constant}_1$ and $c_p = \text{constant}_2$. As a result, Eqs. (3.88) and (3.92) may be written as,

$$u_2 - u_1 = c_v(T_2 - T_1), \tag{3.99}$$

$$h_2 - h_1 = c_p(T_2 - T_1). \tag{3.100}$$

An ideal gas with constant specific heats is referred to as a perfect gas.

(2) The values for c_v and c_p as a function of temperature have been tabulated for various gases in the back of some textbooks, e.g., Table A-20 in Moran et al., 8th ed.

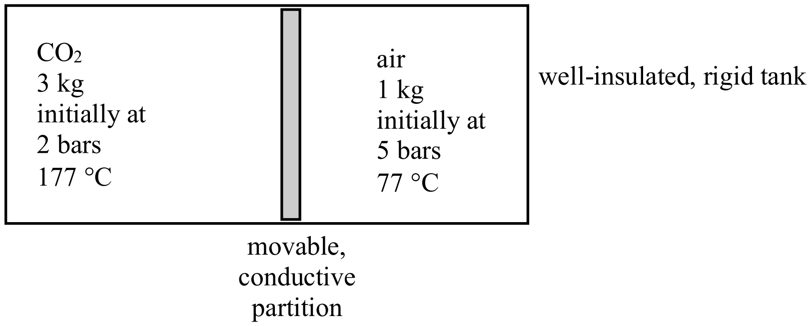
(3) Since air is a frequently used substance in engineering, the values for several properties, such as specific internal energy and specific enthalpy, taking into account the temperature dependence of c_v and c_p have been tabulated in some textbooks, e.g., Table A-22 in Moran et al., 8th ed.

(4) Plots of the variation in the specific heat at constant pressure made dimensionless by the universal gas constant ($\bar{c}_p/\bar{R}_U$) for various gases are shown in Figure 3.29. Note that for air the specific heat does not vary more than approximately 12% over a few hundred Kelvin in the range $T > 270\,\text{K}$. Hence, a perfect gas assumption for air flows in this range and a temperature change of less than a few hundred Kelvin is reasonable. Obviously it would be more accurate to account for the variation due to temperature. The specific heat temperature dependence can be predicted from statistical mechanics models. The variations in specific heat are due to the activation of different energy storage modes (e.g., vibration and rotation) within the gas molecules at different temperatures. More detail on this topic can be found in Callen, H.B., *Thermodynamics and an Introduction to Thermostatistics*, Wiley.

One kilogram of air, initially at 5 bars (abs) and 77 °C, and 3 kg of carbon dioxide (CO_2), initially at 2 bars (abs) and 177 °C, are confined to opposite sides of a rigid, well-insulated container. The partition is free to move and allows conduction from one gas to the other without energy storage in the partition itself. Determine:

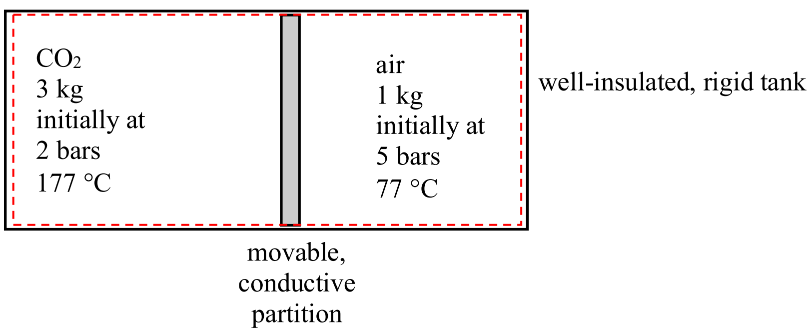
a) the final equilibrium temperature
b) and the final pressure

You may assume that the specific heats for both the air and CO_2 remain constant over the range of temperatures: $c_{v,\text{air}}$=0.726 kJ/(kg·K); $c_{v,CO2}$=0.750 kJ/(kg·K); $c_{p,\text{air}}$=1.013 kJ/(kg·K); $c_{p,CO2}$=0.939 kJ/(kg·K)



SOLUTION:

Apply the 1st Law to the following CV.



$$\Delta E_{\text{CV}} = Q_{\substack{\text{into}\\ \text{CV}}} + W_{\substack{\text{on}\\ \text{CV}}} \tag{1}$$

where

$$\Delta E_{\text{CV}} = \Delta U_{\text{CV}} = U_f - U_i = m_{CO_2}\left(u_{CO_2,f} - u_{CO_2,i}\right) + m_{\text{air}}\left(u_{\text{air},f} - u_{\text{air},i}\right)$$

$$Q_{\substack{\text{into}\\ \text{CV}}} = 0 \quad \text{(well-insulated tank)}$$

$$W_{\substack{\text{on}\\ \text{CV}}} = 0 \quad \text{(rigid tank)}$$

Assuming constant specific heats (perfect gases) and simplifying COE gives:

$$m_{CO_2} c_{v,CO_2}\left(T_{CO_2,f} - T_{CO_2,i}\right) + m_{\text{air}} c_{v,\text{air}}\left(T_{\text{air},f} - T_{\text{air},i}\right) = 0$$

Since the partition is conductive, $T_{CO_2,f} = T_{\text{air},f} = T_f$ resulting in:

$$m_{CO_2} c_{v,CO_2}\left(T_f - T_{CO_2,i}\right) + m_{\text{air}} c_{v,\text{air}}\left(T_f - T_{\text{air},i}\right) = 0$$

$$\boxed{\therefore T_f = \frac{m_{CO_2} c_{v,CO_2} T_{CO_2,i} + m_{\text{air}} c_{v,\text{air}} T_{\text{air},i}}{m_{CO_2} c_{v,CO_2} + m_{\text{air}} c_{v,\text{air}}}} \tag{2}$$

Using the given values:

m_{CO2} = 3 kg
$c_{v,CO2}$ = 0.750 kJ/(kg·K)
$T_{CO2,i}$ = 177 °C = 450 K
m_{air} = 1 kg
$c_{v,air}$ = 0.726 kJ/(kg·K)
$T_{air,i}$ = 77 °C = 350 K
$\Rightarrow$ $\boxed{T_f = 426\text{ K} = 153\text{ °C}}$

The final pressure in each compartment will be the same otherwise the partition would continue to move. Use the ideal gas law to determine the final densities of the gases in terms of the final temperature and pressure.

$$p = \rho RT \Rightarrow \rho = \frac{p}{RT} \tag{3}$$

In addition, the total volume of the tank is the sum of the final volumes occupied by each gas.

$$V_{\text{tank}} = \underbrace{\frac{m_{CO_2}}{\rho_{CO_2,f}}}_{=V_{CO_2,f}} + \underbrace{\frac{m_{\text{air}}}{\rho_{\text{air},f}}}_{=V_{\text{air},f}} \tag{4}$$

Combine Eqns. (3) and (4) and simplify.

$$V_{\text{tank}} = m_{CO_2}\frac{R_{CO_2}T_f}{p_f} + m_{\text{air}}\frac{R_{\text{air}}T_f}{p_f}$$

$$p_f = m_{CO_2}\frac{R_{CO_2}T_f}{V_{\text{tank}}} + m_{\text{air}}\frac{R_{\text{air}}T_f}{V_{\text{tank}}}$$

$$\boxed{p_f = \frac{T_f}{V_{\text{tank}}}\left(m_{CO_2}R_{CO_2} + m_{\text{air}}R_{\text{air}}\right)} \tag{5}$$

The tank volume is known from the initial masses, pressures, and temperatures.

$$\boxed{V_{\text{tank}} = \underbrace{\frac{m_{CO_2}}{\rho_{CO_2,i}}}_{=V_{CO_2,i}} + \underbrace{\frac{m_{\text{air}}}{\rho_{\text{air},i}}}_{=V_{\text{air},i}} = m_{CO_2}\frac{R_{CO_2}T_{CO_2,i}}{p_{CO_2,i}} + m_{\text{air}}\frac{R_{\text{air}}T_{\text{air},i}}{p_{\text{air},i}}} \tag{6}$$

Using the given data:

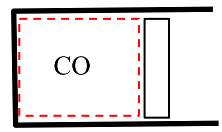
m_{CO2} = 3 kg
R_{CO2} = 0.1889 kJ/(kg·K)
$T_{CO2,i}$ = 177 °C = 450 K
$p_{CO2,i}$ = 2 bar = 202.7 kPa
m_{air} = 1 kg
R_{air} = 0.287 kJ/(kg·K)
$T_{air,i}$ = 77 °C = 350 K
$p_{air,i}$ = 5 bar = 506.63 kPa
T_f = 426 K (from previous part of the problem)
$\Rightarrow$ $\boxed{V_{\text{tank}} = 1.46\text{ m}^3}$ and $\boxed{p_f = 250\text{ kPa} = 2.46\text{ bar}}$

A piston-cylinder assembly contains carbon monoxide modeled as an ideal gas with a constant specific heat ratio of $k = 1.4$. The carbon monoxide undergoes a polytropic expansion with $n = k$ from an initial state, where the temperature is 200 °F and pressure of 40 psia, to a final state where the volume is twice the initial volume. Determine:

a. the final temperature, in °F, and final pressure, in psia, and
b. the work done by the gas and heat transfer into the gas, each in Btu/lb_m.

SOLUTION:

Treat the CO as an ideal gas. Hence,



$$pV = mRT \quad \text{or} \quad pv = RT \,. \tag{1}$$

Since the process is polytropic,

$$pV^n = c \;\text{ or }\; p = \frac{c}{V^n} \;\Rightarrow\; \frac{p_2}{p_1} = \left(\frac{V_1}{V_2}\right)^n , \tag{2}$$

where $n = k = 1.4$, for this case, and c is a constant, which can be determined from the initial state. Note that the mass of CO remains constant, so combining Eqs. (1) and (2) gives,

$$\frac{p_2}{p_1}\frac{V_2}{V_1} = \frac{T_2}{T_1} \Rightarrow \left(\frac{V_1}{V_2}\right)^n \frac{V_2}{V_1} = \frac{T_2}{T_1} \Rightarrow \left(\frac{V_1}{V_2}\right)^{n-1} = \frac{T_2}{T_1}. \tag{3}$$

Using the given parameters and Eqs. (2) and (3),

p_1 = 40 psia
T_1 = 200 °F = 660 °R
n = k = 1.4
V_2 = $2V_1$
$\Rightarrow$ $\boxed{p_2 = 15.16 \text{ psia}}$, $\boxed{T_2 = 500.2\ °\text{R} = 40.19\ °\text{F}}$

The work done by the gas may be found using,

$$W_{\text{by gas}} = \int_{V_1}^{V_2} p\,dV = c\int_{V_1}^{V_2} V^{-n}\,dV = \frac{c}{1-n}\left(V_2^{1-n} - V_1^{1-n}\right) = \frac{p_1V_1^n}{1-n}\left(V_2^{1-n} - V_1^{1-n}\right) \quad (n \neq 1), \tag{4}$$

$$W_{\text{by gas}} = \frac{p_1 m^n v_1^n}{1-n} m^{1-n}\left(v_2^{1-n} - v_1^{1-n}\right), \tag{5}$$

$$\frac{W_{\text{by gas}}}{m} = \frac{p_1 v_1^n}{1-n}\left(v_2^{1-n} - v_1^{1-n}\right). \tag{6}$$

where the initial conditions have been used to determine the constant c. The specific volume v_2 may be found using Eq. (1),

$$v = \frac{RT}{p}. \tag{7}$$

Using the given data,

M_{CO} = 28.01 lb_m/lb_mmol
$\bar{R}_{CO}$ = 1.986 Btu/(lb_mmol.°R)
p_1 = 40 psia = 5760 $lb_m/(ft.s^2)$
T_1 = 660 °R
v_2 = $2v_1$
n = k = 1.4
$\Rightarrow$ R_{CO} = 0.07090 Btu/(lb_m.°R), $v_1 = 8.124 \times 10^{-3}$ ft^3/lb_m, $v_2 = 1.625 \times 10^{-2}$ ft^3/lb_m
$\Rightarrow$ $\boxed{W_{\text{by gas}}/m = 28.33 \text{ Btu/lb}_m}$.

The heat added to the gas may be found using the 1st Law applied to the gas,

$$\underbrace{\Delta E_{\text{sys}}}_{=\Delta U_{\text{sys}}} = Q_{\text{into sys}} - W_{\text{by sys}} \;\Rightarrow\; m\Delta u_{\text{sys}} = Q_{\text{into sys}} - W_{\text{by sys}} \;\Rightarrow\; \frac{Q_{\text{into sys}}}{m} = \Delta u_{\text{sys}} + \frac{W_{\text{by sys}}}{m}, \tag{8}$$

where kinetic and potential energies have been neglected. The change in specific internal energy of the gas, assuming ideal gas behavior, may be found either through ideal gas tables at the appropriate temperatures (e.g., Table A-23E of Moran et al., 7th ed.),

$\bar{u}_1$ = 3275.8 Btu/lb$_m$mol (at T_1 = 660 °R) $\Rightarrow$ u_1 = 117.0 Btu/lb$_m$

$\bar{u}_2$ = 2479.2 Btu/lb$_m$mol (at T_2 = 500 °R) $\Rightarrow$ u_2 = 88.51 Btu/lb$_m$

Note that $u = \bar{u}/M$ where M_{CO} = 28.01 lb$_m$/lb$_m$mol. Hence, from Eq. (8) and the previously calculated value for specific work,

$\boxed{Q_{\text{into gas}}/m = \text{-0.16 Btu/lb}_m}$.

If we instead assume that the CO behaves as a perfect gas (an ideal gas with constant specific heats),

$$u_2 - u_1 = c_v\left(T_2 - T_1\right), \tag{9}$$

where c_v = 0.178 Btu/(lb$_m$.°R) (from Table A-20E in Moran et al., 7th ed., at a temperature of 100 °F). Thus,

$Q_{\text{into gas}}/m$ = -0.11 Btu/lb$_m$.

We get approximately the same result using either method.

It can be shown that the heat transfer is, in fact, identically zero for a polytropic expansion of an ideal gas when $n = k$, as is the case here. Combining Eqs. (6) and (7) gives,

$$\frac{W_{\text{by gas}}}{m} = \frac{p_1 v_1^n}{1-n}\left(v_2^{1-n} - v_1^{1-n}\right) = \frac{\left(RT_1/v_1\right)v_1^n}{1-n}\left(v_2^{1-n} - v_1^{1-n}\right) = \frac{RT_1 v_1^{n-1}}{1-n}\left(v_2^{1-n} - v_1^{1-n}\right) = \frac{RT_1}{1-n}\left[\left(\frac{v_2}{v_1}\right)^{1-n} - 1\right], \tag{10}$$

$$\frac{W_{\text{by gas}}}{m} = \frac{RT_1}{1-n}\left[\left(\frac{v_2}{v_1}\right)^{1-n} - 1\right] = \frac{RT_1}{1-n}\left[\left(\frac{v_2}{v_1}\right)\left(\frac{v_2}{v_1}\right)^{-n} - 1\right] = \frac{RT_1}{1-n}\left[\left(\frac{v_2}{v_1}\right)\left(\frac{p_2}{p_1}\right) - 1\right], \tag{11}$$

where Eq. (2) has been used in the last step. Continuing to simplify,

$$\frac{W_{\text{by gas}}}{m} = \frac{RT_1}{1-n}\left[\left(\frac{RT_2}{RT_1}\right) - 1\right] = \frac{R}{1-n}\left(T_2 - T_1\right). \tag{12}$$

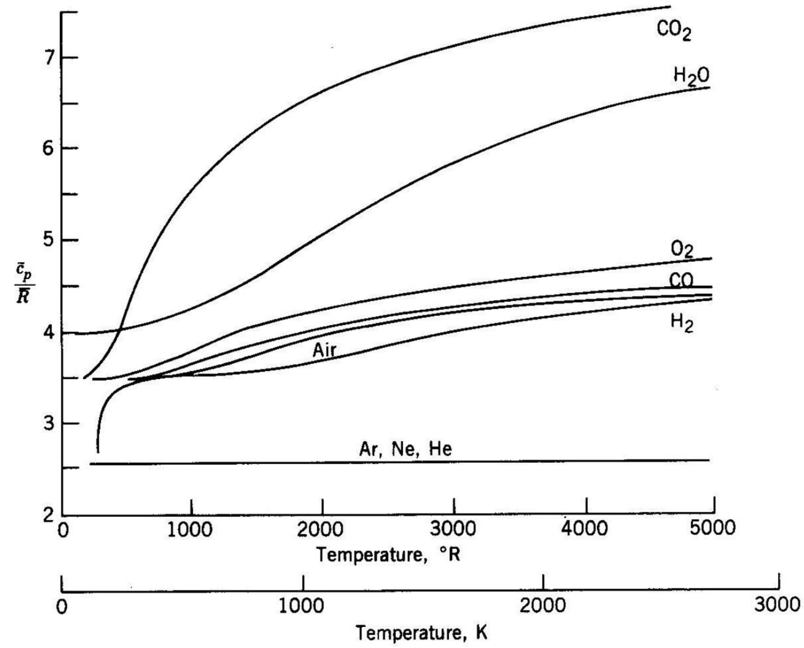
Since we're told that $n = k$ for this polytropic process,

$$\frac{W_{\text{by gas}}}{m} = \frac{R}{1-k}\left(T_2 - T_1\right) = c_v\left(T_2 - T_1\right), \tag{13}$$

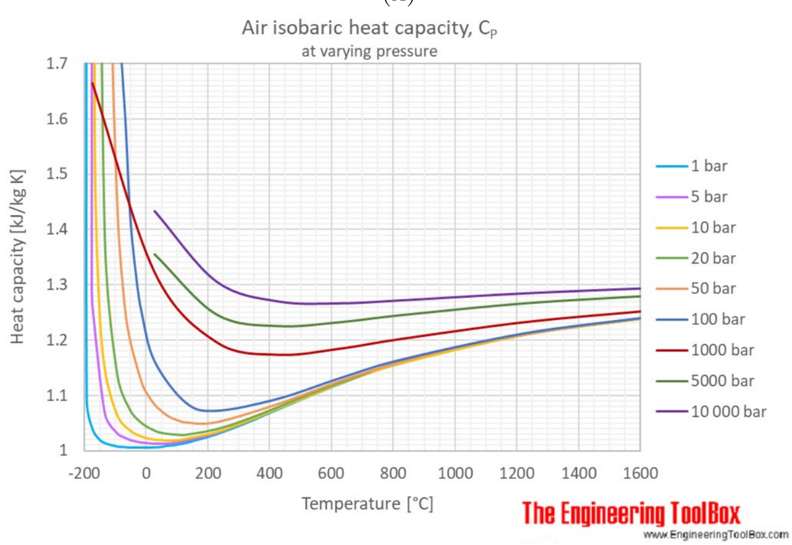
where the relationship between the specific heat at constant volume, the gas constant, and the specific heat ratio has been used for an ideal gas. Note that the right hand side of this expression is the change in the specific internal energy, assuming a perfect gas (ideal gas with constant specific heats),

$$\frac{W_{\text{by gas}}}{m} = u_2 - u_1. \tag{14}$$

Thus, from the 1st Law, we must have $Q = 0$. Our previous answers were close to zero, but there is some numerical error. Plus, the $Q = 0$ result assumes constant specific heats, which isn't exactly true.



(A)



(B)

Figure 3.29. Plots of (A) $\bar{c}_p/\bar{R}_U$ as a function of temperature for various gases. (Figure from ???.) (B) c_p as a function of temperature for air at various pressures. (Figure from *The Engineering ToolBox*.)

3.6. The Second Law of Thermodynamics

The First Law must be satisfied for a process to occur, but it doesn't indicate if the process will occur. For example, consider a system consisting of a block sliding down an inclined surface under the action of gravity. The block is initially at rest and is at rest at the final state so the change in kinetic energy is zero. Assume there are also no heat or work interactions with the surroundings. In this scenario,

$$\Delta U + mg\Delta z = 0 \implies \Delta U = -mg\Delta z. \tag{3.101}$$

Hence, if the block moves down the inclined plane ($\Delta z < 0$), the block's internal energy increases ($\Delta U > 0$). This scenario is reasonable based on our experience with blocks on planes. However, the First Law also states that if the block moves up the plane ($\Delta z > 0$), then the internal energy of the block would decrease ($\Delta U < 0$). We never see this process occurring spontaneously in practice, but the First Law doesn't preclude it from happening.

The Second Law of Thermodynamics is frequently used to predict the direction or possibility of a process, such as in the case of the block on an inclined plane described previously. In addition, the Second Law is used for other purposes, including,

- establishing conditions for equilibrium,
- establishing theoretical limits of a process, and
- evaluating factors limiting attainment of the theoretical performance.

The Second Law can also be used for,

- defining a temperature scale independent of a substance or class of substances, and
- evaluation of thermodynamic properties.

The Second Law of Thermodynamics has been stated in several ways. The three common statements are,

(1) the Clausius Statement,
(2) the Kelvin-Planck Statement, and
(3) entropy.

We'll discuss the first two statements now, but leave the discussion of entropy for later.

3.6.1. The Clausius Statement of the Second Law of Thermodynamics

It is impossible for any system to operate in such a way that the sole result is the transfer of heat from a cooler object to a hotter object (refer to Figure 3.30 for an illustration).

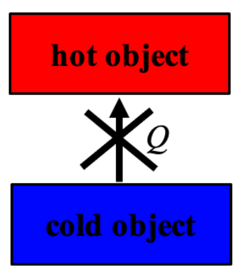


Figure 3.30. An illustration of the Clausius statement of the Second Law of Thermodynamics.

Note:

- It is certainly possible to transfer heat from cold objects to hot objects, e.g., refrigerators and heat pumps do this, but their operation doesn't violate the Clausius statement since the heat transfer isn't the sole effect. Refrigerators and heat pumps require the input of work to make the heat transfer occur.

3.6.2. The Kelvin-Planck Statement of the Second Law of Thermodynamics

It is impossible for any system to operate in a thermodynamic cycle and deliver a net amount of energy by work to its surroundings while receiving energy by heat transfer from a single thermal reservoir (refer to Figure 3.31 for an illustration).

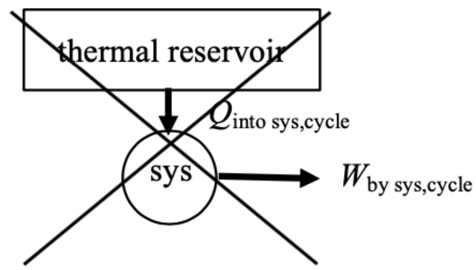


Figure 3.31. An illustration of the Kelvin-Planck statement of the Second Law of Thermodynamics.

Notes:

(1) A thermal reservoir is a body that is sufficiently large so that its temperature remains essentially unchanged despite a transfer of energy to or from it via heat transfer. Thermal reservoirs can be either energy sinks or energy sources, i.e., energy can be transferred to them or from them via heat transfer, respectively, without changing the thermal reservoir's temperature. The defining characteristic of a thermal reservoir is its temperature.

(2) Recall from earlier work that the thermal efficiency of a power cycle, for example, is,

$$\eta = 1 - \frac{Q_{C,\text{cycle}}}{Q_{H,\text{cycle}}}. \tag{3.102}$$

The Kelvin-Planck statement indicates that $Q_{C,\text{cycle}} \neq 0$ which implies that the efficiency cannot be 100%.

(3) Mathematically, the Kelvin-Planck statement may be written as,

$$W_{\substack{\text{by sys,} \\ \text{cycle}}} \leq 0 \quad \text{(single reservoir)}. \tag{3.103}$$

In other words, when there is energy exchange via heat transfer with just a single reservoir (hence the "single reservoir" comment in parentheses), the work done by the system over a cycle cannot be positive (as given by Kelvin-Planck). We can put work into the system or produce no work, but no work can be done by the system.

(4) The Clausius and Kelvin-Planck statements are equivalent. This fact can be shown by considering the following scenario (Figure 3.32).
   (a) Assume the system on the left transfers energy Q_C from the cold reservoir to the hot reservoir, which we know is a violation of the Clausius Statement of the Second Law.
   (b) The system on the right operates over a cycle and produces work. This power cycle does not violate the Second Law.
   (c) The combined system within the dotted line consists of a cold reservoir and two devices (left and right dashed objects). This combined system (dotted line) executes a cycle while receiving energy by heat transfer from a single hot reservoir ($Q_H - Q_C$) and produces work. Thus, this dotted line cycle violates the Kelvin-Planck Statement of the Second Law.
   (d) Thus, we observe that a violation of the Clausius Statement results in a violation of the Kelvin-Planck Statement. A similar argument can be performed in reverse to demonstrate that a violation of the Kelvin-Planck Statement results in a violation of the Clausius Statement.

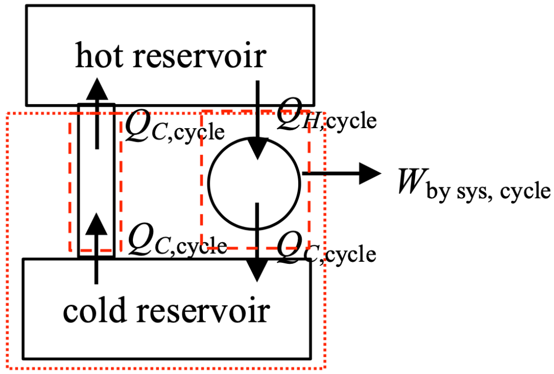


Figure 3.32. A schematic illustrating the equivalence of the Clausius and Kelvin-Planck statements of the Second Law.

Since a violation of one results in a violation of the other, we conclude that the statements are equivalent.

3.6.3. Reversible and Irreversible Processes

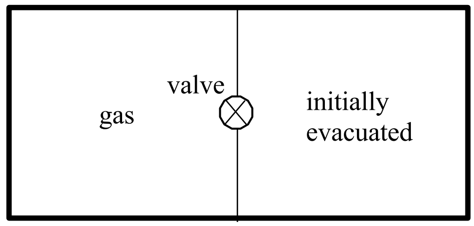
A reversible process is one in which the system is in a state of equilibrium at all points in its path. In a reversible process, the system and the surroundings can be restored exactly to their initial states.

An irreversible process is one where the system is not in a state of equilibrium at all points in its path. The system and surroundings cannot be returned to their exact initial states in an irreversible process. Note that all natural processes are irreversible.

Notes:

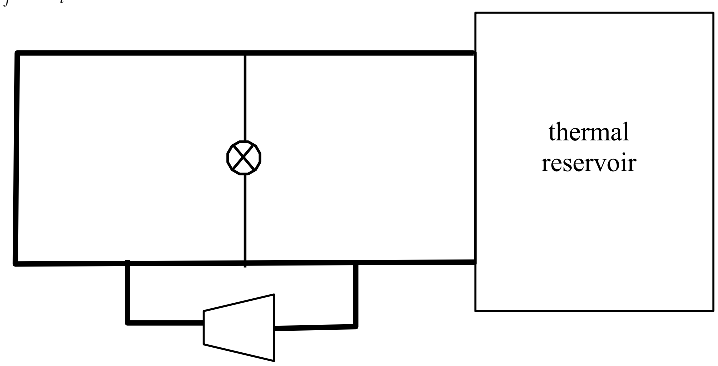
(1) Examples of reversible processes include frictionless pendulums and adiabatic expansion/compression occurring slowly in a frictionless piston-cylinder.
(2) Thermal reservoirs are considered to be reversible.
(3) All real-world processes are irreversible.
(4) Examples of irreversibilities include: heat transfer through a finite temperature gradient, unrestrained expansion of a gas or liquid, spontaneous chemical reaction, spontaneous mixing, friction, electric current flow through a resistance, and inelastic deformation.
(5) Irreversibility can occur within a system, in the surroundings, or both. If a system has no dissipative internal processes, then it's considered internally reversible. For an internally reversible process, Eq. (3.103) becomes $W_{\text{by cycle}} = 0$ (single reservoir). If the surroundings have no dissipative processes, it is considered externally reversible. For an internally irreversible process, then Eq. (3.103) becomes $W_{\text{by cycle}} < 0$ (single reservoir).
(6) Proofs to determine if a process is irreversible typically rely on proof by contradiction. The process is assumed to be reversible and then combined with one or more other reversible processes to form a thermodynamic cycle. Next, it is shown that the cycle violates the Kelvin-Planck statement of the Second Law.

A rigid insulated tank is divided into halves by a partition. On one side of the partition is a gas. The other side is initially evacuated. A valve in the partition is opened and the gas expands to fill the entire volume. Using the Kelvin-Planck statement of the 2nd Law, demonstrate that this process is irreversible.



SOLUTION:
Note that since the tank is well insulated and rigid, there is no heat transfer into the tank nor is there any work. Furthermore, the kinetic energy at the beginning and end of the process is zero and there is no change in potential energy. Hence, from the 1st Law,

$$\Delta E_{\text{sys}} = Q_{\substack{\text{into}\\\text{sys}}} + W_{\substack{\text{on}\\\text{sys}}} \Rightarrow U_f = U_i \,. \tag{1}$$



Process 1. Assume that the process is reversible, meaning that the system can start with the gas in both chambers and the gas can spontaneously move from the right chamber into the left. Note that the pressure in the left chamber will now be larger than the pressure in the right chamber.

Process 2. Let part of the gas pass from the left chamber through a turbine into the right chamber until the pressure in both chambers is the same. Since there has been some work done,

$$U < U_i \tag{2}$$

Process 3. Remove part of the tank insulation and add energy into the system from a thermal reservoir until the system's energy returns to U_i. Since the system is back to its original state, we have completed a cycle.

The net result of this cycle is to draw energy from a single reservoir by heat transfer and produce an equivalent amount of work. Such a cycle violates the Kelvin-Planck statement of the 2nd Law. Since extracting energy from a turbine (Process 2) and heat transfer from a thermal reservoir (Process 3) are possible, Process 1 must be impossible. Thus, gas spontaneously expanding from one tank into another, lower pressure tank must be an irreversible process.

3.6.4. Performance Measures for Cycles

Recall that the power cycle thermal efficiency is given by,

$$\eta := \frac{W_{\text{by sys}}}{Q_H} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}. \tag{3.104}$$

The refrigeration cycle coefficient of performance is,

$$COP_{\text{ref}} := \frac{Q_C}{W_{\text{on sys}}} = \frac{Q_C}{Q_H - Q_C} = \frac{1}{Q_H/Q_C - 1}, \tag{3.105}$$

and the heat pump cycle coefficient of performance is,

$$COP_{\text{HP}} := \frac{Q_H}{W_{\text{on sys}}} = \frac{Q_H}{Q_H - Q_C} = \frac{1}{1 - Q_C/Q_H}. \tag{3.106}$$

Notes:

- The subscript "cycle" has been removed in the previous equations for convenience; however, the evaluations for work and heat are still over a cycle.
- From the Kelvin-Planck statement of the Second Law of Thermodynamics, in the power cycle, we cannot have $Q_C = 0$, which implies that $\eta < 1$.
- From the Clausius statement of the Second Law, in the refrigeration and heat pump cycles, we cannot have $W_{\text{on sys}} = 0$, implying that COP_{ref} and COP_{HP} must remain finite.

Corollaries to the Second Law of Thermodynamics
It can be shown (refer to the proof at the end of this section) that the following corollaries to the Second Law are also true.

(1) The thermal efficiency of an irreversible power cycle will always be less than the thermal efficiency of a reversible power cycle when the two power cycles operate between the same two thermal reservoirs, i.e.,

$$\eta_{\text{irreversible}} < \eta_{\text{reversible}} \quad \text{(same thermal reservoirs)}. \tag{3.107}$$

(2) All reversible power cycles operating between the same two thermal reservoirs have the same thermal efficiency, i.e.,

$$\eta_{\text{reversible},1} = \eta_{\text{reversible},2} \quad \text{(same thermal reservoirs)}. \tag{3.108}$$

(3) The coefficient of performance for a reversible refrigeration cycle (or heat pump cycle) will be larger than the coefficient of performance or an irreversible refrigeration cycle (or heat pump cycle) when operating between the same two thermal reservoirs, i.e.,

$$COP_{\text{irreversible}} < COP_{\text{reversible}} \quad \text{(same thermal reservoirs)}. \tag{3.109}$$

(4) All reversible refrigeration (or heat pump) cycles operating between the same two reservoirs will have the same coefficient of performance, i.e.,

$$COP_{\text{reversible},1} = COP_{\text{reversible},2} \quad \text{(same thermal reservoirs)}. \tag{3.110}$$

The proof for the first corollary is presented now. Consider the situation shown in the left-hand schematic of Figure 3.33. A reversible and irreversible system receive the same energy Q_{H} from a hot reservoir. The irreversible system produces work $W_{\text{by,I}}$ and discharges energy $Q_{\text{C,I}}$ into a cold reservoir while the reversible system produces work $W_{\text{by,R}}$ and discharges energy $Q_{\text{C,R}}$ into the same cold reservoir. From the First Law applied separately to the irreversible and reversible systems and assuming both operate over a cycle,

$$0 = (Q_{\text{H}} - Q_{\text{C,I}}) - W_{\text{by,I}} \implies W_{\text{by,I}} = Q_{\text{H}} - Q_{\text{C,I}}, \tag{3.111}$$

$$0 = (Q_{\text{H}} - Q_{\text{C,R}}) - W_{\text{by,R}} \implies W_{\text{by,R}} = Q_{\text{H}} - Q_{\text{C,R}}, \tag{3.112}$$

where the change in the total energy of each of the two systems is zero since both are operating over a cycle. Choose the reversible system such that $Q_{\text{C,R}} > Q_{\text{C,I}}$ and, thus, $W_{\text{by,R}} < W_{\text{by,I}}$. Now switch the direction of the reversible system (indicated by the dashed arrows in the figure) and consider the combined system indicated by the dashed, red line in the figure. Note that this combined system includes the hot reservoir since

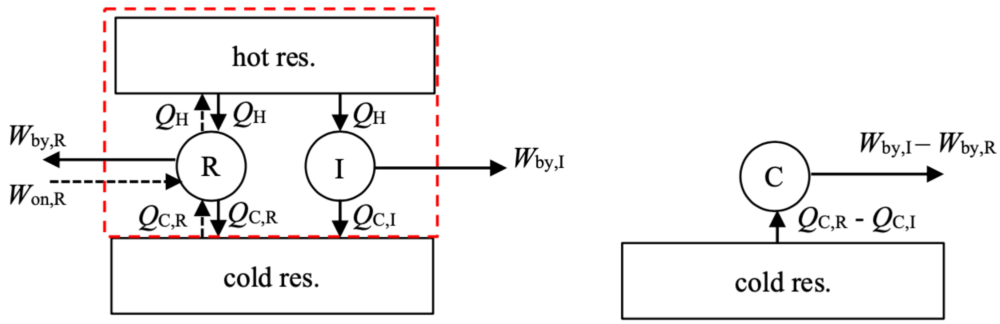


FIGURE 3.33. Schematic used to prove the first and second corollaries to the Second Law.

there is no net energy transfer to/from it since Q_H is removed by the irreversible system, but then replaced by the reversible one. The energy received by the combined system from the cold reservoir is $Q_{\mathrm{C,R}} - Q_{\mathrm{C,I}} > 0$ and the net work done by the combined system is $W_{\mathrm{by,I}} - W_{\mathrm{by,R}} > 0$. This combined system is shown in the right-hand schematic of Figure 3.33. Since the combined system operates over a cycle and interacts with a single thermal reservoir, from the Kelvin-Plank Statement of the Second Law (Eq. (3.103)),

$$W_{\mathrm{by,I}} - W_{\mathrm{by,R}} < 0 \implies W_{\mathrm{by,I}} < W_{\mathrm{by,R}}. \tag{3.113}$$

Note that the inequality has been used since the combined system includes irreversibilities. Since the thermal efficiency is given by,

$$\eta = \frac{W_{\mathrm{by}}}{Q_H}, \tag{3.114}$$

and Q_H is the same for the irreversible and reversible systems, we conclude that,

$$\eta_R > \eta_I. \tag{3.115}$$

Thus, we have proven the first corollary to the Second Law.

The second corollary can be proven by replacing the irreversible system in Figure 3.33 with a reversible one so that there are two reversible systems (call these "R1" and "R2"). Following the same arguments as before, we will arrive at the following statement using the Kelvin-Plank Statement of the Second Law,

$$W_{\mathrm{by,R2}} - W_{\mathrm{by,R1}} = 0 \implies W_{\mathrm{by,R2}} = W_{\mathrm{by,R1}}. \tag{3.116}$$

Here, the equals sign is used since the combined system is reversible. Since the works and heat transfers are the same, the efficiencies of the two reversible systems must be identical.

Proofs for the third and fourth corollaries are not provided here, but follow similar arguments.

3.6.5. Kelvin Absolute Temperature Scale

Note that from the Second Law Corollaries, the reversible cycle performance measures depend solely on the interaction with the thermal reservoirs, namely (Q_C/Q_H) in Eqs. (3.104) - (3.106) since all reversible cycle efficiencies are identical. Since it is the temperature difference between the reservoirs that drives this heat transfer, we can conclude that,

$$\left.\frac{Q_C}{Q_H}\right|_{\text{rev. cycle}} = fcn\left(\frac{T_C}{T_H}\right), \tag{3.117}$$

where T_C and T_H are the temperatures of the cold and hot reservoirs, respectively. Note that since the left hand side of the equation is dimensionless, the right hand side must also be dimensionless. The function fcn is determined by how we define temperature. In the Kelvin absolute temperature scale, we define the temperatures such that the function is a simple linear one, i.e.,

$$\left.\frac{Q_C}{Q_H}\right|_{\text{rev. cycle}} = \frac{T_C}{T_H}. \tag{3.118}$$

By definition, the ratio of temperatures on the Kelvin scale is equal to the ratio of the heat fluxes. Equation (3.118) only provides a ratio of temperatures; it doesn't actually set a value for the temperature. To complete the thermodynamic scale, we arbitrarily set the value of T on the Kelvin scale at the triple point of water to be,

$$T_{\text{triple pt of H2O}} = 273.16\,\text{K}. \tag{3.119}$$

Notes:

(1) Since the performance of a reversible cycle is independent of the details of the cycle, e.g., working fluid, cycle components, etc., it also means that $(Q_C/Q_H)_{\text{rev,cycle}}$ and thus T_C/T_H are also independent of the details of the cycle. This means that the Kelvin absolute temperature scale is independent of any substance or cycle details.

(2) Since $Q_C > 0$ (to satisfy the Kelvin-Planck statement of the Second Law), it also means that $T_C > 0$. Thus, the minimum temperature limit on the Kelvin scale is zero Kelvin, which can never be reached as stipulated by the Second Law.

(3) We can substitute Eq. (3.118) into Eqs. (3.104) - (3.106) to determine reversible, i.e., ideal, cycle performance measures,

   (a) Power cycle *reversible* thermal efficiency,

$$\eta_{\text{rev}} = 1 - \frac{T_C}{T_H}. \tag{3.120}$$

   (b) Refrigeration cycle *reversible* coefficient of performance,

$$COP_{\text{ref,rev}} = \frac{T_C}{T_H - T_C}. \tag{3.121}$$

   (c) Heat pump cycle *reversible* coefficient of performance,

$$COP_{\text{HP,rev}} = \frac{T_H}{T_H - T_C}. \tag{3.122}$$

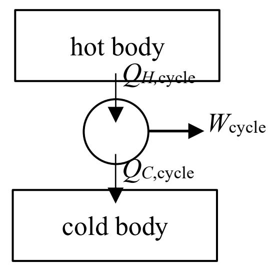
   Interestingly, the maximum performance of these cycles is independent of the details of the cycle (design, working materials, etc.). The only factors that matter are the (absolute) temperatures of the thermal reservoirs.

   For a power cycle, the maximum efficiency increases as T_H increases or T_C decreases. For example, if a combustion process is used to supply heat to the system, the hotter the combustion gases (T_H), the more efficient the reversible cycle. In most practical power cycles, the cycle discharges heat to the environment so there is often less control over T_C. Similar arguments may be made for refrigeration and heat pump cycles.

(4) We can still calculate the efficiency and COPs of any cycle, reversible or irreversible, using Eqs. (3.104) - (3.106). However, for a reversible cycle, we can also make use of Eqs. (3.120) - (3.122).

An internally reversible power cycle with a thermal efficiency of 40% receives 50 kJ of energy by heat transfer from a hot reservoir at 600 K and rejects energy by heat transfer to a cold reservoir at a temperature T_C. Determine the energy rejected and the temperature T_C.

SOLUTION:



We can determine the heat transfer to the cold reservoir using the power cycle thermal efficiency in terms of the heat transfers,

$$\eta = 1 - \frac{Q_{C,\text{cycle}}}{Q_{H,\text{cycle}}} \Rightarrow Q_{C,\text{cycle}} = (1-\eta) Q_{H,\text{cycle}}. \tag{1}$$

Using the given data,

η = 0.40,
$Q_{H,\text{cycle}}$ = 50 kJ,
$\Rightarrow$ $\boxed{Q_{C,\text{cycle}} = 30\text{ kJ}}$.

The temperature of the reservoir can be found by noting that for a reversible cycle,

$$\left.\frac{Q_H}{Q_C}\right|_{\substack{\text{rev,}\\\text{cycle}}} = \frac{T_H}{T_C} \Rightarrow T_C = T_H \left.\frac{Q_C}{Q_H}\right|_{\substack{\text{rev,}\\\text{cycle}}}. \tag{2}$$

Using the parameters given above in addition to T_H = 600 K,

$\boxed{T_C = 360\text{ K}}$.

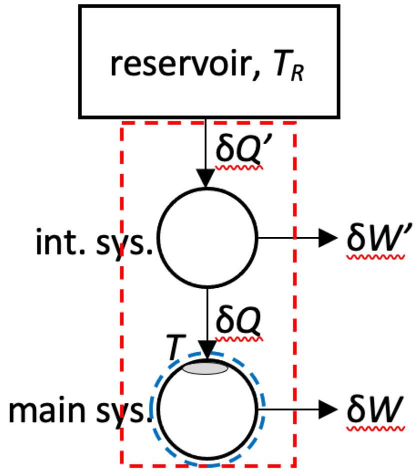


Figure 3.34. Schematic used in proving the Clausius Inequality.

3.6.6. Clausius Inequality

It can be shown using the Second Law (given later in this section) that for a system undergoing a thermodynamic cycle,

$$\left(\int_b \frac{\delta Q}{T}\right)_{\text{cycle}} \leq 0, \tag{3.123}$$

where δQ_{into} is the heat into a system over a small boundary area and T is the absolute temperature at that part of the boundary. The integral over "b" refers to an integral over the system's boundary surface. The equality (=) holds for internally reversible processes (no irreversibilities in the system) and the inequality applies when internal irreversibilities are present (< 0).

Notes:

(1) Equation (3.123) may be also written as,

$$\left(\int_b \frac{\delta Q_{\text{into}}}{T}\right)_{\text{cycle}} = -\sigma_{\text{cycle}}, \tag{3.124}$$

where,

$$\sigma_{\text{cycle}} > 0 \quad \text{internally irreversible system,} \tag{3.125}$$

$$\sigma_{\text{cycle}} = 0 \quad \text{internally reversible system,} \tag{3.126}$$

$$\sigma_{\text{cycle}} < 0 \quad \text{impossible (violates the Second Law).} \tag{3.127}$$

The parameter σ_{cycle} is the entropy produced by irreversibilities in the system over the cycle.

The proof for the Clausius Inequality is as follows. Consider Figure 3.34, which consists of a reservoir with absolute temperature T_R, an intermediate system, and a main system. The main system receives energy with magnitude δQ via heat transfer at a location on the system boundary where the absolute temperature is T. The main system uses this energy to perform work δW. In order to avoid potential irreversibility caused by heat transfer across a finite temperature difference (from T_R to T), the energy transfer occurs through an intermediate system, which operates reversibly. This intermediate system receives energy $\delta Q'$ from the reservoir, produces work $\delta W'$, and discharges the energy δQ, which goes into the main system. Since this intermediate system operates reversibly, we have from the definition of the absolute temperature scale (refer to Section 3.6.5),

$$\frac{\delta Q'}{\delta Q} = \frac{T_R}{T} \implies \delta Q' = T_R \frac{\delta Q}{T}. \tag{3.128}$$

Although the figure only shows a single intermediate system and a single transfer of energy into the main system, we can imagine additional intermediate systems and energy transfers into the main system at various locations on the main system boundary. Now apply the First Law of Thermodynamics to the combined system shown by the red dashed line in the figure,

$$dE_C = \int_b \delta Q' - \delta W_C, \tag{3.129}$$

where δW_C is the sum of all the intermediate system works and the main system's work. The boundary integral on the heat transfer term refers to all of the heat transfers from the reservoir into the intermediate systems. Let the main system and the intermediate systems operate over a cycle so that $dE_C = 0$ and, thus,

$$0 = \int_b \delta Q' - \delta W_C. \tag{3.130}$$

Note that the subscript "cycle" has not been included for convenience. Substituting Eq. (3.128) and rearranging,

$$\delta W_C = \int_b T_R \frac{\delta Q}{T} = T_R \int_b \frac{\delta Q}{T}. \tag{3.131}$$

where the reservoir temperature T_R is moved outside the integral since it is a constant. The combined system operates over a cycle (since its components operate over a cycle) and interacts with a single reservoir and, thus, from the Kelvin-Plank Statement of the Second Law (Eq. (3.103)),

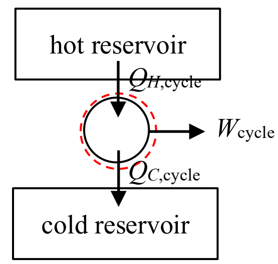
$$W_C \leq 0 \implies \int_b \frac{\delta Q}{T} \leq 0, \tag{3.132}$$

which is the same as Eq. (3.123). Recall that the "<" occurs when the combined system has internal irreversibilities and the "=" occurs when the combined system is internally reversible. Since the intermediate systems are reversible, if there are any irreversibilities, they must occur in the main system.

A system executes a power cycle while receiving 1050 kJ by heat transfer at a temperature of 525 K and discharging 700 kJ by heat transfer at 350 K. There are no other heat transfers.

a. Determine if the cycle is internally reversible, irreversible, or impossible.
b. Determine the thermal efficiency. Compare this value with the maximum possible efficiency.

SOLUTION:



To determine if the cycle is internally reversible, irreversible, or impossible, consider the Clausius Inequality,

$$\int_{A_{\text{sys}}} \frac{\delta Q_{\text{into,cycle}}}{T} \leq 0, \quad (1)$$

where,

Q_H/T_H = (1050 kJ)/(525 K) = 2 kJ/kg,
Q_C/T_C = (700 kJ)/(350 K) = 2 kJ/kg,

$$\Rightarrow \left(\int_b \frac{\delta Q_{\text{into}}}{T}\right)_{\text{cycle}} = \frac{Q_H}{T_H} - \frac{Q_C}{T_C} = 2\text{ kJ/K} - 2\text{ kJ/kg} = 0.$$

(Note $Q_C < 0$ since heat leaves the system. We're also assuming that the temperature at the boundaries to the system where the heat is added/removes is the same as the adjacent reservoir.)

Thus, we see that $\boxed{\text{the cycle is internally reversible.}}$

The thermal efficiency is,

$$\eta = 1 - \frac{Q_C}{Q_H}, \quad (2)$$

$$\Rightarrow \boxed{\eta = 0.33}.$$

The maximum possible efficiency is,

$$\eta_{\max} = 1 - \frac{T_C}{T_H}, \quad (3)$$

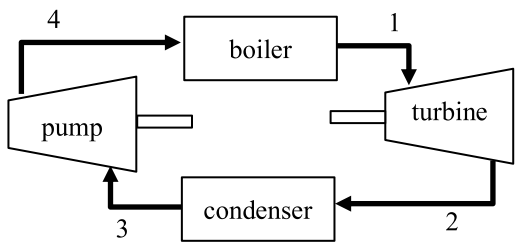
$$\Rightarrow \boxed{\eta_{\max} = 0.33}.$$

The cycle is operating at the maximum possible efficiency since it is internally reversible.

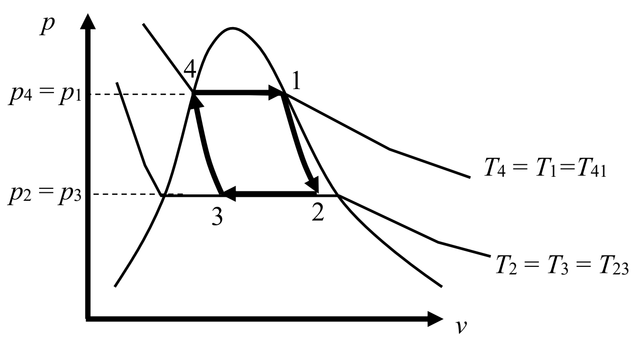
Consider the vapor power plant cycle shown in the figure. The working fluid is water. Water flows through the boiler and condenser at constant pressure and through the turbine and pump adiabatically. Kinetic and potential energy effects can be ignored. The process data are:

Process 4 – 1: constant pressure at 1 MPa (abs) from saturated liquid to saturated vapor,
Process 2 – 3: constant pressure at 20 kPa (abs) from $x_2 = 0.88$ to $x_3 = 0.18$.

a. Determine if the cycle is internally reversible, irreversible, or impossible.
b. Determine the thermal efficiency of the cycle.
c. Compare the thermal efficiency from (b) to the maximum possible efficiency.



SOLUTION:
First sketch the cycle on a p-v diagram for convenience.

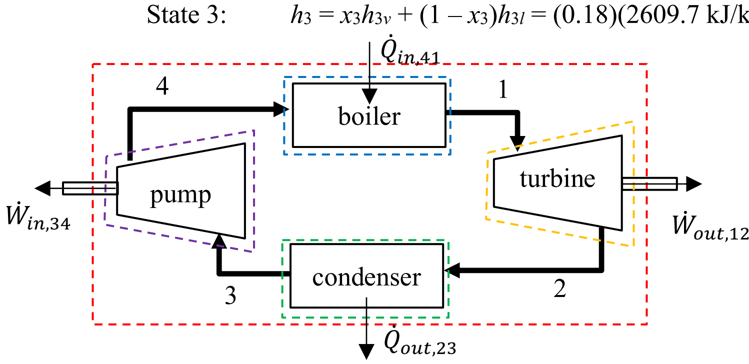


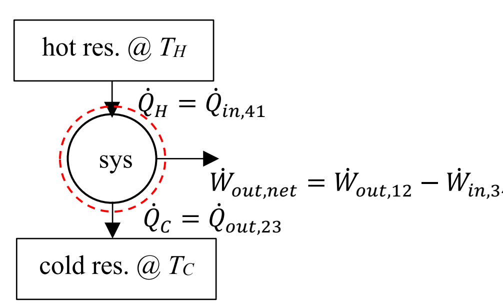
Using the thermodynamic property tables for water in a saturated state (e.g., Table A-3 in Moran et al., 7th ed.):

$T_{41} = 179.9°C = 453.05$ K @ 1 MPa = 10 bar (abs),
$T_{23} = 60.06°C = 333.21$ K @ 20 kPa = 0.2 bar (abs).

In addition, also using Table A-3 in Moran et al., 7th ed. at the given saturation temperature,

State 4: $h_4 = h_l = 762.81$ kJ/kg (saturated liquid state)
State 1: $h_1 = h_v = 2778.1$ kJ/kg (saturated vapor state)
State 2: $h_2 = x_2 h_{2v} + (1 - x_2)h_{2l} = (0.88)(2609.7 \text{ kJ/kg}) + (1 - 0.88)(251.4 \text{ kJ/kg}) = 2326.7$ kJ/kg
State 3: $h_3 = x_3 h_{3v} + (1 - x_3)h_{3l} = (0.18)(2609.7 \text{ kJ/kg}) + (1 - 0.18)(251.4 \text{ kJ/kg}) = 675.89$ kJ/kg





Applying the 1st Law to a control volume surrounding the boiler, assuming steady flow, negligible changes in kinetic and potential energy across the control volume, and no work other than pressure,

$$\dot{Q}_{in,41} = \dot{m}(h_1 - h_4) \Rightarrow \frac{\dot{Q}_{in,41}}{\dot{m}} = h_1 - h_4. \tag{1}$$

Similarly, for the condenser,

$$\dot{Q}_{out,23} = \dot{m}(h_2 - h_3) \Rightarrow \frac{\dot{Q}_{out,23}}{\dot{m}} = h_2 - h_3. \tag{2}$$

Apply the 1st Law to control volumes surrounding the turbine and pump, assuming steady flow, negligible changes in kinetic and potential energy across the control volume, and adiabatic conditions,

$$\dot{W}_{out,12} = \dot{m}(h_1 - h_2) \Rightarrow \frac{\dot{W}_{out,12}}{\dot{m}} = h_1 - h_2. \tag{3}$$

$$\dot{W}_{in,34} = \dot{m}(h_4 - h_3) \Rightarrow \frac{\dot{W}_{in,34}}{\dot{m}} = h_4 - h_3. \tag{4}$$

Substituting the specific enthalpy values found previously,

$$\frac{\dot{Q}_{in,41}}{\dot{m}} = 2015.29 \text{ kJ/kg}$$

$$\frac{\dot{Q}_{out,23}}{\dot{m}} = 1650.81 \text{ kJ/kg}$$

$$\frac{\dot{W}_{out,12}}{\dot{m}} = 451.4 \text{ kJ/kg}$$

$$\frac{\dot{W}_{in,34}}{\dot{m}} = 86.92 \text{ kJ/kg}$$

To determine if the cycle is internally reversible, irreversible, or impossible, consider the Clausius Inequality applied to the entire cycle (or, alternately, the Entropy Equation with $dS/dt = 0$ because the cycle is at steady state),

$$\int_b \frac{\delta \dot{Q}_{into,cycle}}{T} = -\dot{\sigma}, \tag{5}$$

where,

$$\int_b \frac{\delta \dot{Q}_{into,cycle}}{T} = -\dot{\sigma} = \dot{m}\left(\frac{\dot{Q}_{in,41}/\dot{m}}{T_H} - \frac{\dot{Q}_{out,23}/\dot{m}}{T_C}\right), \tag{6}$$

$$\frac{\dot{\sigma}}{\dot{m}} = -\left(\frac{2015.29 \text{ kJ/kg}}{453.05 \text{ K}} - \frac{1650.81 \text{ kJ/kg}}{333.21 \text{ K}}\right), \tag{7}$$

$$\frac{\dot{\sigma}}{\dot{m}} = 0.507 \ \frac{\text{kJ}}{\text{kg.K}}. \tag{8}$$

Thus, we see that $\boxed{\text{the cycle is internally irreversible.}}$ Note that in this analysis it is assumed that the temperatures at which the heat enters and leaves the CV are $T_H = T_{41}$ and $T_C = T_{23}$ since we're not given any information about their temperatures. In reality, this would not be the case since there must be some temperature difference between the hot/cold reservoir and the boiler/condenser to drive the heat transfer, i.e., $T_H > T_{41}$ and $T_C < T_{23}$. If there was a temperature difference between T_H and T_{41} and T_C an T_{23}, then this would create even more irreversibility due to the larger temperature gradient.

The thermal efficiency of the cycle is,

$$\eta = 1 - \frac{\dot{Q}_C}{\dot{Q}_H} = 1 - (1650.81 \text{ kJ/kg})/(2015.29 \text{ kJ/kg}) \tag{9}$$

$$\Rightarrow \boxed{\eta = 0.18 = 18\%}.$$

We could have also calculated the efficiency using the work generated,

$$\eta = \frac{\dot{W}_{out,net}/\dot{m}}{\dot{Q}_H/\dot{m}} = \frac{(\dot{W}_{out,12} - \dot{W}_{in,34})/\dot{m}}{\dot{Q}_{in,41}/\dot{m}} = \frac{(451.4 \text{ kJ/kg} - 86.92 \text{ kJ/kg})}{2015.29 \text{ kJ/kg}} = 0.18 \tag{10}$$

which is the same result as found previously.

The maximum possible efficiency is,

$$\eta_{max} = \eta_{int.rev.} = 1 - (333.21 \text{ K})/(453.05 \text{ K}), \tag{11}$$

$$\Rightarrow \boxed{\eta_{max} = 0.26 = 26\%}.$$

Since the cycle is irreversible, the actual efficiency is less than the maximum possible efficiency.

3.6.7. Entropy

Recall from Eq. (3.124) that the Clausius Inequality is given by,

$$\left(\oint_b \frac{\delta Q_{\text{into}}}{T}\right)_{\text{cycle}} = -\sigma_{\text{cycle}}, \tag{3.133}$$

where,

$$\sigma_{\text{cycle}} > 0 \quad \text{internally irreversible system,} \tag{3.134}$$

$$\sigma_{\text{cycle}} = 0 \quad \text{internally reversible system,} \tag{3.135}$$

$$\sigma_{\text{cycle}} < 0 \quad \text{impossible (violates the Second Law).} \tag{3.136}$$

Consider a system undergoing a cycle in which two different internally reversible cycles are considered: path A then C and path B then C. From Eq. (3.133),

$$\left(\oint_b \frac{\delta Q_{\text{into}}}{T}\right)_{\text{cycle}} = 0, \tag{3.137}$$

since the cycle is internally reversible. Expanding the integral along the cycle's two paths,

$$\int_{A,b} \frac{\delta Q_{\text{into}}}{T} + \int_{C,b} \frac{\delta Q_{\text{into}}}{T} = 0, \tag{3.138}$$

and,

$$\int_{B,b} \frac{\delta Q_{\text{into}}}{T} + \int_{C,b} \frac{\delta Q_{\text{into}}}{T} = 0. \tag{3.139}$$

Combining the previous two equations,

$$\int_{A,b} \frac{\delta Q_{\text{into}}}{T} = \int_{B,b} \frac{\delta Q_{\text{into}}}{T}. \tag{3.140}$$

Thus, the result of the integral is independent of the (reversible) path, which means that the integral can be considered a change in properties between the end points. Recall that property values are independent of path. This system property is defined as the <u>entropy</u> and is defined as,

$$S_2 - S_1 \coloneqq \int_{1,b}^{2} \frac{\delta Q_{\text{into}}}{T}\bigg|_{\substack{\text{internally}\\\text{reversible}}}, \tag{3.141}$$

or in differential form,

$$dS \coloneqq \frac{\delta Q_{\text{into}}}{T}\bigg|_{\substack{\text{internally}\\\text{reversible}}}. \tag{3.142}$$

Note that the integrals in Eqs. (3.139) - (3.142) are performed over the system boundary while traversing the path from state 1 to state 2.

Notes:

(1) Entropy S is an extensive property. The specific entropy s is the entropy per unit mass, i.e., $s = S/m$. The dimensions of entropy are [energy]/[temperature], e.g., $\text{kJ}\,\text{K}^{-1}$ or $\text{Btu}\,{}^\circ\text{R}^{-1}$. Typical units for specific entropy are $\text{kJ}/(\text{kg}\cdot\text{K})$ or $\text{Btu}/(\text{lb}_\text{m}\cdot{}^\circ\text{R})$.

(2) Since entropy is a property, the change in entropy between two states is independent of the path between the two states. The internally reversible path in Eq. (3.141) is used just to define entropy. Even if a process path between two states isn't reversible, as shown in Figure 3.35, we can still calculate the entropy at each of the endpoints since we can imagine reversible paths from some reference state to each endpoint. A later note discusses irreversible process paths in more detail.

(3) From Eq. (3.141), adding heat in an internally reversible manner into a system increases its entropy. Removing heat in an internally reversible manner reduces its entropy.

(4) The change in entropy for a process can be positive, negative, or zero; it just depends on the heat transfer (which occurs reversibly).

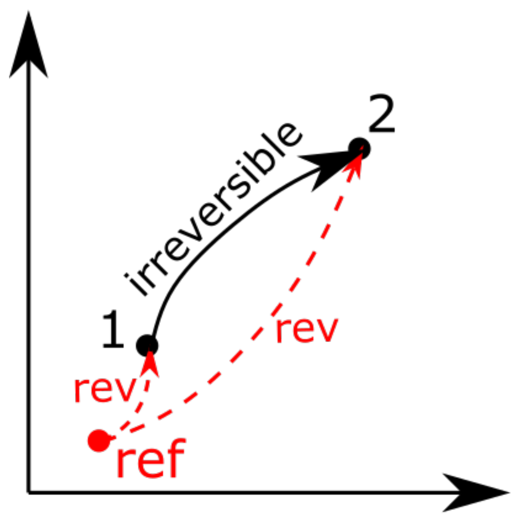


Figure 3.35. An illustration showing how the path for defining the property of entropy involves a reversible path from a reference point to a given state, but the entropy difference between two end states in an irreversible process can still be calculated.

(5) The specific entropy for a two-phase mixture may be found using the quality,

$$s = xs_v + (1-x)s_l. \tag{3.143}$$

(6) The specific entropy for a compressed liquid may be approximated as the specific entropy of a saturated liquid at the same temperature,

$$s_{CL}(T,p) \approx s_l(T). \tag{3.144}$$

(7) As with p - v and T - v diagrams, T - s and h - s (aka Mollier) diagrams are often helpful in visualizing processes (Figure 3.36).

(8) For a process that is both adiabatic and internally reversible, Eq.(3.141) indicates that the process is also isentropic, i.e., $S_2 - S_1 = 0$ or $dS = 0$. Many real processes are often idealized as being isentropic. An isentropic process need not be adiabatic and internally reversible, however. It's possible to have internally irreversible processes in which heat is removed that end up producing no net entropy. We'll discuss this scenario a little later in the notes.

(9) We can re-arrange Eq. (3.141) to give,

$$\delta Q_{\substack{\text{into,}\\ \text{internally}\\ \text{reversible}}} = TdS, \tag{3.145}$$

$$Q_{\substack{\text{into,12,}\\ \text{internally}\\ \text{reversible}}} = \int_1^2 TdS. \tag{3.146}$$

Thus, the area under an *internally reversible* process path on a T - S diagram is equal to the energy entering a system via heat transfer (Figure 3.37).

(10) Now let's consider the possibility of processes that may not be internally reversible. Let's consider a cycle that consists of a process from state 1 to state 2, and a return process from state 2 to state 1. The return process (2 to 1) will be assumed to be internally reversible. The first process (from 1 to 2) can be either internally reversible or internally irreversible. Combining Eqs. (3.133) and

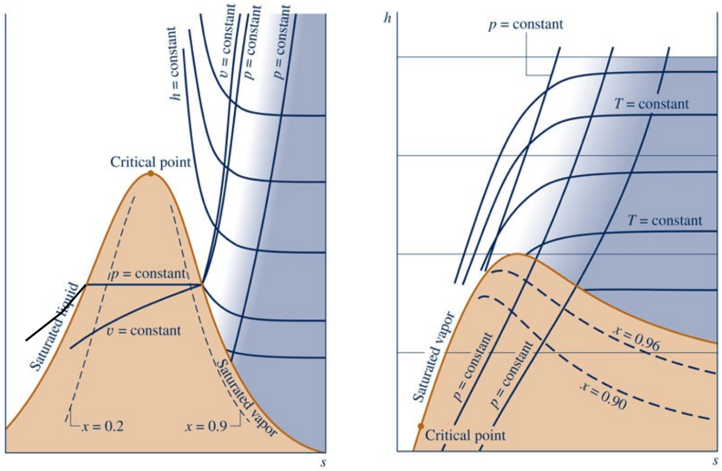


Figure 3.36. Example T - s and h - s (aka Mollier) diagrams. Figures are from Moran et al., 7th ed.

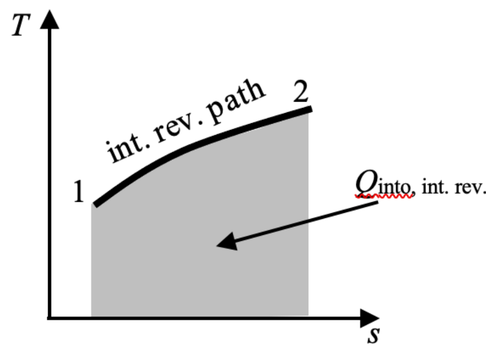


Figure 3.37. A schematic showing that the area under an *internally reversible* process path in a T - S plot is equal to the energy added to the system via heat transfer.

(3.141) gives,

$$\left(\int_b \frac{\delta Q_{\text{into}}}{T}\right)_{\text{cycle}} = -\sigma_{\text{cycle}} \implies \underbrace{\int_{1,b}^{2} \frac{\delta Q_{\text{into}}}{T} + \int_{2,b}^{1} \frac{\delta Q_{\text{into}}}{T}\bigg|_{\substack{\text{internally}\\ \text{reversible}}}}_{\text{the cycle}} = -\sigma_{\text{cycle}}, \tag{3.147}$$

$$\int_{1,b}^{2} \frac{\delta Q_{\text{into}}}{T} + \sigma_{\text{cycle}} = -\int_{2,b}^{1} \frac{\delta Q_{\text{into}}}{T}\bigg|_{\substack{\text{internally}\\ \text{reversible}}} = \int_{1,b}^{2} \frac{\delta Q_{\text{into}}}{T}\bigg|_{\substack{\text{internally}\\ \text{reversible}}} = S_2 - S_1, \tag{3.148}$$

$$\boxed{\underbrace{S_2 - S_1}_{\substack{\text{change in system}\\ \text{entropy between states}}} = \underbrace{\int_{1,b}^{2} \frac{\delta Q_{\text{into}}}{T}}_{\substack{\text{entropy transferred}\\ \text{into the system}\\ \text{through the boundary}\\ \text{via heat transfer}}} + \underbrace{\sigma_{12}}_{\substack{\text{entropy produced}\\ \text{in the system during}\\ \text{the process from 1 to 2}}}} \tag{3.149}$$

where,

$$\sigma_{12} > 0 \quad \text{irreversibilities in the system in going from 1 to 2,} \tag{3.150}$$

$$\sigma_{12} = 0 \quad \text{reversible system in going from 1 to 2,} \tag{3.151}$$

$$\sigma_{12} < 0 \quad \text{impossible in going from 1 to 2.} \tag{3.152}$$

Note that $\sigma_{12} = 0$ since we assumed that the return process was internally reversible.

Equation (3.149) may also be written on a rate basis,

$$\boxed{\underbrace{\frac{dS}{dt}}_{\substack{\text{rate of change in}\\ \text{system entropy}}} = \underbrace{\int_b \frac{\dot{Q}_{\text{into}}}{T}}_{\substack{\text{rate at which entropy}\\ \text{is transferred into}\\ \text{the system through}\\ \text{the boundary}\\ \text{via heat transfer}}} + \underbrace{\dot{\sigma}}_{\substack{\text{rate at which entropy}\\ \text{is produced}\\ \text{in the system}}}} \tag{3.153}$$

where,

$$\dot{\sigma} > 0 \quad \text{irreversibilities in the system,} \tag{3.154}$$

$$\dot{\sigma} = 0 \quad \text{reversible system,} \tag{3.155}$$

$$\dot{\sigma} < 0 \quad \text{impossible.} \tag{3.156}$$

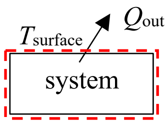
Note that the process in going from 1 to 2 in Eqs. (3.149) and (3.153) need not be internally reversible. If it is internally reversible, then $\sigma_{12} = 0$ and we recover Eq. (3.141). If the process from 1 to 2 is internally irreversible, then $\sigma_{12} > 0$. The change in entropy can be positive, negative, or zero depending on the contributions due to heat transfer and entropy production due to internal irreversibilities; however, the entropy production term must be positive or equal to zero. Note that it is possible to have an isentropic process ($S_2 = S_1$) for an internally irreversible process ($\sigma_{12} > 0$) as long as heat is removed from the system during the process ($\delta Q_{\text{into}} < 0$) and the boundary integral term exactly balances the entropy production term in magnitude.

(11) An isolated system is one that has no interaction with the surroundings, i.e., no work input/output and no heat transfer with the surroundings. In such a case, the change in total energy of the system must be zero from the first law, i.e., $\Delta E_{\text{sys}} = 0$. In addition, the change in entropy of the system will equal the entropy production within the system due to irreversibilities, i.e., $\Delta S_{\text{sys}} = \sigma \geq 0$. Since all real processes have irreversibilities, the change in entropy for a real, isolated system must be positive.

During a process involving a 1 kg mass of material, 9000 J of heat leaves the system and enters the surroundings. The temperature at the surface of the system is 300 K. Determine if the process is internally reversible, internally irreversible, or impossible for the following conditions:

a. The change in the specific entropy of the system is -30.0 J/(kg.K).
b. The change in the specific entropy of the system is -20.0 J/(kg.K).
c. The change in the specific entropy of the system is -40.0 J/(kg.K).

SOLUTION:



The Entropy Equation applied to the control volume shown in the figure,

$$\Delta S = \int_b \frac{\delta Q_{into}}{T} + \sigma \Rightarrow \frac{\sigma}{m} = \Delta s + \frac{1}{m}\frac{Q_{out}}{T_{surf}}. \quad (1)$$

We're given that m = 1 kg, Q_{out} = 9000 J, and T_{surf} = 300 K. Now consider the different cases,

a. $\frac{\sigma}{m} = -30.0 \ \frac{\text{J}}{\text{kg.K}} + \frac{1}{(1 \text{ kg})}\frac{(9000 \text{ J})}{(300 \text{ K})} = 0 \quad \Rightarrow$ $\boxed{\text{This process is internally reversible.}}$

b. $\frac{\sigma}{m} = -20.0 \ \frac{\text{J}}{\text{kg.K}} + \frac{1}{(1 \text{ kg})}\frac{(9000 \text{ J})}{(300 \text{ K})} = 10.0 \ \frac{\text{J}}{\text{kg.K}} \Rightarrow$ $\boxed{\text{This process is internally irreversible.}}$

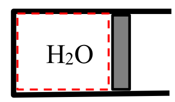
c. $\frac{\sigma}{m} = -40.0 \ \frac{\text{J}}{\text{kg.K}} + \frac{1}{(1 \text{ kg})}\frac{(9000 \text{ J})}{(300 \text{ K})} = -10.0 \ \frac{\text{J}}{\text{kg.K}} \Rightarrow$ $\boxed{\text{This process is impossible.}}$

One kilogram of water contained in a piston-cylinder assembly, initially at 160 °C, 150 kPa (abs), undergoes an isothermal compression process to saturated liquid. For the process, the work done by the water is -471.5 kJ. Determine for the process:

a. the heat transfer into the water, and
b. the change in entropy of the water.

SOLUTION:

The heat transfer into the water may be found from the 1st Law:



$$\Delta E_{\text{sys}} = Q_{\underset{\text{sys}}{\text{into}}} - W_{\underset{\text{sys}}{\text{by}}} \Rightarrow Q_{\underset{\text{sys}}{\text{into}}} = \Delta E_{\text{sys}} + W_{\underset{\text{sys}}{\text{by}}}, \tag{1}$$

where $\Delta E_{\text{sys}} = \Delta U$ (neglected changes in kinetic and potential energy) and $W_{\text{by sys}}$ = -471.5 kJ. The change in internal energy is,

$$\Delta U = m(u_2 - u_1), \tag{2}$$

where,

m = 1 kg

u_1 = 2595.2 kJ/kg (@ 160 °C, 150 kPa = 1.5 bar $\Rightarrow$ superheated vapor; found from Table A-4 in Moran et al., 7th ed.)

u_2 = 674.86 kJ/kg (saturated liquid at 160 °C; from Table A-2 in Moran et al., 7th ed.)

Thus,

ΔU = -1920.34 kJ/kg,

$\boxed{Q_{\text{into}} = -2390 \text{ kJ/kg}}$ (heat leaves the water).

The change in the water's entropy may be found using the thermodynamic tables.

s_1 = 7.4665 kJ/(kg.K) (@160 °C, 150 kPa = 1.5 bar $\Rightarrow$ superheated vapor; found from Table A-4 in Moran et al., 7th ed.)

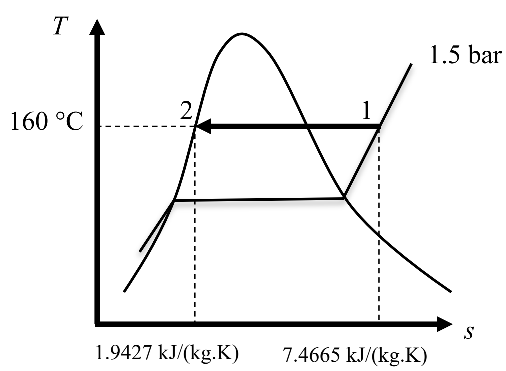
s_2 = 1.9427 kJ/(kg.K) (saturated liquid at 160 °C; from Table A-2 in Moran et al., 7th ed.)

Thus,

$S_2 - S_1 = m(s_2 - s_1)$,

$\Rightarrow \boxed{S_2 - S_1 = -5.5238 \text{ kJ/K}}$.

A plot of the process on a T-s diagram is shown in the following figure.



Although not specifically asked for, we can check to see if this process is internally reversible by checking to see if the entropy production term is zero,

$$S_2 - S_1 = \int_{1,b}^{2} \frac{\delta Q_{\text{into}}}{T} + \sigma_{12} \Rightarrow \sigma_{12} = S_2 - S_1 - \int_{1,b}^{2} \frac{\delta Q_{\text{into}}}{T}, \tag{3}$$

Since the process is isothermal, T remains constant so that we can write the previous equation as,

$$\sigma_{12} = S_2 - S_1 - \frac{Q_{\text{into},12}}{T} \quad . \tag{4}$$

Substituting the values,

$S_2 - S_1 =$ -5.5238 kJ/K,
$Q_{\text{into},12} =$ -2392 kJ,
T = 160 °C = 433 K,
=> $\sigma_{12} = 4.5*10^{-4}$ kJ/K,

which is close enough to zero (within numerical error) for the process to be considered internally reversible.

3.7. Combining the First and Second Laws

Recall that for a system, the First Law of Thermodynamics, written on a per mass and differential basis, is,

$$de_{\text{sys}} = \delta q_{\text{into sys}} + \delta w_{\text{on sys}}. \tag{3.157}$$

If we consider a pure, simple, compressible substance (no kinetic or potential energies to consider and only pdV work), then Eq. (3.157) may be written as,

$$du = \delta q_{\text{into}} - pdv. \tag{3.158}$$

Note that the subscript "sys" has been dropped for convenience. Using the Second Law (Eq. (3.146)) to substitute for the heat transfer term and noting that when we use Eq. (3.146), we're assuming an internally reversible process,

$$du = Tds - pdv \implies \boxed{Tds = du + pdv}, \tag{3.159}$$

or, if written in terms of the enthalpy ($dh = du + pdv + vdp$),

$$\boxed{Tds = dh - vdp} \tag{3.160}$$

Note that pdv work is considered internally reversible work if done quasi-statically. The boxed equations are known as the Tds equations and are useful in relating changes in entropy to other system properties.

Notes:

(1) Even though the Tds equations were derived making use of an internally reversible process, since the equations only involve properties, the process used to go between the states is irrelevant. The Td equations hold for reversible and irreversible process paths.

(2) Recall that during a change of phase, e.g., in the vapor dome, the pressure and temperature of a substance remain constant $\implies dp = 0$ and $T =$ constant. Thus, Eq. (3.160) indicates that,

$$ds = \frac{dh}{T} \implies \Delta s = \frac{\Delta h}{T} \quad \text{(during a phase change)}. \tag{3.161}$$

(3) For an incompressible substance, $dv = 0$ and $du = c(T)dT$ so that Eq. (3.159) becomes,

$$Tds = c(T)dT \implies ds = c(T)\frac{dT}{T} \implies \boxed{s_2 - s_1 = \int_{T_1}^{T_2} \frac{c(T)}{T}dT \quad \text{(incompressible substance)}}. \tag{3.162}$$

The change in specific entropy depends only on the (absolute) temperature. If we can further assume that c is constant (a reasonable assumption in many instances when the change in temperature is less than a few hundred Kelvin or degrees Rankine), then Eq. (3.162) becomes,

$$\boxed{s(T_2) - s(T_1) = c\ln\left(\frac{T_2}{T_1}\right) \quad \text{(incompressible substance, constant specific heat)}}. \tag{3.163}$$

(4) For an ideal gas, $pv = RT(vdp + pdv = RdT)$, $du = c_v(T)dT$, and $dh = c_p(T)dT$, so that Eq. (3.159) becomes,

$$Tds = c_v(T)dT + RdT - vdp = c_v(T)dT + RdT - RT\frac{dp}{p} = [c_v(T) + R]dT - RT\frac{dp}{p}. \tag{3.164}$$

Recall that for an ideal gas that $c_p(T) = c_v(T) + R$ so that the previous equation becomes,

$$ds = c_p(T)\frac{dT}{T} - R\frac{dp}{p}. \tag{3.165}$$

Integrating this equation gives,

$$\boxed{s(T_2, p_2) - s(T_1, p_1) = \int_{T_1}^{T_2} c_p(T)\frac{dT}{T} - R\ln\left(\frac{p_2}{p_1}\right) \quad \text{(ideal gas)}}. \tag{3.166}$$

Similarly, Eq. (3.160) becomes,

$$Tds = dh - vdp = c_p(T)dT - RdT + pdv = [c_p(T) - R]dT + RT\frac{dv}{v}, \tag{3.167}$$

$$ds = c_v(T)\frac{dT}{T} + R\frac{dv}{v} = c_v(T)\frac{dT}{T} - R\frac{d\rho}{\rho}, \tag{3.168}$$

$$\boxed{s(T_2, v_2) - s(T_1, v_1) = \int_1^2 \frac{c_v(T)}{T}dT + R\ln\left(\frac{dv}{v}\right) = \int_1^2 \frac{c_v(T)}{T}dT - R\ln\left(\frac{\rho_2}{\rho_1}\right)} \tag{3.169}$$

Note that $\rho = 1/v \implies d\rho = -dv/v^2 \implies d\rho/\rho = -dv/v$ has been used in the previous equations. Since evaluating the integral involving $c_p(T)$ in Eq. (3.166) is inconvenient, let's define a new variable,

$$s^\circ(T) = \int_{T'}^{T} \frac{c_p(T)}{T}dT, \tag{3.170}$$

where T' is an arbitrary reference temperature. Values of $s^\circ(T)$ are frequently presented in thermodynamic tables for various ideal gases and on a per mole basis (e.g., Tables A-22 and A-23 in Moran et al., 8th ed.; refer to Figure 3.38). The integral involving $c_p(T)$ can now be written as,

$$s^\circ(T_2) - s^\circ(T_1) = \int_{T'}^{T_2} \frac{c_p(T)}{T}dT - \int_{T'}^{T_1} \frac{c_p(T)}{T}dT = \int_{T_1}^{T_2} \frac{c_p(T)}{T}dT, \tag{3.171}$$

and Eq. (3.166) may be written as,

$$\boxed{s(T_2, p_2) - s(T_1, p_1) = s^\circ(T_2) - s^\circ(T_1) - R\ln\left(\frac{p_2}{p_1}\right)} \quad \text{(ideal gas).} \tag{3.172}$$

TABLE A-22

Ideal Gas Properties of Air

T(K), h and u(kJ/kg), s° (kJ/kg · K)

				when $\Delta s = 0$[1]						when $\Delta s = 0$	
T	h	u	s°	p_r	v_r	T	h	u	s°	p_r	v_r
200	199.97	142.56	1.29559	0.3363	1707.	450	451.80	322.62	2.11161	5.775	223.6
210	209.97	149.69	1.34444	0.3987	1512.	460	462.02	329.97	2.13407	6.245	211.4
220	219.97	156.82	1.39105	0.4690	1346.	470	472.24	337.32	2.15604	6.742	200.1
230	230.02	164.00	1.43557	0.5477	1205.	480	482.49	344.70	2.17760	7.268	189.5
240	240.02	171.13	1.47824	0.6355	1084.	490	492.74	352.08	2.19876	7.824	179.7

FIGURE 3.38. Ideal gas properties for air, including s° values. This table is from Moran et al., 8th ed.

(5) If we assume we're dealing with a perfect gas (c_v and c_p are constants), Eqs. (3.166) and (3.169) become,

$$\boxed{s_(T_2, p_2) - s(T_1, p_1) = c_p\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) \quad \text{(perfect gas)}} \tag{3.173}$$

$$\boxed{s_(T_2, v_2) - s(T_1, v_1) = c_v\ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right) \quad \text{(perfect gas)}} \tag{3.174}$$

$$\boxed{s_(T_2, \rho_2) - s(T_1, \rho_1) = c_v\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{\rho_2}{\rho_1}\right) \quad \text{(perfect gas)}}. \tag{3.175}$$

(6) For an isentropic process involving an ideal gas, Eq. (3.172) simplifies to,

$$0 = s^\circ(T_2) - s^\circ(T_1) - R\ln\left(\frac{p_2}{p_1}\right). \tag{3.176}$$

Re-arranging this equation gives,

$$\frac{p_2}{p_1} = \exp\left[\frac{s^\circ(T_2) - s^\circ(T_1)}{R}\right] = \frac{\exp\left[\frac{s^\circ(T_2)}{R}\right]}{\exp\left[\frac{s^\circ(T_1)}{R}\right]}. \tag{3.177}$$

For convenience, define a new parameter, p_r (note that this is not the reduced pressure used in determining the compressibility factor Z), so that Eq. (3.177) becomes,

$$p_r(T) \coloneqq \exp\left[\frac{s^\circ(T)}{R}\right], \tag{3.178}$$

so that Eq. (3.177) becomes,

$$\frac{p_2}{p_1} = \frac{p_r(T_2)}{p_r(T_1)}. \tag{3.179}$$

The parameter $p_r(T)$ is often pre-tabulated as a function of temperature for common ideal gases, such as air (for example, refer to Table A-22 in Moran et al., 8th ed., shown in Figure 3.38). A similar approach can be used for determining the relationship between specific volume and temperature. Using the ideal gas law,

$$\frac{v_2}{v_1} = \frac{\frac{RT_2}{p_2}}{\frac{RT_1}{p_1}} = \frac{\frac{RT_2}{p_r(T_2)}}{\frac{RT_1}{p_r(T_1)}} = \frac{v_r(T_2)}{v_r(T_1)}, \tag{3.180}$$

where,

$$v_r(T) \coloneqq \frac{RT}{p_r(T)}. \tag{3.181}$$

As with $p_r(T)$, $v_r(T)$ is often pre-tabulated for common ideal gases, such as air (refer to Table A-22 in Moran et al., 8th ed.; Figure 3.38). Using the quantities $p_r(T)$ and $v_r(T)$ in tabulated form makes it easier to perform calculations involving an ideal gas with variable specific heats. The influence of the variations of the specific heats with temperature are pre-calculated in the variables p_r and v_r.

(7) For an isentropic process involving a perfect gas, Eqs. (3.173) - (3.175) reduce to,

$$0 = c_p\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) \implies \frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{c_p}{R}} = \left(\frac{T_2}{T_1}\right)^{\frac{c_p}{c_p - c_v}} = \left(\frac{T_2}{T_1}\right)^{\frac{c_p/c_v}{c_p/c_v - 1}}, \tag{3.182}$$

$$\boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}}}, \tag{3.183}$$

$$0 = c_v\ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right) \implies \frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{\frac{-c_v}{R}} = \left(\frac{T_2}{T_1}\right)^{\frac{-c_v}{c_p - c_v}} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{1 - c_p/c_v}}, \tag{3.184}$$

$$\boxed{\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{1-k}}}, \tag{3.185}$$

$$0 = c_v\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{\rho_2}{\rho_1}\right) \implies \frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{c_v}{R}} = \left(\frac{T_2}{T_1}\right)^{\frac{c_v}{c_p - c_v}} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{c_p/c_v - 1}}, \tag{3.186}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{k-1}}}. \tag{3.187}$$

Combining Eqs. (3.183) and (3.185) shows,

$$\boxed{\frac{p_2}{p_1} = \left(\frac{v_2}{v_1}\right)^{-k}} \implies p = \frac{c}{v^k} \implies pv^k = c, \tag{3.188}$$

where c is a constant. Hence, an isentropic process involving a perfect gas is a polytropic process with $n = k$, where n is the exponent in the polytropic process equation. Note that if $n = 1$ for an ideal gas, then the polytropic process is isothermal ($pv = RT$). If $n = 0$, then the process is isobaric (constant pressure). If $n = \pm\infty$, then the process is isometric (constant volume).

(8) There are many engineering situations in which the isentropic assumption is a reasonable approximation to real life. For example, in high speed gas flows through nozzles and diffusers, the heat transfer is often small ($\rightarrow$ adiabatic) and the irreversibilities due to viscous dissipation are concentrated in a thin boundary layer adjacent to the walls of the devices. Hence, most of the flow can be considered adiabatic and reversible, meaning that it can also be considered isentropic. This approximation is used frequently in high-speed gas flows. Since air is often the fluid of interest, we often make the additional assumption that the flow involves an ideal or perfect gas. Hence, the previous equations described in Notes 6 and 7 are frequently used.

A vendor claims that an adiabatic air compressor takes in air at standard atmospheric conditions and delivers the air at 650 kPa (gage) and 285 °C. Is this possible? Justify your answer.

SOLUTION:

Use the 2nd Law of Thermodynamics to determine whether or not the process can occur.

At the inlet (state 1), we have standard atmospheric conditions:

$$p_1 = 101 \text{ kPa}, \; T_1 = 293 \text{ K} \tag{1}$$

At the outlet (state 2):

$$p_2 = (650 + 101) \text{ kPa}, \; T_2 = (285 + 273) \text{ K} \tag{2}$$

The change in entropy between the states may be found using the following relation, which assumes that air behaves as a perfect gas:

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) \tag{3}$$

Substitute the given values:

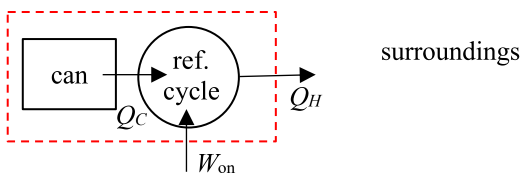
$$s_2 - s_1 = \left(1004 \text{ J/}(\text{kg}\cdot\text{K})\right)\ln\left(\frac{558 \text{ K}}{293 \text{ K}}\right) - \left(287 \text{ J/}(\text{kg}\cdot\text{K})\right)\ln\left(\frac{751 \text{ kPa}}{101 \text{ kPa}}\right) \tag{4}$$

$$\boxed{\therefore s_2 - s_1 = 71 \text{ J/}(\text{kg}\cdot\text{K})} \tag{5}$$

Since for an adiabatic process we must have $\Delta s \geq 0$, the process is feasible.

The temperature of a 12 oz (0.354 l) can of soft drink is reduced from 20 °C to 5 °C by a refrigeration cycle. The cycle receives energy by heat transfer from the soft drink and discharges energy by heat transfer at 20 °C to the surroundings. There are no other heat transfers. Determine the minimum theoretical work input required. You may ignore the aluminum can in your calculations.

SOLUTION:
A schematic of the situation is shown below.



Apply the 1st Law to a system consisting of the can and the refrigeration equipment,

$$\Delta E_{\text{sys}} = Q_{\substack{\text{into}\\ \text{sys}}} + W_{\substack{\text{on}\\ \text{sys}}}, \tag{1}$$

where,

$$\Delta E_{\text{sys}} = \Delta E_{\text{can}} + \Delta E_{\text{ref. cycle}}, \tag{2}$$

where changes in kinetic and potential energies are ignored for the can so $\Delta E_{\text{can}} = \Delta U_{\text{can}}$. Furthermore, since the refrigeration equipment operates over a cycle, $\Delta E_{\text{ref. cycle}} = 0$. Hence, Eq. (1) becomes,

$$\Delta U_{\text{can}} = -Q_H + W_{\text{on}} \quad \Rightarrow \quad W_{\text{on}} = \Delta U_{\text{can}} + Q_H\,. \tag{3}$$

The change in the can internal energy is,

$$\Delta U_{\text{can}} = mc(T_{\text{can},f} - T_{\text{can},i}), \tag{4}$$

where the soft drink is modeled as an incompressible substance since it's a liquid. The parameter m is the mass of the can and c is its specific heat.

The heat transferred out of the system may be found by applying the 2nd Law,

$$\Delta S_{\text{sys}} = \int_{1,b}^{2} \frac{\delta Q_{\text{into}}}{T} + \sigma = \frac{-Q_H}{T_H} + \sigma \quad \Rightarrow \quad Q_H = T_H\sigma - T_H \Delta S_{\text{sys}}, \tag{5}$$

where,

$$\Delta S_{\text{sys}} = \Delta S_{\text{can}} + \Delta S_{\text{ref. cycle}}. \tag{6}$$

Since the refrigeration equipment operates on a cycle, $\Delta S_{\text{ref. cycle}} = 0$. The absolute temperature at the boundary of the system where the heat is transferred out of the system is T_H and σ is the entropy produced during the process due to irreversibilities.

Substituting Eq. (5) into Eq. (3) and simplifying gives,

$$W_{\text{on}} = \Delta U_{\text{can}} + T_H\sigma - T_H \Delta S_{\text{sys}}\,. \tag{7}$$

Since we're interested in the minimum amount of work required during the process, consider the case when $\sigma = 0$ (an internally reversible process). Recall that $\sigma > 0$ when irreversibilities are present. Since the soft drink is assumed to be an incompressible substance,

$$\Delta S_{\text{can}} = mc \ln\left(\frac{T_{\text{can},f}}{T_{\text{can},i}}\right). \tag{8}$$

Substituting Eqs. (4) and (8) into Eq. (7) (with $\sigma = 0$) gives,

$$W_{\text{on,min}} = mc\left(T_{\text{can},f} - T_{\text{can},i}\right) - T_H mc \ln\left(\frac{T_{\text{can},f}}{T_{\text{can},i}}\right), \tag{9}$$

$$\boxed{W_{\text{on,min}} = mc\left[\left(T_{\text{can},f} - T_{\text{can},i}\right) - T_H \ln\left(\frac{T_{\text{can},f}}{T_{\text{can},i}}\right)\right]}. \tag{10}$$

Using the following parameters,

m = (1000 kg/m^3)(0.354 l)(10^{-3} m^3/l) = 0.354 kg (assume the density of liquid water)
c = 4.2 kJ/(kg.K) (assume the specific heat of liquid water)
$T_{\text{can},f}$ = 5 °C = 278 K
$T_{\text{can},i}$ = 20 °C = 293 K
T_H = 20 °C = 293 K
$\Rightarrow$ $\boxed{W_{\text{in,min}} = 0.591 \text{ kJ}}$

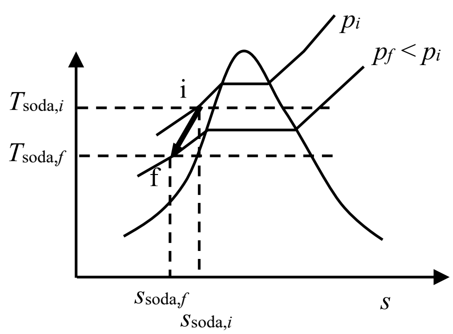
If we assume the soft drink is a compressed liquid instead of being incompressible, then the change in entropy is,

$$\Delta S_{\text{soda}} = m\Delta s_{\text{soda}} \approx m\left[s_l\left(T_{\text{soda,f}}\right) - s_l\left(T_{\text{soda,i}}\right)\right], \tag{11}$$

where $s_{\text{CL}}(T, p) \approx s_l(T)$. Treating the soda as water and using Table A-2 in Moran et al., 8th ed.,

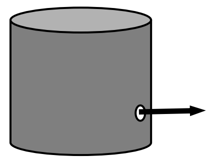
$s_l(T_{\text{soda},f} = 5\ °\text{C}) = 0.0761$ kJ/(kg.K),
$s_l(T_{\text{soda},i} = 20\ °\text{C}) = 0.2966$ kJ/(kg.K),
=> $\Delta S_{\text{soda}} = -0.0781$ kJ/K,

which is identical to the result found using the incompressible substance model.



A rigid, insulated tank with a volume of 21.61 ft^3 is filled initially with air at 100 psia and 535 °R. A leak develops and air slowly escapes until the pressure of the air remaining in the tank is 15 psia. Determine the mass of air remaining in the tank and its temperature.

SOLUTION:



Assume the leaking process is adiabatic because the tank is insulated, and reversible since the air is leaking slowly. Since the process is adiabatic and reversible, it's also isentropic. Furthermore, treat the air as an ideal gas. For an ideal gas undergoing an isentropic process,

$$\frac{p_2}{p_1} = \frac{p_r(T_2)}{p_r(T_1)} \Rightarrow p_r(T_2) = p_r(T_1)\left(\frac{p_2}{p_1}\right), \tag{1}$$

where p_1 = 30 psia, p_2 = 15 psia, and $p_r(T_1 = 535\ °\text{R}) = 1.3423$ (interpolating in Table A-22E, Moran et al., 8th ed.),

=> $p_r(T_2) = 0.6711$.

Interpolating in Table A-22E, we find that $\boxed{T_2 = 439\ °\text{R}}$.

Note that if we assume air is a perfect gas (an ideal gas with constant specific heats), then for an isentropic process,

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} \Rightarrow T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}, \tag{2}$$

=> $\boxed{T_2 = 439\ °\text{R}}$, which is identical to the result found assuming ideal gas behavior.

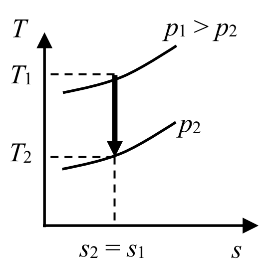
The mass of air remaining in the tank may be found using the ideal gas law,

$$p_2V_2 = m_2RT_2 \Rightarrow m_2 = (p_2V_2)/(RT_2), \tag{3}$$

where V_2 = 21.61 ft^3, $\bar{R}_u$ = 1545 ft.lb$_f$/(lbmol.°R), and M = 28.97 lb$_m$/lbmol,

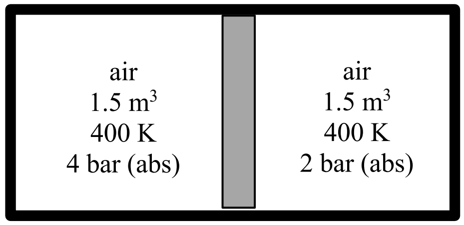
=> $R_{air} = \bar{R}_u/M$ = (1545 ft.lb$_f$/(lbmol.°R)/28.97 lb$_m$/lbmol) = 53.33 ft.lb$_f$/(lb$_m$.°R),

=> $\boxed{m_2 = 1.99\ \text{lb}_m}$.



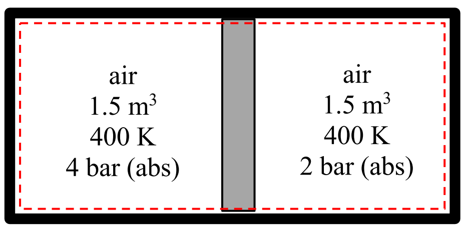
An insulated box is initially divided into halves by a frictionless, thermally conducting piston. On one side of the piston is 1.5 m^3 of air at 400 K, 4 bar (abs). On the other side is 1.5 m^3 of air at 400 K, 2 bar (abs). The piston is released and equilibrium is attained, with the piston experiencing no change of state. Determine:

a. the final temperature,
b. the final pressure, and
c. the entropy produced.



SOLUTION:

Apply the 1st Law to a system consisting of the two chambers of air and the piston,



$$\Delta E_{\text{sys}} = Q_{\substack{\text{into}\\\text{sys}}} + W_{\substack{\text{on}\\\text{sys}}} \Rightarrow \Delta U_{\text{sys}} = 0 \tag{1}$$

where $\Delta E_{\text{sys}} = \Delta U_{\text{sys}}$ (no changes in kinetic and potential energies), $Q_{\text{into sys}} = 0$ (adiabatic) and $W_{\text{on sys}} = 0$ (rigid tank). Noting also that,

$$\Delta U_{\text{sys}} = \Delta U_1 + \Delta U_2 + \Delta U_{\text{piston}}, \tag{2}$$

where ΔU_1 is the change in the internal energy of the air on the left hand side, ΔU_2 is the change in internal energy of the air on the right hand side, and ΔU_{piston} is the change in internal energy of the piston ($\Delta U_{\text{piston}} = 0$), Eq. (2) may be written as,

$$\Delta U_1 + \Delta U_2 = 0 \Rightarrow m_1\Delta u_1 + m_2\Delta u_2 = 0 \Rightarrow m_1(u_{1,f} - u_{1,i}) + m_2(u_{2,f} - u_{2,i}) = 0. \tag{3}$$

Since the initial temperatures of the air are the same ($T_{1,i} = T_{2,i} = T_i = 400$ K), $u_{1,i} = u_{2,i} = u_i$ (assuming air behaves as an ideal gas) and since at final equilibrium the temperatures are also the same ($T_{1,f} = T_{2,f} = T_f$), $u_{1,f} = u_{2,f} = u_f$,

$$(m_1 + m_2)u_f - (m_1 + m_2)u_i = 0,$$

$$\Rightarrow u_f = u_i \Rightarrow \boxed{T_f = T_i = 400\text{ K}}. \tag{4}$$

The final pressures in each chamber will also be equal at equilibrium,

$$p_{1,f} = p_{2,f} = p_f. \tag{5}$$

From the ideal gas law applied to the whole system,

$$p_f = \frac{(m_1 + m_2)RT_f}{(V_{1,f} + V_{2,f})}, \tag{6}$$

where $V_{1,f} + V_{2,f} = V_{1,i} + V_{2,i} = V_{\text{total}} = 3.0$ m^3. The masses in each chamber may also be found using the ideal gas law and the initial conditions,

$$m_1 = \frac{p_{1,i}V_{1,i}}{RT_{1,i}} \quad \text{and} \quad m_1 = \frac{p_{1,i}V_{1,i}}{RT_{1,i}}. \tag{7}$$

where,

$p_{1,i}$ = 4 bar = 4*10^5 Pa
$V_{1,i}$ = $V_{2,i}$ = 1.5 m^3
R_{air} = 287 J/(kg.K)
$T_{1,i}$ = $T_{2,i}$ = 400 K
$p_{2,f}$ = 2 bar = 2*10^5 Pa
$\Rightarrow$ m_1 = 5.23 kg and m_2 = 2.61 kg
$\Rightarrow$ $\boxed{p_f = 3*10^5 \text{ Pa} = 3 \text{ bar}}$

The entropy produced during the process may be determined using the *Tds* equation for an ideal gas,

$$s_{1,f} - s_{1,i} = s^0\left(T_{1,f}\right) - s^0\left(T_{1,i}\right) - R\ln\left(\frac{p_{1,f}}{p_{1,i}}\right), \tag{8}$$

$$s_{2,f} - s_{2,i} = s^0\left(T_{2,f}\right) - s^0\left(T_{2,i}\right) - R\ln\left(\frac{p_{2,f}}{p_{2,i}}\right), \tag{9}$$

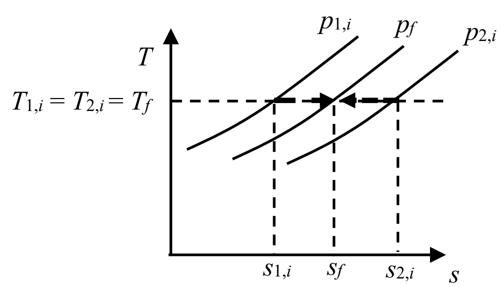
where, since $T_f = T_i$, the s^0 terms cancel. Using the given and calculated pressures,

$s_{1,f} - s_{1,i}$ = 8.26*10^1 J/(kg.K),
$s_{2,f} - s_{2,i}$ = -1.16*10^2 J/(kg.K).

The total change in entropy for the system is,

$$S_f - S_i = m_1\Delta s_1 + m_2\Delta s_2, \tag{10}$$

$\Rightarrow$ $\boxed{\Delta S = 1.27*10^2 \text{ kJ/K}}$



3.8. Common Thermodynamic Cycles

Important: The reader should read the Conservation of Mass, First Law, and Second Law sections of Chapter 4 prior to reading this section since this section involves applying these relations to control volumes, which have mass crossing the control volume boundaries.

In this section we review several thermodynamic cycles encountered in practice, including:

- the Carnot cycle (not common in practice, but real cycles are often compared to this ideal cycle),
- the Rankine cycle and improvements to the Rankine cycle,
- the Otto, Diesel, and dual cycles, and
- the Brayton cycle and improvements to the Brayton cycle.

3.8.1. Carnot Cycle

The Carnot cycle is one particular type of internally reversible cycle and serves as a point of comparison for other real and internally reversible cycles. Carnot cycles include power, refrigeration, and heat pump cycles. This section, however, focuses specifically on Carnot power cycles. The Carnot refrigeration and heat pump cycles are similar to the power cycle, but operate in the opposite direction.

The Carnot power cycle consists of the following four internally reversible processes:

(1) *Process 1 – 2*: Isothermal expansion at T_H.
(2) *Process 2 – 3*: Adiabatic expansion ($Q_{23} = 0$).
(3) *Process 3 – 4*: Isothermal compression at T_C.
(4) *Process 4 – 1*: Adiabatic compression ($Q_{41} = 0$).

As mentioned previously, a Carnot refrigeration or heat pump cycle would operate in reverse, i.e., Process 1 - 2 would be an isothermal compression, Process 2 - 3 would be an adiabatic compression, etc.

Notes:

(1) Figure 3.39 illustrates the four Carnot power cycle processes for a piston-cylinder arrangement. The corresponding processes are shown on the p-v and T-s diagrams for a substance remaining entirely in the vapor phase throughout the cycle. Note that Carnot cycles can also involve working fluids that change phase.
(2) One can also have a Carnot cycle involving a system of components, as shown in Figure 3.40. The same four processes described at the beginning of this section occur in this system. In the particular case shown in the p-v and T-s plots, the working fluid is a saturated liquid vapor mixture.
(3) Since each of the processes in a Carnot cycle is internally reversible, the entire Carnot cycle is also internally reversible. Furthermore, in a Carnot cycle, the absolute temperatures at which the energy is added to or removed from the working fluid via heat transfer are T_H and T_C. Hence, the efficiency and coefficients of performance of a Carnot cycle are given by,

$$\eta_{\text{rev}} = 1 - \frac{T_C}{T_H}, \tag{3.189}$$

$$COP_{\text{ref,rev}} = \frac{T_C}{T_H - T_C}, \tag{3.190}$$

$$COP_{\text{HP,rev}} = \frac{T_H}{T_H - T_C}. \tag{3.191}$$

(4) In order to be internally reversible, the heat addition and removal processes with the thermal reservoirs must not occur over a finite temperature gradient, i.e., the temperature of the system must equal the temperature of the thermal reservoir during the heat addition and heat removal processes. In a real system, there must be some finite temperature difference to drive the heat transfer process, which is one reason a real cycle would not be reversible. Similarly, in order to be internally reversible, the working fluid must have no viscosity, there should be no friction in the system, and all of the processes must be in quasi-equilibrium.

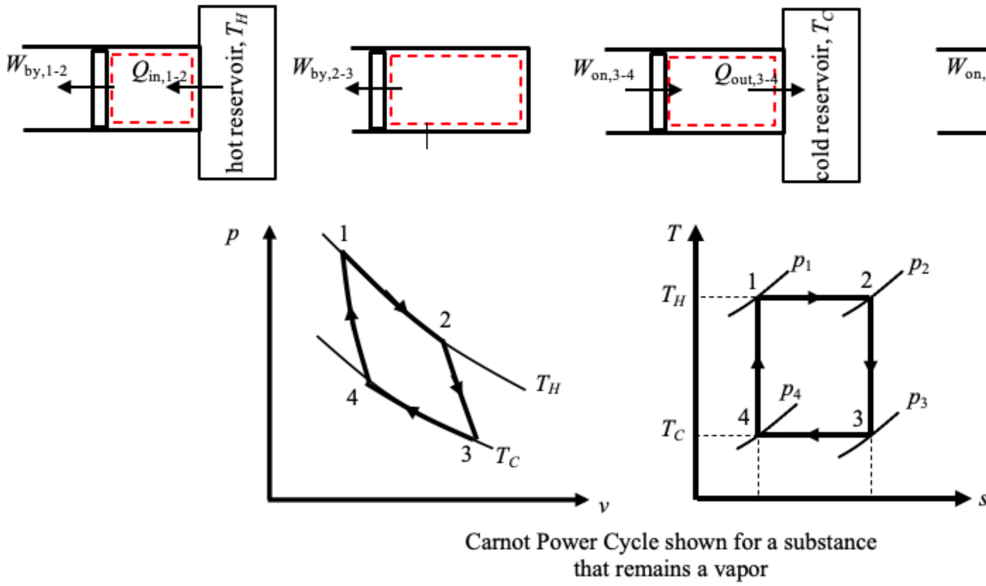


Figure 3.39. A sketch of the four processes of a Carnot power cycle for a piston-cylinder system. The corresponding processes are sketched on p-v and T-s plots for a working fluid that remains entirely in the vapor phase throughout the cycle.

(5) It's possible for a system to have the same set of processes as the Carnot cycle (isothermal expansion, adiabatic expansion, isothermal compression, adiabatic compression), but not be reversible. For example, the heat transfer may occur over a finite temperature difference or the working fluid may be viscous. In that case, the efficiency of the cycle will be less than the Carnot efficiency.

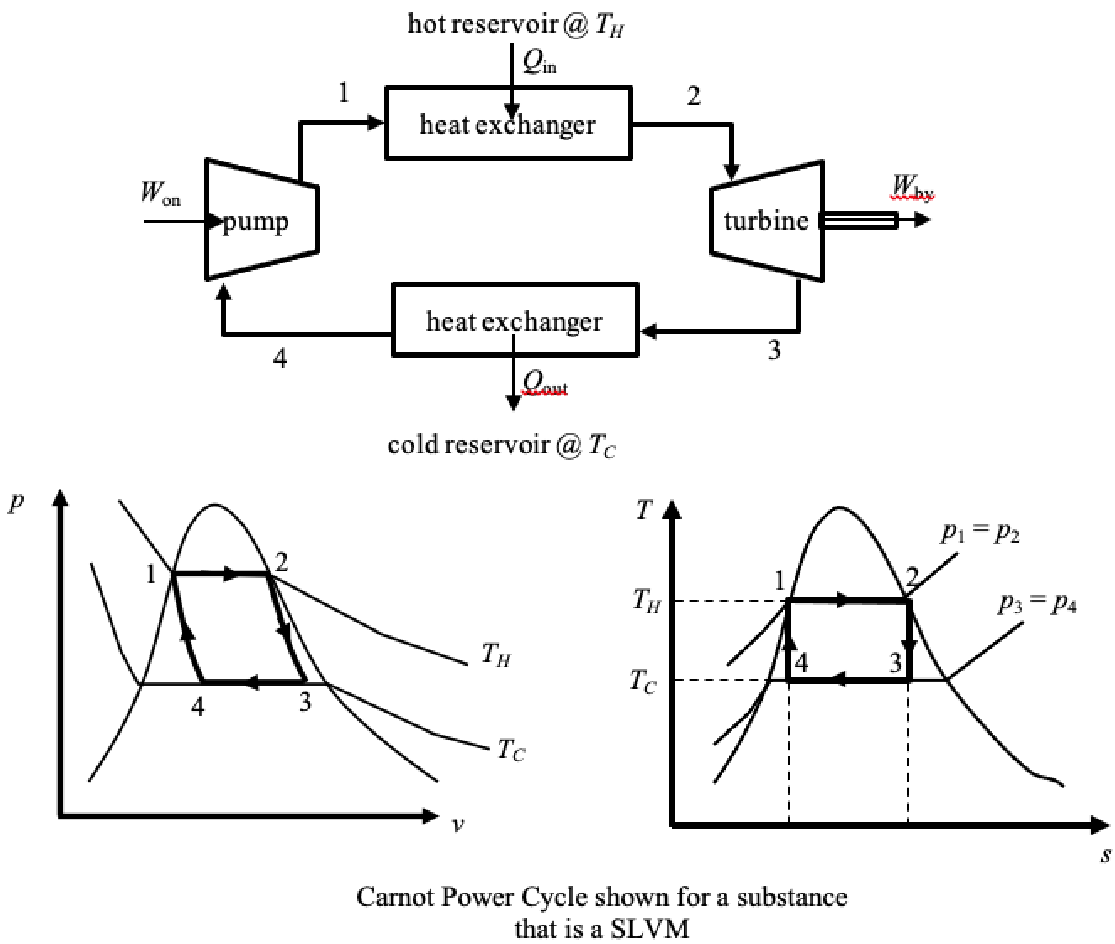
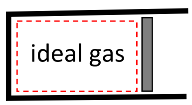


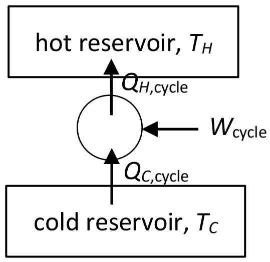
Figure 3.40. A sketch of a Carnot power cycle and p-v, and T-s plots for a simple vapor power cycle in which the working fluid remains as a saturated liquid-vapor mixture.

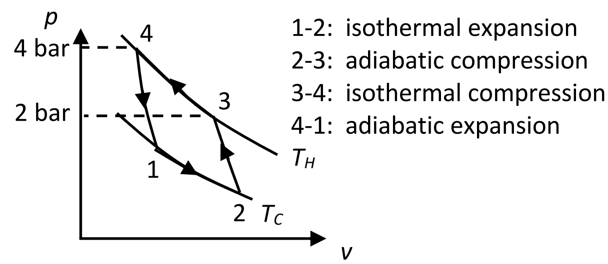
An ideal gas within a piston-cylinder assembly undergoes a Carnot refrigeration cycle. The isothermal compression occurs at 325 K from 2 bar (abs) to 4 bar (abs). The isothermal expansion occurs at 250 K. Determine:

a. the coefficient of performance,
b. the heat transfer to the gas during the isothermal compression, in kJ per kmol of gas,
b. the heat transfer to the gas during the isothermal expansion, in kJ per kmol of gas, and
c. the magnitude of the net work input over the cycle, in kJ per kmol of gas.

SOLUTION:







Since a Carnot cycle is reversible, the coefficient of performance is,

$$\mathrm{COP}_{\substack{\text{ref,}\\ \text{rev}}} = \frac{T_C}{T_H - T_C}\ , \tag{1}$$

with T_H = 325 K and T_C = 250 K => $\boxed{\mathrm{COP}_{\text{ref,rev}} = 3.33}$.

The heat transfer to the gas during isothermal compression (process 3-4) can be found using the 1st Law for the process. From the 1st Law applied to the system identified in the figure, specifically for the process from 3 to 4,

$$\Delta E_{\text{sys},34} = Q_{\text{into sys},34} + W_{\text{on sys},34} \Rightarrow Q_{\text{into sys},34} = W_{\text{by sys},34}\ , \tag{2}$$

where $\Delta E_{\text{sys},34}$ = 0 since we're assuming no change in kinetic or potential energy during the process, and since the process is isothermal and we're working with an ideal gas in which $u = u(T)$, $\Delta U_{34} = 0$. The work done by the system during this compression is,

$$W_{\text{by sys},34} = \int_{V_3}^{V_4} p\,dV = \int_{V_3}^{V_4} \frac{mRT_H}{V}dV = mRT_H \int_{V_3}^{V_4} \frac{dV}{V} = mRT_H \ln\left(\frac{V_4}{V_3}\right) = mRT_H \ln\left(\frac{p_3}{p_4}\right), \tag{3}$$

where the ideal gas law has been used a couple of times ($pV = mRT$) couple with the fact that the process is isothermal with $T = T_H$ and m = constant. Since we're asked to find the heat in kJ per kmol, re-write Eq. (3) on a per mole basis (recall $m = n/M$ and $R = MR_u$, where M is molecular weight),

$$W_{\text{by sys},34} = (n/M)(M\bar{R}_u)T_H \ln\left(\frac{p_3}{p_4}\right) \Rightarrow \frac{W_{\text{by sys},34}}{n} = \bar{R}_u T_H \ln\left(\frac{p_3}{p_4}\right). \tag{4}$$

Combining Eqs. (4) and (2) gives,

$$\frac{Q_{\text{into sys},34}}{n} = \bar{R}_u T_H \ln\left(\frac{p_3}{p_4}\right). \tag{5}$$

Using the given data,

R_u = 8.314 kJ/(kmol.K)
T_H = 325 K
p_3 = 2 bar (abs)
p_4 = 4 bar (abs)
=> $\boxed{Q_{\text{into sys},34}/n = -1870\ \text{kJ/kmol}}$ (heat is leaving the system during this compression process)

To determine the heat transfer during isothermal expansion, make use of the definition of the absolute temperature scale,

$$\left.\frac{Q_C}{Q_H}\right|_{\substack{\text{rev,}\\ \text{cycle}}} = \frac{T_C}{T_H} \Rightarrow Q_C = Q_H \frac{T_C}{T_H} \Rightarrow Q_{12} = -Q_{34} \frac{T_C}{T_H}, \tag{6}$$

where $Q_C = Q_{\text{into},12}$ and $Q_H = -Q_{\text{into},34}$. Using the given data,

=> $\boxed{Q_{\text{into},12}/n = 1440 \text{ kJ/kmol.}}$

The magnitude of the net work input over the cycle may be found using the definition of the coefficient of performance for a refrigeration cycle,

$$COP_{\text{ref}} \equiv \frac{Q_C}{W_{\text{on sys,net}}} \Rightarrow W_{\text{on sys,net}} = \frac{Q_C}{COP_{\text{ref}}}. \tag{7}$$

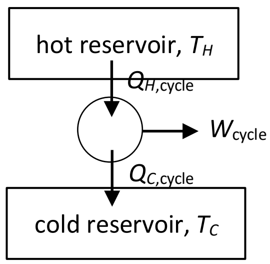
Using the given data,

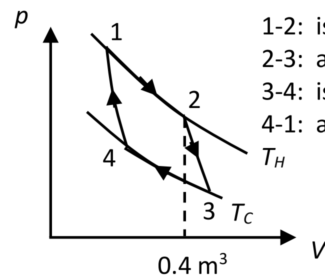
=> $\boxed{W_{\text{on sys,net}}/n = 433 \text{ kJ/kmol}}$.

Two kilograms of air within a piston-cylinder assembly execute a Carnot power cycle with maximum and minimum temperatures of 750 K and 300 K, respectively. The heat transfer to the air during the isothermal expansion is 60 kJ. At the end of the isothermal expansion the volume is 0.4 m^3. Assuming the ideal gas model for the air, determine:

a. the thermal efficiency,
b. the pressure and volume at the beginning of the isothermal expansion, in kPa (abs) and m^3, respectively,
c. the work and heat transfer for each of the four processes, in kJ, and
d. sketch the cycle on a *p-V* diagram.

SOLUTION:





Since the cycle is a Carnot power cycle, which is reversible, the efficiency is given by,

$$\eta_{\text{rev}} = 1 - \frac{T_C}{T_H}. \tag{1}$$

Using the given temperatures (T_C = 300 K and T_H = 750 K) => $\boxed{\eta_{\text{rev}} = 0.6}$.

The pressure and volume at the beginning of the isothermal expansion (state 1) may be found combining the 1st Law applied to the system shown in the figure, with the given heat transfer, and the definition of the power cycle efficiency. Applying the 1st Law to the system for the process from 1 to 2,

$$\Delta E_{12} = Q_{\text{into},12} + W_{\text{on},12} \Rightarrow W_{\text{by},12} = Q_{\text{into},12}\ , \tag{2}$$

where $W_{\text{by},12} = -W_{\text{on},12}$. Furthermore, $\Delta E_{12} = 0$ since the change in kinetic and potential energy in going from state 1 to state 2 is zero, and since the process is isothermal and the system is an ideal gas ($u = u(T)$) => $\Delta U_{12} = 0$. The heat added to the system in going from 1 to 2 is given (= 60 kJ) and thus the work done by the system is also known. The work done by the system may also be found using,

$$W_{\text{by},12} = \int_{V_1}^{V_2} p\,dV = \int_{V_1}^{V_2} \frac{mRT_H}{V}dV = mRT_H \int_{V_1}^{V_2} \frac{dV}{V} = mRT_H \ln\left(\frac{V_2}{V_1}\right), \tag{3}$$

where the ideal gas law has been used ($pV = mRT$) along with the fact that the process from 1 to 2 is isothermal with $T = T_H$. The mass is constant too. Since the volume at the end of the isothermal expansion process is given (V_2 = 0.4 m^3), we can solve for V_1,

$$W_{\text{by},12} = mRT_H \ln\left(\frac{V_2}{V_1}\right) \Rightarrow V_1 = V_2 \exp\left(-\frac{W_{\text{by},12}}{mRT_H}\right), \tag{4}$$

with,

V_2 = 0.4 m^3,
$W_{\text{by},12}$ = 60 kJ,
m = 2 kg,
R_{air} = 0.287 kJ/(kg.K),
T_H = 750 K,
=> $\boxed{V_1 = 0.348\ \text{m}^3}$.

The pressure at state 1 may be found using the ideal gas law,

$$p_1 = \frac{mRT_H}{V_1} \quad (T_1 = T_H), \tag{5}$$

=> $\boxed{p_1 = 1.24 \text{ MPa (abs)}}$.

The work and heat for process 1-2 have already been calculated. The heat for processes 2-3 and 4-1 are zero, i.e., $\boxed{Q_{23} = Q_{41} = 0}$, since these processes are adiabatic.

The work done on the system during process 2-3 may be found using the 1st Law,

$$\Delta E_{23} = Q_{\text{into},23} + W_{\text{on},23} \Rightarrow W_{\text{on},23} = \Delta U_{23} = m\left(u_3 - u_2\right), \tag{6}$$

where $u_2 = u_2(T_2 = T_H)$ and $u_3 = u_3(T_3 = T_C)$ since the air is being treated as an ideal gas. These specific internal energies may be looked up in a table, e.g., Table A-22 from Moran et al., 8th ed.,

$u_2 = 551.99$ kJ/kg and $u_3 = 214.07$ kJ/kg => $\boxed{W_{\text{on},23} = \Delta U_{23} = 676 \text{ kJ}}$. (7)

If a perfect gas model is used, then,

$\Delta U_{23} = mc_v(T_3 - T_2)$ => $\Delta U_{23} = 673$ kJ, (8)

where $c_v = 0.748$ kJ/(kg.K) (Table A-20, Moran et al., 8th ed.). This result is less than 1% different than the ideal gas result.

The work done on the system during process 4-1 will be identical in magnitude, but opposite in sign to the work done on the system during process 2-3 since both processes are adiabatic and operate between the same temperatures. Hence, $\boxed{W_{\text{on},41} = -676 \text{ kJ}}$.

For process 3-4, the heat transferred into the system may be found using the definition of the absolute temperature scale (since the process is assumed reversible),

$$\left.\frac{Q_C}{Q_H}\right|_{\substack{\text{rev,}\\ \text{cycle}}} = \frac{T_C}{T_H} \Rightarrow Q_C = Q_H\left(\frac{T_C}{T_H}\right) \Rightarrow Q_{\text{out of},34} = Q_{\text{into},12}\left(\frac{T_C}{T_H}\right), \tag{9}$$

Using the given data, $\boxed{Q_{\text{out of},34} = 24 \text{ kJ}}$.

Since the process from 3 to 4 is isothermal, we're dealing with an ideal gas, and there is no change in kinetic or potential energy, the 1st Law gives,

$$\Delta E_{34} = Q_{\text{into},34} + W_{\text{on},34} \Rightarrow W_{\text{on},34} = Q_{\text{out of},34}, \tag{10}$$

=> $\boxed{W_{\text{on},34} = 24 \text{ kJ}}$.

As a check, we can substitute the heat and work values into the definition for the power cycle efficiency,

$$\eta \equiv \frac{W_{\text{by sys,net}}}{Q_{\text{into sys},12}}, \tag{11}$$

where $W_{\text{by sys,net}} = W_{\text{by},12} + W_{\text{by},23} + W_{\text{by},34} + W_{\text{by},41}$ = (60 kJ) + (676 kJ) + (-24 kJ) + (-676 kJ) = 36 kJ,

=> $\eta = 0.6$, which is exactly the value expected from part (a).

Also, from the 1st Law applied to the entire cycle (recall $\Delta E_{\text{cycle}} = 0$), we must have,

$Q_{\text{into,net}} = W_{\text{by,net}}$,

where $Q_{\text{into,net}} = Q_{\text{into},12} + Q_{\text{into},23} + Q_{\text{into},34} + Q_{\text{into},41}$ = (60 kJ) + (0) + (-24 kJ) + (0) = 36 kJ, which is precisely $W_{\text{by,net}}$ found previously.

3.8.2. The Rankine Cycle and Improvements

The Rankine Cycle and its variations are commonly used vapor power cycles for large-scale power generation, such as in natural gas and coal-fired power plants, nuclear power plants, and solar power plants. The standard Rankine cycle consists of the following four processes (refer to Figure 3.41):

- *Process 1 – 2*: Expansion of the working fluid from saturated vapor through the turbine.
- *Process 2 – 3*: Heat transfer from the working fluid as it flows at constant pressure through the condenser to a saturated liquid state.
- *Process 3 - 4*: Compression of the working fluid in the pump in the compressed liquid region.
- *Process 4 - 1*: Heat transfer to the working fluid as it flows at constant pressure through the boiler.

If the turbine and pump have 100% isentropic efficiencies, then the cycle is referred to as an "ideal standard Rankine cycle", state 2 is identified as state "2s", and state 4 is identified as state "4s".

Notes:

(1) Using Eq. (3.189) *as a guide*, we observe that the thermal efficiency of a power cycle generally increases as the *average* temperature at which the heat is added in the boiler increases and the *average* temperature at which the heat is rejected in the condenser decreases.

(a) An (internally reversible) Carnot cycle has a larger thermal efficiency than an ideal (internally reversible) Rankine cycle operating between the same two thermal reservoirs since the *average* temperature at which heat is added in the boiler is smaller for the Rankine cycle due to the portion of the path from States 4 - 1 in the condensed liquid phase.

(b) Increasing the average temperature at which heat is added may be achieved by increasing the boiler pressure, thus shifting the path from 4 - 1 to a larger temperature isotherm, or by moving State 1 into a superheated vapor (SHV) phase along the same isotherm. Increasing the boiler pressure can be costly due to the increased stress on the pipe system; however, moving State 1 into a SHV phase while at the same pressure is relatively easy. Moving into the SHV region is known as a Rankine Cycle with Superheat and is discussed in a following note.

(c) The smallest possible condenser temperature corresponds to just larger than the temperature of the surroundings since the surroundings are where the heat is being rejected. Recall that heat is transferred from a hotter object to a colder one so the working fluid temperature would need to be slightly larger than the surrounding temperature. In practice, the cold reservoir usually corresponds to the atmospheric air or a large body of water, such as the ocean, a lake, or a river.

(d) The typical thermal efficiency of a standard Rankine cycle is on the order of 20 - 40%.

(e) The ratio of the power required by the pump to the power generated by the turbine is known as the back work ratio, *bwr*,

$$bwr := \frac{\dot{W}_P}{\dot{W}_T}. \tag{3.192}$$

The *bwr* in a typical Rankine cycle is generally small - on the order of 1 - 3%, for example.

(f) The vibration and mechanical stress generated when a pump impeller encounters alternating regions of vapor (small density) and liquid (large density) can damage the pump. Thus, states 3 and 4 are always either in a saturated liquid or compressed liquid phase. In practice, state 3 is often in the compressed liquid region rather than in a saturated liquid phase to provide a margin of safety to keep the working fluid liquid.

(g) Assuming internally reversible, adiabatic pump operation, i.e., 100% isentropic efficiency, the power required by the pump is (refer to Eq. (4.152), assuming negligible changes in kinetic and potential energies across the pump),

$$\dot{W}_P = \dot{m}v(p_4 - p_3). \tag{3.193}$$

This power assumes that the working fluid can be modeled as an incompressible substance through the pump, which is reasonable since, as the previous note states, the working fluid is a compressed liquid through the pump.

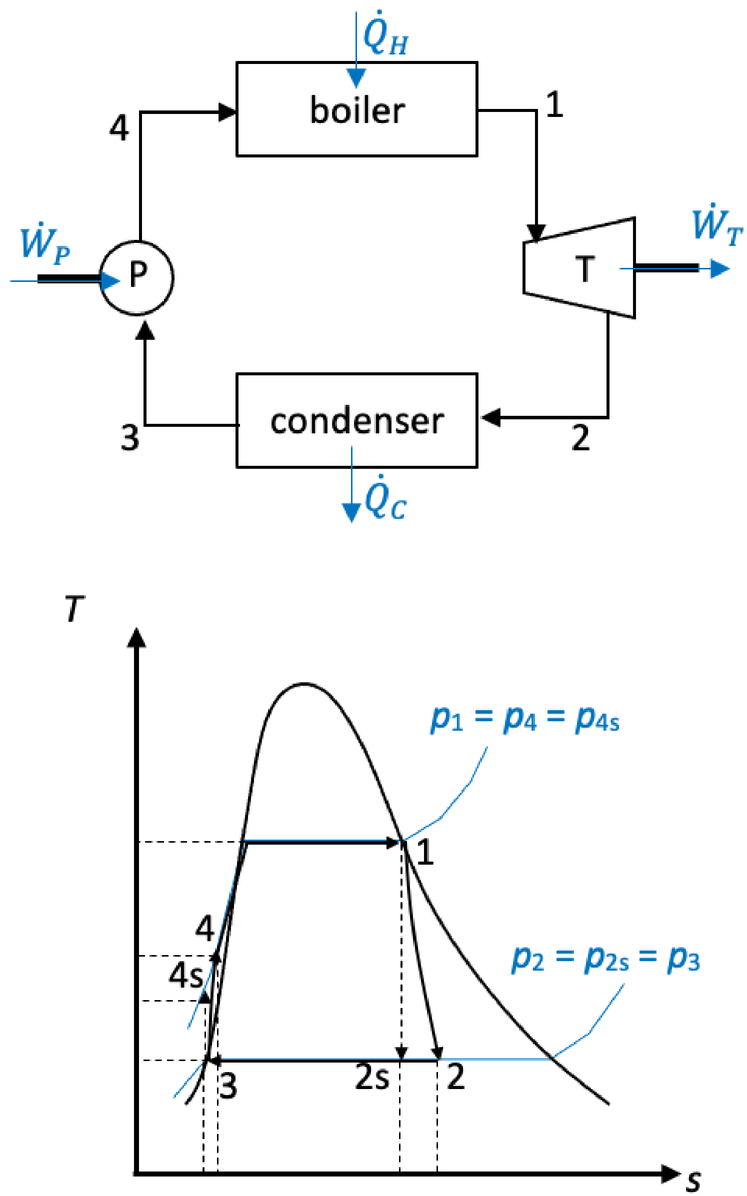


Figure 3.41. A sketch showing the components of a standard vapor power Rankine cycle. The corresponding processes are sketched on a T - s plot. The "2s" and "4s" states correspond to, respectively, flow through the turbine and pump with 100% isentropic efficiencies.

(h) Again for mechanical reasons, turbines perform best with superheated vapor or a saturated liquid vapor mixture at a large quality. Liquid droplets impacting high speed turbine blades can cause damage.

(2) As discussed previously, one improvement to the standard Rankine cycle is the Rankine Cycle with Superheat. The components of this cycle are identical to the standard Rankine cycle, but the T - s diagram is different (Figure 3.42). Specifically, in order to raise the average temperature at which energy is added via heat transfer from the hot reservoir, the working fluid leaves the boiler in a superheated vapor phase (State 3).

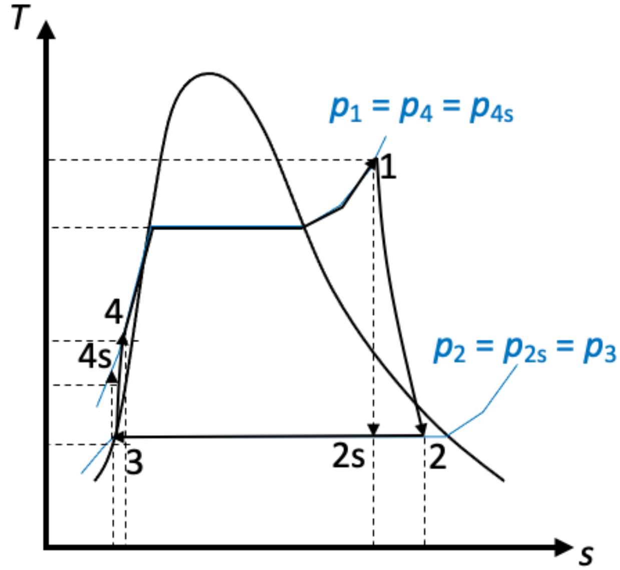


Figure 3.42. A T - s plot for a Rankine cycle with superheating.

(a) The larger average temperature at which heat is added in a Rankine cycle with superheating results in a larger thermal efficiency as compared to a standard Rankine cycle.
(b) Another advantage of superheating is that the working fluid passes through the turbine either as a superheated vapor or a high quality saturated liquid-vapor mixture.

(3) An illustration of a Rankine Cycle with Reheat and the corresponding T - s plot are shown in Figure 3.43). In this cycle, the working fluid leaves the boiler (also often called a steam generator) at State 1, passes through a first-stage turbine (State 2), then re-enters the boiler for additional heating (State 3; hence, the name "reheat"). The re-energized working fluid then passes through a second-stage turbine (State 4) to complete the remainder of the cycle.
(a) The reheating process increases the average temperature at which heat is added in the cycle, thus, increasing the thermal efficiency as compared to a standard Rankine cycle.
(b) The first stage turbine typically exits in the superheated vapor phase (State 2). In addition, the quality at the exit of the second stage turbine (State 4) is larger than that in a standard Rankine cycle and may even be in a superheated vapor phase.

(4) The Rankine Cycle with Supercritical Reheat has the same components as the reheat cycle, but heating in the steam generator occurs in the supercritical phase (above the critical point on the vapor dome). A representative T - s plot is shown in Figure 3.44).
(a) The larger average temperature at which heat is added in a Rankine cycle with supercritical reheating results in a larger efficiency as compared to a standard Rankine cycle.
(b) The first stage turbine typically exits in the superheated phase (State 2). In addition, the quality at the exit of the second stage turbine (State 4) is larger than that in a standard Rankine cycle and may even be in a superheated vapor phase.
(c) The large pressures and temperatures in a Rankine cycle with supercritical reheat requires the use of more expensive components, including high pressure piping and steam generator, and turbine materials that can withstand high temperatures. Thus, the capitol cost of this type of facility is much higher than it would be for a standard Rankine cycle facility. However, a Rankine cycle with supercritical reheating can achieve thermodynamic efficiencies of up to nearly 50%.

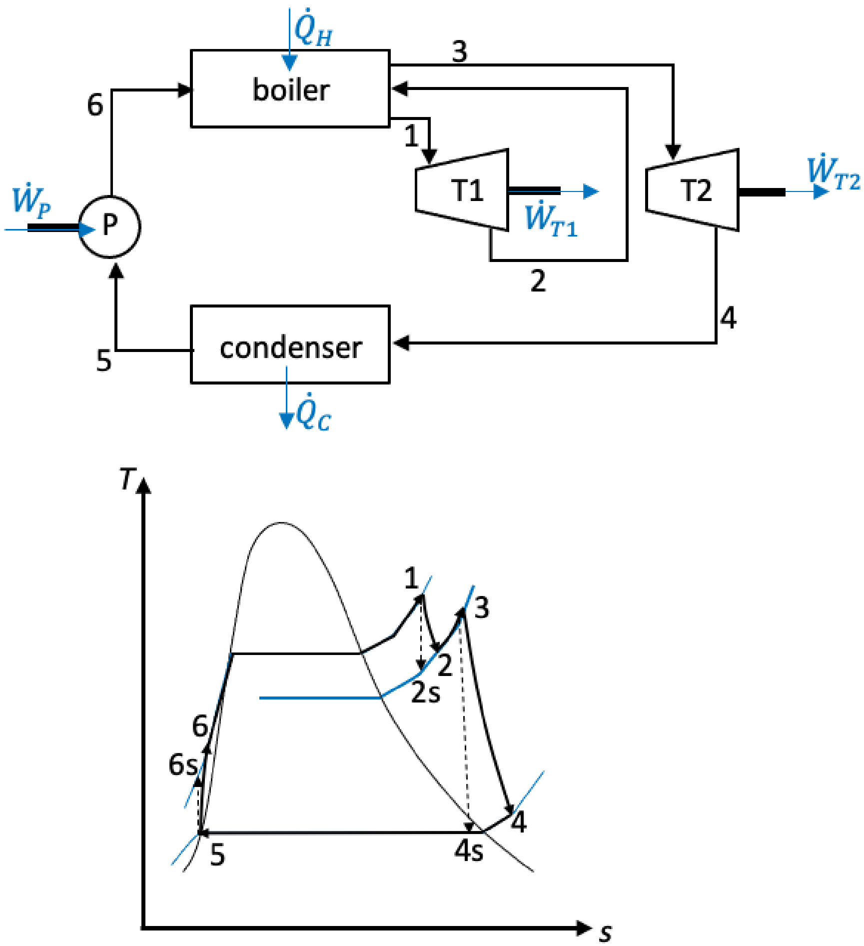


Figure 3.43. An illustration of a Rankine cycle with reheat and the corresponding T-s plot.

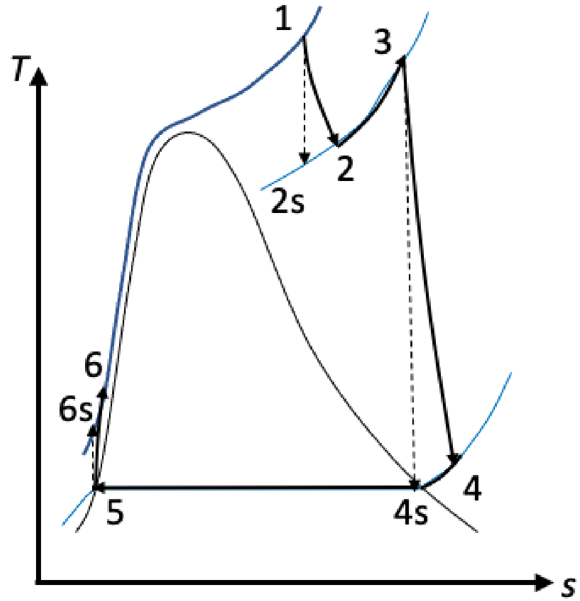
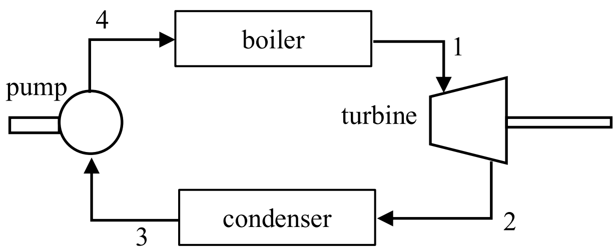


Figure 3.44. A T - s plot for a Rankine cycle with supercritical reheating.

Consider a steam-power plant cycle in which saturated water vapor enters the turbine at 12.0 MPa (abs) and saturated liquid exits the condenser at a pressure of 0.012 MPa (abs). The net power output of the cycle is 122 MW.

a. Assuming that the isentropic efficiencies of the turbine and pump are 80%, determine the following:
   i. the mass flow rate of the water, in kg/h,
   ii. the rate of heat transfer into the boiler, in MW
   iii. the rate of heat transfer from the condenser, in MW, and
   iv. the thermal efficiency of the power plant cycle.
b. Draw a *T-s* diagram for the cycle, clearly indicating the process paths, states, and isobar values.



SOLUTION:
First determine the properties at each of the states.

At State 1:
We're given that the water is in a saturated vapor phase and p_1 = 12.0 MPa (abs) = 120 bar (abs). Using the Saturated Property Tables for water,
$T_1 = T_{1,sat}$ = 324.68 °C, $h_1 = h_{1g}$ = 2685.4 kJ/kg, and $s_1 = s_{1g}$ = 5.4939 kJ/(kg.K)

At State 2:
We're given that the turbine has an isentropic efficiency of 80%. In addition, since the pressure is assumed to remain constant across the condenser, $p_2 = p_3$ = 0.012 MPa (abs) = 0.12 bar (abs). At this pressure, interpolating from the Saturated Property Tables for water,
$T_2 = T_{2,sat}$ = 48.66 °C, h_{2f} = 203.73 kJ/kg, h_{2g} = 2588.9 kJ/kg, s_{2f} = 0.68576 kJ/(kg.K), and s_{2g} = 8.10048 kJ/(kg.K).
The isentropic efficiency of the turbine is given by,

$$\eta_{turbine,isen} \equiv \frac{\dot{W}_{by\ CV}}{\dot{W}_{by\ CV,isen}} = \frac{h_1-h_2}{h_1-h_{2s}}, \tag{1}$$

Thus,

$$h_2 = h_1 - \eta_{turbine,isen}(h_1 - h_{2s}), \tag{2}$$

To find h_{2s}, assume the turbine operates isentropically from 1 to 2, so that $s_{2s} = s_1$ = 5.4939 kJ/(kg.K). Thus,

$$x_{2s} = \frac{h_{2s}-h_{2f}}{h_{2g}-h_{2f}} = \frac{s_{2s}-s_{f2}}{s_{g2}-s_{f2}} \Rightarrow h_{2s} = h_{2f} + \left(h_{2g} - h_{2f}\right)\left(\frac{s_{2s}-s_{2f}}{s_{2g}-s_{2f}}\right). \tag{3}$$

Using the values found previously, h_{2s} = 1748.33 kJ/kg. Substituting into Eq. (2) gives,
h_2 = 1937.41 kJ/kg.
The quality for this state is,

$$x_2 = \frac{h_2-h_{2f}}{h_{2g}-h_{2f}} \Rightarrow x_2 = 0.7269. \tag{4}$$

The specific entropy at state 2 is then,

$$\Rightarrow s_2 = (1-x_2)s_{2f} + x_2 s_{2g} \Rightarrow s_2 = 6.07521 \text{ kJ/(kg.k)}. \tag{5}$$

Note that $s_2 > s_1$, as expected for adiabatic operation of the turbine.

At State 3:

We're given that the water is in a saturated liquid phase and p_3 = 0.012 MPa (abs) = 0.12 bar (abs). Using the Saturated Property Tables for water and interpolating,

T_3 = 48.66 °C, h_3 = 203.73 kJ/kg, v_3 = 0.0010012 m^3/kg, and s_3 = 0.68576 kJ/(kg.K)

At State 4:

We're given that the pump has an isentropic efficiency of 80%. In addition, since the pressure is assumed to remain constant across the boiler, $p_1 = p_4$ = 12.0 MPa (abs) = 120 bar (abs).

The isentropic efficiency of the pump is given by,

$$\eta_{pump,isen} \equiv \frac{\dot{W}_{into\,CV,isen}}{\dot{W}_{into\,CV}} = \frac{h_{4s}-h_3}{h_4-h_3}, \quad (6)$$

$$h_4 = h_3 + \frac{(h_{4s}-h_3)}{\eta_{pump,isen}}, \quad (7)$$

Assuming an isentropic process from State 3 to State 4s (i.e., $s_{4s} = s_3$ = 0.68576 kJ/(kg.K)) and since the water can be treated as an incompressible substance at State 4s,

$$Tds = dh - vdp \Rightarrow dh = vdp \Rightarrow h_{4s} - h_3 = v_3(p_{4s} - p_3), \quad (8)$$

Using the parameters calculated previously, along with p_{4s} = 120 bar (abs),

h_{4s} = 215.86 kJ/kg.

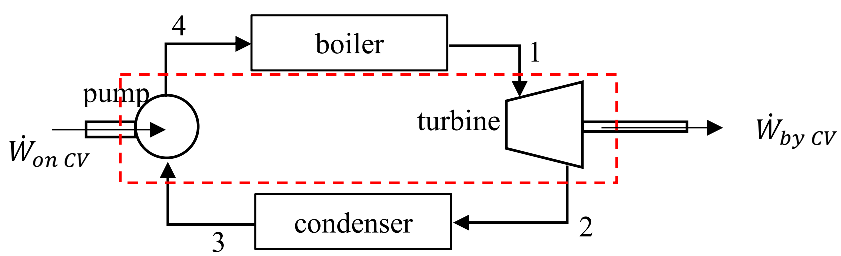
Using Eq. (7),

h_4 = 218.89 kJ/kg.

Applying Conservation of Mass to individual control volumes surrounding each component and assuming steady flow gives,

$$\dot{m} = \dot{m}_1 = \dot{m}_2 = \dot{m}_3 = \dot{m}_4. \quad (9)$$

Apply the 1st Law to a control volume surrounding the turbine and pump.



$$\frac{dE_{CV}}{dt} = \dot{Q}_{into\,CV} - \dot{W}_{net,by\,CV} + \sum_{in}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right), \quad (10)$$

where,

$$\frac{dE_{CV}}{dt} = 0 \text{ (Assuming steady state operation.)}, \quad (11)$$

$$\dot{Q}_{into\,CV} = 0 \text{ (Assuming adiabatic operation.)}, \quad (12)$$

$$\dot{W}_{net,by\,CV} = \dot{W}_{by\,CV} - \dot{W}_{on\,CV}, \quad (13)$$

$$\sum_{in}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) = \dot{m}(h_3 - h_4 + h_1 - h_2). \quad (14)$$

(The changes in kinetic and potential energies are assumed to be negligibly small.)

Substitute and simplify,

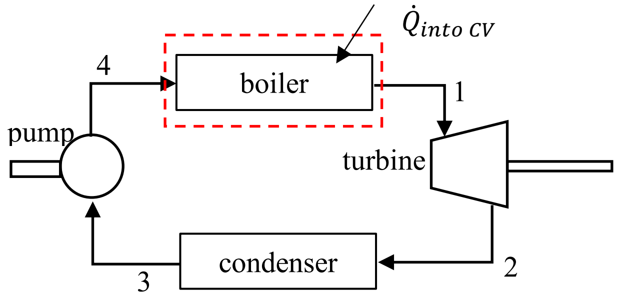
$$\dot{W}_{net,by\,CV} = \dot{m}(h_3 - h_4 + h_1 - h_2), \quad (15)$$

$$\boxed{\dot{m} = \frac{\dot{W}_{net,by\,CV}}{(h_3 - h_4 + h_1 - h_2)}}. \quad (16)$$

Using the parameters calculated previously in addition to the given net power output of 122 MW,

$\boxed{\dot{m} = 599 \times 10^3 \text{ kg/h}}$.

The rate of heat transfer in the boiler is found by applying the 1st Law to a control volume surrounding the boiler.



$$\frac{dE_{CV}}{dt} = \dot{Q}_{into\ CV} - \dot{W}_{by\ CV} + \sum_{in}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right), \tag{17}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \ \text{(Assuming steady state operation.)}, \tag{18}$$

$$\dot{W}_{by\ CV} = 0 \ \text{(A boiler is a passive device.)}, \tag{19}$$

$$\sum_{in}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) = \dot{m}(h_4 - h_1). \tag{20}$$

(The changes in kinetic and potential energies are assumed to be negligibly small.)

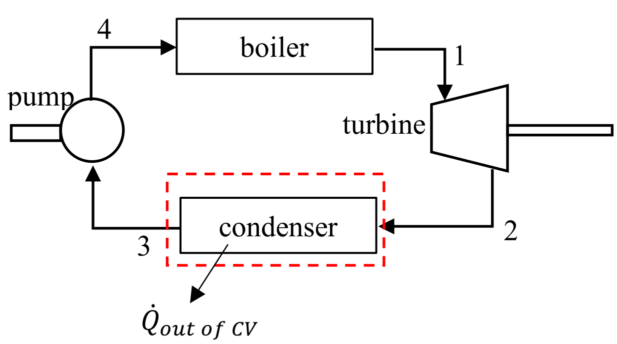
Substitute and simplify,

$$\boxed{\dot{Q}_{into\ CV} = \dot{m}(h_1 - h_4)}, \tag{21}$$

Using the parameters calculated previously in addition to the given net power output of 122 MW,

$\boxed{\dot{Q}_{into\ CV} = 411\text{ MW}}$.

To find the heat transfer from the condenser, apply the 1st Law to a control volume surrounding the condenser.



$$\frac{dE_{CV}}{dt} = -\dot{Q}_{out\ of\ CV} - \dot{W}_{by\ CV} + \sum_{in}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right), \tag{22}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \ \text{(Assuming steady state operation.)}, \tag{23}$$

$$\dot{W}_{by\ CV} = 0 \ \text{(A condenser is a passive device.)}, \tag{24}$$

$$\sum_{in}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out}\dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) = \dot{m}(h_2 - h_3). \tag{25}$$

(The changes in kinetic and potential energies are assumed to be negligibly small.)

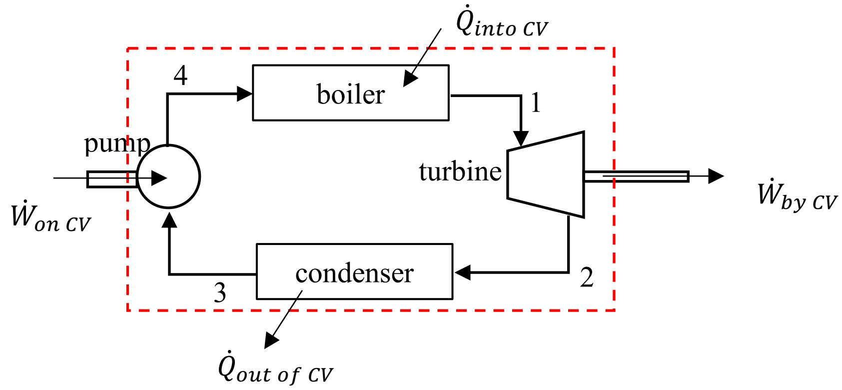
Substitute and simplify,

$$\boxed{\dot{Q}_{out\ of\ CV} = \dot{m}(h_2 - h_3)}, \tag{26}$$

Using the parameters calculated previously in addition to the given net power output of 122 MW,

$\boxed{\dot{Q}_{out\ of\ CV} = 289\text{ MW}}$.

Alternately, the rate of heat transfer out from the boiler could be found by applying the 1st Law to a control volume that surrounds the entire cycle,



$$\frac{dE_{CV}}{dt} = \dot{Q}_{net,into\ CV} - \dot{W}_{net,by\ CV} + \textstyle\sum_{in} \dot{m}\left(h + \frac{1}{2}V^2 + gz\right) - \sum_{out} \dot{m}\left(h + \frac{1}{2}V^2 + gz\right), \tag{27}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \text{ (Assuming steady state operation.)}, \tag{28}$$

$$\dot{Q}_{net,into\ CV} = \dot{Q}_{into\ CV} - \dot{Q}_{out\ of\ CV}, \tag{29}$$

$$\textstyle\sum_{in} \dot{m}\left(h + \frac{1}{2}V^2 + gz\right) - \sum_{out} \dot{m}\left(h + \frac{1}{2}V^2 + gz\right) = 0, \tag{30}$$

(Since there is no mass transfer across the CV surface.)

Substitute and simplify,

$$\dot{W}_{net,by\ CV} = \dot{Q}_{into\ CV} - \dot{Q}_{out\ of\ CV}, \tag{31}$$

$$\boxed{\dot{Q}_{out\ of\ CV} = \dot{Q}_{into\ CV} - \dot{W}_{net,by\ CV}}. \tag{32}$$

Using the value found previously for the heat transfer into the control volume and given net power done by the cycle,

$$\boxed{\dot{Q}_{out\ of\ CV} = 289 \text{ MW}},$$

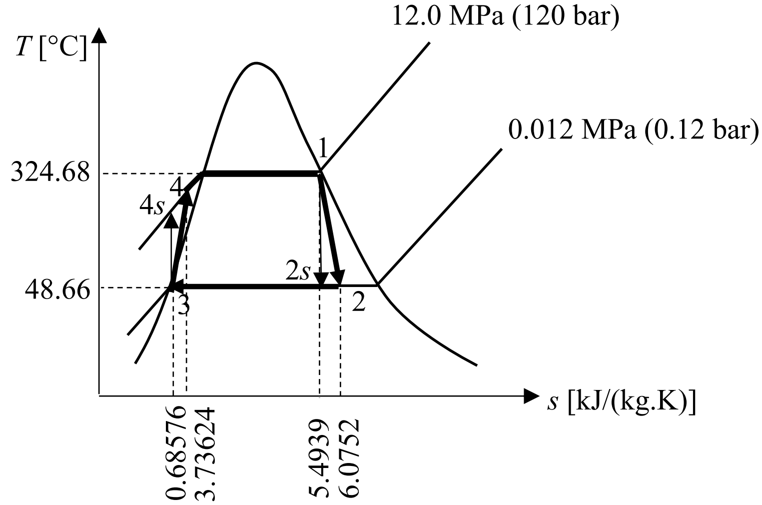
which is the same value found previously.

The thermal efficiency of the power plant is,

$$\boxed{\eta \equiv \frac{\dot{W}_{net,by\ CV}}{\dot{Q}_{into\ CV}}}, \tag{33}$$

Using the parameters found previously,

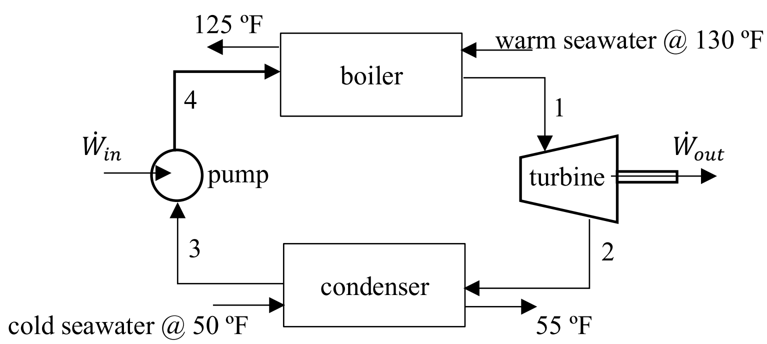
$$\boxed{\eta = 0.297}.$$



On the island of Hawaii lava flows continuously into the ocean. It is proposed to anchor a floating power plant offshore of the lava flow that uses ammonia as the working fluid. The plant would exploit the temperature variation between the warm water near the surface at 130 ºF and seawater at 50 ºF from a depth of 500 ft to produce power. Using the properties of pure water for the seawater and modeling the power plant as a Rankine cycle, determine:

a. the plant's thermal efficiency, and
b. the mass flow rate of ammonia in lb_m/min, for a net power output of 300 hp.
c. the mass flow rates of seawater through the boiler and condenser, in lb_m/min.

For a related story, see:
https://www.scientificamerican.com/article/hawaii-first-to-harness-deep-ocean-temperatures-for-power/



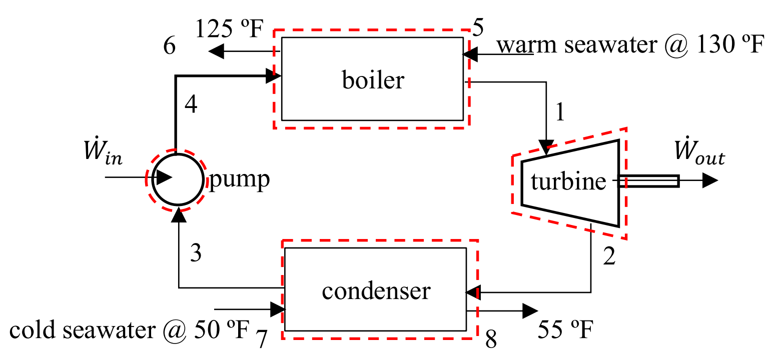
Working fluid: ammonia
State 1:
T_1 = 120 ºF
saturated vapor

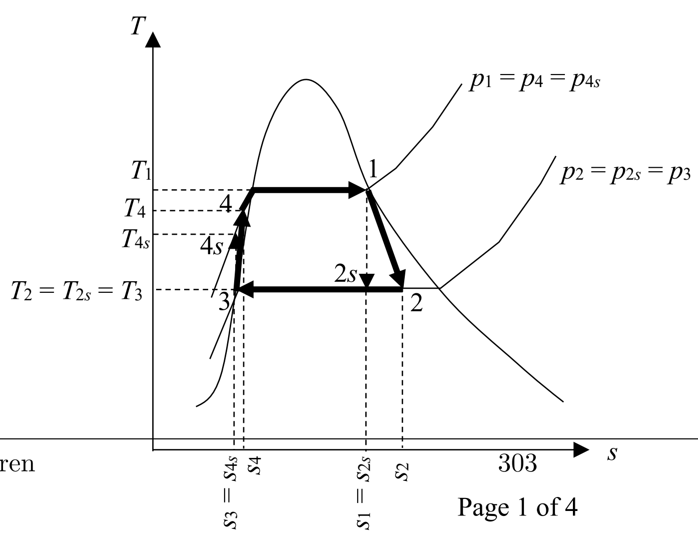
State 2:
T_2 = 60 ºF

State 3:
$p_3 = p_2$
saturated liquid

isentropic turbine efficiency = 0.80
isentropic pump efficiency = 0.85

SOLUTION:





First find the temperatures, specific enthalpies, and specific entropies at each of the states using the property tables for ammonia.

State	p [psia]	T [ºF]	Phase	x [-]	h [Btu/lb$_m$]	s [Btu/(lb$_m$. ºR)]
1	286.47 (= p_{sat})	120	sat. vapor	1	632.95	1.1405
2	107.66 (= p_3)	60	SLVM$^{o,=}$	0.932429	591.626	1.16038
2s	107.66 (= p_2)	60	SLVM^{o}	0.912476	581.295	1.1405 (= s_1)
3	107.66 (= p_{sat})	60	sat. liquid	0	108.87	0.2314
4	286.47 (= p_1)	61.08	CL$^+$	N/A	109.881	0.23374
4s	286.47 (= p_4)	60	CL$^+$	N/A	109.729	0.2314 (= s_3)

^{o}For a SLVM,

$$x = \frac{s - s_f}{s_g - s_f}, \tag{1}$$

$$h = (1 - x)h_f + xh_g. \tag{2}$$

State 2s:

T_{2s} = 60 ºF, $s_{2s} = s_1$ = 1.1405 Btu/(lb$_m$.ºR)]; s_{f2s} = 0.2314 Btu/(lb$_m$.ºR), s_{g2s} = 1.2277 Btu/(lb$_m$.ºR)
=> x_{2s} = 0.912476.
h_{f2s} = 108.87 Btu/lb$_m$, h_{g2s} = 626.61 Btu/lb$_m$ => h_{2s} = 581.295 Btu/lb$_m$.

State 2:

T_2 = 60 ºF, h_2 = 591.626 Btu/lb$_m$ (see below); h_{f2} = 108.87 Btu/lb$_m$, h_{g2} = 626.61 Btu/lb$_m$
=> x_2 = 0.932429.
s_{f2} = 0.2314 Btu/(lb$_m$.ºR), s_{g2} = 1.2277 Btu/(lb$_m$.ºR) => s_2 = 1.16038 Btu/(lb$_m$.ºR).

$^{=}$To find the conditions at State 2, make use of the turbine isentropic efficiency,

$$\eta_{turb.,isen.} = \frac{\dot{W}_{out}}{\dot{W}_{out,isen}} = \frac{h_1 - h_2}{h_1 - h_{2s}} \Rightarrow h_2 = h_1 - \eta_{turb.,isen.}(h_1 - h_{2s}) = 591.626 \text{ Btu/lb}_m, \tag{3}$$

where h_1 = 632.95 Btu/lb$_m$, h_{2s} = 581.295 Btu/lb$_m$, and $\eta_{turb.,isen.}$= 0.80.

$^{+}$ For a compressed liquid,

$$h_{CL}(T,p) \approx h_f(T) + [p - p_{sat}(T)]v_f(T) \quad \text{and} \quad s_{CL}(T,p) \approx s_f(T) \tag{4}$$

State 4s:

p_{4s} = 286.47 psia, $s_{4s} = s_3$ = 0.2314 Btu/(lb$_m$.ºR) =>
T_{4s} = 60 ºF, $p_{sat,4s}$ = 107.66 psia, v_{f4s} = 0.02597 ft^3/lb$_m$, h_{f4s} = 108.87 Btu/lb$_m$ => h_{4s} = 109.729 Btu/lb$_m$.

$^{=}$To find the conditions at State 4, make use of the pump isentropic efficiency,

$$\eta_{pump,isen.} = \frac{\dot{W}_{in,isen}}{\dot{W}_{in}} = \frac{h_{4s} - h_3}{h_4 - h_3} \Rightarrow h_4 = h_3 + \frac{h_{4s} - h_3}{\eta_{pump,isen.}} = 109.881 \text{ Btu/lb}_m, \tag{5}$$

where h_3 = 108.87 Btu/lb$_m$, h_{4s} = 109.729 Btu/lb$_m$, and $\eta_{pump,isen.}$= 0.85. The temperature corresponding to this specific enthalpy is, after some linear interpolation, T_4 = 61.08 ºF, and the specific entropy is, s_4 = 0.23374 Btu/(lb$_m$.ºR).

Now apply the 1st Law to a control volume surrounding the turbine,

$$\frac{dE_{CV}}{dt} = \sum_{in} \dot{m}(h + ke + pe) - \sum_{out} \dot{m}(h + ke + pe) + \dot{Q}_{in} - \dot{W}_{out}, \tag{6}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \quad \text{(assuming steady state operation)}, \tag{7}$$

$$\sum_{in} \dot{m}(h + ke + pe) - \sum_{out} \dot{m}(h + ke + pe) = \dot{m}(h_1 - h_2), \tag{8}$$

(neglecting kinetic and potential energy changes; from COM $\dot{m}_1 = \dot{m}_2 = \dot{m}$)

$$\dot{Q}_{in} = 0 \quad \text{(assuming adiabatic operation)}, \tag{9}$$

$$\dot{W}_{out} = ?. \tag{10}$$

Substitute and solve for the power,

$$\frac{\dot{W}_{out}}{\dot{m}} = h_1 - h_2. \tag{11}$$

Using the data from the table and the given mass flow rate,

$$\frac{\dot{W}_{out}}{\dot{m}} = 41.324 \text{ Btu/lb}_m.$$

Apply the 1st Law to a control volume surrounding the pump,

$$\frac{dE_{CV}}{dt} = \sum_{in} \dot{m}(h + ke + pe) - \sum_{out} \dot{m}(h + ke + pe) + \dot{Q}_{in} + \dot{W}_{in}, \tag{12}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \quad \text{(assuming steady state operation)}, \tag{13}$$

$$\sum_{in} \dot{m}(h + ke + pe) - \sum_{out} \dot{m}(h + ke + pe) = \dot{m}(h_3 - h_4), \tag{14}$$

(neglecting kinetic and potential energy changes; from COM $\dot{m}_3 = \dot{m}_4 = \dot{m}$)

$$\dot{Q}_{in} = 0 \quad \text{(assuming adiabatic operation)}, \tag{15}$$

$$\dot{W}_{in} = ?. \tag{16}$$

Substitute and solve for the power,

$$\frac{\dot{W}_{in}}{\dot{m}} = h_4 - h_3. \tag{17}$$

Using the data from the table and the given mass flow rate,

$$\frac{\dot{W}_{in}}{\dot{m}} = 1.011 \text{ Btu/lb}_m.$$

Apply the 1st Law to a CV surrounding the boiler,

$$\frac{dE_{CV}}{dt} = \sum_{in} \dot{m}(h + ke + pe) - \sum_{out} \dot{m}(h + ke + pe) + \dot{Q}_{in} - \dot{W}_{out}, \tag{18}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \quad \text{(assuming steady state operation)}, \tag{19}$$

$$\sum_{in} \dot{m}(h + ke + pe) - \sum_{out} \dot{m}(h + ke + pe) = \dot{m}(h_4 - h_1), \tag{20}$$

(neglecting kinetic and potential energy changes; from COM $\dot{m}_1 = \dot{m}_4 = \dot{m}$)

$$\dot{Q}_{in} = ?, \tag{21}$$

$$\dot{W}_{out} = 0 \quad \text{(the steam generator is a passive device)}. \tag{22}$$

Substitute and solve for the rate of heat transfer,

$$\frac{\dot{Q}_{in}}{\dot{m}} = h_1 - h_4. \tag{23}$$

Using the data from the table and the given mass flow rate,

$$\frac{\dot{Q}_{in}}{\dot{m}} = 523.069 \text{ Btu/lb}_m.$$

Using the power in and power out results,

$$\frac{\dot{W}_{out,net}}{\dot{m}} = \frac{\dot{W}_{out}}{\dot{m}} - \frac{\dot{W}_{in}}{\dot{m}} = 40.313 \text{ Btu/lb}_m \tag{24}$$

The thermal efficiency for the power cycle is,

$$\boxed{\eta = \frac{\dot{W}_{out,net}/\dot{m}}{\dot{Q}_{in}/\dot{m}} = 0.0771 = 7.71\%} \tag{25}$$

This thermal efficiency is less than the Carnot cycle thermal efficiency of

$$\eta_{Carnot} = 1 - \frac{T_C}{T_H} = 0.136 = 13.6\%, \quad (26)$$

where T_C = 509.67 ºR (= 50 ºF) and T_H = 589.67 ºR (= 130 ºF). The Rankine cycle efficiency is smaller than the Carnot cycle efficiency because of irreversibilities in the cycle. Even if the Rankine cycle was ideal (isentropic conditions across the pump and turbine), it would still have a smaller efficiency than the Carnot cycle because the average temperature during heat addition is smaller than that for a Carnot cycle, i.e., the average temperature from State 4 to State 1 is smaller than the average temperature from State 4 to State 1 in a Carnot cycle.

The mass flow rate in the cycle can be determined using Eq. (24) and the given net power output of $\dot{W}_{out,net} = 300$ hp,

$$\frac{\dot{W}_{out,net}}{\dot{m}} = 40.313 \text{ Btu/lb}_\text{m} \Rightarrow \boxed{\dot{m} = \frac{300 \text{ hp}}{40.313 \text{ Btu/lbm}} = 316 \text{ lb}_\text{m}\text{/min}}. \quad (27)$$

Now determine the mass flow rate of the cooling water for the boiler. Apply a control volume around the boiler and apply the 1st Law,

$$\frac{dE_{CV}}{dt} = \sum_{in} \dot{m}(h + ke + pe) - \sum_{out} \dot{m}(h + ke + pe) + \dot{Q}_{in} + \dot{W}_{in}, \quad (28)$$

where,

$$\frac{dE_{CV}}{dt} = 0 \qquad \text{(assuming steady state operation)}, \quad (29)$$

$$\sum_{in} \dot{m}(h + ke + pe) - \sum_{out} \dot{m}(h + ke + pe) = \dot{m}(h_4 - h_1) + \dot{m}_{b,sw}(h_5 - h_6), \quad (30)$$

(neglecting kinetic and potential energy changes; from COM $\dot{m}_4 = \dot{m}_1 = \dot{m}$ and $\dot{m}_5 = \dot{m}_6 = \dot{m}_{b,sw}$)

$$\dot{Q}_{in} = 0 \quad \text{(assuming adiabatic operation)}, \quad (31)$$

$$\dot{W}_{in} = 0 \quad \text{(the device is passive)}. \quad (32)$$

Substitute and solve for the boiler seawater mass flow rate,

$$0 = \dot{m}(h_4 - h_1) + \dot{m}_{b,sw}(h_5 - h_6), \quad (33)$$

$$\dot{m}_{b,sw} = \dot{m}\left(\frac{h_4 - h_1}{h_6 - h_5}\right). \quad (34)$$

The mass flow rate for the cycle was found in Eq. (27). The specific enthalpies for States 4 and 1 are given in the table at the start of this solution. Not enough information is given to determine the specific enthalpies for States 5 and 6 using the compressed liquid approximation; however, since the temperature is small and seawater can be reasonably assumed to be incompressible, let,

$$\Delta h = \Delta u + v\Delta p = \Delta u = c\Delta T. \quad (35)$$

where $\Delta p = 0$ since the pressure of the surrounding seawater at the inlet and outlet to the boiler is approximately the same. The specific heat for seawater is found from a property table to be c = 0.999 Btu/(lbm.ºR). Using T_5 = 130 ºF and T_6 = 125 ºF along with the previously determined values,

$$\boxed{\dot{m}_{b,sw} = 33000 \text{ lb}_\text{m}\text{/min}}.$$

Performing a similar 1st Law analysis for a CV surrounding the condenser, but with c = 1.005 Btu/(lbm.ºR),

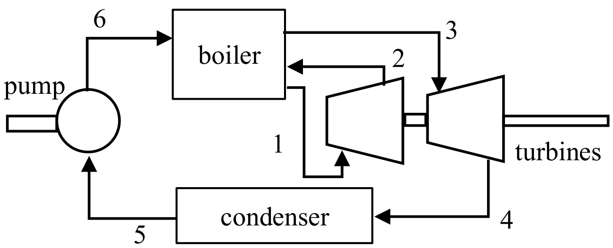
$$\dot{m}_{c,sw} = \dot{m}\left(\frac{h_2 - h_3}{h_8 - h_7}\right), \quad (36)$$

$$\boxed{\dot{m}_{c,sw} = 30300 \text{ lb}_\text{m}\text{/min}}.$$

The efficiency of this power plant is small (7.71%). Even for an ideal Carnot cycle the efficiency is small (13.6%). The small cycle thermal efficiency coupled with the large mass flow rates required for pumping the seawater (decreasing the net power out of the cycle even further) and the material and structural costs for operating in corrosive seawater make for a weak incentive to construct and operate this powerplant from a financial point of view.

Consider a vapor power cycle with reheat where the working fluid is water. The pump and turbines operate adiabatically. At the exit of both turbines, the water exits as saturated water vapor. The mass flow rate through the system is 2.1 kg/s.

a. Determine the net power developed by the cycle, in kW.
b. Determine the thermal efficiency of the power cycle.
c. Sketch the cycle on a *T*-*s* plot, indicating states, paths, and isobars. You needn't include numerical values for the properties.

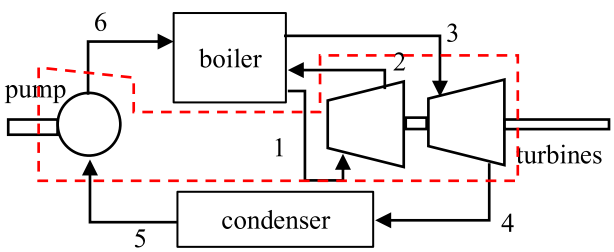


State	*p* [bar (abs)]	*h* [kJ/kg]	*x*
1	160	3355.6	-
2	15	2791.0	1
3	15	3169.8	-
4	1.5	2693.1	1
5	1.5	466.97	0
6	160	486.74	-

SOLUTION:
Applying Conservation of Mass to individual control volumes surrounding each component and assuming steady flow gives,

$$\dot{m} = \dot{m}_1 = \dot{m}_2 = \dot{m}_3 = \dot{m}_4 = \dot{m}_5 = \dot{m}_6. \qquad (1)$$

Apply the 1st Law to a control volume that surrounds both turbines and the pump.



$$\frac{dE_{CV}}{dt} = \dot{Q}_{into\ CV} - \dot{W}_{by\ CV} + \sum_{in} \dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out} \dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right), \qquad (2)$$

where,

$$\frac{dE_{CV}}{dt} = 0 \text{ (Assuming steady state operation.)}, \qquad (3)$$

$$\dot{Q}_{into\ CV} = 0 \text{ (Assuming adiabatic operation.)}, \qquad (4)$$

$$\sum_{in} \dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out} \dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) = \dot{m}(h_5 - h_6 + h_1 - h_2 + h_3 - h_4). \qquad (5)$$

(The changes in kinetic and potential energies are assumed to be negligibly small.)

Substitute and simplify,

$$\dot{W}_{by\ CV} = \dot{m}(h_5 - h_6 + h_1 - h_2 + h_3 - h_4), \qquad (6)$$

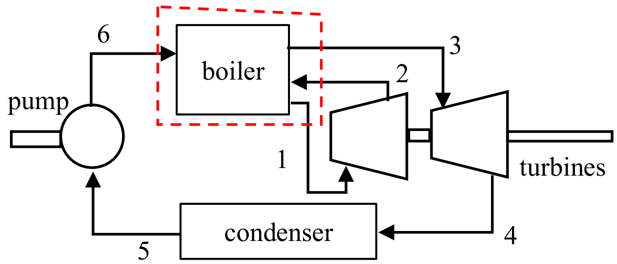
Using the given data,

$$\boxed{\dot{W}_{by\ CV} = 2150 \text{ kW}}.$$

The thermal efficiency of the power cycle is given by,

$$\eta \equiv \frac{\dot{W}_{net}}{\dot{Q}_{into}}. \qquad (7)$$

To find the heat added to the power cycle, apply the 1st Law to a control volume that surrounds the boiler,



$$\frac{dE_{CV}}{dt} = \dot{Q}_{into\ CV} - \dot{W}_{by\ CV} + \sum_{in} \dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out} \dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right), \quad (8)$$

where,

$$\frac{dE_{CV}}{dt} = 0 \text{ (Assuming steady state operation.)}, \quad (9)$$

$$\dot{W}_{by\ CV} = 0 \text{ (The boiler is a passive device.)}, \quad (10)$$

$$\sum_{in} \dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) - \sum_{out} \dot{m}\left(h + \tfrac{1}{2}V^2 + gz\right) = \dot{m}(h_6 - h_1 + h_2 - h_3). \quad (11)$$

(The changes in kinetic and potential energies are assumed to be negligibly small.)

Substitute and simplify,

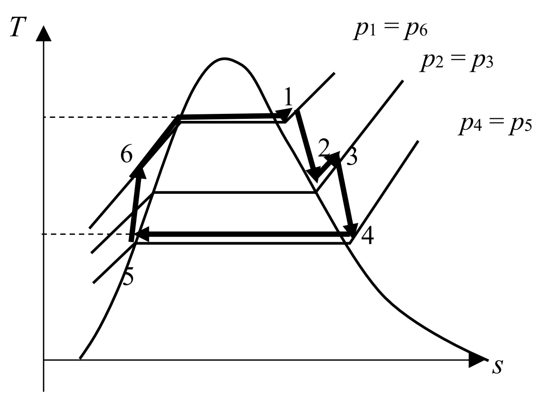
$$\dot{Q}_{into\ CV} = \dot{m}(h_1 - h_6 + h_3 - h_2), \quad (12)$$

Using the given data,

$\underline{\dot{Q}_{into\ CV} = 6820\text{ kW}}$.

Substituting values into Eq. (7) gives the power cycle thermal efficiency,

$\boxed{\eta = 0.315}$.



3.8.3. Vapor Compression Refrigeration and Heat Pump Cycles

The objective of a vapor compression refrigeration cycle is to remove energy from a cold reservoir and move it to the hot reservoir. For example, the cold reservoir may be the interior of a refrigerator or freezer and the hot reservoir is the kitchen, or it could be the interior of a home and the hot reservoir is the outdoors. In this latter case, the device running the refrigeration cycle is known as an air conditioner.

The objective of a heat pump cycle is to move energy from the cold reservoir into the hot reservoir. For example, the cold reservoir might be the outdoors and the hot reservoir would be the interior of a home. Another example is a heat pump water heater where the cold reservoir is the room containing the water heater and the hot reservoir is the water within the water heater tank.

The component schematics for refrigeration and heat pump cycles are identical. Similarly, the cycles traced out by the two cycles on a T - s plot have the same features. A schematic of the components and the T - s diagram are shown in Figure 3.45.

The temperature of the working fluid, typically a refrigerant, must be smaller than the temperature of the cold reservoir in order to have heat transfer into the working fluid in the evaporator (Process 4-1). Once the energy is absorbed, the compressor increases the pressure and temperature of the working fluid to raise its temperature to value larger than the hot reservoir temperature (Process 1-2). This increase in temperature is required in order for heat transfer to be from the working fluid in the condenser to the hot reservoir (Process 2-3). In order to return the working fluid to a cold temperature in the evaporator, a throttling device is used to decrease the pressure and temperature (Process 3-4).

Notes:

(1) Expansion of the working fluid from State 3 to State 4 is achieved through a throttling device rather than using a turbine. Although expansion through a throttling device is inherently non-isentropic, expansion through a turbine would also be inefficient and produce little power due to the low quality of the saturated vapor-liquid mixture and, thus, low specific enthalpies. In addition, a turbine is a more complex device and would present additional engineering and maintenance challenges. In a heat pump or air conditioner, the throttling device is in the form of an expansion valve. In a refrigerator, the throttling device is simply a long, narrow tube called a capillary tube. The pressure decreases across both devices (going from State 3 to State 4) since a higher pressure at the inlet side is required in order to push the viscous working fluid through the valve/tube.

(2) Real compressors operate best on superheated vapor rather than saturated liquid-vapor mixtures and, thus, State 1 is often in the superheated vapor region. Compression of superheated vapor is known as “dry compression” while compression of a saturated liquid-vapor mixture is called “wet compression”.

(3) For a refrigeration cycle, the rate of heat transfer from the cold reservoir is known as the refrigeration capacity. One ton of refrigeration capacity = the rate of heat transfer required to freeze one ton of water in 24 hrs with an enthalpy of fusion of 334 kJ/kg = 200 Btu/min ≈ 211 kJ/min = 3.517 kW.

(4) Recall that the “efficiency” of refrigeration and heat pump cycles are quantified using coefficients of performance,

$$COP_{\text{ref}} = \frac{\dot{Q}_C}{\dot{W}_C}, \tag{3.194}$$

$$COP_{\text{HP}} = \frac{\dot{Q}_H}{\dot{W}_C}. \tag{3.195}$$

(5) A refrigeration/heat pump cycle with a 100% isentropic efficiency is called an “ideal” refrigeration/heat pump cycle. Note that the throttling device is still non-isentropic, however. In an actual heat pump cycle, the location of State 1 is in the superheated vapor region (discussed in a previous note) and the location of State 3 may be in the compressed liquid region.

(6) The coefficients of performance for the refrigeration/heat pump cycle shown in Figure 3.45 are less than the coefficients of performance for refrigeration/heat pump Carnot cycles operating between

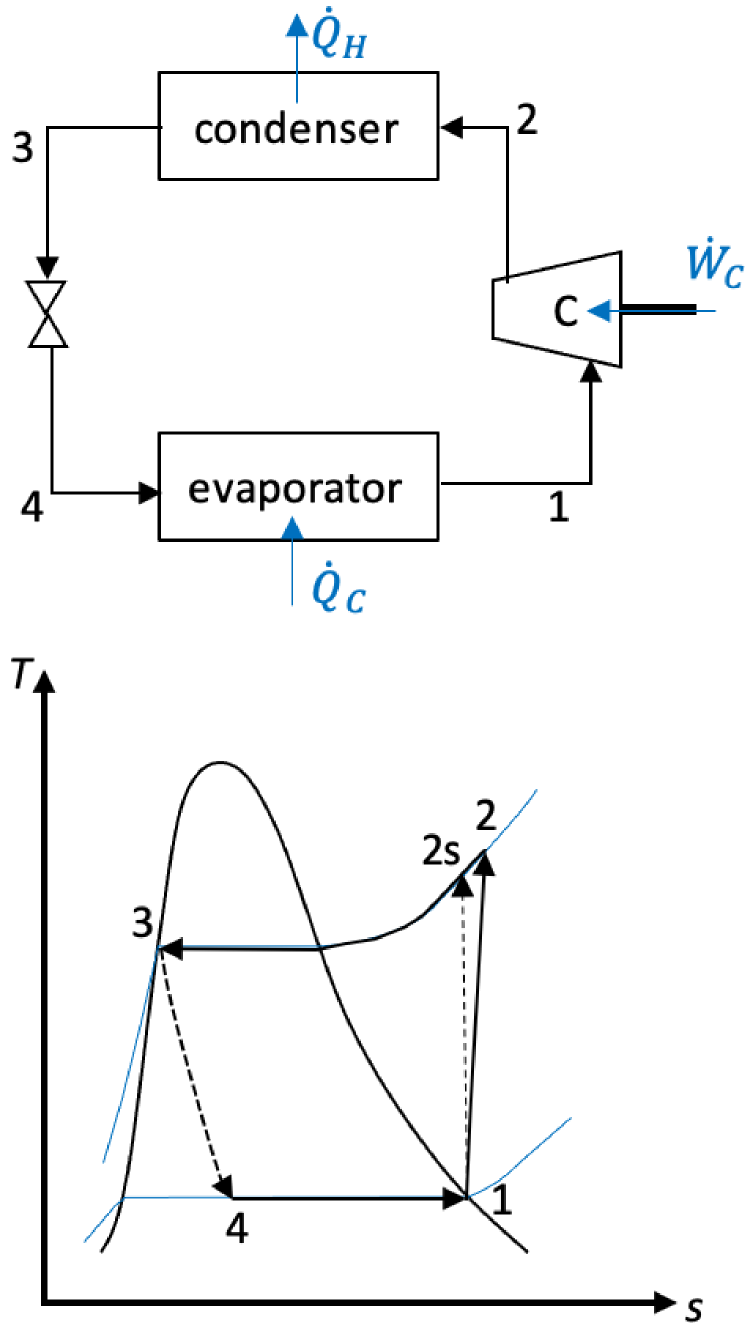


Figure 3.45. A schematic showing the components of refrigeration and heat pump cycles, and the corresponding T-s plot. The dashed line for Process 3-4 (flow through the throttling device) indicates that this process occurs abruptly and in an uncontrolled manner, i.e., the path is not well defined for this process.

identical thermal reservoirs. Recall that the Carnot cycle coefficients of performance are,

$$COP_{\text{ref,rev}} = \frac{T_C}{T_H - T_C}, \tag{3.196}$$

$$COP_{\text{HP,rev}} = \frac{T_H}{T_H - T_C}. \tag{3.197}$$

Even if the compressor has 100% isentropic efficiency, the fact that the temperature isn't constant in the condenser and evaporator (where heat transfer occurs), the non-isentropic expansion in the throttling device, and other real-world effects (viscosity, heat transfer across a finite temperature difference) results in decreased thermal efficiency.

(7) Using Eq. (3.196) *as a guide*, we see that as the hot reservoir temperature (T_H) decreases, the coefficient of performance for a refrigeration cycle increases. Thus, keeping a refrigerator/freezer in a cool basement or a garage during the winter will improve the device's thermodynamic efficiency.

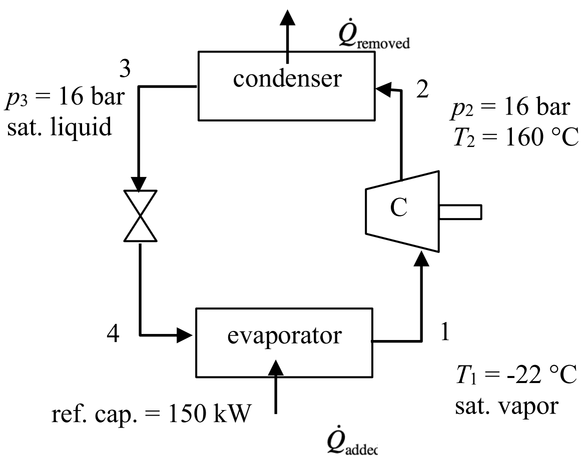
(8) Using Eq. (3.197) *as a guide*, we see that as the cold reservoir temperature (T_C) decreases, the coefficient of performance for a heat pump decreases. Home heat pumps using atmospheric air as the cold reservoir often have electric back-up heaters in order to account for the decreased efficiency when the air temperature becomes very cold. For example, if $T_H = 20\,^\circ\text{C}$ and $T_C = 10\,^\circ\text{C}$, then $COP_{\text{HP,rev}} = 29.3$. However, if $T_C = 0\,^\circ\text{C}$, then $COP_{\text{HP,rev}} = 14.7$. Of course a real heat pump won't have the same coefficients of performance as a Carnot cycle heat pump, but the principle that the coefficient of performance decreases as the cold reservoir temperature decreases still holds. Heat pumps using the Earth or a large body of water as the cold reservoir are less susceptible to this issue since the cold reservoir temperature in those cases remains nearly constant.

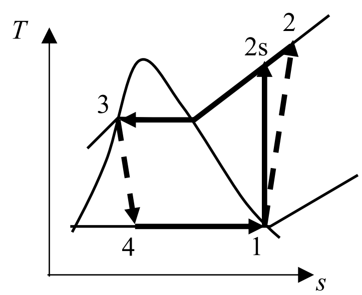
(9) A heat pump can be designed to operate as an air conditioner (i.e., a refrigeration cycle) through the use of a reversing valve. To learn more on this topic, the reader is encourage to look online for more detailed descriptions on practical heat pump design.

In a vapor-compression refrigeration cycle, ammonia exits the evaporator as saturated vapor at -22 °C. The refrigerant enters the condenser at 16 bar (abs) and 160 °C, and saturated liquid exits at 16 bar (abs). There is no significant heat transfer between the compressor and its surroundings, and the refrigerant passes through the evaporator with a negligible change in pressure. If the refrigerating capacity is 150 kW, determine:

a. the mass flow rate of refrigerant,
b. the power input to the compressor,
c. the coefficient of performance, and
d. the isentropic compressor efficiency.

SOLUTION:





First determine the properties at the various states using Tables from Moran et al., 7th ed.

State 1: T_1 = -22 °C, saturated vapor (Table A-13)
$\Rightarrow$ p_1 = 1.7390 bar, h_1 = 1415.08 kJ/kg, s_1 = 5.6457 kJ/(kg.K)

State 2: p_2 = 16 bar, T_2 = 160 °C $\Rightarrow$ superheated vapor (Table A-15)
$\Rightarrow$ h_2 = 1798.45 kJ/kg, s_2 = 5.7475 kJ/(kg.K)

State 3: p_3 = 16 bar, saturated liquid (Table A-14)
$\Rightarrow$ T_3 = 41.03 °C, h_3 = 376.46 kJ/kg, s_3 = 1.3729 kJ/(kg.K)

State 4: throttling process from 3 to 4, constant pressure from 4 to 1
$\Rightarrow$ $h_4 = h_3$ = 376.46 kJ/kg, $p_4 = p_1$ = 1.7390 bar

The mass flow rate may be determined by applying the 1st Law to the evaporator and making use of the refrigeration capacity (= $\dot{Q}_{\text{added}}$ = 150 kW),

$$\dot{Q}_{\text{added}} = \dot{m}\left(h_1 - h_4\right) \Rightarrow \dot{m} = \frac{\dot{Q}_{\text{added}}}{\left(h_1 - h_4\right)}, \tag{1}$$

$\Rightarrow$ $\boxed{\dot{m} = 0.144 \text{ kg/s}}$.

The power input into the compressor is found by applying the 1st Law to the compressor,

$$\dot{W}_{\text{on comp}} = \dot{m}\left(h_2 - h_1\right), \tag{2}$$

$\Rightarrow$ $\boxed{\dot{W}_{\text{on comp}} = 55.4 \text{ kW}}$.

The coefficient of performance for the refrigeration cycle is defined as,

$$\text{COP}_{\text{ref}} \equiv \frac{\dot{Q}_{\text{added}}}{\dot{W}_{\text{on}}}, \tag{3}$$

$\Rightarrow$ $\boxed{\text{COP}_{\text{ref}} = 2.71}$.

The isentropic efficiency of the compressor is defined as,

$$\eta_{\text{comp}} \equiv \frac{\dot{W}_{\text{on comp},s}}{\dot{W}_{\text{on comp}}} = \frac{\dot{W}_{\text{on comp},s}/\dot{m}}{\dot{W}_{\text{on comp}}/\dot{m}} = \frac{h_{2s} - h_1}{h_2 - h_1}, \tag{4}$$

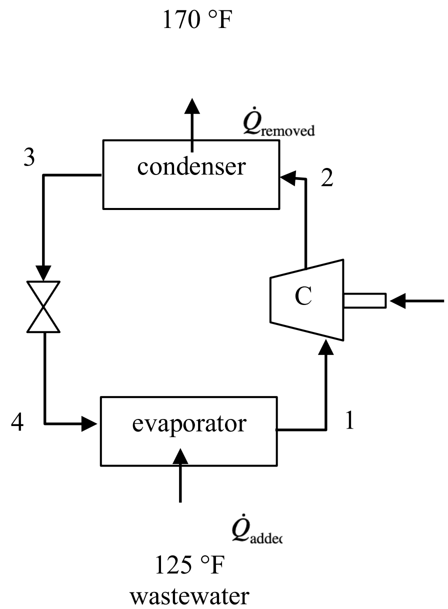
where

$p_{2s} = p_2 = 16$ bar and $s_{2s} = s_1 = 5.6457$ kJ/(kg.K) $\Rightarrow$ $h_{2s} = 1755.38$ kJ/kg, $T_{2s} = 143$ °C (interpolating from Table A-15),

$\Rightarrow$ $\boxed{\eta_{\text{comp}} = 0.888}$.

A process requires a heat transfer rate of $3*10^6$ Btu/h at 170 °F. It is proposed that a Refrigerant 134a vapor-compression heat pump be used to develop the process heating using a wastewater stream at 125 °F as the lower-temperature source. The compressor isentropic efficiency is 80%. Sketch the *T-s* diagram for the cycle and determine the:

a. specific enthalpy at the compressor exit, in Btu/lb_m,
b. temperatures at each of the principal states in °F,
c. mass flow rate of the refrigerant in lb_m/h,
d. compressor power, in Btu/h, and
e. coefficient of performance and compare with the coefficient of performance for a Carnot heat pump cycle operating between reservoirs at the process temperature and the wastewater temperature, respectively.



State	p [psia]	h [Btu/lb_m]
1	180	116.74
2	400	?
3	400	76.11
4	180	76.11

SOLUTION:

The specific enthalpy at state 2 may be found since the compressor efficiency is known ($\eta_{\text{comp}} = 0.80$),

$$\eta_{\text{comp}} \equiv \frac{\dot{W}_{\text{on comp},s}}{\dot{W}_{\text{on comp}}} = \frac{\dot{W}_{\text{on comp},s}/\dot{m}}{\dot{W}_{\text{on comp}}/\dot{m}} = \frac{h_{2s}-h_1}{h_2-h_1} \Rightarrow h_2 = h_1 + \frac{h_{2s}-h_1}{\eta_{\text{comp}}}, \tag{1}$$

where,

$s_{2s} = s_1 = 0.2154$ Btu/(lb_m.°R) using Table A-11E from Moran et al., 7th ed. with $p_1 = 180$ psia and knowing state 1 is in a saturated vapor state ($\boxed{T_1 = 117.74\text{ °F}}$)
$\Rightarrow h_{2s} = 123.32$ Btu/lb_m knowing $p_{2s} = p_2 = 400$ psia (Table A-12E and interpolation; $T_{2s} = 186$ °F, SHV)
$\Rightarrow \boxed{h_2 = 124.97\text{ Btu/lb}_m}$ (using Table A-12E and interpolation $\boxed{T_2 = 191.63\text{ °F}}$, SHV)

Knowing $p_3 = p_2 = 400$ psia and $h_3 = 76.11$ Btu/lb_m (saturated liquid state, Table A-11E) $\Rightarrow \boxed{T_3 = 179.95\text{ °F}}$.
Knowing $p_4 = p_1 = 180$ psia and $h_4 = 76.11$ Btu/lb_m (SLVM Table A-11E) $\Rightarrow \boxed{T_4 = 117.74\text{ °F}}$.

Apply the 1st Law to the condenser to determine the mass flow rate,

$$\dot{Q}_{\text{removed}} = \dot{m}(h_2 - h_3) \Rightarrow \dot{m} = \frac{\dot{Q}_{\text{removed}}}{(h_2 - h_3)}, \tag{2}$$

$\Rightarrow \boxed{\dot{m} = 6.13*10^4\text{ lb}_m\text{/h}}$.

The compressor power is found by applying the 1st Law to the compressor,

$$\dot{W}_{\text{on comp}} = \dot{m}\left(h_2 - h_1\right), \tag{3}$$

$$\Rightarrow \boxed{\dot{W}_{\text{on comp}} = 5.05 * 10^5 \text{ Btu/h}}.$$

The coefficient of performance for the heat pump cycle is,

$$\text{COP}_{\text{hp}} \equiv \frac{\dot{Q}_{\text{removed}}}{\dot{W}_{\text{on comp}}}, \tag{4}$$

$$\Rightarrow \boxed{\text{COP}_{\text{hp}} = 5.95}.$$

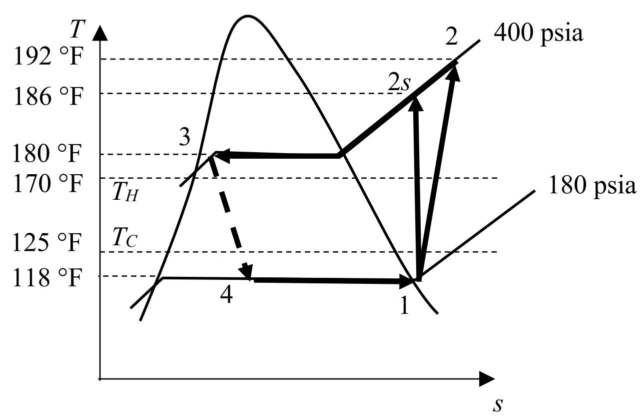
The COP_{hp} for the corresponding Carnot cycle operating between T_C = 125 °F (= 585 °R) and T_H = 170 °F (= 630 °R) is,

$$\text{COP}_{\text{hp,rev}} = \frac{T_H}{T_H - T_C}, \tag{5}$$

$$\Rightarrow \boxed{\text{COP}_{\text{hp,rev}} = 14}.$$

The Carnot cycle COP is larger than the actual COP, as expected. Much of the cause for irreversibility in the actual system is due to the fact that the system temperatures from 2-3 and from 4-1 are substantially different than the hot and cold reservoir temperatures of T_H = 170 °F and T_C = 125 °F leading to large, finite temperature differences. Such large differences are needed for practical heat transfer rates between the condenser, evaporator, and the surroundings.

A sketch of the states and processes are shown on the following *T*-*s* diagram.



3.8.4. The Otto, Diesel, and Dual Cycles

The Otto, Diesel, and dual cycles are idealizations of the cycles observed in internal combustion, piston-cylinder engines. Before analyzing these cycles, it is useful to first describe the components and processes involved in a four-stroke internal combustion (IC) engine, which is the most common type of engine.

Figure 3.46 illustrates the components in a typical IC engine piston-cylinder arrangement. The piston moves vertically within the cylinder as the crankshaft turns. When the piston is at its lowest point it's considered to be at bottom dead center, bdc and when the piston is at its highest point it's at the top dead center, tdc. The volumes within the cylinder at these two points are, respectively, V_{bdc} and V_{tdc}. The piston stroke is the vertical distance between the bdc and tdc. At the top of the cylinder are valves, which open and close during the piston movement. Opening an intake valve allows for fresh air (and possibly fuel) to enter the cylinder while opening an outlet valve allows the piston to push out exhaust gases after combustion has occurred.

The figure also shows a spark plug at the top center of the cylinder. Spark plugs are used to ignite the air/fuel mixture in spark ignition IC engines. In a compression ignition IC engine, the spark plug is removed and instead fuel is injected at this location. The high temperature generated during compression of the air/fuel mixture in a compression ignition engine is enough to initiate combustion.

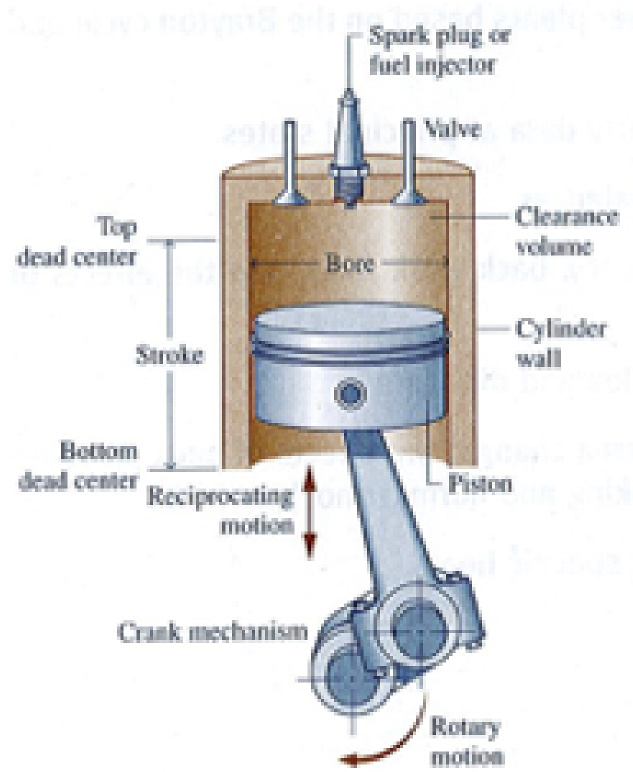


Figure 3.46. An illustration of the components in a typical IC engine piston-cylinder arrangement. This figure is from Moran, M.J., Shapiro, H.N., Boettner, D.D., and Bailey, M.B., *Fundamentals of Engineering Thermodynamics*, Wiley, 7th ed.

The processes involved in a four-stroke IC engine are shown in Figure 3.47. As shown in the figure, in the intake stroke the air/fuel mixture is drawn into the cylinder as the piston moves downward and an intake valve is opened (the exhaust valve is closed). Next, the air/fuel mixture is compressed within the cylinder during the compression stroke. During this process both valves are closed. Near the end of the compression stroke the spark plug fires in a spark-ignition engine causing rapid combustion of the fuel. If the engine instead operates via compression ignition, then combustion initiates a bit later in the compression stroke. The valves are closed during this process. Once ignition begins, the piston is forced downward by the rapid expansion of the combustion gases. This is the power stroke. After reaching the bottom dead center, the exhaust stroke begins in which the piston moves back upwards while the exhaust valve is opened (the intake valve is closed), which forces the combustion products out of the cylinder. The cycle then repeats after the piston reaches the top dead center. Note that the crankshaft completes two revolutions during this four stroke process.

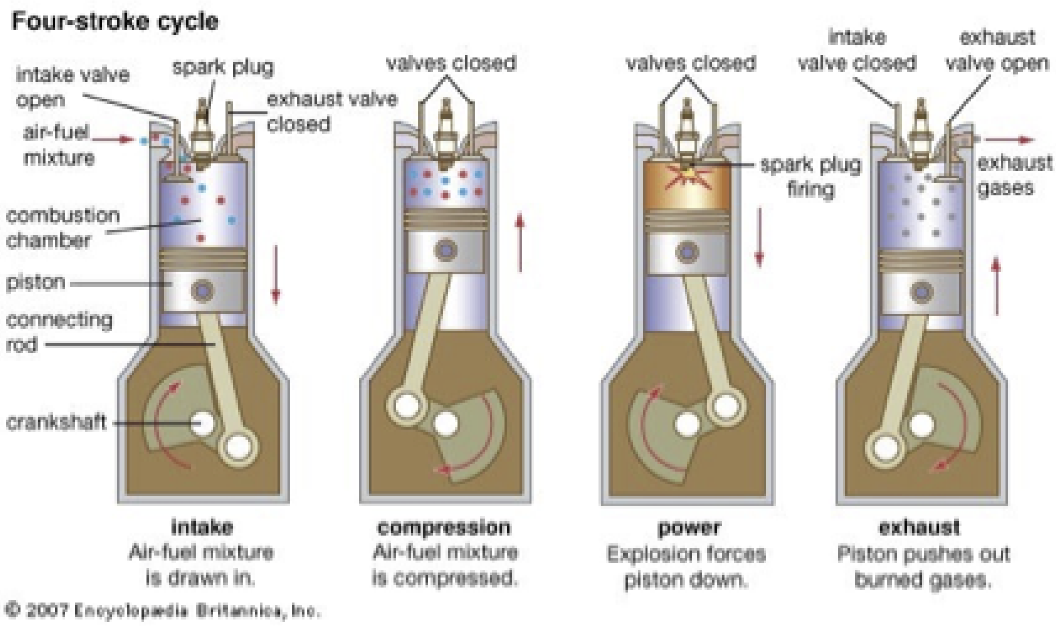


Figure 3.47. Illustrations of the different processes in a four stroke, internal combustion engine. Note that the crankshaft turns twice during the four-stroke process. This figure is from Encyclopedia Britannica, https://www.britannica.com/technology/four-stroke-cycle.

A representative p-v plot for a four-stroke cycle is shown in Figure 3.48. The intake stroke starts at top dead center, with the inlet valve opened and exhaust valve closed, and moves at nearly constant pressure to bottom dead center, at which point the intake valve closes. The compression stroke compresses the air/fuel mixture until reaching the top dead center, moving from the bottom right to the top left in the plot (decreasing volume and increasing pressure). Combustion initiates near the end of the compression stroke. Note that the crankshaft has now completed one rotation. Next, the power stroke begins as the combustion gases push the piston downward, with the piston moving (and cylinder volume increasing) from top dead center to bottom dead center (increasing volume and decreasing pressure). Near the end of the power stroke the exhaust valve opens. Lastly, the exhaust stroke occurs and the piston moves back upwards at nearly constant pressure to push the combustion gases out through the exhaust valve. When the piston reaches top dead center another crankshaft rotation has occurred. The cycle then repeats.

We'll use an air standard analysis to study the Otto, Diesel, and dual cycles. An air standard analysis is a highly simplified model to provide qualitative understanding of cycles that use air as the working fluid. The numerical values in an air standard analysis generally won't be accurate when compared to experimental values; however, the air standard model can still provide valuable insight into the influence of various parameters on performance measures.

The assumptions made in an internal combustion engine air standard analysis include:

(1) The air mass in the system remains constant.

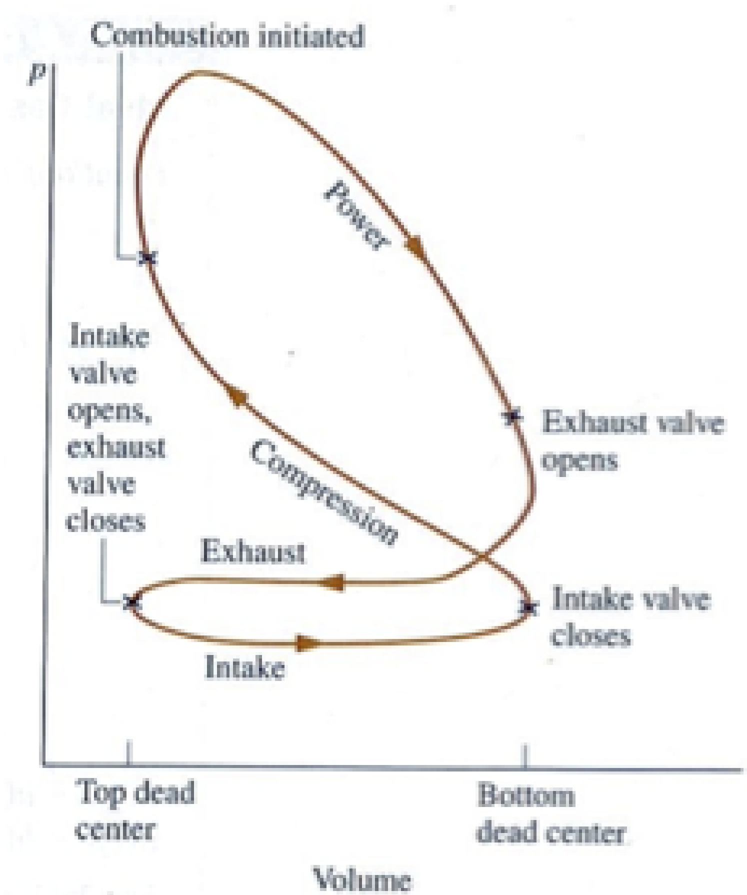


Figure 3.48. A representative p-v plot for a four-stroke, IC engine. This figure is from Moran, M.J., Shapiro, H.N., Boettner, D.D., and Bailey, M.B., *Fundamentals of Engineering Thermodynamics*, Wiley, 7th ed.

(2) There are no intake or exhaust processes.
(3) Air is modeled as an ideal gas.
(4) Combustion is modeled as a heat addition process and the working fluid remains as air.
(5) The exhaust process is modeled as a constant volume heat removal process.
(6) All processes are internally reversible.

In a cold air standard analysis, we further assume constant specific heats, i.e., a perfect gas assumption.

Two definitions used in the analysis of IC cycles include:

- The mean effective pressure, mep, defined as
$$\text{mep} := \frac{W_{\text{out,net}}}{V_{\text{bdc}} - V_{\text{tdc}}} \tag{3.198}$$
The mep can be used to compare the work output between different cycles having the same working volume.
- The compression ratio, r,
$$r := \frac{V_{\text{bdc}}}{V_{\text{tdc}}} = \frac{v_{\text{bdc}}}{v_{\text{tdc}}} > 1. \tag{3.199}$$
As is shown in the following analyses, increasing the compression ratio increases the thermal efficiency of the cycle.

3.8.4.1. Otto Cycle

The Otto cycle is an idealization of the cycle shown in Figure 3.48 for a spark ignition IC engine. The processes in an air-standard Otto cycle include (refer to Figure 3.49):

- *Process 1 - 2*: isentropic compression of the working fluid as the piston moves from bottom dead center to top dead center (compression stroke),
- *Process 2 - 3*: *constant volume* heat addition to the working fluid while the piston is at top dead center (combustion)
- *Process 3 - 4*: isentropic expansion of the working fluid as the piston moves from top dead center to bottom dead center (power stroke)
- *Process 4 - 1* constant volume heat removal from the working fluid while the piston is at bottom dead center (exhaust and intake strokes)

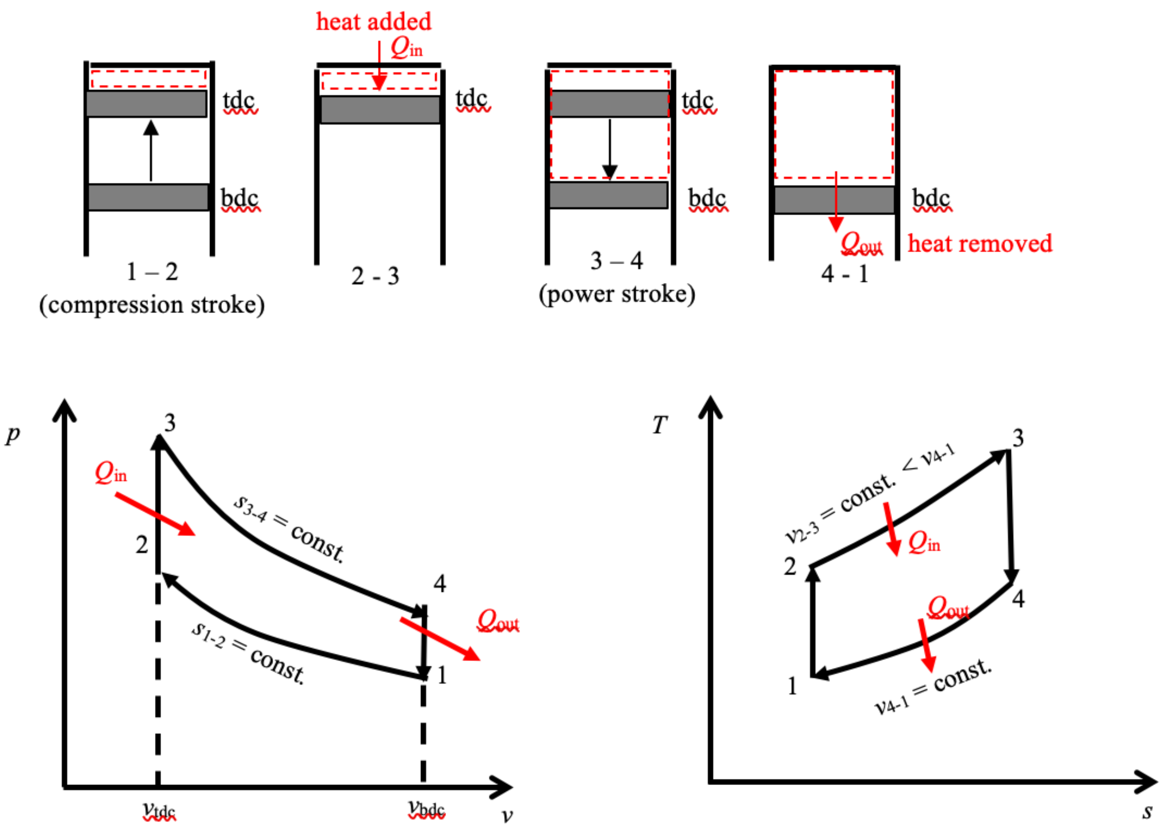


FIGURE 3.49. Illustrations of the processes involved in an air standard Otto cycle including p-v and T-s plots. Compare this p-v plot to the one shown in Figure 3.48.

The cycle can be analyzed using the First Law applied to the air in the cylinder, neglecting changes in kinetic and potential energy. For Process 1 - 2,

$$\Delta U_{12} = m(u_2 - u_1) = Q_{\text{in},12} + W_{\text{on},12}, \tag{3.200}$$

$$\implies W_{\text{on},12} = m(u_2 - u_1). \tag{3.201}$$

where $Q_{\text{in},12} = 0$ since the process is isentropic and internally reversible.

For Process 2 - 3,

$$\Delta U_{23} = m(u_3 - u_2) = Q_{\text{in},23} - W_{\text{by},23}, \tag{3.202}$$

$$\implies Q_{\text{in},23} = m(u_3 - u_2). \tag{3.203}$$

where $W_{\text{by},23} = 0$ since the process is at constant volume.

For Process 3 - 4,

$$\Delta U_{34} = m(u_4 - u_3) = Q_{\text{in},34} - W_{\text{by},34}, \tag{3.204}$$

$$\implies W_{\text{by},34} = m(u_3 - u_4). \tag{3.205}$$

where $Q_{\text{in},34} = 0$ since the process is isentropic and internally reversible.

For Process 4 - 1,

$$\Delta U_{41} = m(u_1 - u_4) = -Q_{\text{out},41} - W_{\text{by},41}, \tag{3.206}$$

$$\implies Q_{\text{out},41} = m(u_4 - u_1). \tag{3.207}$$

where $W_{\text{by},41} = 0$ since the process is at constant volume.

The thermal efficiency of the cycle is,

$$\eta_{\text{Otto}} = \frac{W_{\text{by,net}}}{Q_{\text{in}}} = \frac{W_{\text{by},34} - W_{\text{on},12}}{Q_{\text{in},23}} = \frac{(u_3 - u_4) - (u_2 - u_1)}{(u_3 - u_2)} = \frac{(u_3 - u_2) - (u_4 - u_1)}{(u_3 - u_2)}, \tag{3.208}$$

or, re-writing further,

$$\eta_{\text{Otto}} = 1 - \frac{u_4 - u_1}{u_3 - u_2} = 1 - \frac{Q_{\text{out},41}}{Q_{\text{in},23}}. \tag{3.209}$$

The compression ratio for the cycle (Eq. (3.199)) is,

$$r = \frac{v_{\text{bdc}}}{v_{\text{tdc}}} = \frac{v_1}{v_2} = \frac{v_4}{v_3}. \tag{3.210}$$

Recall that the compression (Process 1-2) and power (Process 3-4) strokes are modeled in the air standard analysis as isentropic processes involving an ideal gas. Thus,

$$r = \frac{v_1}{v_2} = \frac{v_r(T_1)}{v_r(T_2)}, \tag{3.211}$$

and,

$$r = \frac{v_4}{v_3} = \frac{v_r(T_4)}{v_r(T_3)}. \tag{3.212}$$

Thus,

$$\frac{v_r(T_1)}{v_r(T_2)} = \frac{v_r(T_4)}{v_r(T_3)}. \tag{3.213}$$

For a cold air standard analysis, which assumes perfect gas behavior, the previous two relations may be written as,

$$\frac{1}{r} = \frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{1-k}} \implies \frac{T_2}{T_1} = r^{k-1}. \tag{3.214}$$

and,

$$r = \frac{v_4}{v_3} = \left(\frac{T_4}{T_3}\right)^{\frac{1}{1-k}} \implies \frac{T_4}{T_3} = r^{1-k} \tag{3.215}$$

where k is the specific heat ratio. Combining,

$$r^{k-1} = \frac{T_2}{T_1} = \frac{T_3}{T_4} \implies \frac{T_4}{T_1} = \frac{T_3}{T_2}. \tag{3.216}$$

Again, assuming perfect gas behavior, Eq. (3.209) may be written as,

$$\eta_{\text{Otto}} = 1 - \frac{c_v(T_4 - T_1)}{c_v(T_3 - T_2)} = 1 - \frac{T_1}{T_2}\left(\frac{\frac{T_4}{T_1} - 1}{\frac{T_3}{T_2} - 1}\right). \tag{3.217}$$

Using Eq. (3.216),

$$\eta_{\text{Otto}} = 1 - \frac{T_1}{T_2} = 1 - \frac{1}{r^{k-1}}. \tag{3.218}$$

Notes:

(1) Typical values for the compression ratio for a spark ignition IC engine are $r = 8$ to 10 with engine thermal efficiencies of $\eta = 30$ to 35% (these are real efficiencies, not reversible efficiencies).
(2) As the compression ratio r increases, the thermal efficiency η increases. In practice, the compression ratio is limited by auto-ignition, which causes engine knock. Auto-ignition occurs when the temperature reaches a sufficiently high value during compression that the air/fuel mixture ignites prior to spark ignition. Higher octane fuels can go to higher compression ratios before knocking occurs.
(3) As the specific heat ratio k increases, the thermal efficiency η increases. The specific heat ratio is determined by the type of fuel used. In addition, the specific heat ratio increases as the temperature decreases.

3.8.4.2. Diesel Cycle

The Diesel cycle is an idealization of the cycle shown in Figure 3.48 for a compression ignition IC engine. The processes in an air-standard Diesel cycle include (refer to Figure 3.50):

- *Process 1 - 2*: isentropic compression of the working fluid as the piston moves from bottom dead center to top dead center (compression stroke),
- *Process 2 - 3*: *constant pressure* heat addition to the working fluid starting from the top dead center (combustion and start of power stroke)
- *Process 3 - 4*: isentropic expansion of the working fluid as the piston continues to move to bottom dead center (power stroke)
- *Process 4 - 1* constant volume heat removal from the working fluid while the piston is at bottom dead center (exhaust and intake strokes)

The cycle can be analyzed using the First Law applied to the air in the cylinder, neglecting changes in kinetic and potential energy. For Process 1 - 2,

$$\Delta U_{12} = m(u_2 - u_1) = Q_{\text{in},12} + W_{\text{on},12}, \tag{3.219}$$
$$\implies W_{\text{on},12} = m(u_2 - u_1). \tag{3.220}$$

where $Q_{\text{in},12} = 0$ since the process is isentropic and internally reversible.

For Process 2 - 3,

$$\Delta U_{23} = m(u_3 - u_2) = Q_{\text{in},23} - W_{\text{by},23}, \tag{3.221}$$
$$\implies Q_{\text{in},23} = m(u_3 - u_2) + p_{23}(v_3 - v_2) = m(h_3 - h_2). \tag{3.222}$$

Note that this process is different than the corresponding process for an Otto cycle. For the Diesel cycle, Process 2 - 3 is at constant pressure while in the Otto cycle the process is at constant volume. Thus, there is some work done during this process for a Diesel cycle.

For Process 3 - 4,

$$\Delta U_{34} = m(u_4 - u_3) = Q_{\text{in},34} - W_{\text{by},34}, \tag{3.223}$$
$$\implies W_{\text{by},34} = m(u_3 - u_4). \tag{3.224}$$

where $Q_{\text{in},34} = 0$ since the process is isentropic and internally reversible.

For Process 4 - 1,

$$\Delta U_{41} = m(u_1 - u_4) = -Q_{\text{out},41} - W_{\text{by},41}, \tag{3.225}$$
$$\implies Q_{\text{out},41} = m(u_4 - u_1). \tag{3.226}$$

where $W_{\text{by},41} = 0$ since the process is at constant volume.

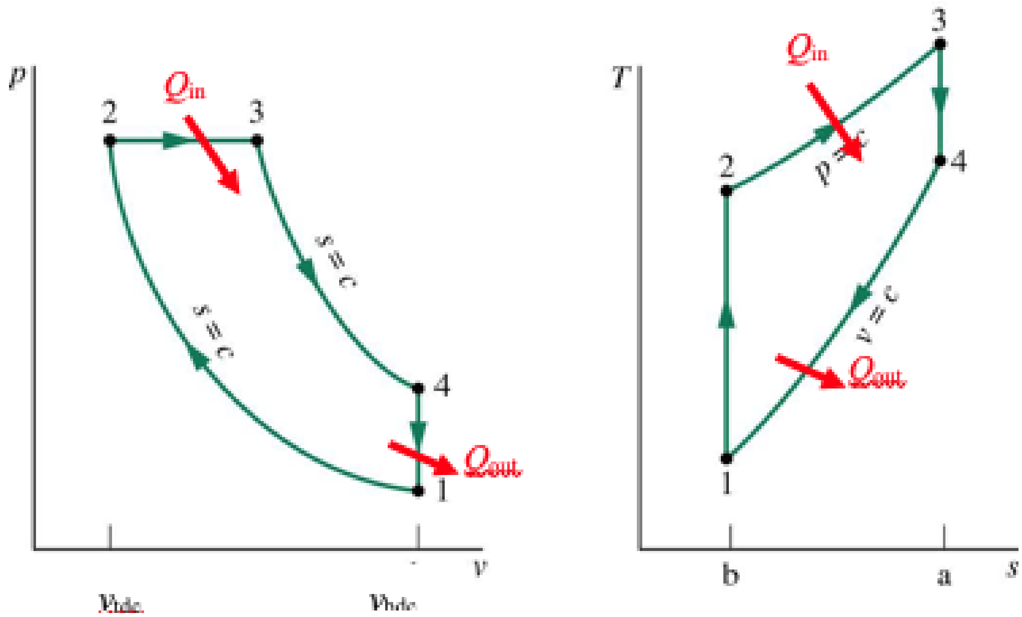


Figure 3.50. Sketches of the p-v and T-s plots for a Diesel cycle. Compare this p-v plot to the one shown in Figure 3.48. These plots are originally from from Moran, M.J., Shapiro, H.N., Boettner, D.D., and Bailey, M.B., *Fundamentals of Engineering Thermodynamics*, Wiley, 7th ed.

The thermal efficiency of the cycle is,

$$\eta_{\rm Diesel} = 1 - \frac{Q_{\rm out,41}}{Q_{\rm in,23}} = 1 - \frac{u_4 - u_1}{h_3 - h_2}. \tag{3.227}$$

The compression ratio for the cycle (Eq. (3.199)) is,

$$r = \frac{v_{\rm bdc}}{v_{\rm tdc}} = \frac{v_1}{v_2}, \tag{3.228}$$

and, since Processes 1 - 2 and 3 - 4 are isentropic processes involving an ideal gas,

$$r = \frac{v_1}{v_2} = \frac{v_r(T_1)}{v_r(T_2)} \quad \text{and} \quad \frac{v_3}{v_4} = \frac{v_r(T_3)}{v_r(T_4)}. \tag{3.229}$$

Note also that $v_4 = v_1 = v_{\rm tdc}$.

For a cold air standard analysis,

$$r = \frac{v_1}{v_2} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{k-1}} \implies \frac{T_2}{T_1} = r^{k-1}, \tag{3.230}$$

$$\frac{v_3}{v_4} = \left(\frac{T_4}{T_3}\right)^{\frac{1}{k-1}} \implies \frac{T_4}{T_3} = \left(\frac{v_3}{v_4}\right)^{k-1}. \tag{3.231}$$

For a Diesel cycle, we also define a cut-off ratio, r_c, which is the ratio of specific volumes during Process 2 - 3, i.e.,

$$r_c \coloneqq \frac{v_3}{v_2} = \frac{T_3}{T_2}, \tag{3.232}$$

where the temperature ratio follows from the Ideal Gas Law since the pressure remains constant during Process 2 - 3. Combining Eqs. (3.231) and (3.232),

$$\frac{T_4}{T_3} = \left(\frac{v_3}{v_4}\right)^{k-1} = \left(\frac{v_3}{v_1}\right)^{k-1} = \left(\frac{v_3}{v_2}\frac{v_2}{v_1}\right)^{k-1}, \tag{3.233}$$

$$\therefore \frac{T_4}{T_3} = \left(\frac{r_c}{r}\right)^{k-1}. \tag{3.234}$$

For the cold air standard analysis, Eq. (3.227) can be written as,

$$\eta_{\text{Diesel}} = 1 - \frac{c_v(T_4 - T_1)}{c_p(T_3 - T_2)}, \tag{3.235}$$

$$= 1 - \frac{1}{k}\frac{T_1}{T_2}\left(\frac{\frac{T_4}{T_1} - 1}{\frac{T_3}{T_2} - 1}\right), \tag{3.236}$$

$$= 1 - \frac{1}{k}\frac{1}{r^{k-1}}\left(\frac{\frac{T_4}{T_3}\frac{T_3}{T_2}\frac{T_2}{T_1} - 1}{r_c - 1}\right), \tag{3.237}$$

$$= 1 - \frac{1}{k}\frac{1}{r^{k-1}}\left[\frac{\left(\frac{r_c}{r}\right)^{k-1} r_c r^{k-1} - 1}{r_c - 1}\right], \tag{3.238}$$

$$\therefore \eta_{\text{Diesel}} = 1 - \frac{1}{r^{k-1}}\left[\frac{r_c^k - 1}{k(r_c - 1)}\right]. \tag{3.239}$$

Notes:

(1) Typical values for the compression ratio for a compression ignition IC engine are $r = 12$ to 24 with engine thermal efficiencies of $\eta = 40$ to 45% (these are real efficiencies, not reversible efficiencies).
(2) As the compression ratio r increases, the thermal efficiency η increases. Diesel engines are not limited by engine knock.
(3) Since they rely on compression ignition, Diesel cycle engines are built for larger pressures. They tend to last longer than spark ignition engines.
(4) The quantity in square brackets in Eq. (3.239) is greater than one for $k > 1$ and $r_c > 1$. Thus, comparing Eq. (3.239) to Eq. (3.218), we observe that the thermal efficiency of the Diesel cycle is less than the thermal efficiency for the Otto cycle *at the same compression ratio* r. Furthermore, the thermal efficiency of the Diesel cycle decreases as r_c increases. As stated previously, however, in practice the compression ratios for compression ignition engines are larger than the compression ratios for spark ignition engines and, thus, the actual efficiencies tend to be larger for compression ignition engines. Additional non-air standard cycle factors, such as air/fuel combustion chemistry, also factor into why compression ignition engines are more efficient in practice.
(5) As the specific heat ratio k increases, the thermal efficiency η increases. The specific heat ratio is determined by the type of fuel used. In addition, the specific heat ratio increases as the temperature decreases.

3.8.4.3. Dual Cycle

The dual cycle combines elements of the combustion processes of the Otto and Diesel cycles to better approximate a real engine cycle. The processes in a dual cycle are (refer to Figure 3.51):

- *Process 1 - 2*: isentropic compression of the working fluid as the piston moves from bottom dead center to top dead center (compression stroke),
- *Process 2 - 3*: *constant volume* heat addition to the working fluid while the piston is at top dead center (beginning of combustion)
- *Process 3 - 4*: *constant pressure* heat addition to the working fluid starting from the top dead center (remaining combustion and start of power stroke)

- *Process 4 - 5*: isentropic expansion of the working fluid as the piston continues to move to bottom dead center (power stroke)
- *Process 5 - 6* constant volume heat removal from the working fluid while the piston is at bottom dead center (exhaust and intake strokes)

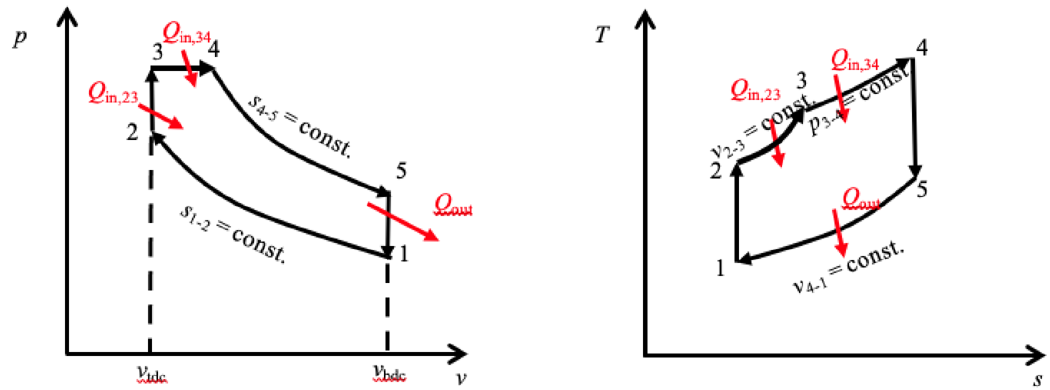


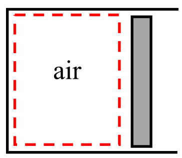
FIGURE 3.51. Sketches of the p-v and T-s plots for a dual cycle. Compare this p-v plot to the one shown in Figure 3.48.

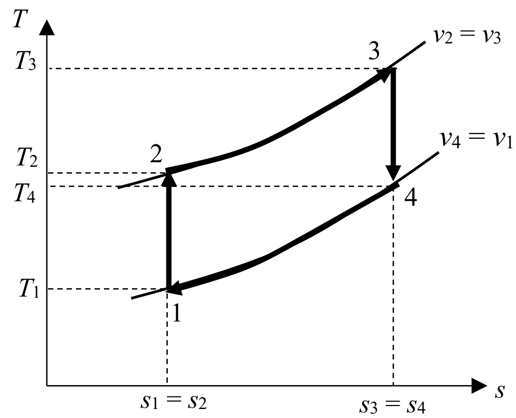
Analysis of the dual cycle won't be described here, but it follows closely the First Law analyses that have already been presented for Otto and Diesel cycles.

An air-standard Otto cycle has a compression ratio of 10. At the beginning of compression, the pressure is 100 kPa (abs) and temperature is 27 ºC. The mass of air is 5 g and the maximum temperature in the cycle is 727 ºC. Determine:

a. the heat rejection, in kJ,
b. the net work, in kJ,
c. the thermal efficiency of the cycle,
d. the mean effective pressure, in kPa (abs), and
e. sketch the process on a *T*-*s* plot, clearly indicating states, paths, and lines of constant specific volume.

SOLUTION:





Note: T_1 = 27 ºC = 300 K and T_3 = 727 ºC = 1000 K.

Assume air is an ideal gas so that,

$$\frac{V_2}{V_1} = \frac{v_r(T_2)}{v_r(T_1)} \Rightarrow v_r(T_2) = v_r(T_1)\frac{V_2}{V_1} \quad (1)$$

where,

$V_2/V_1 = 1/10 = 0.1$ (given),
$v_r(T_1 = 300\text{ K}) = 621.2$ (using the Ideal Gas Table for air),
⇨ $v_r(T_2) = 62.12$.

Using the Ideal Gas Table for air,
$T_2 = 730$ K and $u_2 = 536.1$ kJ/kg.
In addition, $u_1 = 214.1$ kJ/kg.

Similarly,

$$\frac{V_4}{V_3} = \frac{v_r(T_4)}{v_r(T_3)} \Rightarrow v_r(T_4) = v_r(T_3)\frac{V_4}{V_3} \quad (2)$$

where,

$V_4/V_3 = 10/1 = 10$ (given),
$v_r(T_3 = 1000\text{ K}) = 25.17$ (using the Ideal Gas Table for air),
⇨ $v_r(T_4) = 251.7$.

Using the Ideal Gas Table for air,
$T_4 = 430$ K and $u_4 = 308.0$ kJ/kg.
In addition, $u_3 = 758.9$ kJ/kg.

Apply the 1st Law to the system (i.e., the air), for process 2-3,

$$\Delta E_{sys,23} = Q_{into\ sys,23} - W_{by\ sys,23}, \quad (3)$$

where,

$$\Delta E_{sys,23} = \Delta U_{sys,23} = m(u_3 - u_2) \text{ (Neglecting changes in kinetic and potential energies.),} \quad (4)$$

$$W_{by\ sys,23} = 0 \text{ (Constant volume process.)} \quad (5)$$

Substitute and simplify,

$$Q_{into\ sys,23} = m(u_3 - u_2). \quad (6)$$

Using the previously determined and given values,
$Q_{\text{into sys,23}} = 1.114$ kJ.

Similarly, for process 4-1,

$$\Delta E_{sys,41} = Q_{into\ sys,41} - W_{by\ sys,41}, \quad (7)$$

where,

$$\Delta E_{sys,41} = \Delta U_{sys,41} = m(u_1 - u_4) \text{ (Neglecting changes in kinetic and potential energies.),} \quad (8)$$

$$W_{by\ sys,41} = 0 \text{ (Constant volume process.)} \quad (9)$$

Substitute and simplify,

$Q_{into\ sys,41} = m(u_1 - u_4)$. (10)

Using the previously determined and given values,

$\boxed{Q_{\text{into sys},41} = -0.4695 \text{ kJ. Thus, 0.470 kJ of energy is rejected via heat transfer from the system.}}$

The net work for the cycle may be found by applying the 1st Law to the system over the entire cycle,

$\Delta E_{sys,cycle} = Q_{into\ sys,cycle} - W_{by\ sys,cycle}$, (11)

where,

$\Delta E_{sys,cycle} = 0$ (The net change in properties over a cycle is zero.) (12)

$Q_{into\ sys,cycle} = Q_{into\ sys,23} + Q_{into\ sys,41}$ (No heat is added in processes 1-2 and 3-4.) (13)

Substitute and simplify,

$W_{by\ sys,cycle} = Q_{into\ sys,23} + Q_{into\ sys,41}$. (14)

Using the previously calculated values,

$\boxed{W_{by\ sys,cycle} = 0.645 \text{ kJ}}$.

Alternately, we could have found the net work by applying the 1st Law to the compression and power strokes of the cycle separately,

$m(u_2 - u_1) = -W_{by\ sys,12}$, (15)

$m(u_4 - u_3) = -W_{by\ sys,34}$, (16)

⇨ $W_{by\ sys,12} = -1.61$ kJ and $W_{by\ sys,34} = 2.2545$ kJ,

⇨ $W_{by\ sys,cycle} = W_{by\ sys,12} + W_{by\ sys,34} = 0.645$ kJ, which is the same answer found previously.

The thermal efficiency for the cycle is,

$\eta \equiv \frac{W_{by\ sys,cycle}}{Q_{into\ sys}}$, (17)

Using the previously calculated values,

$\boxed{\eta = 0.578}$.

The mean effective pressure is given by,

$mep \equiv \frac{W_{by\ sys,cycle}}{V_1 - V_2}$ => $mep \equiv \frac{W_{by\ sys,cycle}}{V_1(1 - V_2/V_1)}$ (18)

where,

$V_1 = \frac{m R_{air} T_1}{p_1}$, (19)

with,

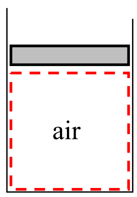
$R_{air} = 0.287$ kJ/(kg.K), $p_1 = 100$ kPa (abs), and $V_2/V_1 = (1/10) = 0.1$,

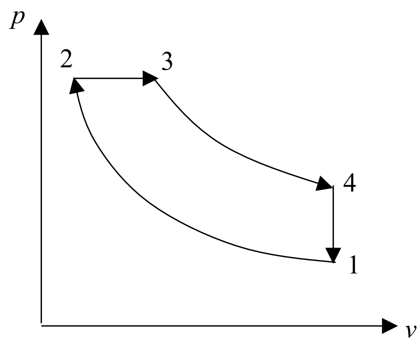
⇨ $V_1 = 4.31 \cdot 10^{-3}\ \text{m}^3$ => $\boxed{mep = 166 \text{ kPa (abs)}}$

The displacement volume of an internal combustion engine is 3 L. The processes within each cylinder of the engine are modeled as an air-standard Diesel cycle with a cutoff ratio of 2.5. The state of the air at the beginning of compression is fixed by p_1 = 95 kPa (abs), T_1 = 22°C, and V_1 = 3.17 L. Determine:

a. the net work per cycle,
b. the power developed by the engine if the cycle repeats 1000 times per minute,
c. and the thermal efficiency of the cycle.

SOLUTION:





First, determine the mass of air in the cylinder using the ideal gas law,

$$m = \frac{p_1 V_1}{R T_1}, \tag{1}$$

Using the given values with R = 0.287 kJ/(kg.K),

$m = 3.5570 \times 10^{-3}$ kg.

Now determine the properties at each state:

State 1:

p_1 = 95 kPa (abs), T_1 = 22°C = 295 K, and V_1 = 3.17 L
=> u_1 = 210.5 kJ/kg and $v_r(T_1 = 295\text{ K}) = 647.9$ (from the Ideal Gas Table (IGT) for air)

State 2:

$V_2 = V_1 - 3.0$ L = 0.17 L (given that the displacement volume is 3 L), (2)

$$\frac{v_2}{v_1} = \frac{V_2}{V_1} = \frac{v_r(T_2)}{v_r(T_1)} \Rightarrow v_r(T_2) = v_r(T_1)\left(\frac{V_2}{V_1}\right), \tag{3}$$

where V_1 = 3.17 L, V_2 = 0.17 L,

⇨ $v_r(T_2) = 34.745$ => T_2 = 896.15 K, u_2 = 671.405 kJ/kg, h_2 = 928.59 kJ/kg (interpolating in the IGT)

The pressure may be found using the ideal gas law,

$$\Rightarrow p_2 = \frac{mRT_2}{V_2} \Rightarrow p_2 = 5381.37 \text{ kPa.} \tag{4}$$

State 3:

The cut-off ratio is given as $r_c = 2.5 = V_3/V_2 = T_3/T_2$ => T_3 = 2240.4 K, V_3 = 0.425 L, (5)
=> h_3 = 2553.87 kJ/kg, u_3 = 1911.76 kJ/kg, $v_r(T_3) = 1.8925$ (interpolating in the IGT)

State 4:

$$\frac{v_4}{v_3} = \frac{V_4}{V_3} = \frac{v_r(T_4)}{v_r(T_3)} \Rightarrow v_r(T_4) = v_r(T_3)\left(\frac{V_4}{V_3}\right) = v_r(T_3)\left(\frac{V_4}{V_1} \cdot \frac{V_1}{V_2} \cdot \frac{V_2}{V_3}\right), \tag{6}$$

where $V_4 = V_1$, V_1 = 3.17 L (given), V_2 = 0.17 L (Eq. (2)), and $V_2/V_3 = 1/r_c = 1/2.5$ (Eq. (5)),
=> $v_r(T_4) = 14.1157$ => T_4 = 1209.8 K and u_4 = 942.17 kJ/kg (interpolating in the IGT)

The work into the air during the compression stroke is found by applying the 1st Law to the air (assuming negligible changes in KE and PE and an adiabatic process),

$$m(u_2 - u_1) = W_{in,12} \tag{7}$$

Using the previously calculated values,

$$W_{in,12} = 1.6394 \text{ kJ}.$$

Now calculate the work done by the air during the heat addition and power strokes using the 1st Law,

$$W_{out,23} = p_2(V_3 - V_2), \tag{8}$$

$$m(u_4 - u_3) = -W_{out,34} \tag{9}$$

Using the previously calculated values,

$$W_{out,23} = 1.3722 \text{ kJ} \quad \text{and} \quad W_{out,34} = 3.449 \text{ kJ}$$

The net work out is,

$$W_{out,net} = W_{out,23} + W_{out,34} - W_{in,12}, \tag{10}$$

$$\boxed{W_{out,net} = 3.18 \text{ kJ} \ \ \text{(This is the work over one cycle.)}}$$

Alternately, we could apply the 1st Law over the whole cycle, keeping in mind that the total energy does not change over the cycle,

$$0 = Q_{in,23} - Q_{out,41} + W_{in,12} - W_{out,23} - W_{out,34}, \tag{11}$$

$$0 = Q_{in,23} - Q_{out,41} - W_{out,net}, \tag{12}$$

$$W_{out,net} = Q_{in,23} - Q_{out,41}. \tag{13}$$

The heat transfer into the system during the combustion process is,

$$m(u_3 - u_2) = Q_{in,23} - p_2(V_3 - V_2), \ \text{(noting that } p_3 = p_2\text{)}, \tag{14}$$

$$Q_{in,23} = m(u_3 - u_2) + p_2(V_3 - V_2) = m(h_3 - h_2). \tag{15}$$

Using the previously calculated values,

$$Q_{in,23} = 5.7811 \text{ kJ}.$$

The heat transfer out of the system is,

$$m(u_4 - u_1) = -Q_{out,41}. \tag{16}$$

$$Q_{out,41} = 2.6025 \text{ kJ}.$$

Using the calculated heat values and Eq. (13),

$W_{out,net} = 3.18$ kJ, which is the same value found previously.

The power is,

$$\dot{W}_{out,net} = \left(\frac{W_{out,net}}{1 \text{ cycle}}\right)\left(\frac{1000 \text{ cycle}}{1 \text{ min}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right), \tag{17}$$

$$\boxed{\dot{W}_{out,net} = 53.0 \text{ kJ/s} = 53.0 \text{ kW}}.$$

The thermal efficiency is,

$$\eta = \frac{W_{out,net}}{Q_{in}}, \tag{18}$$

Using $W_{out,net} = 3.18$ kJ and $Q_{in} = 5.7811$ kJ,

⇨ $\boxed{\eta = 0.550 = 55.0\%}$.

3.8.5. The Brayton Cycle and Improvements

The Brayton cycle is a thermodynamic model for gas turbine engines. In a simple gas turbine engine (components shown schematically in Figure 3.52), air enters a compressor, which does work on the air to increase its pressure and temperature. Downstream of the compressor is a combustor. In the combustor, fuel is added to the air and the air/fuel mixture is ignited, increasing the working fluid temperature significantly. The combustion products move downstream through a turbine, which extracts power from the working fluid. Part of the power from the turbine is used to drive the compressor. In an energy generation application, the remainder of the power is used to drive a generator to create electricity. If the gas turbine engine is used as a jet engine, then only a small portion of the working fluid energy is extracted for excess power. Instead, the remainder of the flow energy is converted to kinetic energy via a nozzle at the engine outlet to provide thrust.

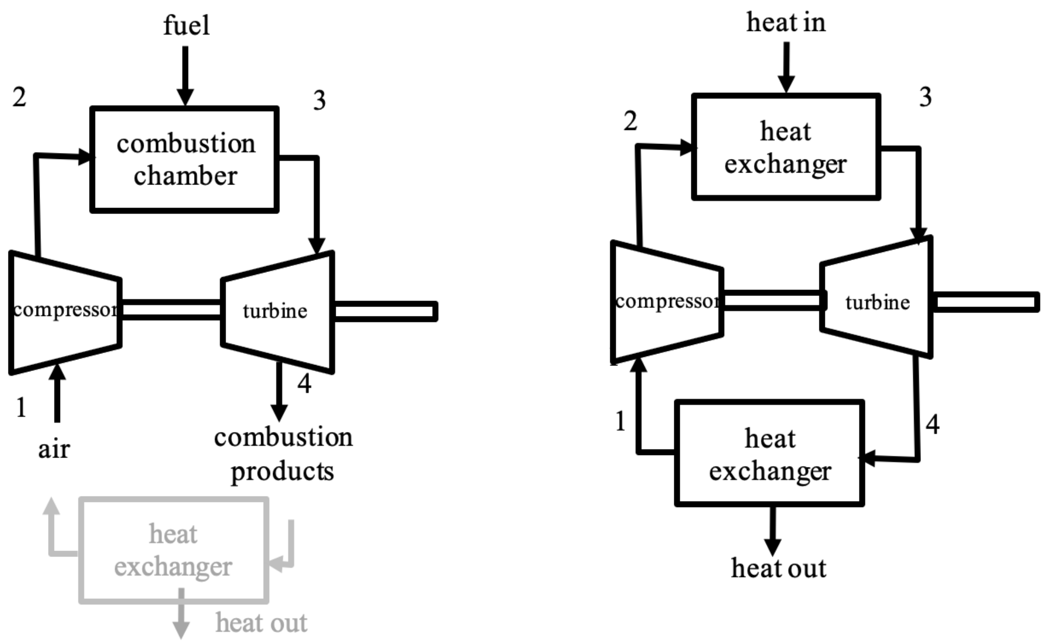


Figure 3.52. Illustrations of the components in open and closed gas turbine cycles.

The left side of Figure 3.52 is an open gas turbine engine cycle while the right side is a closed cycle. In a closed cycle, the working fluid (usually air) is recycled throughout the cycle. In an open cycle, however, fresh air enters the cycle and then is discharged to the atmosphere downstream of the turbine. The figure shows a grayed-out heat exchanger since discharging the exhaust to the atmosphere then pulling in fresh air from the atmosphere is effectively the same as running the air through a heat exchanger. Because the atmosphere is large, the high temperature air leaving the turbine eventually comes into thermal equilibrium with the surrounding atmosphere.

Like the internal combustion engine analyses described in the previous section, the basic study of a gas turbine engine cycle uses an air standard analysis. The assumptions made in this air standard analysis include:

(1) Air is modeled as an ideal gas.
(2) Combustion is modeled as a heat addition process and the working fluid remains as air. The combustion chemistry and changes to the working fluid are ignored.

In a cold air standard analysis, we further assume constant specific heats, i.e., a perfect gas assumption.

The processes in an ideal Brayton cycle analysis, used to model a gas turbine engine cycle, include (Figure 3.53):

- *Process 1 - 2*: isentropic compression of the working fluid through the compressor,
- *Process 2 - 3*: constant pressure heat addition to the working fluid through the heat exchanger (combustion),
- *Process 3 - 4*: isentropic expansion of the working fluid through the turbine,
- *Process 4 - 1*: constant pressure heat transfer from the working fluid as it flows through the heat exchanger.

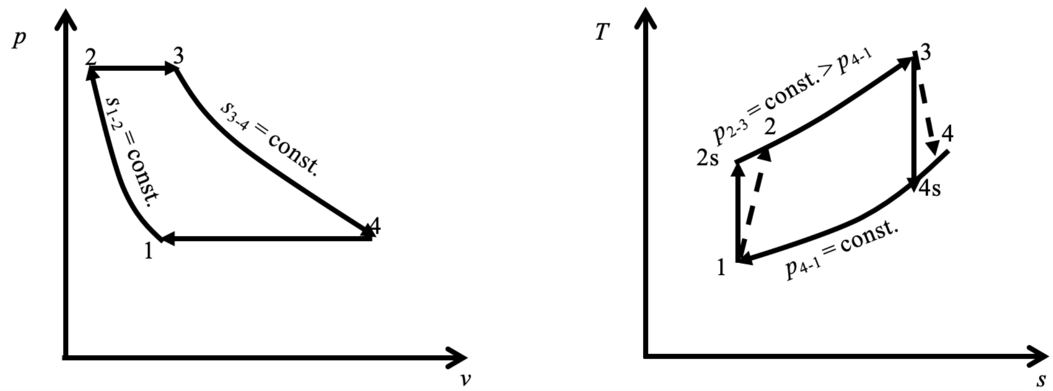


FIGURE 3.53. Illustrations of the processes in a Brayton cycle shown on p - v and T - s plots.

A more detailed analysis of the Brayton cycle involves applying the First Law to control volumes surrounding the various components. For example, for control volumes surrounding the turbine and the compressor,

$$\frac{\dot{W}_{\text{by turbine}}}{\dot{m}} = h_3 - h_4 \quad \text{and} \quad \frac{\dot{W}_{\text{on compressor}}}{\dot{m}} = h_2 - h_1. \tag{3.240}$$

Applying the First Law to control volumes surrounding the heat exchangers,

$$\frac{\dot{Q}_{\text{added}}}{\dot{m}} = h_3 - h_2 \quad \text{and} \quad \frac{\dot{Q}_{\text{removed}}}{\dot{m}} = h_4 - h_1. \tag{3.241}$$

The thermal efficiency of this power cycle is,

$$\eta_{\text{Brayton}} = \frac{\dot{W}_{\text{net, out}}}{\dot{Q}_{\text{added}}} = \frac{\dot{W}_{\text{by turb}}/\dot{m} - \dot{W}_{\text{on comp}}/\dot{m}}{\dot{Q}_{\text{added}}/\dot{m}}, \tag{3.242}$$

$$= \frac{(h_3 - h_4) - (h_2 - h_1)}{(h_3 - h_2)} = \frac{(h_3 - h_2) - (h_4 - h_1)}{(h_3 - h_2)}, \tag{3.243}$$

$$= 1 - \frac{(h_4 - h_1)}{(h_3 - h_2)} = 1 - \frac{\dot{Q}_{\text{removed}}}{\dot{Q}_{\text{added}}}. \tag{3.244}$$

The back work ratio (bwr) for the cycle is,

$$\text{bwr} = \frac{\dot{W}_{\text{on compressor}}}{\dot{W}_{\text{by turbine}}} = \frac{h_2 - h_1}{h_3 - h_2}. \tag{3.245}$$

The bwr for a typical gas turbine is 40 – 80%. The typical bwr for a vapor power plant (Rankine cycle) is 1 – 3%. The difference is due to the fact that the specific volume for a gas is large, but is small for a liquid. Recall that for a steady, internally reversible, process with one inlet and one outlet and negligible changes in kinetic and potential energies: $\dot{W}_{\text{by}}/\dot{m} = \int_1^2 v dp$.

Notes:

(1) Across the compressor and turbine, the typical pressure ratios ($p_2/p_1 = p_3/p_4$) are 5 – 20 and typical engine thermal efficiencies are 35 – 60%.

(2) A larger pressure ratio across the compressor (p_2/p_1) gives a larger efficiency since a larger pressure at State 2 corresponds to a larger average temperature through the combustor where heat addition occurs (recall that $\eta_{\text{rev}} = 1 - T_C/T_H$). Alternately, increasing the temperature leading into the turbine (T_3) also leads to a larger efficiency; however, the temperature at the turbine inlet is typically limited by metallurgical considerations.

(3) Consider the case when T_3 is fixed (for example, due to metallurgical factors), but p_2/p_1 is varied, as shown in Figure 3.54. Cycle A has a larger thermal efficiency than Cycle B since the pressure ratio for Cycle A is larger.

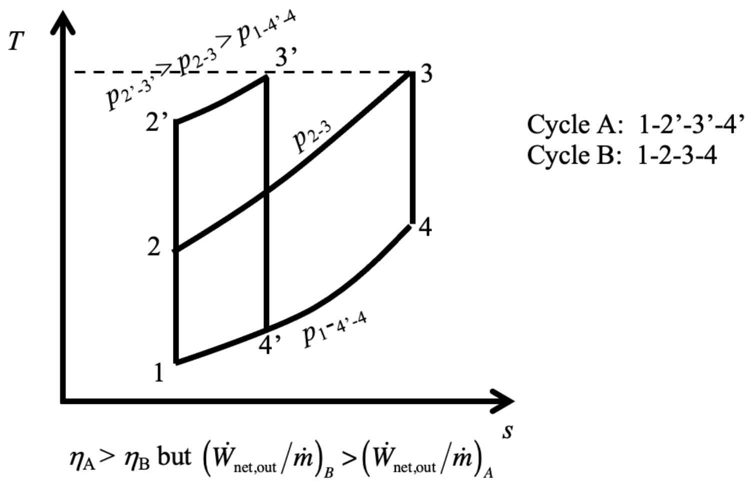


FIGURE 3.54. Two Brayton cycles, but different pressure ratios. Both cycles have the same temperature leaving the combustor ($T_3 = T_3'$), but Cycle A (1-2'-3'-4') has a larger compressor pressure ratio than Cycle B (1-2-3-4), i.e., $p_2'/p_1 > p_2/p_1$.

The area enclosed by Cycle B is larger than the area for Cycle A; hence, Cycle B has a larger work per unit mass flow rate. In order for Cycle A to produce the same work, a larger mass flow rate would be required, potentially requiring a larger set of components, which might be unacceptable for use on an aircraft where weight is a significant design factor. Hence, in aircraft applications, aircraft engine designers typically design for maximum work per unit mass flow rate, i.e., $\left(\dot{W}_{\text{net,out}}/\dot{m}\right)_{\text{max,fixed}T_3}$.

(4) A larger value for the specific heat ratio k results in a larger thermal efficiency. The specific heat ratio is governed by the type of fuel used.

(5) Recall that from the First Law, the Second Law, and the Tds equations combined together, for a steady state flow with a single inlet-outlet and negligible changes in kinetic and potential energies:

$$\frac{\dot{W}_{\text{out}}}{\dot{m}} = w_{\text{out}} = -\int_1^2 v\,dp \quad \text{and} \quad \frac{\dot{W}_{\text{in}}}{\dot{m}} = w_{\text{in}} = \int_1^2 v\,dp \tag{3.246}$$

Thus, the specific work extracted by the turbine can be increased if the specific volume of the working fluid can be increased, and the specific work into the compressor can be decreased if the specific volume can be decreased. These are the ideas behind the concept of "reheating" and "intercooling". Intercooling between successive compressor stages is used to decrease the specific volume of the working fluid. From the Ideal Gas Law,

$$v = \frac{RT}{p} \tag{3.247}$$

For $p =$ constant in an intercooling heat exchanger (refer to Figure 3.55), if T decreases, then v also decreases and, from Eq. (3.246), w_{in} decreases. Thus, intercooling between compressor stages can reduce the specific work required to drive the compressor.

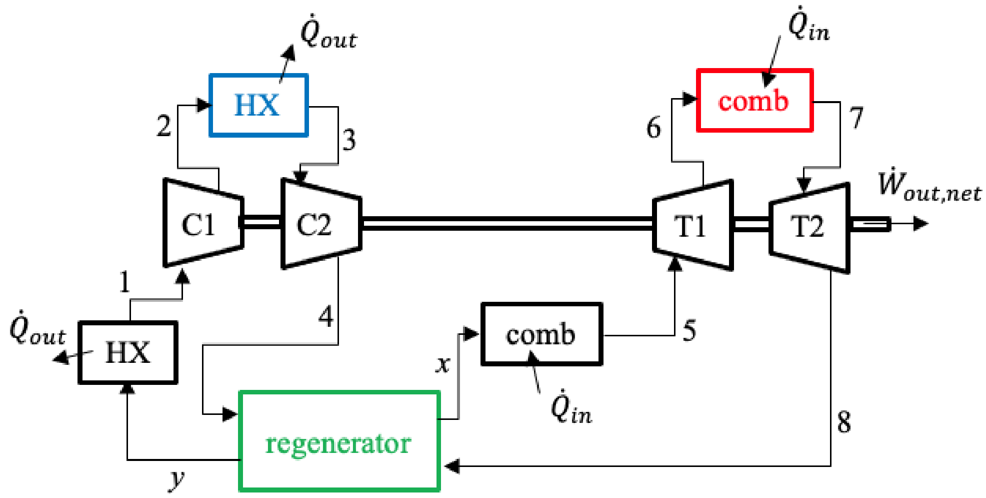


FIGURE 3.55. A gas turbine engine cycle with intercooling between compressor stages, reheating between the turbine stages, and a regenerator.

Reheating between successive turbine stages is used to increase the specific volume of the working fluid (Figure 3.55). Again, from the Ideal Gas Law, if the pressure remains constant in the reheating heat exchanger and the temperature of the working fluid increases, then the specific volume will increase. From Eq. (3.246), w_{out} will increase. Similar to intercooling, reheating between turbine stages can increase the specific work obtained from the turbine.

Another method for improving a Brayton Cycle is to use regenerative heating of the working fluid. Regeneration is when the working fluid is preheated in a heat exchanger using the hot combustion gas in order to reduce the amount of heat (and fuel) needed in the combustor. From the First Law applied to the combustor (Figure 3.55),

$$\dot{Q}_{\text{in}} = \dot{m}(h_5 - h_x), \tag{3.248}$$

where $h_x > h_4$ due to heat transfer with the hot combustion gases. Thus, $\dot{Q}_{\text{in}}$ in the combustor decreases with the use of the regenerator and, as a result, the cycle thermal efficiency increases since $\dot{Q}_{\text{in}}$ decreases.

To determine the effectiveness of using the exhaust working fluid to preheat the working fluid leading into the combustor, we can define a regenerator effectiveness,

$$\eta_{\text{ref}} \coloneqq \frac{h_x - h_4}{h_8 - h_4}. \tag{3.249}$$

The effectiveness is defined in this manner since the largest temperature that State x can reach is the temperature at State 8 (Figure 3.56). Thus, we compare the actual specific heat transfer into the working fluid between States 4 and x to the ideal specific heat transfer, assuming State x has the same temperature as State 8 (and, thus, the same specific enthalpy since specific enthalpy is a function only of temperature for an ideal gas).

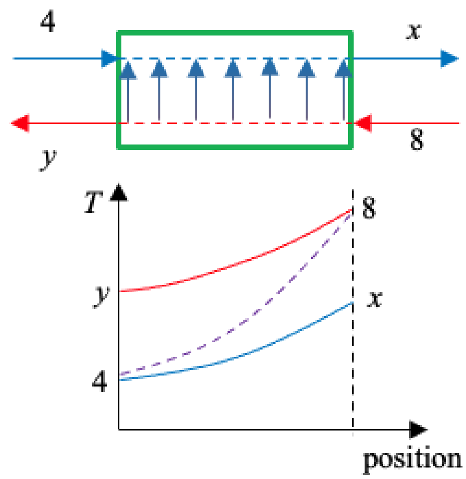
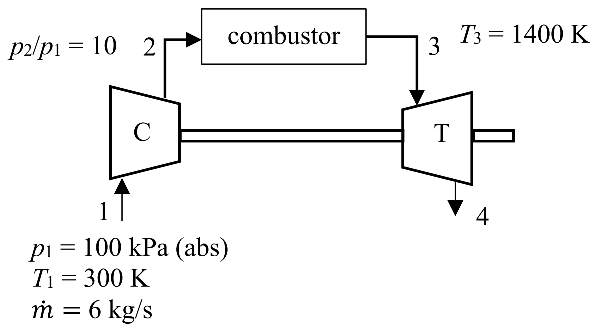


Figure 3.56. Top: A schematic of a counterflow regenerator. Bottom: A plot of the working fluid temperatures as a function of position in the regenerator. State 4 to state x is the flow leading into the combustor. State 8 to state y is the flow leading into the heat removal heat exchanger. The cooler working fluid traveling from State 4 to State x heats up due to heat transfer from the warmer fluid traveling from State 8 to State y. The dashed line in the plot shows the ideal temperature profile for the fluid traveling from State 4 to State x.

Air enters the compressor of an ideal cold air-standard Brayton cycle at 100 kPa (abs) and 300 K, with a mass flow rate of 6 kg/s. The compressor pressure ratio is 10 and the turbine inlet temperature is 1400 K. For a specific heat ratio of 1.4, calculate:
1. the thermal efficiency of the cycle,
2. the back work ratio, and
3. the net power developed.

SOLUTION:



Calculate the thermal efficiency for the Brayton cycle,

$$\eta = 1 - \left(\frac{p_2}{p_1}\right)^{\frac{1-k}{k}}, \tag{1}$$

where,

$k = 1.4$,

$p_2/p_1 = 10$,

⇨ $\boxed{\eta = 0.482}$.

The back work ratio (bwr) is,

$$bwr \equiv \frac{\dot{W}_{into\ comp}}{\dot{W}_{by\ turb}}, \tag{2}$$

where,

$$\dot{W}_{into\ comp} = \dot{m}(h_2 - h_1) = \dot{m}c_p(T_2 - T_1), \tag{3}$$

$$\dot{W}_{by\ turb} = \dot{m}(h_3 - h_4) = \dot{m}c_p(T_3 - T_4), \tag{4}$$

so that Eq. (2) becomes,

$$bwr = \frac{T_2 - T_1}{T_3 - T_4}. \tag{5}$$

Note that Eqs. (3) and (4) were derived by applying the 1st Law to CVs that surround the compressor and turbine, respectively, and assuming steady flow, one inlet and one outlet, adiabatic conditions, and neglecting changes in kinetic and potential energies. In addition, the air is assumed to be a perfect gas (constant specific heats).

The temperature ratios T_2/T_1 and T_4/T_3 may be found by noting that the flow through the compressor and turbine are assumed to be adiabatic an reversible => isentropic. Since the air is also assumed to be a perfect gas, the temperature and pressure ratios are related by,

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}. \tag{6}$$

$$\frac{T_4}{T_3} = \left(\frac{p_4}{p_3}\right)^{\frac{k-1}{k}}. \tag{7}$$

For the given values,

$p_2/p_1 = 10$, $k = 1.4$ => $T_2/T_1 = 1.9307$,

$p_4/p_3 = 1/10$, $k = 1.4$ => $T_4/T_3 = 0.51795$,

(Since the pressure remains constant in the combustor, $p_3 = p_2$. In addition, the pressure at 4 will be the same as the pressure at 1, i.e., $p_4 = p_1$, since both are either open to the atmosphere or are connected via another heat exchanger.)

Given that $T_1 = 300$ K and $T_3 = 1400$ K,

⇨ $T_2 = 579.2$ K,

⇨ $T_4 = 725.1$ K.

Substituting these temperature values into Eq. (5) gives,

$\boxed{bwr = 0.414}$.

The net power developed is,

$$\dot{W}_{by,net} = \dot{W}_{by\ turb} - \dot{W}_{into\ comp} = \dot{W}_{by\ turb}\left(1 - \frac{\dot{W}_{into\ comp}}{\dot{W}_{by\ turb}}\right) = \dot{W}_{by\ turb}(1 - bwr), \tag{8}$$

$$\dot{W}_{by,net} = \dot{m}c_p(T_3 - T_4)(1 - bwr), \tag{9}$$

where Eq. (4) has been used to derive Eq. (9). Using the given values,

$c_p = 1.005$ kJ/(kg.K), (value for air at 300 K, since it's a cold air-standard analysis)

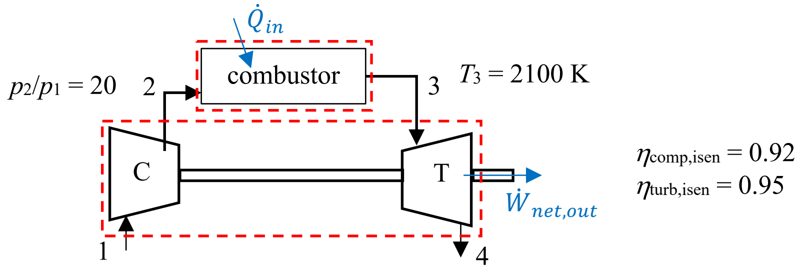
$\dot{m} = 6$ kg/s,

⇨ $\boxed{\dot{W}_{by,net} = 2390 \text{ kW}}$.

Air enters the compressor of an air-standard Brayton cycle with a volumetric flow rate of 60 m^3/s at 0.8 bar (abs) and 280 K. The compressor pressure ratio is 20 and the maximum cycle temperature is 2100 K. The compressor and turbine isentropic efficiencies are 92% and 95%, respectively. Determine:

a. the net power developed from the cycle,
b. the rate of heat addition in the combustor, and
c. the thermal efficiency of the cycle.

SOLUTION:



$p_1 = 0.8$ bar (abs) $= 80$ kPa (abs)
$T_1 = 280$ K
$\dot{V}_1 = 60$ m^3/s

Apply the 1st Law to a control volume surrounding the compressor and turbine,

$$0 = \dot{m}(h_1 - h_2 + h_3 - h_4) - \dot{W}_{out,net}, \quad (1)$$

(assuming steady state, negligible KE and PE, and adiabatic conditions)

$$\dot{W}_{out,net} = \dot{m}(h_1 - h_2 + h_3 - h_4). \quad (2)$$

The mass flow rate can be found from the conditions at State 1 and using the ideal gas law,

$$\dot{m} = \rho_1 \dot{V}_1 = \left(\frac{p_1}{RT_1}\right)\dot{V}_1, \quad (3)$$

⇨ $\dot{m} = 59.731$ kg/s

Now determine the specific enthalpies at each of the states.

State 1:

$T_1 = 280$ K => $h_1 = 280.1$ kJ/kg, $p_r(T_1) = 1.0889$ (from the Ideal Gas Table)

State 3:

$T_3 = 2100$ K => $h_3 = 2377$ kJ/kg, $p_r(T_3) = 2559$ (from the IGT)

State 2:

$$\eta_{comp,isen} = \frac{h_{2s} - h_1}{h_2 - h_1} \Rightarrow h_2 = h_1 + \frac{h_{2s} - h_1}{\eta_{comp,isen}}. \quad (4)$$

State 2s: $p_2/p_1 = p_{2s}/p_1 = 20$,

$$\frac{p_{2s}}{p_1} = \frac{p_r(T_{2s})}{p_r(T_1)} \Rightarrow p_r(T_{2s}) = p_r(T_1)\left(\frac{p_{2s}}{p_1}\right) \text{ (isentropic compression of an ideal gas),} \quad (5)$$

=> $p_r(T_{2s}) = 21.778$ => $T_{2s} = 649.33$ K, $h_{2s} = 659.29$ kJ/kg (from IGT)

Using Eq. (4) and the given and computed values,

$h_2 = 692.26$ kJ/kg.

State 4:

$$\eta_{turb,isen} = \frac{h_4 - h_3}{h_{4s} - h_3} \Rightarrow h_4 = h_3 + \eta_{turb,isen}(h_{4s} - h_3). \quad (6)$$

State 4s: $p_3/p_4 = p_3/p_{4s} = p_2/p_1 = 20$,

$$\frac{p_3}{p_{4s}} = \frac{p_r(T_3)}{p_r(T_{4s})} \Rightarrow p_r(T_{4s}) = p_r(T_3)\left(\frac{p_{4s}}{p_3}\right) \text{ (isentropic expansion of an ideal gas)}, \quad (7)$$

=> $p_r(T_{4s}) = 127.95$ => $T_{4s} = 1029.19$ K, $h_{4s} = 1079.57$ kJ/kg (from IGT)

Using Eq. (6) and the given and computed values,

$h_4 = 1144.44$ kJ/kg.

Using the state data, mass flow rate, and Eq. (2),

$\boxed{\dot{W}_{out,net} = 49.0 \text{ MJ}}$.

The rate of heat addition into the combustor is found by applying the 1st Law to a control volume surrounding the combustor,

$$0 = \dot{m}(h_2 - h_3) + \dot{Q}_{in}, \quad (8)$$

(assuming steady state, negligible KE and PE, and a passive device)

$$\dot{Q}_{in} = \dot{m}(h_3 - h_2). \quad (9)$$

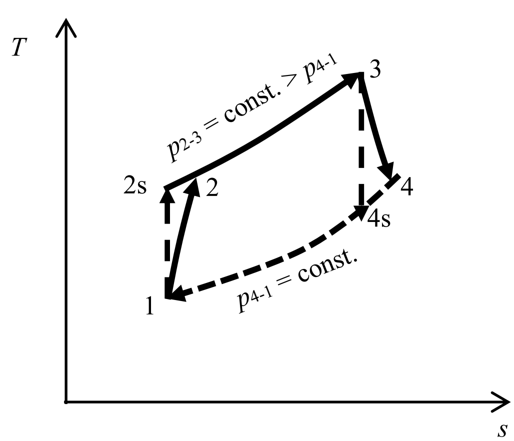
Using the previously calculated quantities,

$\boxed{\dot{Q}_{in} = 101 \text{ MW}}$.

The cycle's thermal efficiency is,

$$\eta = \frac{\dot{W}_{out,net}}{\dot{Q}_{in}},$$

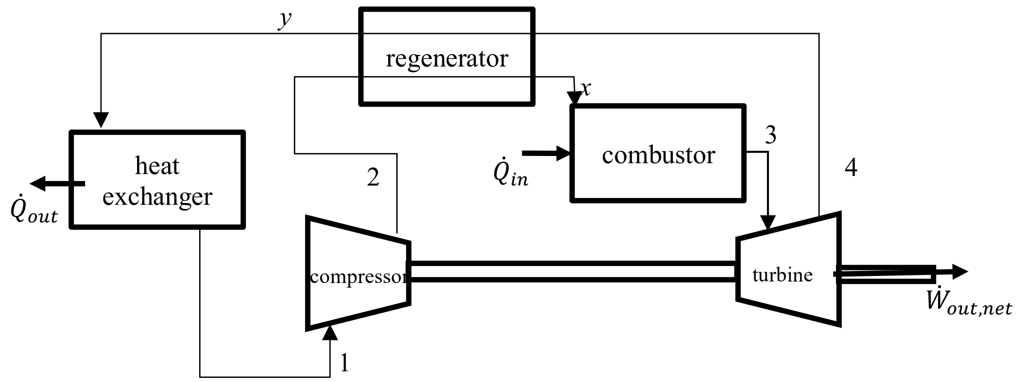
=> $\boxed{\eta = 0.487 = 48.7\%}$.



Air enters the compressor of a regenerative air-standard Brayton cycle with a volumetric flow rate of 60 m^3/s at 0.8 bar (abs) and 280 K. The compressor pressure ratio is 20 and the maximum cycle temperature is 2100 K. The compressor and turbine have isentropic efficiencies of 92% and 95%, respectively. For a regenerator effectiveness of 85%, determine:

a. the net power developed,
b. the rate of heat addition in the combustor,
c. the thermal efficiency of the cycle.

SOLUTION:



To determine the net power developed, apply the 1st Law to a CV surrounding the compressor and turbine,

$\dot{W}_{out,net} = \dot{m}(h_1 + h_3 - h_2 - h_4)$ (assuming SSSF, adiabatic, and negligible KE and PE). (1)

The rate of heat transfer in the combustor is found by applying the 1st Law to a CV surrounding the combustor,

$\dot{Q}_{in} = \dot{m}(h_3 - h_x)$ (assuming SSSF, passive device, and negligible KE and PE). (2)

Now find the properties at the various states.

State 1:

$\dot{V} = 60$ m^3/s, $p_1 = 0.8$ bar (abs) = 80 kPa (abs), $T_1 = 280$ K

=> $h_1 = 280.1$ kJ/kg and $p_r(T_1) = 1.0889$ (from the Ideal Gas Table for air)

Also, from the ideal gas law,

=> $\rho_1 = \frac{p_1}{RT_1} = 0.9955$ kg/m^3 => $\dot{m} = \rho\dot{V} = 59.731$ kg/s (3)

State 3:

$T_3 = 2100$ K,

=> $h_3 = 2377$ kJ/kg and $p_r(T_3) = 2559$ (from the Ideal Gas Table for air)

State 2:

$p_2/p_1 = 20 = p_{2s}/p_1$ and $\eta_{\text{comp,isen}} = 0.92$ (given),

$\eta_{comp,isen} = \frac{w_{in,isen}}{w_{in}} = \frac{h_{2s}-h_1}{h_2-h_1}$ => $h_2 = h_1 + \frac{h_{2s}-h_1}{\eta_{comp,isen}}$. (4)

For an ideal gas undergoing an isentropic process,

$\frac{p_{2s}}{p_1} = \frac{p_r(T_{2s})}{p_r(T_1)}$ => $p_r(T_{2s}) = p_r(T_1)\left(\frac{p_{2s}}{p_1}\right)$, (5)

=> $p_r(T_{2s}) = 21.778$ => $T_{2s} = 649.33$ K, $h_{2s} = 659.29$ kJ/kg (IGT),

=> $h_2 = 692.26$ kJ/kg.

State 4:

$p_3/p_4 = 20 = p_3/p_{4s}$ ($= p_2/p_1$) and $\eta_{\text{turb,isen}} = 0.95$ (given),

$$\eta_{turb,isen} = \frac{w_{out}}{w_{out,isen}} = \frac{h_3 - h_4}{h_3 - h_{4s}} \Rightarrow h_4 = h_3 - \eta_{turb,isen}(h_3 - h_{4s}). \tag{6}$$

For an ideal gas undergoing an isentropic process,

$$\frac{p_{4s}}{p_3} = \frac{p_r(T_{4s})}{p_r(T_3)} \Rightarrow p_r(T_{4s}) = p_r(T_3)\left(\frac{p_{4s}}{p_3}\right), \tag{7}$$

$\Rightarrow p_r(T_{4s}) = 127.95 \Rightarrow T_{4s} = 1029.19$ K, $h_{4s} = 1079.57$ kJ/kg (IGT),

$\Rightarrow h_4 = 1144.44$ kJ/kg.

State x:

From the definition of the regenerator effectiveness,

$$\eta_{reg} = \frac{h_x - h_2}{h_4 - h_2} \Rightarrow h_x = h_2 + \eta_{reg}(h_4 - h_2), \tag{8}$$

Using the previously calculated specific enthalpy values and the given $\eta_{reg} = 0.85$,

$\Rightarrow h_x = 1076.61$ kJ/kg.

Using these state data, Eq. (1) gives,

$\boxed{\dot{W}_{out,net} = 49.0 \text{ MW}}$,

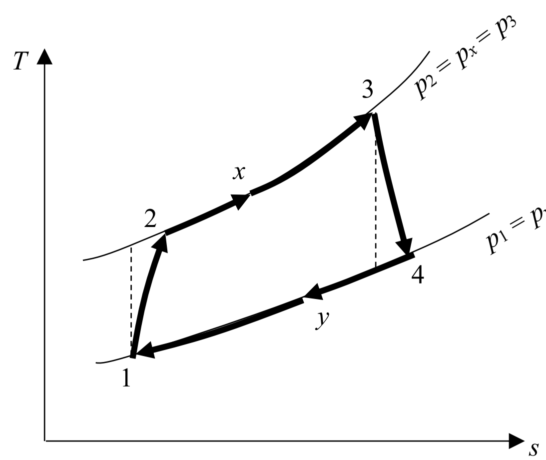
and Eq. (2) gives,

$\boxed{\dot{Q}_{in} = 77.7 \text{ MW}}$.

The thermal efficiency for the cycle is,

$$\boxed{\eta_{cycle} = \frac{\dot{W}_{out,net}}{\dot{Q}_{in}} = 0.631 = 63.1\%.} \tag{9}$$

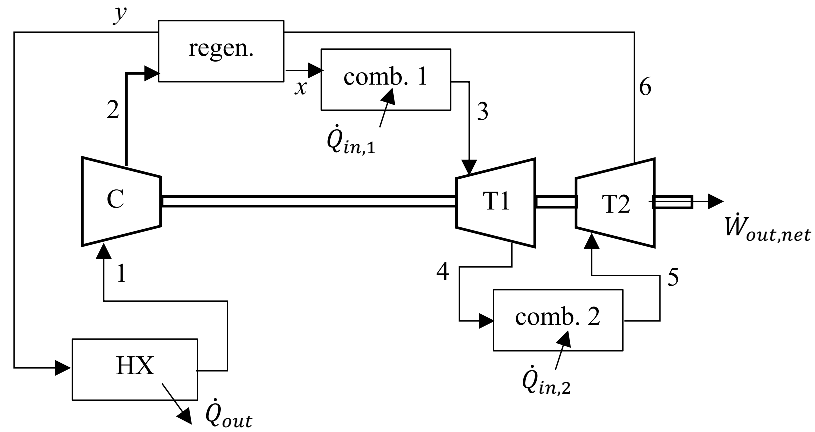
Note that as η_{reg} increases, then h_x approaches h_4 and $\dot{Q}_{in}$ decreases. As a result, the thermal efficiency for the cycle would increase. In the limit of $\eta_{reg} = 100\%$, $\dot{Q}_{in,min} = 73.6$ MW and $\eta_{cycle,max} = 66.6\%$. In contrast, without the regenerator ($\eta_{reg} = 0$), then $\dot{Q}_{in,max} = 100.6$ MW and $\eta_{cycle,min} = 48.7\%$. Thus, we observe that including a regenerator can substantially improve the cycle's thermal efficiency.



Air enters the compressor of a cold air-standard Brayton cycle with regeneration and reheat at 100 kPa (abs), 300 K, with a mass flow rate of 6 kg/s. The compressor pressure ratio is 10 and the inlet temperature for each turbine stage is 1400 K. The pressure ratios across each turbine stage are equal. The turbine stages and compressor each have isentropic efficiencies of 80% and the regenerator effectiveness is 80%. For a specific heat ratio of 1.4, calculate:

a. the thermal efficiency of the cycle,
b. the back work ratio, and
c. the net power developed by the cycle.

SOLUTION:



Since we're assuming a cold air-standard analysis, state the properties of the air in the analysis:

R = 0.287 kJ/(kg.K) and $c_{p@300\text{ K}}$ = 1.005 kJ/(kg.K).

The net power from the cycle is found by applying the 1st Law to a CV surrounding the compressor and turbines (assuming SSSF, adiabatic operation, negligible KE and PE),

$$\dot{W}_{out,net} = \dot{m}(h_1 - h_2 + h_3 - h_4 + h_5 - h_6), \quad (1)$$

and, since we're performing a cold air-standard analysis, meaning the air is a perfect gas,

$$\dot{W}_{out,net} = \dot{m}c_p(T_1 - T_2 + T_3 - T_4 + T_5 - T_6). \quad (2)$$

The power into the compressor is found by applying the 1st Law to a CV surrounding just the compressor (assuming SSSF, adiabatic operation, negligible KE and PE),

$$\dot{W}_{in} = \dot{m}(h_2 - h_1), \quad (3)$$

$$\dot{W}_{in} = \dot{m}c_p(T_2 - T_1) \quad \text{(assuming perfect gas behavior).} \quad (4)$$

The back work ratio (bwr) is,

$$bwr = \frac{\dot{W}_{in}}{\dot{W}_{out}} = \frac{\dot{W}_{in}}{\dot{W}_{out,net} + \dot{W}_{in}}. \quad (5)$$

The rate at which heat is added into the two combustors is found by applying the 1st Law to CVs surrounding each combustor (assuming SSSF, passive devices, negligible KE and PE),

$$\dot{Q}_{in} = \dot{Q}_{in,1} + \dot{Q}_{in,2} = \dot{m}(h_3 - h_x) + \dot{m}(h_5 - h_4), \quad (6)$$

$$\dot{Q}_{in} = \dot{m}c_p(T_3 - T_x + T_5 - T_4) \quad \text{(assuming perfect gas behavior).} \quad (7)$$

The thermal efficiency of the cycle is,

$$\eta = \frac{\dot{W}_{out,net}}{\dot{Q}_{in}}. \quad (8)$$

Now find the temperatures at the various states.

State 1:
$\dot{m} = 6$ kg/s, $p_1 = 100$ kPa (abs), $T_1 = 300$ K (given)

State 2s: (assuming the process is isentropic and involves a perfect gas)

$$\frac{T_{2s}}{T_1} = \left(\frac{p_{2s}}{p_1}\right)^{\frac{k-1}{k}} \Rightarrow T_{2s} = 579.21 \text{ K using } p_{2s}/p_1 = p_2/p_1 = 10. \tag{9}$$

State 2:

$$\eta_{comp,isen} = \frac{w_{in,isen}}{w_{in}} = \frac{h_{2s}-h_1}{h_2-h_1} = \frac{c_p(T_{2s}-T_1)}{c_p(T_2-T_1)} \Rightarrow T_2 = T_1 + \frac{T_{2s}-T_1}{\eta_{comp,isen}}, \tag{10}$$

=> $T_2 = 649.01$ K.

State 3:
$T_3 = 1400$ K (given)

State 4s:

$$\frac{T_{4s}}{T_3} = \left(\frac{p_{4s}}{p_3}\right)^{\frac{k-1}{k}} \Rightarrow T_{4s} = 1007.56 \text{ K using } p_3/p_{4s} = 3.162. \tag{11}$$

Note that since we're told that the pressure drops across both turbine stages are equal,

$$\frac{p_2}{p_1} = \frac{p_3}{p_6} = \left(\frac{p_3}{p_4}\right)\underbrace{\left(\frac{p_4}{p_5}\right)}_{=1}\underbrace{\left(\frac{p_5}{p_6}\right)}_{=p_3/p_4} = \left(\frac{p_3}{p_4}\right)^2 \Rightarrow \frac{p_3}{p_4} = \sqrt{\frac{p_2}{p_1}} \Rightarrow \frac{p_3}{p_4} = \frac{p_5}{p_6} = 3.162. \tag{12}$$

State 4:

$$\eta_{turb,isen} = \frac{w_{out}}{w_{out,isen}} = \frac{h_3-h_4}{h_3-h_{4s}} = \frac{c_p(T_3-T_4)}{c_p(T_3-T_{4s})} \Rightarrow T_4 = T_3 - \eta_{turb,isen}(T_3 - T_{4s}), \tag{13}$$

=> $T_4 = 1086.05$ K.

State 5:
$T_5 = 1400$ K (given)

State 6s:

$$\frac{T_{6s}}{T_5} = \left(\frac{p_{6s}}{p_5}\right)^{\frac{k-1}{k}} \Rightarrow T_{6s} = 1007.56 \text{ K using } p_{6s}/p_5 = 3.162 \text{ (Eq. (12))} \tag{14}$$

State 6:

$$\eta_{turb,isen} = \frac{w_{out}}{w_{out,isen}} = \frac{h_5-h_6}{h_5-h_{6s}} = \frac{c_p(T_5-T_6)}{c_p(T_5-T_{6s})} \Rightarrow T_6 = T_5 - \eta_{turb,isen}(T_5 - T_{6s}), \tag{15}$$

=> $T_6 = 1086.05$ K.

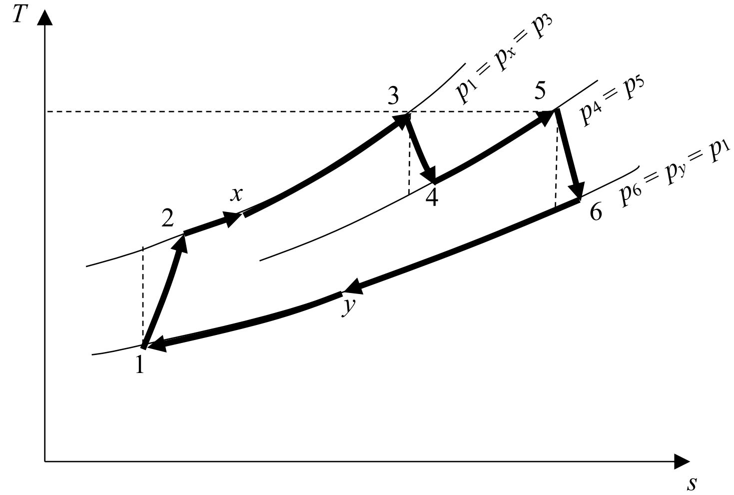
State x:

$$\eta_{reg} = \frac{h_x-h_2}{h_6-h_2} = \frac{c_p(T_x-T_2)}{c_p(T_6-T_2)} \Rightarrow T_x = T_2 + \eta_{reg}(T_6 - T_2). \tag{16}$$

=> $T_x = 998.64$ K using $\eta_{reg} = 0.80$ (given) and the previously calculated values.

Using the calculated temperatures and Eqs. (2), (4), (5), (7), and (8),

$\boxed{\dot{W}_{out,net} = 1680 \text{ kW}}$,
$\boxed{\dot{W}_{in} = 2100 \text{ kW}}$,
$\boxed{\dot{Q}_{in} = 4300 \text{ kW}}$,
$\boxed{bwr = 0.556 = 55.6\%}$,
$\boxed{\eta = 0.390 = 39.0\%}$.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CHAPTER 4

Integral Analysis

4.1. Lagrangian and Eulerian Perspectives

There are two common ways to study a moving fluid:

(1) Look at a particular location and observe how all the fluid passing that location behaves. This is called the Eulerian point of view.
(2) Look at a particular piece of fluid and observe how it behaves as it moves from location to location. This is called the Lagrangian point of view.

For example, Let's say we want to study migrating birds. We could either:

(1) stand in a fixed spot and make measurements as birds fly by (Eulerian point of view; Figure 4.1), or
(2) tag some of the birds and make measurements as they fly along (Lagrangian point of view; Figure 4.1).

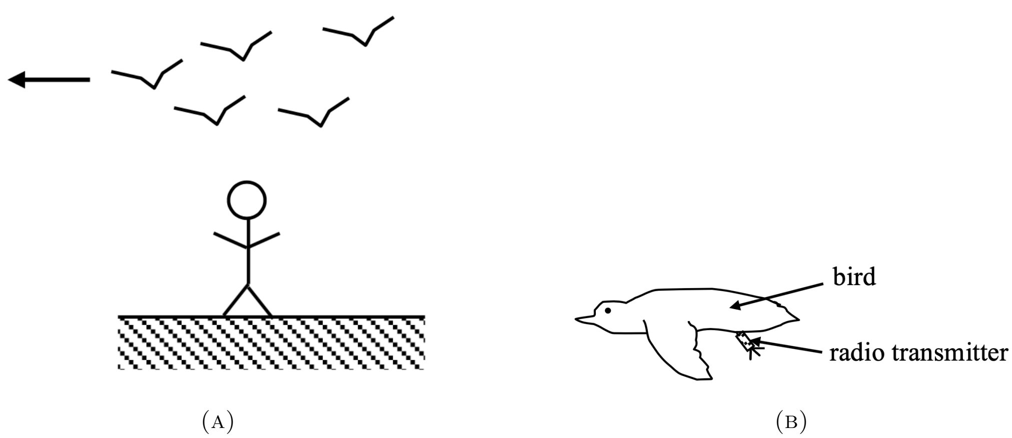


Figure 4.1. Studying birds using (A) Eulerian and (B) Lagrangian approaches.

4.1.1. Lagrangian (aka Material, Particle, Substantial) Derivative

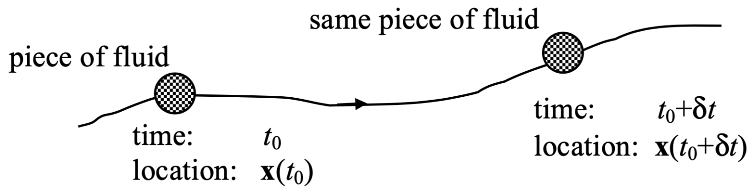


Figure 4.2. An illustration showing the path of a fluid particle.

If we follow a piece of fluid (Lagrangian viewpoint), how will some property of that particular piece of fluid change with respect to time? Let's say we're interested in looking at the time rate of change of temperature, T, that the particle observes as it moves from location to location (Figure 4.2. The particle may experience a temperature change because the temperature of the entire field of fluid may be changing with respect to time (i.e., the temperature field may be unsteady). In addition, the temperature field may have spatial gradients (different temperatures at different locations, i.e., non-uniform) so that as the particle moves from point to point it will experience a change in temperature. Thus, there are two effects that can cause a time rate of change of temperature that the particle experiences: unsteady effects, also known as local or Eulerian effects, and spatial gradient effects, also known as convective effects. We can describe this in mathematical terms by writing the temperature of the entire field as a function of time, t, and location, $\mathbf{x}$,

$$T = T(t, \mathbf{x}). \tag{4.1}$$

Note that the location of the fluid particle is a function of time: $\mathbf{x} = \mathbf{x}(t)$ so that,

$$T = T(t, \mathbf{x}(t)). \tag{4.2}$$

Taking the time derivative of the temperature, expanding the location vector into its x, y, and z components, and using the chain rule gives,

$$\left.\frac{dT}{dt}\right|_{\substack{\text{following a}\\ \text{fluid particle}}} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\underbrace{\frac{dx}{dt}}_{=u_x} + \frac{\partial T}{\partial y}\underbrace{\frac{dy}{dt}}_{=u_y} + \frac{\partial T}{\partial z}\underbrace{\frac{dz}{dt}}_{=u_z}. \tag{4.3}$$

Note that dx/dt, dy/dt, and dz/dt are the particle velocities u_x, u_y, and u_z respectively. Writing this in a more compact form,

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z}, \tag{4.4}$$

$$= \frac{\partial T}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,T. \tag{4.5}$$

The notation, D/Dt, indicating a Lagrangian (also sometimes referred to as the material, particle, or substantial) derivative, has been used in Eq. (4.4) to indicate that we're following a particular piece of fluid. More generally, we have,

$$\underbrace{\frac{D}{Dt}(\dots)}_{\substack{\text{Lagrangian rate of}\\ \text{change (changes as we}\\ \text{follow a fluid particle)}}} = \underbrace{\frac{\partial}{\partial t}(\dots)}_{\substack{\text{local or Eulerian}\\ \text{rate of change (changes}\\ \text{due to unsteady effects)}}} + \underbrace{(\mathbf{u}\cdot\boldsymbol{\nabla})(\dots)}_{\substack{\text{convective rate}\\ \text{of change (changes due to}\\ \text{a change in particle position)}}}, \tag{4.6}$$

$$= \frac{\partial}{\partial t}(\dots) + u_x\frac{\partial}{\partial x}(\dots) + u_y\frac{\partial}{\partial y}(\dots) + u_z\frac{\partial}{\partial z}(\dots). \tag{4.7}$$

where $(\dots)$ represents any field quantity of interest.

Notes:

(1) The Lagrangian derivatives in cylindrical and spherical coordinates are,

$$\text{cylindrical:}\quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}, \tag{4.8}$$

$$\text{spherical:}\quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}. \tag{4.9}$$

$$\tag{4.10}$$

(2) The acceleration experienced by a fluid particle is given by,

Cartesian:
$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + u_r \frac{\partial \mathbf{u}}{\partial r} + \frac{u_\theta}{r}\frac{\partial \mathbf{u}}{\partial \theta} + u_z \frac{\partial \mathbf{u}}{\partial z}, \tag{4.11}$$

cylindrical:
$$\begin{cases} a_r = \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \\ a_\theta = \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} - \frac{u_r u_\theta}{r} \\ a_z = \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \end{cases} \tag{4.12}$$

spherical:
$$\begin{cases} a_r = \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{1}{r}\left(u_\theta^2 + u_\phi^2\right) \\ a_\theta = \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{1}{r}\left(u_r u_\theta - u_\phi^2 \cot\theta\right) \\ a_\phi = \frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r}\left(u_r u_\phi + u_\theta u_\phi \cot\theta\right) \end{cases} \tag{4.13}$$

a. A fluid velocity field is given by:

$$\mathbf{u} = 2t\hat{\mathbf{e}}_x$$

Will a fluid particle accelerate in this flow? Why?

b. Now consider the following flow:

$$\mathbf{u} = x\hat{\mathbf{e}}_x$$

Will a fluid particle accelerate in this flow? Why?

SOLUTION:

Part (a):

The acceleration is given by:

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{=2\hat{\mathbf{e}}_x} + \underbrace{u_x}_{2t}\underbrace{\frac{\partial \mathbf{u}}{\partial x}}_{=\mathbf{0}} + \underbrace{u_y\frac{\partial \mathbf{u}}{\partial y}}_{=\mathbf{0}} + \underbrace{u_z\frac{\partial \mathbf{u}}{\partial z}}_{=\mathbf{0}}$$

Hence, for the given flow:

$\boxed{\mathbf{a} = 2\hat{\mathbf{e}}_x}$ Yes, fluid particles will accelerate due to the local (or Eulerian) derivative.

Part (b):

The acceleration is given by:

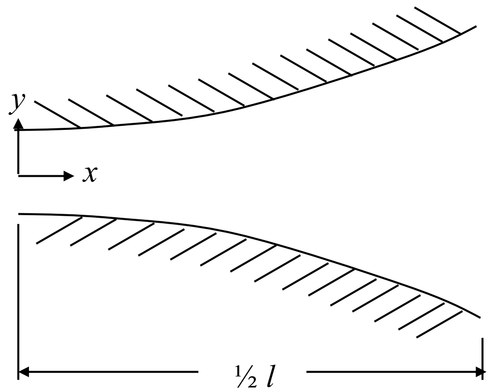
$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{=\mathbf{0}} + \underbrace{u_x}_{x}\underbrace{\frac{\partial \mathbf{u}}{\partial x}}_{=\hat{\mathbf{e}}_x} + \underbrace{u_y\frac{\partial \mathbf{u}}{\partial y}}_{=\mathbf{0}} + \underbrace{u_z\frac{\partial \mathbf{u}}{\partial z}}_{=\mathbf{0}}$$

Hence, for the given flow:

$\boxed{\mathbf{a} = x\hat{\mathbf{e}}_x}$ Yes, fluid particles will accelerate due to the convective derivative.

For the diffuser shown below, determine:

a. the acceleration of a fluid particle for any x and t, and
b. the value of c (other than c=0) for which the acceleration is zero for any x at t=2 s. Assume V_0=10 ft/s and l=5 ft.
c. Explain how the acceleration can be zero if the flow rate is increasing with time.



$$\mathbf{u} = V_0(1-\mathrm{e}^{-ct})(1-\frac{x}{l})\hat{\mathbf{i}}$$

where V_0, c, and l are constants

SOLUTION:

The acceleration of a fluid particle is the Lagrangian derivative of the velocity.

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + u\frac{\partial \mathbf{u}}{\partial x} \tag{1}$$

Substitute the given velocity field.

$$\mathbf{a} = \frac{\partial}{\partial t}\left[V_0\left(1-\mathrm{e}^{-ct}\right)\left(1-\frac{x}{l}\right)\hat{\mathbf{i}}\right] + \left[V_0\left(1-\mathrm{e}^{-ct}\right)\left(1-\frac{x}{l}\right)\right]\frac{\partial}{\partial x}\left[V_0\left(1-\mathrm{e}^{-ct}\right)\left(1-\frac{x}{l}\right)\hat{\mathbf{i}}\right]$$

$$= V_0 c\mathrm{e}^{-ct}\left(1-\frac{x}{l}\right)\hat{\mathbf{i}} + \left[V_0\left(1-\mathrm{e}^{-ct}\right)\left(1-\frac{x}{l}\right)\right]\left[-\frac{V_0}{l}\left(1-\mathrm{e}^{-ct}\right)\hat{\mathbf{i}}\right]$$

$$\boxed{\therefore \mathbf{a} = V_0\left(1-\frac{x}{l}\right)\left[c\mathrm{e}^{-ct} - \frac{V_0}{l}\left(1-\mathrm{e}^{-ct}\right)^2\right]\hat{\mathbf{i}}} \tag{2}$$

At t = 2 s:

$$\mathbf{a}(x, t=2) = \mathbf{0} = V_0\left(1-\frac{x}{l}\right)\left[c\mathrm{e}^{-2c} - \frac{V_0}{l}\left(1-\mathrm{e}^{-2c}\right)^2\right]\hat{\mathbf{i}}$$

$$c\mathrm{e}^{-2c} = \frac{V_0}{l}\left(1-\mathrm{e}^{-2c}\right)^2 \tag{3}$$

Solve numerically for c when V_0 = 10 ft/s and l = 5 ft.

$\Rightarrow \boxed{c = 0.124\ \mathrm{s}^{-1}}$

The acceleration of a fluid particle can be zero even though the flow rate is increasing because the local acceleration ($\partial u/\partial t$) exactly balances the convective deceleration ($u\partial u/\partial x$).

A fluid velocity field is given by,

$$\mathbf{u} = (cy^2)\hat{\mathbf{i}} + (cx^2)\hat{\mathbf{j}},$$

where c is a constant. Determine

a. the components of the acceleration and
b. the points in the flow field where the acceleration is zero.

SOLUTION:

The acceleration of a fluid element is given by,

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + u_x \frac{\partial \mathbf{u}}{\partial x} + u_y \frac{\partial \mathbf{u}}{\partial y} \tag{1}$$

where,

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{0} \quad \text{(steady flow)}$$

$$u_x \frac{\partial \mathbf{u}}{\partial x} = \left(cy^2\right)\left(2cx\hat{\mathbf{j}}\right) = 2c^2xy^2\hat{\mathbf{j}}$$

$$u_y \frac{\partial \mathbf{u}}{\partial y} = \left(cx^2\right)\left(2cy\hat{\mathbf{i}}\right) = 2c^2x^2y\hat{\mathbf{i}}$$

$$\boxed{\therefore \mathbf{a} = 2c^2x^2y\hat{\mathbf{i}} + 2c^2xy^2\hat{\mathbf{j}}} \tag{2}$$

Set the acceleration equal to zero,

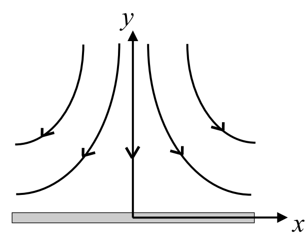
$$\mathbf{a} = \mathbf{0} = 2c^2x^2y\hat{\mathbf{i}} + 2c^2xy^2\hat{\mathbf{j}}$$

$$\boxed{\therefore \text{either } x = 0 \text{ or } y = 0} \text{ (This is locus of points where the total acceleration is zero.)} \tag{3}$$

The velocity field near a planar stagnation point (see the figure below) is given as,

$$\mathbf{u} = U_0\left(\frac{x}{L}\right)\hat{\mathbf{e}}_x - U_0\left(\frac{y}{L}\right)\hat{\mathbf{e}}_y \qquad \text{where } U_0 \text{ and } L \text{ are positive constants}$$

Determine the acceleration of a fluid particle along the line $x = 0$.



SOLUTION:

The acceleration of a fluid particle is,

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + u_x \frac{\partial \mathbf{u}}{\partial x} + u_y \frac{\partial \mathbf{u}}{\partial y}, \tag{1}$$

where,

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{0} \quad \text{(steady flow)}$$

$$u_x \frac{\partial \mathbf{u}}{\partial x} = \left[U_0\left(\frac{x}{L}\right)\right]\left[U_0\left(\frac{1}{L}\right)\hat{\mathbf{e}}_x\right] = U_0^2\left(\frac{x}{L^2}\right)\hat{\mathbf{e}}_x$$

$$u_y \frac{\partial \mathbf{u}}{\partial y} = \left[-U_0\left(\frac{y}{L}\right)\right]\left[-U_0\left(\frac{1}{L}\right)\hat{\mathbf{e}}_y\right] = U_0^2\left(\frac{y}{L^2}\right)\hat{\mathbf{e}}_y$$

$$\therefore \mathbf{a} = U_0^2\left(\frac{x}{L^2}\right)\hat{\mathbf{e}}_x + U_0^2\left(\frac{y}{L^2}\right)\hat{\mathbf{e}}_y \tag{2}$$

Along the line $x = 0$,

$$\boxed{\therefore \mathbf{a}(0, y) = U_0^2\left(\frac{y}{L^2}\right)\hat{\mathbf{e}}_y} \tag{3}$$

The market price, P (in dollars), of used cars of a certain model is found to be:

$$P = \$1000 + \left(\$0.02 / \text{mile}\right)x - \left(\$2 / \text{day}\right)t$$

where x is the distance in miles west of Detroit, MI and t is the time in days. If a car of this model is driven from Detroit at t=0 towards the west at a rate of 400 miles per day, determine:

a. whether the value of the car is increasing or decreasing, and
b. how much of this change is due to depreciation and how much is due to moving into a better market.

SOLUTION:

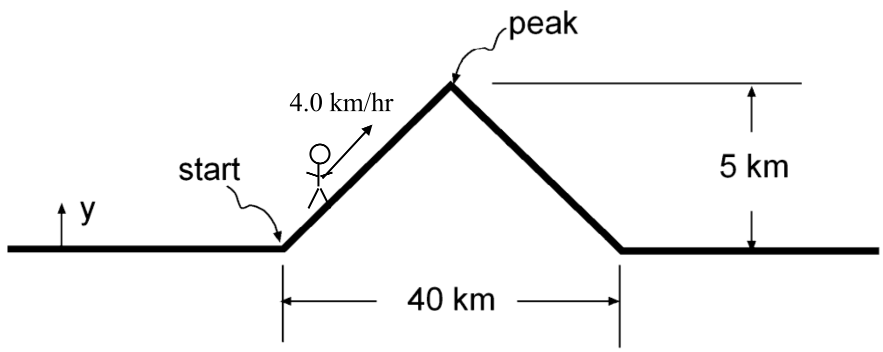
To determine if the value of the car is decreasing, take the Lagrangian derivative of the market price.

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} = \left(-\$2/\text{day}\right) + \underbrace{\left(400 \text{ miles/day}\right)\left(\$0.02/\text{mile}\right)}_{=\$8/\text{day}} \quad \text{(where } u \text{ is the speed of the car)} \tag{1}$$

$$\boxed{\therefore \frac{DP}{Dt} = \$6/\text{day} \quad \text{Hence, the value of the car is increasing.}}$$

The car depreciates at a rate of -\$2/day (this is $\partial P/\partial t$). The change in the car's value increases at a rate of \$8/day due to moving into a different market (this is $u\partial P/\partial x$).

You are climbing up the side of Triangle Mountain, so named because the sides are relatively straight, making the cross-section of the mountain look like a triangle. The mountain is 5 km high and has a base of 40 km, as shown.



You are worried about how hot you will get on your trip and the rate at which the temperature will change with time. The temperature decreases with altitude at a rate of 5 °C/km. Also, the temperature changes in time as the sun heats the ground. The temperature (in °C) as you climb the mountain is given by:

$$T(y,t) = 25\ ^\circ\text{C} - \left(0.005\ ^\circ\text{C/m}\right)y + \left(5\ ^\circ\text{C}\right)\sin\left[\frac{2\pi\left(t-t_o\right)}{24\text{ hrs}}\right]$$

where t is measured in hours from midnight, $t_o = 9$ hrs, and y is the altitude measured in meters from the base of the mountain.

You start ascending the mountain at 6:00 A.M. and travel at a speed of 4.0 km/hr up the mountain side.

a. Derive an expression for the time derivative of temperature you experience as you climb up the mountain.
b. Calculate the rate of change in temperature at the moment you reach the mountain peak (in °C/hr).

SOLUTION:

Write the Lagrangian time derivative, keeping in mind that the two variables of interest are t and y:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_y\frac{\partial T}{\partial y} \tag{1}$$

where

$$\frac{\partial T}{\partial t} = 5\ ^\circ\text{C}\cdot\frac{2\pi}{24\text{ hrs}}\cos\left[\frac{2\pi\left(t-t_0\right)}{24\text{ hrs}}\right], \tag{2}$$

$$u_y = \left(4.0\ \text{km/hr}\right)\left[\frac{5\text{ km}}{\sqrt{\left(20\text{ km}\right)^2+\left(5\text{ km}\right)^2}}\right] = 0.97\ \text{km/hr}, \tag{3}$$

$$\frac{\partial T}{\partial y} = -0.005\ ^\circ\text{C/m}. \tag{4}$$

Substitute and simplify.

$$\boxed{\frac{DT}{Dt} = \frac{5\pi\ ^\circ\text{C}}{12\text{ hrs}}\cos\left[\frac{2\pi\left(t-t_0\right)}{24\text{ hrs}}\right] - \left(4.85\ ^\circ\text{C/hr}\right)} \tag{5}$$

You reach the peak in 5.15 hrs (= 5 km / (0.97 km/hr)), which means you reach the peak at 11:09 A.M (t = 11.15 hrs). Evaluating Eqn. (5) using t_0 = 9 hrs gives, at the moment you reach the peak:

$$\boxed{DT/Dt = -3.74\ ^\circ\text{C/hr}} \tag{6}$$

4.2. The Reynolds Transport Theorem

Recall that we can look at the behavior of small pieces of fluid in two ways: the Eulerian perspective or the Lagrangian perspective. Often we're interested in the behavior of an entire system of fluid (many pieces of fluid) rather than just an individual piece. How do we analyze this situation? We can use Eulerian and Lagrangian approaches for analyzing a macroscopic amount of fluid but we need to first develop an important tool called the Reynolds Transport Theorem.

Why do we want to do this? It turns out that the behavior of fluids (most substances in fact) can be described in terms of a few fundamental laws. These laws include:

- Conservation of Mass,
- Newton's 2nd Law,
- the angular momentum principle,
- the First Law of Thermodynamics, and
- the Second Law of Thermodynamics.

These laws are typically easiest to apply to a particular system of fluid particles (Lagrangian perspective). However, the Lagrangian forms of the laws are typically difficult to use in practical applications since we can't easily keep track of many individual bits of fluid. It's much easier to apply the laws to a particular volume in space instead (referred to as a control volume, an Eulerian perspective). For example, tracking the behavior of individual bits of gas flowing through a rocket nozzle would be difficult. It's much easier to just look at the behavior of the gas flowing into, out of, and within the volume enclosed by the rocket nozzle. The Reynolds Transport Theorem is a tool that will allow us to convert from a system point of view (Lagrangian) to a control volume point of view (Eulerian).

Let's consider a system of fluid particles that is coincident (occupying the same region in space) as our control volume (CV) at some time, t (Figure 4.3). At some later time, $t + \delta t$, the system may have moved relative

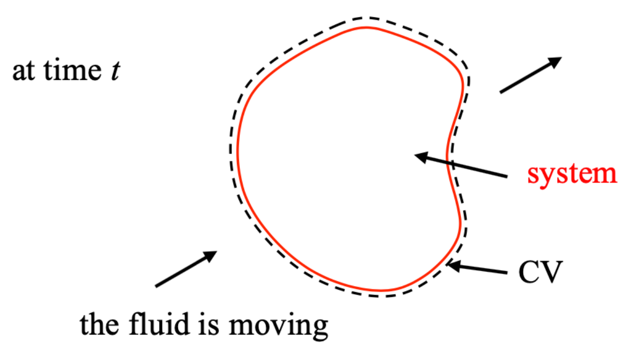


Figure 4.3. A sketch of a system and control volume of fluid that are coincident at time t.

to the CV (Figure 4.4).

Let B be some transportable property (i.e., some property that can be transported from one location to another, e.g., mass, momentum, energy) and β be the corresponding amount of B per unit mass (Figure 4.5), i.e.,

$$B_{\text{sys}} = \int_{V_{\text{sys}}} \beta \rho dV, \tag{4.14}$$

$$B_{CV} = \int_{CV} \beta \rho dV, \tag{4.15}$$

where B_{sys} and B_{CV} refer to the total amount of B in the system and control volume, respectively. Note that at time t, the total amounts of B in the system and control volume are equal since the system and CV

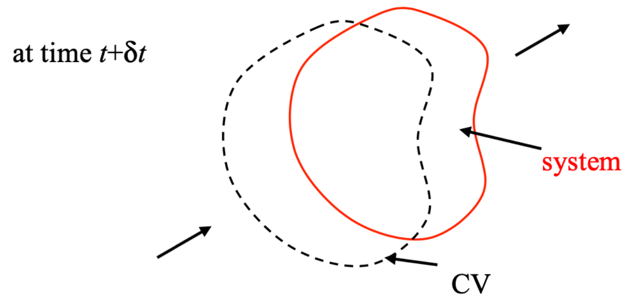


Figure 4.4. A sketch of a system and control volume of fluid at time $t + \delta t$.

are coincident,

$$B_{\text{sys}}(t) = B_{CV}(t). \tag{4.16}$$

However, at time, $t + \delta t$, the system and CV no longer occupy the same region in space so that, in general,

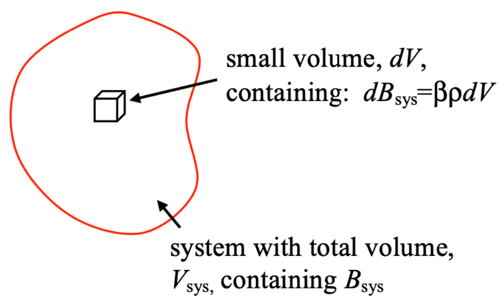


Figure 4.5. A sketch of a system of fluid illustrating a small amount the transportable quantity B, i.e., dB.

$B_{\text{sys}}(t+\delta t) \neq B_{CV}(t+\delta t)$. Note that B may be changing with time so that, in general, $B_{\text{sys}}(t+\delta t) \neq B_{\text{sys}}(t)$. In Figure 4.6, B_{out} is the amount of B that has left the CV and B_{in} is the amount of B that has entered the CV.

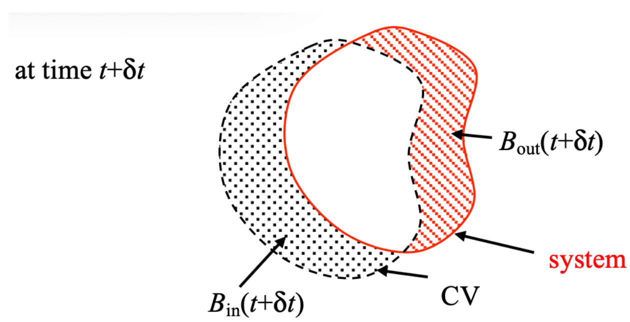


Figure 4.6. A sketch of a system and control volume showing B_{out} and B_{in} at $t + \delta t$.

Utilizing the figure shown above, we see that,

$$B_{CV}(t + \delta t) = B_{\text{sys}}(t + \delta t) - B_{\text{out}}(t + \delta t) + B_{\text{in}}(t + \delta t). \tag{4.17}$$

Subtracting $B_{\text{sys}}(t)$ from both sides and dividing through by δt gives,

$$\frac{B_{CV}(t+\delta t)-B_{\text{sys}}(t)}{\delta t}=\frac{B_{\text{sys}}(t+\delta t)-B_{\text{sys}}(t)}{\delta t}-\frac{B_{\text{out}}(t+\delta t)}{\delta t}+\frac{B_{\text{in}}(t+\delta t)}{\delta t}. \tag{4.18}$$

Now let's substitute $B_{CV}(t)=B_{\text{sys}}(t)$ on the left hand side of the equation, subtract $B_{\text{out}}(t)/\delta t$ and $B_{\text{in}}(t)/\delta t$ on the right-hand side (note that $B_{\text{out}}(t)=B_{\text{in}}(t)=0$), and then take the limit of the entire equation as $\delta t\to 0$,

$$\lim_{\delta t\to 0}\frac{B_{CV}(t+\delta t)-B_{\text{sys}}(t)}{\delta t}=\lim_{\delta t\to 0}\frac{B_{\text{sys}}(t+\delta t)-B_{\text{sys}}(t)}{\delta t} \tag{4.19}$$

$$-\lim_{\delta t\to 0}\frac{B_{\text{out}}(t+\delta t)-B_{\text{out}}(t)}{\delta t}+\lim_{\delta t\to 0}\frac{B_{\text{in}}(t+\delta t)-B_{\text{in}}(t)}{\delta t}, \tag{4.20}$$

$$\frac{dB_{CV}}{dt}=\frac{DB_{\text{sys}}}{Dt}-\frac{dB_{\text{out}}}{dt}+\frac{dB_{\text{in}}}{dt}. \tag{4.21}$$

Note that the D/Dt notation has been used to signify that the first term on the right-hand side of Eq. (4.21) represents the time rate of change as we follow a particular system of fluid (Lagrangian perspective). Rearranging the equation and substituting in for B_{CV} and B_{sys} using Eqs. (4.15) and (4.14),

$$\frac{D}{Dt}\left(\int_{V_{\text{sys}}}\beta\rho dV\right)=\frac{d}{dt}\left(\int_{V_{CV}}\beta\rho dV\right)+\frac{d(B_{\text{out}}-B_{\text{in}})}{dt}. \tag{4.22}$$

The last term on the right-hand side of Eq. (4.22) represents the net rate at which B is leaving the control volume through the control surface (CS). Let's examine this term more closely by zooming in on a small piece of the control surface and observing how much B leaves through this surface in time δt (Figure 4.7).

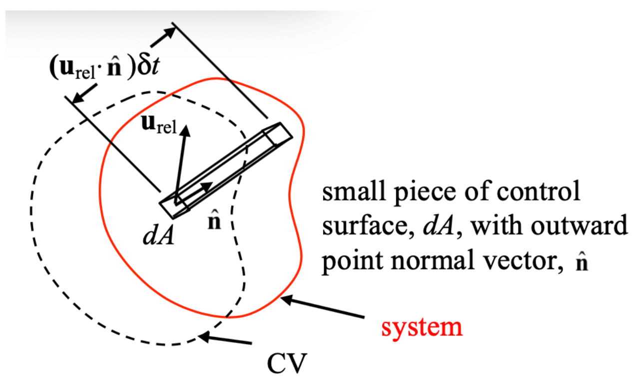


Figure 4.7. A sketch of a system and control volume of fluid at time $t+\delta t$. The parallelpiped is the small volume of fluid that has left the control volume through the area $d\mathbf{A}$ over time δt. Note that in the figure, the area $d\mathbf{A}=dA\hat{\mathbf{n}}$.

The component of the fluid velocity out of the control volume through surface, $d\mathbf{A}$, is given by,

$$u_{\text{out through }d\mathbf{A}}=\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}. \tag{4.23}$$

where $\mathbf{u}_{\text{rel}}=\mathbf{u}_{\text{sys}}-\mathbf{u}_{CS}$ is the velocity of the fluid relative to the control surface. The volume of fluid leaving through surface $d\mathbf{A}$ in time δt is then,

$$\delta V=(\mathbf{u}_{\text{rel}}\cdot d\mathbf{A})\,\delta t. \tag{4.24}$$

Thus, the volumetric flow rate, dQ, (volume per unit time) through surface $d\mathbf{A}$ is given by,

$$dQ=\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}. \tag{4.25}$$

Note that the mass flow rate, $d\dot{m}$ through the small area is,

$$d\dot{m}=\rho dQ=\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}. \tag{4.26}$$

Now use Eq. (4.25) to write the net rate at which B leaves the control volume,

$$\frac{d\left(B_{\text{out}}-B_{\text{in}}\right)}{dt}=\int_{CS}\beta\rho dQ=\int_{CS}\beta\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right). \tag{4.27}$$

Combining Eq. (4.27) with Eq. (4.22) gives,

$$\boxed{\underbrace{\frac{D}{Dt}\left(\int_{V_{\text{sys}}}\beta\rho dV\right)}_{\substack{\text{rate of increase of}\\ B\text{ within the system}}}=\underbrace{\frac{d}{dt}\left(\int_{V_{CV}}\beta\rho dV\right)}_{\substack{\text{rate of increase of}\\ B\text{ within the CV}}}+\underbrace{\int_{CS}\beta\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)}_{\substack{\text{net rate at which }B\text{ leaves}\\ \text{the CV through the CS}}}}. \tag{4.28}$$

This is the Reynolds Transport Theorem!

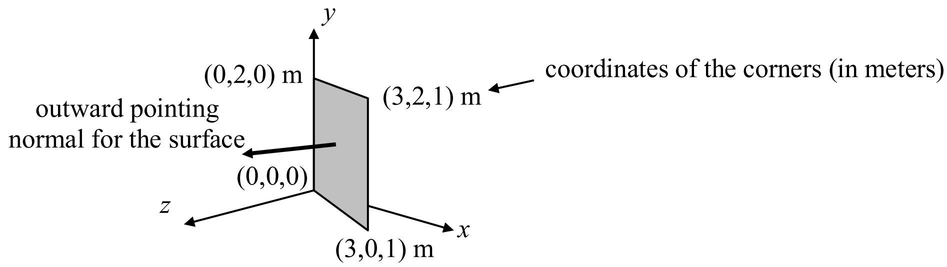
Consider a fluid flowing with the following velocity profile:

$$\mathbf{u}_{\text{fluid}} = Ax\hat{\mathbf{i}} + By\hat{\mathbf{j}} + Cxy\hat{\mathbf{k}}$$

where $A = 1\ \text{s}^{-1}$, $B = 2\ \text{s}^{-1}$, and $C = 3\ (\text{m}\cdot\text{s})^{-1}$.

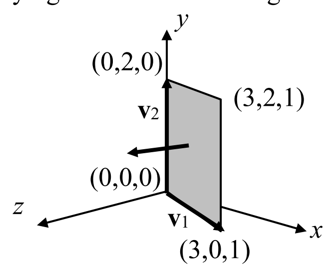
a. Determine the magnitude of the volumetric flow rate through the fixed surface shown in the figure below.

b. What is the magnitude of the average velocity through the surface?



SOLUTION:

First determine the unit normal vector for the surface. The unit normal vector to the surface may be found by normalizing the cross product of the two vectors lying on the surface's edges emanating from the origin.



$$\mathbf{v}_1 = 3\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 1\hat{\mathbf{k}} \tag{1}$$

$$\mathbf{v}_2 = 0\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \tag{2}$$

$$\hat{\mathbf{n}} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\left|\mathbf{v}_1 \times \mathbf{v}_2\right|} = \frac{(3,0,1)\times(0,2,0)}{\left|(3,0,1)\times(0,2,0)\right|} = \frac{(-2,0,6)}{\left|(-2,0,6)\right|} \tag{3}$$

$$\therefore \hat{\mathbf{n}} = \tfrac{1}{\sqrt{10}}(-1,0,3) \tag{4}$$

Now find the volumetric flow rate, Q, through the surface. Note that the fluid velocity varies in all three directions so the flow rate must be found through integration. Also note that $\mathbf{u}_{\text{rel}} = \mathbf{u}_{\text{fluid}} - \mathbf{u}_{\text{surface}} = \mathbf{u}_{\text{fluid}}$ since $\mathbf{u}_{\text{surface}} = \mathbf{0}$.

$$Q = \int_A \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \int_{s=0}^{s=\sqrt{10}\text{ m}} \int_{y=0}^{y=2\text{ m}} \mathbf{u}_{\text{rel}} \cdot \underbrace{\hat{\mathbf{n}}\,dyds}_{=d\mathbf{A}} \quad \text{where } s \text{ is the distance in the } \mathbf{v}_1 \text{ direction} \tag{5}$$

$$Q = \int_{s=0}^{s=\sqrt{10}\text{ m}} \int_{y=0}^{y=2\text{ m}} \left(Ax\hat{\mathbf{i}} + By\hat{\mathbf{j}} + Cxy\hat{\mathbf{k}}\right)\cdot \tfrac{1}{\sqrt{10}}\left(-\hat{\mathbf{i}} + 3\hat{\mathbf{k}}\right)dyds = \tfrac{1}{\sqrt{10}} \int_{s=0}^{s=\sqrt{10}\text{ m}} \int_{y=0}^{y=2\text{ m}} \left(-Ax + 3Cxy\right)dyds \tag{6}$$

$$Q = \tfrac{1}{\sqrt{10}} \int_{s=0}^{s=\sqrt{10}\text{ m}} \left[-Axy + \tfrac{3}{2}Cxy^2\right]_0^{2\text{ m}} ds = \tfrac{1}{\sqrt{10}} \int_{s=0}^{s=\sqrt{10}\text{ m}} \left[(-2\text{ m})Ax + \left(6\text{ m}^2\right)Cx\right]ds \tag{7}$$

Now relate s to x.

$$s = \tfrac{\sqrt{10}}{3}x \Rightarrow ds = \tfrac{\sqrt{10}}{3}dx \tag{8}$$

Substitute Eqn. (8) into Eqn. (7) and solve.

$$Q = \tfrac{1}{\sqrt{10}} \int_{x=0}^{x=3\text{ m}} \left[(-2\text{ m})Ax + \left(6\text{ m}^2\right)Cx\right]\tfrac{\sqrt{10}}{3}dx = \tfrac{1}{3}\left[(-1\text{ m})Ax^2 + \left(3\text{ m}^2\right)Cx^2\right]_0^{3\text{ m}} \tag{9}$$

$$Q = \left(-3\text{ m}^3\right)A + \left(9\text{ m}^4\right)C \tag{10}$$

$$\boxed{\therefore Q = 24\ \text{m}^3/\text{s}} \tag{11}$$

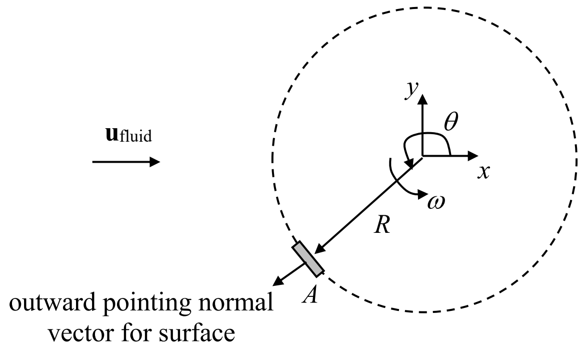
The average velocity through the surface is:

$$\bar{u} = \frac{Q}{A} = \frac{24\ \text{m}^3/\text{s}}{(2\text{ m})\left(\sqrt{10}\text{ m}\right)} \Rightarrow \boxed{\therefore \bar{u} = \tfrac{12}{\sqrt{10}}\ \text{m/s}} \tag{12}$$

Consider a fluid velocity field given by:

$$\mathbf{u}_{\text{fluid}} = C\hat{\mathbf{i}}$$

where C is a constant. Determine the volumetric flow rate through a surface of area A that is attached to an arm of radius R that rotates with constant angular speed ω in the x-y plane as shown in the figure below. Express your result in terms of (a subset of) C, R, ω, A, and θ.



SOLUTION:

The unit normal vector for the surface is:

$$\hat{\mathbf{n}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}} \tag{1}$$

The flow rate through the surface, Q, is:

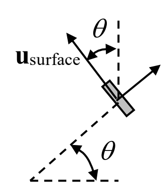
$$Q = \int_A \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \mathbf{u}_{\text{rel}} \cdot A\hat{\mathbf{n}} \tag{2}$$

Note that since the fluid velocity is uniform, integration over the area isn't required.

The velocity of the fluid relative to the surface, $\mathbf{u}_{\text{rel}}$, is:

$$\mathbf{u}_{\text{rel}} = \mathbf{u}_{\text{fluid}} - \mathbf{u}_{\text{surface}} = C\hat{\mathbf{i}} - \omega R\left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}\right) \tag{3}$$

$$\therefore \mathbf{u}_{\text{rel}} = \left(C + \omega R\sin\theta\right)\hat{\mathbf{i}} - \omega R\cos\theta\hat{\mathbf{j}} \tag{4}$$



Substitute Eqns. (4) and (1) into Eqn. (2) and simplify.

$$Q = \left[\left(C + \omega R\sin\theta\right)\hat{\mathbf{i}} - \omega R\cos\theta\hat{\mathbf{j}}\right] \cdot A\left[\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right] \tag{5}$$

$$Q = A\left[\left(C + \omega R\sin\theta\right)\cos\theta - \omega R\sin\theta\cos\theta\right] \tag{6}$$

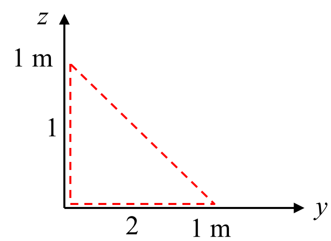
$$\boxed{\therefore Q = AC\cos\theta}$$

The velocity field in the region shown is given by:

$$\mathbf{u} = az\hat{\mathbf{j}} + b\hat{\mathbf{k}}$$

where $a = 10\ \text{s}^{-1}$ and $b = 5$ m/s. For depth w into the page, an element of area 1 may be represented by $wdz(-\mathbf{j})$ and an element of area 2 by $wdy(-\mathbf{k})$. (Note that both are drawn *outward* from the control volume, hence the minus signs.)

1. Find an expression for $\mathbf{u}\cdot d\mathbf{A}_1$.
2. Evaluate $\int_{A_1} \mathbf{u}\cdot d\mathbf{A}_1$.
3. Find an expression for $\mathbf{u}\cdot d\mathbf{A}_2$.
4. Find an expression for $\mathbf{u}\ (\mathbf{u}\cdot d\mathbf{A}_2)$.
5. Evaluate $\int_{A_2} \mathbf{u}\left(\mathbf{u}\cdot d\mathbf{A}_2\right)$.



SOLUTION:

$$\mathbf{u}\cdot d\mathbf{A}_1 = \left(az\hat{\mathbf{j}} + b\hat{\mathbf{k}}\right)\cdot wdz\left(-\hat{\mathbf{j}}\right) = -awzdz \tag{1}$$

$$\int_{A_1} \mathbf{u}\cdot d\mathbf{A}_1 = \int_{z=0}^{z=1\text{ m}} -awzdz = -aw\tfrac{1}{2}z^2\Big|_0^{1\text{ m}} = -\left(10\ \text{s}^{-1}\right)\left(w\right)\left(\tfrac{1}{2}\ \text{m}^2\right) = \left(-5\frac{\text{m}^2}{\text{s}}\right)w \tag{2}$$

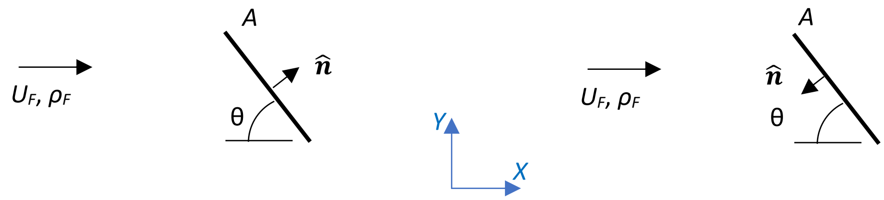
$$\mathbf{u}\cdot d\mathbf{A}_2 = \left(az\hat{\mathbf{j}} + b\hat{\mathbf{k}}\right)\cdot wdy\left(-\hat{\mathbf{k}}\right) = -bwdy \tag{3}$$

$$\mathbf{u}\left(\mathbf{u}\cdot d\mathbf{A}_2\right) = \left(az\hat{\mathbf{j}} + b\hat{\mathbf{k}}\right)\left(-bwdy\right) \tag{4}$$

$$\int_{A_2} \mathbf{u}\left(\mathbf{u}\cdot d\mathbf{A}_2\right) = \int_{y=0}^{y=1\text{ m}} \left(az\hat{\mathbf{j}} + b\hat{\mathbf{k}}\right)\left(-bwdy\right) = \hat{\mathbf{j}}bwaz\,y\Big|_0^{1\text{ m}} - \hat{\mathbf{k}}b^2w\,y\Big|_0^{1\text{ m}} = \left(-25\tfrac{\text{m}^3}{\text{s}^2}\right)w\hat{\mathbf{k}} \tag{5}$$

Note that $z = 0$ along $d\mathbf{A}_2$.

Determine the mass flow rate through the fixed surfaces shown below. Assume the velocities are uniform over the area.



SOLUTION:
The mass flow rate is,

$$\dot{m} = \rho \boldsymbol{u}_{rel} \cdot \boldsymbol{A}, \quad (1)$$

where,

$$\boldsymbol{u}_{rel} = \boldsymbol{u}_{fluid} - \boldsymbol{u}_{CS} = (U_F \hat{\boldsymbol{\imath}}) - (\boldsymbol{0}) \text{ (in both cases)}, \quad (2)$$

$$|\boldsymbol{A}| = A \text{ (in both cases)}, \quad (3)$$

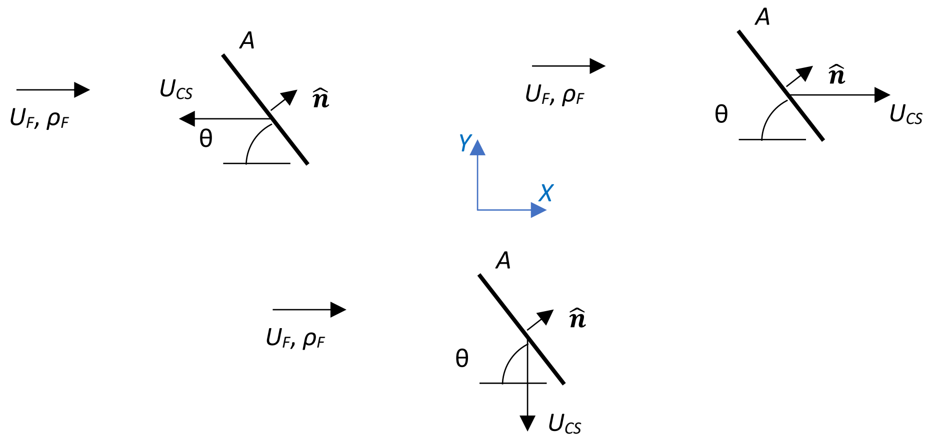
$$\hat{\boldsymbol{n}}_1 = \sin\theta\, \hat{\boldsymbol{\imath}} + \cos\theta\, \hat{\boldsymbol{\jmath}} \text{ (left)}, \quad (4)$$

$$\hat{\boldsymbol{n}}_2 = -\sin\theta\, \hat{\boldsymbol{\imath}} - \cos\theta\, \hat{\boldsymbol{\jmath}} \text{ (right)}. \quad (5)$$

$$\dot{m}_1 = \rho_F U_F A \sin\theta \text{ (left)}. \quad (6)$$

$$\dot{m}_2 = -\rho_F U_F A \sin\theta \text{ (right)}. \quad (7)$$

Determine the mass flow rate through the moving surfaces shown below. Assume the velocities are uniform over the area.



SOLUTION:

The mass flow rate is,

$$\dot{m} = \rho \boldsymbol{u}_{rel} \cdot \boldsymbol{A}, \tag{1}$$

where,

$$\boldsymbol{u}_{rel,1} = \boldsymbol{u}_{fluid} - \boldsymbol{u}_{CS} = (U_F \hat{\boldsymbol{\imath}}) - (-U_{CS} \hat{\boldsymbol{\imath}}) \quad \text{(upper left)}, \tag{2}$$

$$\boldsymbol{u}_{rel,2} = \boldsymbol{u}_{fluid} - \boldsymbol{u}_{CS} = (U_F \hat{\boldsymbol{\imath}}) - (U_{CS} \hat{\boldsymbol{\imath}}) \quad \text{(upper right)}, \tag{3}$$

$$\boldsymbol{u}_{rel,3} = \boldsymbol{u}_{fluid} - \boldsymbol{u}_{CS} = (U_F \hat{\boldsymbol{\imath}}) - (-U_{CS} \hat{\boldsymbol{\jmath}}) \quad \text{(bottom)}. \tag{4}$$

and, in all cases,

$$\boldsymbol{A} = \sin\theta\, \hat{\boldsymbol{\imath}} + \cos\theta\, \hat{\boldsymbol{\jmath}}, \tag{5}$$

Substituting and performing the dot products,

$$\dot{m}_1 = \rho_F A (U_F + U_{CS}) \sin\theta, \tag{6}$$

$$\dot{m}_2 = \rho_F A (U_F - U_{CS}) \sin\theta, \tag{7}$$

$$\dot{m}_3 = \rho_F A (U_F \sin\theta + U_{CS} \cos\theta). \tag{8}$$

4.3. Conservation of Mass

In words and in mathematical terms, COM for a system is:

$$\text{The mass of a system remains constant.} \implies \frac{D}{Dt}\underbrace{\int_{V_{\text{sys}}} \rho dV}_{\text{system mass}} = 0. \tag{4.29}$$

where D/Dt is the Lagrangian derivative (implying that we're using the rate of change as we follow the system), V is the volume, and ρ is the density. Using the Reynolds Transport Theorem, Eq. (4.29) can be converted into an expression for a control volume,

$$\frac{D}{Dt}\int_{V_{\text{sys}}} \rho dV = \frac{d}{dt}\int_{CV} \rho dV + \int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0, \tag{4.30}$$

$$\boxed{\underbrace{\frac{d}{dt}\int_{CV} \rho dV}_{\substack{\text{rate of increase of} \\ \text{mass inside the CV}}} + \underbrace{\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0}_{\substack{\text{net rate at which mass leaves} \\ \text{the CV through the CS}}}} \tag{4.31}$$

This equation is Conservation of Mass for a control volume!

Notes:

(1) Carefully draw your control volume. Don't neglect to draw a control volume or draw a control volume and then use a different one.
(2) Make sure you understand what each term in Conservation of Mass represents.
(3) Carefully evaluate the dot product in the mass flux term.
(4) You must integrate the terms in conservation of mass when the density or velocity are not uniform.
(5) Note that the first term in Eq. (4.31) is the rate of increase of mass in the CV, which can be re-written as,

$$\frac{d}{dt}\int_{CV} \rho dV = \frac{d}{dt}(M_{CV}) = \frac{dM_{CV}}{dt}, \tag{4.32}$$

where M_{CV} is the mass inside the control volume. Similarly, the second term in Eq. (4.31), which is the net rate at which mass leaves the CV through the CS, may be written as,

$$\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \sum_{\text{all outlets}} \dot{m} - \sum_{\text{all inlets}} \dot{m}, \tag{4.33}$$

where $\dot{m}$ is a mass flow rate. Combining Eqs. (4.31) - (4.33) gives:

$$\boxed{\frac{dM_{CV}}{dt} = \sum_{\text{all inlets}} \dot{m} - \sum_{\text{all outlets}} \dot{m}}. \tag{4.34}$$

Note that the net mass flow rate term has been moved to the right side of the equation.
(6) The term "steady state" means that none of the properties within the control volume change with time, i.e.,

$$\frac{d}{dt}(\cdots)_{CV} = 0, \tag{4.35}$$

where the dots can be any property.
(7) The term "steady flow" means that the mass flow rates into and out of the control volume do not change with time, i.e.,

$$\dot{m}_i = constant_i, \tag{4.36}$$

where the subscript "i" refers to each inlet/outlet.
(8) It's possible to have a steady state, but not a steady flow. For example, consider a simple system consisting of a rigid tank with a single inlet (i) and a single outlet (o) with the mass flow rates $\dot{m}_i = \dot{m}_o = A\sin(\omega t)$ (not steady flow). From Conservation of Mass, the mass inside the tank would not vary with time and, thus, the control volume would be at steady state.

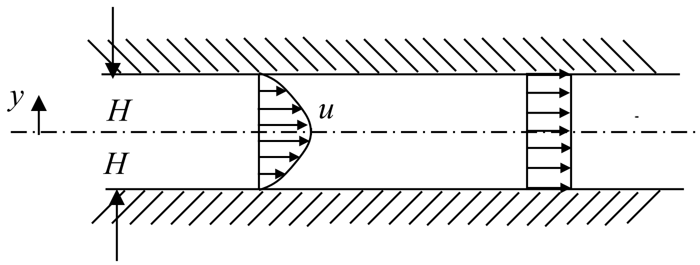
(9) It's possible to have steady flow, but not steady state. For example, consider the same rigid tank system, but this time $\dot{m}_i > \dot{m}_o$, where each mass flow rate is a constant (steady flow). From Conservation of Mass, the mass within the control volume increases with time and, thus, would not be at steady state.

Let's consider a few examples to see how COM is applied.

Consider the flow of an incompressible fluid between two parallel plates separated by a distance $2H$. If the velocity profile is given by:

$$u = u_c\left(1 - \frac{y^2}{H^2}\right)$$

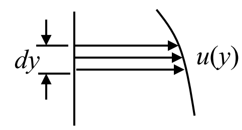
where u_c is the centerline velocity, determine the average velocity of the flow, $\bar{u}$. Assume the depth into the page is w.



SOLUTION:

The volumetric flow rate using the average velocity profile must give the same volumetric flow rate using the real velocity profile.

$$Q_{\text{real}} = \int_A \mathbf{u} \cdot d\mathbf{A} = \int_A dQ = \int_{y=-H}^{y=+H} \underbrace{u_c\left(1 - {y^2}\big/{H^2}\right)\overbrace{dyw}^{=dA}}_{=dQ} = \tfrac{4}{3}u_c wH \tag{1}$$



The velocity, $u(y)$, is nearly constant over the small distance dy so we can write the volumetric flowrate over this small area as $dQ = u(y)dA = u(y)(dyw)$.

$$Q_{\text{average}} = \int_A \mathbf{u} \cdot d\mathbf{A} = \bar{u}\left(2Hw\right) \quad \text{(There is no need to integrate since the velocity is uniform over } y\text{.)} \tag{2}$$

$$Q_{\text{real}} = Q_{\text{average}} \implies \tfrac{4}{3}u_c wH = \bar{u}\left(2Hw\right) \tag{3}$$

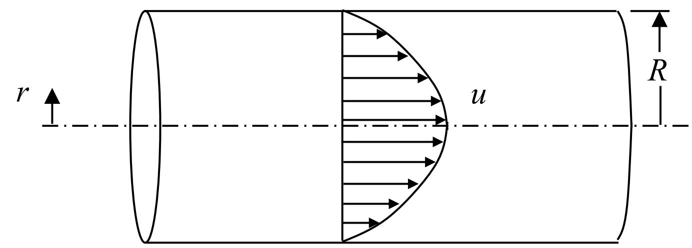
$$\boxed{\therefore \bar{u} = \tfrac{2}{3}u_c} \tag{4}$$

Video solution: https://www.youtube.com/watch?v=nZ1c0eldyc8

An incompressible flow in a pipe has a velocity profile given by:

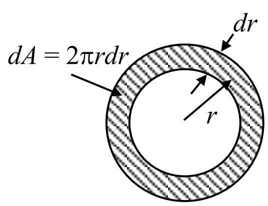
$$u(r) = u_c\left(1 - \frac{r^2}{R^2}\right)$$

where u_c is the centerline velocity and R is the pipe radius. Determine the average velocity in the pipe.



SOLUTION:

The volumetric flow rate using the average velocity profile must give the same volumetric flow rate using the real velocity profile.



The velocity, $u(r)$, is nearly constant over the small annulus with radius dr so we can write the volumetric flow rate over this small area as $dQ = u(r)dA = u(r)(2\pi r dr)$.

$$Q_{\text{real}} = \int_A dQ = \int_A \boldsymbol{u} \cdot d\boldsymbol{A} = \int_{r=0}^{r=R} \underbrace{u_c\left(1 - \frac{r^2}{R^2}\right)\underbrace{(2\pi r dr)}_{=dA}}_{=dQ} = \frac{1}{2}\pi u_c R^2 \tag{1}$$

$$Q_{\text{average}} = \int_A \mathbf{u} \cdot d\mathbf{A} = \bar{u}\left(\pi R^2\right) \quad \text{(There is no need to integrate since the velocity is uniform over } r\text{.)} \tag{2}$$

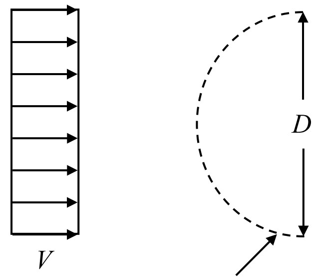
$$Q_{\text{real}} = Q_{\text{average}} \implies \tfrac{1}{2}\pi u_c R^2 = \bar{u}\left(\pi R^2\right) \tag{3}$$

$$\boxed{\therefore \bar{u} = \tfrac{1}{2}u_c} \tag{4}$$

$$Q_{\text{real}} = \int_A \mathbf{u} \cdot d\mathbf{A} = \int_A dQ = \int_{y=-H}^{y=+H} \underbrace{u_c\left(1 - r^2/R^2\right)\overbrace{(2\pi r dr)}^{dA}}_{=dQ} = \tfrac{1}{2}\pi u_c R^2$$

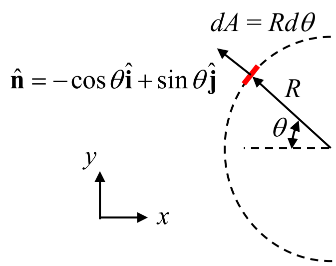
Video solution: https://www.youtube.com/watch?v=Y1fRo2Avm_I

Calculate the mass flux through the control surface shown below. Assume a unit depth into the page.



SOLUTION:

The mass flux through the surface is given by:

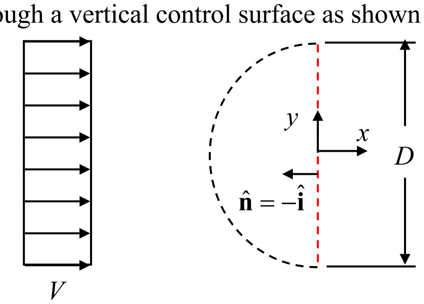


$$\dot{m} = \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \int_{\theta=-\pi/2}^{\theta=\pi/2} \rho \underbrace{V\hat{\mathbf{i}}}_{=\mathbf{u}_{rel}} \cdot \underbrace{\left(-\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}_{=\hat{\mathbf{n}}} \underbrace{\left(Rd\theta\right)}_{=dA}$$

$$= -\rho VR \int_{\theta=-\pi/2}^{\theta=\pi/2} \cos\theta d\theta = -\rho VR \sin\theta\Big|_{-\pi/2}^{\pi/2} = -2\rho VR$$

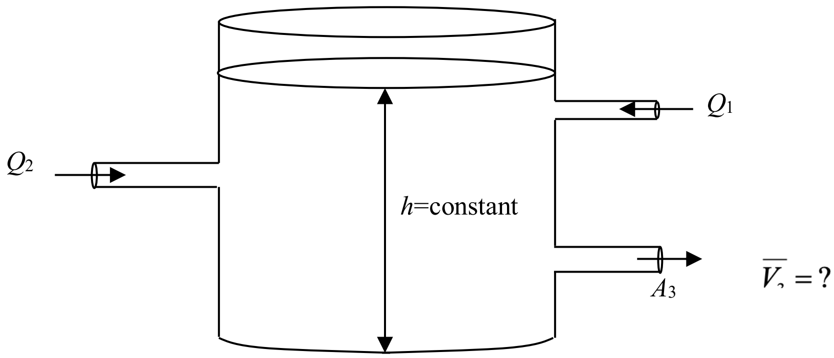
$$\boxed{\therefore \dot{m} = -\rho VD}$$

We could have also figured out the mass flux by noticing that any mass passing through the curved control surface must also pass through a vertical control surface as shown below.



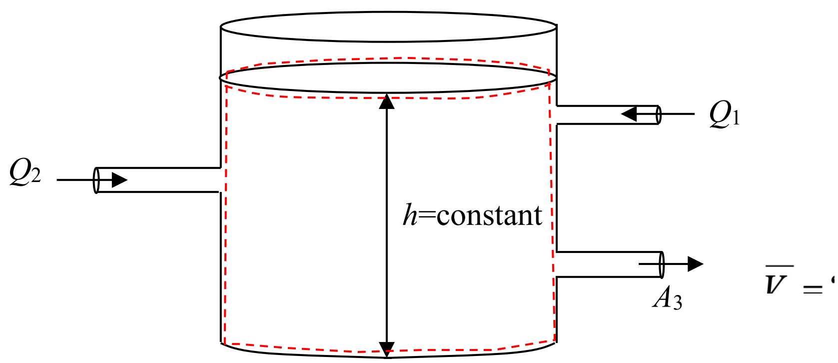
$$\dot{m} = \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \int_{y=-R}^{y=R} \rho \underbrace{V\hat{\mathbf{i}}}_{=\mathbf{u}_{rel}} \cdot \underbrace{\left(-\hat{\mathbf{i}}\right)}_{=\hat{\mathbf{n}}} \underbrace{\left(dy\right)}_{=dA} = -2\rho VR = -\rho VD \quad \text{(The same answer as before!)}$$

Water enters a cylindrical tank through two pipes at volumetric flow rates of Q_1 and Q_2. If the level in the tank remains constant, calculate the average velocity of the flow leaving the tank through a pipe with an area, A_3.



SOLUTION:

Apply conservation of mass to the fixed control volume shown below.



$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow, the mass in the control volume isn't changing with time)}$$

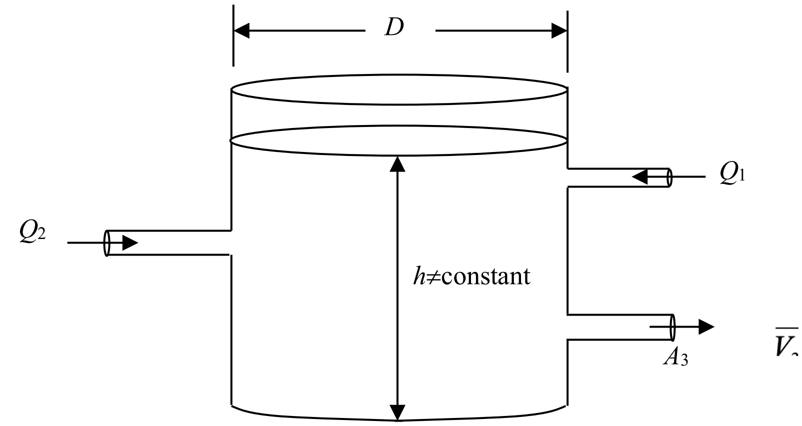
$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho Q_2 - \rho Q_1 + \rho \bar{V}_3 A_3$$

Substitute and re-arrange.

$$-\rho Q_2 - \rho Q_1 + \rho \bar{V}_3 A_3 = 0$$

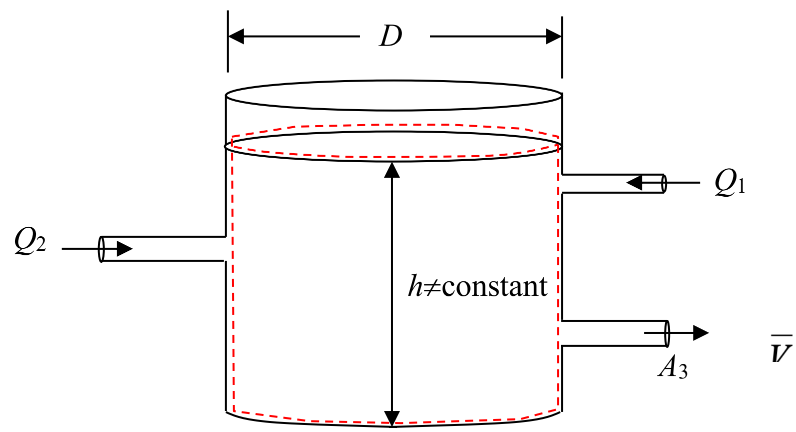
$$\boxed{\therefore \bar{V}_3 = \frac{Q_1 + Q_2}{A_3}} \tag{2}$$

Water enters a cylindrical tank with diameter, D, through two pipes at volumetric flow rates of Q_1 and Q_2 and leaves through a pipe with area, A_3, with an average velocity, $\overline{V}_3$. The level in the tank, h, does not remain constant. Determine the time rate of change of the level in the tank.



SOLUTION:

Apply conservation of mass to a control volume that deforms to follow the free surface of the liquid.



$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = \frac{d}{dt}\underbrace{\left(\rho h \frac{\pi D^2}{4}\right)}_{=M_{CV}} = \rho \frac{dh}{dt}\frac{\pi D^2}{4}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho Q_2 - \rho Q_1 + \rho \overline{V}_3 A_3$$

Substitute and re-arrange.

$$\rho \frac{dh}{dt}\frac{\pi D^2}{4} - \rho Q_2 - \rho Q_1 + \rho \overline{V}_3 A_3 = 0$$

$$\boxed{\frac{dh}{dt} = \frac{Q_2 + Q_1 - \overline{V}_3 A_3}{\pi D^2 / 4}} \tag{2}$$

We could have also chosen a fixed control volume through which the free surface moves. Using this time of control volume, conservation of mass is given by:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(the mass of fluid in the fixed control volume remains constant)}$$

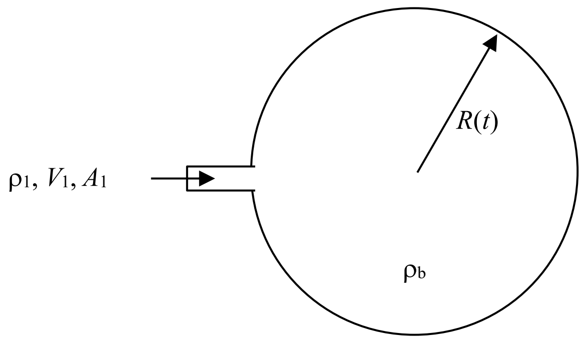
$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho Q_2 - \rho Q_1 + \rho \bar{V}_3 A_3 + \underbrace{\rho \frac{\overbrace{dh}^{=V_{\text{top}}}}{dt}\frac{\pi D^2}{4}}_{\dot{m}_{\text{top}}}$$

Substitute and re-arrange.

$$\frac{dh}{dt} = \frac{\rho Q_2 + \rho Q_1 - \bar{V}_3 A_3}{\pi D^2 / 4}$$

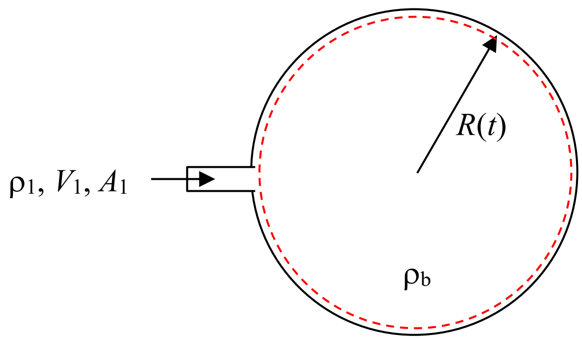
$$\boxed{\frac{dh}{dt} = \frac{Q_2 + Q_1 - \bar{V}_3 A_3}{\pi D^2 / 4}} \quad \text{(This is the same answer as before!)} \tag{4}$$

A spherical balloon is filled through an area, A_1, with air flowing at velocity, V_1, and constant density, ρ_1. The radius of the balloon, $R(t)$, can change with time, t. The average density within the balloon at any given time is $\rho_b(t)$. Determine the relationship between the rate of change of the density within the balloon and the rest of the variables.



SOLUTION:

Apply conservation of mass to a control volume that deforms to follow the interior surface of the balloon.



$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{d}{dt}\underbrace{\left(\rho_b \frac{4}{3}\pi R^3\right)}_{=M_{\text{CV}}} = \frac{4}{3}\pi R^3 \frac{d\rho_b}{dt} + 4\pi\rho_b R^2 \frac{dR}{dt}$$

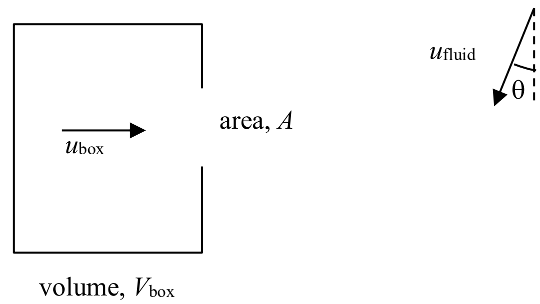
$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho_1 V_1 A_1$$

Substitute and simplify.

$$\frac{4}{3}\pi R^3 \frac{d\rho_b}{dt} + 4\pi\rho_b R^2 \frac{dR}{dt} - \rho_1 V_1 A_1 = 0$$

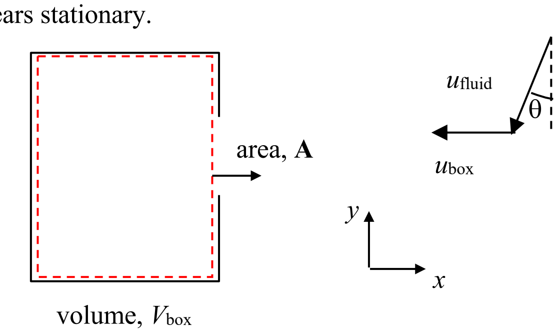
$$\boxed{\frac{d\rho_b}{dt} = \frac{\rho_1 V_1 A_1 - 4\pi\rho_b R^2 \frac{dR}{dt}}{\frac{4}{3}\pi R^3}} \tag{2}$$

A box with a hole of area, A, moves to the right with velocity, u_{box}, through an incompressible fluid as shown in the figure. If the fluid has a velocity of u_{fluid} which is at an angle, θ, to the vertical, determine how long it will take to fill the box with fluid. Assume the box volume is V_{box} and that it is initially empty.



SOLUTION:

Apply conservation of mass to a control volume fixed to the interior of the box. Change our frame of reference so the box appears stationary.



$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{d\left(\rho V_{\text{CV}}\right)}{dt} = \rho \frac{dV_{\text{CV}}}{dt}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \rho\left[\underbrace{\left(-u_{\text{box}}\hat{\mathbf{i}} - u_{\text{fluid}}\sin\theta\hat{\mathbf{i}} - u_{\text{fluid}}\cos\theta\hat{\mathbf{j}}\right)}_{=\mathbf{u}_{\text{rel}}} \cdot \underbrace{A\hat{\mathbf{i}}}_{=d\mathbf{A}}\right] = -\rho\left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A$$

Substitute and simplify.

$$\rho\frac{dV_{\text{CV}}}{dt} - \rho\left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A = 0$$

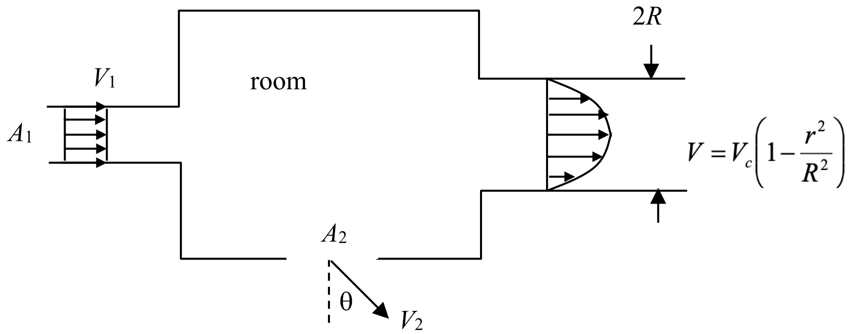
$$\frac{dV_{\text{CV}}}{dt} = \left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A$$

$$\int_{V_{\text{CV}}=0}^{V_{\text{CV}}=V_{\text{CV}}} dV_{\text{CV}} = \left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A\int_{t=0}^{t=T} dt$$ (Note that u_{box}, u_{fluid}, θ, and A don't change with time.)

$$V_{\text{CV}} = \left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)AT$$

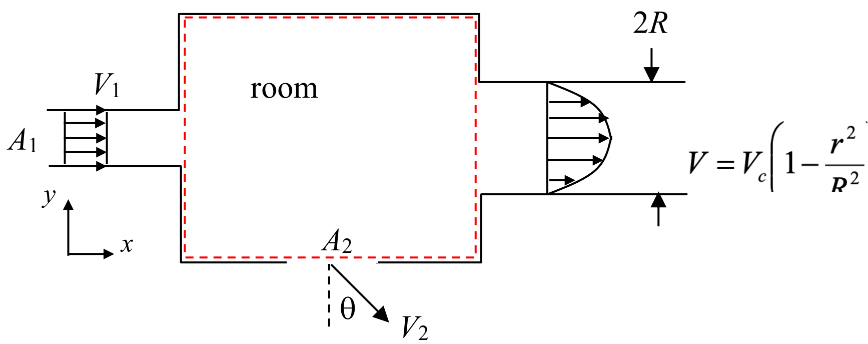
$$\boxed{\therefore T = \frac{V_{\text{CV}}}{\left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A}} \tag{2}$$

Determine the rate at which fluid mass collects inside the room shown below in terms of ρ, V_1, A_1, V_2, A_2, V_c, R, and θ. Assume the fluid moving through the system is incompressible.



SOLUTION:

Apply conservation of mass to a control volume fixed to the interior of the room.



$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{dM_{\text{CV}}}{dt}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \rho\left(V_1\hat{\mathbf{i}} \cdot -A_1\hat{\mathbf{i}}\right) + \rho\left[\left(V_2 \sin\theta\hat{\mathbf{i}} - \cos\theta\hat{\mathbf{j}}\right) \cdot -A_2\hat{\mathbf{j}}\right] + \rho\int_{r=0}^{r=R} V\hat{\mathbf{i}} \cdot dA\hat{\mathbf{i}}$$

$$= -\rho V_1 A_1 + \rho V_2 A_2 \cos\theta + \rho\int_{r=0}^{r=R} V_c\left(1 - r^2/R^2\right) 2\pi r dr$$

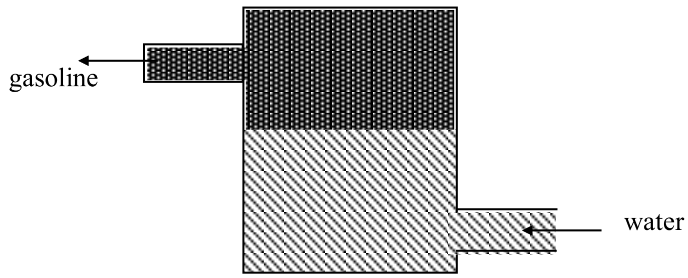
$$= -\rho V_1 A_1 + \rho V_2 A_2 \cos\theta + \tfrac{\pi}{2}\rho V_c R^2$$

Substitute and simplify.

$$\frac{dM_{\text{CV}}}{dt} - \rho V_1 A_1 + \rho V_2 A_2 \cos\theta + \tfrac{\pi}{2}\rho V_c R^2 = 0$$

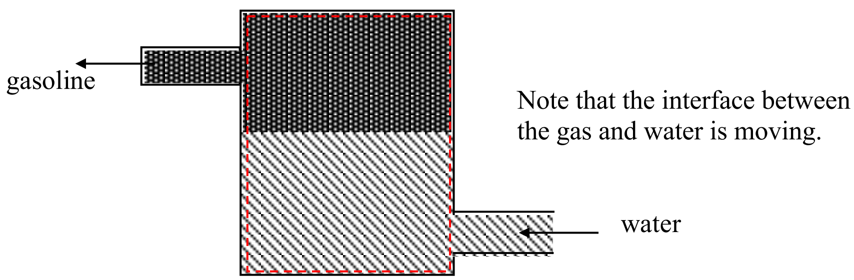
$$\boxed{\therefore \frac{dM_{\text{CV}}}{dt} = \rho V_1 A_1 - \rho V_2 A_2 \cos\theta - \tfrac{\pi}{2}\rho V_c R^2} \tag{2}$$

Water enters a rigid, sealed, cylindrical tank at a steady rate of 100 L/hr and forces gasoline (with a specific gravity of 0.68) out as is indicated in the drawing. The tank has a total volume of 1000 L. What is the time rate of change of the mass of gasoline contained in the tank?



SOLUTION:

Apply conservation of mass to the control volume shown below.



$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV + \int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV = \frac{d}{dt}\left(M_{\mathrm{gas}} + \underbrace{M_{\mathrm{H20}}}_{=\rho_{\mathrm{H20}}V_{\mathrm{H20}}}\right) = \frac{dM_{\mathrm{gas}}}{dt} + \rho_{\mathrm{H20}}\frac{dV_{\mathrm{H20}}}{dt} \quad \text{(Gas and water are incompressible.)}$$

$$\int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = \rho_{\mathrm{gas}} Q_{\mathrm{gas}} - \rho_{\mathrm{H20}} Q_{\mathrm{H20}}$$

Substitute and simplify.

$$\frac{dM_{\mathrm{gas}}}{dt} + \rho_{\mathrm{H20}}\frac{dV_{\mathrm{H20}}}{dt} + \rho_{\mathrm{gas}} Q_{\mathrm{gas}} - \rho_{\mathrm{H20}} Q_{\mathrm{H20}} = 0$$

$$\frac{dM_{\mathrm{gas}}}{dt} = \rho_{\mathrm{H20}}\left(Q_{\mathrm{H20}} - \frac{dV_{\mathrm{H20}}}{dt}\right) - \rho_{\mathrm{gas}} Q_{\mathrm{gas}} \tag{2}$$

Note that the time rate of change of the water volume, dV_{H20}/dt, is equal to the water's volumetric flow rate, Q_{H20}. Furthermore, since both liquids are incompressible and the total tank volume remains constant, $Q_{\mathrm{gas}} = Q_{\mathrm{H20}}$. Utilizing these facts to simply Eqn. (2) gives:

$$\boxed{\frac{dM_{\mathrm{gas}}}{dt} = -\rho_{\mathrm{gas}} Q_{\mathrm{H20}} = -SG_{\mathrm{gas}}\rho_{\mathrm{H20}} Q_{\mathrm{H20}}} \tag{3}$$

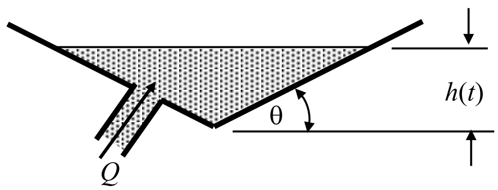
Using the given parameters:

SG_{gas} = 0.68
ρ_{H20} = 1000 kg/m^3
Q_{H20} = 100 L/hr = 0.1 m^3/hr
$\Rightarrow$ $\boxed{dM_{\mathrm{gas}}/dt = \text{-68 kg/hr = 0.019 kg/s}}$

The (symmetric) V-shaped container shown in the figure has width, b, into the page and is filled from the inlet pipe at volume flow rate, Q. Derive expressions for:

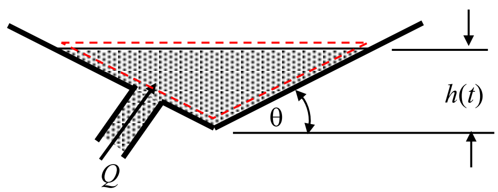
a. the rate of change of the surface height, dh/dt

b. the time required for the surface to rise from h_1 to h_2.



SOLUTION:

Apply conservation of mass to the deformable control volume shown in the figure below.



$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{d}{dt}\left(\rho 2 \cdot \tfrac{1}{2} h \frac{h}{\tan\theta} \cdot b\right) = \frac{\rho b}{\tan\theta}\frac{d}{dt}\left(h^2\right) = \frac{2\rho b h}{\tan\theta}\frac{dh}{dt}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho Q$$

Substitute and simplify.

$$\frac{2\rho b h}{\tan\theta}\frac{dh}{dt} - \rho Q = 0$$

$$\boxed{\frac{dh}{dt} = \frac{\tan\theta}{2hb} Q} \tag{1}$$

Solve the differential equation to determine the time required for a specified change in the liquid level.

$$\frac{dh}{dt} = \frac{\tan\theta}{2hb} Q$$

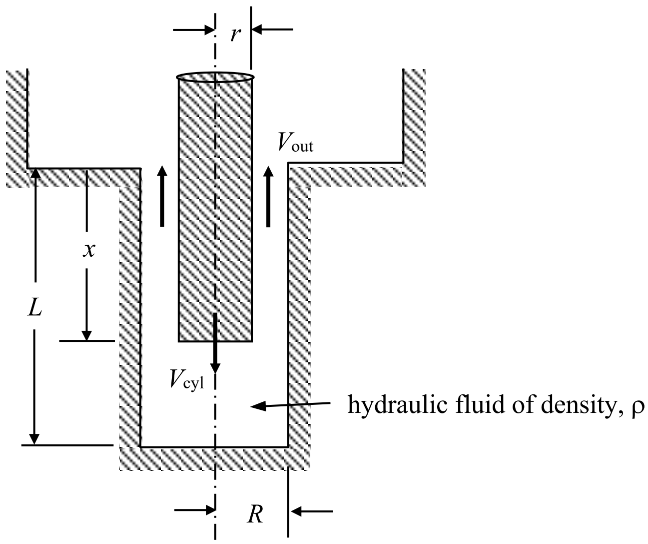
$$\int_{h=h_1}^{h=h_2} h dh = \frac{\tan\theta}{2b} Q \int_{t=t_1}^{t=t_2} dt$$

$$\tfrac{1}{2}\left(h_2^2 - h_1^2\right) = \frac{\tan\theta}{2b} Q\left(t_2 - t_1\right)$$

$$\boxed{t_2 - t_1 = \frac{b\left(h_2^2 - h_1^2\right)}{Q\tan\theta}} \tag{2}$$

The motion of a hydraulic cylinder is cushioned at the end of its stroke by a piston that enters a hole as shown. The cavity and cylinder are filled with hydraulic fluid of uniform density, ρ.

a. Obtain an expression for the average velocity, V_{out}, at which hydraulic fluid escapes from the cylindrical hole assuming that the cylinder moves at a constant velocity, V_{cyl}.

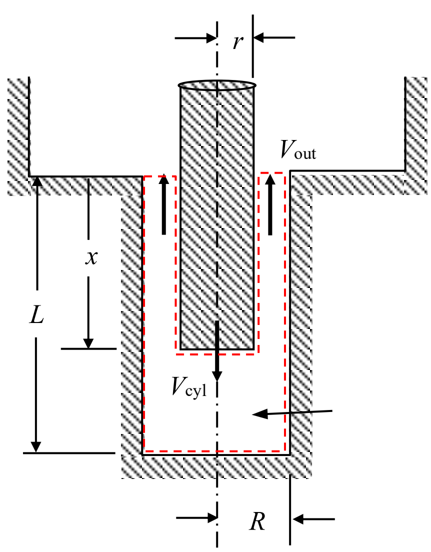


b. Determine the velocity, V_{out}, with relative uncertainty, for the following conditions.

$\rho = 900 \pm 5$ kg/m^3
$L = 100 \pm 0.1$ mm
$R = 15 \pm 0.1$ mm
$r = 10 \pm 0.1$ mm
$V_{cyl} = 100 \pm 1$ mm/s

SOLUTION:

Apply conservation of mass to a control volume that deforms to follow the piston as shown below.



$$\frac{d}{dt}\int_{\text{CV}}\rho dV+\int_{\text{CS}}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=0$$

where

$$\frac{d}{dt}\int_{\text{CV}}\rho dV=\rho\frac{d}{dt}\underbrace{\left[\pi R^2L-\pi r^2x\right]}_{=V_{\text{CV}}}=-\rho\pi r^2\frac{dx}{dt}=-\rho\pi r^2V_{\text{cyl}}$$

$$\int_{\text{CS}}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=\rho V_{\text{out}}\pi\left(R^2-r^2\right)$$

Substitute and solve for V_{out}.

$$-\rho\pi r^2V_{\text{cyl}}+\rho V_{\text{out}}\pi\left(R^2-r^2\right)=0$$

$$V_{\text{out}}=V_{\text{cyl}}\frac{r^2}{R^2-r^2}$$

$$\boxed{\therefore V_{\text{out}}=V_{\text{cyl}}\frac{1}{\left(R/r\right)^2-1}}$$

The total relative uncertainty in V_{out} is given by:

$$u_{V_{\text{out}}}=\left[u^2_{V_{\text{out}},V_{\text{cyl}}}+u^2_{V_{\text{out}},R}+u^2_{V_{\text{out}},r}\right]^{1/2}$$

where

$$u_{V_{\text{out}},V_{\text{cyl}}}=\frac{1}{V_{\text{out}}}\frac{\partial V_{\text{out}}}{\partial V_{\text{cyl}}}\delta V_{\text{cyl}}=\frac{\left(R/r\right)^2-1}{V_{\text{cyl}}}\frac{1}{\left(R/r\right)^2-1}\delta V_{\text{cyl}}=\frac{\delta V_{\text{cyl}}}{V_{\text{cyl}}}=u_{V_{\text{cyl}}}$$

$$u_{V_{\text{out}},R}=\frac{1}{V_{\text{out}}}\frac{\partial V_{\text{out}}}{\partial R}\delta R=\frac{\left(R/r\right)^2-1}{V_{\text{cyl}}}\left\{-\frac{2R/r^2\,V_{\text{cyl}}}{\left[\left(R/r\right)^2-1\right]^2}\right\}\delta R=-\frac{2R/r^2}{\left(R/r\right)^2-1}\delta R=\frac{-2}{1-\left(r/R\right)^2}\frac{\delta R}{R}=\frac{-2u_R}{1-\left(r/R\right)^2}$$

$$u_{V_{\text{out}},r}=\frac{1}{V_{\text{out}}}\frac{\partial V_{\text{out}}}{\partial r}\delta r=\frac{\left(R/r\right)^2-1}{V_{\text{cyl}}}\left\{-\frac{-2R^2/r^3\,V_{\text{cyl}}}{\left[\left(R/r\right)^2-1\right]^2}\right\}\delta r=\frac{2R^2/r^3}{\left(R/r\right)^2-1}\delta r=\frac{2}{1-\left(r/R\right)^2}\frac{\delta r}{r}=\frac{2u_r}{1-\left(r/R\right)^2}$$

Substitute and simplify.

$$u_{V_{\text{out}}}=\left[u^2_{V_{\text{cyl}}}+\frac{4u_R^2}{\left[1-\left(r/R\right)^2\right]^2}+\frac{4u_r^2}{\left[1-\left(r/R\right)^2\right]^2}\right]^{1/2}$$

Using the given data:

$V_{\text{out}} = 80 \text{ mm/s}$

$u_{V_{\text{cyl}}} = {}^{1 \text{ mm}}\!\big/_{100 \text{ mm}} = 1.0 * 10^{-2}$

$u_R = {}^{0.1 \text{ mm}}\!\big/_{15 \text{ mm}} = 6.7 * 10^{-3}$

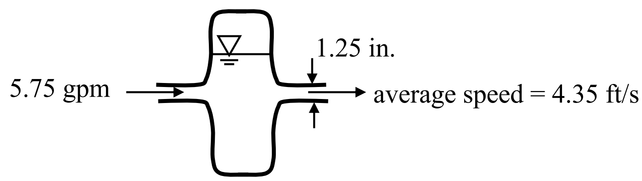
$u_r = {}^{0.1 \text{ mm}}\!\big/_{10 \text{ mm}} = 1.0 * 10^{-2}$

$r/R = {}^{10 \text{ mm}}\!\big/_{15 \text{ mm}} = 6.7 * 10^{-1}$

$\therefore u_{V_{\text{out}}} = 4.5 * 10^{-2} \Rightarrow \delta V_{\text{out}} = 3.4 \text{ mm/s}$

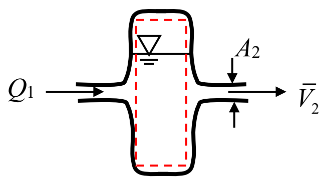
$\boxed{\therefore V_{\text{out}} = 80.0 \pm 3.6 \text{ mm/s}}$

A hydraulic accumulator is designed to reduce pressure pulsations in a machine tool hydraulic system. For the instant shown, determine the rate at which the accumulator gains or loses hydraulic oil (with a specific gravity of 0.88).



SOLUTION:

Apply conservation of mass to the control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{dM_{CV}}{dt} \tag{2}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho Q_1 + \rho \bar{V}_2 A_2 \tag{3}$$

Substitute and simplify.

$$\frac{dM_{CV}}{dt} - \rho Q_1 + \rho \bar{V}_2 A_2 = 0 \tag{4}$$

$$\boxed{\frac{dM_{CV}}{dt} = \rho\left(Q_1 - \bar{V}_2 A_2\right)} \tag{5}$$

Using the given data,

ρ = (0.88)(1.94 slug/ft^3) = 1.71 slug/ft^3

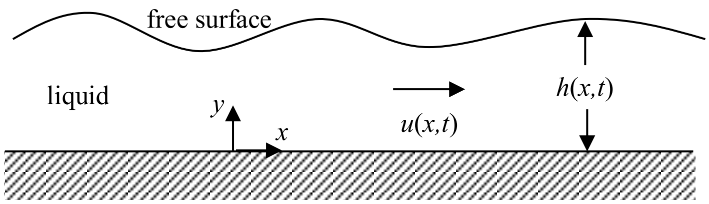
Q_1 = 5.75 gpm = 1.28*10^{-2} ft^3/s

$\bar{V}_2$ = 4.35 ft/s

A_2 = π(1.25 in.)^2/4 = 8.52*10^{-3} ft^2

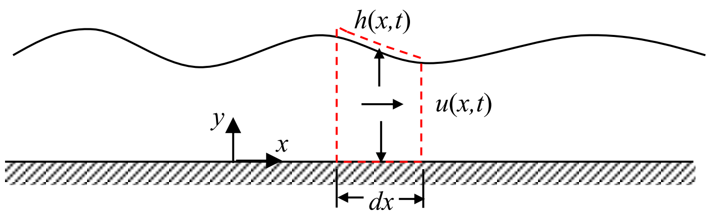
⇒ $\boxed{dM_{CV}/dt = -4.15*10^{-2}\text{ slug/s}}$ The accumulator is losing oil.

Construct from first principles an equation for the conservation of mass governing the planar flow (in the *xy* plane) of a compressible liquid lying on a flat horizontal plane. The depth, $h(x,t)$, is a function of position, x, and time, t. Assume that the velocity of the fluid in the positive x-direction, $u(x,t)$, is independent of y. Also assume that the wavelength of the wave is much greater than the wave amplitude so that the horizontal velocities are much greater than the vertical velocities.



SOLUTION:

Apply conservation of mass to the fixed control volume shown below. Assume a unit depth into the page.



Note that the mass flow rate at the center of the control volume is ρuh.

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV + \int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV = \frac{\partial}{\partial t}\underbrace{\left(\rho h dx\right)}_{=M_{\mathrm{CV}}} = \frac{\partial}{\partial t}\left(\rho h\right)dx$$

$$\int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = \underbrace{-\left[\left(\rho uh\right) + \frac{\partial}{\partial x}\left(\rho uh\right)\left(-\tfrac{1}{2}dx\right)\right]}_{=\dot{m}_{\mathrm{left}}} + \underbrace{\left[\left(\rho uh\right) + \frac{\partial}{\partial x}\left(\rho uh\right)\left(\tfrac{1}{2}dx\right)\right]}_{=\dot{m}_{\mathrm{right}}}$$

$$= \frac{\partial}{\partial x}\left(\rho uh\right)dx$$

Substitute and simplify.

$$\frac{\partial}{\partial t}\left(\rho h\right)dx + \frac{\partial}{\partial x}\left(\rho uh\right)dx = 0$$

$$\boxed{\frac{\partial}{\partial t}\left(\rho h\right) + \frac{\partial}{\partial x}\left(\rho uh\right) = 0} \tag{2}$$

If the fluid is incompressible, then Eqn. (2) simplifies to:

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(uh\right) = 0} \tag{3}$$

In order to avoid getting wet, is it better to walk or run in the rain? Assume that the rain falls at an angle θ from the vertical. Clearly state your assumptions and provide justification for your conclusion.

4.4. The Linear Momentum Equations (LMEs)

In this section we'll consider Newton's Second law applied to a control volume of fluid. Recall that linear momentum is a vector quantity, it has both magnitude and direction, and is given by mass*velocity. In words and in mathematical terms, Newton's Second Law for a system is:

The rate of change of a system's linear momentum is equal to the net force acting on the system.

$$\implies \underbrace{\frac{D}{Dt}\int_{V_{\text{sys}}} \mathbf{u}_{XYZ}\rho dV}_{\text{LM of system}} = \mathbf{F}_{\text{on sys}}. \tag{4.37}$$

where D/Dt is the Lagrangian derivative (implying that we're using the rate of change as we follow the system), V is the volume, and ρ is the density. The quantity $\mathbf{u}_{XYZ}$ represents the velocity of a small piece of fluid in the system with respect to an inertial (aka non-accelerating) coordinate system XYZ (Figure 4.8). Recall that Newton's Second law holds strictly for inertial coordinate systems. Note that a coordinate system moving at a constant velocity in a straight line is non-accelerating and, thus, is inertial.

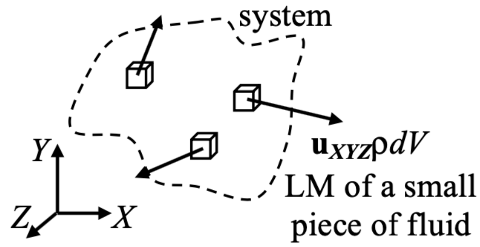


Figure 4.8. A system of fluid illustrating the linear momentum associated with a small piece of fluid.

The term, $\mathbf{F}_{\text{on sys}}$, represents the net forces acting on the system. These forces can be of two different types. The first are body forces, $\mathbf{F}_{\text{body}}$, and the second are surface forces, $\mathbf{F}_{\text{surface}}$. Body forces are those forces that act on each piece of fluid in the system, including the interior system volume. Examples include gravitational and electromagnetic forces. Surface forces are those forces acting only at the surface of the system. Examples of surface forces include pressure and shear forces. Expanding the force term,

$$\mathbf{F}_{\text{on sys}} = \mathbf{F}_{\text{body,on sys}} + \mathbf{F}_{\text{surface,on sys}}. \tag{4.38}$$

Using the Reynolds Transport Theorem to convert the left-hand side of Eq. (4.37) from a system point of view to an expression for a control volume gives,

$$\frac{D}{Dt}\int_{V_{\text{sys}}} \mathbf{u}_{XYZ}\rho dV = \frac{d}{dt}\int_{CV} \mathbf{u}_{XYZ}\rho dV + \int_{CS} \mathbf{u}_{XYZ}\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right). \tag{4.39}$$

Since the Reynolds Transport Theorem is applied to a coincident system and control volume, the forces acting on the system will also act on the control volume. Thus,

$$\boxed{\underbrace{\frac{d}{dt}\int_{CV} \mathbf{u}_{XYZ}\rho dV}_{\substack{\text{rate of increase}\\ \text{of LM in CV}}} + \underbrace{\int_{CS} \mathbf{u}_{XYZ}\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right)}_{\substack{\text{net rate at which LM}\\ \text{leaves the CV}}} = \underbrace{\mathbf{F}_{B,\text{on CV}}}_{\substack{\text{net body force}\\ \text{on the CV}}} + \underbrace{\mathbf{F}_{S,\text{on CV}}}_{\substack{\text{net surface force}\\ \text{on the CV}}}} \tag{4.40}$$

This is the Linear Momentum Equation for a control volume!

Notes:

(1) Recall that the LME is a vector expression. There are actually three equations built into Eq. (4.40). For example, in a rectangular coordinate system (Cartesian coordinates) we have,

$$\frac{d}{dt}\int_{CV} u_X \rho dV + \int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X} \tag{4.41}$$

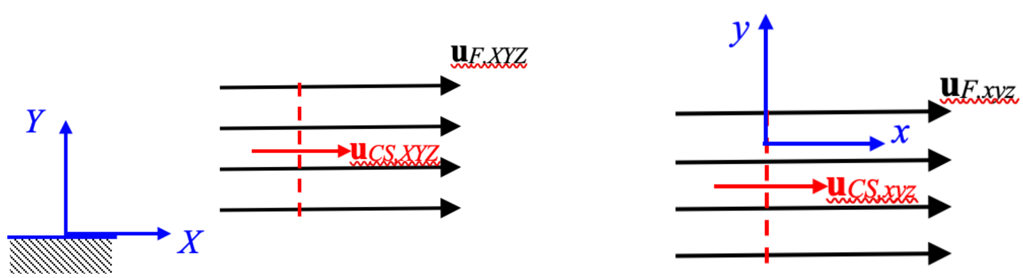
$$\frac{d}{dt}\int_{CV} u_Y \rho dV + \int_{CS} u_Y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,Y} + F_{S,Y} \tag{4.42}$$

$$\frac{d}{dt}\int_{CV} u_Z \rho dV + \int_{CS} u_Z \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,Z} + F_{S,Z} \tag{4.43}$$

(2) When applying the Linear Momentum Equations, Conservation of Mass is often used too. This point is illustrated in the examples at the end of this section.

(3) Note that the velocity $\mathbf{u}_{XYZ}$ in the CV term in Eq. (4.40) is the velocity of fluid *within the CV* using inertial coordinate system XYZ. The velocity $\mathbf{u}_{XYZ}$ in the CS term of Eq. (4.40) is the velocity of fluid *as it crosses the CS* using inertial coordinate system XYZ. The subscript on the integral is important!

(4) It is important to distinguish between the two velocities $\mathbf{u}_{XYZ}$ and $\mathbf{u}_{\text{rel}}$ in the CS term in Eq. (4.40). The velocity $\mathbf{u}_{XYZ}$ represents the fluid velocity with respect to an inertial coordinate system XYZ, e.g., a coordinate system fixed in space or moving at a constant velocity in a straight line. The velocity $\mathbf{u}_{\text{rel}}$ is the velocity of the fluid as it crosses the control surface, e.g., $\mathbf{u}_{\text{rel}} = \mathbf{u}_{\text{fluid}} - \mathbf{u}_{CS}$. The velocity $\mathbf{u}_{XYZ}$ *must* be measured using the inertial coordinate system XYZ; however, the relative velocity $\mathbf{u}_{\text{rel}}$ can be measured using *any* coordinate system since it is a difference of two velocities.



(A) (B)

Figure 4.9. Sketches corresponding to the relative velocity example. (A) Using a coordinate system fixed to the ground. (B) Using a coordinate system fixed to the moving control surface.

To illustrate this point, consider a fluid flowing in a straight line with velocity $\mathbf{u}_{F,XYZ}$ using the fixed coordinate system XYZ shown in Figure 4.9. Also shown in the figure is a portion of a control surface, which moves at a speed $\mathbf{u}_{CS,XYZ}$ using the same fixed coordinate system. Now let's evaluate the linear momentum flow rate term in Eq. (4.40),

$$\int_{CS} \mathbf{u}_{XYZ} \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right). \tag{4.44}$$

The velocity $\mathbf{u}_{XYZ}$ is the velocity of the fluid at the control surface using our coordinate system. Hence,

$$\mathbf{u}_{XYZ} = \mathbf{u}_{F,XYZ}. \tag{4.45}$$

The velocity of the fluid at the control surface relative to the control surface is,

$$\mathbf{u}_{\text{rel}} = \mathbf{u}_{\text{fluid}} - \mathbf{u}_{CS} = \mathbf{u}_{F,XYZ} - \mathbf{u}_{CS,XYZ}, \tag{4.46}$$

where both the fluid and control surface velocities are measured using the XYZ coordinate system, for convenience. Substituting back into Eq. (4.44) gives,

$$\int_{CS} \mathbf{u}_{XYZ} \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \int_{CS} \mathbf{u}_{F,XYZ} \left[\rho(\mathbf{u}_{F,XYZ} - \mathbf{u}_{CS,XYZ}) \cdot d\mathbf{A}\right]. \tag{4.47}$$

Now let's re-evaluate the momentum flow rate term in Eq (4.40) using a coordinate system *that is fixed to the moving control surface*, which we'll call coordinate system xyz (Figure 4.9),

$$\int_{CS} \mathbf{u}_{xyz} \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right). \tag{4.48}$$

Note that this coordinate system is still inertial since it's moving in a straight line at a constant speed. Using this new coordinate system, the fluid velocity is,

$$\mathbf{u}_{F,xyz} = \mathbf{u}_{F,XYZ} - \mathbf{u}_{CS,XYZ}, \tag{4.49}$$

and the control surface velocity is,

$$\mathbf{u}_{CS,xyz} = \mathbf{u}_{CS,XYZ} - \mathbf{u}_{CS,XYZ} = \mathbf{0}. \tag{4.50}$$

The control surface doesn't appear to be moving using this coordinate system. The fluid velocity relative to the control surface velocity using this new coordinate system is,

$$\mathbf{u}_{\text{rel}} = \mathbf{u}_{\text{fluid}} - \mathbf{u}_{CS} = \mathbf{u}_{F,xyz} - \mathbf{u}_{CS,xyz} = (\mathbf{u}_{F,XYZ} - \mathbf{u}_{CS,XYZ}) - \mathbf{0}, \tag{4.51}$$

$$\mathbf{u}_{\text{rel}} = \mathbf{u}_{F,XYZ} - \mathbf{u}_{CS,XYZ}. \tag{4.52}$$

Substituting Eqs. (4.49) and (4.52) into Eq. (4.48) gives,

$$\int_{CS} \mathbf{u}_{xyz} \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \int_{CS} \mathbf{u}_{F,xyz} \left[\rho(\mathbf{u}_{F,XYZ} - \mathbf{u}_{CS,XYZ}) \cdot d\mathbf{A}\right]. \tag{4.53}$$

Note that the relative velocity $\mathbf{u}_{\text{rel}}$ is the same regardless of the coordinate system used (compare Eqs. (4.46) and (4.52)). However, the value for the fluid velocity at the control surface, $\mathbf{u}_{F,XYZ}$ or $\mathbf{u}_{F,xyz}$ *does* depend on the choice of coordinate system, i.e., $\mathbf{u}_{F,XYZ} \neq \mathbf{u}_{F,xyz}$. Furthermore, this fluid velocity at the control surface can be different than the relative velocity, in general, i.e., $\mathbf{u}_{XYZ} \neq \mathbf{u}_{\text{rel}}$. The only time the two will be the same is if the control surface velocity is zero in the chosen frame of reference. With this in mind, it's often most convenient to fix the coordinate system to the control surface. Several examples are provided in which problems are worked using a fixed coordinate system or one moving at a constant speed in a straight line. The same answer is obtained regardless of the choice of coordinate system, but it is almost always easiest to use a coordinate system fixed to the control surface.

(5) So far we've only discussed the LME for inertial (aka non-accelerating) coordinate systems. We can also apply the LME to non-inertial (aka accelerating) coordinate systems, but we need to add additional acceleration terms. We'll consider accelerating coordinate systems later in this chapter.

(6) In order to avoid mistakes when analyzing problems with the LME, be sure to do the following:
   (a) Unambiguously draw the control volume that the LME is being applied to.
   (b) Clearly indicate the coordinate system that is being used. Identify if the coordinate system is inertial or non-inertial.
   (c) Draw a free body diagram (FBD) of the relevant forces. Include both body and surface forces.
   (d) State any significant assumptions that may be used to simplify the LME, e.g., steady state, incompressible fluid, etc.
   (e) Write the significant components of the LME and then indicate the value of each term in the equation.
   (f) Carefully evaluate the velocity terms. This step is where many mistakes are made.
   (g) You must integrate the terms in the linear momentum equation when the density or velocity are not uniform.
   (h) Don't forget to include pressure and shear forces in the surface force term.
   (i) Don't forget to include the weight of everything inside the control volume when gravitational body forces are significant.

While these things may seem trivial and unnecessary, writing them down in a clear and concise manner can greatly reduce the likelihood of mistakes and better communicate your analysis to others.

(7) Note that the first term on the left-hand side of Eq. (4.40) is the rate of increase of linear momentum in the CV, which can be re-written as,

$$\frac{d}{dt}\int_{CV}\mathbf{u}_{XYZ}\rho dV = \frac{d}{dt}\left(\mathbf{L}_{CV,XYZ}\right) = \frac{d\mathbf{L}_{CV,XYZ}}{dt}, \tag{4.54}$$

where $\mathbf{L}_{CV,XYZ}$ is the linear momentum contained within the CV with respect to the inertial coordinate system XYZ. Similarly, the second term on the left-hand side of Eq. (4.40), which is the net rate at which linear momentum leaves the CV through the CS, may be written as,

$$\int_{CS}\mathbf{u}_{XYZ}\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \sum_{\text{all outlets}}\dot{\mathbf{L}}_{XYZ} - \sum_{\text{all inlets}}\dot{\mathbf{L}}_{XYZ}, \tag{4.55}$$

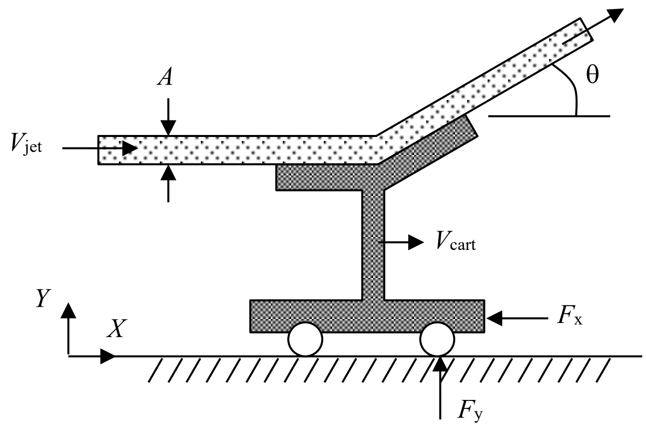
where $\dot{\mathbf{L}}_{XYZ}$ is the rate at which linear momentum, evaluated using the inertial coordinate system XYZ, passes through the control surface. Combining Eqs. (4.40), (4.54), and (4.55) gives,

$$\frac{d\mathbf{L}_{CV,XYZ}}{dt} = \sum_{\text{all inlets}}\dot{\mathbf{L}}_{XYZ} - \sum_{\text{all outlets}}\dot{\mathbf{L}}_{XYZ}. \tag{4.56}$$

Let's consider a few examples to see how LME using an inertial coordinate system is applied.

A jet of water is deflected by a vane mounted on a cart. The water jet has an area, A, everywhere and is turned an angle θ with respect to the horizontal. The pressure everywhere within the jet is atmospheric. The incoming jet velocity with respect to the ground (axes XY) is V_{jet}. The cart has mass M. Determine:

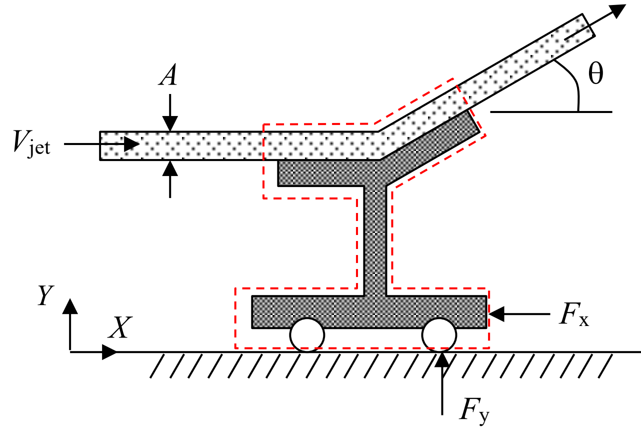
a. the force components, F_x and F_y, required to hold the cart stationary,
b. the horizontal force component, F_x, if the cart moves to the right at the constant velocity, V_{cart} ($V_{cart}<V_{jet}$)



SOLUTION:

Part (a):

Apply conservation of mass and the linear momentum equation to a control volume surrounding the cart. Use an inertial frame of reference fixed to the ground (XY).



First apply conservation of mass to the control volume to determine V_{out}.

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(the mass within the control volume doesn't change)}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \underbrace{\left(\rho \overbrace{V_{\text{jet}}\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right)}_{\text{left side}} + \underbrace{\left[\rho \overbrace{V_{\text{out}}\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{A\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$= -\rho V_{\text{jet}} A + \rho V_{\text{out}} A \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1}$$

$$= -\rho V_{\text{jet}} A + \rho V_{\text{out}} A$$

(Note that the jet area remains constant.)

Substitute and re-arrange.

$$-\rho V_{\text{jet}} A + \rho V_{\text{out}} A = 0$$

$$V_{\text{out}} = V_{\text{jet}} \tag{2}$$

Now apply the linear momentum equation in the X-direction:

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{CS}} u_X \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = F_{B,X} + F_{S,X} \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV = 0 \quad \text{(the momentum within the control volume doesn't change with time)}$$

$$\int_{\text{CS}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \underbrace{\overbrace{\left(V_{\text{jet}}\right)}^{=u_X}\left(\rho \overbrace{V_{\text{jet}}\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right)}_{\text{left side}} + \underbrace{\overbrace{\left(V_{\text{jet}}\cos\theta\right)}^{=u_X}\left[\rho \overbrace{V_{\text{jet}}\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{A\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$= -\rho V_{\text{jet}}^2 A + \rho V_{\text{jet}}^2 A \cos\theta \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1}$$

$$= \rho V_{\text{jet}}^2 A\left(\cos\theta - 1\right)$$

$$F_{B,X} = 0 \quad \text{(no body forces in the } x\text{-direction)}$$

$$F_{S,X} = -F_x \quad \text{(all of the pressure forces cancel out)}$$

Substitute and re-arrange.

$$\rho V_{\text{jet}}^2 A\left(\cos\theta - 1\right) = -F_x$$

$$\boxed{F_x = \rho V_{\text{jet}}^2 A\left(1 - \cos\theta\right)} \tag{4}$$

Now look at the Y-direction:

$$\frac{d}{dt}\int_{\text{CV}} u_Y \rho dV + \int_{\text{CS}} u_Y \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = F_{B,Y} + F_{S,Y} \tag{5}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_Y \rho dV = 0 \quad \text{(the momentum within the control volume doesn't change with time)}$$

$$\int_{\text{CS}} u_Y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \underbrace{\overbrace{\left(V_{\text{jet}} \sin\theta\right)}^{=u_Y}\left[\rho \overbrace{V_{\text{jet}}\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{A\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$= \rho V_{\text{jet}}^2 A \sin\theta \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1}$$

$$= \rho V_{\text{jet}}^2 A \sin\theta$$

$F_{B,Y} = -Mg$ (assume that the fluid weight in the CV is negligible compared to the cart weight)

$F_{S,Y} = F_y$ (all of the pressure forces cancel out)

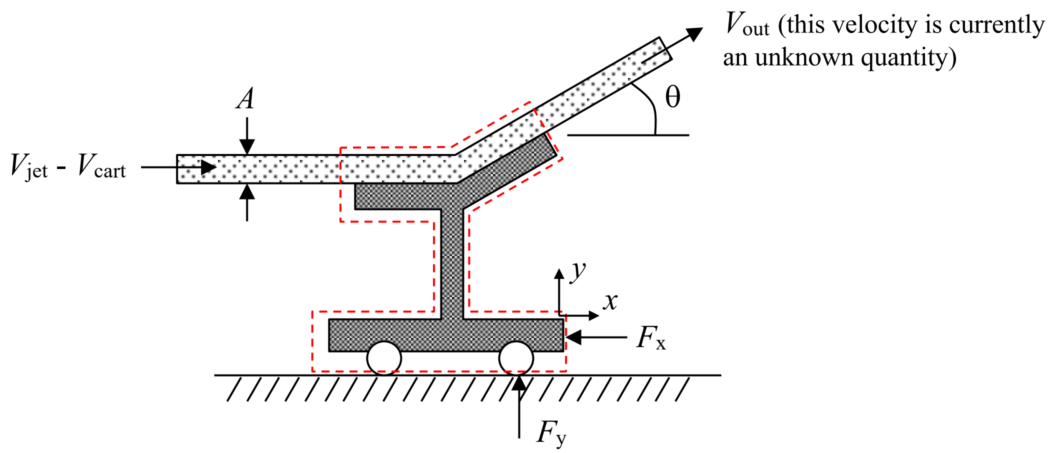
Substitute and re-arrange.

$$\rho V_{\text{jet}}^2 A \sin\theta = -Mg + F_y$$

$$\boxed{F_y = \rho V_{\text{jet}}^2 A \sin\theta + Mg} \tag{6}$$

Part (b):

Apply the linear momentum equation to a control volume surrounding the cart. Use a frame of reference fixed to the cart (*xy*). Note that this is an inertial frame of reference since the cart moves in a straight line at a constant speed. In addition, in this frame of reference, the cart appears stationary and the jet velocity at the left is equal to V_{jet}-V_{cart}.



First apply conservation of mass to the control volume to determine V_{out}

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0 \tag{7}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(the mass within the control volume doesn't change)}$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = \underbrace{\left[\rho\overbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right]}_{\text{left side}} + \underbrace{\left[\rho\overbrace{V_{\text{out}}\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{A\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$= -\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)A + \rho V_{\text{out}}A\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{=1}$$

$$= -\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)A + \rho V_{\text{out}}A$$

(Note that the jet area remains constant.)

Substitute and re-arrange.

$$-\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)A + \rho V_{\text{out}}A = 0$$

$$V_{\text{out}} = V_{\text{jet}} - V_{\text{cart}} \tag{8}$$

Now apply the linear momentum equation in the *x*-direction:

$$\frac{d}{dt}\int_{CV}u_x\rho dV + \int_{CS}u_x\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = F_{B,x} + F_{S,x} \tag{9}$$

where,

$$\frac{d}{dt}\int_{CV}u_x\rho dV = 0 \quad \text{(the momentum within the control volume doesn't change with time)}$$

$$\int_{CS} u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\underbrace{\overbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)}^{=u_x}\left[\overbrace{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right]}_{\text{left side}}+\underbrace{\overbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)\cos\theta}^{=u_X}\left[\overbrace{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{A\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$=-\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A+\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\cos\theta\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{=1}$$

$$=\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(\cos\theta-1\right)$$

$F_{B,x}=0$ (no body forces in the x-direction)

$F_{S,x}=-F_x$ (all of the pressure forces cancel out)

Substitute and re-arrange.

$$\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(\cos\theta-1\right)=-F_x$$

$$\boxed{F_x=\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(1-\cos\theta\right)} \tag{10}$$

Now solve the problem using an inertial frame of reference fixed to the ground (frame XY). From Eqn. (8) we know that using a frame of reference fixed to the cart, the jet velocity on the right-hand side is:

$$\mathbf{V}_{\substack{\text{out,}\\ \text{relative to cart}}}=\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right) \tag{11}$$

Hence, relative to the ground the jet velocity on the right-hand side is:

$$\mathbf{V}_{\substack{\text{out,}\\ \text{relative to}\\ \text{ground}}}=\mathbf{V}_{\substack{\text{out,}\\ \text{relative to}\\ \text{cart}}}+\mathbf{V}_{\text{cart}}=\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)+V_{\text{cart}}\hat{\mathbf{i}} \tag{12}$$

Now consider conservation of linear momentum in the X direction.

$$\frac{d}{dt}\int_{CV}u_X\rho dV+\int_{CS}u_X\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=F_{B,X}+F_{S,X} \tag{13}$$

where,

$\frac{d}{dt}\int_{CV}u_X\rho dV=0$ (the momentum within the control volume doesn't change with time)

$$\int_{CS} u_X\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\underbrace{\overbrace{V_{\text{jet}}}^{=u_X}\left[\overbrace{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right]}_{\text{left side}}+\underbrace{\overbrace{\left[\left(V_{\text{jet}}-V_{\text{cart}}\right)\cos\theta+V_{\text{cart}}\right]}^{=u_X}\left[\overbrace{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{A\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$=-\rho V_{\text{jet}}\left(V_{\text{jet}}-V_{\text{cart}}\right)A+\rho\left[\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\cos\theta+V_{\text{cart}}\left(V_{\text{jet}}-V_{\text{cart}}\right)\right]A\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{=1}$$

$$=\rho\left[-V_{\text{jet}}^2+V_{\text{jet}}V_{\text{cart}}+\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\cos\theta+V_{\text{cart}}V_{\text{jet}}-V_{\text{cart}}^2\right]A$$

$$=\rho\left[\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\cos\theta-\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\right]A$$

$$=\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\left(\cos\theta-1\right)A$$

$F_{B,X}=0$ (no body forces in the x-direction)

$F_{S,X}=-F_x$ (all of the pressure forces cancel out)

Substitute and re-arrange.

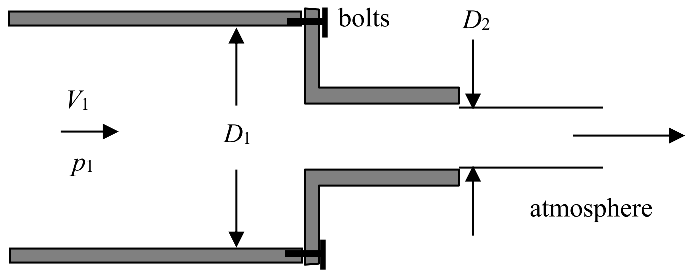
$$\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(\cos\theta-1\right)=-F_x$$

$$\boxed{F_x=\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2A\left(1-\cos\theta\right)} \quad \text{(Same answer as before!)} \tag{14}$$

Note that using a frame of reference that is fixed to the control volume is easier than using one fixed to the ground. This is often the case.

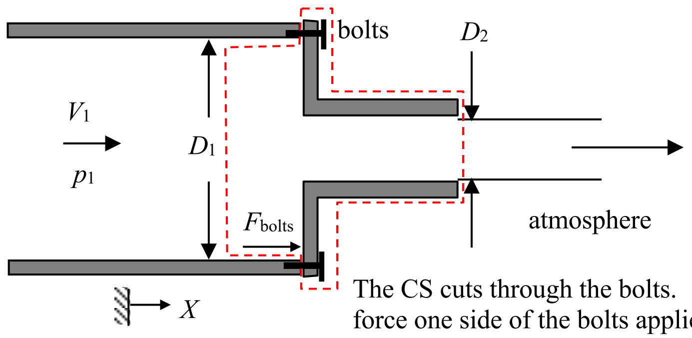
Video solution: https://www.youtube.com/watch?v=yCiOfNZU4Vo

A fluid enters a horizontal, circular cross-sectioned, sudden contraction nozzle. At section 1, which has diameter D_1, the velocity is uniformly distributed and equal to V_1. The gage pressure at 1 is p_1. The fluid exits into the atmosphere at section 2, with diameter D_2. Determine the force in the bolts required to hold the contraction in place. Neglect gravitational effects and assume that the fluid is inviscid.



SOLUTION:

Apply the linear momentum equation in the X-direction to the fixed control volume shown below.



$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{Cs}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X} \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{Cs}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \rho \overbrace{V_1}^{=u_X}\left(\overbrace{V_1\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{-\frac{\pi D_1^2}{4}\hat{\mathbf{i}}}^{=\mathbf{A}}\right) + \rho \overbrace{V_2}^{=u_X}\left(\overbrace{V_2\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}} \cdot \overbrace{\frac{\pi D_2^2}{4}\hat{\mathbf{i}}}^{=\mathbf{A}}\right) = -\rho V_1^2 \frac{\pi D_1^2}{4} + \rho V_2^2 \frac{\pi D_2^2}{4}$$

(Note that V_2 is unknown for now.)

$$F_{B,X} = 0$$

$$F_{S,X} = p_{1,\text{gage}} \frac{\pi D_1^2}{4} + F_{\text{bolts}}$$

(Note that $p_{2,\text{gage}} = 0$ since $p_{2,\text{abs}} = p_{\text{atm}}$. We could have also worked the problem using absolute pressures everywhere. The pressure force on the left hand side would be $p_{1,\text{abs}}\pi D_1^2/4$ and the pressure force on the right hand side would be $p_{\text{atm}}\pi D_1^2/4$ (note that the diameter is D_1 and not D_2).)

Substitute and re-arrange.

$$-\rho V_1^2 \frac{\pi D_1^2}{4} + \rho V_2^2 \frac{\pi D_2^2}{4} = p_{1,\text{gage}} \frac{\pi D_1^2}{4} + F_{\text{bolts}}$$

$$F_{\text{bolts}} = -\rho V_1^2 \frac{\pi D_1^2}{4} + \rho V_2^2 \frac{\pi D_2^2}{4} - p_{1,\text{gage}} \frac{\pi D_1^2}{4} \tag{2}$$

To determine V_2, apply conservation of mass to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{Cs}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0$$

$$\int_{\text{Cs}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho V_1 \frac{\pi D_1^2}{4} + \rho V_2 \frac{\pi D_2^2}{4}$$

Substitute and simplify.

$$-\rho V_1 \frac{\pi D_1^2}{4} + \rho V_2 \frac{\pi D_2^2}{4} = 0$$

$$V_2 = V_1 \left(\frac{D_1}{D_2}\right)^2 \tag{4}$$

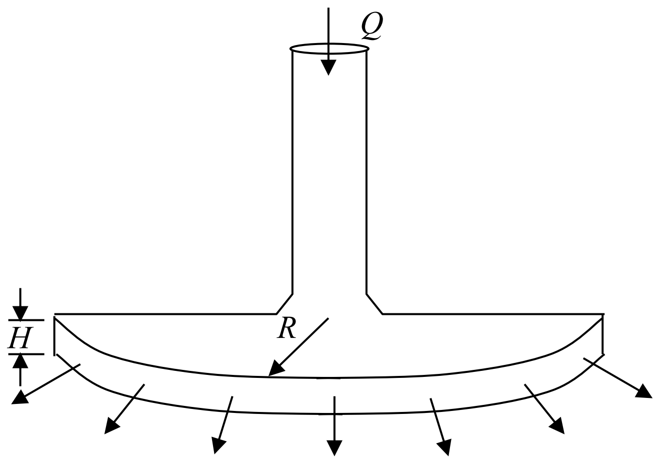
Substitute Eqn. (4) into Eqn. (2) and simplify.

$$F_{\text{bolts}} = -\rho V_1^2 \frac{\pi D_1^2}{4} + \rho V_1^2 \left(\frac{D_1}{D_2}\right)^4 \frac{\pi D_2^2}{4} - p_{1,\text{gage}} \frac{\pi D_1^2}{4}$$

$$\boxed{F_{\text{bolts}} = \rho V_1^2 \frac{\pi D_1^2}{4}\left[\left(\frac{D_1}{D_2}\right)^2 - 1\right] - p_{1,\text{gage}} \frac{\pi D_1^2}{4}} \tag{5}$$

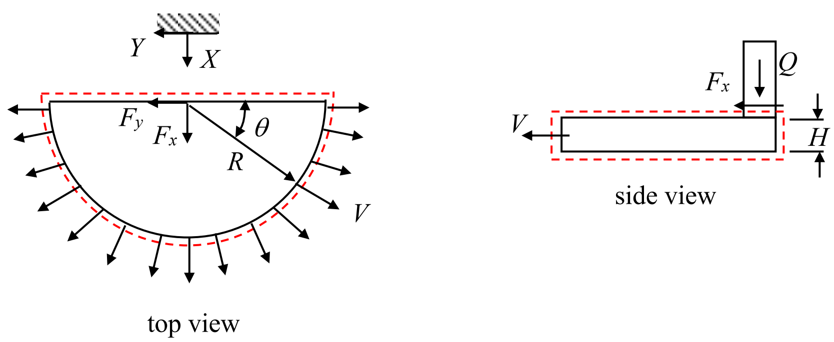
Note that F_{bolts} was assumed to be positive when acting in the $+X$ direction (causing compression in the bolts). If $F_{\text{bolts}} < 0$ then the bolts will be in tension.

Water is sprayed radially outward through 180° as shown in the figure. The jet sheet is in the horizontal plane and has thickness, H. If the jet volumetric flow rate is Q, determine the resultant horizontal anchoring force required to hold the nozzle stationary.



SOLUTION:

Apply the linear momentum equation in the X direction to the fixed control volume shown below.



$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{CS}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X} \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \int_{\theta=0}^{\theta=\pi} \overbrace{\left(V\sin\theta\right)}^{=u_X}\left(\rho V \overbrace{R d\theta H}^{=dA}\right) = \rho V^2 RH \int_{\theta=0}^{\theta=\pi} \sin\theta d\theta = -\rho V^2 RH \cos\theta\Big|_0^{\pi}$$

$$= -\rho V^2 RH\left(-1-1\right)$$

$$= 2\rho V^2 RH$$

(Note that there is no X-momentum at the control volume inlet. Also, V is an unknown quantity at the moment.)

$F_{B,X} = 0$

$F_{S,X} = F_x$ (All of the pressure forces cancel and only the anchoring force remains.)

Substitute.

$$F_x = 2\rho V^2 RH \tag{2}$$

To determine V, apply conservation of mass to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \underbrace{-\rho Q}_{\text{inlet}} + \underbrace{\int_{\theta=0}^{\theta=\pi} \rho V \overbrace{Rd\theta H}^{=dA}}_{\text{outlet}} = -\rho Q + \rho V \pi RH$$

Substitute and simplify.

$$-\rho Q + \rho V \pi RH = 0$$

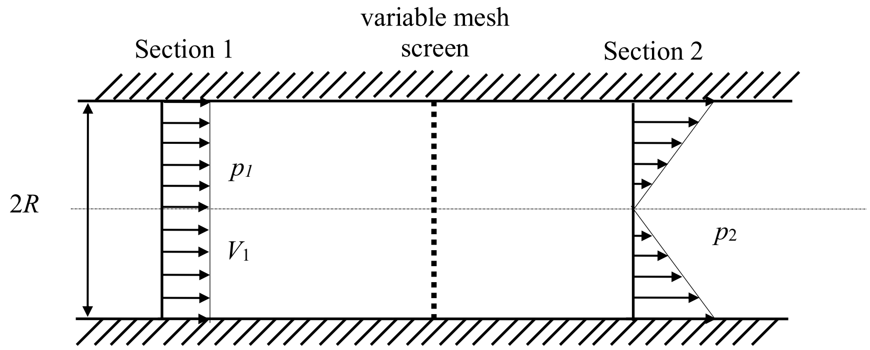
$$V = \frac{Q}{\pi RH} \tag{4}$$

Substitute Eqn. (4) into Eqn. (2).

$$\boxed{F_x = 2\rho\left(\frac{Q}{\pi RH}\right)^2 RH} \tag{5}$$

Note that $F_y = 0$ due to symmetry.

A variable mesh screen produces a linear and axi-symmetric velocity profile as shown in the figure. The static pressure upstream and downstream of the screen are p_1 and p_2 respectively (and are uniformly distributed). If the flow upstream of the mesh is uniformly distributed and equal to V_1, determine the force the mesh screen exerts on the fluid. Assume that the pipe wall does not exert any force on the fluid.



SOLUTION:

First, note that the linear velocity profile at the outlet may be written as,

$$V = V_{\max}\frac{r}{R}, \tag{1}$$

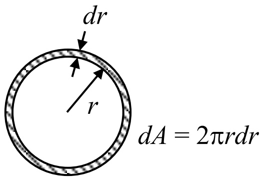
where $V_{\max}$ is the flow velocity at $r = R$. Now apply Conservation of Mass to the fixed control volume shown in the figure to find $V_{\max}$ in terms of the upstream properties,

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CV}\rho\mathbf{u}_{rel}\cdot d\mathbf{A} = 0, \tag{2}$$

where,

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady state)}, \tag{3}$$

$$\int_{CV}\rho\mathbf{u}_{rel}\cdot d\mathbf{A} = \underbrace{-\rho V_1\pi R^2}_{\text{left side}} + \underbrace{\int_{r=0}^{r=R}\rho\overbrace{\left(V_{\max}\frac{r}{R}\right)}^{=V}\overbrace{(2\pi r dr)}^{=dA}}_{\text{right side}} \tag{4}$$

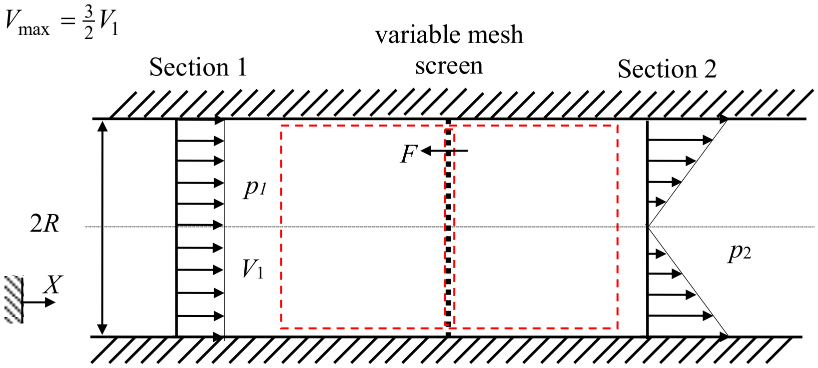


$$= -\rho V_1\pi R^2 + \rho\tfrac{2}{3}\pi V_{\max}R^2$$

Substitute and simplify,

$$-\rho V_1\pi R^2 + \rho\tfrac{2}{3}\pi V_{\max}R^2 = 0 \tag{5}$$

$$V_{\max} = \tfrac{3}{2}V_1 \tag{6}$$



The control volume weaves in and out of the mesh so that the mesh is not part of the control volume, and instead exerts a force, F, on the control volume.

Video solution: https://www.youtube.com/watch?v=hHmJwDEXufE

Now apply the Linear Momentum Equation in the X-direction to the fixed control volume shown in the figure,

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{CV}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X}\,, \tag{7}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV = 0 \quad \text{(steady state)}, \tag{8}$$

$$\int_{CS} u_X(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A}) = \underbrace{V_1(-\rho V_1 \pi R^2)}_{\text{left side}} + \underbrace{\int_{r=0}^{r=R} \left(\frac{3}{2}V_1\frac{r}{R}\right)\left[\rho\left(\frac{3}{2}V_1\frac{r}{R}\right)\underbrace{(2\pi r dr)}_{=dA}\right]}_{\text{right side}}, \tag{9}$$

$$= -\rho V_1^2 \pi R^2 + \frac{9\pi}{2}\frac{\rho V_1^2}{R^2}\int_0^R r^3 dr, \tag{10}$$

$$= -\rho V_1^2 \pi R^2 + \frac{9}{8}\rho V_1^2 \pi R^2 = \frac{1}{8}\rho V_1^2 \pi R^2\ , \tag{11}$$

$$F_{B,X} = 0\,, \tag{12}$$

$$F_{S,X} = -F + p_1 \pi R^2 - p_2 \pi R^2\,. \tag{13}$$

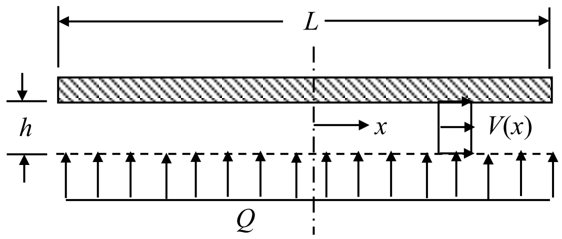
Substitute and simplify,

$$\frac{1}{8}\rho V_1^2 \pi R^2 = -F + p_1 \pi R^2 - p_2 \pi R^2, \tag{14}$$

$$\boxed{F = (p_1 - p_2)\pi R^2 - \frac{1}{8}\rho V_1^2 \pi R^2}, \tag{15}$$

This is the force the mesh applies to the control volume (i.e., the fluid). The fluid applies an equal and opposite force to the mesh.

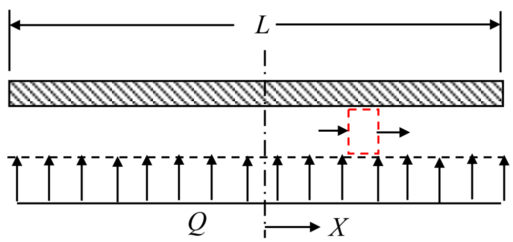
Incompressible fluid of negligible viscosity is pumped, at total volume flow rate Q, through a porous surface into the small gap between closely spaced parallel plates as shown. The fluid has only horizontal motion in the gap. Assume uniform flow across any vertical section. Obtain an expression for the pressure variation as a function of x.



Assume a depth w into the page.

SOLUTION:

Apply conservation of mass to the following differential control volume.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\left[\rho Vh(w) + \frac{d}{dx}(\rho Vhw)\left(-\tfrac{1}{2}dx\right)\right] + \left[\rho Vhw + \frac{d}{dx}(\rho Vhw)\left(\tfrac{1}{2}dx\right)\right] - \rho\frac{Q}{Lw}wdx$$

$$= \frac{d}{dx}(\rho Vhw)dx - \rho\frac{Q}{Lw}wdx = \rho hw\frac{dV}{dx}dx - \rho\frac{Q}{Lw}wdx$$

Substituting and simplifying gives:

$$\rho hw\frac{dV}{dx}dx = \rho\frac{Q}{Lw}wdx$$

$$\frac{dV}{dx} = \frac{Q}{Lhw} \tag{1}$$

$$\int_{V=0}^{V=V} dV = \int_{x=0}^{x=x}\frac{Q}{Lhw}dx$$

$$\frac{Vhw}{Q} = \frac{x}{L} \quad \text{or} \quad V = \left(\frac{Q}{hw}\right)\left(\frac{x}{L}\right) \tag{2}$$

Now apply the linear momentum equation in the X-direction to the same control volume.

$$\frac{d}{dt}\int_{CV} u_X \rho dV + \int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{BX} + F_{SX}$$

where

$$\frac{d}{dt}\int_{CV} u_X \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\left[\rho V^2 hw + \frac{d}{dx}\left(\rho V^2 hw\right)\left(-\tfrac{1}{2}dx\right)\right] + \left[\rho V^2 hw + \frac{d}{dx}\left(\rho V^2 hw\right)\left(\tfrac{1}{2}dx\right)\right]$$

$$= \frac{d}{dx}\left(\rho V^2 hw\right)dx = 2\rho V \frac{dV}{dx} hwdx$$

(Assume unit depth into the page. Note that the flux of mass from the porous surface has no X-momentum.)

$$F_{BX} = 0$$

$$F_{SX} = \left[phw + \frac{d}{dx}\left(phw\right)\left(-\tfrac{1}{2}dx\right)\right] - \left[phw + \frac{d}{dx}\left(phw\right)\left(\tfrac{1}{2}dx\right)\right]$$

$$= -\frac{d}{dx}\left(phw\right)dx = -\frac{dp}{dx}hwdx$$

Substituting and simplifying gives:

$$2\rho V \frac{dV}{dx} hwdx = -\frac{dp}{dx}hwdx$$

$$2\rho V \frac{dV}{dx} = -\frac{dp}{dx}$$

Substituting Eqns. (1) and (2) gives:

$$2\rho\left(\frac{Q}{Lhw}\right)^2 x = -\frac{dp}{dx}$$

$$\int_{p=p}^{p=p_{\text{atm}}} dp = -2\rho\left(\frac{Q}{Lhw}\right)^2 \int_{x=x}^{x=\frac{1}{2}L} xdx$$

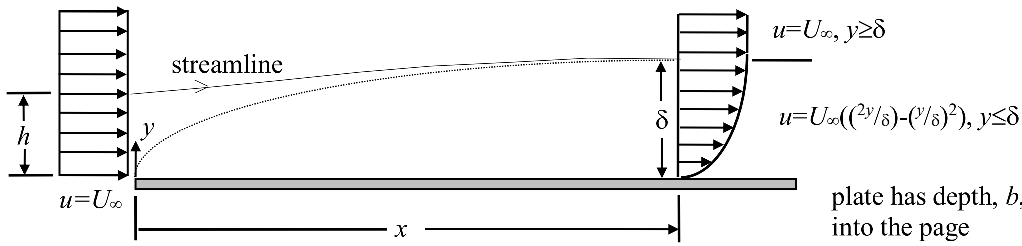
$$p_{\text{atm}} - p = -\rho\left(\frac{Q}{Lhw}\right)^2 \left(\tfrac{1}{4}L^2 - x^2\right)$$

$$p - p_{\text{atm}} = \rho\left(\frac{Q}{hw}\right)^2 \left(\tfrac{1}{4} - \left(\frac{x}{L}\right)^2\right)$$

$$\boxed{\frac{p - p_{\text{atm}}}{\rho\left(Q/hw\right)^2} = \frac{1}{4} - \left(\frac{x}{L}\right)^2} \qquad (3)$$

Video solution: https://www.youtube.com/watch?v=FTA2D2VkzHo

An incompressible, viscous fluid with density, ρ, flows past a solid flat plate which has a depth, b, into the page. The flow initially has a uniform velocity U_∞, before contacting the plate. After contact with the plate at a distance x downstream from the leading edge, the flow velocity profile is altered due to the no-slip condition. The velocity profile at location x is estimated to have a parabolic shape, $u=U_\infty((2y/\delta)-(y/\delta)^2)$, for $y \leq \delta$ and $u=U_\infty$ for $y \geq \delta$ where δ is termed the "boundary layer thickness."



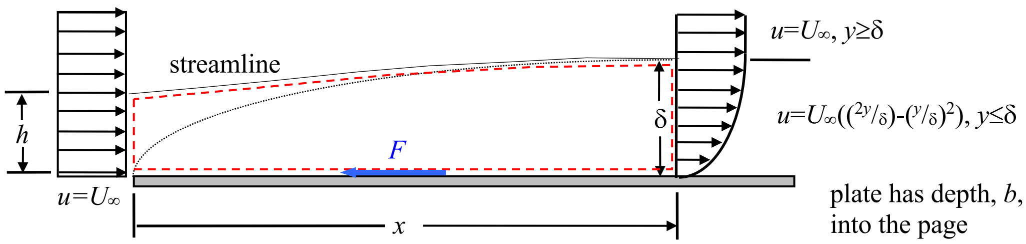
1. Determine the upstream height from the plate, h, of a streamline which has a height, δ, at the downstream distance x. Express your answer in terms of δ.
2. Determine the force the plate exerts on the fluid over the distance x. Express your answer in terms of ρ, U_∞, b, and δ. You may assume that the pressure everywhere is p_∞. The force the drag exerts on the plate is called the "skin friction" drag.

BRIEF SOLUTION:

1. Apply conservation of mass to a control volume that is adjacent to the plate, crosses perpendicularly to the stream at the leading edge of the plate, follows a streamline, and crosses perpendicularly to the stream at the location where the boundary layer has thickness, δ. Note that there is no flow across a streamline.
2. Apply the linear momentum equation to the same control volume used in Step 1. Be sure to include the force the plate exerts on the control volume.

DETAILED SOLUTION:

Apply conservation of mass to the fixed control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho U_\infty hb + \int_{y=0}^{y=\delta}\rho U_\infty\left[2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right]dyb = -\rho U_\infty hb + \rho U_\infty\left(\delta - \frac{1}{3}\delta\right)b$$

$$= -\rho U_\infty hb + \tfrac{2}{3}\rho U_\infty \delta b$$

(Note that there is no flow across the streamline.)

Substitute into conservation of mass and solve for h.

$$\boxed{h = \tfrac{2}{3}\delta} \quad (1)$$

Now apply the linear momentum equation in the x-direction on the same control volume.

$$\frac{d}{dt}\int_{CV} u\rho dV + \int_{CS} u\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u\rho dV = 0 \quad \text{(steady flow)}$$

$$\begin{aligned}\int_{CS} u\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) &= -\rho U_\infty^2 hb + \int_{y=0}^{y=\delta} \rho U_\infty^2 \left[2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right]^2 dyb \\ &= -\rho U_\infty^2 hb + \rho U_\infty^2 b \int_0^\delta \left[4\frac{y^2}{\delta^2} - 4\frac{y^3}{\delta^3} + \frac{y^4}{\delta^4}\right] dy \\ &= -\rho U_\infty^2 hb + \rho U_\infty^2 b\left[\tfrac{4}{3}\delta - \delta + \tfrac{1}{5}\delta\right] \\ &= -\rho U_\infty^2 hb + \tfrac{8}{15}\rho U_\infty^2 b\delta\end{aligned}$$

$$F_{B,x} = 0$$

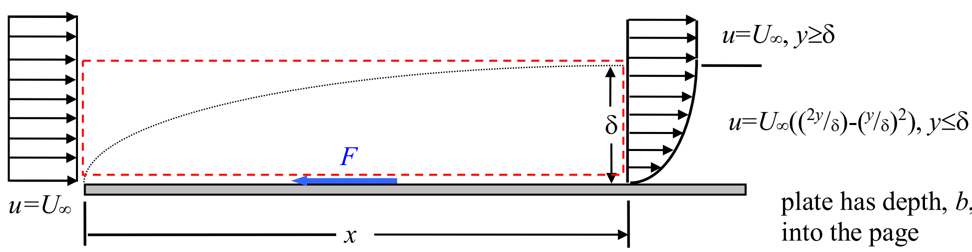
$$F_{S,x} = -F \quad \text{(the pressure everywhere is } p_\infty\text{)}$$

Substitute and simplify, making use of Eqn. (1).

$$-\rho U_\infty^2\left(\tfrac{2}{3}\delta\right)b + \tfrac{8}{15}\rho U_\infty^2 b\delta = -F$$

$$\boxed{F = \tfrac{2}{15}\rho U_\infty^2 b\delta} \tag{2}$$

We could have also determined the force using a different control volume as shown below.



Determine the mass flow rate out of the control volume through the top using conservation of mass.

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho U_\infty \delta b + \int_{y=0}^{y=\delta} \rho U_\infty \left[2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right] dyb + \dot{m}_{\text{top}} = -\tfrac{1}{3}\rho U_\infty \delta b + \dot{m}_{\text{top}}$$

Substitute and solve for the mass flow rate.

$$\dot{m}_{\text{top}} = \tfrac{1}{3}\rho U_\infty \delta b \tag{3}$$

Now apply the linear momentum equation in the x-direction to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} u\rho dV + \int_{\text{CS}} u\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\rho U_\infty^2 \delta b + \int_{y=0}^{y=\delta} \rho U_\infty^2 \left[2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right]^2 dyb + \dot{m}_{\text{top}} U_\infty$$

$$= -\tfrac{7}{15}\rho U_\infty^2 \delta b + \dot{m}_{\text{top}} U_\infty$$

(Note that the horizontal component of the velocity at the top is U_∞ since it's outside of the boundary layer.)

$$F_{B,x} = 0$$

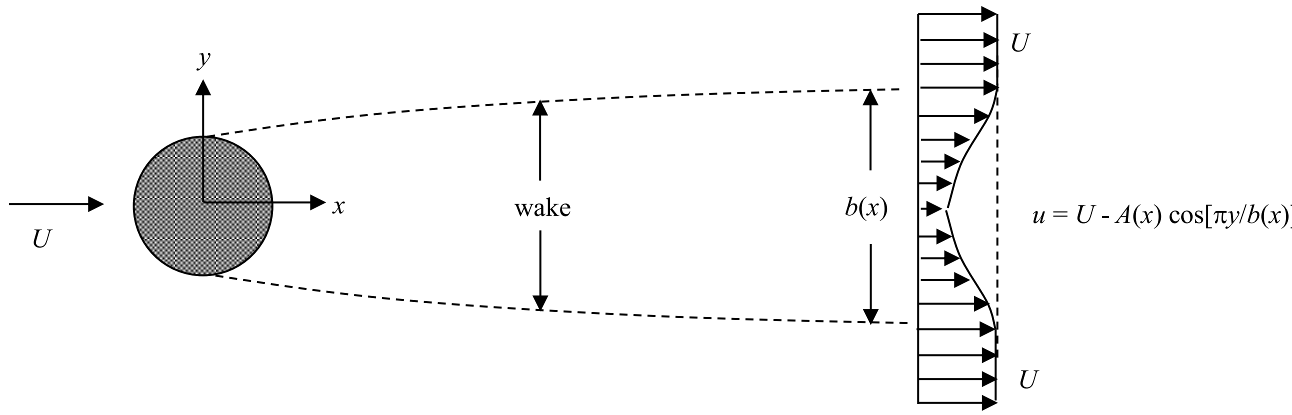
$$F_{S,x} = -F \quad \text{(the pressure everywhere is } p_\infty\text{)}$$

Substitute and simplify making use of Eqn. (3).

$$-\tfrac{7}{15}\rho U_\infty^2 \delta b + \left(\tfrac{1}{3}\rho U_\infty \delta b\right) U_\infty = -F$$

$$F = \tfrac{2}{15}\rho U_\infty^2 \delta b \quad \text{(This is the same answer as before!)}$$

Wake surveys are made in the two-dimensional wake behind a cylindrical body which is externally supported in a uniform stream of incompressible fluid approaching the cylinder with velocity, U.



The surveys are made at x locations sufficiently far downstream of the body so that the pressure across the wake is the same as the ambient pressure in the fluid far from the body. The surveys indicate that, to a first approximation, the velocity in the wake varies with lateral position, y, according to:

$$\frac{u}{U} = 1 - \frac{A(x)}{U}\cos\left[\pi \frac{y}{b(x)}\right] \quad \text{where} \quad -\frac{1}{2} < \frac{y}{b(x)} < +\frac{1}{2}$$

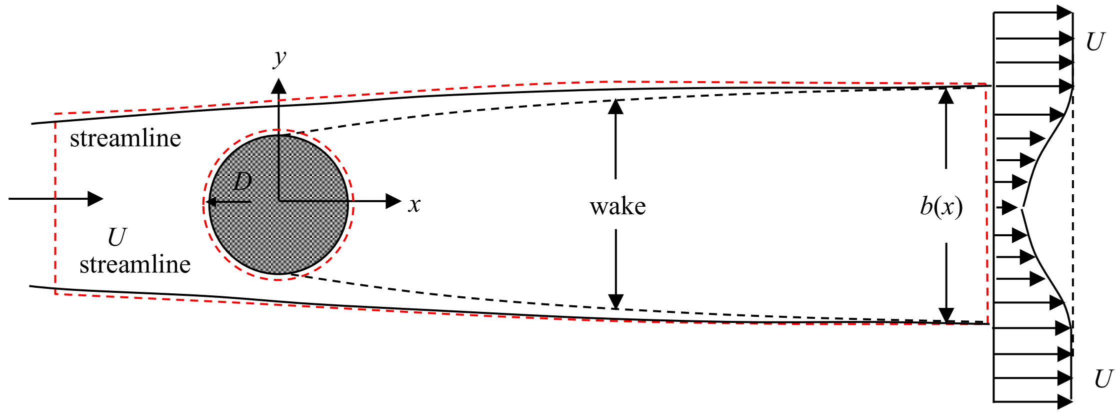
The quantities $A(x)$ and $b(x)$ are the centerline velocity defect and wake width, respectively, both of which vary with position, x. If the drag on the body per unit distance normal to the plane of the sketch is denoted by D and the density of the fluid by ρ, find the relation for $b(x)$ in terms of $A(x)$, U, ρ, and D.

BRIEF SOLUTION:

1. First apply the linear momentum equation to determine a relation between the various quantities. Use a control volume that surrounds the cylinder, crosses the flow perpendicularly far upstream of the cylinder where the velocity is uniform (call this cross stream distance, h), crosses the flow perpendicularly downstream of the cylinder where the wake width is $b(x)$, and follows streamlines between the upstream to downstream locations along the sides of the control volume. Note that there is no flow across a streamline. Be sure to include the force the cylinder exerts on the control volume.
2. Apply conservation of mass to the same control volume described above to relate the upstream cross flow width, h, to the downstream cross flow width, $b(x)$.

DETAILED SOLUTION:

Apply linear momentum equation in the x-direction to the control volume shown below. Use the fixed frame of reference indicated in the figure. Note that there is no flow across a streamline.



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = U\left(-\rho U h\right) + \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} u \rho u dy$$

$$= -\rho U^2 h + \rho U^2 \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} \left[1 - \frac{A}{U}\cos\left(\pi \frac{y}{b}\right)\right]^2 dy$$

$$= -\rho U^2 h + \rho U^2 \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} \left[1 - \frac{2A}{U}\cos\left(\pi \frac{y}{b}\right) + \frac{A^2}{U^2}\cos^2\left(\pi \frac{y}{b}\right)\right] dy$$

$$= -\rho U^2 h + \rho U^2 \left[b - \underbrace{\frac{2Ab}{\pi U}\sin\left(\pi \frac{y}{b}\right)\Bigg|_{-\frac{1}{2}b}^{+\frac{1}{2}b}}_{=\frac{4Ab}{\pi U}} + \frac{bA^2}{\pi U^2}\left\{ \underbrace{\frac{\pi}{2}\frac{y}{b}\Bigg|_{-\frac{1}{2}b}^{+\frac{1}{2}b}}_{=\pi/2} + \underbrace{\frac{1}{4}\sin\left(2\pi \frac{y}{b}\right)\Bigg|_{-\frac{1}{2}b}^{+\frac{1}{2}b}}_{=0} \right\} \right]$$

$$= -\rho U^2 h + \rho U^2 \left[b - \frac{4Ab}{\pi U} + \frac{bA^2}{2U^2} \right]$$

$$= \rho U^2 \left[-h + b - \frac{4Ab}{\pi U} + \frac{bA^2}{2U^2} \right]$$

$F_{B,x} = 0$ (no body forces in the x direction)

$F_{S,x} = -D$ (no pressure forces in the x direction)

Substitute and simplify.

$$\rho U^2\left[-h+b-\frac{4Ab}{\pi U}+\frac{bA^2}{2U^2}\right]=-D \tag{2}$$

Now apply conservation of mass to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}}\rho dV+\int_{\text{CS}}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=0 \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}}\rho dV=0 \quad \text{(steady flow)}$$

$$\begin{aligned}\int_{\text{CS}}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}&=-\rho Uh+\int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b}\rho u dy\\&=-\rho Uh+\rho U\int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b}\left[1-\frac{A}{U}\cos\left(\pi\frac{y}{b}\right)\right]dy\\&=-\rho Uh+\rho U\left[b-\underbrace{\frac{bA}{\pi U}\sin\left(\pi\frac{y}{b}\right)\Big|_{-\frac{1}{2}b}^{+\frac{1}{2}b}}_{=\frac{2bA}{\pi U}}\right]\\&=-\rho Uh+\rho Ub\left[1-\frac{2A}{\pi U}\right]\end{aligned}$$

Substitute and simplify.

$$0=-\rho Uh+\rho Ub\left[1-\frac{2A}{\pi U}\right]$$

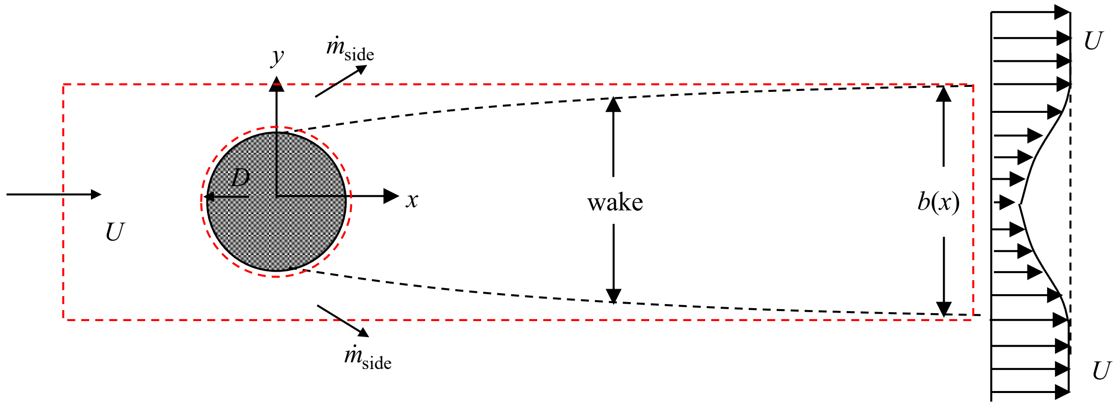
$$h=b\left[1-\frac{2A}{\pi U}\right] \tag{4}$$

Substitute Eqn. (4) into Eqn. (2) and solve for $b(x)$.

$$\rho U^2\left[-b\left(1-\frac{2A}{\pi U}\right)+b-\frac{4Ab}{\pi U}+\frac{bA^2}{2U^2}\right]=-D$$

$$\boxed{\therefore b(x)=\frac{D}{\rho U^2A(x)\left[\frac{2}{\pi U}-\frac{A(x)}{2U^2}\right]}} \tag{5}$$

The rectangular control volume shown below could also have used. Note that there will be some mass flow rate through the sides as indicated in the figure below (since the upstream mass flux is larger than the downstream mass flux). The horizontal velocity through the sides will be U everywhere since the boundaries are outside the wake.



The linear momentum equation in the x-direction is:

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{6}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = U\left(-\rho U b\right) + \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} u \rho u dy + 2\dot{m}_{\text{side}} U$$

$$= \rho U^2 \left[-\frac{4Ab}{\pi U} + \frac{bA^2}{2U^2}\right] + 2\dot{m}_{\text{side}} U$$

$F_{B,x} = 0$ (no body forces in the x direction)

$F_{S,x} = -D$ (no pressure forces in the x direction)

Substitute and simplify.

$$\rho U^2 \left[-\frac{4Ab}{\pi U} + \frac{bA^2}{2U^2}\right] + 2\dot{m}_{\text{side}} U = -D \tag{7}$$

Now apply conservation of mass to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{8}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho U b + \int_{y=-\frac{1}{2}b}^{y=+\frac{1}{2}b} \rho u dy + 2\dot{m}_{\text{side}}$$

$$= -\rho U b + \rho U b \left[1 - \frac{2A}{\pi U}\right] + 2\dot{m}_{\text{side}}$$

$$= -\rho b \frac{2A}{\pi} + 2\dot{m}_{\text{side}}$$

Substitute and simplify.

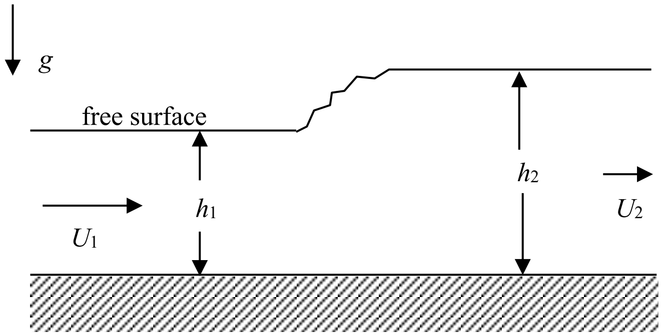
$$0 = -\rho b \frac{2A}{\pi} + 2\dot{m}_{\text{side}}$$

$$\dot{m}_{\text{side}} = \frac{\rho b A}{\pi} \tag{9}$$

Substitute Eqn. (9) into Eqn. (7)and solve for $b(x)$.

$$\therefore b(x) = \frac{D}{\rho U^2 A(x)\left[\frac{2}{\pi U} - \frac{A(x)}{2U^2}\right]} \quad \text{(This is the same result as before!)} \tag{10}$$

A hydraulic jump is a sudden increase in the depth of a liquid stream (which in this case we assume is flowing over a horizontal stream bed with atmospheric pressure air everywhere above the liquid):



The depth increases suddenly from h_1 to h_2 downstream of the jump. The jump itself is often turbulent and involves viscous losses so that the total pressure downstream is less than that of the upstream flow.

a. Find the ratio of the depths, h_2/h_1, in terms of the upstream velocity, U_1, the depth, h_1, and g, the acceleration due to gravity. Assume the flows upstream and downstream have uniform velocity parallel to the stream bed and that the shear stress between the liquid and the stream bed is zero. The liquid is incompressible.
b. What inequality on the value of $U_1^2/(gh_1)$ must hold for a hydraulic jump like this to occur?

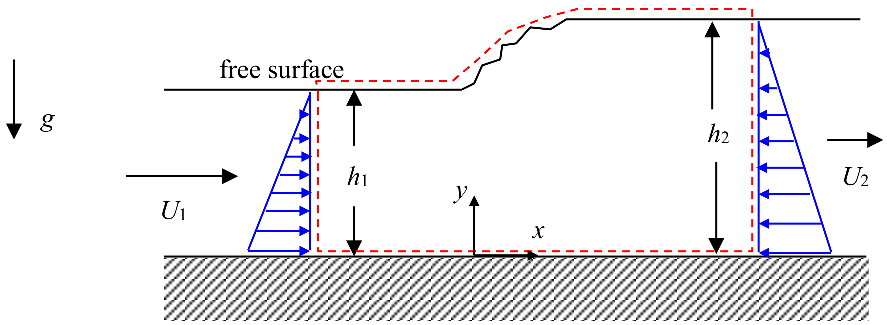
BRIEF SOLUTION:

1. Apply conservation of mass to relate the upstream and downstream depths, h_1 and h_2, to the upstream and downstream velocities, U_1 and U_2. Use a control volume that perpendicularly crosses the upstream and downstream flows where the velocities are uniform, follows the free surface, and is adjacent to the floor.
2. Apply the linear momentum equation to the same control volume as in Step 2. Be sure to include the pressure forces acting on the upstream and downstream faces. Note that the pressure increases linearly with depth in the fluid.

Video solution: https://www.youtube.com/watch?v=ycny6ub1xzw

DETAILED SOLUTION:

First apply conservation of mass to the fixed control volume shown below.



Note: Since the streamlines are parallel at the inlet and outlet of the CV, the pressure gradient normal to the streamlines will be hydrostatic.

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho U_1 h_1 b + \rho U_2 h_2 b$$

Substitute and simplify.

$$h_2 U_2 = h_1 U_1 \tag{1}$$

Now apply conservation of linear momentum in the x-direction to the same control volume.

$$\frac{d}{dt}\int_{CV} u\rho dV + \int_{CS} u\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\rho U_1^2 h_1 b + \rho U_2^2 h_2 b$$

$$F_{B,x} = 0$$

$$F_{S,x} = \int_{y=0}^{y=h_1} \rho g\left(h_1 - y\right) dyb - \int_{y=0}^{y=h_2} \rho g\left(h_2 - y\right) dyb = \tfrac{1}{2}\rho g h_1^2 b - \tfrac{1}{2}\rho g h_2^2 b$$

(hydrostatic pressure forces on left and right sides)

Substitute and simplify making use of Eqn. (1). Solve for the ratio h_2/h_1.

$$-\rho U_1^2 h_1 b + \rho U_2^2 h_2 b = \tfrac{1}{2}\rho g h_1^2 b - \tfrac{1}{2}\rho g h_2^2 b$$

$$-U_1^2 h_1 + U_2^2 h_2 = \tfrac{1}{2} g\left(h_1^2 - h_2^2\right)$$

$$U_1^2 h_1\left(U_2/U_1 - 1\right) = \tfrac{1}{2} g\left(h_1 - h_2\right)\left(h_1 + h_2\right) \quad \text{(using Eqn. (1))}$$

$$U_1^2 h_1\left(h_1/h_2 - 1\right) = \tfrac{1}{2} g\left(h_1 - h_2\right)\left(h_1 + h_2\right) \quad \text{(using Eqn. (1))}$$

$$U_1^2 h_1\left(\frac{h_1 - h_2}{h_2}\right) = \tfrac{1}{2} g\left(h_1 - h_2\right)\left(h_1 + h_2\right)$$

$$U_1^2 \frac{h_1}{h_2} = \tfrac{1}{2} g\left(h_1 + h_2\right) \Rightarrow \frac{U_1^2}{h_2/h_1} = \tfrac{1}{2} g h_1\left(1 + h_2/h_1\right)$$

$$\frac{2U_1^2}{gh_1} = h_2/h_1\left(1 + h_2/h_1\right) \Rightarrow \left(h_2/h_1\right)^2 + h_2/h_1 - \frac{2U_1^2}{gh_1} = 0$$

$$h_2/h_1 = \frac{-1 \pm \sqrt{1 + \dfrac{8U_1^2}{gh_1}}}{2}$$

We can neglect the negative sign in front of the second term since it is unrealistic.

$$\boxed{\therefore h_2/h_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2U_1^2}{gh_1}}} \tag{2}$$

For the hydraulic jump to occur, we need $h_2/h_1 > 1$.

$$1 < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2U_1^2}{gh_1}} \Rightarrow \boxed{\frac{U_1^2}{gh_1} > 1} \tag{3}$$

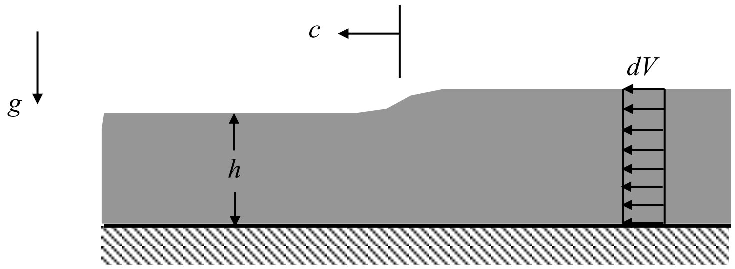
The dimensionless parameter in Eqn. (3) is the square of the flow's Froude number, Fr.

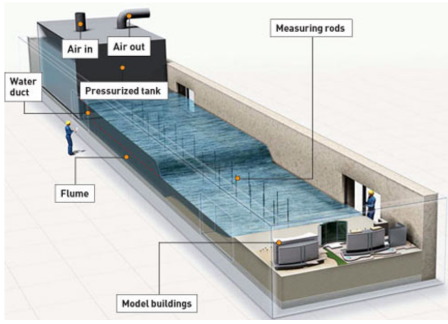
$$\text{Fr} \equiv \frac{U_1}{\sqrt{gh_1}} \tag{4}$$

where Fr < 1 is referred to as subcritical flow, Fr = 1 is critical flow, and Fr > 1 is supercritical flow. For the hydraulic jump to occur, we must have supercritical flow, *i.e.* Fr > 1.

In an attempt to model the speed of a tsunami wave in the deep ocean, consider the propagation of a small amplitude, solitary wave front moving with speed, c, from right to left as shown in the figure below. Neglect the effects of surface tension. The liquid is initially at rest but after the wave passes by, the fluid behind the wave has a small velocity, dV, in the same direction as the wave.

Derive an expression for the wave speed, c. You may neglect the shear forces the channel bed and the atmosphere exert on the liquid. Hint: Consider choosing a steady frame of reference when analyzing the problem.



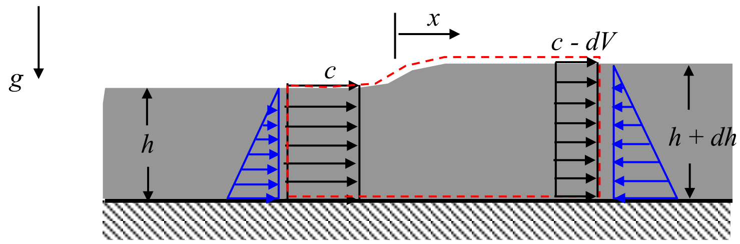


BRIEF SOLUTION:

1. Use a frame of reference fixed to the wave so that the flow appears steady.
2. Apply the linear momentum equation using a control volume that is perpendicular to the upstream and downstream flows where the velocities are uniform, along the bottom of the channel, and along the free surface. Be sure to include the pressure forces on the upstream and downstream boundaries. Note that the pressure increases linearly with depth from the free surface.
3. Apply conservation of mass to the same control volume.

DETAILED SOLUTION:

Apply the linear momentum equation in the x-direction to the control volume shown below. Use a frame of reference that is fixed to the wave. Since a constant wave velocity is assumed, the frame of reference will be inertial.



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CV} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CV} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = c\left(-\dot{m}\right) + \left(c - dV\right)\dot{m} = -\dot{m}dV \quad \text{where} \quad \dot{m} = \rho ch$$

$$F_{B,x} = 0$$

$$F_{S,x} = \tfrac{1}{2}\rho g h^2 - \tfrac{1}{2}\rho g\left(h + dh\right)^2 = -\rho g h dh - \underbrace{\tfrac{1}{2}\rho g dh^2}_{H.O.T.} = -\rho g h dh$$

Substitute and simplify.

$$-\rho ch dV = -\rho g h dh$$

$$cdV = gdh \tag{1}$$

Apply conservation of mass to the same control volume.

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CV} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CV} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho ch + \rho\left(c - dV\right)\left(h + dh\right) = -\rho ch + \rho ch + \rho c dh - \rho h dV - \underbrace{\rho dV dh}_{=H.O.T.}$$

$$= \rho c dh - \rho h dV$$

Substitute and simplify.

$$\rho c dh - \rho h dV = 0 \Rightarrow c dh = h dV$$

$$dh = \frac{hdV}{c} \tag{2}$$

Substitute Eqn. (2) into Eqn. (1) and simplify.

$$cdV = g\frac{hdV}{c}$$

$$\boxed{\therefore c = \sqrt{gh}} \tag{3}$$

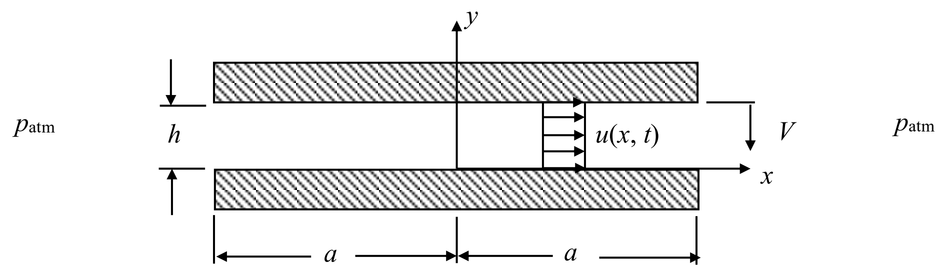
As an example, consider the speed of a traveling wave in the deep ocean resulting from an undersea earthquake for example (the wave amplitude is small compared to its wavelength). Assuming an ocean depth of 1610 m (1 mile), the speed of the wave will be 126 m/s (280 mph)!

Two parallel plates of width, $2a$, (and unit depth) are separated by a gap of height, h, which changes with time. The upper plate approaches the lower plate at a constant speed, V. The space between the plates is filled with a frictionless, incompressible gas of density, ρ. Assume that the velocity is uniform across the gap width (y direction) so that $u=u(x, t)$.

Obtain algebraic expressions for:

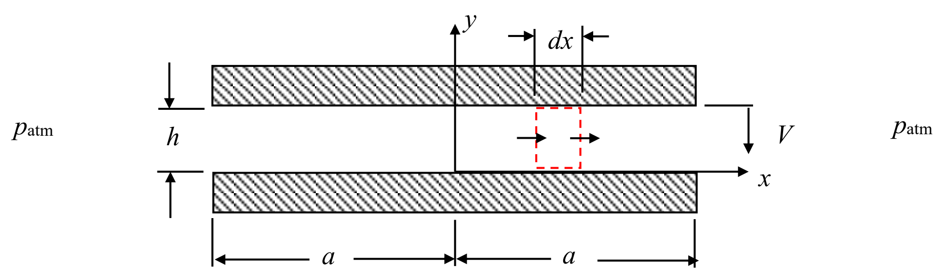
a. the velocity distribution, $u(x, t)$.

b. the pressure distribution in the gap, $p(x, t)$. The pressure outside of the gap is atmospheric pressure. Note: You do not need to use Bernoulli's equation to solve this problem.



SOLUTION:

Apply conservation of mass to the control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{d}{dt}(\rho h dx) = \rho\frac{dh}{dt}dx = -\rho V dx \tag{1}$$

$$\begin{aligned}\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} &= -\left[(\rho uh) + \frac{\partial}{\partial x}(\rho uh)\left(-\tfrac{1}{2}dx\right)\right] + \left[(\rho uh) + \frac{\partial}{\partial x}(\rho uh)\left(\tfrac{1}{2}dx\right)\right] \\ &= \frac{\partial}{\partial x}(\rho uh)dx = \rho\frac{\partial u}{\partial x}h dx\end{aligned} \tag{2}$$

Substitute and simplify.

$$-\rho V dx + \rho\frac{\partial u}{\partial x}h dx = 0$$

$$\frac{\partial u}{\partial x} = \frac{V}{h}$$

$$u = V\frac{x}{h} + f(t)$$ where $f(t)$ is an unknown function of time (Note: $u = u(x, t)$.)

Since the velocity at the center line of the plate is always zero, *i.e.* $u(x = 0, t) = 0$, then $f(t) = 0$.

$$\boxed{\therefore u = V\frac{x}{h}} \text{ (Note: } h = h(t) \Rightarrow u = u(x, t).) \tag{3}$$

Now apply the linear momentum equation in the x-direction to the same control volume using the given fixed frame of reference.

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = \frac{d}{dt}\left(u\rho h dx\right) = \rho dx\left(u\frac{dh}{dt} + h\frac{\partial u}{\partial t}\right) = \rho dx\left(-uV + h\frac{\partial u}{\partial t}\right) \tag{4}$$

$$\begin{aligned}\int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) &= -\left[\left(u\rho uh\right) + \frac{\partial}{\partial x}\left(u\rho uh\right)\left(-\tfrac{1}{2}dx\right)\right] + \left[\left(u\rho uh\right) + \frac{\partial}{\partial x}\left(u\rho uh\right)\left(\tfrac{1}{2}dx\right)\right] \\ &= \frac{\partial}{\partial x}\left(u\rho uh\right)dx = 2\rho u\frac{\partial u}{\partial x}hdx\end{aligned} \tag{5}$$

$$F_{B,x} = 0 \tag{6}$$

$$\begin{aligned}F_{S,x} &= \left[\left(ph\right) + \frac{\partial}{\partial x}\left(ph\right)\left(-\tfrac{1}{2}dx\right)\right] - \left[\left(ph\right) + \frac{\partial}{\partial x}\left(ph\right)\left(\tfrac{1}{2}dx\right)\right] \\ &= -\frac{\partial}{\partial x}\left(ph\right)dx = -\frac{\partial p}{\partial x}hdx\end{aligned} \tag{7}$$

Substitute and simplify.

$$\rho dx\left(-uV + h\frac{\partial u}{\partial t}\right) + 2\rho u\frac{\partial u}{\partial x}hdx = -\frac{\partial p}{\partial x}hdx \tag{8}$$

Substitute for u using the expression derived from conservation of mass.

$$\left[-\left(V\frac{x}{h}\right)V + h\left(V^2\frac{x}{h^2}\right)\right] + 2\left(V\frac{x}{h}\right)\left(V\frac{1}{h}\right)h = -\frac{1}{\rho}\frac{\partial p}{\partial x}h \tag{9}$$

$$2V^2\frac{x}{h} = -\frac{1}{\rho}\frac{\partial p}{\partial x}h$$

$$\frac{\partial p}{\partial x} = -2\rho V^2\frac{x}{h^2}$$

$$p = -\rho V^2\frac{x^2}{h^2} + f\left(t\right) \tag{10}$$

The pressure at $x = a$ is p_{atm} for all times, *i.e.* $p(x = a, t) = p_{atm}$:

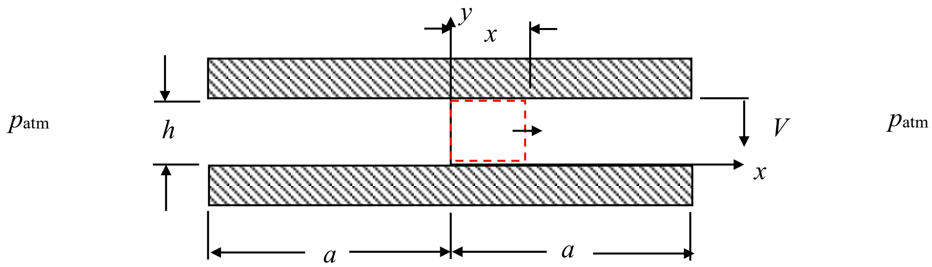
$$p_{atm} = -\rho V^2\frac{a^2}{h^2} + f\left(t\right) \Rightarrow f\left(t\right) = p_{atm} + \rho V^2\frac{a^2}{h^2} \tag{11}$$

Substituting and simplifying gives:

$$p = -\rho V^2\frac{x^2}{h^2} + p_{atm} + \rho V^2\frac{a^2}{h^2}$$

$$\boxed{\frac{p - p_{atm}}{\frac{1}{2}\rho V^2} = 2\left[\left(\frac{a}{h}\right)^2 - \left(\frac{x}{h}\right)^2\right]} \quad \text{(Note: } h = h(t) \Rightarrow p = p(x, t).\text{)} \tag{12}$$

Now let's work the problem using the control volume shown below.



Conservation of Mass:

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{d}{dt}\left[\rho hx\right] = \rho\frac{dh}{dt}x = -\rho Vx \tag{13}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = \rho uh \quad \text{(Mass flux only through right side due to symmetry.)} \tag{14}$$

Substitute and simplify.

$$-\rho Vx + \rho uh = 0$$

$$\boxed{u = V\left(\frac{x}{h}\right)} \quad \text{This is the same result as before!} \tag{15}$$

Linear Momentum Equation in the x-direction:

$$\frac{d}{dt}\int_{CV}u_x\rho dV + \int_{CS}u_x\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV}u_x\rho dV = \frac{d}{dt}\int_{x=0}^{x=x}\rho uhdx = \frac{d}{dt}\int_{x=0}^{x=x}\rho\left(V\frac{x}{h}\right)hdx = \rho V\frac{d}{dt}\left(\int_{x=0}^{x=x}xdx\right) = 0 \tag{16}$$

(The result from conservation of mass has been used in simplifying the previous expression.)

$$\int_{CS}u_x\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \rho u^2 h = \rho V^2\frac{x^2}{h} \quad \text{(Momentum flux only through right side due to symmetry.)} \tag{17}$$

(The result from conservation of mass has been used in simplifying the previous expression.)

$$F_{B,x} = 0 \tag{18}$$

$$F_{S,x} = p_{x=0}h - p_{x=x}h \tag{19}$$

Substitute and simplify.

$$\rho V^2\frac{x^2}{h} = p_{x=0}h - p_{x=x}h$$

$$p_{x=x} = p_{x=0} - \rho V^2\frac{x^2}{h^2} \tag{20}$$

Since the pressure at $x = a$ is p_{atm}, *i.e.* $p(x = a, t) = p_{atm}$:

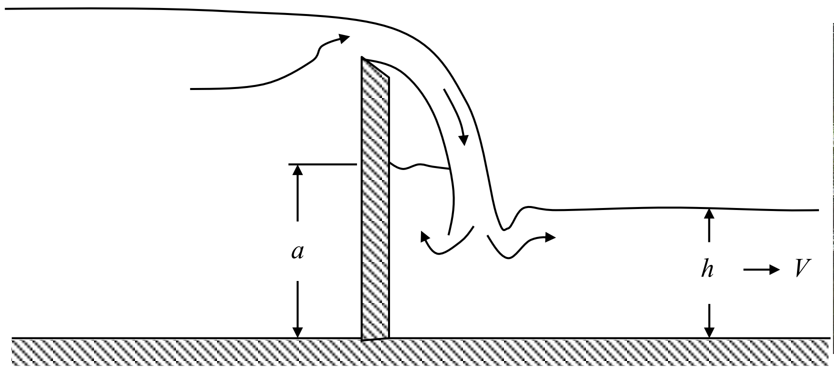
$$p_{atm} = p_{x=0} - \rho V^2\frac{a^2}{h^2} \Rightarrow p_{x=0} = p_{atm} + \rho V^2\frac{a^2}{h^2} \tag{21}$$

$$p_{x=x} = p_{atm} + \rho V^2\frac{a^2}{h^2} - \rho V^2\frac{x^2}{h^2}$$

$$\boxed{\therefore \frac{p - p_{atm}}{\frac{1}{2}\rho V^2} = 2\left[\left(\frac{a}{h}\right)^2 - \left(\frac{x}{h}\right)^2\right]} \quad \text{This is the same result as before!} \tag{22}$$

Note that since $a > x$, $(p - p_{atm}) > 0$. Thus, a downward force must be applied to move the top plate downward. Furthermore, as h decreases, this force increases since $(p - p_{atm})$ increases.

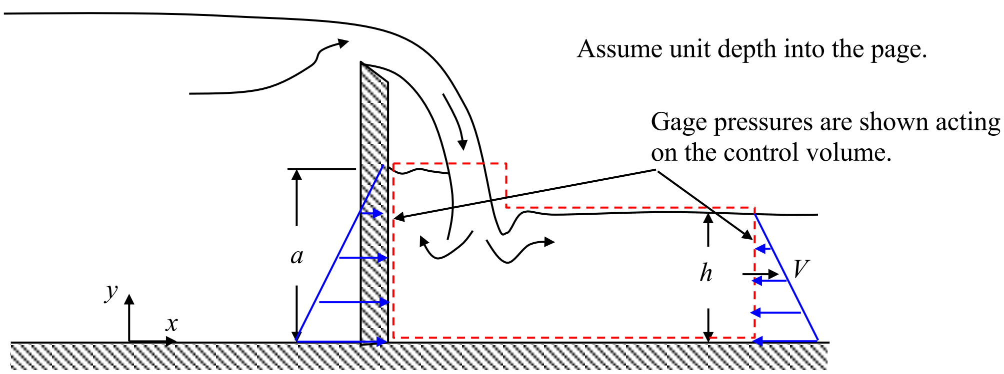
A weir discharges into a channel of constant breadth as shown in the figure. It is observed that a region of still water backs up behind the jet to a height a. The velocity and height of the flow in the channel are given as V and h, respectively, and the density of the water is ρ. You may assume that friction and the horizontal momentum of the fluid falling over the weir are negligible.



What is the height a in terms of the other parameters?

SOLUTION:

Apply the linear momentum equation in the x-direction to the control volume shown below. Use the fixed frame of reference shown in the figure.



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \rho V^2 h \quad \text{(assume incoming flow has negligible horizontal velocity)}$$

$$F_{B,x} = 0$$

$$F_{S,x} = \tfrac{1}{2}\rho g a^2 - \tfrac{1}{2}\rho g h^2 \quad \text{(net horizontal pressure forces)}$$

Substitute and simplify.

$$\rho V^2 h = \tfrac{1}{2}\rho g a^2 - \tfrac{1}{2}\rho g h^2 \tag{1}$$

$$a^2 = h^2 + \frac{2V^2 h}{g}$$

$$a = h\sqrt{1 + \frac{2V^2}{gh}}$$

$$\boxed{\therefore \frac{a}{h} = \sqrt{1 + 2\mathrm{Fr}^2}} \tag{2}$$

where $\mathrm{Fr} = V/(gh)^{1/2}$ is a dimensionless parameter known as the Froude number.

4.4.1. The LME using a Non-inertial Coordinate System

Recall that Newton's Second law holds strictly for inertial (non-accelerating) coordinate systems. Now let's consider coordinate systems that are non-inertial (accelerating). First examine how we can describe the motion of a particle in an accelerating coordinate system, call it frame xyz, in terms of a non-accelerating coordinate system, call it frame XYZ (Figure 4.10).

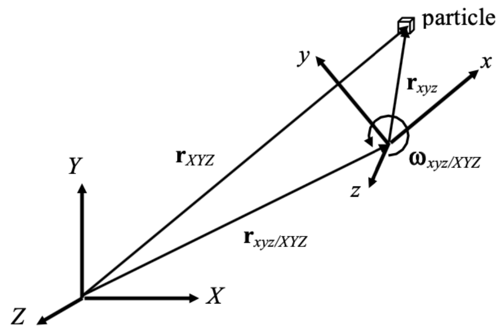


Figure 4.10. A schematic illustrating a particle's movement in two coordinate systems.

The position of a particle in XYZ is given by $\mathbf{r}_{XYZ}$ and in xyz the particle's position is given by $\mathbf{r}_{xyz}$. The two position vectors are related by the position vector of the origin of xyz in XYZ, $\mathbf{r}_{xyz/XYZ}$,

$$\mathbf{r}_{XYZ} = \mathbf{r}_{xyz/XYZ} + \mathbf{r}_{xyz}. \tag{4.57}$$

The velocity of the particle in XYZ can be found by taking the time derivative of the position vector, $\mathbf{r}_{XYZ}$, with respect to XYZ (as indicated by the subscript XYZ in the following equation),

$$\left.\frac{d\mathbf{r}_{XYZ}}{dt}\right|_{XYZ} = \left.\frac{d\mathbf{r}_{xyz/XYZ}}{dt}\right|_{XYZ} + \left.\frac{d\mathbf{r}_{xyz}}{dt}\right|_{XYZ}. \tag{4.58}$$

The time derivative of $\mathbf{r}_{xyz/XYZ}$ is simply the velocity of the origin of xyz with respect to XYZ, $\mathbf{u}_{xyz/XYZ}$,

$$\left.\frac{d\mathbf{r}_{xyz/XYZ}}{dt}\right|_{XYZ} = \mathbf{u}_{xyz/XYZ}. \tag{4.59}$$

We must be careful, however, when evaluating the time derivative of $\mathbf{r}_{xyz}$ in XYZ since both the magnitude of $\mathbf{r}_{xyz}$ and the basis vectors of xyz can change with time (the basis vectors of xyz can change due to rotation of the xyz with respect to XYZ). To calculate the time derivative of $\mathbf{r}_{xyz}$ in XYZ, let's first write $\mathbf{r}_{xyz}$ as a magnitude, r_{xyz}, multiplied by the basis vectors of xyz, $\hat{\mathbf{e}}_{xyz}$, then use the product rule to evaluate the time derivative,

$$\left.\frac{d\mathbf{r}_{xyz}}{dt}\right|_{XYZ} = \left.\frac{d(r_{xyz}\hat{\mathbf{e}}_{xyz})}{dt}\right|_{XYZ} = \left.\frac{dr_{xyz}}{dt}\right|_{XYZ}\hat{\mathbf{e}}_{xyz} + r_{xyz}\left.\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\right|_{XYZ}. \tag{4.60}$$

Note that,

$$\left.\frac{dr_{xyz}}{dt}\right|_{XYZ}\hat{\mathbf{e}}_{xyz} = \mathbf{u}_{xyz}, \tag{4.61}$$

is the velocity of the particle in xyz.

The time derivative of the xyz basis vectors is found from geometric considerations. Consider the drawing shown in Figure 4.11 illustrating the change in the x-basis vector as a function of time. For simplicity, we'll assume that the rotation only occurs in the xy plane, i.e., $\Delta\theta_x = \Delta\theta_y = 0$. The time derivative of the basis vector is,

$$\frac{d\hat{\mathbf{e}}_x}{dt} = \lim_{\Delta t\to 0}\frac{\hat{\mathbf{e}}_x(t+\Delta t) - \hat{\mathbf{e}}_x(t)}{\Delta t}. \tag{4.62}$$

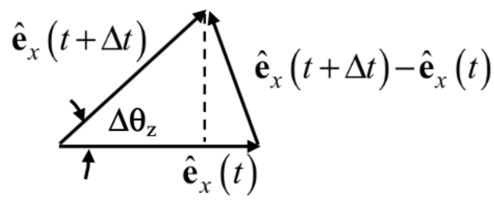


Figure 4.11. A schematic showing how the $\hat{\mathbf{e}}_x$ basis vector changes due to rotation in the xy plane.

Note from the figure that,

$$\hat{\mathbf{e}}_x(t+\Delta t) - \hat{\mathbf{e}}_x(t) = [\hat{\mathbf{e}}_x(t)\cos\Delta\theta_z + \hat{\mathbf{e}}_y(t)\sin\Delta\theta_z] - \hat{\mathbf{e}}_x(t), \tag{4.63}$$

$$= \hat{\mathbf{e}}_x(t)(\cos\Delta\theta_z - 1) + \hat{\mathbf{e}}_y(t)\sin\Delta\theta_z. \tag{4.64}$$

In addition, as $\Delta t \to 0$, $\Delta\theta_z \to 0$ and,

$$(\cos\Delta\theta_z - 1) \approx [1-(\Delta\theta_z)^2/2] - 1 = -\frac{1}{2}(\Delta\theta_z)^2 \quad \text{and} \quad \sin\Delta\theta_z \approx \Delta\theta_z, \tag{4.65}$$

so that,

$$\frac{d\hat{\mathbf{e}}_x}{dt} = \lim_{\Delta t\to 0}\frac{\hat{\mathbf{e}}_x(t+\Delta t)-\hat{\mathbf{e}}_x(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{-\frac{1}{2}(\Delta\theta_z)^2\hat{\mathbf{e}}_x(t)+\Delta\theta_z\hat{\mathbf{e}}_y}{\Delta t}, \tag{4.66}$$

$$= \frac{d\theta_z}{dt}\hat{\mathbf{e}}_y \quad (\text{since } \Delta\theta_z \ll 1), \tag{4.67}$$

$$\therefore \frac{d\hat{\mathbf{e}}_x}{dt} = \omega_z\hat{\mathbf{e}}_y \quad \text{where} \quad \omega_z = \frac{d\theta_z}{dt}. \tag{4.68}$$

In general, it can be shown that,

$$\left.\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\right|_{XYZ} = \boldsymbol{\omega}_{xyz/XYZ} \times \hat{\mathbf{e}}_{xyz}, \tag{4.69}$$

so that,

$$r_{xyz}\left.\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\right|_{XYZ} = r_{xyz}(\boldsymbol{\omega}_{xyz/XYZ} \times \hat{\mathbf{e}}_{xyz}) = \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}. \tag{4.70}$$

Combining Eqs. (4.58) - (4.61) and (4.70), we find that the velocity of a fluid particle in the inertial coordinate system XYZ is,

$$\boxed{\underbrace{\mathbf{u}_{XYZ}}_{\substack{\text{velocity of particle}\\ \text{in } XYZ}} = \underbrace{\mathbf{u}_{xyz/XYZ}}_{\substack{\text{velocity of } xyz\\ \text{w/r/t } XYZ}} + \underbrace{\mathbf{u}_{xyz}}_{\substack{\text{velocity of particle}\\ \text{in } xyz}} + \underbrace{\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}}_{\substack{\text{velocity of particle in } XYZ\\ \text{due to rotation of } xyz\\ \text{w/r/t } XYZ}}}, \tag{4.71}$$

where $\mathbf{u}_{xyz}$ is the particle velocity in non-inertial coordinate system xyz, $\boldsymbol{\omega}_{xyz/XYZ}$ is the angular velocity of xyz with respect to XYZ, and $\mathbf{r}_{xyz}$ is the position vector of the particle from the origin of xyz.

The acceleration of a particle in XYZ in terms of xyz quantities can be found in a similar manner,

$$\underbrace{\left.\frac{d\mathbf{u}_{XYZ}}{dt}\right|_{XYZ}}_{=\mathbf{a}_{XYZ}} = \underbrace{\left.\frac{d\mathbf{u}_{xyz/XYZ}}{dt}\right|_{XYZ}}_{=\mathbf{a}_{xyz/XYZ}} + \underbrace{\left.\frac{d\mathbf{u}_{xyz}}{dt}\right|_{XYZ}}_{=\frac{d}{dt}(u_{xyz}\hat{\mathbf{e}}_{xyz})\big|_{XYZ}} + \underbrace{\left.\frac{d}{dt}(\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\right|_{XYZ}}_{=\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\frac{d(r_{xyz}\hat{\mathbf{e}}_{xyz})}{dt}\big|_{XYZ}}, \tag{4.72}$$

where the results from Eqs. (4.60), (4.61), (4.69), and (4.70) are used to simplify the last two expressions in Eq. (4.72),

$$\left.\frac{d}{dt}(u_{xyz}\hat{\mathbf{e}}_{xyz})\right|_{XYZ} = \frac{du_{xyz}}{dt}\hat{\mathbf{e}}_{xyz} + u_{xyz}\frac{d\hat{\mathbf{e}}_{xyz}}{dt}, \tag{4.73}$$

$$= \mathbf{a}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz}. \tag{4.74}$$

and,

$$\boldsymbol{\omega}_{xyz/XYZ} \times \left.\frac{d(r_{xyz}\hat{\mathbf{e}}_{xyz})}{dt}\right|_{XYZ} = \boldsymbol{\omega}_{xyz/XYZ} \times (\mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}), \tag{4.75}$$

$$= \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}). \tag{4.76}$$

Substituting Eqs. (4.74) and (4.76) into Eq. (4.72) and simplifying gives,

$$\boxed{\begin{aligned} \underbrace{\mathbf{a}_{XYZ}}_{\substack{\text{rectilinear acceleration}\\ \text{of particle in } XYZ}} &= \underbrace{\mathbf{a}_{xyz/XYZ}}_{\substack{\text{rectilinear acceleration}\\ \text{of } xyz \text{ w/r/t } XYZ}} + \underbrace{\mathbf{a}_{xyz}}_{\substack{\text{rectilinear acceleration}\\ \text{of particle in } xyz}} + \underbrace{(\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz})}_{\substack{\text{tangential acceleration of}\\ \text{particle in } XYZ \text{ due to}\\ \text{rotational acceleration of } xyz}} \\ &+ \underbrace{(2\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz})}_{\substack{\text{Coriolis acceleration of}\\ \text{particle in } XYZ \text{ due to}\\ \text{rectilinear motion of particle}\\ \text{in } xyz}} + \underbrace{[\boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})]}_{\substack{\text{centripetal acceleration of particle}\\ \text{in } XYZ \text{ due to rotation of } xyz}}. \end{aligned}} \tag{4.77}$$

Now let's use these relations to determine an expression for the LME using a non-inertial coordinate system. Recall that the Lagrangian statement for the LME is (refer to Eq. (4.37)),

$$\frac{D}{Dt}\int_{V_{\text{sys}}} \mathbf{u}_{XYZ}\rho dV = \mathbf{F}_{\text{on sys}}. \tag{4.78}$$

Substitute Eq. (4.71) into Eq. (4.78) and re-arrange,

$$\mathbf{F}_{\text{on sys}} = \frac{D}{Dt}\int_{V_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV, \tag{4.79}$$

$$= \frac{D}{Dt}\int_{V_{\text{sys}}} \mathbf{u}_{xyz}\rho dV + \frac{D}{Dt}\int_{V_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV. \tag{4.80}$$

Now use the Reynolds Transport Theorem to convert the first term on the right-hand side to a control volume and re-arrange,

$$\begin{aligned} \mathbf{F}_{B,CV} + \mathbf{F}_{S,CV} - \frac{D}{Dt}\int_{V_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV \\ = \frac{d}{dt}\int_{CV} \mathbf{u}_{xyz}\rho dV + \int_{CS} \mathbf{u}_{xyz}\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right). \end{aligned} \tag{4.81}$$

The remaining Lagrangian term can be simplified by changing the volume integral to a mass integral and noting that the mass of the system doesn't change with time,

$$\frac{D}{Dt}\int_{V_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV = \frac{D}{Dt}\int_{M_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})dm, \tag{4.82}$$

$$= \int_{M_{\text{sys}}} \frac{D}{Dt}(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})dm, \tag{4.83}$$

$$= \int_{V_{\text{sys}}} \frac{D}{Dt}(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV. \tag{4.84}$$

Since $\mathbf{u}_{xyz/XYZ}$ and $\boldsymbol{\omega}_{xyz/XYZ}$ are functions only of time (these variables describe the motion of the coordinate system xyz and not the fluid), and because $D\mathbf{r}_{xyz}/Dt = \mathbf{u}_{xyz}$[1], we can replace the Lagrangian time derivative with an Eulerian time derivative and substitute in our result from Eq. (4.77),

$$\int_{V_{\text{sys}}} \frac{D}{Dt}(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV = \int_{V_{\text{sys}}} \frac{d}{dt}(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV, \tag{4.85}$$

$$= \int_{V_{\text{sys}}} [\mathbf{a}_{xyz/XYZ} + \dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz} + 2\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})]\rho dV. \tag{4.86}$$

Substituting Eq. (4.86) back into Eq. (4.81) and noting that when we apply the Reynolds Transport Theorem the control volume and system volume are coincident (so that the system volume integral in Eq. (4.86) can be replaced by a control volume integral), we find that the LME can be applied using a non-inertial coordinate, xyz, if the following form is used,

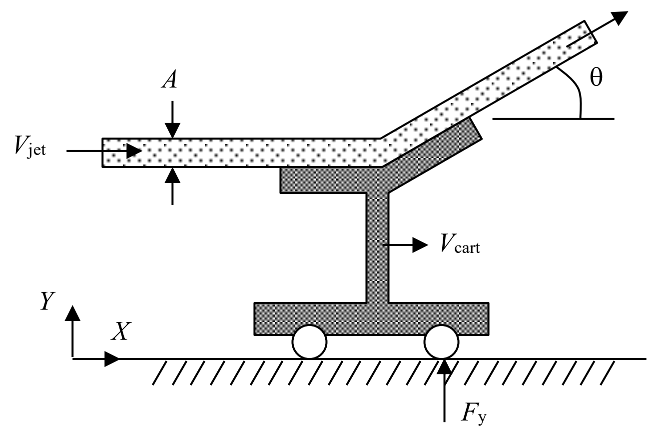
$$\boxed{\begin{aligned}&\mathbf{F}_{B,CV} + \mathbf{F}_{S,CV}\\ &- \int_{CV} \{\mathbf{a}_{xyz/XYZ} + (\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz}) + (2\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz}) + [\boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})]\}\rho dV\\ &= \frac{d}{dt}\int_{CV} \mathbf{u}_{xyz}\rho dV + \int_{CS} \mathbf{u}_{xyz}\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right).\end{aligned}} \tag{4.87}$$

This is the Linear Momentum Equation using a non-inertial (aka accelerating) coordinate system!

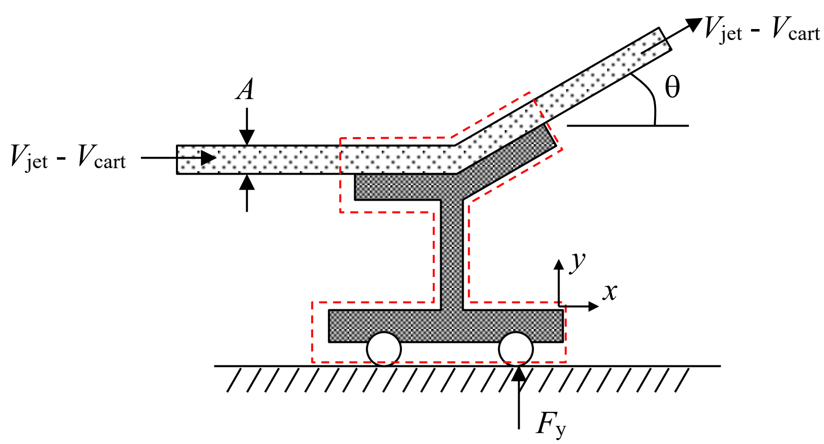
Let's consider a few examples to see how this form of the LME is applied.

[1] $\frac{D\mathbf{r}_{xyz}}{Dt} = \underbrace{\frac{\partial \mathbf{r}_{xyz}}{\partial t}}_{=0} + u_x \underbrace{\frac{\partial \mathbf{r}_{xyz}}{\partial x}}_{=\hat{\mathbf{e}}_x} + u_y \underbrace{\frac{\partial \mathbf{r}_{xyz}}{\partial y}}_{=\hat{\mathbf{e}}_y} + u_z \underbrace{\frac{\partial \mathbf{r}_{xyz}}{\partial z}}_{=\hat{\mathbf{e}}_z} = \mathbf{u}_{xyz}$ where $\mathbf{r}_{xyz} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$

A jet of water is deflected by a vane mounted on a cart. The water jet has an area, A, everywhere and is turned an angle θ with respect to the horizontal. The pressure everywhere within the jet is atmospheric. The incoming jet velocity with respect to the ground (axes XY) is V_{jet}. The cart has mass M. Determine the horizontal acceleration of the cart at the instant when the cart moves with velocity V_{cart} ($V_{cart}<V_{jet}$) if no horizontal forces are applied



SOLUTION:
Apply the linear momentum equation to a control volume surrounding the cart. Use a frame of reference fixed to the cart (xy). Note that this is not an inertial frame of reference since the cart is accelerating. As before, in this frame of reference the cart appears stationary and the jet velocity at the left is equal to V_{jet}-V_{cart}. From conservation of mass, the velocity on the right of the control volume is $V_{jet} - V_{cart}$.



Apply the linear momentum equation in the x-direction:

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = F_{B,x} + F_{S,x} - \int_{CV} a_{x/X} \rho dV \qquad (1)$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV \approx 0$$

(The cart has zero velocity in this frame of reference. The fluid in the control volume does accelerate in this frame of reference; however, its mass is assumed to be much smaller than the cart mass. Hence, the rate of change of the control volume momentum in this frame of reference is assumed to be zero.)

Video solution: https://www.youtube.com/watch?v=a9HZTu9x13I

$$\int_{CS} u_x(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \underbrace{\overbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)}^{=u_x}\left[\overbrace{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right]}_{\text{left side}} + \underbrace{\overbrace{\left(V_{\text{jet}}-V_{\text{cart}}\right)\cos\theta}^{=u_X}\left[\overbrace{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{A\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$= -\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2 A + \rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2 A\cos\theta\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{=1}$$

$$= \rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2 A(\cos\theta-1)$$

$F_{B,x} = 0$ (no body forces in the x-direction)

$F_{S,x} = 0$ (all of the pressure forces cancel out)

$\int_{CV} a_{x/X}\rho dV = Ma$ (the mass within the CV is approximately equal to the cart mass)

Substitute and re-arrange.

$$\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2 A(\cos\theta-1) = -Ma$$

$$\boxed{a = \frac{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2 A(1-\cos\theta)}{M}} \tag{2}$$

Now solve the problem using an inertial frame of reference fixed to the ground (frame XY). The linear momentum equation in the X direction gives:

$$\frac{d}{dt}\int_{CV} u_X \rho dV + \int_{CS} u_X \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = F_{B,X} + F_{S,X} \tag{3}$$

where,

$$\frac{d}{dt}\int_{CV} u_X \rho dV \approx Ma$$

(The mass within the control volume is approximately equal to the cart mass since the fluid mass is assumed to be negligible.)

$$\int_{CS} u_X(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \underbrace{\overbrace{V_{\text{jet}}}^{=u_X}\left[\overbrace{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)\hat{\mathbf{i}}}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right]}_{\text{left side}} + \underbrace{\overbrace{\left[\left(V_{\text{jet}}-V_{\text{cart}}\right)\cos\theta+V_{\text{cart}}\right]}^{=u_X}\left[\overbrace{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{u}_{\text{rel}}}\cdot\overbrace{A\left(\cos\theta\hat{\mathbf{i}}+\sin\theta\hat{\mathbf{j}}\right)}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$= -\rho V_{\text{jet}}\left(V_{\text{jet}}-V_{\text{cart}}\right)A + \rho\left[\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\cos\theta + V_{\text{cart}}\left(V_{\text{jet}}-V_{\text{cart}}\right)\right]A\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{=1}$$

$$= \rho\left[-V_{\text{jet}}^2 + V_{\text{jet}}V_{\text{cart}} + \left(V_{\text{jet}}-V_{\text{cart}}\right)^2\cos\theta + V_{\text{cart}}V_{\text{jet}} - V_{\text{cart}}^2\right]A$$

$$= \rho\left[\left(V_{\text{jet}}-V_{\text{cart}}\right)^2\cos\theta - \left(V_{\text{jet}}-V_{\text{cart}}\right)^2\right]A$$

$$= \rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2(\cos\theta-1)A$$

$F_{B,X} = 0$ (no body forces in the x-direction)

$F_{S,X} = 0$ (all of the pressure forces cancel out)

Substitute and re-arrange.

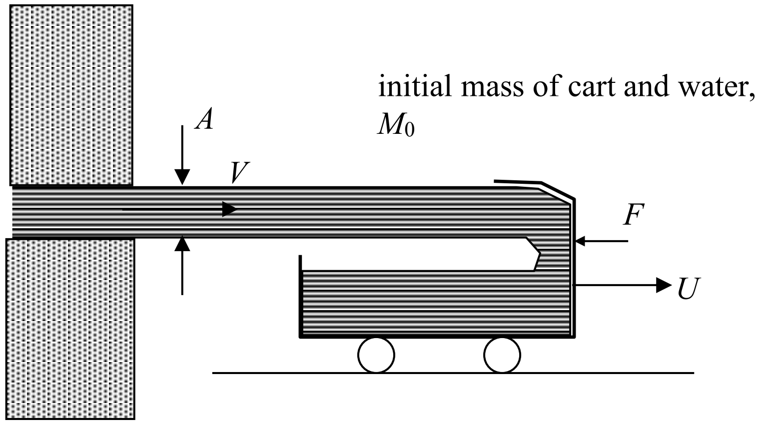
$$Ma + \rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2 A(\cos\theta-1) = 0$$

$$\boxed{a = \frac{\rho\left(V_{\text{jet}}-V_{\text{cart}}\right)^2 A(1-\cos\theta)}{M}} \quad \text{(Same answer as before!)} \tag{4}$$

Using a frame of reference that is fixed to the control volume is easier than using one fixed to the ground.

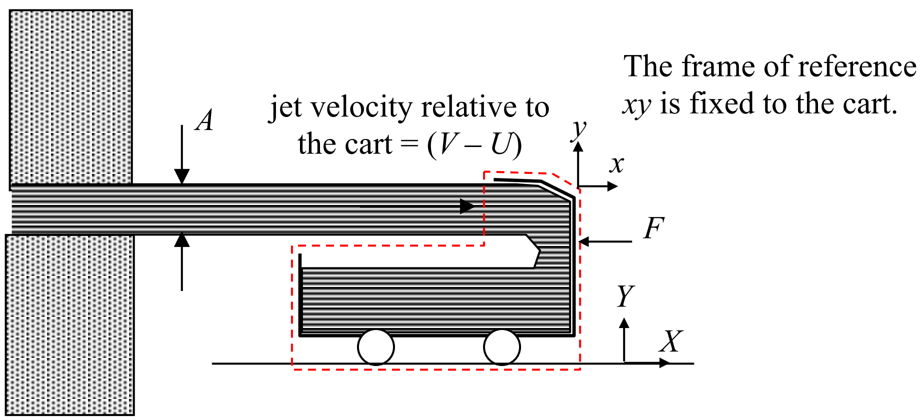
Video solution: https://www.youtube.com/watch?v=EBTjxf5tSfs

The tank shown rolls along a level track. Water received from a jet is retained in the tank. The tank is to accelerate from rest toward the right with constant acceleration, a. Neglect wind and rolling resistance. Find an algebraic expression for the force (as a function of time) required to maintain the tank acceleration at constant a.



SOLUTION:

First apply conservation of mass to a control volume surrounding the cart (shown below) in order to determine how the cart mass changes with time.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{dM_{CV}}{dt}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho\left(V-U\right)A$$

Substitute and re-arrange.

$$\frac{dM_{CV}}{dt} - \rho\left(V-U\right)A = 0$$

$$\frac{dM_{CV}}{dt} = \rho\left(V-U\right)A \tag{2}$$

Since the cart acceleration is constant (= a), we may write:

$$U = at \quad \text{(Note that } U(t=0)=0 \text{ since the cart starts from rest.)} \tag{3}$$

Note that Eqn. (3) is only true when a = constant. Otherwise, if $a = a(t)$ one must write the velocity as:

$$U = U_0 + \int_0^t a\,dt \tag{4}$$

Substitute Eqn. (3) into Eqn. (2) and solve the resulting differential equation.

$$\frac{dM_{\text{CV}}}{dt} = \rho\left(V - at\right)A \tag{5}$$

$$\int_{M_{\text{CV}}=M_0}^{M_{\text{CV}}=M_{\text{CV}}} dM_{\text{CV}} = \int_{t=0}^{t=t} \rho\left(V - at\right)A\,dt$$

$$M_{\text{CV}} - M_0 = \rho\left(Vt - \tfrac{1}{2}at^2\right)A$$

$$M_{\text{CV}} = M_0 + \rho\left(Vt - \tfrac{1}{2}at^2\right)A \tag{6}$$

Now apply the linear momentum equation in the x direction to the same control volume. Note that the frame of reference xy is not inertial since the cart is accelerating.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho\,dV + \int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{\text{CV}} a_{x/X}\rho\,dV \tag{7}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho\,dV \approx 0 \quad \text{(most of the mass inside the CV has zero velocity in the given frame of reference)}$$

$$\int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\rho\left(V-U\right)^2 A$$

$$F_{B,x} = 0$$

$$F_{S,x} = -F$$

$$\int_{\text{CV}} a_{x/X}\rho\,dV = aM_{\text{CV}}$$

Substitute and re-arrange.

$$-\rho\left(V-U\right)^2 A = -F - aM_{\text{CV}}$$

$$F = \rho\left(V-U\right)^2 A - aM_{\text{CV}} \tag{8}$$

Now substitute Eqns. (3) and (6) into Eqn. (8).

$$\boxed{F = \rho\left(V-at\right)^2 A - a\left[M_0 + \rho\left(Vt - \tfrac{1}{2}at^2\right)A\right]} \tag{9}$$

Now let's solve the problem using a frame of reference fixed to the ground (*XYZ* - inertial).

$$\frac{d}{dt}\int_{\mathrm{CV}} u_X \rho dV + \int_{\mathrm{CS}} u_X \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} u_X \rho dV = \frac{d}{dt}\left(M_{\mathrm{CV}} U\right) = M_{\mathrm{CV}}\frac{dU}{dt} + U\frac{dM_{\mathrm{CV}}}{dt}$$

$$\int_{\mathrm{CS}} u_X \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = (V)\left[-\rho(V-U)A\right] = -\rho V(V-U)A$$

$$F_{B,X} = 0$$

$$F_{S,X} = -F$$

Substitute and utilize Eqn. (5) to simplify.

$$M_{\mathrm{CV}}\frac{dU}{dt} + U\frac{dM_{\mathrm{CV}}}{dt} - \rho V(V-U)A = -F$$

$$M_{\mathrm{CV}}\frac{dU}{dt} + U\rho(V-U)A - \rho V(V-U)A = -F$$

$$F = -aM_{\mathrm{CV}} - U\rho(V-U)A + \rho V(V-U)A$$

$$F = \rho(V-U)^2 A - aM_{\mathrm{CV}} \tag{10}$$

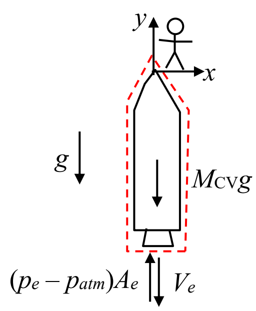
Eqn. (10) is identical to Eqn. (8) as expected!

A model solid propellant rocket has a mass of 69.6 gm, of which 12.5 gm is fuel. The rocket produces 1.3 lb_f of thrust for a duration of 1.7 sec. For these conditions, calculate the maximum speed and height attainable in the absence of air resistance. Plot the rocket speed and the distance traveled as functions of time.

SOLUTION:

Assume that the mass flow rate from the rocket is constant. Also assume that the thrust remains constant over the burn duration.

Apply the linear momentum equation in the y-direction to the CV shown using a frame of reference attached to the rocket.



$$\frac{d}{dt}\int_{CV} u_y \rho dV + \int_{CS} u_y\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = F_{B,y} + F_{S,y} - \int_{CV} a_{y/Y}\rho dV$$

where,

$$\frac{d}{dt}\int_{CV} u_y \rho dV \approx 0$$ (Most of the fluid has zero velocity in this frame of reference.)

$$\int_{CS} u_y\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = -V_e\left(\rho_e V_e A_e\right) = -\rho_e V_e^2 A_e$$

$$F_{B,y} = -M_{CV}g$$ (weight)

$$F_{S,y} = \left(p_e - p_{atm}\right)A_e$$ (The exit pressure may be different from atmospheric pressure.)

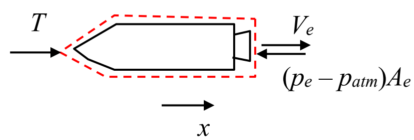
$$\int_{CV} a_{y/Y}\rho dV = aM_{CV}$$ (We're using an accelerating frame of reference.)

Substituting and simplifying:

$$-\rho_e V_e^2 A_e = -M_{CV}g + \left(p_e - p_{atm}\right)A_e - M_{CV}a$$

$$a = -g + \frac{\rho_e V_e^2 A_e + \left(p_e - p_{atm}\right)A_e}{M_{CV}} \tag{1}$$

Note that the thrust, T, is the force required to hold the rocket stationary (neglecting gravity).



$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV \approx 0 \quad \text{(Most of the fluid has zero } x\text{-velocity.)}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = V_e\left(\rho_e V_e A_e\right) = \rho_e V_e^2 A_e$$

$$F_{B,x} = 0$$

$$F_{S,x} = -\left(p_e - p_{atm}\right)A_e + T$$

Substituting and simplifying:

$$\rho_e V_e^2 A_e = -\left(p_e - p_{atm}\right)A_e + T$$

$$T = \rho_e V_e^2 A_e + \left(p_e - p_{atm}\right)A_e \tag{2}$$

Substitute Eqn. (2) into Eqn. (1):

$$a = -g + \frac{T}{M_{CV}} \tag{3}$$

Apply COM to the same CV:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = 0$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{dM_{CV}}{dt}$$

$$\int_{\text{CS}} \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = \rho_e V_e A_e = \dot{m}$$

Substituting and simplifying:

$$\frac{dM_{CV}}{dt} + \dot{m} = 0 \tag{4}$$

Assuming the mass flow rate is a constant, solve Eqn. (4) subject to initial conditions:

$$\int_{M_0}^{M_{CV}} dM_{CV} = -\dot{m}\int_0^t dt$$

$$M_{CV} = M_0 - \dot{m}t \tag{5}$$

where M_0 is the initial mass of the CV.

Substitute Eqn. (5) into Eqn. (3) and solve the differential equation for the velocity:

$$a = \frac{dU}{dt} = -g + \frac{T}{M_0 - \dot{m}t}$$

$$\int_0^U dU = \int_0^t -g\,dt + \int_0^t \frac{T\,dt}{M_0 - \dot{m}t}$$

$$U = -gt - \frac{T}{\dot{m}}\ln\left(\frac{M_0 - \dot{m}t}{M_0}\right)$$

$$U = -gt - \frac{T}{\dot{m}}\ln\left(1 - \frac{\dot{m}t}{M_0}\right) \tag{6}$$

Solve the differential equation given in Eqn. (6) for the height of the rocket.

$$U = \frac{dh}{dt} = -gt - \frac{T}{\dot{m}}\ln\left(1 - \frac{\dot{m}t}{M_0}\right)$$

$$\int_0^h dh = \int_0^t -gt\,dt - \int_0^t \frac{T}{\dot{m}}\ln\left(1 - \frac{\dot{m}t}{M_0}\right)dt$$

$$h = -\tfrac{1}{2}gt^2 + \frac{T}{\dot{m}}\left[\frac{M_0}{\dot{m}}\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - t\ln\left(1 - \frac{\dot{m}t}{M_0}\right) + t\right] \tag{7}$$

Note that Eqns. (3), (5), (6), and (7) are written specifically for when the fuel is burning. When the fuel has been expended, the rocket equations of motion are:

$$a = -g \tag{8}$$

$$U = U_{t=t'} - g\left(t - t'\right) \tag{9}$$

$$h = -\tfrac{1}{2}g\left(t - t'\right)^2 + U_{t=t'}\left(t - t'\right) + h_{t=t'} \tag{10}$$

where t' is the time at which the fuel has been expended.

For the given problem we're told:

M_0 = 69.6 g
M_{fuel} = 12.5 g
T = 1.3 lb$_f$ = 5.79 N
t' = 1.7 sec

giving a mass flow rate of:

$$\dot{m} = \frac{M_{\text{fuel}}}{t'} = 7.35 \text{ g/sec} = 7.35{*}10^{-3} \text{ kg/sec}$$

The maximum velocity will occur at the moment the fuel has been expended (neglecting the velocities as the rocket falls back to the ground). The maximum height will occur when the velocity is zero.

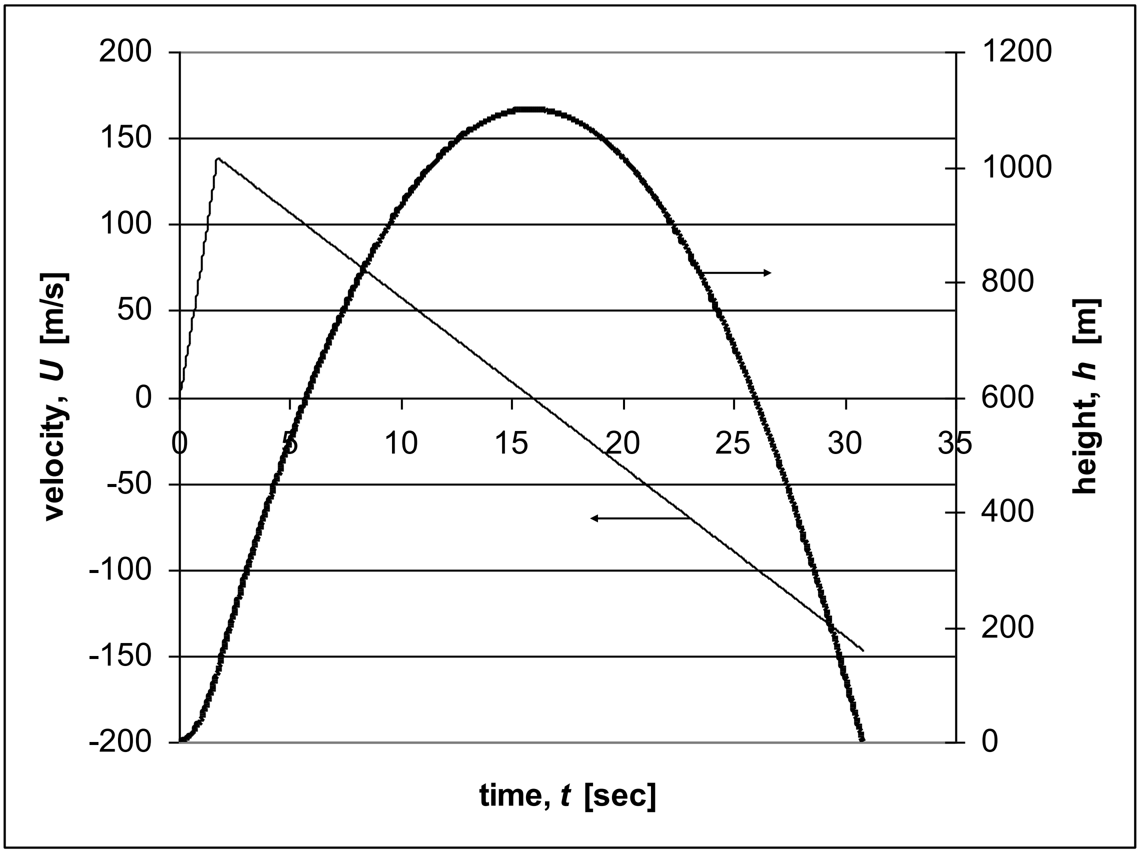
$U_{\max}$ = $U(t = t' = 1.7$ sec) = 139.2 m/s   ($h(t = t')$ = 114 m)
$h_{\max}$ = $h(t = t_m = 15.9$ sec) = 1100 m

The maximum height occurs when:

$$U = U_{t=t'} - g\left(t_m - t'\right) = 0$$

$$t_m = t' + \frac{U_{t=t'}}{g}$$

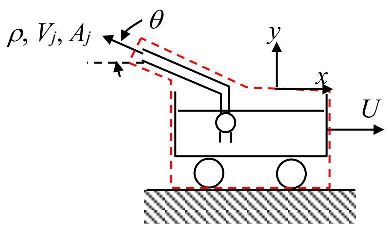
The rocket speed and height are plotted below:



A cart with frictionless wheels holds a water tank, motor, pump, and nozzle. The cart is on horizontal ground and initially still. At time zero the cart has a mass M_0 and the pump is started to produce a jet of water with constant area A_j, velocity V_j at an angle θ with respect to the horizontal. Find and solve the equations governing the mass and velocity of the cart as a function of time.

SOLUTION:

Apply the linear momentum equation in the x-direction to a control volume surrounding the cart. Use a frame of reference fixed to the control volume (non-inertial).



The frame of reference is fixed to the (accelerating) control volume and, hence, is non-inertial.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{\text{CV}} a_{x/X} \rho dV$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV \approx 0$$

(Using the given FOR, the rate of change of the CV linear momentum is nearly zero since most of the mass in the CV has a constant (=0) horizontal velocity.)

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \left(-V_j \cos\theta\right)\left(\rho V_j A_j\right) = -\rho V_j^2 A_j \cos\theta$$

$$F_{B,x} = 0$$

$$F_{S,x} = 0$$

$$\int_{\text{CV}} a_{x/X} \rho dV = M_{\text{CV}} \frac{dU}{dt}$$ (Note that the CV mass changes with time.)

Substitute and solve for the cart acceleration.

$$\frac{dU}{dt} = \frac{\rho V_j^2 A_j \cos\theta}{M_{\text{CV}}} \tag{1}$$

Determine the mass inside the control volume using conservation of mass applied to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{dM_{\text{CV}}}{dt}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \rho V_j A_j$$

Substitute and solve for M_{CV}.

$$\frac{dM_{\text{CV}}}{dt} = -\rho V_j A_j$$

$$\int_{M_0}^{M_{\text{CV}}} dM_{\text{CV}} = -\rho V_j A_j \int_0^t dt$$ (Note that $\rho V_j A_j$ is constant with respect to time.)

$$\boxed{M_{\text{CV}} = M_0 - \rho V_j A_j t} \tag{2}$$

Substitute Eqn. (2) into Eqn. (1) and solve for U.

$$\frac{dU}{dt} = \frac{\rho V_j^2 A_j \cos\theta}{M_0 - \rho V_j A_j t}$$

$$\int_0^U dU = \int_0^t \frac{\rho V_j^2 A_j \cos\theta dt}{M_0 - \rho V_j A_j t}$$

$$U = \frac{\rho V_j^2 A_j \cos\theta}{-\rho V_j A_j} \ln\left(\frac{M_0 - \rho V_j A_j t}{M_0}\right)$$

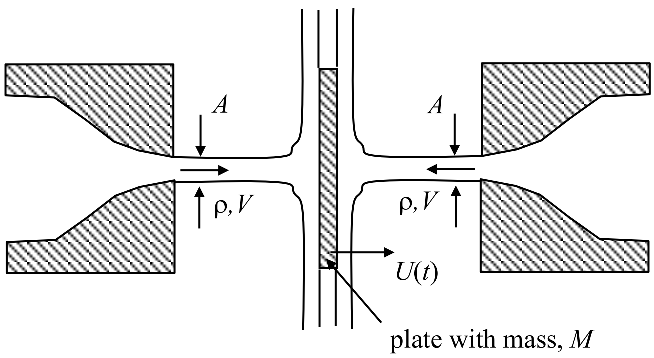
$$\boxed{\therefore U = -V_j \cos\theta \ln\left(1 - \frac{\rho V_j A_j t}{M_0}\right)} \tag{3}$$

A flat plate of mass, M, is located between two equal and opposite jets of liquid as shown in the figure. At time t=0, the plate is set into motion. Its initial speed is U_0 to the right; subsequently its speed is a function of time, $U(t)$. The motion is without friction and parallel to the jet axes. The mass of liquid that adheres to the plate is negligible compared to M.

Obtain algebraic expressions (as functions of time for t>0) for:

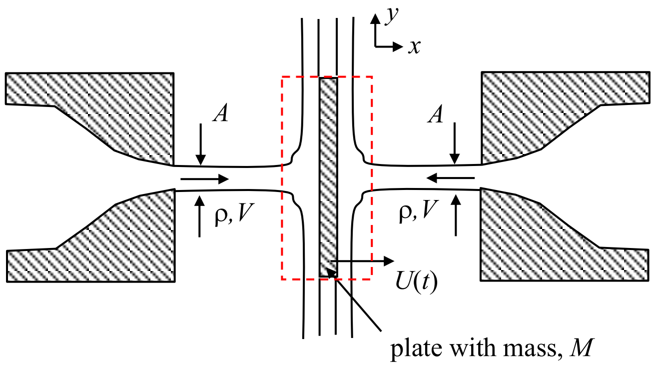
a. the velocity of the plate and
b. the acceleration of the plate.
c. What is the maximum displacement of the plate from its original position?

Express all of your answers in terms of (a subset of) U_0, V, A, ρ, M, and t.



SOLUTION:

Apply the linear momentum equation in the x-direction to a control volume that surrounds the plate as shown in the figure below. Use a frame of reference (FOR) that is fixed to the control volume (non-inertial).



$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{\text{CV}} a_{x/X} \rho dV \quad (1)$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV \approx 0 \quad \text{(The CV's } x\text{-linear momentum is approximately zero in the given FOR.)} \quad (2)$$

$$\begin{aligned}\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) &= \underbrace{\left[(V-U)\right]\left[-\rho(V-U)A\right]}_{\text{left side}} + \underbrace{\left[-(V+U)\right]\left[-\rho(V+U)A\right]}_{\text{right side}} \\ &= -\rho(V-U)^2 A + \rho(V+U)^2 A \\ &= \rho\left(-V^2 + 2UV - U^2 + V^2 + 2UV + U^2\right)A \\ &= 4\rho UVA\end{aligned} \tag{3}$$

$$F_{B,x} = F_{S,x} = 0 \quad \text{(No body or surface forces in the } x\text{-direction. The pressure everywhere is } p_{\text{atm}}.\text{)} \tag{4}$$

$$\int_{CV} a_{x/X} \rho dV \approx M \frac{dU}{dt} \quad \text{(Assume the plate mass is much larger than the water mass in the CV.)} \tag{5}$$

Substitute and simplify.

$$4\rho UVA = -M\frac{dU}{dt} \tag{6}$$

$$\therefore \frac{dU}{dt} = -\frac{4\rho UVA}{M} \tag{7}$$

$$\int_{U=U_0}^{U=U} \frac{dU}{U} = -\frac{4\rho VA}{M}\int_{t=0}^{t=t} dt \tag{8}$$

$$\ln\left(\frac{U}{U_0}\right) = -\frac{4\rho VAt}{M} \tag{9}$$

$$\boxed{\therefore \frac{U}{U_0} = \exp\left(-\frac{4\rho VAt}{M}\right)} \tag{10}$$

The acceleration is found by differentiating the velocity.

$$\boxed{a = \frac{dU}{dt} = -\frac{4\rho U_0 VA}{M}\exp\left(-\frac{4\rho VAt}{M}\right)} \tag{11}$$

The displacement of the plate is found by integrating the velocity in time.

$$U = \frac{dx}{dt} = U_0 \exp\left(-\frac{4\rho VAt}{M}\right) \tag{12}$$

$$\int_{x=0}^{x=x} dx = U_0 \int_{t=0}^{t=t} \exp\left(-\frac{4\rho VAt}{M}\right)dt \tag{13}$$

$$\therefore x = \frac{MU_0}{4\rho VA}\left[1 - \exp\left(-\frac{4\rho VAt}{M}\right)\right] \tag{14}$$

The maximum displacement occurs as $t \to \infty$.

$$\boxed{\therefore x_{\max} = \frac{MU_0}{4\rho VA}} \tag{15}$$

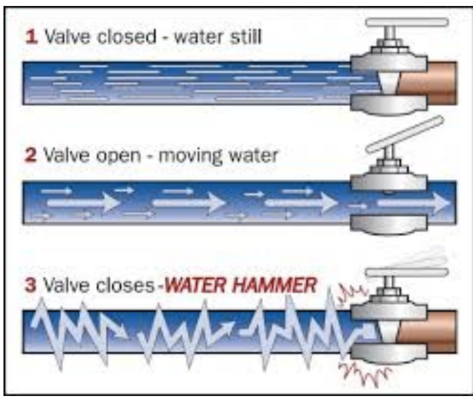
The pressure waves created by a rapid change of flow in a water line are referred to as water-hammers. To analyze the behavior of this phenomenon, consider a fluid flowing at speed U in a *rigid* pipe. The flow is stopped by a sudden closure of a valve. The pressure and the density of the fluid near the valve are suddenly increased by an amount Δp and $\Delta\rho$, respectively, and a pressure wave propagates upstream of the valve with speed, a.

a. Show that the increase in pressure, Δp, and the wave speed, a, are related by:

$$\Delta p = \rho U\left(U+a\right)$$

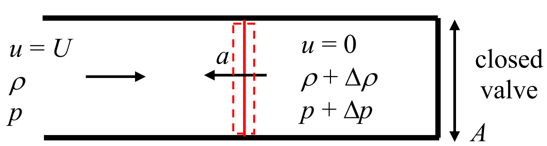
$$a\left(U+a\right)=\frac{\Delta p}{\Delta\rho}$$

b. The bulk modulus $K = \rho\,(dp/d\rho)$ is $43\text{x}10^6$ $\text{lb}_\text{f}/\text{ft}^2$ for water. Compute the wave speed a in a rigid pipe and Δp due to a sudden stoppage of water flowing with a speed of 1ft/s. You may assume that the pressure change across the wave is sufficiently weak to be considered an acoustic wave for the given conditions.

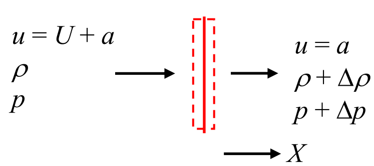


SOLUTION:

Apply conservation of mass and the linear momentum equation to a control volume surrounding the pressure wave.



Change the frame of reference so that wave appears stationary.



Apply conservation of mass to the control volume.

$$\frac{d}{dt}\int_{\text{CV}}\rho\,dV+\int_{\text{CS}}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}}\rho\,dV=0 \quad \text{(steady in the given frame of reference)} \tag{2}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho(U+a)A + (\rho+\Delta\rho)aA \tag{3}$$

Combine and simplify.

$$-\rho(U+a)A + (\rho+\Delta\rho)aA = 0 \tag{4}$$

$$\rho(U+a) = (\rho+\Delta\rho)a \tag{5}$$

Apply the linear momentum in the x-direction using an inertial frame of reference.

$$\frac{d}{dt}\int_{CV} u_X \rho\, dV + \int_{CS} u_X(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = F_{B,X} + F_{S,X} \tag{6}$$

where

$$\frac{d}{dt}\int_{CV} u_X \rho\, dV = 0 \quad \text{(steady in the given frame of reference)} \tag{7}$$

$$\int_{CS} u_X(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = -\rho(U+a)^2 A + (\rho+\Delta\rho)a^2 A \tag{8}$$

$$F_{B,X} = 0 \tag{9}$$

$$F_{S,X} = pA - (p+\Delta p)A \tag{10}$$

Combine and simplify.

$$-\rho(U+a)^2 A + (\rho+\Delta\rho)a^2 A = pA - (p+\Delta p)A \tag{11}$$

$$-\rho(U+a)^2 + (\rho+\Delta\rho)a^2 = -\Delta p \tag{12}$$

$$-\rho(U+a)^2 + \rho(U+a)a = -\Delta p \quad \text{(making use of Eq. (5))} \tag{13}$$

$$\rho(U+a)\left[(U+a)-a\right] = \Delta p \tag{14}$$

$$\boxed{\Delta p = \rho U(U+a)} \tag{15}$$

Note that if $U \ll a$, which is typically the case, then Eq. (15) becomes,

$$\Delta p = \rho U a \tag{16}$$

Re-arranging Eq. (15) to solve for ρ gives,

$$\rho = \frac{\Delta p}{\rho U(U+a)} \tag{17}$$

Substitute this relation into Eq. (5) and simplify.

$$(U+a) = \left(1 + \frac{\Delta\rho}{\rho}\right)a \tag{18}$$

$$(U+a) = \left[1 + \frac{U(U+a)\Delta\rho}{\Delta p}\right]a \tag{19}$$

$$\frac{(U+a)}{a} = 1 + U(U+a)\frac{\Delta\rho}{\Delta p} \tag{20}$$

$$\frac{\Delta\rho}{\Delta p} = \frac{1}{U(U+a)}\left[\frac{(U+a)}{a} - 1\right] \tag{21}$$

$$\frac{\Delta p}{\Delta\rho} = U(U+a)\left[\frac{a}{(U+a)-a}\right] \tag{22}$$

$$\boxed{\frac{\Delta p}{\Delta\rho} = a(U+a)} \tag{23}$$

Again, if $U \ll a$, then this relation becomes,

$$\frac{\Delta p}{\Delta \rho} = a^2 \tag{24}$$

In addition, if the wave is weak, meaning that the change in pressure and density across the wave are infinitesimally small, i.e., a sound wave, then Eq. (24) becomes,

$$\frac{dp}{d\rho} = a^2 \tag{25}$$

The bulk modulus is defined as,

$$K \equiv \rho \frac{dp}{d\rho}, \tag{26}$$

Since the wave is assumed to be an acoustic wave for the given conditions (refer to Eq. (25)),

$$a^2 = \frac{dp}{d\rho} \quad \Rightarrow \quad a = \sqrt{\frac{K}{\rho}} \tag{27}$$

The pressure change across the wave is found from Eq. (15). Using the given data,

K = 43*10^6 lb_f/ft^2

ρ = 1.94 slug/ft^3

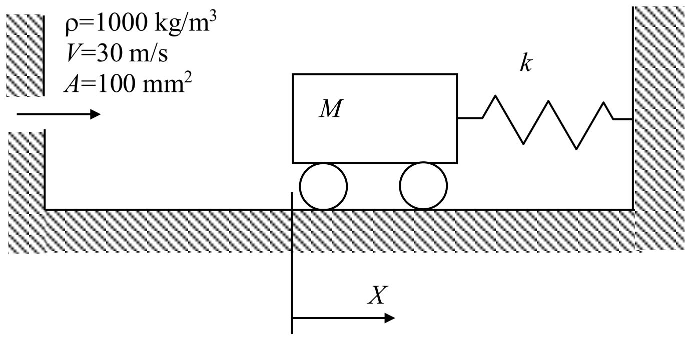
U = 1 ft/s

$\Rightarrow$ $\boxed{a = 4710 \text{ ft/s and } \Delta p = 9.14*10^3 \text{ psf} = 63.4 \text{ psi}}$

Note that $U \ll a$ and $d\rho/\rho \ll 1$, consistent with the assumption of an acoustic wave.

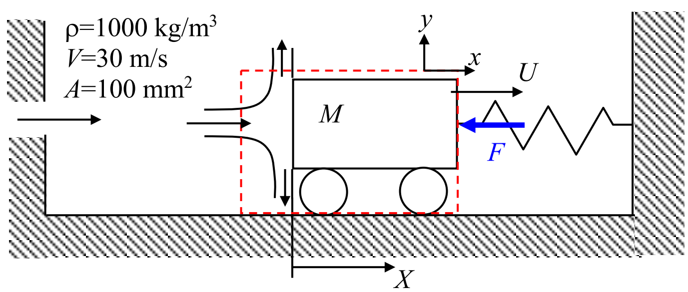
A block of mass, M=10 kg, with rectangular cross-section is arranged to slide with negligible friction along a horizontal plane. As shown in the sketch, the block is fastened to a spring that has stiffness such that $F=kx$ where k=500 N/m. The block is initially stationary. At time, t=0, a liquid jet begins to impinge on the block (the jet properties are also shown in the sketch). For t>0, the block moves laterally with speed, $U(t)$.

a. Obtain a differential equation valid for t>0 that could be solved for $U(t)$ and $X(t)$. Do not solve.
b. State appropriate boundary conditions for the differential equation of part (a).
c. Evaluate the final displacement of the block.



SOLUTION:

Apply the linear momentum equation in the x-direction to a control volume surrounding the block. Use a frame of reference that is fixed to the control volume (non-inertial).



$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{\text{CV}} a_{x/X} \rho dV$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV \approx 0$$

(Although the fluid mass in the CV will change its velocity with time (the block mass using the given FOR is always zero), this time rate of change of momentum within the CV will be very small compared to the other terms in COLM and can be reasonably neglected.)

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = (V-U)\left[-\rho(V-U)A\right] = -\rho(V-U)^2 A$$

$$F_{B,x} = 0$$

$$F_{S,x} = -kX$$

$$\int_{\text{CV}} a_{x/X} \rho dV = \frac{dU}{dt} M_{\text{CV}} \approx M\frac{dU}{dt}$$

(Assume the block mass is much greater than the water mass in the CV.)

Substitute and simplify.

$$-\rho\left(V-U\right)^2 A = -kX - M\frac{dU}{dt} \tag{1}$$

Note that:

$$U = \frac{dX}{dt} \text{ and } \frac{dU}{dt} = \frac{d^2X}{dt^2}$$

so that Eqn. (1) becomes:

$$-\rho\left(V-\frac{dX}{dt}\right)^2 A = -kX - M\frac{d^2X}{dt^2}$$

$$\boxed{\frac{d^2X}{dt^2} - \frac{\rho A}{M}\left(V-\frac{dX}{dt}\right)^2 + \frac{k}{M}X = 0} \tag{2}$$

Note that this is a non-linear 2nd order ODE.

The initial conditions for Eqn. (2) are:

$$\boxed{X\left(t=0\right)=0} \tag{3}$$

$$\boxed{\frac{dX}{dt}\left(t=0\right)=0} \tag{4}$$

The final position of the block occurs when the acceleration and velocity of the block are zero. From Eqn. (2) we have:

$$-\frac{\rho A}{M}V^2 + \frac{k}{M}X_f = 0$$

$$\boxed{X_f = \frac{\rho A}{k}V^2} \tag{5}$$

Note that we could have also worked this problem using an inertial frame of reference. Choose one that is fixed to the ground. Linear momentum in the X-direction using this new frame of reference gives:

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{CS}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV \approx M\frac{dU}{dt}$$ (Assume the block mass is much greater than the water mass in the CV.)

$$\begin{aligned}\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) &= \left(V\right)\left[-\rho\left(V-U\right)A\right] + \left(U\right)\underbrace{\left[\rho\left(V-U\right)A\right]}_{=\dot{m}_{\text{sides}}} \\ &= -V\rho\left(V-U\right)A + U\rho\left(V-U\right)A \\ &= -\rho\left(V-U\right)^2 A\end{aligned}$$

$$F_{B,x} = 0$$

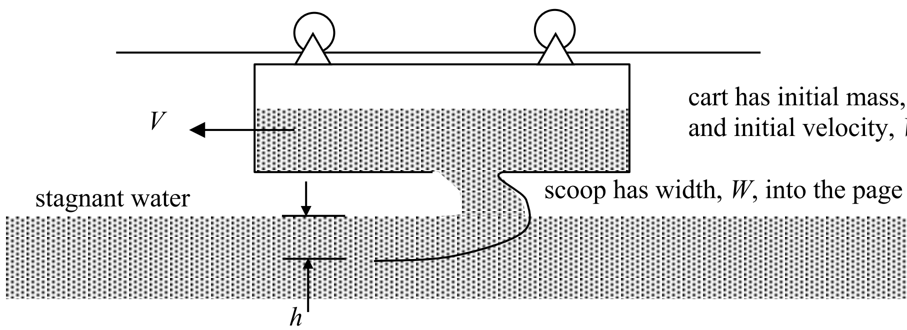
$$F_{S,x} = -kX$$

Substitute and simplify.

$$M\frac{dU}{dt} - \rho\left(V-U\right)^2 A = -kX$$

$$\frac{d^2X}{dt^2} - \frac{\rho A}{M}\left(V-\frac{dX}{dt}\right)^2 + \frac{k}{M}X = 0$$ (This is the same as Eqn. (2)!)

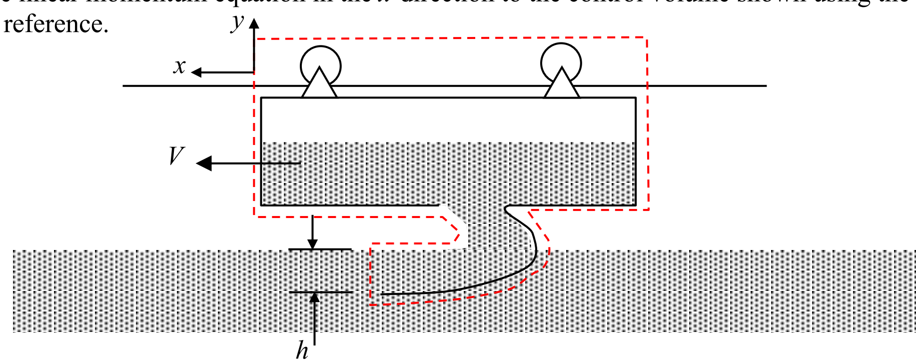
A cart hangs from a wire as shown in the figure below. Attached to the cart is a scoop of width W (into the page) which is submerged into the water a depth, h, from the free surface. The scoop is used to fill the cart tank with water of density, ρ.



a. Show that at any instant $V=V_0M_0/M$ where M is the mass of the cart and the fluid within the cart.
b. Determine the velocity, V, as a function of time.

SOLUTION:

Apply the linear momentum equation in the x-direction to the control volume shown using the indicated frame of reference.



$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{S,x} + F_{B,x} - \int_{\text{CV}} a_{x/X}\rho dV$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV \approx 0$$

(The x-linear momentum within the CV is approximately zero in the given frame of reference.)

$$\int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -V\rho\left(-V\right)hW = \rho V^2 hW$$

$F_{S,x} = 0$ (The pressure forces on the front and rear portions of the scoop cancel each other out.)

$F_{B,x} = 0$

$$\int_{\text{CV}} a_{x/X}\rho dV = \frac{dV}{dt}M$$

Substitute and simplify:

$$\rho V^2 hW = -\frac{dV}{dt}M \tag{1}$$

Apply conservation of mass to the same control volume in order to determine the mass as a function of time.

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{dM}{dt}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho VhW$$

Substitute and simplify:

$$\frac{dM}{dt} - \rho VhW = 0$$

$$\frac{dM}{dt} = \rho VhW \tag{2}$$

Substitute Eqn. (2) into Eqn. (1):

$$\frac{dM}{dt}V = -\frac{dV}{dt}M$$

$$\int_{M_0}^{M}\frac{dM}{M} = -\int_{V_0}^{V}\frac{dV}{V}$$

$$\ln\frac{M}{M_0} = -\ln\frac{V}{V_0} \Rightarrow \frac{M}{M_0} = \frac{V_0}{V}$$

$$\ln\frac{M}{M_0} = -\ln\frac{V}{V_0} \Rightarrow \frac{M}{M_0} = \frac{V_0}{V}$$

$$\boxed{\therefore V = V_0\frac{M_0}{M}} \tag{3}$$

To determine the cart velocity as a function of time, combine Eqns. (1) and (3):

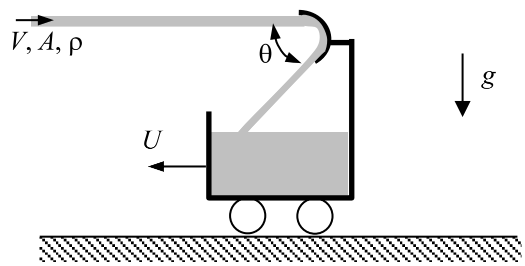
$$\rho V^2 hW = -\frac{dV}{dt}\frac{V_0}{V}M_0$$

$$\int_0^t dt = -\frac{M_0V_0}{\rho hW}\int_{V_0}^{V}\frac{dV}{V^3}$$

$$t = \frac{M_0V_0}{2\rho hW}\left(\frac{1}{V^2} - \frac{1}{V_0^2}\right) \Rightarrow V = \left(\frac{2\rho hWt}{M_0V_0} + \frac{1}{V_0^2}\right)^{-1/2}$$

$$\boxed{\frac{V}{V_0} = \frac{1}{\sqrt{\frac{2\rho hWV_0t}{M_0}+1}}} \tag{4}$$

A cart travels at velocity, U, toward a liquid jet that has a velocity, V, relative to the ground, a density, ρ, and a constant area, A. The mass of the cart and its contents at time $t = 0$ is M_0 and the cart's initial velocity is U_0 toward the jet. The resistance between the cart's wheels and the surface is negligible.

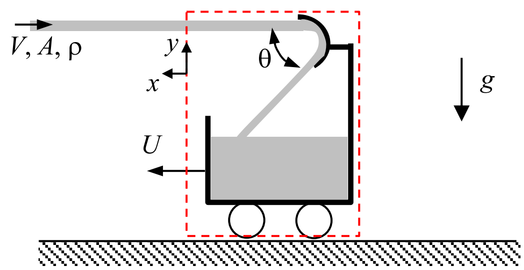


a. Determine the mass flow rate into the cart in terms of (a subset of) ρ, A, V, U, g, and θ.

b. Determine the acceleration of the cart, dU/dt, in terms of (a subset of) ρ, A, V, U, g, θ, and $M(t)$ where $M(t)$ is the mass of the cart and water at time t. You needn't solve any integrals or differential equations that appear in your answer.

SOLUTION:

Apply conservation of mass to a control volume surrounding the cart.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{dM}{dt} \tag{2}$$

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho\left(U+V\right)A \tag{3}$$

Note that the rate at which liquid mass *enters* the CV is $\boxed{\dot{m}_{\substack{\text{into}\\\text{cart}}} = -\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = \rho\left(U+V\right)A}$ (4)

Substitute and simplify.

$$\frac{dM}{dt} - \rho\left(U+V\right)A = 0 \tag{5}$$

$$\frac{dM}{dt} = \rho\left(U+V\right)A \tag{6}$$

Note that $U = U(t)$.

Now apply the linear momentum equation in the x-direction to the same control volume. Use a frame of reference fixed to the cart (non-inertial).

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{\text{CV}} a_{x/X} \rho dV \tag{7}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV \approx 0 \quad \text{(Most of the material in the CV has zero horz. velocity in this FOR.)} \tag{8}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \left[-\left(U+V\right)\right]\left[-\rho\left(U+V\right)A\right] = \rho\left(U+V\right)^2 A \tag{9}$$

$$F_{B,x} = 0 \tag{10}$$

$$F_{S,x} = 0 \tag{11}$$

$$\int_{\text{CV}} a_{x/X} \rho dV = \frac{dU}{dt} M \tag{12}$$

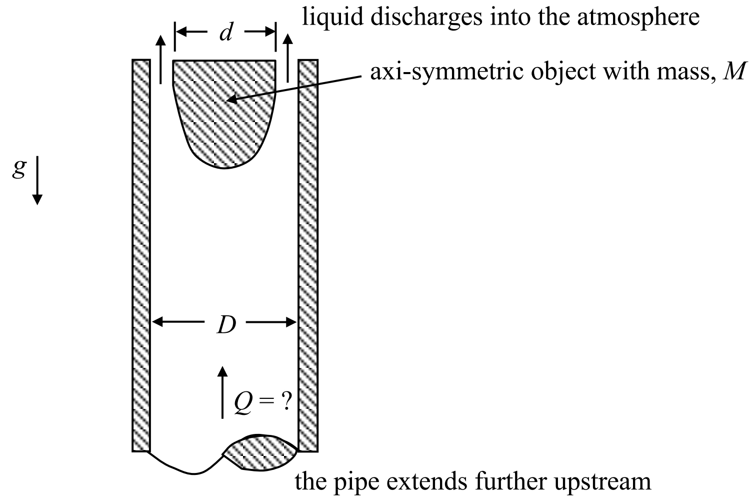
Substitute and simplify.

$$\rho\left(U+V\right)^2 A = -\frac{dU}{dt} M \tag{13}$$

$$\boxed{\frac{dU}{dt} = -\frac{\rho\left(U+V\right)^2 A}{M}} \tag{14}$$

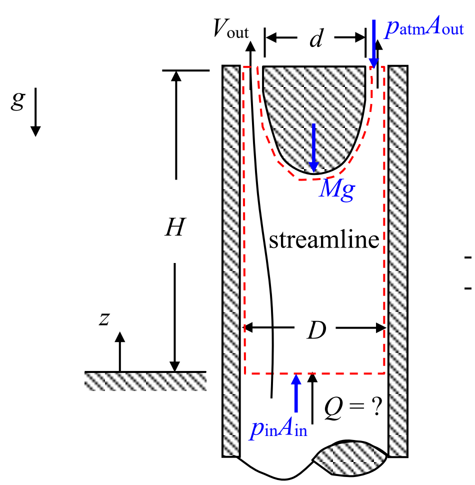
Note that $M = M(t)$ and $U = U(t)$. To solve for the motion of the cart, one would need to solve Eqns. (6) and (14) simultaneously subject to the initial conditions $M(t = 0) = M_0$ and $U(t = 0) = U_0$.

The axi-symmetric object shown below is placed in the end of a vertical circular pipe of inner diameter, D. A liquid with density, ρ, is pumped upward through the pipe and discharges to the atmosphere. Neglecting viscous effects, determine the volume flow rate, Q, of the liquid needed to support the object in the position shown in terms of d, D, g, ρ, and M.



SOLUTION:

Apply conservation of mass to the control volume shown below.



- Let: $A_{\text{in}} = \pi D^2/4$ and $A_{\text{out}} = \pi(D^2-d^2)/4$.
- Choose H such that it is much larger than the size of the object ($\Rightarrow V_{CV} \approx A_{\text{in}}H$).

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho Q + \rho V_{\text{out}} A_{\text{out}}$$

Substituting and simplifying gives:

$$-\rho Q + \rho V_{\text{out}} A_{\text{out}} = 0$$

$$V_{\text{out}} = \frac{Q}{A_{\text{out}}} \tag{2}$$

Apply the linear momentum equation in the z-direction to the same control volume. Use the fixed frame of reference shown in the figure.

$$\frac{d}{dt}\int_{\text{CV}} u_z \rho dV + \int_{\text{CS}} u_z \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,z} + F_{S,z} \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_z \rho dV = 0 \quad \text{(steady flow)}$$

$$\begin{aligned}\int_{\text{CS}} u_z \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) &= -\underbrace{\dot{m}}_{=\rho Q}\left(\frac{Q}{A_{\text{in}}}\right) + \dot{m} V_{\text{out}} \\ &= \rho Q\left(\frac{Q}{A_{\text{out}}} - \frac{Q}{A_{\text{in}}}\right) \\ &= \rho Q^2\left(\frac{A_{\text{in}} - A_{\text{out}}}{A_{\text{in}} A_{\text{out}}}\right)\end{aligned}$$

(Note that Eqn. (2) was used in simplifying the momentum flux term.)

$$F_{B,z} = -\rho V_{\text{CV}} g \approx -\rho A_{\text{in}} H g \quad (H \text{ is chosen to be much larger than the object size.})$$

$$F_{S,z} = p_{\text{in}} A_{\text{in}} - Mg \quad (\text{use gage pressures so } p_{\text{out}} = p_{\text{atm}} = 0)$$

Substitute and simplify.

$$\rho Q^2\left(\frac{A_{\text{in}} - A_{\text{out}}}{A_{\text{in}} A_{\text{out}}}\right) = -\rho A_{\text{in}} H g + p_{\text{in}} A_{\text{in}} - Mg \tag{4}$$

To determine p_{in}, apply Bernoulli's equation along a streamline from the inlet to the outlet.

$$\left(p + \tfrac{1}{2}\rho V^2 + \rho g z\right)_{\text{in}} = \left(p + \tfrac{1}{2}\rho V^2 + \rho g z\right)_{\text{out}} \tag{5}$$

where

$$p_{\text{in}} = ? \qquad p_{\text{out}} = 0 \quad \text{(gage pressure)}$$

$$V_{\text{in}} = \frac{Q}{A_{\text{in}}} \qquad V_{\text{out}} = \frac{Q}{A_{\text{out}}} \quad \text{(from Eqn. (2))}$$

$$z_{\text{in}} = 0 \qquad z_{\text{out}} = H$$

Substitute and simplify.

$$p_{\text{in}} = \tfrac{1}{2}\rho Q^2\left(\frac{1}{A_{\text{out}}^2} - \frac{1}{A_{\text{in}}^2}\right) + \rho g H \tag{6}$$

Substitute Eqn. (6) into Eqn. (4) and simplify.

$$\rho Q^2\left(\frac{A_{\text{in}}-A_{\text{out}}}{A_{\text{in}}A_{\text{out}}}\right)=-\rho A_{\text{in}}Hg+\left[\tfrac{1}{2}\rho Q^2\left(\frac{1}{A_{\text{out}}^2}-\frac{1}{A_{\text{in}}^2}\right)+\rho gH\right]A_{\text{in}}-Mg$$

$$=\tfrac{1}{2}\rho Q^2\left(\frac{1}{A_{\text{out}}^2}-\frac{1}{A_{\text{in}}^2}\right)A_{\text{in}}-Mg$$

$$\rho Q^2\left[\left(\frac{A_{\text{in}}-A_{\text{out}}}{A_{\text{in}}A_{\text{out}}}\right)-\tfrac{1}{2}\left(\frac{A_{\text{in}}^2-A_{\text{out}}^2}{A_{\text{in}}^2A_{\text{out}}^2}\right)A_{\text{in}}\right]=-Mg$$

$$\rho Q^2\left[\left(\frac{A_{\text{in}}-A_{\text{out}}}{A_{\text{in}}A_{\text{out}}}\right)-\tfrac{1}{2}\left(\frac{A_{\text{in}}^2-A_{\text{out}}^2}{A_{\text{in}}A_{\text{out}}^2}\right)\right]=-Mg$$

$$\rho Q^2\left(\frac{2A_{\text{in}}A_{\text{out}}-2A_{\text{out}}^2-A_{\text{in}}^2+A_{\text{out}}^2}{2A_{\text{in}}A_{\text{out}}^2}\right)=-Mg$$

$$\rho Q^2\left(\frac{-A_{\text{in}}^2+2A_{\text{in}}A_{\text{out}}-A_{\text{out}}^2}{2A_{\text{in}}A_{\text{out}}^2}\right)=-Mg$$

$$\rho Q^2\left[\frac{\left(A_{\text{in}}-A_{\text{out}}\right)^2}{2A_{\text{in}}A_{\text{out}}^2}\right]=Mg$$

$$\boxed{\therefore Q=\sqrt{\frac{Mg}{\rho\dfrac{\left(A_{\text{in}}-A_{\text{out}}\right)^2}{2A_{\text{in}}A_{\text{out}}^2}}}=\frac{1}{\left(\dfrac{A_{\text{in}}}{A_{\text{out}}}-1\right)}\sqrt{\frac{2MgA_{\text{in}}}{\rho}}}\quad\text{where }A_{\text{in}}=\frac{\pi D^2}{4}\quad\text{and}\quad A_{\text{out}}=\frac{\pi\left(D^2-d^2\right)}{4}\qquad(7)$$

4.5. Angular Momentum Equation (AME)

In words and in mathematical terms, the angular momentum principle for a system is,

$$\begin{gathered}\text{The rate of change of a system's angular momentum (AM)}\\ \text{is equal to the net moment (aka torque) acting on the system.}\end{gathered} \tag{4.88}$$

$$\implies \frac{D}{Dt}\int_{V_{\text{sys}}}(\mathbf{h}_{XYZ}+\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ})\rho dV=\mathbf{M}_{\text{on sys}}, \tag{4.89}$$

where D/Dt is the Lagrangian derivative (implying that we're using the rate of change as we follow the system), V is the volume, and ρ is the density. The quantity $\mathbf{h}_{XYZ}$ is the intrinsic specific angular momentum of a small piece of fluid resulting from the spin of fluid molecules contained within that small piece of fluid (Figure 4.12). In typical fluids (e.g., non-polar, non-magnetic fluids), the angular momentum vectors of the individual molecules are randomly oriented so that the sum of the intrinsic angular momentum vectors in a region containing many molecules is zero. Hence, we will neglect this contribution to the angular momentum of the fluid in the remainder of these notes. The quantity, $\mathbf{u}_{XYZ}$, represents the velocity of a small piece of fluid in the system with respect to an inertial (aka non-accelerating) coordinate system XYZ (recall that Newton's Second law holds strictly for inertial coordinate systems) and $\mathbf{r}_{XYZ}$ is the distance from the inertial coordinate system origin to the fluid element. Note that a coordinate system moving at a constant velocity in a straight line is non-accelerating and, thus, is inertial.

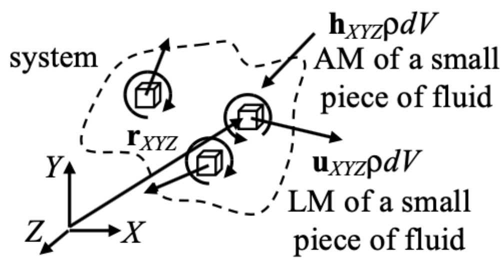


Figure 4.12. A schematic showing the angular momentum of a small fluid particle in a system.

The term, $\mathbf{M}_{\text{on sys}}$, represents the net moments (or torques) acting on the system. These moments can be due to both body and surface forces, i.e.,

$$\mathbf{M}_B=\mathbf{r}_{xyz}\times\mathbf{F}_B, \tag{4.90}$$

$$\mathbf{M}_S=\mathbf{r}_{xyz}\times\mathbf{F}_S. \tag{4.91}$$

Note that if the fluid is magnetic, it is also possible to have an additional body moment that would induce the fluid molecules to change their intrinsic angular momentum ($\mathbf{h}$). As stated before, we won't consider such fluids in these notes. The study of magnetic fluids is known as magnetohydrodynamics.

Using the Reynolds Transport Theorem to convert the left-hand side of Eq. (4.89) from a system point of view to an expression for a control volume gives,

$$\frac{D}{Dt}\int_{V_{\text{sys}}}(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ})\rho dV=\frac{d}{dt}\int_{CV}(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ})\rho dV+\int_{CS}(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ})\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right). \tag{4.92}$$

Since the Reynolds Transport Theorem is applied to a coincident system and control volume, the moments acting on the system will also act on the control volume. Thus,

$$\boxed{\underbrace{\frac{d}{dt}\int_{CV}(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ})\rho dV}_{\substack{\text{rate of increase of}\\ \text{AM inside of CV}}}+\underbrace{\int_{CS}(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ})\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)}_{\substack{\text{net rate at which the CV AM changes}\\ \text{due to fluid leaving through the CS}}}=\underbrace{\mathbf{M}_{B,CV}}_{\substack{\text{net moment due}\\ \text{to body forces}\\ \text{acting on CV}}}+\underbrace{\mathbf{M}_{S,CV}}_{\substack{\text{net moment due}\\ \text{to surface forces}\\ \text{acting on CV}}}.} \tag{4.93}$$

This is the Angular Momentum Equation (also known as the Moment of Momentum Equation) using an inertial coordinate system!

Notes:

(1) Recall that the AME is a vector expression. There are actually three equations built into Eq. (4.93).

4.5.1. AME for a Rotating (but not accelerating in translation) Coordinate System

Recall that Newton's Second law holds strictly for inertial (non-accelerating) coordinate systems. Often it is more convenient to use a rotating (non-inertial) coordinate system when applying the AME.

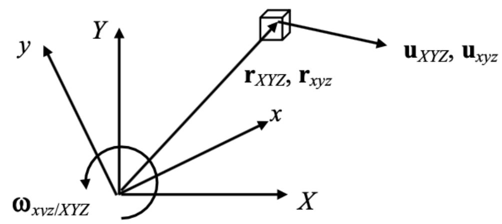


Figure 4.13. A schematic for deriving the kinematics of a particle using a rotating coordinate system.

The Lagrangian statement for the AME is (refer to Eq. (4.89)),

$$\frac{D}{Dt}\int_{V_{\text{sys}}}(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ})\rho dV = \mathbf{M}_{\text{on sys}}. \tag{4.94}$$

The mass of the system remains constant so the Lagrangian derivative can be brought inside the integral,

$$\frac{D}{Dt}\int_{V_{\text{sys}}}(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ})\rho dV = \int_{V_{\text{sys}}}\frac{D}{Dt}(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ})\rho dV. \tag{4.95}$$

Since we're considering a coordinate system that is only rotating and not accelerating in translation (Figure 4.13), the position and velocity vectors may be written as (refer to Eqs. (4.57) and (4.71)),

$$\mathbf{r}_{XYZ} = \mathbf{r}_{xyz}, \tag{4.96}$$

$$\mathbf{u}_{XYZ} = \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}. \tag{4.97}$$

Substituting into Eq. (4.95) gives,

$$\begin{aligned}
&\int_{V_{\text{sys}}}\frac{D}{Dt}[\mathbf{r}_{xyz}\times(\mathbf{u}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz})]\rho dV \\
&\quad= \int_{V_{\text{sys}}}\left\{\frac{D}{Dt}(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}) + \frac{D}{Dt}[\mathbf{r}_{xyz}\times(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz})]\right\}\rho dV && (4.98)\\
&\quad= \int_{V_{\text{sys}}}\frac{D}{Dt}(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz})\rho dV \\
&\quad+ \int_{V_{\text{sys}}}\left[\frac{D\mathbf{r}_{xyz}}{Dt}\times(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}) + \mathbf{r}_{xyz}\times\frac{D}{Dt}(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz})\right]\rho dV. && (4.99)
\end{aligned}$$

Again, since the mass of the system is constant, the Lagrangian derivative can be brought outside of the first integral. In addition, we know from previous work that $D\mathbf{r}_{xyz}/Dt = \mathbf{u}_{xyz}$ and,

$$\frac{D}{Dt}\left(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right) = \dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ}\times\left(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right). \tag{4.100}$$

Substituting and simplifying,

$$\begin{aligned}
&\int_{V_{\text{sys}}} \frac{D}{Dt}(\mathbf{r}_{xyz} \times \mathbf{u}_{xyz})\rho dV \\
&+ \int_{V_{\text{sys}}} \left[\frac{D\mathbf{r}_{xyz}}{Dt} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}) + \mathbf{r}_{xyz} \times \frac{D}{Dt}(\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\right] \rho dV, \\
&= \frac{D}{Dt}\int_{V_{\text{sys}}} (\mathbf{r}_{xyz} \times \mathbf{u}_{xyz})\rho dV \\
&+ \int_{V_{\text{sys}}} \left\{\begin{aligned}&\mathbf{u}_{xyz} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}) \\ &+ \mathbf{r}_{xyz} \times \left[\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\right] \rho dV\end{aligned}\right\}. \qquad (4.101)
\end{aligned}$$

To simplify things further, we can re-arrange the first term in the second integral and incorporate it into the second term,

$$\begin{aligned}
&\mathbf{u}_{xyz} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}) + \\
&\quad \mathbf{r}_{xyz} \times \left[\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\right] \\
&\quad = \mathbf{r}_{xyz} \times \left[\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz} + 2\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\right]. \qquad (4.102)
\end{aligned}$$

Now use the Reynolds Transport Theorem to convert the first term on the right-hand side to a control volume perspective and re-arrange. Also note that since the CV and system are coincident, the moments acting on the CV will be the same as the moments acting on the system and the CV mass will be the same as the system mass,

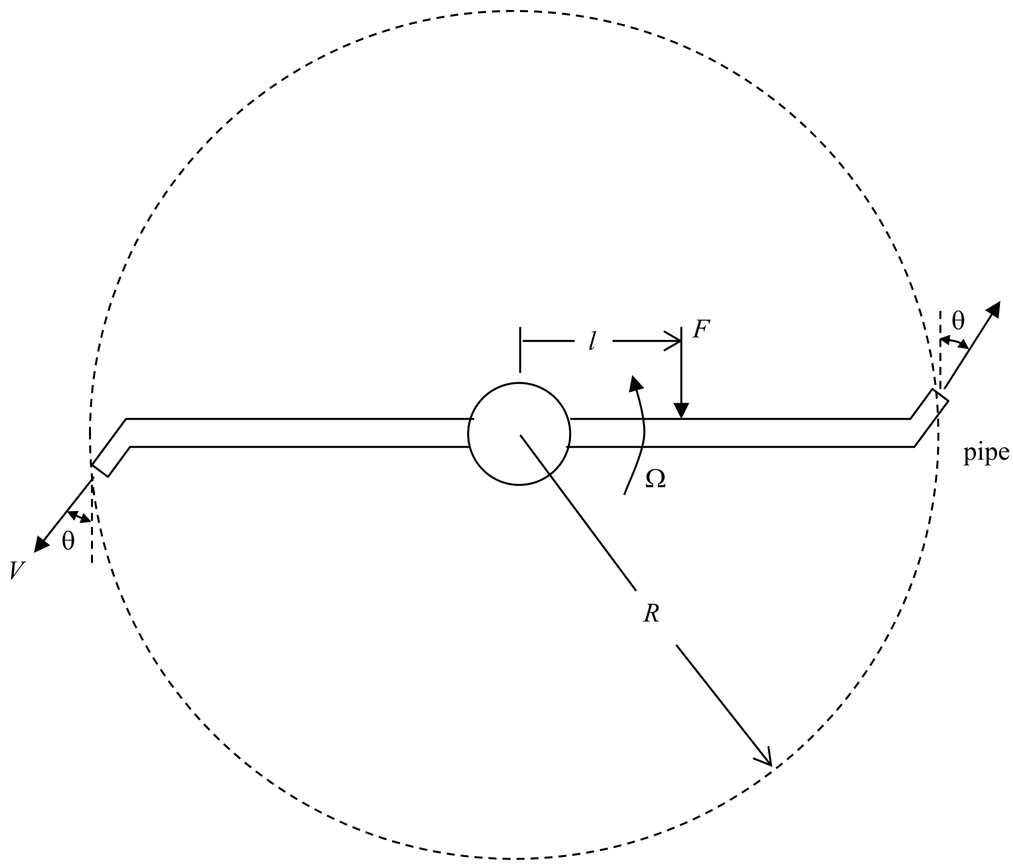
$$\boxed{\begin{aligned}
&\mathbf{M}_{B,CV} + \mathbf{M}_{S,CV} \\
&- \int_{CV} \left\{\mathbf{r}_{xyz} \times \left[\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz} + 2\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\right]\right\} \rho dV \\
&= \frac{d}{dt}\int_{CV} (\mathbf{r}_{xyz} \times \mathbf{u}_{xyz})\rho dV + \int_{CS} (\mathbf{r}_{xyz} \times \mathbf{u}_{xyz})\,(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}).
\end{aligned}} \qquad (4.103)$$

This is the Angular Momentum Equation using a rotating coordinate system!

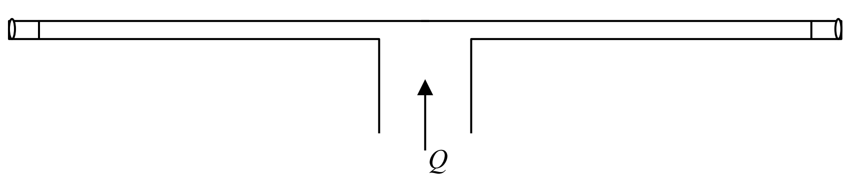
Let's consider some examples to see how this form of the AME is applied.

A lawn sprinkler is constructed from pipe with an inner diameter of d with each arm having a length of R. Water flows through the sprinkler at a volumetric flow rate of Q. A force, F, is applied a distance, l, from the sprinkler hub on one of the sprinkler arms. If the water stream leaving the sprinkler arm is at an angle θ with respect to the tangent of the circle traced out by the sprinkler arms, determine:

a. the force, F, required to hold the sprinkler stationary,
b. the force, F, required to have the sprinkler rotate at a constant angular velocity, Ω,
c. the angular acceleration of the sprinkler if the sprinkler's moment of inertia (including the fluid inside the sprinkler) is I and it is rotating with angular velocity Ω, and
d. the maximum angular velocity, Ω_{max}, of the sprinkler if no force is applied



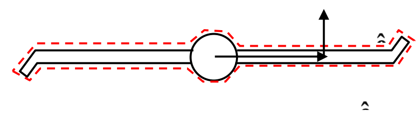
TOP VIEW



SIDE VIEW

SOLUTION:
First consider the case where the sprinkler does not rotate ($\Omega = 0$). For the fixed frame of reference and control volume shown, the moment of momentum equation is:

$$\frac{d}{dt}\int_{CV}(\mathbf{r}\times\mathbf{u})\rho dV + \int_{CS}(\mathbf{r}\times\mathbf{u})(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}) = \mathbf{M}_S + \mathbf{M}_B \quad (1)$$



where,

$$\frac{d}{dt}\int_{CV}(\mathbf{r}\times\mathbf{u})\rho dV = \mathbf{0} \quad \text{(steady flow)}$$

$$\int_{CS}(\mathbf{r}\times\mathbf{u})(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}) = \underbrace{2}_{\substack{\text{two}\\\text{arms}}}\left[R\hat{\mathbf{e}}_r \times V\left(\sin\theta\hat{\mathbf{e}}_r - \cos\theta\hat{\mathbf{e}}_\theta\right)\right]\left(\rho V\frac{\pi d^2}{4}\right)$$

$$= -2\rho R V^2\cos\theta\frac{\pi d^2}{4}\hat{\mathbf{e}}_z$$

$$\mathbf{M}_S = l\hat{\mathbf{e}}_r \times -F\hat{\mathbf{e}}_\theta = -Fl\hat{\mathbf{e}}_z$$

$$\mathbf{M}_B = \mathbf{0}$$

Substitute and simplify:

$$-2\rho R V^2\cos\theta\frac{\pi d^2}{4}\hat{\mathbf{e}}_z = -Fl\hat{\mathbf{e}}_z$$

$$\boxed{\therefore F = 2\rho V^2\frac{\pi d^2}{4}\cos\theta\left(\frac{R}{l}\right)} \quad (2)$$

Note that from conservation of mass on the same control volume:

$$Q = 2V\frac{\pi d^2}{4} \quad \Rightarrow \quad V = \frac{2Q}{\pi d^2} \quad (3)$$

If the sprinkler is rotating, then use the same control volume (attached to and surrounding the sprinkler arms) but use a coordinate system that rotates with the control volume. The moment of momentum equation for a rotating coordinate system is,

$$\frac{d}{dt}\int_{\text{CV}}(\mathbf{r}\times\mathbf{u})_{xyz}\,\rho dV+\int_{\text{CS}}(\mathbf{r}\times\mathbf{u})_{xyz}(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A})=\mathbf{M}_S+\mathbf{M}_B-\int_{\text{CV}}\mathbf{r}_{xyz}\times\left(\ddot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+2\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{u}_{xyz}+\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho dV$$

where,

$$\frac{d}{dt}\int_{\text{CV}}(\mathbf{r}\times\mathbf{u})_{xyz}\,\rho dV=\mathbf{0}\quad\text{(the flow is steady in the rotating frame of reference)}$$

$$\int_{\text{CS}}(\mathbf{r}\times\mathbf{u})_{xyz}(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A})=2\left[R\hat{\mathbf{e}}_r\times V\left(\sin\theta\hat{\mathbf{e}}_r-\cos\theta\hat{\mathbf{e}}_\theta\right)\right]\left(\rho V\frac{\pi d^2}{4}\right)=-2\rho V^2\frac{\pi d^2}{4}R\cos\theta\hat{\mathbf{e}}_z$$

$$\mathbf{M}_S=l\hat{\mathbf{e}}_r\times-F\hat{\mathbf{e}}_\theta=-Fl\hat{\mathbf{e}}_z$$

$$\mathbf{M}_B=\mathbf{0}$$

$$\int_{\text{CV}}\mathbf{r}_{xyz}\times\left(\ddot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+2\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{u}_{xyz}+\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho dV=$$

$$2\int_{r=0}^{r=R}r\hat{\mathbf{e}}_r\times\left(\underbrace{\dot{\Omega}\hat{\mathbf{e}}_z\times r\hat{\mathbf{e}}_r}_{=\dot{\Omega}r\hat{\mathbf{e}}_\theta}+\underbrace{2\Omega\hat{\mathbf{e}}_z\times V\hat{\mathbf{e}}_r}_{=2\Omega V\hat{\mathbf{e}}_\theta}+\underbrace{\Omega\hat{\mathbf{e}}_z\times\underbrace{\Omega\hat{\mathbf{e}}_z\times r\hat{\mathbf{e}}_r}_{=\Omega r\hat{\mathbf{e}}_\theta}}_{=-\Omega^2 r\hat{\mathbf{e}}_r}\right)\rho dr\frac{\pi d^2}{4}$$

$$=2\int_0^R\left(\dot{\Omega}r^2\hat{\mathbf{e}}_z+2\Omega Vr\hat{\mathbf{e}}_z\right)\rho\frac{\pi d^2}{4}dr=\dot{\Omega}\hat{\mathbf{e}}_z\underbrace{2\int_0^R\rho\frac{\pi d^2}{4}r^2dr}_{=I}+\rho\frac{\pi d^2}{4}4\Omega V\hat{\mathbf{e}}_z\int_0^R rdr$$

$$=I\dot{\Omega}\hat{\mathbf{e}}_z+2\rho V\frac{\pi d^2}{4}\Omega R^2\hat{\mathbf{e}}_z$$

where I is the sprinkler's moment of inertia. Substitute and simplify,

$$-2\rho V^2\frac{\pi d^2}{4}R\cos\theta\hat{\mathbf{e}}_z=-I\dot{\Omega}\hat{\mathbf{e}}_z-2\Omega R^2\rho V\frac{\pi d^2}{4}\hat{\mathbf{e}}_z-Fl\hat{\mathbf{e}}_z$$

$$\boxed{I\dot{\Omega}=-Fl+2\rho V\frac{\pi d^2}{4}R\left(V\cos\theta-\Omega R\right)}\tag{4}$$

From Eqn. (4) the force required to maintain the sprinkler at a constant angular velocity is,

$$\boxed{F=2\rho V\frac{\pi d^2}{4}\frac{R}{l}\left(V\cos\theta-\Omega R\right)}\quad\text{(Note that if }\Omega=0\text{ this simplifies to Eqn. (2).)}\tag{5}$$

If the force, F, is removed, the angular acceleration is,

$$\boxed{\dot{\Omega}=2\rho V\frac{\pi d^2}{4}\frac{R}{I}\left(V\cos\theta-\Omega R\right)}\tag{6}$$

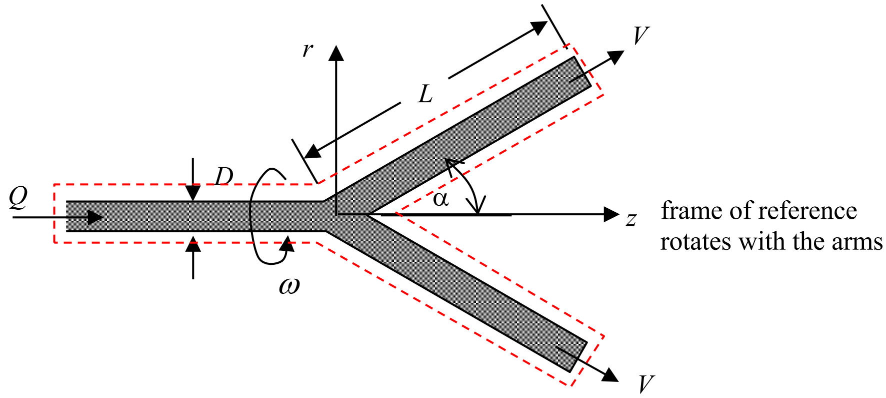
The maximum angular velocity of the sprinkler if no force is applied is,

$$\boxed{\Omega_{\max}=\frac{V\cos\theta}{R}}\tag{7}$$

A pipe branches symmetrically into two legs of length, L, and the whole system rotates with angular speed, ω, around its axis. Each branch is inclined at angle, α, to the axis of rotation. Liquid enters the pipe steadily, with zero angular momentum, at the volume flow rate Q. The pipe diameter, D, is much smaller than L.

a. Obtain an expression for the external torque required to turn the pipe.

b. What additional torque would be required to impart angular acceleration $\dot{\omega}$?



SOLUTION:
Apply the angular momentum equation to the CV shown above using a frame of reference rotating with the arms (this is an accelerating frame of reference). Let V be the velocity of the fluid in the pipe arms ($V \pi D^2/4 = {}^1/_2 Q$).

$$\frac{d}{dt}\int_{CV}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\rho dV+\int_{CS}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)$$
$$=\mathbf{M}_B+\mathbf{M}_S-\int_{CV}\mathbf{r}_{xyz}\times\left(\ddot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+2\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{u}_{xyz}+\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho dV$$

where

$$\frac{d}{dt}\int_{CV}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\rho dV=\mathbf{0}\quad\text{(steady problem in the given frame of reference)}$$

$$\int_{CS}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\underbrace{\mathbf{0}}_{\text{incoming flow}}+\underbrace{\left[L\left(\cos\alpha\hat{\mathbf{e}}_z+\sin\alpha\hat{\mathbf{e}}_r\right)\times V\left(\cos\alpha\hat{\mathbf{e}}_z+\sin\alpha\hat{\mathbf{e}}_r\right)\right]\left(\tfrac{1}{2}\rho Q\right)}_{\text{top arm}}$$
$$+\underbrace{\left[L\left(\cos\alpha\hat{\mathbf{e}}_z-\sin\alpha\hat{\mathbf{e}}_r\right)\times V\left(\cos\alpha\hat{\mathbf{e}}_z-\sin\alpha\hat{\mathbf{e}}_r\right)\right]\left(\tfrac{1}{2}\rho Q\right)}_{\text{bottom arm}}$$
$$=\mathbf{0}$$

$\mathbf{M}_B=\mathbf{0}$ (neglect gravity)

$\mathbf{M}_S=\mathbf{T}$ (this is the torque we must apply to the rotating arm)

$$\int_{CV} \mathbf{r}_{xyz} \times \left(\ddot{\boldsymbol{\theta}}_{xyz/XYZ} \times \mathbf{r}_{xyz} + 2\dot{\boldsymbol{\theta}}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \dot{\boldsymbol{\theta}}_{xyz/XYZ} \times \dot{\boldsymbol{\theta}}_{xyz/XYZ} \times \mathbf{r}_{xyz} \right) \rho dV =$$

$$\underbrace{\int_{s=0}^{s=L} s\left(\cos\alpha\hat{\mathbf{e}}_z + \sin\alpha\hat{\mathbf{e}}_r\right) \times \begin{bmatrix} \dot{\omega}\hat{\mathbf{e}}_z \times s\left(\cos\alpha\hat{\mathbf{e}}_z + \sin\alpha\hat{\mathbf{e}}_r\right) + \\ 2\omega\hat{\mathbf{e}}_z \times V\left(\cos\alpha\hat{\mathbf{e}}_z + \sin\alpha\hat{\mathbf{e}}_r\right) + \\ \omega\hat{\mathbf{e}}_z \times \omega\hat{\mathbf{e}}_z \times s\left(\cos\alpha\hat{\mathbf{e}}_z + \sin\alpha\hat{\mathbf{e}}_r\right) \end{bmatrix} \rho \frac{\pi D^2}{4} ds}_{\text{top arm}} +$$

$$\underbrace{\int_{s=0}^{s=L} s\left(\cos\alpha\hat{\mathbf{e}}_z - \sin\alpha\hat{\mathbf{e}}_r\right) \times \begin{bmatrix} \dot{\omega}\hat{\mathbf{e}}_z \times s\left(\cos\alpha\hat{\mathbf{e}}_z - \sin\alpha\hat{\mathbf{e}}_r\right) + \\ 2\omega\hat{\mathbf{e}}_z \times V\left(\cos\alpha\hat{\mathbf{e}}_z - \sin\alpha\hat{\mathbf{e}}_r\right) + \\ \omega\hat{\mathbf{e}}_z \times \omega\hat{\mathbf{e}}_z \times s\left(\cos\alpha\hat{\mathbf{e}}_z - \sin\alpha\hat{\mathbf{e}}_r\right) \end{bmatrix} \rho \frac{\pi D^2}{4} ds}_{\text{bottom arm}}$$

$$= \rho\frac{\pi D^2}{4} \left\{ \begin{aligned} &\int_{s=0}^{s=L} s\left(\cos\alpha\hat{\mathbf{e}}_z + \sin\alpha\hat{\mathbf{e}}_r\right) \times \left[s\dot{\omega}\sin\alpha\hat{\mathbf{e}}_\theta + 2\omega V\sin\alpha\hat{\mathbf{e}}_\theta - s\omega^2\sin\alpha\hat{\mathbf{e}}_r \right] ds + \\ &\int_{s=0}^{s=L} s\left(\cos\alpha\hat{\mathbf{e}}_z - \sin\alpha\hat{\mathbf{e}}_r\right) \times \left[-s\dot{\omega}\sin\alpha\hat{\mathbf{e}}_\theta - 2\omega V\sin\alpha\hat{\mathbf{e}}_\theta + s\omega^2\sin\alpha\hat{\mathbf{e}}_r \right] ds \end{aligned} \right\}$$

$$= \rho\frac{\pi D^2}{4} \int_0^L \begin{bmatrix} -s^2\dot{\omega}\sin\alpha\cos\alpha\hat{\mathbf{e}}_r + s^2\dot{\omega}\sin^2\alpha\hat{\mathbf{e}}_z - 2s\omega V\sin\alpha\cos\alpha\hat{\mathbf{e}}_r + 2s\omega V\sin^2\alpha\hat{\mathbf{e}}_z - s^2\omega^2\sin\alpha\cos\alpha\hat{\mathbf{e}}_\theta \\ +s^2\dot{\omega}\sin\alpha\cos\alpha\hat{\mathbf{e}}_r + s^2\dot{\omega}\sin^2\alpha\hat{\mathbf{e}}_z + 2s\omega V\sin\alpha\cos\alpha\hat{\mathbf{e}}_r + 2s\omega V\sin^2\alpha\hat{\mathbf{e}}_z + s^2\omega^2\sin\alpha\cos\alpha\hat{\mathbf{e}}_\theta \end{bmatrix} ds$$

$$= \rho\frac{\pi D^2}{4} \int_0^L \left[2s^2\dot{\omega}\sin^2\alpha\hat{\mathbf{e}}_z + 4s\omega V\sin^2\alpha\hat{\mathbf{e}}_z \right] ds$$

$$= \rho\frac{\pi D^2}{4} \left[\tfrac{2}{3}\dot{\omega}L^3\sin^2\alpha + 2\omega VL^2\sin^2\alpha \right] \hat{\mathbf{e}}_z$$

Combining together all of the terms in the angular momentum equation,

$$\mathbf{0} = \mathbf{T} - \rho\frac{\pi D^2}{4} \left[\tfrac{2}{3}\dot{\omega}L^3\sin^2\alpha + 2\omega VL^2\sin^2\alpha \right] \hat{\mathbf{e}}_z$$

$$\mathbf{T} = \rho\frac{\pi D^2}{4} \left[\tfrac{2}{3}\dot{\omega}L^3\sin^2\alpha + 2\omega VL^2\sin^2\alpha \right] \hat{\mathbf{e}}_z$$

or, since $V\,\pi D^2/4 = {}^1/_2 Q$,

$$\boxed{\mathbf{T} = \left[\rho\frac{\pi D^2}{6}\dot{\omega}L^3\sin^2\alpha + \rho Q\omega L^2\sin^2\alpha \right] \hat{\mathbf{e}}_z}$$

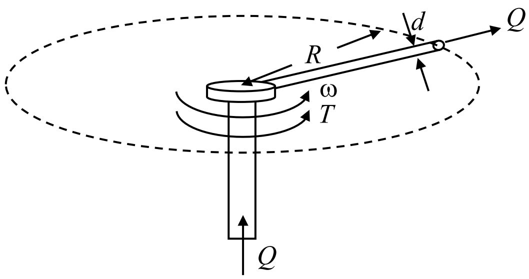
The torque required to turn the pipe at a constant angular velocity, ω, is,

$$\boxed{\mathbf{T} = \rho Q\omega L^2\sin^2\alpha\hat{\mathbf{e}}_z}$$

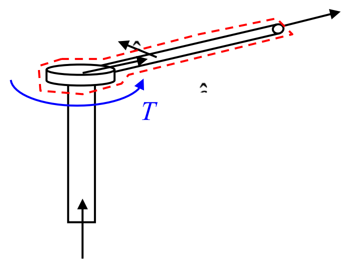
The additional torque required to impart an angular acceleration $\dot{\omega}$ is,

$$\boxed{\mathbf{T}_{\text{additional}} = \rho\frac{\pi D^2}{6}\dot{\omega}L^3\sin^2\alpha\hat{\mathbf{e}}_z}$$

A single tube, with diameter of d and length of R rotates at constant angular speed, ω as shown in the figure. Water is pumped through the tube at volume flow rate, Q. Find the torque, T, that must be applied to maintain the steady rotation of the tube.



SOLUTION:
Apply the angular momentum equation to the control volume shown below. Use a coordinate system that rotates with the rotating tube.



$$\frac{d}{dt}\int_{CV}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\rho\,dV+\int_{CS}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)$$
$$=\mathbf{M}_B+\mathbf{M}_S-\int_{CV}\mathbf{r}_{xyz}\times\left(\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+2\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho\,dV$$

where,

$$\frac{d}{dt}\int_{CV}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\rho dV=\mathbf{0}\quad\text{(steady flow)}$$

$$\int_{CS}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\left(R\hat{\mathbf{e}}_r\times\frac{Q}{A}\hat{\mathbf{e}}_r\right)\left(\rho Q\right)=\mathbf{0}$$

(There is no contribution from the incoming flow since its moment arm is zero. The quantity A is the cross-sectional area of the tube.)

$$\mathbf{M}_B=0$$

$$\mathbf{M}_S=T\hat{\mathbf{e}}_z$$

$$\begin{aligned}\int_{CV}\mathbf{r}_{xyz}\times\left(\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+2\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho\,dV&=\int_{r=0}^{r=R}r\hat{\mathbf{e}}_r\times\left(\mathbf{0}+2\omega\hat{\mathbf{e}}_z\times\frac{Q}{A}\hat{\mathbf{e}}_r+\omega\hat{\mathbf{e}}_z\times\omega\hat{\mathbf{e}}_z\times r\hat{\mathbf{e}}_r\right)\rho\underbrace{A\,dr}_{=dV}\\&=\int_{r=0}^{r=R}r\hat{\mathbf{e}}_r\times\left(2\omega\frac{Q}{A}\hat{\mathbf{e}}_\theta-\omega^2 r\hat{\mathbf{e}}_r\right)\rho A\,dr\\&=2\rho\omega Q\hat{\mathbf{e}}_z\int_{r=0}^{r=R}r\,dr\\&=\rho\omega QR^2\hat{\mathbf{e}}_z\end{aligned}$$

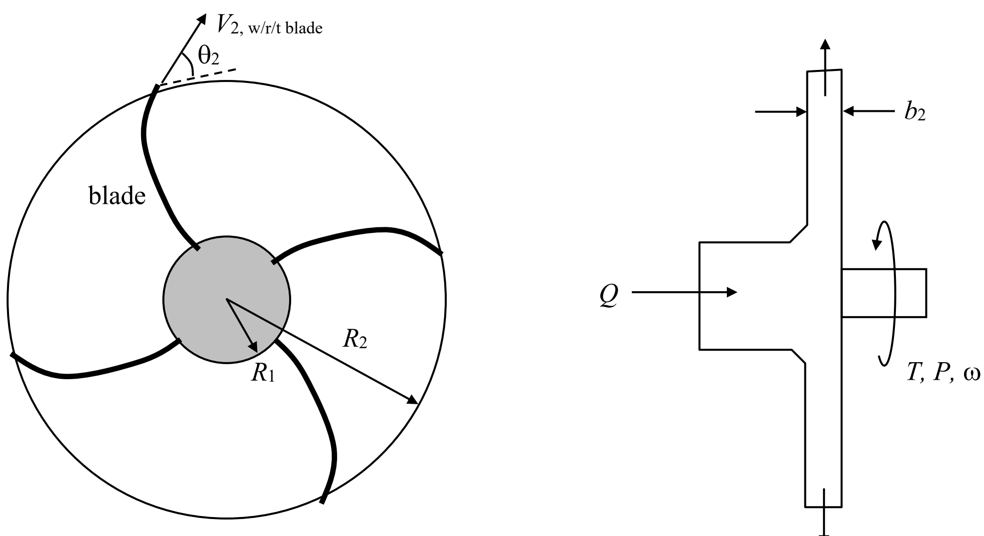
Substitute and simplify.

$$\mathbf{0}=T\hat{\mathbf{e}}_z-\rho\omega QR^2\hat{\mathbf{e}}_z \tag{1}$$

$$\boxed{\therefore T=\rho\omega QR^2} \tag{2}$$

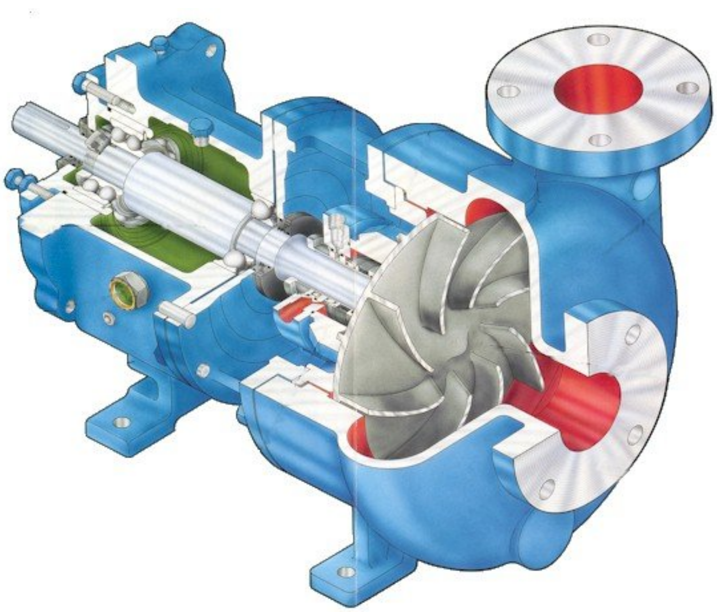
The centrifugal pump shown in the figure below has a flow rate, Q. The flow exits the impeller at an angle θ_2, measured from the tangent of the impeller, with a velocity $V_{2,\text{w/r/t blade}}$, relative to the rotating blades. The thickness of the discharge stream at the impeller perimeter is b_2. The fluid, with density ρ, enters axially at section 1.

Assuming steady, incompressible flow at constant shaft angular velocity, ω, derive a formula for the power, P, required to drive the impeller in terms of ρ, ω, Q, θ_2, R_2, and b_2.



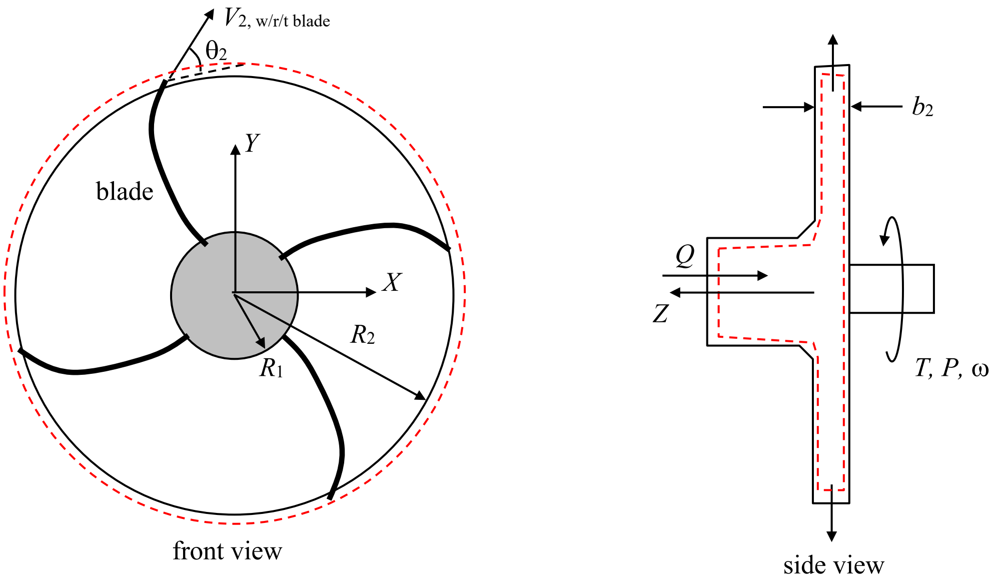
front view

side view



SOLUTION:

Apply the moment of momentum equation to a control volume surrounding the impeller as shown in the figure below. Use a coordinate system fixed to the ground. Consider only the Z direction.



$$\frac{d}{dt}\int_{CV}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)_Z \rho dV + \int_{CV}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)_Z\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = M_{B,Z} + M_{S,Z} \tag{1}$$

where,

$$\frac{d}{dt}\int_{CV}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)_Z \rho dV = 0 \quad \text{(steady flow)} \tag{2}$$

$$\int_{CV}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)_Z\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = R_2 V_{\substack{2,\text{tangential}\\ \text{w/r/t ground}}}\dot{m} \tag{3}$$

$$M_{B,Z} = 0 \tag{4}$$

$$M_{S,Z} = T \tag{5}$$

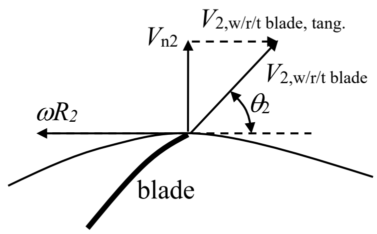
Substitute and simplify,

$$T = \dot{m}R_2 V_{\substack{2,\text{tangential}\\ \text{w/r/t ground}}} \tag{6}$$

The power is the $P = \mathbf{T}\cdot\boldsymbol{\omega}$ so,

$$P = \omega T = \dot{m}\omega R_2 V_{\substack{2,\text{tangential}\\ \text{w/r/t ground}}} \tag{7}$$

The velocity of the water at the point of discharge relative to the ground may be found by considering a velocity vector diagram as shown below.



$$\tan\theta_2 = \frac{V_{n2}}{V_{2,\substack{\text{w/r/t blade,}\\\text{tangential}}}} \Rightarrow V_{2,\substack{\text{w/r/t blade,}\\\text{tangential}}} = \frac{V_{n2}}{\tan\theta_2} \tag{8}$$

$$V_{2,\substack{\text{w/r/t ground,}\\\text{tangential}}} = \omega R_2 - V_{2,\substack{\text{w/r/t blade,}\\\text{tangential}}} = \omega R_2 - \frac{V_{n2}}{\tan\theta_2} \tag{9}$$

The velocity component V_{n2} may be found by applying conservation of mass to the same control volume,

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CV}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{10}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow)} \tag{11}$$

$$\int_{\text{CV}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho Q + \rho V_{n2}\left(2\pi R_2 b_2\right) \tag{12}$$

Substitute and simplify,

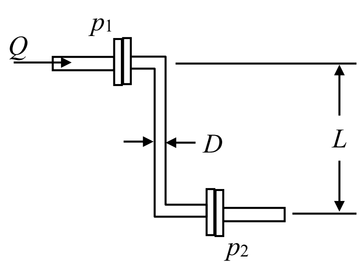
$$V_{n2} = \frac{Q}{2\pi R_2 b_2} \tag{13}$$

Combine Eqs. (7), (9), and (13) to get,

$$P = \dot{m}\omega R_2 V_{\substack{2,\text{tangential}\\\text{w/r/t ground}}} = \rho Q \omega R_2 \left(\omega R_2 - \frac{V_{n2}}{\tan\theta_2}\right) \tag{14}$$

$$\boxed{\therefore P = \rho Q \omega R_2 \left(\omega R_2 - \frac{Q}{2\pi R_2 b_2 \tan\theta_2}\right)} \tag{15}$$

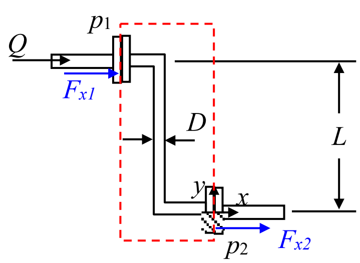
Crude oil (SG=0.95) from a tanker dock flows through a pipe of 0.4 m diameter in the configuration shown below. The pipe is oriented horizontally so gravity may be neglected. The flow rate is 0.58 m^3/sec, and the gage pressures are shown in the diagram. Determine the force and torque that are exerted by the supports on the pipe assembly.



$D = 0.4$ m
$L = 20$ m
$Q = 0.58$ m^3/sec
$p_1 = 345$ kPa (gage)
$p_2 = 332$ kPa (gage)

SOLUTION:

Apply the linear momentum equation in the x-direction to the control volume shown below. Use the fixed frame of reference indicated in the diagram.



F_{x1} and F_{x2} are the horizontal reaction forces that the supports exert on the pipe assembly. Since there is no momentum flux in the y-direction, gravity is not a factor, and assuming that there is no "pre-load" on the assembly in the y-direction, the vertical reaction forces, i.e., F_{y1} and F_{y2}, will be zero.

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0 \quad \text{(same momentum flux coming in as going out)}$$

$$F_{B,x} = 0$$

$$F_{S,x} = p_1 \frac{\pi D^2}{4} - p_2 \frac{\pi D^2}{4} + F_{x1} + F_{x2} = \left(p_1 - p_2\right)\frac{\pi D^2}{4} + F_{x1} + F_{x2}$$

Substitute and simplify,

$$F_{x2} + F_{x1} = \left(p_2 - p_1\right)\frac{\pi D^2}{4} \tag{1}$$

Now apply the moment of momentum equation to the same control volume using the same coordinate system,

$$\frac{d}{dt}\int_{CV}(\mathbf{r}\times\mathbf{u})\rho dV+\int_{CS}(\mathbf{r}\times\mathbf{u})(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A})=\mathbf{M}_B+\mathbf{M}_S$$

where,

$$\frac{d}{dt}\int_{CV}(\mathbf{r}\times\mathbf{u})\rho dV=\mathbf{0}\quad\text{(steady flow)}$$

$$\int_{CS}(\mathbf{r}\times\mathbf{u})(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A})=\left[\left(L\hat{\mathbf{e}}_y-W\hat{\mathbf{e}}_x\right)\times\frac{Q}{\frac{\pi D^2}{4}}\hat{\mathbf{e}}_x\right](-\rho Q)=\rho\frac{Q^2}{\frac{\pi D^2}{4}}L\hat{\mathbf{e}}_z$$

$$\mathbf{M}_B=\mathbf{0}$$

$$\mathbf{M}_S=\left[L\hat{\mathbf{e}}_y\times\left(p_1\frac{\pi D^2}{4}+F_{x1}\right)\hat{\mathbf{e}}_x\right]+\left[-W\hat{\mathbf{e}}_x\times F_{y1}\hat{\mathbf{e}}_y\right]=-L\left(p_1\frac{\pi D^2}{4}+F_{x1}\right)\hat{\mathbf{e}}_z \tag{2}$$

Substitute and simplify,

$$\rho\frac{Q^2}{\frac{\pi D^2}{4}}L\hat{\mathbf{e}}_z=-L\left(p_1\frac{\pi D^2}{4}+F_{x1}\right)\hat{\mathbf{e}}_z$$

$$\rho\frac{Q^2}{\frac{\pi D^2}{4}}=-\left(p_1\frac{\pi D^2}{4}+F_{x1}\right)$$

$$\boxed{F_{x1}=-\rho\frac{Q^2}{\frac{\pi D^2}{4}}-p_1\frac{\pi D^2}{4}} \tag{3}$$

The other reaction force is determined by combining Eqs. (1) and (3),

$$\boxed{F_{x2}=p_2\frac{\pi D^2}{4}+\rho\frac{Q^2}{\frac{\pi D^2}{4}}} \tag{4}$$

The torque exerted on the pipe assembly is given by Eq. (2) (copied below for convenience),

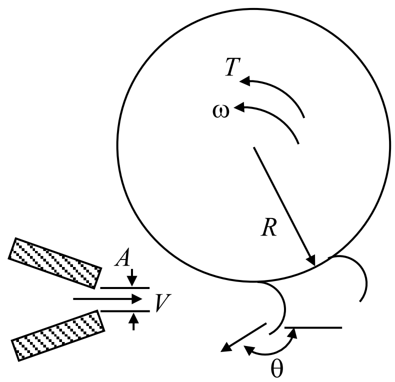
$$\boxed{\mathbf{M}_S=-L\left(p_1\frac{\pi D^2}{4}+F_{x1}\right)\hat{\mathbf{e}}_z} \tag{5}$$

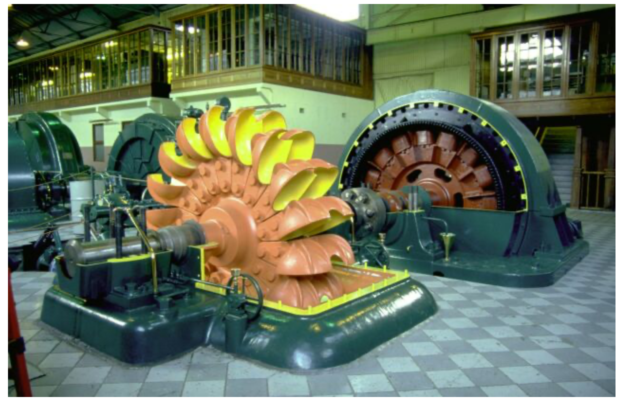
Substitute the given parameters,

L = 20 m
ρ = 950 kg/m^3
p_1 = 345 kPa (gage)
p_2 = 332 kPa (gage)
D = 0.4 m
Q = 0.58 m^3/s
$\Rightarrow$ $\boxed{F_{x1} = -46\text{ kN},\ F_{x2} = 44\text{ N, and } M_z = -866\text{ kN·m}}$

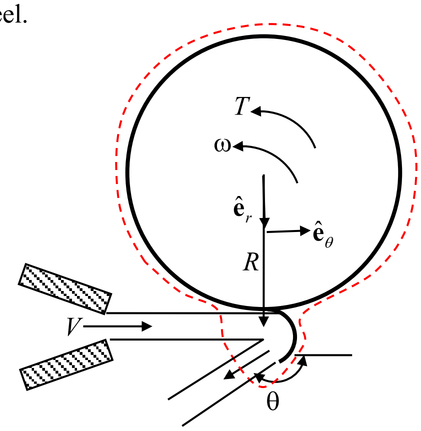
A Pelton wheel is a form of water turbine well adapted to situations of high head and low flow rate. The wheel consists of a series of vanes mounted on a rotor, as shown. One or more jets are arranged to strike the buckets tangentially. In practice it is possible to deflect the jet stream through angles, θ, of up to 165°. Consider the Pelton wheel and single jet arrangement shown below.

a. Obtain an expression for the torque exerted by the water stream on the wheel and the corresponding power output.
b. Determine the value of U/V, where U is the bucket speed and V is the water jet speed, required to maximize the power produced by the wheel.





SOLUTION:
Apply the moment of momentum equation to the control volume shown below. Use a coordinate system that rotates with the wheel.



T is the torque required to maintain the wheel at a steady angular velocity.

$$\frac{d}{dt}\int_{CV}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\rho\, dV+\int_{CS}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)$$
$$=\mathbf{M}_B+\mathbf{M}_S-\int_{CV}\mathbf{r}_{xyz}\times\left(\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+2\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho\, dV$$

where,

$$\frac{d}{dt}\int_{CV}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\rho dV=\mathbf{0}$$

$$\int_{CS}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\left[R\hat{\mathbf{e}}_r\times\left(V-\omega R\right)\hat{\mathbf{e}}_\theta\right]\left[-\rho\left(V-\omega R\right)A\right]+\left[R\hat{\mathbf{e}}_r\times\left(V-\omega R\right)\left(\sin\theta\hat{\mathbf{e}}_r+\cos\theta\hat{\mathbf{e}}_\theta\right)\right]\left[\rho\left(V-\omega R\right)A\right]$$
$$=\left[-R\left(V-\omega R\right)+R\left(V-\omega R\right)\cos\theta\right]\hat{\mathbf{e}}_z\left[\rho\left(V-\omega R\right)A\right]$$
$$=R\left(\cos\theta-1\right)\left[\rho\left(V-\omega R\right)^2A\right]\hat{\mathbf{e}}_z$$

(Note that the change in the incident and reflected jet angles as the wheel rotates is being neglected. We are assuming that there is always a bucket deflecting the jet at the given angle.)

$$\mathbf{M}_B=\mathbf{0}$$
$$\mathbf{M}_S=T\hat{\mathbf{e}}_z$$

$$\int_{CV}\mathbf{r}_{xyz}\times\left(\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+2\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho\,dV=\int_{CV}r\hat{\mathbf{e}}_r\times\left(\mathbf{0}+2\omega\hat{\mathbf{e}}_z\times\mathbf{0}+\omega\hat{\mathbf{e}}_z\times\omega\hat{\mathbf{e}}_z\times r\hat{\mathbf{e}}_r\right)\rho\,dV$$
$$=\int_{CV}r\hat{\mathbf{e}}_r\times\left(\mathbf{0}+\mathbf{0}-\omega^2r\hat{\mathbf{e}}_r\right)\rho\,dV$$
$$=\mathbf{0}$$

Substitute and simplify,

$$R\left(\cos\theta-1\right)\left[\rho\left(V-\omega R\right)^2A\right]\hat{\mathbf{e}}_z=T\hat{\mathbf{e}}_z$$
$$T=R\left(\cos\theta-1\right)\left[\rho\left(V-\omega R\right)^2A\right] \tag{1}$$

From the problem statement, $U\equiv\omega R$, so that Eq. (1) becomes,

$$\boxed{T=R\rho V^2A\left(\cos\theta-1\right)\left(1-\frac{U}{V}\right)^2}\quad\text{(Note: } T\le 0.\text{)} \tag{2}$$

The power added to the wheel (since the torque is acting on the wheel) is,

$$\dot{W}=\boldsymbol{\omega}\cdot\mathbf{T}=\underbrace{\omega R}_{=U}\rho V^2A\left(\cos\theta-1\right)\left(1-\frac{U}{V}\right)^2$$
$$\boxed{\therefore\dot{W}=\rho V^3A\left(\cos\theta-1\right)\left[\frac{U}{V}\left(1-\frac{U}{V}\right)^2\right]}\quad\text{(Note: } \dot{W}\le 0\ \Rightarrow\ \text{power extracted from the wheel)} \tag{3}$$

To maximize the power with respect to U/V, differentiate Eq. (3) with respect to U/V and set equal to zero. Re-writing Eq. (3) in dimensionless form gives,

$$\frac{\dot{W}}{\rho V^3A\left(\cos\theta-1\right)}=\frac{U}{V}\left(1-\frac{U}{V}\right)^2 \tag{4}$$

Let $\dot{W}'$ equal the left hand side of Eq. (4) and $U'=U/V$ so that Eq. (4) is,

$$\dot{W}'=U'\left(1-U'\right)^2=U'\left(1-2U'+U'^2\right)=U'-2U'^2+U'^3 \tag{5}$$

Differentiate Eq. (5) with respect to U' and set equal to zero to find the inflection points,

$$\frac{d\dot{W}'}{dU'}=0=1-4U'+3U'^2=3\left(U'-1\right)\left(U'-\tfrac{1}{3}\right) \tag{6}$$
$$\therefore U'\big|_{\text{inflection}}=1,\tfrac{1}{3} \tag{7}$$

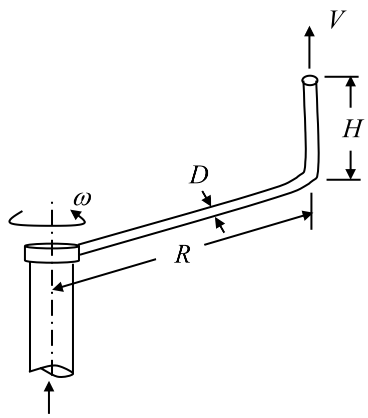
Substituting Eq. (7) into Eq. (5) shows that the maximum power occurs when,

$$\boxed{U'\big|_{\max\dot{W}}=\left.\frac{U}{V}\right|_{\max\dot{W}}=\tfrac{1}{3}}\ \Rightarrow\ \frac{\dot{W}_{\max}}{\rho V^3A\left(\cos\theta-1\right)}=\frac{4}{27} \tag{8}$$

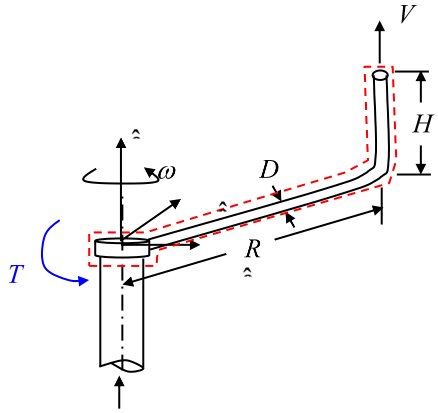
The minimum power occurs when $U'=1$.

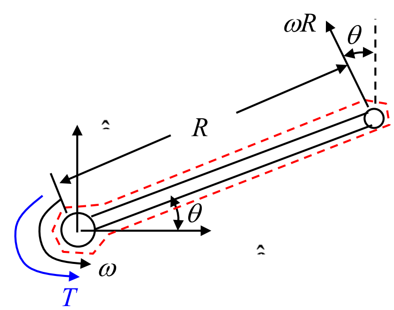
A single-arm water distribution tube is shown in the diagram. The arm consists of a uniform diameter, L-shaped tube with horizontal and vertical sections. Water is pumped through the tube at a constant volumetric flow rate and the tube's speed of rotation is constant. Neglecting friction and aerodynamic drag, estimate the torque required to rotate the tube. Solve the problem *twice* using:

a. a fixed coordinate system, and
b. a coordinate system that rotates with the tube.



SOLUTION:
Apply the momentum of momentum equation in the Z-direction on the control volume shown in the figures below. Use the fixed coordinate system XYZ (this is an inertial frame of reference).





top view

$$\frac{d}{dt}\int_{\text{CV}}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)_Z \rho dV + \int_{\text{CS}}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)_Z\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = M_{B,Z} + M_{S,Z} \tag{1}$$

where,

$$\frac{d}{dt}\int_{\text{CV}}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)_Z \rho dV = 0 \quad \text{(steady flow – angular momentum in } Z\text{-direction doesn't change)} \tag{2}$$

$$\int_{\text{CS}}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)_Z\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \left[\left(R\cos\theta\hat{\mathbf{e}}_X + R\sin\theta\hat{\mathbf{e}}_Y + H\hat{\mathbf{e}}_Z\right)\times\left(-\omega R\sin\theta\hat{\mathbf{e}}_X + \omega R\cos\theta\hat{\mathbf{e}}_Y + V\hat{\mathbf{e}}_Z\right)\right]_Z\left(\rho V\tfrac{\pi}{4}D^2\right) \tag{3}$$

$$M_{B,Z} = 0 \tag{4}$$

$$M_{S,Z} = T \tag{5}$$

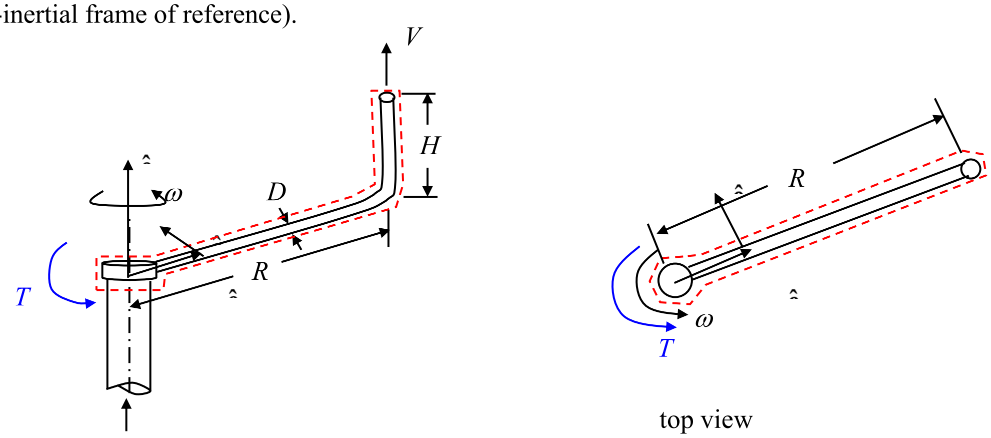
Substitute and simplify,

$$\left[\left(R\cos\theta\hat{\mathbf{e}}_X + R\sin\theta\hat{\mathbf{e}}_Y + H\hat{\mathbf{e}}_Z\right)\times\left(-\omega R\sin\theta\hat{\mathbf{e}}_X + \omega R\cos\theta\hat{\mathbf{e}}_Y + V\hat{\mathbf{e}}_Z\right)\right]_Z\left(\rho V\tfrac{\pi}{4}D^2\right) = T \tag{6}$$

$$\left[\left(VR\sin\theta - \omega RH\cos\theta\right)\hat{\mathbf{e}}_X - \left(VR\cos\theta + \omega RH\sin\theta\right) + \omega R^2\hat{\mathbf{e}}_Z\right]_Z\left(\rho V\tfrac{\pi}{4}D^2\right) = T \tag{7}$$

$$\boxed{\therefore T = \left(\rho V\tfrac{\pi}{4}D^2\right)\omega R^2}$$

Now use a coordinate system that rotates with the control volume as shown in the figures below (this is a non-inertial frame of reference).



$$\frac{d}{dt}\int_{\mathrm{CV}}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)_z\rho\,dV + \int_{\mathrm{CS}}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)_z\left(\rho\mathbf{u}_{\mathrm{rel}}\cdot d\mathbf{A}\right) =$$
$$M_{B,z} + M_{S,z} - \int_{\mathrm{CV}}\left\{\mathbf{r}_{xyz}\times\left[\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz} + 2\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ}\times\left(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\right]\right\}_z\rho\,dV \tag{8}$$

where,

$$\frac{d}{dt}\int_{\mathrm{CV}}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)_z\rho dV = 0 \quad \text{(steady flow)} \tag{9}$$

$$\int_{\mathrm{CS}}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)_z\left(\rho\mathbf{u}_{\mathrm{rel}}\cdot d\mathbf{A}\right) = \left[\left(R\hat{\mathbf{e}}_r + H\hat{\mathbf{e}}_z\right)\times\left(V\hat{\mathbf{e}}_z\right)\right]_z\left(\rho V\tfrac{\pi}{4}D^2\right) = \left[-RV\hat{\mathbf{e}}_\theta\right]_z\left(\rho V\tfrac{\pi}{4}D^2\right) = 0 \tag{10}$$

$$M_{B,Z} = 0 \tag{11}$$

$$M_{S,Z} = T \tag{12}$$

$$\int_{\mathrm{CV}}\left\{\mathbf{r}_{xyz}\times\left[\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz} + 2\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ}\times\left(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\right]\right\}_z\rho\,dV =$$
$$\int_{r=0}^{r=R}\left\{\left(r\hat{\mathbf{e}}_r\right)\times\left[\mathbf{0}\times r\hat{\mathbf{e}}_r + \underbrace{2\omega\hat{\mathbf{e}}_z\times V\hat{\mathbf{e}}_r}_{=2\omega V\hat{\mathbf{e}}_\theta} + \underbrace{\omega\hat{\mathbf{e}}_z\times\underbrace{\left(\omega\hat{\mathbf{e}}_z\times r\hat{\mathbf{e}}_r\right)}_{=\omega r\hat{\mathbf{e}}_\theta}}_{=-\omega^2 r\hat{\mathbf{e}}_r}\right]\right\}_z\left(\rho dr\tfrac{\pi}{4}D^2\right) +$$
$$\int_{z=0}^{z=H}\left\{\left(R\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z\right)\times\left[\mathbf{0}\times R\hat{\mathbf{e}}_r + \underbrace{2\omega\hat{\mathbf{e}}_z\times V\hat{\mathbf{e}}_z}_{=\mathbf{0}} + \underbrace{\omega\hat{\mathbf{e}}_z\times\underbrace{\left(\omega\hat{\mathbf{e}}_z\times\left(R\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z\right)\right)}_{=\omega R\hat{\mathbf{e}}_\theta}}_{=-\omega^2 R\hat{\mathbf{e}}_r}\right]\right\}_z\left(\rho dz\tfrac{\pi}{4}D^2\right) \tag{13}$$

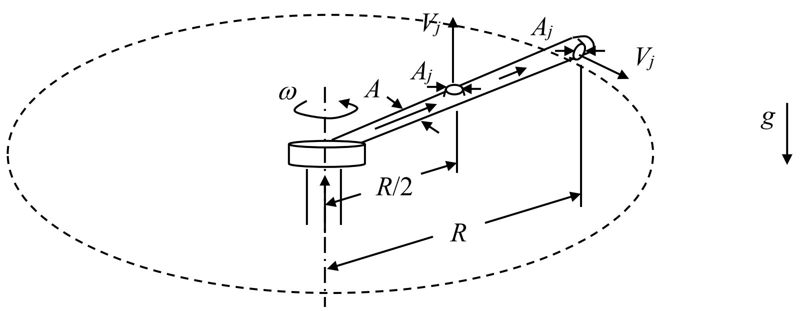
Substitute and simplify,

$$0 = T - \int_{r=0}^{r=R} \left\{ \left(r\hat{\mathbf{e}}_r \right) \times \left(2\omega V\hat{\mathbf{e}}_\theta - \omega^2 r\hat{\mathbf{e}}_r \right) \right\}_z \left(\rho dr \tfrac{\pi}{4} D^2 \right) + \int_{z=0}^{z=H} \left\{ \left(R\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z \right) \times \left(-\omega^2 R\hat{\mathbf{e}}_r \right) \right\}_z \left(\rho dz \tfrac{\pi}{4} D^2 \right) \tag{14}$$

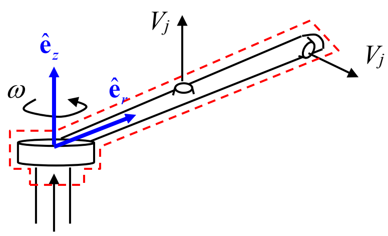
$$T = \int_{r=0}^{r=R} \left(2\omega Vr \right) \left(\rho dr \tfrac{\pi}{4} D^2 \right) \tag{15}$$

$$\boxed{\therefore T = \left(\rho V \tfrac{\pi}{4} D^2 \right) \omega R^2} \text{ (This is the same answer as before!)} \tag{16}$$

A sprinkler arm with two jets is shown below. Water flows steadily from each jet (with area A_j) with speed V_j relative to the arm. One jet leaves vertically at a distance of $R/2$ from the center of the sprinkler; the other leaves in the plane of rotation at R. The inside area of the arm is A. Gravity acts in the vertical direction. Estimate the components of the torque (<u>all</u> components) that must be applied at the pivot to make the arm turn at a constant rotational speed ω.



SOLUTION:
Apply the moment of momentum equation to a control volume surrounding the sprinkler arm as shown in the figure below. The control volume rotates with the sprinkler arm. Use a (polar) coordinate system fixed to the control volume.



$Q = 2V_jA_j$ (from COM)

$$\begin{aligned}\frac{d}{dt}\int_{\text{CV}}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\rho dV + \int_{\text{CS}}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)\\ = \mathbf{M}_B + \mathbf{M}_S - \int_{\text{CV}}\mathbf{r}_{xyz}\times\left(\ddot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz} + 2\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{u}_{xyz} + \dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho dV\end{aligned} \tag{1}$$

where,

$$\frac{d}{dt}\int_{\text{CV}}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\rho dV = \mathbf{0} \quad \text{(steady flow)} \tag{2}$$

$$\begin{aligned}\int_{\text{CS}}\left(\mathbf{r}_{xyz}\times\mathbf{u}_{xyz}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) &= \left(\tfrac{1}{2}R\hat{\mathbf{e}}_r\times V_j\hat{\mathbf{e}}_z\right)\left(\rho V_jA_j\right) + \left(R\hat{\mathbf{e}}_r\times -V_j\hat{\mathbf{e}}_\theta\right)\left(\rho V_jA_j\right)\\ &= -RV_j\left(\tfrac{1}{2}\hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_z\right)\left(\rho V_jA_j\right)\end{aligned} \tag{3}$$

(Note that there is no contribution at the inlet of the arm since the position vector $\mathbf{r}_{xyz}$ and the flow velocity $\mathbf{u}_{xyz}$ are both in the z direction. Hence, the resulting cross product is zero.)

$$\mathbf{M}_B = \tfrac{1}{2}R\hat{\mathbf{e}}_r\times -\rho RAg\hat{\mathbf{e}}_z = \tfrac{1}{2}\rho R^2Ag\hat{\mathbf{e}}_\theta \tag{4}$$

$$\mathbf{M}_S = \mathbf{T} \tag{5}$$

$$
\begin{aligned}
&\int\limits_{\text{CV}} \mathbf{r}_{xyz}\times\left[\ddot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+2\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{u}_{xyz}+\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\left(\dot{\boldsymbol{\theta}}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\right]\rho dV \\
&=\int\limits_{r=0}^{r=\frac{1}{2}R} r\hat{\mathbf{e}}_r\times\left[\mathbf{0}+2\omega\hat{\mathbf{e}}_z\times\frac{Q}{A}\hat{\mathbf{e}}_r+\omega\hat{\mathbf{e}}_z\times\left(\omega\hat{\mathbf{e}}_z\times r\hat{\mathbf{e}}_r\right)\right]\rho\underbrace{Adr}_{=dV}+\int\limits_{r=\frac{1}{2}R}^{r=R} r\hat{\mathbf{e}}_r\times\left[\mathbf{0}+2\omega\hat{\mathbf{e}}_z\times\frac{\frac{1}{2}Q}{A}\hat{\mathbf{e}}_r+\omega\hat{\mathbf{e}}_z\times\left(\omega\hat{\mathbf{e}}_z\times r\hat{\mathbf{e}}_r\right)\right]\rho\underbrace{Adr}_{=dV} \\
&=\int\limits_{r=0}^{r=\frac{1}{2}R} r\hat{\mathbf{e}}_r\times\left(2\omega\frac{Q}{A}\hat{\mathbf{e}}_\theta-\omega^2 r\hat{\mathbf{e}}_r\right)\rho Adr+\int\limits_{r=\frac{1}{2}R}^{r=R} r\hat{\mathbf{e}}_r\times\left(\omega\frac{Q}{A}\hat{\mathbf{e}}_\theta-\omega^2 r\hat{\mathbf{e}}_r\right)\rho Adr \\
&=2\omega\rho Q\hat{\mathbf{e}}_z\int\limits_{r=0}^{r=\frac{1}{2}R} rdr+\omega\rho Q\hat{\mathbf{e}}_z\int\limits_{r=\frac{1}{2}R}^{r=R} rdr \\
&=\tfrac{1}{4}\omega\rho QR^2\hat{\mathbf{e}}_z+\tfrac{3}{8}\omega\rho QR^2\hat{\mathbf{e}}_z \\
&=\tfrac{5}{8}\omega\rho QR^2\hat{\mathbf{e}}_z \qquad \left(\text{Recall that } Q=2V_jA_j\right) \\
&=\tfrac{5}{4}\omega\rho V_jA_jR^2\hat{\mathbf{e}}_z
\end{aligned}
\tag{6}
$$

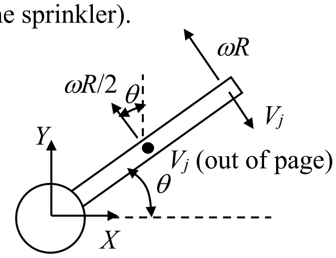
Note that half the flow leaves through the first jet (at $R/2$), so the flow rate from $R/2$ to R has a flow rate of $Q/2$.

Substitute and simplify,

$$-RV_j\left(\tfrac{1}{2}\hat{\mathbf{e}}_\theta+\hat{\mathbf{e}}_z\right)\left(\rho V_jA_j\right)=\tfrac{1}{2}\rho R^2Ag\hat{\mathbf{e}}_\theta+\mathbf{T}-\tfrac{5}{4}\omega\rho V_jA_jR^2\hat{\mathbf{e}}_z \tag{7}$$

$$\boxed{\mathbf{T}=\left[-\tfrac{1}{2}R\rho V_j^2A_j-\tfrac{1}{2}\rho R^2Ag\right]\hat{\mathbf{e}}_\theta+\left[-R\rho V_j^2A_j+\tfrac{5}{4}\omega\rho V_jA_jR^2\right]\hat{\mathbf{e}}_z} \tag{8}$$

Now try solving the problem using an inertial coordinate system as shown in the figure below (looking down from the top of the sprinkler).



$$\frac{d}{dt}\int\limits_{\text{CV}}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)\rho dV+\int\limits_{\text{CS}}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\mathbf{M}_B+\mathbf{M}_S \tag{9}$$

where,

$$\frac{d}{dt}\int\limits_{\text{CV}}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)\rho dV=\mathbf{0}\quad\text{(The sprinkler is rotating at a constant speed.)} \tag{10}$$

$$
\begin{aligned}
\int\limits_{\text{CS}}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)&=\left\{\tfrac{1}{2}R\left(\cos\theta\hat{\mathbf{e}}_X+\sin\theta\hat{\mathbf{e}}_Y\right)\times\left[V_j\hat{\mathbf{e}}_Z+\omega\tfrac{1}{2}R\left(-\sin\theta\hat{\mathbf{e}}_X+\cos\theta\hat{\mathbf{e}}_Y\right)\right]\right\}\left(\rho V_jA_j\right)+ \\
&\qquad\left\{R\left(\cos\theta\hat{\mathbf{e}}_X+\sin\theta\hat{\mathbf{e}}_Y\right)\times\left[\left(\omega R-V_j\right)\left(-\sin\theta\hat{\mathbf{e}}_X+\cos\theta\hat{\mathbf{e}}_Y\right)\right]\right\}\left(\rho V_jA_j\right) \\
&=\left(\rho V_jA_j\right)\left\{\begin{array}{l}\tfrac{1}{2}R\left[-V_j\cos\theta\hat{\mathbf{e}}_Y+V_j\sin\theta\hat{\mathbf{e}}_X+\omega\tfrac{1}{2}R\left(\cos^2\theta+\sin^2\theta\right)\hat{\mathbf{e}}_Z\right]+ \\ R\left(\omega R-V_j\right)\left(\cos^2\theta+\sin^2\theta\right)\hat{\mathbf{e}}_Z\end{array}\right\} \\
&=\left(\rho V_jA_j\right)\left\{\tfrac{1}{2}RV_j\left(-\cos\theta\hat{\mathbf{e}}_Y+\sin\theta\hat{\mathbf{e}}_X\right)+\left[\tfrac{1}{4}\omega R^2+R\left(\omega R-V_j\right)\right]\hat{\mathbf{e}}_Z\right\} \\
&=\left(\rho V_jA_j\right)\left[\tfrac{1}{2}RV_j\left(-\cos\theta\hat{\mathbf{e}}_Y+\sin\theta\hat{\mathbf{e}}_X\right)+\left(\tfrac{5}{4}\omega R^2-RV_j\right)\hat{\mathbf{e}}_Z\right]
\end{aligned}
\tag{11}
$$

$$\therefore\int\limits_{\text{CS}}\left(\mathbf{r}_{XYZ}\times\mathbf{u}_{XYZ}\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\tfrac{1}{2}R\rho V_j^2A_j\left(\sin\theta\hat{\mathbf{e}}_X-\cos\theta\hat{\mathbf{e}}_Y\right)+\left(-R\rho V_j^2A_j+\tfrac{5}{4}\omega\rho V_jA_jR^2\right)\hat{\mathbf{e}}_Z \tag{12}$$

$$\mathbf{M}_B=\tfrac{1}{2}R\left(\cos\theta\hat{\mathbf{e}}_X+\sin\theta\hat{\mathbf{e}}_Y\right)\times-\rho RAg\hat{\mathbf{e}}_Z=-\tfrac{1}{2}\rho R^2Ag\left(\sin\theta\hat{\mathbf{e}}_X-\cos\theta\hat{\mathbf{e}}_Y\right) \tag{13}$$

$$\mathbf{M}_S=\mathbf{T} \tag{14}$$

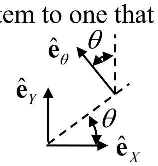
Substitute and simplify,

$$\tfrac{1}{2}R\rho V_j^2 A_j\left(\sin\theta\hat{\mathbf{e}}_X - \cos\theta\hat{\mathbf{e}}_Y\right) + \left(-R\rho V_j^2 A_j + \tfrac{5}{4}\omega\rho V_j A_j R^2\right)\hat{\mathbf{e}}_Z = -\tfrac{1}{2}\rho R^2 Ag\left(\sin\theta\hat{\mathbf{e}}_X - \cos\theta\hat{\mathbf{e}}_Y\right) + \mathbf{T} \qquad (15)$$

$$\mathbf{T} = \left(\tfrac{1}{2}R\rho V_j^2 A_j + \tfrac{1}{2}\rho R^2 Ag\right)\left(\sin\theta\hat{\mathbf{e}}_X - \cos\theta\hat{\mathbf{e}}_Y\right) + \left(-R\rho V_j^2 A_j + \tfrac{5}{4}\omega\rho V_j A_j R^2\right)\hat{\mathbf{e}}_Z \qquad (16)$$

If we compare the fixed coordinate system to one that rotates with the sprinkler as used in the first part of this problem, we find,

$$\hat{\mathbf{e}}_\theta = -\sin\theta\hat{\mathbf{e}}_X + \cos\theta\hat{\mathbf{e}}_Y \qquad (17)$$



Thus, Eq. (16) may be written as,

$$\boxed{\mathbf{T} = \left[-\tfrac{1}{2}R\rho V_j^2 A_j - \tfrac{1}{2}\rho R^2 Ag\right]\hat{\mathbf{e}}_\theta + \left[-R\rho V_j^2 A_j + \tfrac{5}{4}\omega\rho V_j A_j R^2\right]\hat{\mathbf{e}}_z} \qquad (18)$$

which is exactly the same as Eq. (8).

4.6. The First Law of Thermodynamics for a Control Volume

The reader should review the Introductory Thermodynamics chapter (Chapter 3) before continuing with this section.

To write the First Law for a control volume, we utilize the Reynolds Transport Theorem (RTT) to convert our system expression to a control volume expression. Let's first rewrite Eq. (3.32) using the Lagrangian derivative notation (we're interested in how things change with respect to time as we follow a particular system of fluid) and write the total energy of a system in terms of an integral,

$$\underbrace{\frac{D}{Dt}\int_{V_{\text{sys}}} e\rho dV}_{=E_{\text{sys}}} = \dot{Q}_{\text{into sys}} + \dot{W}_{\text{on sys}}. \tag{4.104}$$

Applying the Reynolds Transport Theorem and noting that the system and control volume are coincident at the time we apply the theorem gives,

$$\boxed{\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS} e\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{on CV}}}. \tag{4.105}$$

This is the First Law of Thermodynamics for a control volume!

Notes:

(1) The specific total energy is $e = u + \frac{1}{2}V^2 + G$ where G is a conservative potential energy function with the specific gravitational force given by $\mathbf{f}_{\text{gravity}} = -\boldsymbol{\nabla}G$. For the remainder of these notes, G will be assumed to be $G = gz$ ($\implies \mathbf{f}_{\text{gravity}} = -g\hat{\mathbf{e}}_z$) where g is the acceleration due to gravity.

Now let's expand the rate of work (power) term into rate of pressure work (pdV power) and the power due to other effects such as shaft work, viscous work, electric work, etc.,

$$\dot{W}_{\text{on CV}} = \dot{W}_{p,\text{on CV}} + \dot{W}_{\text{other,on CV}}. \tag{4.106}$$

In particular, we can write the rate of pressure work term for fluid crossing the boundary in the following

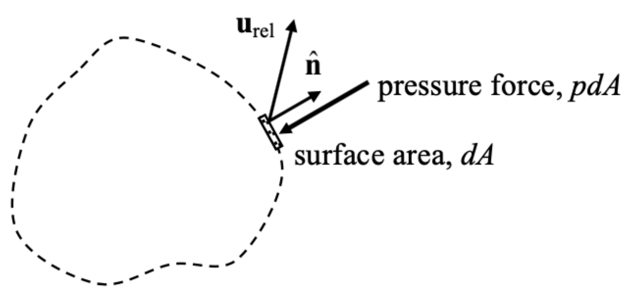


Figure 4.14. A schematic illustrating the rate of pressure work at the control surface.

way (Figure 4.14),

$$\begin{aligned} d\dot{W}_{p,\text{on CV}} &= d\mathbf{F}_{\text{p,on CV}}\cdot\mathbf{u}_{\text{rel}}, &(4.107)\\ &= (-pd\mathbf{A})\cdot\mathbf{u}_{\text{rel}}, &(4.108)\\ &= -p(\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}), &(4.109)\\ &= -\frac{p}{\rho}\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right). &(4.110)\end{aligned}$$

The rate of pressure work as fluid crosses the boundary over the entire CS is,

$$\boxed{\dot{W}_{p,\text{on CV}} = \int_{CS} -\frac{p}{\rho}\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)}. \tag{4.111}$$

Equation (4.111) is the rate at which pressure work is performed on the fluid flowing through the control surface.

Substituting Eqs. (4.111) and (4.106) into Eq. (4.105), expanding the specific total energy term in the surface integral, and bringing the rate of pressure work term to the left-hand side gives,

$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS}\left(u + \frac{p}{\rho} + \frac{1}{2}V^2 + gz\right)(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{other,on CV}}. \tag{4.112}$$

The quantity $(u + p/\rho)$ appears often in thermal-fluid systems and is given the special name of specific enthalpy, h,

$$h \coloneqq u + \frac{p}{\rho} = u + pv, \tag{4.113}$$

where $v = 1/\rho$ is the specific volume. Note that just as with internal energy, tables of thermodynamic properties typically list the value of the specific enthalpy for various substances at various conditions.

Substituting Eq. (4.113) into Eq. (4.112) gives,

$$\boxed{\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS}\left(h + \frac{1}{2}V^2 + gz\right)(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{other,on CV}}}. \tag{4.114}$$

Notes:

(1) An alternate way to write Eq. (4.114) is,

$$\frac{dE_{CV}}{dt} = \sum_{\text{all inlets}} \dot{m}\left(h + \frac{1}{2}V^2 + gz\right) - \sum_{\text{all outlets}} \dot{m}\left(h + \frac{1}{2}V^2 + gz\right) + \dot{Q}_{\text{into CV}} + \dot{W}_{\text{other,on CV}}. \tag{4.115}$$

The previous equation can be integrated in time,

$$\Delta E_{CV} = \sum_{\text{all inlets}} m\left(h + \frac{1}{2}V^2 + gz\right) - \sum_{\text{all outlets}} m\left(h + \frac{1}{2}V^2 + gz\right) + Q_{\text{into CV}} + W_{\text{other,on CV}}, \tag{4.116}$$

where m is the total mass entering/leaving the control volume. It has been assumed that the specific enthalpies, kinetic energy, and potential energies at the inlets and outlets don't change with time. This form of the First Law is useful for evaluating conditions at the end of an unsteady process. Note that if there are no inlets and outlets, then Eq. (4.116) simplifies to the system form of the First Law (Eq. (3.32)).

(2) The specific enthalpy term in Eq. (4.114) accounts for the rate of pressure work as fluid crosses the control surface, e.g., at inlets and outlets of the control volume. If there is pressure work caused by a moving, solid boundary through which no fluid flows, e.g., a moving piston, then that work would be included in the $\dot{W}_{\text{other,on CV}}$ term.

(3) During problem solving, we often must estimate the relative magnitudes of the terms in the total specific enthalpy term, i.e., $h_T = h + \frac{1}{2}V^2 + gz$. For example, consider a simple system operating at steady state with a single inlet and a single outlet. The inlet and outlet mass flow rates will be the same. The change in the total enthalpy between the inlet and outlet is (refer to Eq. (4.115)),

$$\dot{m}\Delta h_T = \dot{m}\left[\Delta h + \Delta\left(\frac{1}{2}V^2\right) + g\Delta z\right]. \tag{4.117}$$

Let's assume that $\Delta h \sim 1\,\mathrm{kJ\,kg^{-1}}$. To have an equivalent change in the kinetic energy, we would need $\Delta V \sim 45\,\mathrm{m\,s^{-1}}$. An equivalent change in the potential energy would require $\Delta z \sim 100\,\mathrm{m}$. Thus, it is often reasonable to neglect changes in kinetic and potential energies if the change in specific enthalpy is large and the changes in velocity and elevation are small.

(4) Let's examine the "other" work term more closely. This term includes work due to anything other than pressure work, such as work due to viscous forces, shaft work, electrical work, etc. In this note, let's examine the work done by viscous stresses. Consider the rate of viscous work done on the CV shown in Figure 4.15,

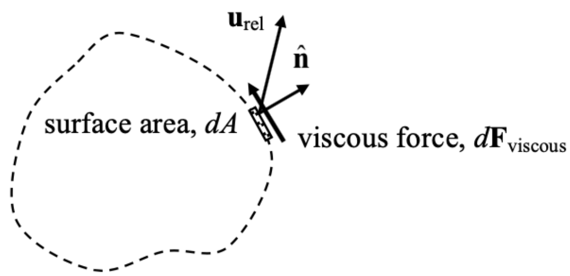


FIGURE 4.15. A schematic illustrating the rate of viscous work at the control surface.

$$d\dot{W}_{\text{viscous,on CV}} = d\mathbf{F}_{\text{viscous,on CV}} \cdot \mathbf{u}_{\text{rel}}, \tag{4.118}$$

so that the total rate of viscous work acting on the CS is,

$$\dot{W}_{\text{viscous,on CV}} = \int_{CS} d\mathbf{F}_{\text{viscous,on CV}} \cdot \mathbf{u}_{\text{rel}}. \tag{4.119}$$

(a) Note that at a solid boundary, $\mathbf{u}_{\text{rel}} = \mathbf{0}$ due to the no-slip condition so that the rate of viscous work is zero at solid surfaces. If the flow is inviscid, then $\mathbf{u}_{\text{rel}} \neq \mathbf{0}$, but $d\mathbf{F}_{\text{viscous,on CV}} = \mathbf{0}$ and so the rate of viscous work is zero for that case too.
(b) If the control volume is oriented such that the velocity vectors are perpendicular to the normal vectors of the CS, then the rate of viscous work done on the CV will be zero,

$$d\mathbf{F}_{\text{viscous,on CV}} \cdot \mathbf{u}_{\text{rel}} = 0, \tag{4.120}$$

since the viscous force will be perpendicular to the velocity vector. Thus, orienting the control surface so that it cuts perpendicularly across streamlines eliminates viscous work on the control volume.
(c) The rate of viscous work may not be negligible if the control volume is chosen as shown in Figure 4.16. Viscous forces along streamline surfaces may be significant if the shear stress there isn't negligible.

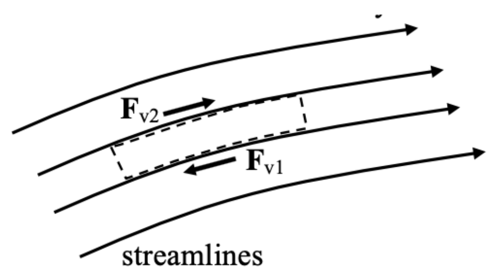


FIGURE 4.16. A schematic illustrating the viscous forces at the control surface if the control surface is tangential to the streamlines.

For the remainder of these notes, it will be assumed that the work on the CV due to viscous stresses is zero since our control surfaces will be chosen such that the surfaces are along solid boundaries or boundaries where viscous stresses are negligible (e.g., negligible velocity gradients), or with normal vectors perpendicular to the flow velocities.

(5) For a flow where the total energy within the CV does not change with time (steady state), Eq. (4.114) simplifies to,

$$\int_{CS} \left(h + \frac{1}{2}V^2 + gz\right)(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{other,on CV}}. \tag{4.121}$$

Note that flows may be unsteady at the local level, e.g., the localized flow within a pump, but may be steady at a larger scale, e.g., the average conditions within the pump housing.

(6) For a steady state, steady flow (meaning that the mass flow rate remains constant) with a single inlet (call it state 1) and outlet (call it state 2), we can write Eq. (4.121) as,

$$\left(h + \alpha\frac{1}{2}\bar{V}^2 + gz\right)_2 - \left(h + \alpha\frac{1}{2}\bar{V}^2 + gz\right)_1 = \dot{q}_{\text{into CV}} + \dot{w}_{\text{other,on CV}}. \tag{4.122}$$

where $q = \dot{Q}/\dot{m}$ and $w = \dot{W}/\dot{m}$ are the specific heat, i.e., the heat transfer per unit mass, and the specific work, i.e., the work per unit mass, respectively. Note that from COM the mass flow rate into the CV equals the mass flow rate out of the CV, i.e., $\dot{m}_{\text{in}} = \dot{m}_{\text{out}} = \dot{m}$.

(a) The average velocity through an area is,

$$\bar{V} := \frac{1}{A}\int_A (\mathbf{V}\cdot d\mathbf{A}). \tag{4.123}$$

(b) The quantity, α, is known as the kinetic energy correction factor. It is a correction factor accounting for the fact that an average velocity profile, $\bar{V}$, may not contain the same kinetic energy as a non-uniform velocity profile. For example, consider the kinetic energy contained in the two flow profiles shown in Figure 4.17. The average flow rate of specific kinetic energy is,

$$\overline{ke} = \int_A \frac{1}{2}V^2\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) \neq \frac{1}{2}\dot{m}\bar{V}^2, \tag{4.124}$$

in general. We define the kinetic energy correction factor, α, as,

$$\boxed{\alpha := \frac{\int_A \frac{1}{2}V^2\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)}{\frac{1}{2}\dot{m}\bar{V}^2}}. \tag{4.125}$$

so that,

$$\overline{ke} = \alpha\frac{1}{2}\dot{m}\bar{V}^2. \tag{4.126}$$

For a laminar flow in a circular pipe, the velocity profile is parabolic (discussed in a different chapter) resulting in $\alpha = 2$. For a turbulent flow, $\alpha \to 1$ as increasing turbulent mixing causes the velocity profile to become more uniform.

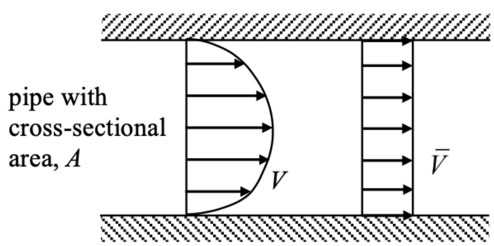


FIGURE 4.17. A schematic of a pipe flow with two different velocity profiles.

(c) The quantity,

$$h_T = h_0 := h + \alpha\frac{1}{2}\bar{V}^2 + gz, \tag{4.127}$$

is referred to as the total specific enthalpy, h_T or the stagnation specific enthalpy, h_0. Note that for gases, the gz term is much smaller than the other terms and, thus, is often neglected.

(d) If the flow is adiabatic ($q = 0$) and the rate of work by forces other than pressure can be neglected ($w_{\text{other}} = 0$), then,

$$h_T = h_0 = \text{constant}. \tag{4.128}$$

(7) Now let's re-write Eq. (4.122) but expand the specific enthalpy terms,

$$\left(u + \frac{p}{\rho} + \alpha\frac{1}{2}\bar{V}^2 + gz\right)_2 - \left(u + \frac{p}{\rho} + \alpha\frac{1}{2}\bar{V}^2 + gz\right)_1 = q_{\text{into CV}} + w_{\text{other,on CV}}. \tag{4.129}$$

Re-arranging terms and dividing through by the gravitational acceleration gives,

$$\left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_2 = \left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_1 - \frac{u_2 - u_1 - q_{\text{into CV}}}{g} + \frac{\dot{W}_{\text{other,on CV}}}{\dot{m}g}. \tag{4.130}$$

Each term in this equation is referred to as a head quantity and has the dimensions of length:

$$\frac{p}{\rho g} \coloneqq \text{pressure head} \tag{4.131}$$

$$\frac{\bar{V}^2}{2g} \coloneqq \text{velocity head} \tag{4.132}$$

$$z \coloneqq \text{elevation head} \tag{4.133}$$

$$\frac{u_2 - u_1 - q_{\text{into CV}}}{g} = H_L \coloneqq \text{head loss} \tag{4.134}$$

$$\frac{\dot{W}_{\text{shaft,on CV}}}{\dot{m}g} = H_S \coloneqq \text{shaft head} \tag{4.135}$$

The head loss is the head lost due to mechanical energy being converted to thermal energy and energy lost via heat transfer to the surroundings. The "other" work term frequently only includes shaft work, particularly in pipe flow systems, and so the shaft head is a convenient definition. It is the head added to the flow due to shaft work.

The equation in this form is known as the Extended Bernoulli Equation,

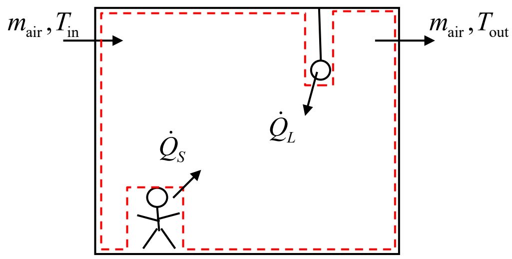
$$\boxed{\left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_2 = \left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_1 - H_L + H_S}, \tag{4.136}$$

where it has been assumed that the only form of "other" work is shaft work.

Let's consider some examples to see how the First Law is applied to control volumes.

Consider a large classroom on a hot summer day with 150 students, each dissipating 60 W of sensible heat. All the lights, with 4.0 kW of rated power, are kept on. The room has no external walls, and thus heat gain through the walls and the roof is negligible. Chilled air is available at 15 °C and the temperature of the return air is not to exceed 25 °C. Determine the required flow rate of air, in kg/s, that needs to be supplied to the room to keep the average temperature of the room constant.

SOLUTION:



Apply the First Law to the CV shown. Assume the conditions in the CV are steady and uniform and that the inlet and outlet flows are also steady and uniform,

$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{other,}\\ \text{on CV}}} \tag{1}$$

where,

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady state in the CV)} \tag{2}$$

$$\int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{m}\left(h_{\text{out}}-h_{\text{in}}\right) \tag{3}$$

The differences in the inlet and outlet kinetic and potential energies are assumed negligible and, since the flow is steady, the inlet and outlet mass flow rates are identical.

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}} = \dot{Q}_{\text{L}} + \dot{Q}_{\text{S}} \quad \text{(The influx of heat is due to the lights and the students.)} \tag{4}$$

$$\dot{W}_{\substack{\text{other,}\\ \text{on CV}}} = 0 \quad \text{(There is no work performed on the CV.)} \tag{5}$$

Substitute and simplify,

$$\dot{m}\left(h_{\text{out}}-h_{\text{in}}\right) = \dot{Q}_{\text{L}} + \dot{Q}_{\text{S}} \Rightarrow \dot{m} = \frac{\dot{Q}_{\text{L}} + \dot{Q}_{\text{S}}}{h_{\text{out}}-h_{\text{in}}} \tag{6}$$

Given that,

$$\dot{Q}_{\text{L}} = 4.0\text{ kW} \tag{7}$$

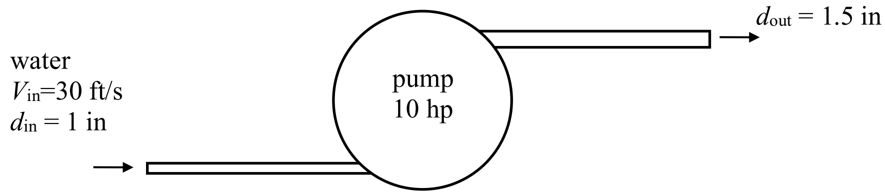
$$\dot{Q}_{\text{S}} = \text{(150 students)(60 W/student)} = 9.0\text{ kW} \tag{8}$$

$$h_{\text{out}} = 298.18\text{ kJ/kg} \quad \text{(thermodynamics tables for air at } T_{\text{out}} = 25\text{ °C} = 298\text{ K)} \tag{9}$$

$$h_{\text{in}} = 288.15\text{ kJ/kg} \quad \text{(thermodynamics tables for air at } T_{\text{in}} = 15\text{ °C} = 288\text{ K)} \tag{10}$$

$$\boxed{\therefore \dot{m} = 1.30\ \text{kg/s}} \tag{11}$$

Determine the maximum pressure increase across the 10 hp pump shown in the figure.



SOLUTION:

Apply the First Law to a control volume surrounding the pump,

$$\frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}}\left(h+\tfrac{1}{2}V^2+gz\right)\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = \dot{Q}_{\text{into CV}} + \dot{W}_{\substack{\text{on CV,}\\ \text{other}}}, \tag{1}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} e\rho\, dV = 0 \quad \text{(steady flow)}, \tag{2}$$

$$\int_{\text{CS}}\left(h+\tfrac{1}{2}V^2+gz\right)\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = \dot{m}\left[\left(h_{\text{out}}-h_{\text{in}}\right)+\tfrac{1}{2}\left(V_{\text{out}}^2-V_{\text{in}}^2\right)\right] \quad \text{(neglecting elevation differences)}, \tag{3}$$

$$\dot{Q}_{\text{into CV}} = 0, \tag{4}$$

$$\dot{W}_{\substack{\text{on CV,}\\ \text{other}}} = \text{given} \tag{5}$$

Assuming the water is an incompressible fluid, re-write the change in specific enthalpy as,

$$h_{\text{out}} - h_{\text{in}} = c\left(T_{\text{out}}-T_{\text{in}}\right) + \frac{1}{\rho}\left(p_{\text{out}}-p_{\text{in}}\right). \tag{6}$$

Substitute into Eq. (1) and simplify,

$$\dot{m}\left[c\left(T_{\text{out}}-T_{\text{in}}\right)+\frac{1}{\rho}\left(p_{\text{out}}-p_{\text{in}}\right)+\tfrac{1}{2}\left(V_{\text{out}}^2-V_{\text{in}}^2\right)\right] = \dot{W}_{\substack{\text{on CV,}\\ \text{other}}} \tag{7}$$

In this particular case we're asked to find the maximum pressure rise across the pump, which would correspond to no temperature change, i.e., $T_{\text{out}} = T_{\text{in}}$. Thus,

$$\dot{m}\left[\frac{1}{\rho}\left(p_{\text{out}}-p_{\text{in}}\right)+\tfrac{1}{2}\left(V_{\text{out}}^2-V_{\text{in}}^2\right)\right] = \dot{W}_{\substack{\text{on CV,}\\ \text{other}}}. \tag{8}$$

The velocity at the outlet may be found by applying conservation of mass to the same control volume,

$$V_{\text{out}} = V_{\text{in}}\left(\frac{d_{\text{in}}}{d_{\text{out}}}\right)^2. \tag{9}$$

Substitute and simplify Eq. (8),

$$\dot{m}\left\{\frac{1}{\rho}\left(p_{\text{out}}-p_{\text{in}}\right)+\tfrac{1}{2}V_{\text{in}}^2\left[\left(\frac{d_{\text{in}}}{d_{\text{out}}}\right)^4-1\right]\right\} = \dot{W}_{\substack{\text{on CV,}\\ \text{other}}}, \tag{10}$$

$$p_{\text{out}} - p_{\text{in}} = \frac{\rho\dot{W}_{\substack{\text{on CV,}\\ \text{other}}}}{\dot{m}} + \frac{1}{2}\rho V_{\text{in}}^2\left[1-\left(\frac{d_{\text{in}}}{d_{\text{out}}}\right)^4\right]. \tag{11}$$

Using the given parameters,

ρ = 62.4 $\text{lb}_\text{m}/\text{ft}^3$ (approximate density of liquid water at standard conditions)

V_{in} = 30 ft/s

d_{in} = 1 in. = (1/12) ft

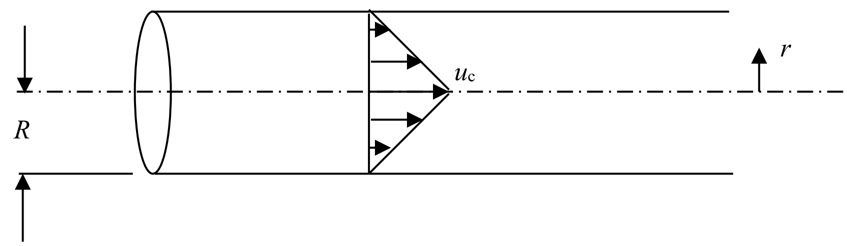
d_{out} = 1.5 in.

$\dot{W}_{\substack{\text{on CV,}\\ \text{other}}}$ = 10 hp = 5500 $\text{ft.lb}_\text{f}/\text{s}$

$\dot{m} = \rho V_{\text{in}}(\pi d_{\text{in}}^2/4) = 10.2\ \text{lb}_\text{m}/\text{s}$

$\Rightarrow \boxed{p_{\text{out}} - p_{\text{in}} = 34300\ \text{lb}_\text{f}/\text{ft}^2 = 238\ \text{lb}_\text{f}/\text{in}^2}$

The velocity profile for a particular pipe flow is linear from zero at the wall to a maximum of u_c at the centerline. Determine the average velocity and the kinetic energy correction factor.



SOLUTION:
The average velocity is found by setting the volumetric flow rate using the average velocity profile equal to the volumetric flow rate using the real profile,

$$Q_{\substack{\text{avg}\\\text{profile}}} = Q_{\substack{\text{real}\\\text{profile}}} \tag{1}$$

$$\bar{u}\pi R^2 = \int_{r=0}^{r=R} u_C\left(1-\frac{r}{R}\right)(2\pi r dr)$$

$$= 2\pi u_C \int_{r=0}^{r=R}\left(r-\frac{r^2}{R}\right)dr$$

$$= 2\pi u_C\left[\frac{1}{2}r^2-\frac{1}{3}\frac{r^3}{R}\right]_{r=0}^{r=R}$$

$$\boxed{\therefore \bar{u} = \tfrac{1}{3}u_C} \tag{2}$$

The kinetic energy correction factor, α, is found by equating the kinetic energy flow rate using the average velocity with the kinetic energy flow rate using the actual velocity profile,

$$\alpha\tfrac{1}{2}\underbrace{\left(\rho\bar{u}\pi R^2\right)}_{=\dot{m}}\bar{u}^2 = \int_{r=0}^{r=R}\tfrac{1}{2}\underbrace{\left[\rho u_C\left(1-\frac{r}{R}\right)(2\pi r dr)\right]}_{=d\dot{m}}\left[u_C\left(1-\frac{r}{R}\right)\right]^2$$

$$= \int_{r=0}^{r=R}\tfrac{1}{2}\rho u_C^3\left(1-\frac{r}{R}\right)^3(2\pi r dr) \tag{3}$$

$$= \pi\rho u_C^3\int_{r=0}^{r=R}\left(1-\frac{r}{R}\right)^3(r dr)$$

where $\bar{u} = \tfrac{1}{3}u_C$. Solving the previous equation for α gives,

$$\boxed{\alpha = \frac{27}{10} = 2.7} \tag{4}$$

Air at 10 °C and 80 kPa (abs) enters the diffuser of a jet engine steadily with a velocity of 200 m/s. The inlet area of the diffuser is 0.4 m^2. The air leaves the diffuser with a velocity that is very small compared with the inlet velocity. Determine

a. the mass flow rate of the air and

b. the temperature of the air leaving the diffuser.

You may assume adiabatic flow through the diffuser.

SOLUTION:

The mass flow rate of the air may be found from,

$$\dot{m} = \rho_1 V_1 A_1 \;\Rightarrow\; \dot{m} = \left(\frac{p_1}{RT_1}\right) V_1 A_1 \tag{1}$$

Using the given data,

p_1 = 80 kPa (abs)

R = 287 J/(kg·K)

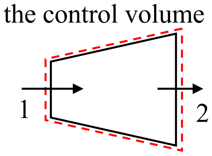
T_1 = 10 ºC = 283 K

V_1 = 200 m/s

A_1 = 0.4 m^2

$\Rightarrow \boxed{\dot{m} = 78.8\text{ kg/s}}$

Now apply the First Law to the control volume shown below.



$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{other,}\\ \text{on CV}}} \tag{2}$$

where,

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady flow)} \tag{3}$$

$$\int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m}_2\left(h_2 + \tfrac{1}{2}V_2^2\right) - \dot{m}_1\left(h_1 + \tfrac{1}{2}V_1^2\right) \tag{4}$$

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}} = 0 \quad \text{(Flow through diffusers is usually assumed to occur adiabatically.)} \tag{5}$$

$$\dot{W}_{\substack{\text{other,}\\ \text{on CV}}} = 0 \tag{6}$$

Note that from conservation of mass for the same control volume,

$$\dot{m}_2 = \dot{m}_1 = \dot{m} \tag{7}$$

Substitute into the First Law and simplify,

$$\dot{m}\left(h_2 + \tfrac{1}{2}V_2^2\right) - \dot{m}\left(h_1 + \tfrac{1}{2}V_1^2\right) = 0 \quad \text{(Note: } V_2 << V_1.\text{)} \tag{8}$$

$$\boxed{h_2 = h_1 + \tfrac{1}{2}V_1^2} \tag{9}$$

From a thermodynamics table for air, h_1 = 283.14 kJ/kg @ 283 K giving,

h_2 = 303.14 kJ/kg $\Rightarrow$ $\boxed{T_2 = 302.9\text{ K} \approx 30\text{ ºC}}$ (from a thermodynamics table for air)

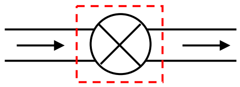
Alternately, if air is assumed to behave as a perfect gas so that $\Delta h = c_p \Delta T$ with c_p = 1000 J/(kg·K), then,

$$\boxed{T_2 = T_1 + \frac{V_1^2}{2c_p}} \;\Rightarrow\; T_2 = 303\text{ K} \quad \text{(Same as before!)} \tag{10}$$

A well-insulated valve is used to throttle steam from 8 MPa (abs) and 500 °C to 6 MPa (abs). Determine the final temperature of the steam.

SOLUTION:

$p_i = 8$ MPa
$T_i = 500$ °C



$p_o = 6$ MPa
$T_o = ?$

Apply the First Law to the CV shown assuming 1D, steady flow, no heat transfer (the valve is well-insulated), and no "other" work done on the CV besides pressure work.

$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{other,}\\ \text{on CV}}} \tag{1}$$

where,

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady flow)} \tag{2}$$

$$\int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m}\left(h_o - h_i\right) \tag{3}$$

(The changes in kinetic and potential energies are assumed negligible and, since the flow is steady and 1D, the mass flow rate is the same at the inlet and outlet.)

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}} = 0 \quad \text{(The valve is well-insulated.)} \tag{4}$$

$$\dot{W}_{\substack{\text{other,}\\ \text{on CV}}} = 0 \quad \text{(No work is performed other than pressure work.)} \tag{5}$$

Substitute and simplify,

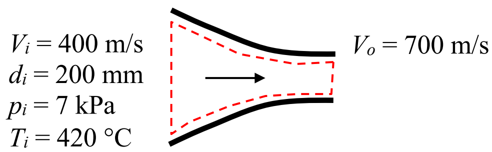
$$\dot{m}\left(h_o - h_i\right) = 0 \;\Rightarrow\; \boxed{h_o = h_i} \tag{6}$$

From steam tables at $p_i = 8$ MPa = 800 bars, $T_i = 500$ °C, $h_i = 3398.05$ kJ/kg. Again, using the steam tables for $h_o = 3398.05$ kJ/kg and $p_o = 6$ MPa = 600 bars, $\boxed{T_o = 490\text{ °C}}$. Hence, the temperature drops by 10 °C across the valve.

Air flows through a nozzle with an inlet diameter of 200 mm, velocity of 400 m/s, pressure of 7 kPa (abs), and temperature of 420 °C. The nozzle exit diameter is adjusted such that the exiting velocity is 700 m/s. Determine:

a. the exit temperature, and
b. the mass flow rate through the nozzle

SOLUTION:



Apply the First Law to the CV shown assuming 1D, steady flow, no heat transfer, and no "other" work done on the CV besides pressure work,

$$\frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\\text{CV}}} + \dot{W}_{\substack{\text{other,}\\\text{on CV}}} \tag{1}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} e\rho dV = 0 \quad \text{(steady state)} \tag{2}$$

$$\int_{\text{CS}}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{m}\left(h_o+\tfrac{1}{2}V_o^2-h_i-\tfrac{1}{2}V_i^2\right) \tag{3}$$

(The change in potential energies is assumed negligible and, since the flow is steady and 1D, the mass flow rate is the same at the inlet and outlet.)

$$\dot{Q}_{\substack{\text{into}\\\text{CV}}} = 0 \quad \text{(Assume little heat transfer occurs over the nozzle surface area.)} \tag{4}$$

$$\dot{W}_{\substack{\text{other,}\\\text{on CV}}} = 0 \quad \text{(No work is performed other than pressure work.)} \tag{5}$$

Substitute and simplify,

$$\dot{m}\left(h_o+\tfrac{1}{2}V_o^2-h_i-\tfrac{1}{2}V_i^2\right)=0 \;\Rightarrow\; \boxed{h_o = h_i + \tfrac{1}{2}\left(V_i^2 - V_o^2\right)} \tag{6}$$

For the given conditions,

h_i = 705.75 kJ/kg (from thermodynamics tables for air, assumed to be an ideal gas, at T_i = 420 °C)
V_i = 400 m/s
V_o = 700 m/s
$\Rightarrow$ h_o = 540.75 kJ/kg $\Rightarrow$ $\boxed{T_o = 264\ °\text{C}}$ (from thermo tables assuming air is an ideal gas)

The mass flow rate is,

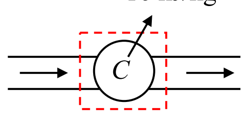
$$\boxed{\dot{m} = \rho_i V_i A_i = \frac{p_i}{RT_i}V_i \tfrac{\pi}{4}d_i^2} \tag{7}$$

Using the given parameters:

p_i = 7 kPa
R = 287 J/(kg·K)
T_i = 420 °C = 673 K
V_i = 400 m/s
d_i = 0.2 m
$\Rightarrow$ $\boxed{\dot{m} = 0.44\ \text{kg/s}}$

Air at 100 kPa and 280 K is compressed steadily to 600 kPa and 400 K. The mass flow rate of the air is 0.02 kg/s and a heat loss per unit of flowing mass of 16 kJ/kg occurs during the process. Assuming the changes in kinetic and potential energies are negligible, determine the necessary power input to the compressor.

SOLUTION:



Apply the First Law to a control volume (CV) surrounding the compressor,

$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{other,}\\ \text{on CV}}} \tag{1}$$

where,

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady state)} \tag{2}$$

$$\int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \dot{m}\left(h_o-h_i\right) \tag{3}$$

(Changes in KE and PE are negligible; the mass flow rate at the inlet and outlet are the same since the flow is steady and 1D.)

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}}\Big/\dot{m} = -16\ \text{kJ/kg} \quad \text{(conduction through the sides of the can)} \tag{4}$$

$$\dot{W}_{\substack{\text{other,}\\ \text{on CV}}} = \dot{W}_{\substack{\text{shaft,}\\ \text{on CV}}} \tag{5}$$

Substitute and simplify,

$$\dot{m}\left(h_o-h_i\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{shaft,}\\ \text{on CV}}} \tag{6}$$

$$\boxed{\dot{W}_{\substack{\text{shaft,}\\ \text{on CV}}} = \dot{m}\left(h_o-h_i\right) - \dot{Q}_{\substack{\text{into}\\ \text{CV}}}} \tag{7}$$

For the given conditions,

h_o = 400.98 kJ/kg (thermodynamic tables for air at p_o = 100 kPa, T_o = 280 K)
h_i = 280.13 kJ/kg (thermodynamic tables for air at p_i = 600 kPa, T_i = 400 K)
$\dot{Q}_{\substack{\text{into}\\ \text{CV}}}\Big/\dot{m}$ = -16 kJ/kg
$\dot{m}$ = 0.02 kg/s

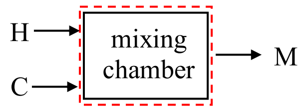
$$\Rightarrow \boxed{\dot{W}_{\substack{\text{shaft,}\\ \text{on CV}}} = 2.74\ \text{kW}}$$

Thus, 2.74 kW must be supplied to the air. The power supplied to the compressor would, in fact, be larger than this due to inefficiencies in the compressor.

Consider an ordinary shower where hot water at 140 °F is mixed with cold water at 50 °F. If it is desired that a steady stream of warm water at 110 °F be supplied, determine the ratio of the mass flow rates of the hot to cold water. Assume the heat losses from the mixing chamber to be negligible and the mixing to take place at a pressure of 20 psia.

SOLUTION:

Apply the First Law to the control volume shown below,



$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{on}\\ \text{CV}}} \tag{1}$$

where,

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = -\dot{m}_H h_H - \dot{m}_C h_C + \dot{m}_M h_M$$

(Neglect changes in potential and kinetic energies since they won't be significant for flow through a shower head.)

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}} = 0 \quad \text{(negligible heat transfer to the surroundings} \Rightarrow \text{assume adiabatic)}$$

$$\dot{W}_{\substack{\text{on}\\ \text{CV}}} = 0 \quad \text{(no work besides pressure work is being done on the CV)}$$

Substitute,

$$-\dot{m}_H h_H - \dot{m}_C h_C + \dot{m}_M h_M = 0 \tag{2}$$

Apply conservation of mass to the same control volume to find,

$$\dot{m}_M = \dot{m}_H + \dot{m}_C \tag{3}$$

Combine Eqs. (2) and (3) and simplify,

$$-\dot{m}_H h_H - \dot{m}_C h_C + \left(\dot{m}_H + \dot{m}_C\right) h_M = 0$$

$$-\frac{\dot{m}_H}{\dot{m}_C} h_H - h_C + \left(\frac{\dot{m}_H}{\dot{m}_C} + 1\right) h_M = 0$$

$$\frac{\dot{m}_H}{\dot{m}_C}\left(h_M - h_H\right) = h_C - h_M$$

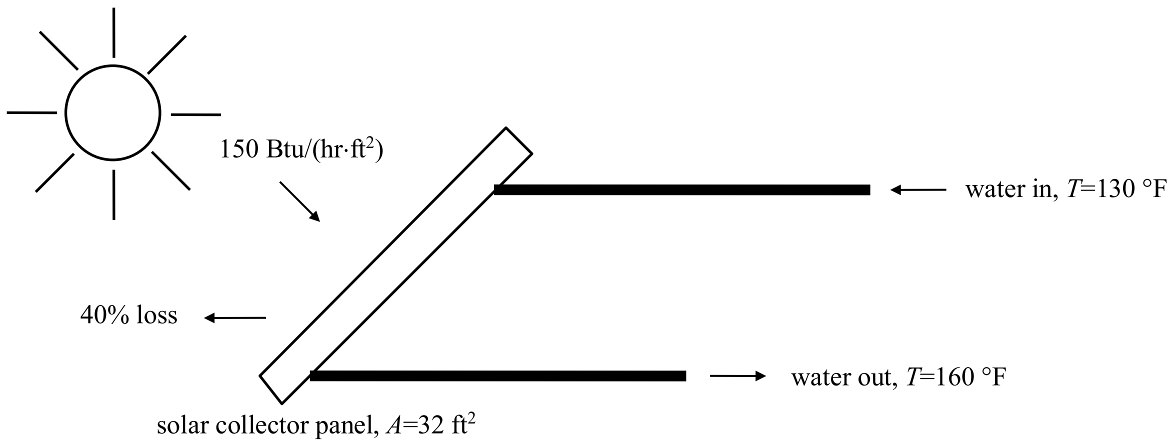
$$\boxed{\therefore \frac{\dot{m}_H}{\dot{m}_C} = \frac{h_C - h_M}{h_M - h_H}} \tag{4}$$

Look up the specific enthalpies for water in a thermodynamics reference (note that since the water is a pure liquid, the mixing pressure is irrelevant):

$h_C = h_{50\ °F} = 18.06\ \text{Btu/lb}_m$
$h_H = h_{140\ °F} = 107.96\ \text{Btu/lb}_m$
$h_M = h_{110\ °F} = 78.02\ \text{Btu/lb}_m$

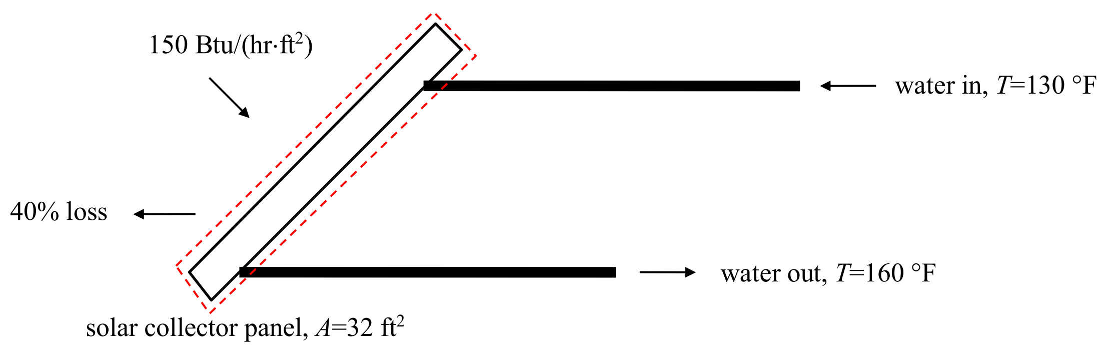
$$\boxed{\therefore \frac{\dot{m}_H}{\dot{m}_C} = 2.0}$$

The figure below shows a solar collector panel with a surface area of 32 ft^2. The panel receives energy from the sun at a rate of 150 Btu/hr per ft^2 of collector surface. Forty percent of the incoming energy is lost to the surroundings. The remainder is used to warm liquid water from 130 to 160 °F. The water passes through the solar collector with a negligible pressure drop. Neglecting kinetic and potential energy effects, determine at steady state the mass flow rate of the water in lb$_m$/min. How many solar collectors would be needed to provide a total of 40 gal of 160 °F water in 30 min?



SOLUTION:

Apply the First Law to the control volume shown below.



$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{into\\CV}} + \dot{W}_{\substack{on\\CV}}$$

where,

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady state)}$$

$$\int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = -\left(\dot{m}h\right)_{in} + \left(\dot{m}h\right)_{out} \quad \text{(Neglect changes in kinetic and potential energies.)}$$

$$\dot{Q}_{\substack{into\\CV}} = \dot{Q}_{\substack{in\ from\\sun}} - \dot{Q}_{\substack{out\ from\\panel}} = \left(1-\alpha\right)\dot{Q}_{\substack{in\ from\\sun}} \quad \text{(where } \alpha \text{ is the fraction of heat lost from the panel)}$$

$$\dot{W}_{\substack{on\\CV}} = 0$$

Substitute and simplify,

$$-\left(\dot{m}h\right)_{\text{in}} + \left(\dot{m}h\right)_{\text{out}} = \left(1-\alpha\right)\dot{Q}_{\substack{\text{in from}\\ \text{sun}}} \tag{1}$$

Applying conservation of mass to the same control volume gives,

$$\dot{m}_{\text{in}} = \dot{m}_{\text{out}} \tag{2}$$

Combine Eqs. (1) and (2) and simplify,

$$\boxed{\dot{m} = \frac{\left(1-\alpha\right)\dot{Q}_{\substack{\text{in from}\\ \text{sun}}}}{h_{\text{out}} - h_{\text{in}}}} \tag{3}$$

Using the given data:

α = 0.40

$\dot{Q}_{\substack{\text{in from}\\ \text{sun}}}$ = [150 Btu/(hr·ft^2)][32 ft^2] = 4800 Btu/hr

h_{out} = 127.96 Btu/lb$_m$ (water @ 160 °F, from thermodynamics tables)

h_{in} = 97.98 Btu/lb$_m$ (water @ 130 °F, from thermodynamics tables)

$\Rightarrow$ $\boxed{\dot{m} = 96.1\ \text{lb}_m/\text{hr} = 1.60\ \text{lb}_m/\text{min}}$

To get 40 gal of 160 °F water in 30 min,

$$\boxed{\text{Vol} = nQ\Delta t} \tag{4}$$

where Vol is the total volume, n is the number of solar collectors, Q is the volumetric flow rate, and Δt is the duration. Using the given data:

Vol = 40 gal = 5.35 ft^3

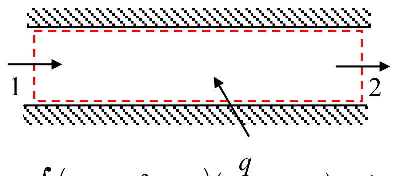
Q = (1.60 lb$_m$/min)/(62.4 lb$_m$/ft^3) = 2.56*10^{-2} ft^3/min

Δt = 30 min

$\Rightarrow$ n = 6.95 $\Rightarrow$ $\boxed{\text{Seven solar collectors would be required.}}$

Carbon dioxide flows through a constant area duct. At the inlet to the duct, the velocity is 120 m/s and the temperature and pressure are 200 °C and 700 kPa (abs), respectively. Heat is added to the flow in the duct and at the exit of the duct the velocity is 240 m/s and the temperature is 450 °C. Find the amount of heat being added to the carbon dioxide per unit mass of gas and the mass flow rate through the duct per unit cross-sectional area of the duct. Assume that the specific heat ratio for carbon dioxide is 1.3 and the gas constant is 189 J/(kg·K).

SOLUTION:
Apply the First Law to the control volume shown below,



$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{other,}\\ \text{on CV}}} \tag{1}$$

where,

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady flow)} \tag{2}$$

$$\int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{m}_2\left(h_2+\tfrac{1}{2}V_2^2\right) - \dot{m}_1\left(h_1+\tfrac{1}{2}V_1^2\right) \tag{3}$$

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}} = ? \tag{4}$$

$$\dot{W}_{\substack{\text{other,}\\ \text{on CV}}} = 0 \tag{5}$$

Note that from conservation of mass for the same control volume,

$$\dot{m}_2 = \dot{m}_1 = \dot{m} \tag{6}$$

Substitute into the First Law and simplify,

$$\dot{m}\left(h_2+\tfrac{1}{2}V_2^2\right) - \dot{m}\left(h_1+\tfrac{1}{2}V_1^2\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} \tag{7}$$

$$q_{\substack{\text{into}\\ \text{CV}}} = \left(h_2-h_1\right)+\tfrac{1}{2}\left(V_2^2-V_1^2\right) \tag{8}$$

Assume that CO_2 behaves as a perfect gas so that $\Delta h = c_p\Delta T$ and Eq. (8) becomes,

$$\boxed{q_{\substack{\text{into}\\ \text{CV}}} = c_p\left(T_2-T_1\right)+\tfrac{1}{2}\left(V_2^2-V_1^2\right)} \tag{9}$$

Note that a more accurate solution would not use the perfect gas assumption, but would instead evaluate the specific enthalpies directly using thermodynamic property tables.

Using the given data:

γ = 1.3 $\quad$ R = 189 J/(kg·K) $\quad$ $c_p = \dfrac{\gamma R}{\gamma - 1}$ = 819 J/(kg·K)

T_2 = 450 ºC = 723 K $\quad$ T_1 = 200 ºC = 473 K
V_2 = 240 m/s $\quad$ V_1 = 120 m/s
$\Rightarrow \boxed{q_{\text{into CV}} = 226 \text{ kJ/kg}}$

The mass flow rate per unit area is simply,

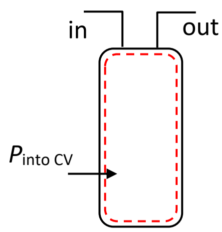
$$\dot{m} = \rho_1 V_1 A \;\Rightarrow\; \frac{\dot{m}}{A} = \rho_1 V_1 = \left(\frac{p_1}{RT_1}\right)V_1 \;\Rightarrow\; \boxed{\frac{\dot{m}}{A} = 940 \text{ kg/s/m}^2} \tag{10}$$

where p_1 = 700 kPa (abs).

If the water in a well-insulated, 50 gal electric water heater initially has the same temperature as the inlet water temperature (55 °F), determine how long it will take for the water at the outlet to reach a comfortable shower temperature of 105 °F if the water flows continuously in the shower at a rate of 2 gal/min (a typical flow rate for a shower). What will be the steady state temperature in the water heater for these conditions? This particular water heater can provide 4500 W of power to the heating element. The inlet supply line has a pressure of 50 psi and the pressure in the tank is 70 psi.

SOLUTION:

Apply the First Law to a control volume surrounding the water heater tank.



$$\frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{on}\\ \text{CV}}} \tag{1}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} e\rho dV = \frac{d}{dt}\left(u_{CV}\rho V_{CV}\right) = \rho V_{CV}\frac{du_{CV}}{dt} \quad \text{(only the internal energy changes in the CV)} \tag{2}$$

$$\int_{\text{CS}}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{m}\left(h_{CV}-h_{\text{in}}\right) \quad \text{(same velocity in and out, same elevation)} \tag{3}$$

Note that the water leaving the control volume is assumed to have the same properties as the water in the CV.

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}} = 0 \quad \text{(the tank is well insulated)} \tag{4}$$

$$\dot{W}_{\substack{\text{on}\\ \text{CV}}} = P \quad \text{(the input electrical power)} \tag{5}$$

Substitute and simplify,

$$\rho V_{CV}\frac{du_{CV}}{dt} + \dot{m}\left(h_{CV}-h_{\text{in}}\right) = P \tag{6}$$

Model water as an incompressible substance with constant specific heat,

$$\frac{du_{CV}}{dt} = c\frac{dT_{CV}}{dt} \quad \text{and} \quad h_{CV}-h_{\text{in}} = c\left(T_{CV}-T_{\text{in}}\right) + \frac{1}{\rho}\left(p_{CV}-p_{\text{in}}\right) \tag{7}$$

Substitute into Eq. (6),

$$\rho V_{CV}c\frac{dT_{CV}}{dt} + \dot{m}c\left(T_{CV}-T_{\text{in}}\right) + \frac{\dot{m}}{\rho}\left(p_{CV}-p_{\text{in}}\right) = P \tag{8}$$

$$\rho V_{CV}c\frac{dT_{CV}}{dt} + \dot{m}cT_{CV} - \dot{m}cT_{\text{in}} + \frac{\dot{m}}{\rho}\left(p_{CV}-p_{\text{in}}\right) = P \tag{9}$$

$$\frac{dT_{CV}}{dt}+\left(\frac{\dot{m}}{\rho V_{CV}}\right)T_{CV}=\frac{\dot{m}cT_{\text{in}}+\frac{\dot{m}}{\rho}\left(p_{\text{in}}-p_{CV}\right)+P}{\rho V_{CV}c} \tag{10}$$

$$\frac{dT_{CV}}{dt}+\alpha T_{CV}=\beta \tag{11}$$

where,

$$\alpha=\frac{\dot{m}}{\rho V_{CV}} \quad \text{and} \quad \beta=\frac{\dot{m}}{\rho V_{CV}}T_{\text{in}}+\frac{\dot{m}\left(p_{\text{in}}-p_{CV}\right)}{\rho^2 cV_{CV}}+\frac{P}{\rho cV_{CV}}=\frac{1}{\rho cV_{CV}}\left[\dot{m}cT_{\text{in}}+\frac{\dot{m}\left(p_{\text{in}}-p_{CV}\right)}{\rho}+P\right] \tag{12}$$

and,

$$T_{CV}\left(t=0\right)=T_0 \tag{13}$$

where T_0 is the initial water temperature in the tank.

Also note that,

$$\frac{\beta}{\alpha}=T_{\text{in}}+\frac{\left(p_{\text{in}}-p_{CV}\right)}{\rho c}+\frac{P}{\dot{m}c} \tag{14}$$

The solution to the ODE given in Eq. (11) subject to the initial condition given in Eq. (13) is,

$$\int_{T_0}^{T}\frac{dT_{CV}}{\beta-\alpha T_{CV}}=\int_0^t dt \tag{15}$$

$$-\frac{1}{\alpha}\ln\left(\frac{\beta-\alpha T}{\beta-\alpha T_0}\right)=t \tag{16}$$

$$\boxed{T=\frac{\beta}{\alpha}-\left(\frac{\beta}{\alpha}-T_0\right)\exp\left(-\alpha t\right)} \tag{17}$$

The time to reach a particular temperature may be found by re-arranging Eq. (17),

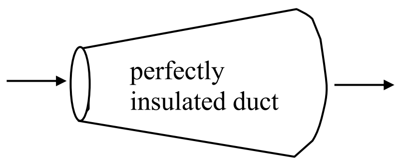
$$\boxed{t=-\frac{1}{\alpha}\ln\left[\frac{\left(\frac{\beta}{\alpha}-T\right)}{\left(\frac{\beta}{\alpha}-T_0\right)}\right]} \tag{18}$$

The steady state temperature is found by letting $t\to\infty$,

$$\boxed{T_{t\to\infty}=\frac{\beta}{\alpha}} \tag{19}$$

Using the given parameters, it is not possible to reach the desired shower temperature. In fact, the steady state temperature is only 70 °F. In order to reach the desired temperature, one would need to decrease the flow rate. A flow rate of approximately 0.6 gpm will give a steady state temperature of 105 °F, but it will take a long time to get there and you'll waste a lot of water!

A steady flow of viscous air at 20 °C and 1 atm enters a perfectly insulated, horizontal, circular duct at a velocity of 3 m/s. The duct diameter increases in the direction of flow as shown in the figure. There are no electrical lines or rotating shafts within the duct.



As the air flows through the duct, the temperature

A. increases

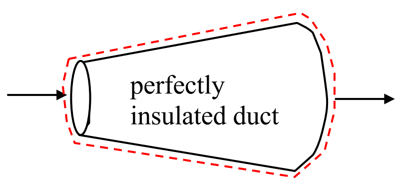
B. decreases

C. remains the same

D. there is insufficient information to determine the trend

SOLUTION:

Apply the First Law to the control volume shown below.



$$\frac{d}{dt}\int_{\mathrm{CV}} e\rho\, dV + \int_{\mathrm{CS}} \left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\mathrm{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\mathrm{into}\\ \mathrm{CV}}} + \dot{W}_{\substack{\mathrm{other}\\ \mathrm{on\ CV}}} \tag{1}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} e\rho dV = 0 \quad \text{(steady flow)} \tag{2}$$

$$\int_{\mathrm{CS}} \left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\mathrm{rel}}\cdot d\mathbf{A}\right) = \dot{m}\left[\Delta h+\tfrac{1}{2}\Delta\left(V^2\right)\right] \quad \text{(no elevation changes)} \tag{3}$$

$$\dot{Q}_{\substack{\mathrm{into}\\ \mathrm{CV}}} = 0 \quad \text{(perfectly insulated duct)} \tag{4}$$

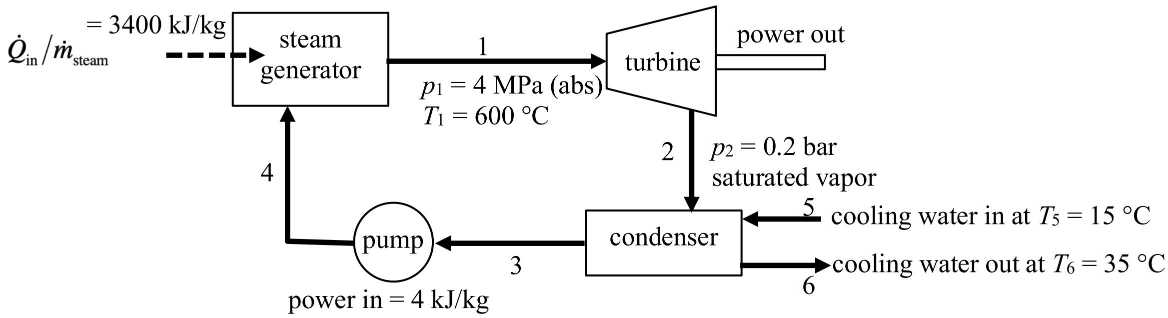
$$\dot{W}_{\substack{\mathrm{other}\\ \mathrm{on\ CV}}} = 0 \quad \text{(the only work is pressure work)} \tag{5}$$

Substitute and simplify.

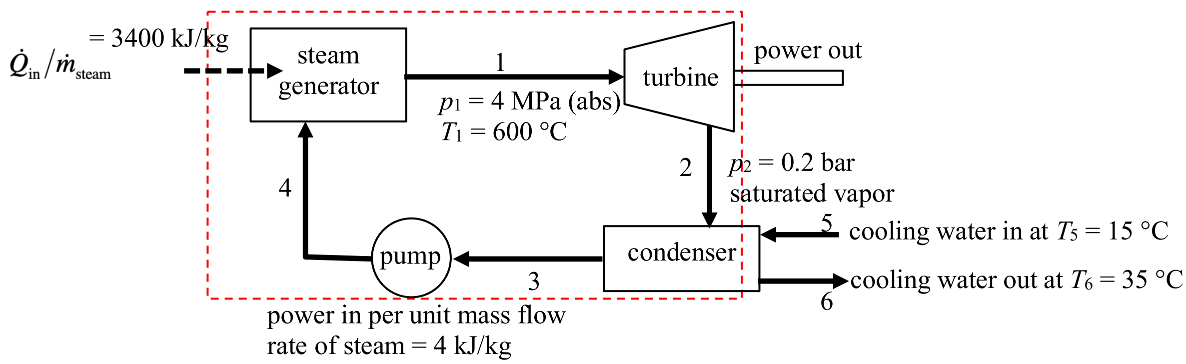
$$\dot{m}\left[\Delta h+\tfrac{1}{2}\Delta\left(V^2\right)\right] = 0 \;\Rightarrow\; \Delta h = -\tfrac{1}{2}\Delta\left(V^2\right) \tag{6}$$

Since $\Delta(V^2) < 0$ (from conservation of mass on the same CV), $\Delta h > 0$. Treating air as a perfect gas, $\Delta h = c_p\Delta T$ (note that if the air is assumed incompressible, then $\Delta h = c\Delta T$). Hence, the temperature must be increasing.

Steady-state operating data for a simple steam power plant are provided in the figure. Stray heat transfer and kinetic and potential energy effects can be ignored. Determine the:
a. thermal efficiency and
b. the ratio of the cooling water mass flow rate to the steam mass flow rate.



SOLUTION:
Consider the following control volume.



The efficiency of the system is the ratio of the rate at which work is produced by the system divided by the rate at which heat is put into the system,

$$\eta = \frac{\dot{W}_{\text{by system}}}{\dot{Q}_{\text{into system}}}, \tag{1}$$

or, since the rate at which heat enters the system is given in terms of heat rate per mass flow rate of steam,

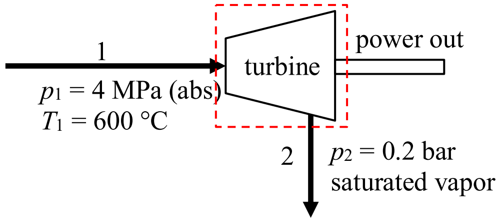
$$\eta = \frac{\dot{W}_{\text{by system}}/\dot{m}_{\text{steam}}}{\dot{Q}_{\text{into system}}/\dot{m}_{\text{steam}}}. \tag{2}$$

Note that the steam mass flow rates along connections 1 – 4 are the same since the system operates at steady conditions,

$$\dot{m}_1 = \dot{m}_2 = \dot{m}_3 = \dot{m}_4 = \dot{m}. \tag{3}$$

The rate at which work is done by the system is the power produced by the turbine minus the power entering the system into the pump,

$$\dot{W}_{\text{by system}} = \dot{W}_{\text{by turbine}} - \dot{W}_{\text{on pump}}. \tag{4}$$



The power produced by the turbine may be found by applying the 1st Law to a control volume surrounding just the turbine,

$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} - \dot{W}_{\substack{\text{by}\\ \text{CV}}}, \tag{5}$$

where,

$$\frac{d}{dt}\int_{CV} e\rho\, dV = 0 \quad \text{(steady operation)}, \tag{6}$$

$$\int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \dot{m}_{steam}\left(h_2 - h_1\right), \tag{7}$$

(kinetic and potential energy changes are negligible; mass flow rate is the same at 1 and 2 due to conservation of mass),

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}} = 0 \quad \text{(adiabatic operation assumed)}. \tag{8}$$

Substitute and simplify,

$$\dot{m}_{steam}\left(h_2 - h_1\right) = -\dot{W}_{\text{by CV}} \Rightarrow \frac{\dot{W}_{\text{by CV}}}{\dot{m}_{steam}} = h_1 - h_2. \tag{9}$$

The specific enthalpies at the inlet may be found using the thermodynamic property tables for water (e.g., Table A-4 in Moran et al., 7th ed.),

h_1 = 3674.4 kJ/kg (superheated vapor, e.g., Table A-4 in Moran et al., 7th ed.)
h_2 = 2609.7 kJ/kg (saturated vapor, e.g., Table A-3 in Moran et al., 7th ed.)

$$\Rightarrow \frac{\dot{W}_{\text{by CV}}}{\dot{m}_{steam}} = 1064.7 \text{ kJ/kg} \tag{10}$$

Substituting this result into Eqs. (4) and (2) gives,

$$\frac{\dot{W}_{\text{by system}}}{\dot{m}_{steam}} = 1060.7 \text{ kJ/kg}, \tag{11}$$

$$\boxed{\eta = 0.312} \tag{12}$$

where

$\dot{Q}_{\text{into system}}/\dot{m}_{steam}$ = 3400 kJ/kg and $\dot{W}_{\text{on pump}}/\dot{m}_{steam}$ = 4 kJ/kg.

To find the ratio of the cooling water mass flow rate to the steam mass flow rate, apply conservation of energy to a control volume surrounding the entire system.

$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} - \dot{W}_{\substack{\text{by}\\ \text{CV}}}, \tag{13}$$

where,

$$\frac{d}{dt}\int_{CV} e\rho\, dV = 0 \quad \text{(steady operation)}, \tag{14}$$

$$\int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \dot{m}_{cw}\left(h_6 - h_5\right) \quad \text{(cooling water)}, \tag{15}$$

(kinetic and potential energy changes are negligible; mass flow rate is the same at 5 and 6 due to conservation of mass),

$$\frac{\dot{Q}_{\text{into CV}}}{\dot{m}_{\text{steam}}} = 3400 \text{ kJ/kg (given)}, \tag{16}$$

$$\frac{\dot{W}_{\text{by system}}}{\dot{m}_{\text{steam}}} = 1060.7 \text{ kJ/kg (given)}. \tag{17}$$

Substitute and simplify,

$$\dot{m}_{\text{cw}}\left(h_6 - h_5\right) = \dot{m}_{\text{steam}}\frac{\dot{Q}_{\text{into CV}}}{\dot{m}_{\text{steam}}} - \dot{m}_{\text{steam}}\frac{\dot{W}_{\text{by system}}}{\dot{m}_{\text{steam}}}, \tag{18}$$

$$\boxed{\frac{\dot{m}_{\text{cw}}}{\dot{m}_{\text{steam}}} = \frac{\frac{\dot{Q}_{\text{into CV}}}{\dot{m}_{\text{steam}}} - \frac{\dot{W}_{\text{by system}}}{\dot{m}_{\text{steam}}}}{\left(h_6 - h_5\right)}}. \tag{19}$$

Using the given and calculated parameters,

$$\frac{\dot{Q}_{\text{into CV}}}{\dot{m}_{\text{steam}}} = 3400 \text{ kJ/kg (given)},$$

$$\frac{\dot{W}_{\text{by system}}}{\dot{m}_{\text{steam}}} = 1060.7 \text{ kJ/kg (calculated, Eq. (10))}$$

$h_5 = 62.99$ kJ/kg (20)

(subcooled water at $T_5 = 15$ °C, use $h_5 \approx h_l(T_5)$, e.g., Table A.2 in Moran et al., 7th ed.)

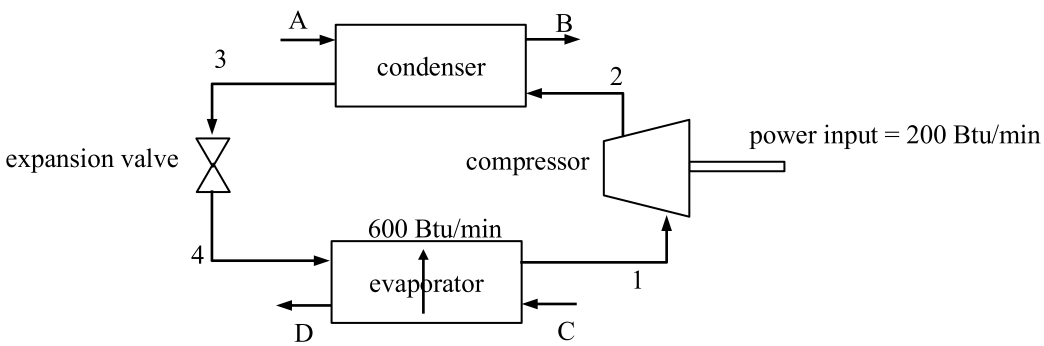
$h_6 = 146.68$ kJ/kg (21)

(subcooled water at $T_6 = 35$ °C, use $h_6 \approx h_l(T_6)$, e.g., Table A.2 in Moran et al., 7th ed.)

$$\Rightarrow \boxed{\frac{\dot{m}_{\text{cw}}}{\dot{m}_{\text{steam}}} = 28.0} \tag{22}$$

A residential air-conditioning system operates at steady state. Refrigerant 22 circulates through the components of the system. If the evaporator removes energy by heat transfer from the room air at a rate of 600 Btu/min, determine:

a. the rate of heat transfer between the compressor and the surroundings, in Btu/min, and
b. the coefficient of performance.

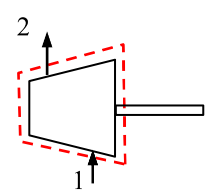


Location	Properties
A	outside air at 90 °F
B	air at a temperature greater than 90 °F
C	return room air at 75 °F
D	supply air to residence at a temperature less than 75 °F
1	Refrigerant 22, 120 psia, saturated vapor
2	Refrigerant 22, 225 psia, specific enthalpy of 130 Btu/lb$_m$
3	Refrigerant 22, 225 psia, 100 °F
4	Refrigerant 22, 62 °F

SOLUTION:

The rate of heat transfer between the compressor and the surroundings may be found by applied the 1st Law to a control volume surrounding the compressor,

$$\frac{dE_{\text{sys}}}{dt} = \sum_{\text{in}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} - \sum_{\text{out}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} + \dot{Q}_{\substack{\text{into}\\ \text{sys}}} + \dot{W}_{\substack{\text{other,}\\ \text{on sys}}}, \tag{1}$$



where,

$$\frac{dE_{\text{sys}}}{dt} = 0 \quad \text{(assuming steady conditions)}, \tag{2}$$

$$\sum_{\text{in}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} - \sum_{\text{out}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} = \left(h_1 - h_2\right)\dot{m}, \tag{3}$$

(assuming negligible differences in kinetic energy and potential energy across the compressor; the mass flow rate remains the same due to conservation of mass and the assumption of steady flow)

$$\dot{W}_{\substack{\text{other,}\\ \text{on sys}}} = \text{given (200 Btu/min)} \tag{4}$$

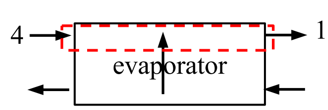
Substitute and simplify,

$$\underline{\dot{Q}_{\substack{\text{out of}\\ \text{sys}}} = \left(h_1 - h_2\right)\dot{m} + \dot{W}_{\substack{\text{other,}\\ \text{on sys}}}} \quad \text{(Note the change in the subscript for the heat transfer rate.)} \tag{5}$$

The specific enthalpy at state 2 is given in the problem statement (h_2 = 130 Btu/lb$_m$). The specific enthalpy at state 1 is found using a thermodynamic table (e.g., Table A.8E in Moran et al., 8th ed.) for Refrigerant 22 for saturated vapor at a pressure of 120 psia: h_1 = 109.88 Btu/lb$_m$.

The mass flow rate is not yet known, but can be determined by applying the 1st Law to the refrigerant in the evaporator where the energy transfer to the refrigerant via heat transfer is known,

$$\frac{dE_{\text{sys}}}{dt} = \sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} + \dot{Q}_{\substack{\text{into}\\ \text{sys}}} + \dot{W}_{\substack{\text{other,}\\ \text{on sys}}}, \tag{6}$$



where,

$$\frac{dE_{\text{sys}}}{dt} = 0 \quad \text{(assuming steady conditions)}, \tag{7}$$

$$\sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} = \left(h_4 - h_1\right)\dot{m}, \tag{8}$$

(assuming negligible differences in kinetic energy and potential energy across the evaporator; the mass flow rate remains the same due to conservation of mass and the assumption of steady flow)

$$\dot{Q}_{\text{into sys}} = \text{given (600 Btu/min)} \tag{9}$$

$$\dot{W}_{\substack{\text{other,}\\ \text{on sys}}} = 0 \quad \text{(no work other than pressure work is done on the evaporator)} \tag{10}$$

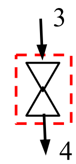
Substitute and simplify,

$$0 = \left(h_4 - h_1\right)\dot{m} + \dot{Q}_{\text{into sys}}, \tag{11}$$

$$\underline{\dot{m} = \frac{\dot{Q}_{\text{into sys}}}{h_1 - h_4}}. \tag{12}$$

The specific enthalpy of the Refrigerant 22 at state 1 was determined previously to be h_1 = 109.88 Btu/lb$_m$. The specific enthalpy at state 4 is not yet known, but it can be found by applying the 1st Law to a control volume surrounding the expansion valve,

$$\frac{dE_{\text{sys}}}{dt} = \sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} + \dot{Q}_{\substack{\text{into}\\ \text{sys}}} + \dot{W}_{\substack{\text{other,}\\ \text{on sys}}}, \tag{13}$$



where,

$$\frac{dE_{\text{sys}}}{dt} = 0 \quad \text{(assuming steady conditions)}, \tag{14}$$

$$\sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} = \left(h_3 - h_4\right)\dot{m}, \tag{15}$$

(assuming negligible differences in kinetic energy and potential energy across the valve; the mass flow rate remains the same due to conservation of mass and the assumption of steady flow)

$$\dot{Q}_{\text{into sys}} = 0 \quad \text{(assumed adiabatic)} \tag{16}$$

$$\dot{W}_{\substack{\text{other,}\\ \text{on sys}}} = 0 \quad \text{(no work other than pressure work)} \tag{17}$$

Substitute and simplify,

$$0 = \left(h_3 - h_4\right)\dot{m} \implies \underline{h_4 = h_3}. \tag{18}$$

The specific enthalpy at state 3 may be found using the thermodynamic tables given the pressure and temperature at that state. Using Table A-8E, we observe that at p_3 = 225 psia, $T_{3,\text{sat}}$ = 104.82 °F. Since T_3 = 100 °F < $T_{3,\text{sat}}$, the refrigerant must be in a compressed liquid phase at state 3. The specific enthalpy for a compressed liquid may be approximated as,

$$h_3(T_3, p_3) \approx h_l(T_3) + v(T_3)[p_3 - p_{\text{sat}}(T_3)], \tag{19}$$

where, from Table A-7E in Moran et al., 8th ed.,

$h_l(T_3)$ = 39.41 Btu/lb$_m$,
$v_l(T_3)$ = 0.01407 ft^3/lb$_m$,
$p_{\text{sat}}(T_3)$ = 210.69 psia,

$$\Rightarrow h_3(T_3, p_3) \approx 39.41\ \text{Btu/lb}_m + (0.01407\ \text{ft}^3/\text{lb}_m)[225\ \text{psia} - 210.69\ \text{psia}](1\ \text{Btu}/778.2\ \text{lb}_f\text{.ft})(144\ \text{in.}^2/\text{ft}^2), \tag{20}$$

$$\Rightarrow \underline{h_3(T_3, p_3) \approx 39.45\ \text{Btu/lb}_m}$$

$$\Rightarrow h_4 = h_3 = 39.45\ \text{Btu/lb}_m \text{ (from Eq. (18))} \tag{21}$$

Substitute values into Eq. (12),

$$\underline{\dot{m} = 8.52\ \text{lb}_\text{m}/\text{min}}. \tag{22}$$

Substitute values into Eq. (5),

$$\boxed{\dot{Q}_{\substack{\text{out of}\\ \text{compressor}}} = 28.58\ \text{Btu/min}}. \tag{23}$$

The coefficient of performance for this system, which is a type of refrigeration cycle since we're interested in removing energy via heat transfer from the room, is given by,

$$\text{COP}_\text{ref} = \frac{\dot{Q}_\text{into sys}}{\dot{W}_\text{on sys}} = \frac{600\ \text{Btu/min}}{200\ \text{Btu/min}} \quad \Rightarrow \quad \boxed{\text{COP}_\text{ref} = 3}. \tag{24}$$

where the system here is the refrigerant.

4.7. The Second Law of Thermodynamics for a Control Volume

As with COM, the LME, the AME, and the First Law, we will convert our system form of the Second Law to a control volume form using the Reynolds Transport Theorem. To do so, we should first write the Second Law in terms of time rates of change (refer to Eq. (3.149)),

$$\underbrace{\frac{DS}{Dt}}_{\substack{\text{rate of increase}\\ \text{of entropy in system}}} = \underbrace{\int_b \frac{\delta \dot{Q}_{\text{into sys}}}{T}}_{\substack{\text{rate at which entropy}\\ \text{enters the system via}\\ \text{heat transfer through}\\ \text{the boundary}}} + \underbrace{\dot{\sigma}}_{\substack{\text{rate at which}\\ \text{entropy is produced}\\ \text{in the system due to}\\ \text{internal irreversibilities}}}, \tag{4.137}$$

where the Lagrangian derivative notation has been used to remind us that we're following a system. Recall that $\dot{\sigma} \geq 0$ with the equality holding only for internally reversible processes. Now let's write the left-hand side of the equation in terms of an integral so that we can have variations in the system specific entropy,

$$\frac{D}{Dt}\int_{V_{\text{sys}}} s\rho dV = \int_b \frac{\delta \dot{Q}_{\text{into sys}}}{T} + \dot{\sigma}. \tag{4.138}$$

where s is the specific entropy. After applying the Reynolds Transport Theorem to convert to a control volume perspective,

$$\boxed{\underbrace{\frac{d}{dt}\int_{CV} s\rho dV}_{\substack{\text{rate of entropy}\\ \text{increase in CV}}} + \underbrace{\int_{CS} s\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right)}_{\substack{\text{net rate at which entropy}\\ \text{leaves the CV through the CS}}} = \underbrace{\int_{CS} \frac{\delta \dot{Q}_{\text{into CV}}}{T}}_{\substack{\text{rate at which entropy}\\ \text{enters the CV via heat transfer}}} + \underbrace{\dot{\sigma}}_{\substack{\text{rate at which entropy}\\ \text{is produced in the CV}\\ \text{due to internal irreversibilities}}}} \tag{4.139}$$

Note that the subscript on the heat transfer integral was changed from "b" for "boundary" to "CS" for control surface. This is the control volume form of the Second Law of Thermodynamics (aka the Entropy Equation)!

Notes:

(1) The rate of entropy increase within the control volume (the first term on the left-hand side of Eq. (4.139)) may also be written as,

$$\frac{dS_{CV}}{dt} = \frac{d}{dt}\int_{CV} s\rho dV. \tag{4.140}$$

The net flow rate of entropy *out* of the control volume due to mass flowing out of and into the control volume may be written as,

$$\sum_{\text{all outlets}} s\dot{m} - \sum_{\text{all inlets}} s\dot{m} = \int_{CS} s\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right). \tag{4.141}$$

Combined together, the Second Law for a control volume may then be written as,

$$\frac{dS_{CV}}{dt} = \sum_{\text{all inlets}} \dot{m}s - \sum_{\text{all outlets}} \dot{m}s + \int_{CS} \frac{\delta \dot{Q}_{\text{into CV}}}{T} + \dot{\sigma}. \tag{4.142}$$

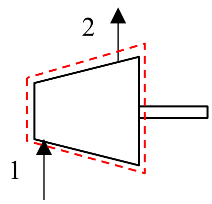
(2) Recall that entropy production is related to the degree of irreversibility in a system/control volume. The larger the change in entropy, the further the system/control volume is from being reversible or ideal. Hence, Eq. (4.139) can be used to determine situations that result in inefficiencies.

Let's consider some examples to see how the Second Law is applied to CVs.

Steam enters a turbine operating at steady state at 1 MPa (abs) and 200 °C and exits at 40 °C with a quality of 83%. Stray heat transfer and changes in kinetic and potential energy are negligible. Determine:

a. the power developed by the turbine per unit mass of steam, and
b. the change in the specific entropy from the inlet to the exit per unit mass of steam.

SOLUTION:
To determine the power developed by the turbine, apply the 1st Law to a control volume surrounding the turbine.



$$\frac{dE_{\text{CV}}}{dt}+\sum_{\text{out}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m}-\sum_{\text{in}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m}=\dot{Q}_{\text{into}}+\dot{W}_{\text{on,other}}\,, \tag{1}$$

where,

$dE_{\text{CV}}/dt = 0$ (steady flow)

$\Delta(^1/_2V^2)$ and $\Delta(gz)$ are assumed to be negligibly small compared to the specific enthalpy

$\dot{Q}_{\text{into}} = 0$ (adiabatic flow)

$\dot{m}_{\text{out}} = \dot{m}_{\text{in}} = \dot{m}$ (from conservation of mass)

$$\Rightarrow \frac{\dot{W}_{\text{on,other}}}{\dot{m}} = h_2 - h_1\,. \tag{2}$$

The specific enthalpies are,

$h_1 = 2827.9$ kJ/kg (@ 200 °C and 1 MPa = 10 bar $\Rightarrow$ superheated vapor; from Table A-4 in Moran et al., 7th ed.)

$$h_2 = x_2h_v + (1 - x_2)h_l \tag{3}$$

$x_2 = 0.83$, $h_v = 2574.3$ kJ/kg, $h_l = 167.57$ kJ/kg
(@ 40 °C, two-phase, liquid-vapor state; from Table A-2 in Moran et al., 7th ed.)

$\Rightarrow h_2 = 2165.1$ kJ/kg.

$\Rightarrow \boxed{\frac{\dot{W}_{\text{on,other}}}{\dot{m}} = -662.8 \text{ kJ/kg}}$ (work is being done by the system).

The rate of entropy production per unit mass of steam may be found using,

$$\frac{dS_{CV}}{dt}=\sum_{\text{in}}s\dot{m}-\sum_{\text{out}}s\dot{m}+\int_b\frac{\dot{Q}_{\text{into}}}{T}+\dot{\sigma}\Rightarrow\dot{\sigma}=\frac{dS_{CV}}{dt}+\sum_{\text{out}}s\dot{m}-\sum_{\text{in}}s\dot{m}-\int_b\frac{\dot{Q}_{\text{into}}}{T}\,, \tag{4}$$

where,

$$\frac{dS_{CV}}{dt}=0 \quad \text{(steady flow assumed),} \tag{5}$$

$$\sum_{\text{out}}s\dot{m}-\sum_{\text{in}}s\dot{m}=\dot{m}\left(s_2-s_1\right) \quad \text{(mass flow rate constant from Cons. of Mass),} \tag{6}$$

$$\int_b\frac{\dot{Q}_{\text{into}}}{T}=0 \quad \text{(adiabatic process),} \tag{7}$$

$$\Rightarrow \quad \dot{\sigma}=\dot{m}\left(s_2-s_1\right)\Rightarrow\frac{\dot{\sigma}}{\dot{m}}=s_2-s_1\,. \tag{8}$$

The change in specific entropy is,

$s_1 = 6.6940$ kJ/(kg.K) (@ 200 °C and 1 MPa = 10 bar ⇒ superheated vapor; from Table A-4 in Moran et al., 7th ed.)

$$s_2 = x_2 s_v + (1 - x_2) s_l \tag{9}$$

$x_2 = 0.83$, $s_v = 8.2570$ kJ/kg, $s_l = 0.5725$ kJ/kg

(@ 40 °C, two-phase, liquid-vapor state; from Table A-2 in Moran et al., 7th ed.)

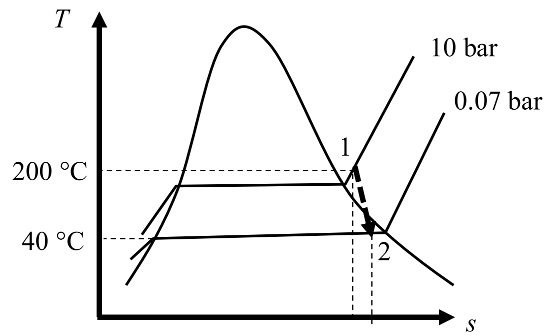
⇒ $s_2 = 6.9506$ kJ/(kg.K).

⇒ $s_2 - s_1 = 0.2566$ kJ/(kg.K).

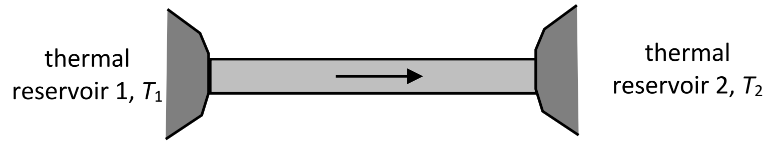
⇒ $\boxed{\dot{\sigma}/\dot{m} = 0.2566 \text{ kJ/(kg.K)}.}$

Note that the process is not internally reversible since $\dot{\sigma} > 0$.

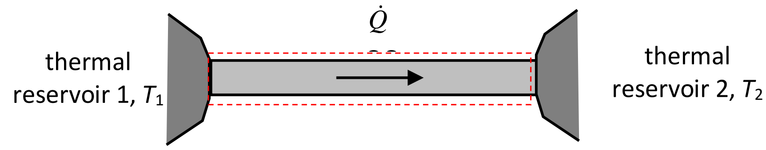
A plot of the process on a *T*-*s* diagram is shown in the following figure.



Consider a steel rod steadily conducting heat from thermal reservoir 1 into thermal reservoir 2. Using the entropy rate balance equation, show that $T_1 > T_2$. Note that T_1 and T_2 are absolute temperatures.



SOLUTION:



Apply the entropy rate balance equation to a control volume surrounding the steel rod,

$$\frac{dS_{CV}}{dt} = \sum_{\text{in}} s\dot{m} - \sum_{\text{out}} s\dot{m} + \int_b \frac{\delta \dot{Q}_{\text{into CV}}}{T} + \dot{\sigma} \text{ ,} \tag{1}$$

where,

$$\frac{dS_{CV}}{dt} = 0 \quad \text{(steady),} \tag{2}$$

$$\sum_{\text{in}} s\dot{m} - \sum_{\text{out}} s\dot{m} = 0 \quad \text{(no flow in or out of the CV),} \tag{3}$$

$$\int_b \frac{\delta \dot{Q}_{\text{into CV}}}{T} = \frac{\dot{Q}}{T_1} - \frac{\dot{Q}}{T_2} \text{ ,} \tag{4}$$

where, from the 1st Law applied to the same CV, the rate of heat transfer through the rod remains constant. In addition, the temperature at the left boundary, where the heat enters the CV, i.e., $\dot{Q}_{\text{into CV,1}} = \dot{Q}$, is T_1 and the temperature at the right boundary, where heat leaves the CV, i.e., $\dot{Q}_{\text{into CV,2}} = -\dot{Q}$, is T_2.

Substitute and simplify,

$$0 = \frac{\dot{Q}}{T_1} - \frac{\dot{Q}}{T_2} + \dot{\sigma} \text{ ,} \tag{5}$$

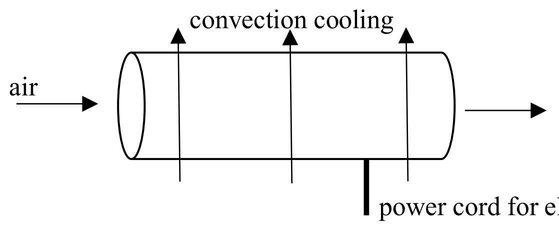
$$\dot{\sigma} = \dot{Q}\left(\frac{T_1 - T_2}{T_1 T_2}\right). \tag{6}$$

Since $\dot{\sigma} \geq 0$, $T_1 \geq T_2$. The equality, $T_1 = T_2$, results in an internally reversible process whereas a finite temperature difference, $T_1 > T_2$, results in an internally irreversible process.

This result also demonstrates the equivalence between entropy generation via heat transfer and the Clausius statement of the 2nd Law.

Electronic components are mounted on the inner surface of a horizontal cylindrical duct with an inner diameter of 0.2 m. To prevent overheating of the electronics, the cylinder is cooled by a stream of air flowing through the cylinder and by convection from its outer surface. Air enters the duct at 25 °C, 1 bar (abs), and a speed of 0.3 m/s, and exits at 40 °C with negligible changes in kinetic energy and pressure. Convective cooling occurs on the cylinder's outer surface to the surroundings, which are at 25 °C, in accord with hA = 3.4 W/K, where h is the heat transfer coefficient and A is the cylinder's surface area. The electronic components require 0.20 kW of electric power. For steady state conditions, determine:

a. the mass flow rate of air through the cylinder, in kg/s,
b. the temperature on the outer surface of the duct, and
c. the rate of entropy production in the air passing through the duct.

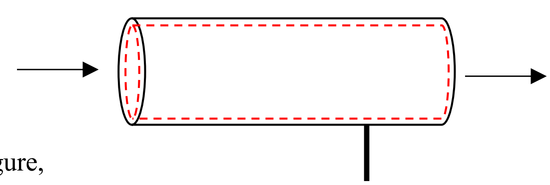


SOLUTION:
The mass flow rate through the cylinder is,

$$\dot{m} = \rho_1 V_1 A_1 = \left(\frac{p_1}{RT_1}\right) V_1 \left(\frac{\pi D_1^2}{4}\right), \tag{1}$$

where the ideal gas law has been used. Using the given data,

p_1 = 1 bar (abs) = 1*10^5 Pa
R_{air} = 287 J/(kg.K)
T_1 = 25 °C = 298 K
V_1 = 0.3 m/s
D_1 = 0.2 m
=> $\boxed{\dot{m} = 0.011 \text{ kg/s}}$.



Now apply the 1st Law the control volume shown in the figure,

$$\frac{dE_{CV}}{dt} = \sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} + \dot{Q}_{\text{into,CV}} + \dot{W}_{\text{other,on CV}}, \tag{2}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \quad \text{(steady conditions)}, \tag{3}$$

$$\sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} = \dot{m}\left(h_1 - h_2\right) = \dot{m}c_p\left(T_1 - T_2\right), \tag{4}$$

(assuming changes in kinetic energy and potential energy are negligible; the mass flow at the inlet and outlet are the same from conservation of mass; the air behaves as a perfect gas),

$$\dot{Q}_{\text{into,CV}} = -hA\left(T_{\text{surface}} - T_{\text{surr}}\right) \quad \text{(Newton's Law of Convection)}, \tag{5}$$

$$\dot{W}_{\text{other,on CV}} = 0.20 \text{ kW} \quad \text{(given power for electronics)}. \tag{6}$$

Substitute and simplify,

$$0 = \dot{m}c_p\left(T_1 - T_2\right) - hA\left(T_{\text{surface}} - T_{\text{surr}}\right) + \dot{W}_{\text{other,on CV}}\,, \tag{7}$$

$$T_{\text{surface}} = T_{\text{surr}} + \frac{\dot{m}c_p\left(T_1 - T_2\right) + \dot{W}_{\text{other,on CV}}}{hA}\,,$$

where,

T_{surr} = 25 °C = 298 K (given),
$\dot{m}$ = 0.011 kg/s (found previously),
c_p = 1.005 kJ/(kg.K) (Table A-20 in Moran et al., 8th ed.)
T_1 = 25 °C = 298 K (given),
T_2 = 40 °C = 313 K (given),
$\dot{W}_{\text{other,on CV}} = 0.20$ kW (given),
hA = 3.4 W/K (given),
=> $\boxed{T_{\text{surface}} = 308\text{ K} = 35.1\text{ °C}}$.

The rate at which entropy is produced in the air passing through the duct is found by applying the entropy equation to the same control volume,

$$\frac{dS_{CV}}{dt} = \sum_{\text{in}} s\dot{m} - \sum_{\text{out}} s\dot{m} + \int_b \frac{\dot{Q}_{\text{into}}}{T} + \dot{\sigma}\,, \tag{8}$$

where

$$\frac{dS_{CV}}{dt} = 0 \quad \text{(steady)}, \tag{9}$$

$$\sum_{\text{in}} s\dot{m} - \sum_{\text{out}} s\dot{m} = \dot{m}\left(s_1 - s_2\right) = \dot{m}\left[c_p \ln\left(\frac{T_1}{T_2}\right) - R\ln\left(\frac{p_1}{p_2}\right)\right], \tag{10}$$

(where a perfect gas assumption has been used to determine the change in specific entropy),

$$\int_b \frac{\dot{Q}_{\text{into}}}{T} = \frac{-hA\left(T_{\text{surface}} - T_{\text{surr}}\right)}{T_{\text{surface}}}\,, \tag{11}$$

(Note that the absolute temperature in the denominator is the temperature where the heat transfer occurs on the control surface boundary.)

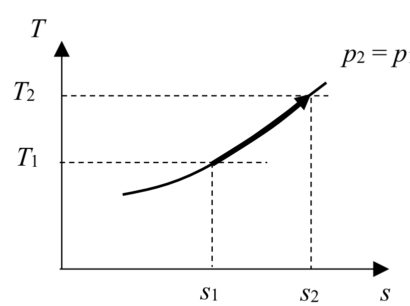
Substitute and simplify,

$$0 = \dot{m}\left[c_p \ln\left(\frac{T_1}{T_2}\right) - R\ln\left(\frac{p_1}{p_2}\right)\right] + \frac{-hA\left(T_{\text{surface}} - T_{\text{surr}}\right)}{T_{\text{surface}}} + \dot{\sigma}\,, \tag{12}$$

$$\dot{\sigma} = \dot{m}\left[c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)\right] + \frac{hA\left(T_{\text{surface}} - T_{\text{surr}}\right)}{T_{\text{surface}}}\,. \tag{13}$$

Using the given data,

T_{surr} = 25 °C = 298 K (given),
$\dot{m}$ = 0.011 kg/s (found previously),
c_p = 1.005 kJ/(kg.K) (Table A-20 in Moran et al., 8th ed.)
T_1 = 25 °C = 298 K (given),
T_2 = 40 °C = 313 K (given),
$p_2 = p_1$ (given),
T_{surface} = 308 K (found previously),
$\dot{W}_{\text{other,on CV}} = 0.20$ kW (given),
hA = 3.4 W/K (given),
=> $\boxed{\dot{\sigma} = 0.653\text{ W/K}}$.

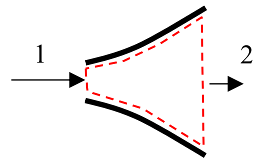


Air enters a diffuser operating at steady state at 4 bar (abs) and 290 K with a speed of 512 m/s. The exit speed is 110 m/s. For adiabatic operation with no internal irreversibilities, determine:

a. the exit temperature, in K, and
b. the exit pressure, in bar (abs).

SOLUTION:
Apply the 1st Law to the control volume shown in the figure,

$$\frac{dE_{CV}}{dt} = \sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} + \dot{Q}_{\text{into CV}} + \dot{W}_{\text{other,on CV}}, \tag{1}$$



where,

$$\frac{dE_{CV}}{dt} = 0 \quad \text{(steady)}, \tag{2}$$

$$\sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} = \dot{m}\left[\left(h_1 + \tfrac{1}{2}V_1^2\right) - \left(h_2 + \tfrac{1}{2}V_2^2\right)\right], \tag{3}$$

(Changes in potential energy are assumed negligible since we're dealing with a gas.)

$$\dot{Q}_{\text{into CV}} \quad \text{(adiabatic process)}, \tag{4}$$

$$\dot{W}_{\text{other,on CV}} = 0 \quad \text{(no “other” work)}, \tag{5}$$

$$\Rightarrow \quad h_2 = h_1 + \tfrac{1}{2}\left(V_1^2 - V_2^2\right). \tag{6}$$

Treating air as an ideal gas,

$h_1(T_1 = 290\text{ K}) = 290.16$ kJ/kg (from Table A-22 in Moran et al., 8th ed.),
$V_1 = 512$ m/s,
$V_2 = 110$ m/s,
=> $h_2 = 415.18$ kJ/kg => $\boxed{T_2 = 414\text{ K}}$ (interpolating in Table A-22). (7)

The exit pressure may be found using the change in entropy relation for an ideal gas,

$$s_2 - s_1 = s_2^0\left(T_2\right) - s_1^0\left(T_1\right) - R\ln\left(\frac{p_2}{p_1}\right), \tag{8}$$

where,

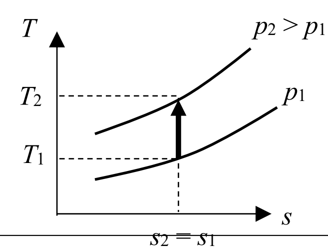
$s_2 - s_1 = 0$ since the process is adiabatic and internally reversible => isentropic (given),
$s^0_2(T_2 = 414\text{ K}) = 2.02676$ kJ/(kg.K) (interpolating in Table A-22),
$s^0_1(T_1 = 290\text{ K}) = 1.66802$ kJ/(kg.K) (Table A-22),
$R_{\text{air}} = 0.287$ kJ/(kg.K),
$p_1 = 4$ bar (abs),
=> $\boxed{p_2 = 14.0\text{ bar (abs)}}$.

If we assumed the air behaves as a perfect gas with $c_p = 1.005$ kJ/(kg.K) (Table A-20 in Moran et al., 8th ed.), then Eq. (6) may be written as,

$$T_2 = T_1 + \tfrac{1}{2c_p}\left(V_1^2 - V_2^2\right) \quad \Rightarrow \quad T_2 = 414\text{ K} \quad \text{(Same answer as before!)} \tag{9}$$

In addition, for a perfect gas undergoing an isentropic process,

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} \quad \Rightarrow \quad p_2 = 13.9\text{ bar (abs)} \quad \text{(within 1\% of the answer found previously)} \tag{10}$$



Note that if the flow had internal irreversibilities, then the entropy equation applied to the control volume gives,

$$\frac{dS_{CV}}{dt} = \sum_{\text{in}}(s\dot{m}) - \sum_{\text{out}}(s\dot{m}) + \int_b \frac{\dot{Q}_{\text{into}}}{T} + \dot{\sigma}\ , \tag{11}$$

where,

$$\frac{dS_{CV}}{dt} = 0 \quad \text{(steady flow)}, \tag{12}$$

$$\int_b \frac{\dot{Q}_{\text{into}}}{T} = 0 \quad \text{(adiabatic flow)}, \tag{13}$$

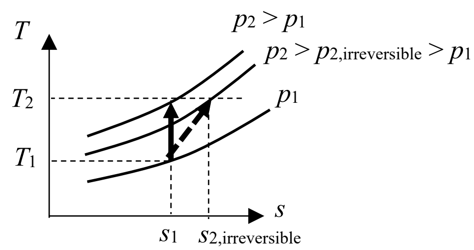
$$\Rightarrow \quad s_2 - s_1 = \frac{\dot{\sigma}}{\dot{m}} \quad \text{where } \dot{\sigma}/\dot{m} > 0 \text{ if the flow is internally irreversible (= 0 if internally reversible)} \tag{14}$$

Substituting into Eq. (8) and solving for p_2 gives,

$$\frac{\dot{\sigma}}{\dot{m}} = s_2^0(T_2) - s_1^0(T_1) - R\ln\left(\frac{p_2}{p_1}\right), \tag{15}$$

$$p_2 = p_1 \exp\left[\frac{s_2^0(T_2) - s_1^0(T_1) - \dot{\sigma}/\dot{m}}{R}\right]. \tag{16}$$

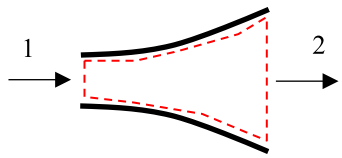
Thus, with internal irreversibilities, the exit pressure is smaller than if the flow was internally reversible.



Nitrogen (N_2) enters a well-insulated diffuser operating at steady state at 0.656 bar (abs), 300 K with a velocity of 282 m/s. The inlet area is $4.8*10^{-3}$ m^2. At the diffuser exit, the pressure is 0.9 bar (abs) and the velocity is 130 m/s. The nitrogen behaves as an ideal gas with a specific heat ratio of 1.4. Determine:

a. the exit temperature, in K,
b. the exit area, in m^2, and
c. the rate of entropy production, in kJ/K per kg of flowing nitrogen.

SOLUTION:



Apply the 1st Law to a control volume surrounding the interior of the diffuser,

$$\frac{dE_{CV}}{dt} = \sum_{\text{in}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} - \sum_{\text{out}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} + \dot{Q}_{\text{into,CV}} + \dot{W}_{\text{other,on CV}}, \tag{1}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \quad \text{(assuming steady flow)}, \tag{2}$$

$$\sum_{\text{in}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} - \sum_{\text{out}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} = \dot{m}\left[\left(h_2-h_1\right)+\tfrac{1}{2}\left(V_2^2-V_1^2\right)\right], \tag{3}$$

(changes in PE are assumed negligible compared to changes in the other terms, especially since we're dealing with a gas; from conservation of mass, $\dot{m} = \dot{m}_2 = \dot{m}_1$)

$$\dot{Q}_{\text{into,CV}} = 0 \quad \text{(the diffuser is well-insulated)} \tag{4}$$

$$\dot{W}_{\text{other, on CV}} = 0 \quad \text{(no other work acting on the control volume)} \tag{5}$$

Substituting and simplifying,

$$h_2 = h_1 + \tfrac{1}{2}\left(V_1^2 - V_2^2\right). \tag{6}$$

Assuming ideal gas behavior,

$\bar{h}_1$ (T_1 = 300 K) = 8723 kJ/kmol (Table A-23 in Moran et al., 8th ed.)

$$\Rightarrow \quad h_1 = \bar{h}_1/M \quad \Rightarrow \quad h_1 = 311.42 \text{ kJ/kg}, \tag{7}$$

where the molecular weight of nitrogen gas (N_2) is 28.01 kg/kmol has been used.

Given V_1 = 282 m/s, V_2 = 130 m/s, and making use of Eqs. (7) and (6),

$$h_2 = 342.73 \text{ kJ/kg} \Rightarrow \bar{h}_2 = 9600. \text{ kJ/kmol} \Rightarrow \boxed{T_2 = 330 \text{ K}}. \tag{8}$$

Interpolation in Table A-23 was used to determine this outlet temperature.

Alternately, if a perfect gas model is assumed, we can write Eq. (6) as,

$$c_pT_2 = c_pT_1 + \tfrac{1}{2}\left(V_1^2 - V_2^2\right) \Rightarrow T_2 = T_1 + \frac{V_1^2 - V_2^2}{2c_p}, \tag{9}$$

where,

$$c_p = \frac{kR}{k-1} = \frac{k\left(\bar{R}_u/M\right)}{k-1}, \tag{10}$$

=> c_p = 1.0389 kJ/(kg.K) with k = 1.4, $\bar{R}_u$ = 8.314 kJ/(kmol.K), and M = 28.01 kg/kmol

=> T_2 = 330 K, which is the same result found previously

The exit area may be found by applying conservation of mass to the same control volume,

$$\frac{dM_{CV}}{dt} = \dot{m}_1 - \dot{m}_2 \Rightarrow \dot{m}_2 = \dot{m}_1 \Rightarrow \rho_2 V_2 A_2 = \rho_1 V_1 A_1 \Rightarrow A_2 = A_1\left(\frac{\rho_1}{\rho_2}\right)\left(\frac{V_1}{V_2}\right), \tag{11}$$

Note that the process is assumed to be steady. The density ratio may be found by making use of the ideal gas law,

$$\frac{\rho_1}{\rho_2} = \frac{p_1/RT_1}{p_2/RT_2} = \left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right). \tag{12}$$

Substituting and simplifying,

$$A_2 = A_1\left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)\left(\frac{V_1}{V_2}\right). \tag{13}$$

Using the given data

$A_1 = 4.8*10^{-3}\text{ m}^2$,
$p_1 = 0.656$ bar (abs),
$p_2 = 0.9$ bar (abs),
$T_2 = 330$ K (calculated for part (a)),
$T_1 = 300$ K,
$V_1 = 282$ m/s,
$V_2 = 130$ m/s,
=> $\boxed{A_2 = 8.35*10^{-3}\text{ m}^2}$.

The rate of entropy production may be found by applying the entropy equation to the same control volume,

$$\frac{dS_{\text{CV}}}{dt} = \sum_{\text{in}} s\dot{m} - \sum_{\text{out}} s\dot{m} + \int_b \frac{\dot{Q}_{\text{into CV}}}{T} + \dot{\sigma}_{\text{CV}}, \tag{14}$$

where,

$$\frac{dS_{\text{CV}}}{dt} = 0 \quad \text{(assuming steady flow)}, \tag{15}$$

$$\sum_{\text{in}} s\dot{m} - \sum_{\text{out}} s\dot{m} = \dot{m}\left(s_1 - s_2\right), \tag{16}$$

$$\int_b \frac{\dot{Q}_{\text{into CV}}}{T} = 0 \quad \text{(assuming adiabatic operation)}, \tag{17}$$

$$\Rightarrow \quad \dot{\sigma}_{\text{CV}} = \dot{m}\left(s_2 - s_1\right) \quad \Rightarrow \quad \frac{\dot{\sigma}_{\text{CV}}}{\dot{m}} = s_2 - s_1 \tag{18}$$

The change in entropy for an ideal gas is,

$$s_2 - s_1 = \frac{\left[\bar{s}_2^0\left(T_2\right) - \bar{s}_1^0\left(T_1\right) - \bar{R}_u \ln\left(\frac{p_2}{p_1}\right)\right]}{M}, \tag{19}$$

where,

$\bar{s}_2^0\left(T_2 = 330\text{ K}\right) = 194.459$ kJ/(kmol.K), (Table A-23 in Moran et al., 8th ed.)

$\bar{s}_1^0\left(T_1 = 300\text{ K}\right) = 191.682$ kJ/(kmol.K), (Table A-23 in Moran et al., 8th ed.)

$\bar{R}_u = 8.314$ kJ/(kmol.K),

$p_2/p_1 = (0.9\text{ bar (abs)})/(0.656\text{ bar (abs)}) = 1.372$,
$M = 28.01$ kg/kmol,
=> $s_2 - s_1 = 5.28*10^{-3}$ kJ/(kg.K).

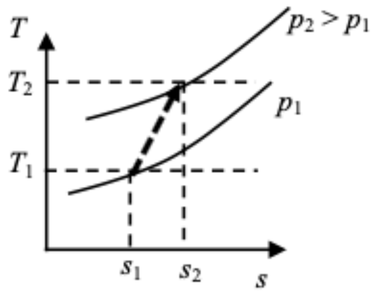
Thus, making use of Eq. (18),

$$\boxed{\frac{\dot{\sigma}_{\text{CV}}}{\dot{m}} = 5.28*10^{-3}\text{ kJ/(kg.K)}}$$

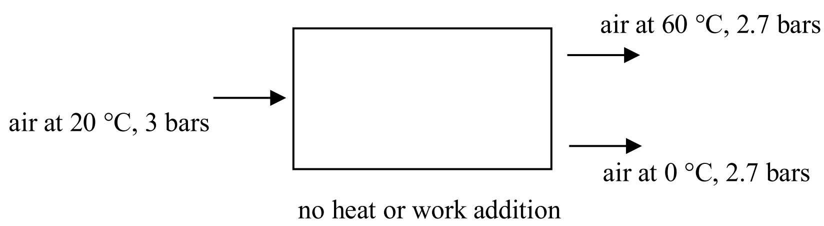
If we assume the nitrogen behaves as an ideal gas, then we could find the change in entropy using,

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right), \tag{20}$$

=> $s_2 - s_1 = 5.15*10^{-3}$ kJ/(kg.K), which is within 2.5% of the previous result.

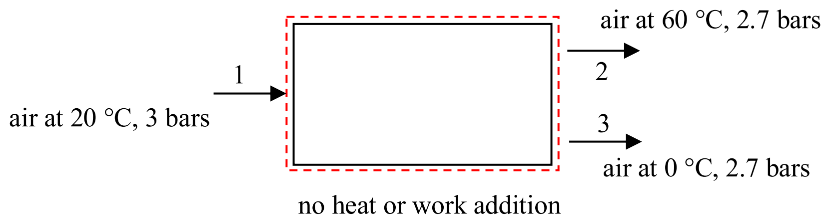


An inventor claims to have developed a device requiring no work input or heat transfer, yet able to produce steady state hot and cold air streams as shown in the figure. Evaluate this claim assuming the ideal gas model for air and ignoring kinetic and potential energy effects.



SOLUTION:

Apply conservation of mass, the First Law, and the Second Law to the control volume shown below.



Conservation of Mass,

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = 0 \tag{1}$$

where,

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = \dot{m}_3 + \dot{m}_2 - \dot{m}_1$$

Substitute and re-arrange,

$$\dot{m}_3 + \dot{m}_2 - \dot{m}_1 = 0$$

$$\frac{\dot{m}_3}{\dot{m}_1} = 1 - \frac{\dot{m}_2}{\dot{m}_1} \tag{2}$$

First Law of Thermodynamics,

$$\frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{on}\\ \text{CV}}} \tag{3}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} e\rho dV = 0 \quad \text{(steady flow}$$

$$\int_{\text{CS}}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{m}_3 h_3 + \dot{m}_2 h_2 - \dot{m}_1 h_1$$

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}} = \dot{W}_{\substack{\text{on}\\ \text{CV}}} = 0 \quad \text{(no heat or work addition)}$$

Substitute and re-arrange,

$$\dot{m}_3 h_3 + \dot{m}_2 h_2 - \dot{m}_1 h_1 = 0$$

$$\frac{\dot{m}_3}{\dot{m}_1}h_3 = h_1 - \frac{\dot{m}_2}{\dot{m}_1}h_2 \tag{4}$$

Substitute Eq. (2) into Eq. (4) and simplify,

$$\left(1-\frac{\dot{m}_2}{\dot{m}_1}\right)h_3 = h_1 - \frac{\dot{m}_2}{\dot{m}_1}h_2$$

$$\frac{\dot{m}_2}{\dot{m}_1} = \frac{h_1 - h_3}{h_2 - h_3} \tag{5}$$

From thermodynamics tables for air at the given inlet and outlet temperatures,

h_1 = 293.2 kJ/kg (T_1 = 20 °C = 293 K)
h_2 = 333.3 kJ/kg (T_2 = 60 °C = 333 K)
h_3 = 273.1 kJ/kg (T_3 = 0 °C = 273 K)

Hence,

$$\frac{\dot{m}_2}{\dot{m}_1} = 0.334 \quad \text{and} \quad \frac{\dot{m}_3}{\dot{m}_1} = 0.666 \tag{6}$$

Note that the outgoing mass flow rates are both positive, consistent with the problem description.

Entropy Equation,

$$\frac{d}{dt}\int_{CV} s\rho dV + \int_{CS} s(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A}) = \int_{CS} \frac{\delta \dot{Q}_{into}}{T} + \dot{\sigma}, \tag{7}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} s\rho dV = 0 \quad \text{(steady state)}$$

$$\int_{\text{CS}} s\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m}_3 s_3 + \dot{m}_2 s_2 - \dot{m}_1 s_1$$

$$\int_{CS} \frac{\delta \dot{Q}_{into}}{T} = 0 \text{ (no heat added to the control volume)}$$

Substitute and re-arrange,

$$\dot{m}_3 s_3 + \dot{m}_2 s_2 - \dot{m}_1 s_1 = \dot{\sigma},$$

$$\frac{\dot{m}_3}{\dot{m}_1} s_3 + \frac{\dot{m}_2}{\dot{m}_1} s_2 - s_1 = \frac{\dot{\sigma}}{\dot{m}_1},$$

$$\left(1 - \frac{\dot{m}_2}{\dot{m}_1}\right) s_3 + \frac{\dot{m}_2}{\dot{m}_1} s_2 - s_1 = \frac{\dot{\sigma}}{\dot{m}_1},$$

$$(s_3 - s_1) + \frac{\dot{m}_2}{\dot{m}_1}(s_2 - s_3) = \frac{\dot{\sigma}}{\dot{m}_1}. \tag{8}$$

Note that for an ideal gas, the specific entropy at a given temperature and pressure can be determined by,

$$s(T_B, p_B) - s(T_A, p_A) = s^0(T_B) - s^0(T_A) - R\ln\frac{p_B}{p_A} \tag{9}$$

Substituting Eq. (9) into Eq. (8) gives,

$$\left(s_3^0 - s_1^0 - R\ln\frac{p_3}{p_1}\right) + \frac{\dot{m}_2}{\dot{m}_1}\left(s_2^0 - s_3^0 - R\ln\frac{p_2}{p_3}\right) = \frac{\dot{\sigma}}{\dot{m}_1}. \tag{10}$$

From thermodynamics tables for air at the given conditions,

$s^0{}_1$ = 1.6783 kJ/(kg·K) (T_1 = 20 °C = 293 K)
$s^0{}_2$ = 1.8069 kJ/(kg·K) (T_2 = 60 °C = 333 K)
$s^0{}_3$ = 1.6073 kJ/(kg·K) (T_3 = 0 °C = 273 K)

and from the given conditions,

p_1 = 3.0 bar
p_2 = 2.7 bar
p_3 = 2.7 bar

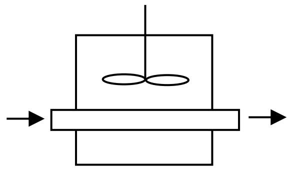
Thus, Eq. (10) simplifies to,

$$\frac{\dot{\sigma}}{\dot{m}_1} = 0.0259 \text{ kJ/(kg.K)}. \tag{11}$$

Thus, the Second Law of Thermodynamics is satisfied.

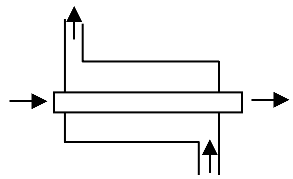
<u>Since conservation of mass, the First Law, and the Second Law are all satisfied, the claim of the inventor is not unreasonable.</u>

Two alternate systems are under consideration for bringing a stream of air from 17 °C to 52 °C at an essentially constant pressure of 1 bar.



Air temperature increases as a consequence of the stirring of a liquid surrounding the line carrying the air.

METHOD 1



Air temperature increases by passing it through one side of a counterflow heat exchanger. On the other side, steam condenses at a pressure of 1 bar from saturated vapor to saturated liquid.

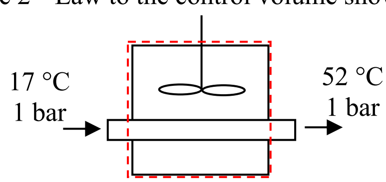
METHOD 2

Both systems operate at steady state. All kinetic and potential energy effects can be ignored and no significant heat transfer with the surroundings occurs. For each of the two systems, calculate the rate of entropy production in kJ/K per kg of air passing through the system.

SOLUTION:

First analyze METHOD 1.

Apply the 2nd Law to the control volume shown below.



$$\frac{d}{dt}\int_{CV} s\rho dV + \int_{CS} s(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A}) = \int_{CS} \frac{\delta \dot{Q}_{into\ CV}}{T} + \dot{\sigma}, \tag{1}$$

where,

$$\frac{d}{dt}\int_{CV} s\rho dV = 0 \quad \text{(steady state)}$$

$$\int_{CS} s(\rho \boldsymbol{u}_{rel} \cdot d\boldsymbol{A}) = \dot{m}_{air}(s_{out} - s_{in})$$

$$\int_{CS} \frac{\delta \dot{Q}_{into\ CV}}{T} = 0 \quad \text{(since adiabatic)}$$

Substitute and simplify,

$$s_{out} - s_{in} = \frac{\dot{\sigma}}{\dot{m}_{air}} \tag{2}$$

where σ is the entropy produced per unit mass flow rate.

From thermodynamics tables for air at the given temperatures (note that the air, treated as an ideal gas, is at the same pressure at the inlet and outlet so that: $s(T_B, p_B) - s(T_A, p_A) = s^0(T_B) - s^0(T_A)$):

$s_{out,air}$ = 1.7825 kJ/(kg·K) (T_{out} = 52 °C = 325 K)
$s_{in,air}$ = 1.6680 kJ/(kg·K) (T_{out} = 17 °C = 290 K)

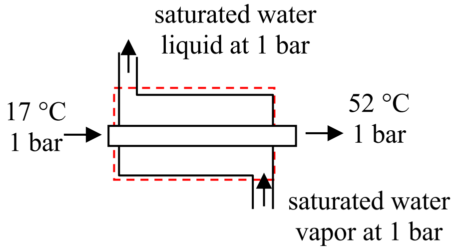
The entropy generation is,

$$\boxed{\frac{\dot{\sigma}}{\dot{m}_{air}} = 0.1145 \text{ kJ/(kg.K)}} \text{ for Method 1} \tag{3}$$

Since the entropy production is positive, the process satisfies the Second Law.

Now consider METHOD 2.

Apply the 2nd Law to the control volume shown below.



$$\frac{d}{dt}\int_{CV} s\rho dV + \int_{CS} s(\rho \boldsymbol{u}_{rel}\cdot d\boldsymbol{A}) = \int_{CS}\frac{\delta \dot{Q}_{into\ CV}}{T} + \dot{\sigma} \tag{4}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} s\rho dV = 0 \quad \text{(steady state)}$$

$$\int_{\text{CS}} s\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \left[\dot{m}\left(s_{\text{out}}-s_{\text{in}}\right)\right]_{\text{H}_2\text{O}} + \left[\dot{m}\left(s_{\text{out}}-s_{\text{in}}\right)\right]_{\text{air}}$$

$$\int_{CS}\frac{\delta\dot{Q}_{into\ CV}}{T} = 0 \quad \text{(no heat transfer into or out of the control volume)}$$

Substitute and simplify,

$$\dot{m}_{H2O}(s_{out}-s_{in})_{H2O} + \dot{m}_{air}(s_{out}-s_{in})_{air} = \dot{\sigma},$$

$$\frac{\dot{m}_{H2O}}{\dot{m}_{air}}(s_{out}-s_{in})_{H2O} + \dot{m}_{air}(s_{out}-s_{in})_{air} = \frac{\dot{\sigma}}{\dot{m}_{air}}, \tag{5}$$

The specific entropies of the outgoing and incoming air were calculated previously. The specific entropies of the saturated liquid and vapor water (both at 1 bar) are found from thermodynamics tables:

$s_{\text{out,H20}}$ = 1.3026 kJ/(kg·K) (saturated liquid water at 1 bar)
$s_{\text{in,H20}}$ = 7.3594 kJ/(kg·K) (saturated water vapor at 1 bar)

Now apply the First Law to the same control volume to determine the mass flow rate ratio,

$$\frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}}\left(h+\tfrac{1}{2}\rho V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\\text{CV}}} + \dot{W}_{\substack{\text{on}\\\text{CV}}} \tag{6}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} e\rho dV = 0 \quad \text{(steady state)}$$

$$\int_{\text{CS}}\left(h+\tfrac{1}{2}\rho V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \left[\dot{m}\left(h_{\text{out}}-h_{\text{in}}\right)\right]_{\text{H}_2\text{O}} + \left[\dot{m}\left(h_{\text{out}}-h_{\text{in}}\right)\right]_{\text{air}}$$

$$\dot{Q}_{\substack{\text{into}\\\text{CV}}} = \dot{W}_{\substack{\text{on}\\\text{CV}}} = 0 \quad \text{(adiabatic and no work)}$$

Substitute and simplify,

$$\dot{m}_{\text{H}_2\text{O}}\left(h_{\text{out}}-h_{\text{in}}\right)_{\text{H}_2\text{O}} + \dot{m}_{\text{air}}\left(h_{\text{out}}-h_{\text{in}}\right)_{\text{air}} = 0$$

$$\frac{\dot{m}_{\text{H}_2\text{O}}}{\dot{m}_{\text{air}}} = \frac{\left(h_{\text{out}}-h_{\text{in}}\right)_{\text{air}}}{\left(h_{\text{in}}-h_{\text{out}}\right)_{\text{H}_2\text{O}}} \tag{7}$$

Using the given data,

$h_{\text{out,air}}$ = 325.31 kJ/kg (T_{out} = 52 °C = 325 K)
$h_{\text{in,air}}$ = 290.16 kJ/kg (T_{out} = 17 °C = 290 K)
$h_{\text{out,H20}}$ = 417.46 kJ/(kg·K) (saturated liquid water at 1 bar)
$h_{\text{in,H20}}$ = 2675.5 kJ/(kg·K) (saturated water vapor at 1 bar)

$$\Rightarrow \frac{\dot{m}_{\text{H}_2\text{O}}}{\dot{m}_{\text{air}}} = 0.0156 \tag{8}$$

Substituting the previous result into Eq. (5) gives,

$$\frac{\dot{\sigma}}{\dot{m}_{air}} = 0.020 \text{ kJ/(kg.K)} \tag{9}$$

Since the entropy production is positive, the data are consistent with the Second Law. Furthermore, the total entropy production per unit mass flow rate of air is 0.020 kJ/(kg·K).

The given results indicate that METHOD 2 is a better approach, thermodynamically speaking, i.e., there is less irreversibility per unit mass flow rate of air, than METHOD 1.

4.7.1. Component Efficiencies

The Entropy Equation can be used to determine the entropy change expected during a given process. In particular, it can be used to calculate the entropy change expected for a system undergoing a process that is internally reversible, i.e., a process that is ideal and, thus, having the largest efficiency. For example, we can apply the Entropy Equation and the First Law together to calculate the efficiencies for various thermo-fuid components, such as compressors, turbines, and nozzles.

4.7.1.1. Compressor Efficiency

The efficiency of a compressor is defined as,

$$\eta_{\text{comp}} \coloneqq \frac{(\dot{W}_{\text{on comp}})_{\text{rev}}}{(\dot{W}_{\text{on comp}})_{\text{actual}}}, \tag{4.143}$$

where the subscripts "rev" and "actual" indicate internally reversible and the actual processes, respectively. Note that the actual power required to operate a compressor will always be larger than or equal to the power required to operate the compressor if the process is internally reversible and, thus, the efficiency will be $\eta_{\text{comp}} \leq 1$.

4.7.1.2. Turbine Efficiency

The efficiency of a turbine is defined as,

$$\eta_{\text{turb}} \coloneqq \frac{(\dot{W}_{\text{by turb}})_{\text{actual}}}{(\dot{W}_{\text{by turb}})_{\text{rev}}}. \tag{4.144}$$

Note that the actual power generated by a turbine will always be less than or equal to the power generated by an internally reversible turbine and, thus, $\eta_{\text{turb}} \leq 1$.

4.7.1.3. Nozzle Efficiency

Since the purpose of a nozzle is to speed up a flow, it is reasonable to define the nozzle efficiency as the ratio of the specific kinetic energy actually produced by the nozzle to the specific kinetic energy that would be produced under internally reversible conditions,

$$\eta_{\text{nozzle}} \coloneqq \frac{(\frac{1}{2}V_{\text{exit}}^2)_{\text{actual}}}{(\frac{1}{2}V_{\text{exit}}^2)_{\text{rev}}}. \tag{4.145}$$

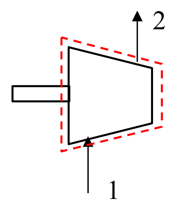
Nozzle efficiencies of 95% or more are common in practice.

Nitrogen (N_2) enters an insulated compressor operating at steady state at 1 bar (abs) and 37 °C with a mass flow rate of 1000 kg/h and exits at10 bar (abs). Kinetic and potential energy changes through the compressor are negligible. The nitrogen can be modeled as an ideal gas with a specific heat ratio of 1.391 and a specific heat at constant pressure of 1.056 kJ/(kg.K).

a. Determine the minimum theoretical power input required to operate the compressor and the corresponding exit temperature.
b. If the exit temperature is 397 °C, determine the power input and the compressor efficiency.

SOLUTION:

To find the power required to operate the compressor, apply the First Law to a control volume surrounding the compressor as shown in the following figure.



$$\frac{dE_{CV}}{dt} = \sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} + \dot{Q}_{\substack{\text{into}\\\text{CV}}} + \dot{W}_{\substack{\text{other,on}\\\text{CV}}}, \tag{1}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \quad \text{(steady flow assumed)}, \tag{2}$$

$$\sum_{\text{in}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}}\left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} = \dot{m}_1 h_1 - \dot{m}_2 h_2 = \dot{m}\left(h_1 - h_2\right), \tag{3}$$

(changes in KE and PE neglected; from COM, the mass flow rates are the same)

$$\dot{Q}_{\substack{\text{into}\\\text{CV}}} = 0 \quad \text{(adiabatic operation since insulated)}. \tag{4}$$

Solving for the power added into the compressor,

$$\dot{W}_{\substack{\text{other,on}\\\text{CV}}} = \dot{m}\left(h_2 - h_1\right). \tag{5}$$

If we further assume that the nitrogen behaves as a perfect gas, i.e., it has constant specific heats, which is a reasonable assumption if the temperature change is only a few hundred degrees, then,

$$\boxed{\dot{W}_{\substack{\text{other,on}\\\text{CV}}} = \dot{m}c_p\left(T_2 - T_1\right)}. \tag{6}$$

To calculate the <u>minimum</u> power required to operate the compressor, assume reversible operation. Since the flow is then adiabatic and reversible, it will also be isentropic. For isentropic operation of a perfect gas,

$$\frac{p_2}{p_1} = \left(\frac{T_{2s}}{T_1}\right)^{\frac{k}{k-1}} \Rightarrow T_{2s} = T_1\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}, \tag{7}$$

where the subscript "*s*" has been added to the temperature at state 2 to indicate isentropic conditions. Using the given parameters,

T_1 = 37 °C = 310 K,
p_2 = 10 bar (abs),
p_1 = 1 bar (abs),
k = 1.391,
$\Rightarrow T_{2s}$ = 592 K (= 319 °C)

Substituting into Eq. (6) gives,

$$\boxed{\left(\dot{W}_{\substack{\text{other,into}\\ \text{CV}}}\right)_{\text{min}} = 82.8\ \text{kW}},$$

with c_p = 1.056 kJ/(kg.K).

Using the actual measured temperature of T_2 = 397 °C = 670 K,

$$\boxed{\dot{W}_{\substack{\text{other,into}\\ \text{CV}}} = 106\ \text{kW}.}$$

The efficiency of the compressor is given by,

$$\boxed{\eta = \frac{\left(\dot{W}_{\substack{\text{other,into}\\ \text{CV}}}\right)_{\text{min}}}{\dot{W}_{\substack{\text{other,into}\\ \text{CV}}}} = 0.78.}$$

If we assume ideal, rather than perfect, gas behavior, then outlet temperature corresponding to an isentropic process is found using,

$$s_2 - s_1 = 0 = s_2^0\left(T_{2s}\right) - s_1^0\left(T_1\right) - R\ln\left(\frac{p_2}{p_1}\right) \Rightarrow \bar{s}_2^0\left(T_{2s}\right) = \bar{s}_1^0\left(T_1\right) + \bar{R}_u \ln\left(\frac{p_2}{p_1}\right), \quad (8)$$

with,

$\bar{s}_1^0$ (T_1 = 310 K) = 192.638 kJ/(kmol.K) (Table A-23 in Moran et al., 8th ed.),

p_2/p_1 = (10 bar)/(1 bar) = 10,

$\bar{R}_u$ = 8.314 kJ/(kmol.K),

=> $\bar{s}_2^0$ = 211.78 kJ/(kmol.K) => $\boxed{T_{2s} = 594\ \text{K}}$ (interpolating in Table A-23).

This result is less than 1% different from the one found earlier assuming perfect gas behavior.

Water vapor at 10 MPa (abs) and 600 °C enters a turbine operating at steady state with a volumetric flow rate of 0.36 m^3/s and exits at 0.1 bar (abs) and a quality of 92%. Stray heat transfer and kinetic and potential energy changes across the turbine are negligible. Determine for the turbine:

a. the mass flow rate,
b. the power developed by the turbine,
c. the rate at which entropy is produced, and
d. the isentropic turbine efficiency.

SOLUTION:

The mass flow rate through the turbine is given by,

$$\dot{m} = \rho Q \,, \tag{1}$$

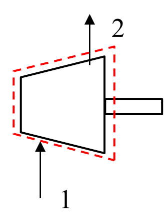
where ρ is the density of the water vapor and Q is the given volumetric flow rate. The water vapor density may be found from the inlet conditions,

$$\rho = 1/v = 26.1 \text{ kg/m}^3, \tag{2}$$

where v = 0.03837 m^3/kg @ 10 MPa (abs), 600 °C ⇒ superheated vapor (using Table A-4 in Moran et al., 7$^{\text{th}}$ ed., for example). Hence, the mass flow rate is,

$$\boxed{\dot{m} = 9.38 \text{ kg/s}}.$$

The power generated by the turbine may be found by applying the First Law to a control volume surrounding the turbine as shown in the following figure.



$$\frac{dE_{CV}}{dt} = \sum_{\text{in}} \left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}} \left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} + \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{other,on}\\ \text{CV}}} \,, \tag{3}$$

where,

$$\frac{dE_{CV}}{dt} = 0 \quad \text{(steady flow assumed)}, \tag{4}$$

$$\sum_{\text{in}} \left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} - \sum_{\text{out}} \left(h + \tfrac{1}{2}V^2 + gz\right)\dot{m} = \dot{m}_1 h_1 - \dot{m}_2 h_2 = \dot{m}\left(h_1 - h_2\right), \tag{5}$$

(changes in KE and PE neglected; from COM, the mass flow rates are the same)

$$\dot{Q}_{\substack{\text{into}\\ \text{CV}}} = 0 \quad \text{(adiabatic operation since insulated).} \tag{6}$$

Solving for the power generated by the turbine,

$$\dot{W}_{\substack{\text{other,on}\\ \text{CV}}} = \dot{m}\left(h_2 - h_1\right). \tag{7}$$

The specific enthalpies may be found using thermodynamic property tables,

h_1 = 3625.3 kJ/kg (@ 10 MPa (abs), 600 °C ⇒ superheated vapor; using Table A-4 in Moran et al., 7$^{\text{th}}$ ed.)

$h_2 = x_2 h_{2v} + (1 - x_2) h_{2l}$ = 2393.3 kJ/kg with h_{2v} = 2584.7 kJ/kg and h_{2l} = 191.83 kJ/kg (@ 0.1 bar (abs), x_2 = 0.92 ⇒ two-phase, liquid-vapor state; using Table A-3 in Moran et al., 7$^{\text{th}}$ ed.)

Using the given parameters,

$$\boxed{\dot{W}_{\substack{\text{other,on}\\ \text{CV}}} = \text{-11.6 MW}} \text{ (work is being done by the turbine).}$$

The rate at which entropy is produced is found by applying the entropy equation to the same control volume,

$$\frac{dS_{\text{CV}}}{dt} = \sum_{\text{in}} s\dot{m} - \sum_{\text{out}} s\dot{m} + \int_b \frac{\dot{Q}_{\text{into}}}{T} + \dot{\sigma} \ , \tag{8}$$

where,

$$\frac{dS_{CV}}{dt} = 0 \quad \text{(steady flow)}, \tag{9}$$

$$\sum_{\text{in}} s\dot{m} - \sum_{\text{out}} s\dot{m} = \dot{m}\left(s_1 - s_2\right) \quad \text{(the mass flow rate is constant from COM)}, \tag{10}$$

$$\int_b \frac{\dot{Q}_{\text{into}}}{T} = 0 \quad \text{(adiabatic operation)}, \tag{11}$$

$$\Rightarrow \quad \dot{\sigma} = \dot{m}\left(s_2 - s_1\right). \tag{12}$$

The specific entropies may be found using thermodynamic property tables,

$s_1 = 6.9029$ kJ/(kg.K) (@ 10 MPa (abs), 600 °C ⇒ superheated vapor; using Table A-4 in Moran et al., 7th ed.)

$s_2 = x_2 s_{2v} + (1 - x_2)s_{2l} = 7.5501$ kJ/(kg.K) with $s_{2v} = 8.1502$ kJ/(kg.K) and $s_{2l} = 0.6493$ kJ/(kg.K) (@ 0.1 bar (abs), $x_2 = 0.92$ ⇒ two-phase, liquid-vapor state; using Table A-4 in Moran et al., 7th ed.)

Using the given parameters,

$\boxed{\dot{\sigma} = 6.07 \text{ kW/K}}$. Note that the positive value indicates that the process is internally irreversible.

The isentropic efficiency of the turbine is defined as,

$$\eta \equiv \frac{\dot{W}_{\substack{\text{other,into}\\ \text{CV}}}}{\left(\dot{W}_{\substack{\text{other,into}\\ \text{CV}}}\right)_{\max}} , \tag{13}$$

where the maximum power generated by the turbine may be found assuming isentropic operation,

$s_2 = s_1 = 6.9029$ kJ/(kg.K). (14)

Since this specific entropy falls between $s_{2l} = 0.6493$ kJ/(kg.K) and $s_{2v} = 8.1502$ kJ/(kg.K) at $p_2 = 0.1$ bar (abs), state 2 for the isentropic case is in the two-phase, liquid-vapor region with a quality given by,

$$s_{2s} = x_{2s}s_{2v} + \left(1 - x_{2s}\right)s_{2l} \Rightarrow x_{2s} = \frac{s_{2s} - s_{2l}}{s_{2v} - s_{2l}} , \tag{15}$$

$\Rightarrow x_{2s} = 0.833$,

where the subscript "s" indicates the conditions for the isentropic case.

The specific enthalpy for this case is,

$$h_{2s} = x_{2s}h_{2v} + \left(1 - x_{2s}\right)h_{2l} , \tag{16}$$

$\Rightarrow h_{2s} = 2186.8$ kJ/kg,

$$\Rightarrow \left(\dot{W}_{\substack{\text{other,into}\\ \text{CV}}}\right)_{\max} = -13.5 \text{ MW}, \quad \text{(from Eq. (7))}$$

$\Rightarrow \boxed{\eta = 0.86}$.

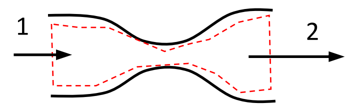
Helium gas at 810 °R, 45 psia, and a speed of 10 ft/s enters an insulated nozzle operating at steady state and exits at 670 °R, 25 psia. Modeling helium as an ideal gas with a specific heat ratio of 1.67, determine:

a. the speed at the nozzle exit, in ft/s,
b. the isentropic nozzle efficiency, and
c. the rate of entropy production within the nozzle, in Btu/(lb$_m$.°R).

SOLUTION:

Apply the First Law to a control volume surrounding the nozzle,

$$\frac{dE_{CV}}{dt} = \sum_{\text{in}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} - \sum_{\text{out}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} + \dot{Q}_{\text{into,CV}} + \dot{W}_{\text{other,on CV}}, \tag{1}$$



where,

$$\frac{dE_{CV}}{dt} = 0 \quad \text{(assumed steady flow)}, \tag{2}$$

$$\sum_{\text{in}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} - \sum_{\text{out}}\left(h+\tfrac{1}{2}V^2+gz\right)\dot{m} = \dot{m}\left[\left(h_1-h_2\right)+\tfrac{1}{2}\left(V_1^2-V_2^2\right)\right], \tag{3}$$

(since the flow is steady, conservation of mass states that the mass flow rate will remain constant; also assuming that the change in potential energy for the gas is negligible when compared to the change in specific enthalpy and specific kinetic energy)

$$\dot{Q}_{\text{into,CV}} = 0 \quad \text{(assumed adiabatic)}, \tag{4}$$

$$\dot{W}_{\text{other,on CV}} = 0 \quad \text{(no work other than pressure work)}. \tag{5}$$

Thus,

$$\dot{m}\left[\left(h_1-h_2\right)+\tfrac{1}{2}\left(V_1^2-V_2^2\right)\right] = 0 \Rightarrow V_2 = \sqrt{V_1^2+2\left(h_1-h_2\right)}\,. \tag{6}$$

Since helium is a noble gas, its specific heat won't change with temperature. Hence, a perfect gas assumption can be used and Eq. (6) becomes,

$$V_2 = \sqrt{V_1^2+2c_p\left(T_1-T_2\right)}\,. \tag{7}$$

Using the given data,

V_1 = 10 ft/s,
T_1 = 810 °R,
T_2 = 670 °R,
$c_p = \dfrac{kR}{k-1} = \dfrac{k}{k-1}\left(\dfrac{\bar{R}_u}{M}\right)$ with k = 1.67, $\bar{R}_u$ = 1545.4 ft.lb$_f$/(lbmol.°R), M = 4.003 lb$_m$/lbmol

=> R = 386.1 ft.lb$_f$/(lb$_m$.°R), c_p = 962.3 ft.lb$_f$/(lb$_m$.°R) (Note: 1 lb$_f$ = 32.2 lb$_m$.ft/s^2) (8)

=> $\boxed{V_2 = 2950 \text{ ft/s}}$.

The isentropic nozzle efficiency is defined as,

$$\eta_{\text{nozzle}} = \frac{\left(\tfrac{1}{2}V^2\right)_{\text{actual}}}{\left(\tfrac{1}{2}V^2\right)_{\text{ideal}}}\,. \tag{9}$$

The ideal specific kinetic energy may be found by assuming an internally reversible process. Since the flow is both internally reversible and adiabatic, it is also isentropic so $s_2 = s_1$. Since helium is a perfect gas,

$$\underbrace{\Delta s}_{=0} = c_p \ln\left(\frac{T_{2s}}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) \Rightarrow \frac{T_{2s}}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{R}{c_p}}, \tag{10}$$

=> T_{2s} = 640 °R,
=> V_{2s} = 3250 ft/s,
=> $\boxed{\eta_{\text{nozzle}} = 0.823}$.

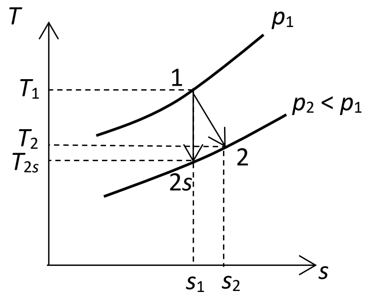
The rate of entropy production is found from,

$$\dot{m}\Delta s = \underbrace{\int_b \frac{\dot{Q}_{\text{into}}}{T}}_{=0\ (\text{adiabatic})} + \dot{\sigma} \Rightarrow \frac{\dot{\sigma}}{\dot{m}} = \Delta s\ , \tag{11}$$

where, again, because helium is a perfect gas,

$$\Delta s = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right), \tag{12}$$

=> $\boxed{\dot{\sigma}/\dot{m} = \Delta s\ = 44.3\ \text{ft}^2/(\text{s}^2.°\text{R})}$.



4.7.2. Heat Transfer and Work in Internally Reversible, State State, Steady Flow Processes

From the First Law, assuming one inlet and one outlet, steady state, and steady flow,

$$\frac{\dot{W}_{\text{other,on CV}}}{\dot{m}} = -\frac{\dot{Q}_{\text{into CV}}}{\dot{m}} + (h_2 - h_1) + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1). \tag{4.146}$$

Making use of the Tds equation to re-write the specific enthalpy term,

$$Tds = dh - vdp \implies \int_1^2 Tds = h_2 - h_1 - \int_1^2 vdp \implies h_2 - h_1 = \int_1^2 Tds + \int_1^2 vdp. \tag{4.147}$$

Substitute Eq. (4.147) into Eq. (4.146) and simplify to get,

$$\frac{\dot{W}_{\text{other,on CV}}}{\dot{m}} = -\frac{\dot{Q}_{\text{into CV}}}{\dot{m}} + \int_1^2 Tds + \int_1^2 vdp + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1). \tag{4.148}$$

If we further assume that *the process is internally reversible*, then,

$$d\dot{S} = \left.\frac{\delta\dot{Q}_{\text{into CV}}}{T}\right|_{\text{int. rev.}} \implies \dot{m}ds = \left.\frac{\delta\dot{Q}_{\text{into CV}}}{T}\right|_{\text{int. rev.}} \implies \frac{\dot{Q}_{\text{into CV,int. rev.}}}{\dot{m}} = \int_1^2 Tds. \tag{4.149}$$

Substituting into Eq. (4.148),

$$\frac{\dot{W}_{\text{other,on CV}}}{\dot{m}} = -\frac{\dot{Q}_{\text{into CV,int. rev.}}}{\dot{m}} + \left(\frac{\dot{Q}_{\text{into CV,int. rev.}}}{\dot{m}}\right) + \int_1^2 vdp + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1), \tag{4.150}$$

$$\boxed{\frac{\dot{W}_{\text{other,on CV}}}{\dot{m}} = \int_1^2 vdp + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1)}, \tag{4.151}$$

or, alternatively,

$$\boxed{\frac{\dot{W}_{\text{other,by CV}}}{\dot{m}} = -\int_1^2 vdp + \frac{1}{2}(V_1^2 - V_2^2) + g(z_1 - z_2)}. \tag{4.152}$$

This is the First Law for an internally reversible, steady state, steady flow with one inlet and one outlet.

Notes:

(1) For an isothermal process ($T =$ constant), we can integrate Eq. (4.149) to get,

$$\dot{m}(s_2 - s_1) = \frac{\dot{Q}_{\text{into CV}}\big|_{\text{int. rev.}}}{T} \implies \frac{\dot{Q}_{\text{into CV}}\big|_{\text{int. rev.}}}{\dot{m}} = T(s_2 - s_1). \tag{4.153}$$

(2) For the case where there is no "other" work, e.g., there is no shaft or electrical work, Eq. (4.152) becomes,

$$\boxed{\int_1^2 vdp + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0}, \tag{4.154}$$

which is known as Bernoulli's Equation. Bernoulli's equation is arguably the most frequently used relation in fluid mechanics. It's also frequently used incorrectly since the assumptions (steady state, steady flow, one inlet and one outlet, internally reversible, and no "other" work) must be satisfied. For an incompressible fluid, $v =$ constant and Bernoulli's equation becomes,

$$\boxed{v(p_2 - p_1) + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0}. \tag{4.155}$$

Recall that $v = 1/\rho$ so the previous equation may also be written as,

$$\boxed{\frac{p_2 - p_1}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0}. \tag{4.156}$$

For an ideal gas,

$$v = \frac{RT}{p} \implies \int_1^2 \frac{RT}{p} dp + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0. \tag{4.157}$$

For an isothermal process involving an ideal gas,

$$RT \ln\left(\frac{p_2}{p_1}\right) + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0. \tag{4.158}$$

For an isentropic process involving an ideal gas,

$$0 = c_p \frac{dT}{T} - R\frac{dp}{p} \implies \frac{dp}{p} = \frac{c_p}{R}\frac{dT}{T}, \tag{4.159}$$

$$\implies \int_1^2 RT \frac{c_p}{R}\frac{dT}{T} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0, \tag{4.160}$$

$$\therefore \underbrace{\int_1^2 c_p dT}_{=h(T_2)-h(T_1)} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0. \tag{4.161}$$

If isentropic flow of a perfect gas is considered (c_p = constant), then the previous equation becomes,

$$c_p(T_2 - T_1) + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0, \tag{4.162}$$

$$(h_2 - h_1) + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0, \tag{4.163}$$

$$\boxed{\Delta h_T = 0}. \tag{4.164}$$

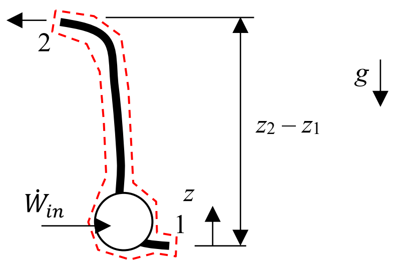
Note that when gases are considered, the potential energy changes are usually very small when compared to the other terms in Bernoulli's equation and can be neglected.

Bernoulli's equation (in the forms given here; there are other forms in which some of the assumptions are relaxed) can be viewed as a statement of the First Law with the assumptions of steady state and steady flow, one inlet and one outlet, internally reversible flow, and a flow with no "other" work.

A 3 hp pump operating at steady state draws in liquid water at 1 atm (abs), 60 °F and delivers it at 5 atm (abs) at an elevation 20 ft above the inlet. There is no significant change in velocity between the inlet and exit. Is it possible to pump 1000 gal in 10 min or less? Explain.

Image: https://www.bobvila.com/articles/some-advice-about-sump-pumps/

SOLUTION:



The mass flow rate required to pump 1000 gal of liquid water in 10 min is,

$$\dot{m} = \rho Q \,, \tag{1}$$

where ρ is the density of liquid water, assumed here to be 62.4 $\text{lb}_\text{m}/\text{ft}^3$, and Q is the volumetric flow rate,

$$Q = (1000 \text{ gal})/(10 \text{ min}) = 100 \text{ gal/min} = 13.37 \text{ ft}^3/\text{min} = 0.223 \text{ ft}^3/\text{s} \tag{2}$$

=> $\dot{m}$ = 13.9 lb_m/s.

Since we're interested in knowing if the pump is capable of pumping at the given flow rate, consider the ideal case, i.e., assume internally reversible, adiabatic flow. Note that the flow is at steady state and has one inlet and outlet. For these conditions, the 1st Law may be written as,

$$\frac{\dot{W}_{\text{other,on CV}}}{\dot{m}} = \int_{p_1}^{p_2} v\,dp + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g\left(z_2 - z_1\right). \tag{3}$$

For the current situation, assume the liquid water is incompressible (also re-write the specific volume as $v = 1/\rho$). Furthermore, we're told that there's no significant change in the velocity between the inlet and outlet, so the change in kinetic energy may be neglected. Re-writing Eq. (3) for these conditions gives,

$$\frac{\dot{W}_{\text{other,on CV}}}{\dot{m}} = \frac{p_2 - p_1}{\rho} + g\left(z_2 - z_1\right), \tag{4}$$

$$\dot{m} = \frac{\dot{W}_{\text{other,on CV}}}{\frac{p_2 - p_1}{\rho} + g\left(z_2 - z_1\right)}. \tag{5}$$

Using the given parameters,

$\dot{W}_{\text{other,on CV}}$ = 3 hp = 1650 ft.lb_f/s,

p_1 = 1 atm (abs) = 2117 $\text{lb}_\text{f}/\text{ft}^2$, p_2 = 5 atm (abs) = 10580 $\text{lb}_\text{f}/\text{in}^2$,

ρ = 62.4 $\text{lb}_\text{m}/\text{ft}^3$,

g = 32.2 ft/s^2, $z_2 - z_1$ = 20 ft,

=> $\dot{m}$ = 10.6 lb_m/s.

Since the ideal mass flow rate is smaller than what is required, it's not possible to pump the water at the desired flow rate.

4.8. Review Questions

(1) What is meant by the Eulerian and Lagrangian perspectives?
(2) Describe the Reynolds Transport Theorem in words? Why is it used?
(3) State, both in words and in mathematics, the Lagrangian forms of Conservation of Mass, Newton's Second Law, the First Law of Thermodynamics, and the Second Law of Thermodynamics.
(4) Why is it important to draw a well-defined control volume when applying conservation of mass, the linear momentum equations, the First Law of Thermodynamics, or the Second Law of Thermodynamics?
(5) What do each of the terms represent in the Lagrangian and Eulerian statements of Conservation of Mass?
(6) Does Conservation of Mass depend upon the coordinate system?
(7) Why is it important to draw a well-defined coordinate system when applying the linear momentum equations?
(8) What does each of the terms represent in the Eulerian form of the Linear Momentum Equation?
(9) What restrictions are placed on the coordinate system when applying the LMEs?
(10) In order to change the momentum of a flow, what must act on the flow?
(11) Give examples of body and surface forces.
(12) Explain what $\mathbf{u}_{\text{rel}}$ is. For what circumstances will $\mathbf{u}_{\text{rel}}$ and $\mathbf{u}_{XYZ}$ be the same?
(13) Why is the dot product used ($\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}$) in determining the flow rate out of a control volume?
(14) Can one apply the non-inertial form of the LME to an inertial frame of reference? How about applying the inertial form of the LME to a non-inertial frame of reference?
(15) What types of frames of reference can be considered inertial? Give examples of frames of reference that are not inertial.
(16) Given the following where M_{CV} is the mass in a control volume, $\dot{m}$ is a mass flow rate, and t is time,
$$\frac{dM_{CV}}{dt} = -\dot{m}, \tag{4.165}$$
will the following always be true (where M_0 is the control volume mass at $t = 0$)? Explain your answer.
$$M_{CV} = M_0 - \dot{m}t \tag{4.166}$$
(17) Describe what each term represents in the Eulerian form of the angular momentum equation.
(18) Why isn't the intrinsic angular momentum of the fluid included in the angular momentum equation?
(19) Consider a precessing, spinning top. Is its angular momentum conserved?
(20) Describe what each term represents in the Eulerian form of the First Law.
(21) What is meant by the term "adiabatic"?
(22) Why are shear work terms often (but not always!) neglected in conservation of energy?
(23) What is the definition of enthalpy?
(24) In the following form of the First Law, where are the terms involving the work due to movement in a gravity field and the work due to pressure forces?
$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS} (h + \frac{1}{2}V^2 + gz)\,(\rho\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{on CV}}. \tag{4.167}$$
(25) Describe what each term represents in the Eulerian form of the Second Law of Thermodynamics.
(26) What is the definition of entropy in terms of heat and temperature?
(27) What is meant by the term "internally reversible"? Give some examples of physical processes that result in irreversibility.
(28) How are adiabatic, internally reversible processes related to isentropic ones?

CHAPTER 5

Differential Analysis

5.1. Introduction to Index Notation (aka Tensor Notation, aka Einstein Notation)

Index notation is a compact way of writing equations and is often used in writing the equations used in fluid mechanics, solid mechanics, and many other fields.

Examples:

(1) The three numbers a_1, a_2, a_3 can be written as a_i, where $i = 1, 2, 3$

(2) A matrix of nine numbers can be written as: $a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(3) $a_i + b_i$ represents three numbers: $a_1 + b_1$, $a_2 + b_2$, $a_3 + b_3$

(4) $A_{ij} = B_{ij}$ represents nine equations: $A_{11} = B_{11}, A_{12} = B_{12}, A_{13} = B_{13}, A_{21} = B_{21}, \cdots, A_{33} = B_{33}$

(5) $A_{ijk} = B_{ijk}$ represents 27 equations: $A_{111} = B_{111}, A_{112} = B_{112}, A_{113} = B_{113}, A_{211} = B_{211}, \cdots, A_{333} = B_{333}$

5.1.1. Free Indices

A free index is an index that appears exactly once in a term. Each term in an equation must have the same free indices. A repeated index is one that appears twice in a term. No index may appear more than twice in a term.

Examples:

(1) $a_{ijk}b_j = c_{ik}$ is a correct equation. There are two free indices: i and k, and one repeated index: j.

(2) $a_{ij}b_{jk} = c_{lm}$ is an incorrect equation. There are two free indices on each side of the equation: i, j and l, m, but they're not the same.

(3) $a_{ij}b_j = c_{ij}$ is an incorrect equation. The only free index on the left-hand side is i while i and j are free indices on the right-hand side.

5.1.2. Summation Convention

If a subscript appears exactly twice in a term, i.e., it's a repeated index, then summation over that subscript from 1 to 3 is implied.

Examples:

(1) $a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$ (This is a single number.)

(2) $a_{ij}b_j = \sum_{j=1}^{3} a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$ (Since i can vary from 1 to 3, $a_{ij}b_j$ is three separate numbers.)

(3) $A_{ij}B_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}B_{ij} = (A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} + \cdots + A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33})$ (This is a single number.)

Notes:

(1) Repeated indices are dummy indices: $a_{ii} = a_{jj} = a_{kk}$

(2) No index may appear more than twice in a term: $a_ib_ic_i$ and A_{iii} are incorrect.

(3) The summation convention is suspended by writing "no sum" or by underlining one of the repeated subscripts, e.g., σ_{ii} (no sum) $= \sigma_{i\underline{i}} = \sigma_{11}, \sigma_{22}, \sigma_{33}$ (three separate numbers).

5.1.3. Kronecker's Delta

Kronecker's Delta, δ_{ij} is the two-index symbol defined by,

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{5.1}$$

Notes:

(1) $\delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0$ and $\delta_{11} = \delta_{22} = \delta_{33} = 1$.

(2) The Kronecker Delta is the identity matrix: $\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Examples:

(1) Show that $\delta_{ij} a_j = a_i$.

$$i = 1: \quad \delta_{1j} a_j = \underbrace{\delta_{11}}_{=1} a_1 + \underbrace{\delta_{12}}_{=0} a_2 + \underbrace{\delta_{13}}_{=0} a_3 = a_1 \tag{5.2}$$

$$i = 2: \quad \delta_{2j} a_j = \underbrace{\delta_{21}}_{=0} a_1 + \underbrace{\delta_{22}}_{=1} a_2 + \underbrace{\delta_{23}}_{=0} a_3 = a_2 \tag{5.3}$$

$$i = 3: \quad \delta_{3j} a_j = \underbrace{\delta_{31}}_{=0} a_1 + \underbrace{\delta_{32}}_{=0} a_2 + \underbrace{\delta_{33}}_{=1} a_3 = a_3 \tag{5.4}$$

(2) Show that $\delta_{ii} = 3$.

$$\delta_{ii} = \underbrace{\delta_{11}}_{=1} + \underbrace{\delta_{22}}_{=1} + \underbrace{\delta_{33}}_{=1} = 3 \tag{5.5}$$

5.1.4. Permutation (aka Alternating) Unit Tensor

The permutation tensor, ϵ_{ijk}, is a three-index symbol defined as,

$$\epsilon_{ijk} = \begin{cases} +1 & \text{for} \quad \epsilon_{123}, \epsilon_{231}, \epsilon_{312} \\ -1 & \text{for} \quad \epsilon_{321}, \epsilon_{213}, \epsilon_{132} \\ 0 & \text{for all other permutations} \end{cases} \tag{5.6}$$

Notes:

(1) It's convenient to remember the pictures in Figure 5.1 for determining the proper sign of the permutation tensor.

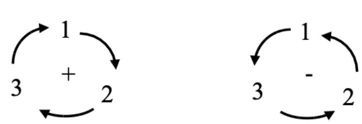


Figure 5.1. Figures to help determine the sign of the permutation tensor.

(2) Switching any two indices changes the sign of the permutation tensor, e.g., $\epsilon_{ijk} = -\epsilon_{ikj}$.

(3) A convenient identity is:

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \tag{5.7}$$

Example:

Show that $\epsilon_{ijk}\epsilon_{ijk} = 6$.

$$\epsilon_{ijk}\epsilon_{ijk} = \underbrace{\delta_{jj}}_{=3}\underbrace{\delta_{kk}}_{=3} - \delta_{jk}\delta_{kj} = 9 - \left(\underbrace{\delta_{1k}\delta_{k1}}_{=1} + \underbrace{\delta_{2k}\delta_{k2}}_{=1} + \underbrace{\delta_{3k}\delta_{k3}}_{=1}\right) \tag{5.8}$$

$$= 9 - 3 \tag{5.9}$$

$$= 6 \tag{5.10}$$

5.1.5. Tensors

A tensor of rank r is a quantity having n^r components in n-dimensional space, e.g., a tensor of rank 2 in 3D has $3^2 = 9$ components. The components of a tensor expressed in two different coordinate systems are related by,

$$T_{ijk\cdots m} = \lambda_{is}\lambda_{jt}\lambda_{ku}\cdots\lambda_{mv}T_{stu\cdots v} \tag{5.11}$$

where λ_{is} are the direction cosines between the $\hat{\boldsymbol{e}}_i$ and $\hat{\boldsymbol{e}}_s$ axes.

Notes:

(1) A tensor of rank 2 is often called a dyad, e.g., A_{ij} (two free subscripts).
(2) A tensor of rank 1 is called a vector, e.g., a_i (one free subscript).
(3) A tensor of rank 0 is called a scalar, e.g., c (zero free subscripts).
(4) The vector notation for a dyad is often written as: $\underline{\underline{\boldsymbol{A}}}$.

5.1.6. Basic Mathematical Operations

- Addition: Two tensors of equal rank can be added to yield a tensor of the same rank,

$$C_{ij\cdots k} = A_{ij\cdots k} + B_{ij\cdots k} \tag{5.12}$$

- Multiplication: If a tensor, A, having rank, a, is multiplied by tensor, B, having rank, b, then a tensor, C, of rank $a + b$ results,

$$C_{ij\cdots krs\cdots t} = A_{ij\cdots k}B_{rs\cdots t} \tag{5.13}$$

  For example, $\underbrace{A_{ij}}_{r=2}\underbrace{B_{rs}}_{r=2} = \underbrace{C_{ijrs}}_{r=4}$.

- Transpose: The transpose of a tensor is,

$$T^T_{ij\cdots k} = T_{k\cdots ji} \tag{5.14}$$

  For example, $A^T_{ij} = A_{ji}$. If the components of a second-order tensor are presented in matrix form, then the transpose is equivalent to swapping the off-diagonal components.

- Symmetric: A symmetric tensor is one that has the property,

$$T_{ij\cdots k} = T_{k\cdots ji} \tag{5.15}$$

  A symmetric tensor is equal to its transpose.

- Anti-symmetric: An anti-symmetric tensor is one that has the property,

$$T_{ij\cdots k} = -T_{k\cdots ji} \tag{5.16}$$

Notes:

(1) A tensor, symmetric in $ij\cdots k$, is often indicated using the notation: $T_{(ij\cdots k)}$.
(2) A tensor, anti-symmetric in $ij\cdots k$, is often indicated using the notation: $T_{[ij\cdots k]}$.
(3) $T_{(ij\cdots k)} = \frac{1}{2}\left(T_{ij\cdots k} + T_{k\cdots ji}\right)$
(4) $T_{[ij\cdots k]} = \frac{1}{2}\left(T_{ij\cdots k} - T_{k\cdots ji}\right)$
(5) $T_{ij\cdots k} = T_{(ij\cdots k)} + T_{[ij\cdots k]}$
(6) $T_{ii} = T_{11} + T_{22} + T_{33} = \mathrm{trace}(T_{ij})$

- Dot Products (aka Inner Products):

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i \tag{5.17}$$
$$\boldsymbol{a} \cdot \underline{\underline{\boldsymbol{B}}} = a_i B_{ij} \tag{5.18}$$
$$\underline{\underline{\boldsymbol{A}}} \cdot \boldsymbol{b} = A_{ij} b_j \tag{5.19}$$
$$\underline{\underline{\boldsymbol{A}}} \cdot \underline{\underline{\boldsymbol{B}}} = A_{ij} B_{jk} \tag{5.20}$$
$$\underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}} = A_{ij} B_{ji} \tag{5.21}$$
$$\underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}}^T = A_{ij} B_{ij} \tag{5.22}$$
$$\underline{\underline{\boldsymbol{T}}} = \boldsymbol{ab} \implies T_{ij} = a_i b_j \quad \text{(Note: } \boldsymbol{ab} = (\boldsymbol{ba})^T.) \tag{5.23}$$
$$\tag{5.24}$$

- Cross-Product:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \implies c_i = \epsilon_{ijk} a_j b_k \tag{5.25}$$

- Gradient:

$$(\boldsymbol{\nabla}\lambda)_i = \frac{\partial \lambda}{\partial x_i} = \lambda_{,i} \tag{5.26}$$

- Divergence:

$$\boldsymbol{\nabla} \cdot \boldsymbol{a} = \frac{\partial a_i}{\partial x_i} = a_{i,i} \tag{5.27}$$

- Curl:

$$(\boldsymbol{\nabla} \times \boldsymbol{a})_i = \epsilon_{ijk} \frac{\partial a_k}{\partial x_j} = \epsilon_{ijk} a_{k,j} \tag{5.28}$$

- Laplacian (scalar):

$$\nabla^2 \lambda = \frac{\partial^2 \lambda}{\partial x_i \partial x_i} = \lambda_{,ii} \tag{5.29}$$

- Gauss's Theorem (aka Divergence Theorem):

$$\int_S (\boldsymbol{a} \cdot \hat{\mathbf{n}})\, dS = \int_V (\boldsymbol{\nabla} \cdot \boldsymbol{a})\, dV \tag{5.30}$$
$$\int_S a_i n_i dS = \int_V a_{i,i} dV \tag{5.31}$$

where S is the surface enclosing the volume V and $\hat{\mathbf{n}}$ is the outward-pointing unit normal vector for the surface area element dS (Figure 5.2).

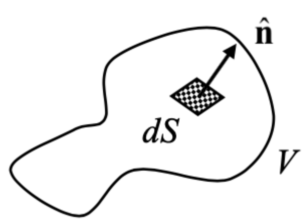


Figure 5.2. Illustration corresponding to Gauss' Theorem.

- Stokes's Theorem:

$$\oint_C (\boldsymbol{a} \cdot d\boldsymbol{l}) = \int_S (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot \hat{\mathbf{n}} dS \tag{5.32}$$
$$\oint_C a_i dl_i = \int_S \epsilon_{ijk} a_{k,j} n_i dS \tag{5.33}$$

where the curve C defines the surface S and $d\boldsymbol{l}$ is a vector tangent to the curve C at a particular point. Note that the shape of the surface on which Stokes' Theorem is to be applied must be known so that the relation between the surface area and contour is well defined.

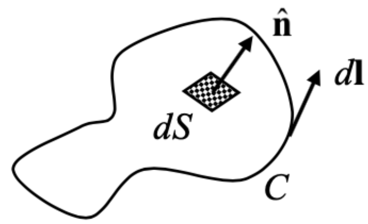


Figure 5.3. Illustration corresponding to Stokes' Theorem.

References:

(1) Borg, S.F., *Matrix-Tensor Methods in Continuum Mechanics*, Van Nostrand, 1963.
(2) Myklestad, N.O., *Cartesian Tensors*, Van Nostrand, 1967.
(3) Synge, J.L. and Schild, A., *Tensor Calculus*, University of Toronto Press, 1949.
(4) Aris, R., *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Prentice-Hall, 1962.

Prove that the following are true using index notation:

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$$

$$\mathbf{t}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}(\mathbf{t}\cdot\mathbf{v}) - \mathbf{v}(\mathbf{t}\cdot\mathbf{u})$$

$$\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$$

SOLUTION:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= \varepsilon_{ijk}a_j b_k c_i \\ &= \varepsilon_{jki}b_k c_i a_j = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} \\ &= \varepsilon_{kij}c_i a_j b_k = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}\end{aligned}$$

$$\begin{aligned}\left[\mathbf{t}\times(\mathbf{u}\times\mathbf{v})\right]_i &= \varepsilon_{ijk}t_j\varepsilon_{klm}u_l v_m \\ &= \varepsilon_{ijk}\varepsilon_{klm}t_j u_l v_m \\ &= \varepsilon_{kij}\varepsilon_{klm}t_j u_l v_m \\ &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)t_j u_l v_m \\ &= t_j u_i v_j - t_j u_j v_i \\ &= \left[\mathbf{u}\left(\mathbf{t}\cdot\mathbf{v}\right) - \mathbf{v}\left(\mathbf{t}\cdot\mathbf{u}\right)\right]_i\end{aligned}$$

$$\begin{aligned}\left[\mathbf{u}\times\mathbf{v}\right]_i &= \varepsilon_{ijk}u_j v_k \\ &= -\varepsilon_{ikj}v_k u_j \\ &= \left[-\mathbf{v}\times\mathbf{u}\right]_i\end{aligned}$$

Show, using index notation, that:

$$\nabla \times \nabla \theta = \mathbf{0}$$

SOLUTION:

$$\begin{aligned}
(\nabla \times \nabla \theta)_i &= \varepsilon_{ijk} \left(\theta_{,k}\right)_{,j} \\
&= \varepsilon_{ijk} \theta_{,kj} && (1) \\
&= -\varepsilon_{ikj} \theta_{,kj} \\
&= -\varepsilon_{ijk} \theta_{,jk} \\
&= -\varepsilon_{ijk} \theta_{,kj} \quad \text{(the order of the differentiation doesn't matter)} && (2)
\end{aligned}$$

The only way for lines (1) and (2) to be equal is if they both equal zero, *i.e.*:

$$\varepsilon_{ijk} \theta_{,kj} = 0 = -\varepsilon_{ijk} \theta_{,kj} \tag{3}$$

Therefore:

$$(\nabla \times \nabla \theta)_i = 0 \;\Rightarrow\; \nabla \times \nabla \theta = \mathbf{0}$$

Solve:

$$a_i = \varepsilon_{ijk} b_{jk}$$

for $b_{[jk]}$.

SOLUTION:

$$\begin{aligned}
\varepsilon_{imn} a_i &= \varepsilon_{imn} \varepsilon_{ijk} b_{jk} \\
&= \left(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}\right) b_{jk} \\
&= b_{mn} - b_{nm} \\
&= 2 \cdot \tfrac{1}{2}\left(b_{mn} - b_{nm}\right) \\
&= 2b_{[mn]}
\end{aligned}$$

$$\therefore b_{[jk]} = \tfrac{1}{2} \varepsilon_{ijk} a_i$$

Prove that the following is true using index notation:

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$$

SOLUTION:

$$\begin{aligned}
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk} u_j v_k\right) \\
&= \varepsilon_{ijk} \frac{\partial}{\partial x_i}\left(u_j v_k\right) \\
&= \varepsilon_{ijk}\left(u_j \frac{\partial v_k}{\partial x_i} + v_k \frac{\partial u_j}{\partial x_i}\right) \\
&= \varepsilon_{ijk} u_j \frac{\partial v_k}{\partial x_i} + \varepsilon_{ijk} v_k \frac{\partial u_j}{\partial x_i} \\
&= u_j \varepsilon_{jki} \frac{\partial v_k}{\partial x_i} + v_k \varepsilon_{kij} \frac{\partial u_j}{\partial x_i} \\
&= v_k \varepsilon_{kij} \frac{\partial u_j}{\partial x_i} - u_j \varepsilon_{jik} \frac{\partial v_k}{\partial x_i} \\
&= \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})
\end{aligned}$$

Given the following:

$$\varepsilon_{ijk}\varepsilon_{pqr}\det\left[a_{mn}\right]=\begin{vmatrix}a_{ip} & a_{iq} & a_{ir}\\ a_{jp} & a_{jq} & a_{jr}\\ a_{kp} & a_{kq} & a_{kr}\end{vmatrix}$$

where "det" is the determinant operator, show that:

$$\varepsilon_{ijk}\varepsilon_{iqr}=\delta_{jq}\delta_{kr}-\delta_{jr}\delta_{kq}$$

SOLUTION:

Using the given relation:

$$\varepsilon_{ijk}\varepsilon_{iqr}\underbrace{\det\left[\delta_{mn}\right]}_{=1}=\begin{vmatrix}\delta_{ii} & \delta_{iq} & \delta_{ir}\\ \delta_{ji} & \delta_{jq} & \delta_{jr}\\ \delta_{ki} & \delta_{kq} & \delta_{kr}\end{vmatrix}\quad \text{(Note: } \det\left[\delta_{mn}\right]=\begin{vmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{vmatrix}=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=1\text{)}$$

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{pqr}&=\delta_{ii}\delta_{jq}\delta_{kr}+\delta_{iq}\delta_{jr}\delta_{ki}+\delta_{ir}\delta_{ji}\delta_{kq}-\delta_{ir}\delta_{jq}\delta_{ki}-\delta_{jr}\delta_{kq}\delta_{ii}-\delta_{kr}\delta_{iq}\delta_{ji}\\ &=3\delta_{jq}\delta_{kr}+\delta_{jr}\delta_{kq}+\delta_{jr}\delta_{kq}-\delta_{kr}\delta_{jq}-3\delta_{jr}\delta_{kq}-\delta_{kr}\delta_{jq}\end{aligned}$$

$$\boxed{\therefore\ \varepsilon_{ijk}\varepsilon_{pqr}=\delta_{jq}\delta_{kr}-\delta_{jr}\delta_{kq}}$$

Let $\underline{\underline{\mathbf{A}}}$ and $\underline{\underline{\mathbf{B}}}$ be second order tensors. Show that if one of the tensors is symmetric and the other is anti-symmetric, then $\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}} = 0$.

SOLUTION:

$$\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}} = A_{ij} B_{ji} \tag{1}$$

Let $\underline{\underline{\mathbf{A}}}$ be the symmetric tensor and $\underline{\underline{\mathbf{B}}}$ be the anti-symmetric tensor so that:

$$A_{ij} = A_{ji} \quad \text{and} \quad B_{ji} = -B_{ij} \tag{2}$$

Substitute the previous expressions back into Eqn. (1):

$$A_{ij} B_{ji} = A_{ji} \left(-B_{ij}\right) = -A_{ji} B_{ij} \tag{3}$$

Note that the subscripts in the previous expression are dummy indices so that the right-hand side of the relation may be written as:

$$A_{ij} B_{ji} = -A_{ji} B_{ij} = -A_{ij} B_{ji} \tag{4}$$

The only way for the previous expression to be true is if both sides equal zero. Thus,

$$A_{ij} B_{ji} = -A_{ij} B_{ji} = 0 \Rightarrow \boxed{\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}} = 0} \tag{5}$$

Using index notation, show that:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

SOLUTION:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= \varepsilon_{ijk}a_j b_k \varepsilon_{ilm} c_l d_m \\ &= \varepsilon_{ijk}\varepsilon_{ilm} a_j b_k c_l d_m \\ &= \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) a_j b_k c_l d_m \\ &= a_l b_m c_l d_m - a_m b_l c_l d_m \\ &= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\end{aligned}$$

Show that:

$$\left(\mathbf{a}\times\mathbf{b}\right)_i = \varepsilon_{ijk} a_j b_k$$

SOLUTION:

Expand the index notation side first.

$$\varepsilon_{ijk} a_j b_k = \begin{cases} \underbrace{\varepsilon_{123}}_{=+1} a_2 b_3 + \underbrace{\varepsilon_{132}}_{=-1} a_3 b_2 & i=1 \\ \underbrace{\varepsilon_{231}}_{=+1} a_3 b_1 + \underbrace{\varepsilon_{213}}_{=-1} a_1 b_3 & i=2 \\ \underbrace{\varepsilon_{312}}_{=+1} a_1 b_2 + \underbrace{\varepsilon_{321}}_{=-1} a_2 b_1 & i=3 \end{cases}$$ (where all other ε terms are zero, *e.g.* $\varepsilon_{122} = \varepsilon_{112} = \ldots = 0$.)

$$\varepsilon_{ijk} a_j b_k = \begin{cases} a_2 b_3 - a_3 b_2 & i=1 \\ a_3 b_1 - a_1 b_3 & i=2 \\ a_1 b_2 - a_2 b_1 & i=3 \end{cases} \tag{1}$$

Now expand the vector notation side.

$$\mathbf{a}\times\mathbf{b} = \left(a_2 b_3 - a_3 b_2\right)\hat{\mathbf{e}}_1 + \left(a_3 b_1 - a_1 b_3\right)\hat{\mathbf{e}}_2 + \left(a_1 b_2 - a_2 b_1\right)\hat{\mathbf{e}}_3 \tag{2}$$

Hence we see that the components of $\varepsilon_{ijk} a_j b_k$ match up with components of ($\mathbf{a}\times\mathbf{b}$), *i.e.*

$$\left(\mathbf{a}\times\mathbf{b}\right)_i = \varepsilon_{ijk} a_j b_k \tag{3}$$

Expand the following expression for i, j, and k = 1, 2, 3.

$$\lambda\left(\frac{\partial u_k}{\partial x_k}\right)^2+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}$$

SOLUTION:

Expand the first term first.

$$\lambda\left(\frac{\partial u_k}{\partial x_k}\right)^2=\lambda\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)^2 \tag{1}$$

Expand the second term next.

$$\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}=\mu\begin{bmatrix}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\right)\frac{\partial u_1}{\partial x_1}+\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\frac{\partial u_1}{\partial x_2}+\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)\frac{\partial u_1}{\partial x_3}\\+\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)\frac{\partial u_2}{\partial x_1}+\left(\frac{\partial u_2}{\partial x_2}+\frac{\partial u_2}{\partial x_2}\right)\frac{\partial u_2}{\partial x_2}+\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)\frac{\partial u_2}{\partial x_3}\\+\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)\frac{\partial u_3}{\partial x_1}+\left(\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\right)\frac{\partial u_3}{\partial x_2}+\left(\frac{\partial u_3}{\partial x_3}+\frac{\partial u_3}{\partial x_3}\right)\frac{\partial u_3}{\partial x_3}\end{bmatrix} \tag{2}$$

Thus:

$$\boxed{\begin{aligned}\lambda\left(\frac{\partial u_k}{\partial x_k}\right)^2+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}&=\lambda\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)^2\\&\quad+\mu\begin{bmatrix}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\right)\frac{\partial u_1}{\partial x_1}+\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\frac{\partial u_1}{\partial x_2}+\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)\frac{\partial u_1}{\partial x_3}\\+\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)\frac{\partial u_2}{\partial x_1}+\left(\frac{\partial u_2}{\partial x_2}+\frac{\partial u_2}{\partial x_2}\right)\frac{\partial u_2}{\partial x_2}+\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)\frac{\partial u_2}{\partial x_3}\\+\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)\frac{\partial u_3}{\partial x_1}+\left(\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\right)\frac{\partial u_3}{\partial x_2}+\left(\frac{\partial u_3}{\partial x_3}+\frac{\partial u_3}{\partial x_3}\right)\frac{\partial u_3}{\partial x_3}\end{bmatrix}\end{aligned}} \tag{3}$$

5.2. The Continuity Equation (aka Conservation of Mass for a Differential Control Volume)

The Continuity Equation, which is Conservation of Mass for a differential fluid element or control volume, can be derived several different ways. Two of these methods are given in this section.

Method 1: Apply the integral approach to the fixed differential control volume shown in Figure 5.4. Assume

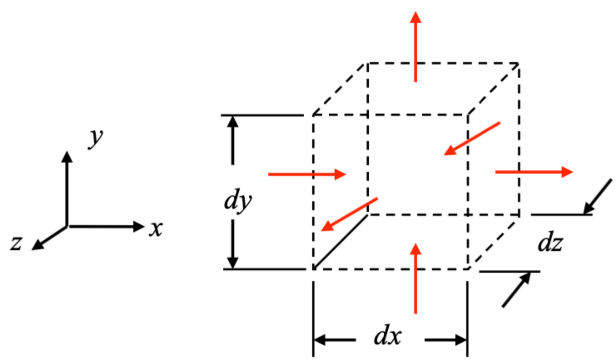


Figure 5.4. The control volume used to derive the Continuity Equation.

that the density and velocity are ρ and $\boldsymbol{u}$, respectively, at the control volume's center. Using a Taylor series approximation, the mass flow rate through the left side of the control volume is given by,

$$\dot{m}_{\text{in through left}} = \dot{m}_{x,\text{center}} + \frac{\partial \dot{m}_{x,\text{center}}}{\partial x}\left(-\frac{1}{2}dx\right), \tag{5.34}$$

$$= (\rho u_x dydz) + \frac{\partial}{\partial x}(\rho u_x dydz)\left(-\frac{1}{2}dx\right), \tag{5.35}$$

$$= \left[\rho u_x + \frac{\partial}{\partial x}(\rho u_x)\left(-\frac{1}{2}dx\right)\right](dydz), \tag{5.36}$$

where $\dot{m}_{x,\text{center}}$ is the mass flow rate in the x-direction at the center of the control volume. A similar approach can be used to find the mass flow rates through the other sides of the control volume,

$$\dot{m}_{\text{out through right}} = \left[\rho u_x + \frac{\partial}{\partial x}(\rho u_x)\left(\frac{1}{2}dx\right)\right](dydz), \tag{5.37}$$

$$\dot{m}_{\text{in through bottom}} = \left[\rho u_y + \frac{\partial}{\partial y}(\rho u_y)\left(-\frac{1}{2}dy\right)\right](dxdz), \tag{5.38}$$

$$\dot{m}_{\text{out through top}} = \left[\rho u_y + \frac{\partial}{\partial y}(\rho u_y)\left(\frac{1}{2}dy\right)\right](dxdz), \tag{5.39}$$

$$\dot{m}_{\text{in through back}} = \left[\rho u_z + \frac{\partial}{\partial z}(\rho u_z)\left(-\frac{1}{2}dz\right)\right](dxdy), \tag{5.40}$$

$$\dot{m}_{\text{out through front}} = \left[\rho u_z + \frac{\partial}{\partial z}(\rho u_z)\left(\frac{1}{2}dz\right)\right](dxdy). \tag{5.41}$$

Thus, the net mass flow rate into the control volume is,

$$\dot{m}_{\text{net, into CV}} = -\left[\frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z)\right](dxdydz). \tag{5.42}$$

The rate at which mass increases within the control volume is,

$$\frac{\partial m_{\text{CV}}}{\partial t} = \frac{\partial}{\partial t}(\rho dxdydz) = \frac{\partial \rho}{\partial t}(dxdydz), \tag{5.43}$$

where ρ is the density at the center of the control volume. Note that since the density varies linearly within the control volume (from the Taylor Series approximation), the average density in the control volume is ρ.

Conservation of Mass states that the rate of increase of mass within the control volume must equal the net rate at which mass enters the control volume,

$$\frac{\partial m_{\text{CV}}}{\partial t} = -\dot{m}_{\text{net, into CV}}, \tag{5.44}$$

$$\frac{\partial \rho}{\partial t}(dxdydz) = -\left[\frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z)\right](dxdydz), \tag{5.45}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0}. \tag{5.46}$$

Written in a more compact form,

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0}, \tag{5.47}$$

or, in index notation,

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0}. \tag{5.48}$$

Method 2: Recall that the integral form of Conservation of Mass is given by,

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0. \tag{5.49}$$

Consider a fixed control volume so that,

$$\frac{d}{dt}\int_{CV} \rho dV = \int_{CV} \frac{\partial \rho}{\partial t} dV \quad \text{and} \quad \boldsymbol{u}_{\text{rel}} = \boldsymbol{u}. \tag{5.50}$$

By utilizing Gauss' Theorem (aka the Divergence Theorem), we can convert the area integral into a volume integral,

$$\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \int_{CV} \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})\, dV. \tag{5.51}$$

Substitute these expressions back into Conservation of Mass to get,

$$\int_{CV} \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})\right] dV = 0. \tag{5.52}$$

Since the choice of control volume is arbitrary, the kernel of the integral must be zero, i.e.,

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0}, \tag{5.53}$$

which is the same result found previously.

Notes:

(1) For a fluid in which the density remains uniform and constant, i.e., ρ = constant, the Continuity Equation simplifies to,

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0} \quad \text{or} \quad \boxed{\frac{\partial u_i}{\partial x_i} = 0}. \tag{5.54}$$

(2) An <u>incompressible fluid</u> is one in which the density of a <u>particular piece of fluid</u> remains constant, i.e.,

$$\boxed{\frac{D\rho}{Dt} = 0}. \tag{5.55}$$

Note that an incompressible fluid does not necessarily imply that the density is the same everywhere in the flow, i.e. it's not necessarily uniform. An example of such a flow would be a stratified flow in the ocean where the density of various layers of ocean water varies due to salinity and temperature variations (Figure 5.5). A fluid with a constant and uniform density, however, is an incompressible

Figure 5.5. The density of fluid particles varies from layer to layer in this stratified flow, but remains constant within a layer.

fluid.

The Continuity Equation for an incompressible fluid can be found by using Eq. (5.55),

$$\frac{D\rho}{Dt} = 0 = \frac{\partial \rho}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho \implies \frac{\partial \rho}{\partial t} = -(\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho. \tag{5.56}$$

Substituting into the Continuity Equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{5.57}$$

$$-(\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{5.58}$$

$$-(\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho + \rho\,(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho = 0, \tag{5.59}$$

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0}. \tag{5.60}$$

Thus, an incompressible fluid has the same Continuity Equation as a fluid with constant and uniform density.

(3) Another useful form of the Continuity Equation is,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \implies \underbrace{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}}_{=\frac{D\rho}{Dt}} + \rho \frac{\partial u_i}{\partial x_i}, \tag{5.61}$$

$$\boxed{\frac{D\rho}{Dt} = -\rho \frac{\partial u_i}{\partial x_i}}. \tag{5.62}$$

(4) The Continuity Equation (Eq. (5.53)) is valid for any continuous substance, e.g., a solid as well as a fluid.

(5) Equation (5.54) is referred to as the conservative form of the Continuity Equation while Eq. (5.62) is the non-conservative form. The conservative form implies that the equation represents an Eulerian viewpoint of the Continuity Equation. The non-conservative form represents the Lagrangian viewpoint.

The y-velocity component of a steady, 2D, incompressible flow is given by:

$$u_y = 3xy - x^2 y$$

Determine the most general velocity component in the x-direction for this flow.

SOLUTION:

Consider the continuity equation:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{1}$$

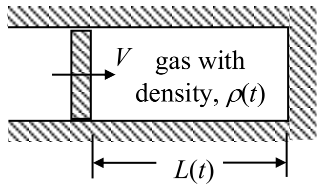
$$\frac{\partial u_x}{\partial x} = -\frac{\partial u_y}{\partial y} = -\frac{\partial}{\partial y}\left(3xy - x^2 y\right) = -3x + x^2$$

Integrate u_x with respect to x.

$$\boxed{u_x = -\tfrac{3}{2}x^2 + \tfrac{1}{3}x^3 + f\left(y\right)} \tag{2}$$

where $f(y)$ is an unknown function of y.

A piston compresses gas in a cylinder by moving at a constant speed, V. The gas density and the piston length are initially ρ_0 and L_0, respectively. Assume that the gas velocity varies linearly from velocity, V, at the piston face to zero velocity at the cylinder wall (at L). If the gas density varies only with time, determine $\rho(t)$.



SOLUTION:

As given in the problem statement, assume the gas velocity, u, varies linearly with distance x from the piston face with the boundary conditions: $u(x = 0) = V$ and $u(x = L(t)) = 0$.

$$\Rightarrow \quad u(x,t) = V\left(1 - \frac{x}{L(t)}\right) \tag{1}$$

However, the piston moves at a constant speed so that:

$$L(t) = L_0 - Vt \tag{2}$$

Substituting Eqn. (2) into Eqn. (1) gives:

$$u(x,t) = V\left(1 - \frac{x}{L_0 - Vt}\right) \tag{3}$$

Apply the continuity equation assuming 1D flow.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{4}$$

$$\frac{d\rho}{dt} = -\rho \frac{\partial u}{\partial x} \quad \text{(Note that } \rho = \rho(t)\text{.)}$$

$$\frac{d\rho}{\rho} = V\left(\frac{1}{L_0 - Vt}\right) dt$$

$$\int_{\rho=\rho_0}^{\rho=\rho} \frac{d\rho}{\rho} = V \int_{t=0}^{t=t} \frac{dt}{L_0 - Vt}$$

$$\ln\left(\frac{\rho}{\rho_0}\right) = -\ln\left(\frac{L_0 - Vt}{L_0}\right)$$

$$\boxed{\therefore \frac{\rho}{\rho_0} = \left(1 - \frac{Vt}{L_0}\right)^{-1}} \tag{5}$$

A velocity field for an incompressible flow is given by

$$\mathbf{u} = (-2xz)\hat{\mathbf{i}} + (2xy + z^2)\hat{\mathbf{j}} + (z^2 - 2xz - 2yz)\hat{\mathbf{k}}$$

Is this flow physically possible?

SOLUTION:

Does the given velocity field satisfy the continuity equation?

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{1}$$

Using the given velocity field:

$$\frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}\left(-2xz\right) = -2z$$

$$\frac{\partial u_y}{\partial y} = \frac{\partial}{\partial y}\left(2xy + z^2\right) = 2x$$

$$\frac{\partial u_z}{\partial z} = \frac{\partial}{\partial z}\left(z^2 - 2xz - 2yz\right) = 2z - 2x - 2y$$

Substitute into Eqn. (1).

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = -2z + 2x + 2z - 2x - 2y = -2y \neq 0$$

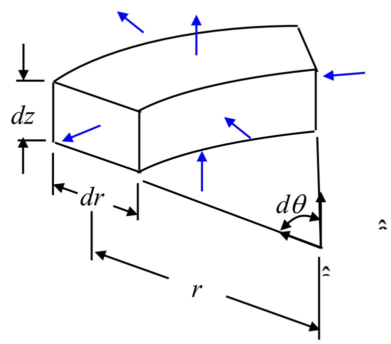
Hence, the given flow field is <u>not</u> physically possible since it does not satisfy the continuity equation.

Derive the continuity equation in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho u_r\right)+\frac{1}{r}\frac{\partial\left(\rho u_\theta\right)}{\partial \theta}+\frac{\partial\left(\rho u_z\right)}{\partial z}=0$$

by considering the mass flux through an infinitesimal control volume which is fixed in space.

SOLUTION:



Let the density and velocity at the center of the control volume be ρ and $\mathbf{u}$, respectively. First determine the mass fluxes through each side of the control volume.

$$\dot{m}_{\text{in,bottom}}=\left[\left(\rho u_z\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)\left(-\tfrac{1}{2}dz\right)\right]\left(rdrd\theta\right)$$

$$\dot{m}_{\text{out,top}}=\left[\left(\rho u_z\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)\left(\tfrac{1}{2}dz\right)\right]\left(rdrd\theta\right)$$

$$\dot{m}_{\text{in,front}}=\left[\left(\rho u_r\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)\left(-\tfrac{1}{2}dr\right)\right]\left[\left(r-\tfrac{1}{2}dr\right)d\theta dz\right]$$

$$\dot{m}_{\text{out,back}}=\left[\left(\rho u_r\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)\left(\tfrac{1}{2}dr\right)\right]\left[\left(r+\tfrac{1}{2}dr\right)d\theta dz\right]$$

$$\dot{m}_{\text{in,RHS}}=\left[\left(\rho u_\theta\right)+\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(-\tfrac{1}{2}d\theta\right)\right]\left(drdz\right)$$

$$\dot{m}_{\text{out,LHS}}=\left[\left(\rho u_\theta\right)+\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(\tfrac{1}{2}d\theta\right)\right]\left(drdz\right)$$

The net mass flux <u>out</u> of the control volume is:

$$\begin{aligned}
\dot{m}_{\text{out,net}}&=\dot{m}_{\text{out,top}}-\dot{m}_{\text{in,bottom}}+\dot{m}_{\text{out,back}}-\dot{m}_{\text{in,front}}+\dot{m}_{\text{out,LHS}}-\dot{m}_{\text{in,RHS}}\\
&=\left[\left(\rho u_z\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)\left(\tfrac{1}{2}dz\right)\right]\left(rdrd\theta\right)-\left[\left(\rho u_z\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)\left(-\tfrac{1}{2}dz\right)\right]\left(rdrd\theta\right)\\
&\quad+\left[\left(\rho u_r\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)\left(\tfrac{1}{2}dr\right)\right]\left[\left(r+\tfrac{1}{2}dr\right)d\theta dz\right]-\left[\left(\rho u_r\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)\left(-\tfrac{1}{2}dr\right)\right]\left[\left(r-\tfrac{1}{2}dr\right)d\theta dz\right]\\
&\quad+\left[\left(\rho u_\theta\right)+\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(\tfrac{1}{2}d\theta\right)\right]\left(drdz\right)-\left[\left(\rho u_\theta\right)+\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(-\tfrac{1}{2}d\theta\right)\right]\left(drdz\right)\\
&=\left[\frac{\partial}{\partial z}\left(\rho u_z\right)\left(dz\right)\right]\left(rdrd\theta\right)+\left[\left(\rho u_r dr\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)rdr\right]\left(d\theta dz\right)+\left[\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(d\theta\right)\right]\left(drdz\right)
\end{aligned}$$

$$\therefore \dot{m}_{\text{out,net}}=\left[\frac{\partial}{\partial r}\left(\rho u_r\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)+\left(\frac{\rho u_r}{r}\right)\right]\left(rdrd\theta dz\right) \tag{1}$$

The rate of increase of mass within the control volume is:

$$\left.\frac{dm}{dt}\right|_{\text{within CV}} = \frac{\partial}{\partial t}\left(\rho r dr d\theta dz\right) = \frac{\partial \rho}{\partial t}\left(r dr d\theta dz\right) \tag{2}$$

From conservation of mass, the rate at which the mass inside the control volume increases plus the net rate at which mass leaves the control volume must be zero, *i.e.*:

$$\left.\frac{dm}{dt}\right|_{\text{within CV}} + \dot{m}_{\text{out,net}} = 0$$

$$\frac{\partial \rho}{\partial t}\left(r dr d\theta dz\right) + \left[\frac{\partial}{\partial r}\left(\rho u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) + \left(\frac{\rho u_r}{r}\right)\right]\left(r dr d\theta dz\right) = 0$$

Hence:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left(\rho u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) + \left(\frac{\rho u_r}{r}\right) = 0} \tag{3}$$

or, by combining the 2nd and last terms on the LHS:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \rho u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) = 0} \tag{4}$$

The x-velocity component of a steady, 2D, incompressible flow is given by:

$$u_x = y - x$$

Determine the most general velocity component in the y-direction for this flow.

SOLUTION:

Consider the continuity equation:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u_y}{\partial y} = -\frac{\partial u_x}{\partial x} = -\frac{\partial}{\partial x}(y - x) = 1$$

Integrate u_y with respect to y.

$$\boxed{u_y = y + f(x)} \tag{2}$$

where $f(x)$ is an unknown function of x.

Double check:

$$\frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}(y - x) = -1 \tag{3}$$

$$\frac{\partial u_y}{\partial y} = \frac{\partial}{\partial y}\left[y + f(x)\right] = 1 \tag{4}$$

$$\Rightarrow \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = -1 + 1 = 0 \quad \text{OK!} \tag{5}$$

5.3. A Review of Stress

5.3.1. Traction Vector (aka Stress Intensity, Stress Vector)

Consider a small area on or within a deformable body subject to both surface and body forces as shown in Figure 5.6. The net force and moment acting on the area $\Delta A \boldsymbol{\nu}$, where ΔA is the magnitude of the area and

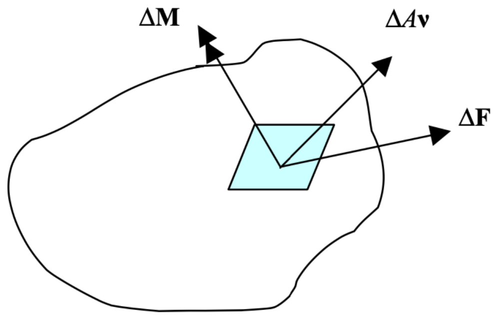


Figure 5.6. The force and moment acting on a differential area within an object. The differential area magnitude is ΔA and its unit normal vector is $\boldsymbol{\nu}$.

$\boldsymbol{\nu}$ is its corresponding unit normal vector, are denoted by $\Delta \boldsymbol{F}$ and $\Delta \boldsymbol{M}$, respectively. The traction vector (aka stress intensity or stress vector) on the surface is defined as,

$$\boldsymbol{T}^{\boldsymbol{\nu}} := \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{F}}{\Delta A}, \tag{5.63}$$

a vector with dimensions of force per unit area, and the couple stress vector on the surface is defined as,

$$\boldsymbol{C}^{\boldsymbol{\nu}} := \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{M}}{\Delta A}, \tag{5.64}$$

a vector with dimensions of torque per unit area.

Notes:

(1) Usually $\boldsymbol{C}^{\boldsymbol{\nu}} = \boldsymbol{0}$ since body moments are rare. An example of a case in which body moments and, thus the couple stress vector, is not zero is in a material comprised of polar or magnetic elements, i.e., molecules or domains, subject to an external electric or magnetic field. In such a case, the material elements will try to orient themselves in a preferred direction. The couple stress vector may also be non-zero in powders in which the component particles have an aspect ratio greater than one and, hence, tend to re-orient when under load.

(2) To completely describe the traction at a "point", we need to know $\boldsymbol{T}^{\boldsymbol{\nu}}$ for all orientations, $\boldsymbol{\nu}$, of the differential surface area at that point.

(3) Let σ_{ij} be the components of the traction vector $\boldsymbol{T}^{\boldsymbol{\nu}}$. Consider, for example, the tractions on the faces of a differential cube as shown in Figure 5.7. The traction on each face in terms of its components is,

$$\boldsymbol{T}^1 = \sigma_{11}\hat{\boldsymbol{e}}_1 + \sigma_{12}\hat{\boldsymbol{e}}_2 + \sigma_{13}\hat{\boldsymbol{e}}_3, \tag{5.65}$$

$$\boldsymbol{T}^2 = \sigma_{21}\hat{\boldsymbol{e}}_1 + \sigma_{22}\hat{\boldsymbol{e}}_2 + \sigma_{23}\hat{\boldsymbol{e}}_3, \tag{5.66}$$

$$\boldsymbol{T}^3 = \sigma_{31}\hat{\boldsymbol{e}}_1 + \sigma_{32}\hat{\boldsymbol{e}}_2 + \sigma_{33}\hat{\boldsymbol{e}}_3, \tag{5.67}$$

$$\tag{5.68}$$

where $\hat{\boldsymbol{e}}_i$ are the unit direction vectors of the axes. The quantity σ_{ij} is known as the stress tensor.

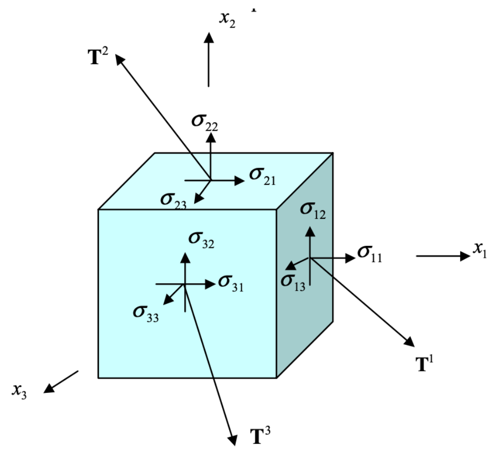


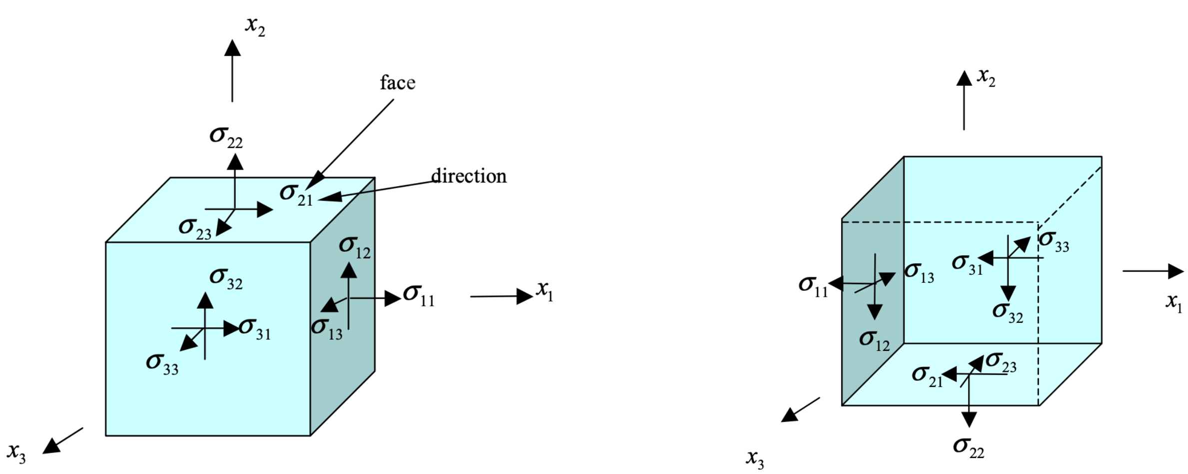
Figure 5.7. The traction vectors and stress components on the faces of a cube.

5.3.2. Stress Sign Convention

The sign convention for stresses are as follows:

(1) A positive face is a face that has a normal vector pointing in a positive direction.
(2) Positive stresses on positive faces point in the positive direction.
(3) Positive stresses on negative faces point in the negative direction.
(4) The first subscript on the stress refers to the face on which the stress acts. The second subscript refers to the direction in which the stress acts.

Figure 5.8 shows positive stresses on all of the cube's faces.



(A) Positive stresses on positive faces. (B) Positive stresses on negative faces.

Figure 5.8. Figures illustrating the stress sign convention.

Notes:

(1) σ_{ii} (no sum) are referred to as normal stresses.
(2) σ_{ij} $(i \neq j)$ are referred to as shear stresses.
(3) negative normal stresses $\Longrightarrow$ compression
(4) positive normal stresses $\Longrightarrow$ tension

5.3.3. Cauchy's Formula

Cauchy's formula is used to determine the traction vector on an arbitrarily-oriented surface with an orientation vector, $\boldsymbol{\nu}$, given the stress tensor. Consider the small tetrahedral element shown in Figure 5.9. The area of each face is,

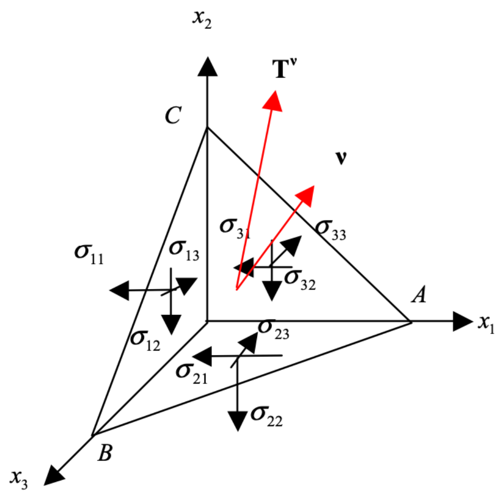


Figure 5.9. The traction vector on the face of a tetrahedral element of material and the stress components on the "back" faces of the element.

$$d\boldsymbol{A}^{\boldsymbol{\nu}} = dA^{\boldsymbol{\nu}}\left(\nu_1\hat{\boldsymbol{e}}_1 + \nu_2\hat{\boldsymbol{e}}_2 + \nu_3\hat{\boldsymbol{e}}_3\right), \tag{5.69}$$

$$|d\boldsymbol{A}^{\boldsymbol{\nu}}| = dA^{\boldsymbol{\nu}}, \tag{5.70}$$

$$|d\boldsymbol{A}^{\hat{e}_1}| = d\boldsymbol{A}^{\boldsymbol{\nu}} \cdot \hat{\boldsymbol{e}}_1 = dA^{\boldsymbol{\nu}}\nu_1, \tag{5.71}$$

$$|d\boldsymbol{A}^{\hat{e}_2}| = d\boldsymbol{A}^{\boldsymbol{\nu}} \cdot \hat{\boldsymbol{e}}_2 = dA^{\boldsymbol{\nu}}\nu_2, \tag{5.72}$$

$$|d\boldsymbol{A}^{\hat{e}_3}| = d\boldsymbol{A}^{\boldsymbol{\nu}} \cdot \hat{\boldsymbol{e}}_3 = dA^{\boldsymbol{\nu}}\nu_3. \tag{5.73}$$

Apply Newton's Second Law to the element in the x_1 direction,

$$(\rho dV)\ddot{x}_1 = \sum F_1 = \left(\boldsymbol{T}^{\boldsymbol{\nu}} \cdot \hat{\boldsymbol{e}}_1\right) dA^{\boldsymbol{\nu}} - \sigma_{11}|d\boldsymbol{A}^{\hat{e}_1}| - \sigma_{21}|d\boldsymbol{A}^{\hat{e}_2}| - \sigma_{31}|d\boldsymbol{A}^{\hat{e}_3}| + f_1\rho dV, \tag{5.74}$$

$$\implies T_1^{\boldsymbol{\nu}} dA^{\boldsymbol{\nu}} = \sigma_{11}dA^{\boldsymbol{\nu}}\nu_1 + \sigma_{21}dA^{\boldsymbol{\nu}}\nu_2 + \sigma_{31}dA^{\boldsymbol{\nu}}\nu_3 + \left(\ddot{x}_1 - f_1\right)\rho dV, \tag{5.75}$$

$$\implies T_1^{\boldsymbol{\nu}} = \sigma_{11}\nu_1 + \sigma_{21}\nu_2 + \sigma_{31}\nu_3 + \left(\ddot{x}_1 - f_1\right)\rho\frac{dV}{dA^{\boldsymbol{\nu}}}, \tag{5.76}$$

but,

$$\lim_{dV\to 0}\left(\frac{dV}{dA^{\boldsymbol{\nu}}}\right) = 0 \implies T_1^{\boldsymbol{\nu}} = \sigma_{11}\nu_1 + \sigma_{21}\nu_2 + \sigma_{31}\nu_3, \tag{5.77}$$

and, finally,

$$T_1^{\boldsymbol{\nu}} = \sigma_{j1}\nu_j. \tag{5.78}$$

A similar approach may be followed to drive expressions for the x_2 and x_3 directions. In general,

$$\boxed{T_i^{\boldsymbol{\nu}} = \sigma_{ji}\nu_j} \quad \underline{\text{Cauchy's Formula}}. \tag{5.79}$$

Cauchy's formula may be used to determine the traction, $\boldsymbol{T}^{\boldsymbol{\nu}}$, on a surface with an orientation, $\boldsymbol{\nu}$, given the stress tensor, σ_{ji}, at the point of interest. For example, consider the surface shown in Figure 5.10, which shows how the traction on that surface is comprised of the stress components on the surface.

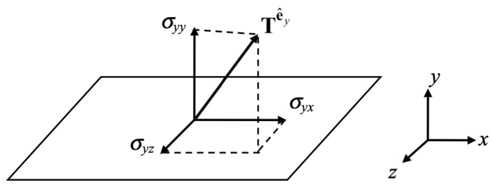


Figure 5.10. An illustration showing how the traction vector on a surface is comprised of the stress components on the surface.

5.3.4. Symmetry of the Stress Tensor

Consider the Angular Momentum Equation applied to the small element of material shown in Figure 5.11 (only the stresses acting on the positive faces and causing rotations about the x_3 axis are shown for clarity). Also note that no body couples are assumed to act on the element. Using Newton's Second Law for rotational

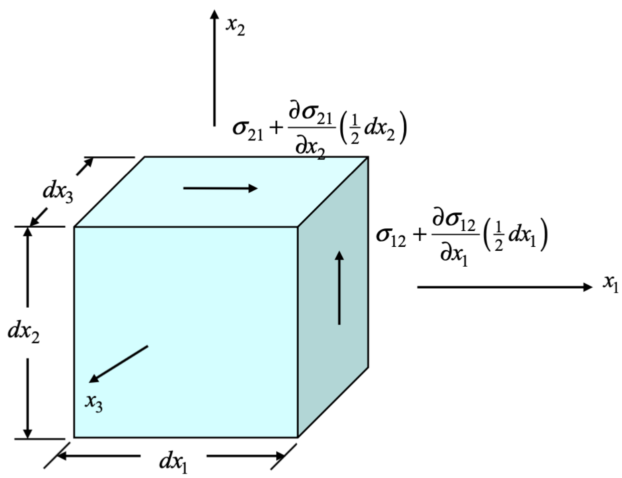


Figure 5.11. An illustration showing the stress components (on positive faces) causing a moment about the x_3 axis.

moments in the x_3-direction,

$$\begin{aligned}\underbrace{I_3}_{\sim[dx_1dx_2dx_3(dx_1^2+dx_2^2)]}\dot{\omega}_3 &= \left[\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1}\left(\frac{1}{2}dx_1\right)\right]dx_2dx_3\left(\frac{1}{2}dx_1\right) + \left[\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1}\left(-\frac{1}{2}dx_1\right)\right]dx_2dx_3\left(\frac{1}{2}dx_1\right)\\ &\quad - \left[\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2}\left(\frac{1}{2}dx_2\right)\right]dx_1dx_3\left(\frac{1}{2}dx_2\right) - \left[\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2}\left(-\frac{1}{2}dx_2\right)\right]dx_1dx_3\left(\frac{1}{2}dx_2\right)\end{aligned} \tag{5.80}$$

Dividing through by the volume $(dx_1dx_2dx_3)$ and taking the limit as $dx_1, dx_2, dx_3 \to 0$ gives,

$$\sigma_{21} = \sigma_{12}. \tag{5.81}$$

A similar approach can be taken about the x_1 and x_2 axes to arrive at the general result,

$$\boxed{\sigma_{ij} = \sigma_{ji}}. \tag{5.82}$$

The stress tensor is symmetric when no couple stresses are present!

Another approach to proving symmetry of the stress tensor is to write the Angular Momentum Equation explicitly for the small element, again assuming no body couples,

$$\underbrace{\int_{V_{\text{sys}}} \epsilon_{ijk} r_j f_k \rho dV}_{\substack{\text{torque due to} \\ \text{body forces}}} + \underbrace{\int_{S_{\text{sys}}} \epsilon_{ijk} r_j \underbrace{\sigma_{lk} n_l}_{=T_k} dA}_{\substack{\text{torque due to} \\ \text{surface forces}}} = \underbrace{\frac{D}{Dt} \int_{V_{\text{sys}}} \epsilon_{ijk} r_j u_k \rho dV}_{\substack{\text{rate at which the element's} \\ \text{angular momentum changes}}} . \tag{5.83}$$

Using the Divergence Theorem, the surface integral may be written as a volume integral,

$$\int_{S_{\text{sys}}} \epsilon_{ijk} r_j \sigma_{lk} n_l dA = \int_{V_{\text{sys}}} \epsilon_{ijk} \frac{\partial}{\partial x_i} \left(r_j \sigma_{lk} \right) dV = \int_{V_{\text{sys}}} \epsilon_{ijk} \left(\underbrace{\delta_{jl} \sigma_{lk}}_{=\sigma_{jk}} + r_j \sigma_{lk,l} \right) dV. \tag{5.84}$$

Substituting this last equation into the previous one and simplifying (note that the element mass remains constant),

$$\int_{V_{\text{sys}}} \epsilon_{ijk} r_j f_k \rho dV + \int_{V_{\text{sys}}} \epsilon_{ijk} \left(\sigma_{jk} + r_j \sigma_{lk,l} \right) dV = \frac{D}{Dt} \int_{V_{\text{sys}}} \epsilon_{ijk} r_j u_k \rho dV, \tag{5.85}$$

$$\int_{V_{\text{sys}}} \epsilon_{ijk} \left[r_j \rho f_k + \sigma_{jk} + r_j \sigma_{lk,l} - \rho \underbrace{\frac{D}{Dt} \left(r_j u_k \right)}_{= r_j \frac{Du_k}{Dt} + \frac{Dr_j}{Dt} u_k} \right] dV = 0. \tag{5.86}$$

Note that $Dr_j/Dt = u_j$ and $\epsilon_{ijk} u_j u_k = 0$ (i.e., $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$). Re-arranging the previous equation gives,

$$\int_{V_{\text{sys}}} \epsilon_{ijk} \left[r_j \left(\rho f_k + \sigma_{lk,l} - \rho \frac{Du_k}{Dt} \right) + \sigma_{jk} \right] dV = 0. \tag{5.87}$$

As the volume of the cube becomes very small ($dV \to 0$), the $\boldsymbol{r}$ vector also becomes very small. Hence,

$$\lim_{dV \to 0} \left\{ \int_{V_{\text{sys}}} \epsilon_{ijk} \left[r_j \left(\rho f_k + \sigma_{lk,l} - \rho \frac{Du_k}{Dt} \right) + \sigma_{jk} \right] dV \right\} = \int_{V_{\text{sys}}} \epsilon_{ijk} \sigma_{jk} dV = 0. \tag{5.88}$$

Since the volume is arbitrary,

$$\epsilon_{ijk} \sigma_{jk} = 0. \tag{5.89}$$

Now apply the permutation tensor to both sides of the equation and utilize an identity,

$$\epsilon_{ilm} \epsilon_{ijk} \sigma_{jk} = 0, \tag{5.90}$$

$$\left(\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{mj} \right) \sigma_{jk} = 0, \tag{5.91}$$

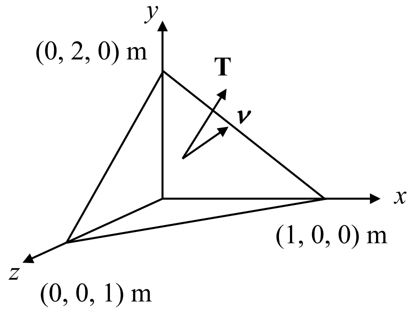
$$\sigma_{lm} - \sigma_{ml} = 0, \tag{5.92}$$

$$\boxed{\therefore \sigma_{ml} = \sigma_{lm}}. \tag{5.93}$$

Thus, the stress tensor is symmetric (again, assuming no body couples).

The material element shown below has the following stress tensor components:

$$[\sigma] = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 0 \\ -4 & 0 & 5 \end{bmatrix} \text{ kPa}$$



a. Find the components of the traction vector, **T**, on the plane described by the unit normal vector, $\boldsymbol{\nu}$.
b. Determine the component of **T** parallel to $\boldsymbol{\nu}$.
c. Determine the component of **T** perpendicular to $\boldsymbol{\nu}$.
d. Determine the angle between **T** and $\boldsymbol{\nu}$.

SOLUTION:
First determine the components of the unit normal vector, $\boldsymbol{\nu}$, by taking the cross product of the vector pointing from (1, 0, 0) to (0, 2, 0) with the vector pointing from (1, 0, 0) to (0, 0, 1) then normalizing.

$$\mathbf{v} = \frac{\left[(0,2,0)-(1,0,0)\right]\times\left[(0,0,1)-(1,0,0)\right]}{\left|\left[(0,2,0)-(1,0,0)\right]\times\left[(0,0,1)-(1,0,0)\right]\right|} = \frac{(-1,2,0)\times(-1,0,1)}{\left|(-1,2,0)\times(-1,0,1)\right|} = \frac{(2,1,2)}{\left|(2,1,2)\right|} \tag{1}$$

$$\therefore \mathbf{v} = \left(\tfrac{2}{3},\tfrac{1}{3},\tfrac{2}{3}\right) \tag{2}$$

Next, use Cauchy's formula to determine the traction vector, **T**.

$$T_i = \sigma_{ji}\nu_j \tag{3}$$

$$\Rightarrow \begin{aligned} T_1 &= \sigma_{11}\nu_1 + \sigma_{21}\nu_2 + \sigma_{31}\nu_3 = (1\text{ kPa})\left(\tfrac{2}{3}\right) + (0\text{ kPa})\left(\tfrac{1}{3}\right) + (-4\text{ kPa})\left(\tfrac{2}{3}\right) \\ T_2 &= \sigma_{12}\nu_1 + \sigma_{22}\nu_2 + \sigma_{32}\nu_3 = (0\text{ kPa})\left(\tfrac{2}{3}\right) + (3\text{ kPa})\left(\tfrac{1}{3}\right) + (0\text{ kPa})\left(\tfrac{2}{3}\right) \\ T_3 &= \sigma_{13}\nu_1 + \sigma_{23}\nu_2 + \sigma_{33}\nu_3 = (-4\text{ kPa})\left(\tfrac{2}{3}\right) + (0\text{ kPa})\left(\tfrac{1}{3}\right) + (5\text{ kPa})\left(\tfrac{2}{3}\right) \end{aligned}$$

$$\boxed{\therefore \mathbf{T} = \left(-2,1,\tfrac{2}{3}\right)\text{ kPa}} \tag{4}$$

The component of **T** parallel to $\boldsymbol{\nu}$ is found by taking the dot product **T** · $\boldsymbol{\nu}$.

$$T_{\parallel} = \mathbf{T}\cdot\mathbf{v} = T_i\nu_i = (-2\text{ kPa})\left(\tfrac{2}{3}\right) + (1\text{ kPa})\left(\tfrac{1}{3}\right) + \left(\tfrac{2}{3}\text{ kPa}\right)\left(\tfrac{2}{3}\right) \tag{5}$$

$$\boxed{\therefore T_{\parallel} = -\tfrac{5}{9}\text{ kPa}} \tag{6}$$

The component of **T** perpendicular to $\boldsymbol{\nu}$ is found by taking the difference between **T** and $T_{\parallel}\boldsymbol{\nu}$.

$$T_{\perp} = \left|\mathbf{T} - T_{\parallel}\mathbf{v}\right| = \left|\left(-2,1,\tfrac{2}{3}\right) - \left(-\tfrac{5}{9}\right)\left(\tfrac{2}{3},\tfrac{1}{3},\tfrac{2}{3}\right)\right|\text{ kPa} = \left|\left(-\tfrac{44}{27},\tfrac{32}{27},\tfrac{28}{27}\right)\right|\text{ kPa} \tag{7}$$

$$\boxed{T_{\perp} = 2.27\text{ kPa}} \tag{8}$$

The angle between **T** and $\boldsymbol{\nu}$ can be determined from the dot product between the two vectors.

$$\mathbf{T}\cdot\mathbf{v} = |\mathbf{T}||\mathbf{v}|\cos\theta \;\Rightarrow\; \cos\theta = \frac{\mathbf{T}\cdot\mathbf{v}}{|\mathbf{T}||\mathbf{v}|} \tag{9}$$

$$\cos\theta = \frac{\mathbf{T}\cdot\mathbf{v}}{|\mathbf{T}||\mathbf{v}|} = \frac{-\tfrac{5}{9}\text{ kPa}}{\left|\left(-2,1,\tfrac{2}{3}\right)\text{ kPa}\right|(1)} = \frac{-\tfrac{5}{9}\text{ kPa}}{\tfrac{7}{3}\text{ kPa}} = -\tfrac{5}{21}$$

$$\boxed{\therefore \theta = 103.8^{\circ}} \tag{10}$$

The velocity profile in a two-dimensional flow is $u_x = U[1-(y/h)^2]$. The stress tensor for the flow is:

$$[\sigma] = \begin{bmatrix} -5 & -2\mu U \frac{y}{h^2} & 0 \\ -2\mu U \frac{y}{h^2} & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

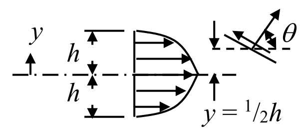
Find the stress normal and tangential to a plane located at $y/h = ½$ with its normal at a 30° angle to the flow direction.

SOLUTION:

The unit normal vector for the plane is:

$$\hat{\mathbf{n}} = \cos\theta \hat{\mathbf{e}}_x + \sin\theta \hat{\mathbf{e}}_y \quad (1)$$

where $\theta = 30°$.



The stress normal to the plane is:

$$\sigma_N = \mathbf{T}^{\hat{\mathbf{n}}} \cdot \hat{\mathbf{n}} = \sigma_{ji} n_j n_i \quad \text{where } \mathbf{T}^{\hat{\mathbf{n}}} \text{ is the traction vector for the plane} \quad (2)$$

$$\sigma_N = \sigma_{11}(n_1)^2 + \sigma_{21}n_2n_1 + \sigma_{31}n_3n_1 + \sigma_{12}n_1n_2 + \sigma_{22}(n_2)^2 + \sigma_{32}n_3n_2 + \sigma_{13}n_1n_3 + \sigma_{23}n_2n_3 + \sigma_{33}(n_3)^2 \quad (3)$$

$$\sigma_N = -5\cos^2\theta - 2\mu U \tfrac{y}{h^2}\sin\theta\cos\theta - 2\mu U \tfrac{y}{h^2}\cos\theta\sin\theta - 5\sin^2\theta \quad (4)$$

$$\sigma_N = -5 - 2\mu U \tfrac{y}{h^2}\sin(2\theta) \quad (5)$$

For $y = ½h$ and $\theta = 30°$:

$$\boxed{\sigma_N = -5 - \frac{\sqrt{3}}{2}\frac{\mu U}{h}} \quad (6)$$

The vector tangent to the plane is:

$$\hat{\mathbf{t}} = -\sin\theta \hat{\mathbf{e}}_x + \cos\theta \hat{\mathbf{e}}_y \quad \text{(Note: } \hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = 0\text{.)} \quad (7)$$

The stress tangent to the plane is:

$$\sigma_T = \mathbf{T}^{\hat{\mathbf{n}}} \cdot \hat{\mathbf{t}} = \sigma_{ji} n_j t_i \quad (8)$$

$$\sigma_T = \sigma_{11}n_1t_1 + \sigma_{21}n_2t_1 + \sigma_{31}n_3t_1 + \sigma_{12}n_1t_2 + \sigma_{22}n_2t_2 + \sigma_{32}n_3t_2 + \sigma_{13}n_1t_3 + \sigma_{23}n_2t_3 + \sigma_{33}n_3t_3 \quad (9)$$

$$\sigma_T = 5\cos\theta\sin\theta + 2\mu U \tfrac{y}{h^2}\sin^2\theta - 2\mu U \tfrac{y}{h^2}\cos^2\theta - 5\sin\theta\cos\theta \quad (10)$$

$$\sigma_T = -2\mu U \tfrac{y}{h^2}\left(\cos^2\theta - \sin^2\theta\right) = -2\mu U \tfrac{y}{h^2}\cos(2\theta) \quad (11)$$

For $y = ½h$ and $\theta = 30°$:

$$\boxed{\sigma_T = -\frac{1}{2}\frac{\mu U}{h}} \quad (12)$$

5.4. The Momentum Equations (aka the Linear Momentum Equations for a Differential Control Volume)

The Momentum Equations, which are the Linear Momentum Equations for a differential fluid element or control volume, can be derived several different ways. Three of these methods are given in this section.

Method 1: Apply the integral approach to the differential control volume shown in Figure 5.12. Assume that

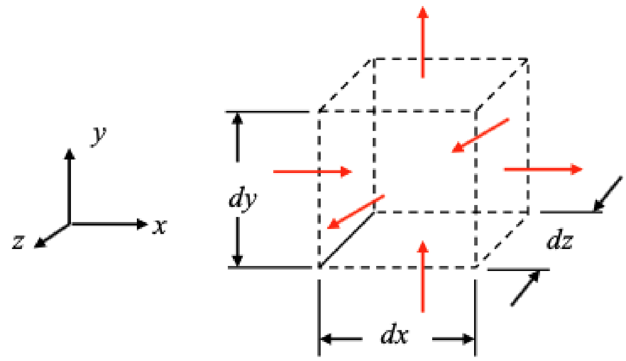


FIGURE 5.12. The differential control volume on which the Linear Momentum Equations are applied.

the density and velocity are ρ and $\boldsymbol{u}$, respectively, at the control volume's center. Consider the x-momentum equation first. The x-momentum flow rate through each of the side of the control volume is,

$$(\dot{m}_x u_x)_{\text{in through left}} = (\dot{m}_x u_x)_{\text{center}} + \frac{\partial (\dot{m}_x u_x)_{\text{center}}}{\partial x}\left(-\frac{1}{2}dx\right), \tag{5.94}$$

$$= (\rho u_x dydz u_x) + \frac{\partial}{\partial x}(\rho u_x dydz u_x)\left(-\frac{1}{2}dx\right), \tag{5.95}$$

$$= \left[\rho u_x u_x + \frac{\partial}{\partial x}(\rho u_x u_x)\left(-\frac{1}{2}dx\right)\right](dydz), \tag{5.96}$$

where $\dot{m}_{x,\text{center}}$ is the mass flow rate in the x-direction at the center of the control volume. Similarly,

$$(\dot{m}_x u_x)_{\text{out through right}} = \left[\rho u_x u_x + \frac{\partial}{\partial x}(\rho u_x u_x)\left(\frac{1}{2}dx\right)\right](dydz), \tag{5.97}$$

$$(\dot{m}_y u_x)_{\text{in through bottom}} = \left[\rho u_y u_x + \frac{\partial}{\partial y}(\rho u_y u_x)\left(-\frac{1}{2}dy\right)\right](dxdz), \tag{5.98}$$

$$(\dot{m}_y u_x)_{\text{out through top}} = \left[\rho u_y u_x + \frac{\partial}{\partial y}(\rho u_y u_x)\left(\frac{1}{2}dy\right)\right](dxdz), \tag{5.99}$$

$$(\dot{m}_z u_x)_{\text{in through back}} = \left[\rho u_z u_x + \frac{\partial}{\partial z}(\rho u_z u_x)\left(-\frac{1}{2}dz\right)\right](dxdy), \tag{5.100}$$

$$(\dot{m}_z u_x)_{\text{out through front}} = \left[\rho u_z u_x + \frac{\partial}{\partial z}(\rho u_z u_x)\left(\frac{1}{2}dz\right)\right](dxdy). \tag{5.101}$$

Thus, the net x-momentum flow rate out of the control volume is,

$$(\dot{m}u_x)_{\text{net, out of CV}} = \left[\frac{\partial}{\partial x}(\rho u_x u_x) + \frac{\partial}{\partial y}(\rho u_y u_x) + \frac{\partial}{\partial z}(\rho u_z u_x)\right](dxdydz). \tag{5.102}$$

The rate at which the x-momentum increases within the control volume is,

$$\frac{\partial}{\partial t}(mu_x)_{\text{within CV}} = \frac{\partial}{\partial t}(u_x \rho dxdydz) = \frac{\partial}{\partial t}(u_x \rho)(dxdydz), \tag{5.103}$$

where ρ and u_x are the density and x-component of the velocity, respectively, at the center of the control volume. Note that since these quantities vary linearly within the control volume (from the Taylor Series approximation), the averages within the control volume are simply ρ and u_x.

The forces acting on the control volume include both body and surface forces. The body force acting on the control volume in the x-direction, $F_{B,x}$, can be written as,

$$F_{B,x} = f_{B,x}\rho\left(dxdydz\right), \tag{5.104}$$

where $f_{B,x}$ is the body force per unit mass acting in the x-direction. For example, for weight, the body force per unit mass acting in the x-direction is simply g_x.

The surface forces acting on the control volume include both normal and tangential forces. Writing the surface force acting in the x-direction, $F_{S,x}$, in terms of stresses,

$$\begin{aligned} F_{S,x} = & \underbrace{-\left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\left(-\frac{1}{2}dx\right)\right](dydz)}_{\text{normal force on left face}} + \underbrace{\left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\left(\frac{1}{2}dx\right)\right](dydz)}_{\text{normal force on right face}} \\ & \underbrace{-\left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\left(-\frac{1}{2}dy\right)\right](dxdz)}_{\text{shear force on bottom face}} + \underbrace{\left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\left(\frac{1}{2}dy\right)\right](dxdz)}_{\text{shear force on top face}} \\ & \underbrace{-\left[\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\left(-\frac{1}{2}dz\right)\right](dxdy)}_{\text{shear force on back face}} + \underbrace{\left[\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\left(\frac{1}{2}dz\right)\right](dxdy)}_{\text{shear force on front face}}, \end{aligned} \tag{5.105}$$

$$\therefore F_{S,x} = \left[\frac{\sigma_{xx}}{\partial x} + \frac{\sigma_{yx}}{\partial y} + \frac{\sigma_{zx}}{\partial z}\right](dxdydz). \tag{5.106}$$

The Linear Momentum Equation in the x-direction states that the rate of increase of x linear momentum within the control volume plus the net rate at which x linear momentum leaves the control volume must equal the net force in the x direction acting on the control volume,

$$\frac{\partial}{\partial t}\left(mu_x\right)_{\text{within CV}} + \left(\dot{m}u_x\right)_{\text{net, out of CV}} = F_{B,x} + F_{S,x}. \tag{5.107}$$

Substituting Eqs. (5.102), (5.103), (5.104), and (5.106) into Eq. (5.107) gives,

$$\begin{aligned} \frac{\partial}{\partial t}\left(u_x\rho\right)(dxdydz) + \left[\frac{\partial}{\partial x}\left(\rho u_x u_x\right) + \frac{\partial}{\partial y}\left(\rho u_y u_x\right) + \frac{\partial}{\partial z}\left(\rho u_z u_x\right)\right](dxdydz) = \\ f_{B,x}\rho\left(dxdydz\right) + \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right](dxdydz), \end{aligned} \tag{5.108}$$

$$\frac{\partial}{\partial t}\left(u_x\rho\right) + \frac{\partial}{\partial x}\left(\rho u_x u_x\right) + \frac{\partial}{\partial y}\left(\rho u_y u_x\right) + \frac{\partial}{\partial z}\left(\rho u_z u_x\right) = \rho f_{B,x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}. \tag{5.109}$$

A similar approach can be taken to determine the y and z-components of the Momentum Equations. All three components of the Momentum Equations can be written in the following compact (index notation) form,

$$\boxed{\frac{\partial}{\partial t}\left(u_i\rho\right) + \frac{\partial}{\partial x_j}\left(\rho u_j u_i\right) = \rho f_{B,i} + \frac{\partial \sigma_{ji}}{\partial x_j}}. \tag{5.110}$$

In vector notation, the Momentum Equations can be written as,

$$\boxed{\frac{\partial}{\partial t}\left(\boldsymbol{u}\rho\right) + \left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\left(\rho\boldsymbol{u}\right) = \rho\boldsymbol{f}_B + \boldsymbol{\nabla}\cdot\underline{\underline{\boldsymbol{\sigma}}}^T}. \tag{5.111}$$

Note that $\underline{\underline{\boldsymbol{\sigma}}}^T = \underline{\underline{\boldsymbol{\sigma}}}$ since the stress tensor is symmetric.

Expanding the left-hand side of Eq. (5.110) and utilizing the Continuity Equation,

$$\begin{aligned}\frac{\partial}{\partial t}(u_i\rho)+\frac{\partial}{\partial x_j}(\rho u_j u_i) &= u_i\frac{\partial \rho}{\partial t}+\rho\frac{\partial u_i}{\partial t}+u_i\frac{\partial}{\partial x_j}(\rho u_j)+\rho u_j\frac{\partial u_i}{\partial x_j}\\ &= u_i\underbrace{\left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)\right]}_{=0\,\text{(Continuity Eq.)}}+\rho\underbrace{\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)}_{=\frac{Du_i}{Dt}}.\end{aligned} \tag{5.112}$$

Substituting back into Eq. (5.110),

$$\boxed{\rho\frac{Du_i}{Dt}=\rho f_{B,i}+\frac{\partial \sigma_{ji}}{\partial x_j}}. \tag{5.113}$$

Method 2: Apply Newton's Second Law directly to a small piece of fluid,

$$\frac{D}{Dt}(u_i\rho dxdydz)=f_{B,i}\rho\,(dxdydz)+\frac{\partial \sigma_{ji}}{\partial x_j}(dxdydz), \tag{5.114}$$

where the determination of the body and surface forces are described in Method 1. Expanding the Lagrangian derivative gives,

$$\frac{D}{Dt}(u_i\rho dxdydz)=\frac{Du_i}{Dt}(\rho dxdydz)+u_i\frac{D}{Dt}(\rho dxdydz), \tag{5.115}$$

but the second term on the right-hand side of this equation is zero since the mass of the fluid element remains constant. Thus, Eq. (5.114) can be simplified to,

$$\boxed{\rho\frac{Du_i}{Dt}=\rho f_{B,i}+\frac{\partial \sigma_{ji}}{\partial x_j}}, \tag{5.116}$$

which is the same result found using Method 1.

Method 3: Recall that the integral form of the Linear Momentum Equations is,

$$\frac{d}{dt}\int_{CV}u_i\rho dV+\int_{CS}u_i\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=F_{B,i}+F_{S,i}. \tag{5.117}$$

Consider a fixed control volume so that,

$$\frac{d}{dt}\int_{CV}u_i\rho dV=\int_{CV}\frac{\partial (u_i\rho)}{\partial t}dV \quad \text{and} \quad \boldsymbol{u}_{\text{rel}}=\boldsymbol{u}. \tag{5.118}$$

Note that the body force can be written as,

$$F_{B,i}=\int_{CV}f_{B,i}\rho dV, \tag{5.119}$$

and the surface forces can be written as,

$$F_{S,i}=\int_{CS}\sigma_{ji}n_j dA. \tag{5.120}$$

Utilizing Gauss' Theorem (aka the Divergence Theorem), we can convert the area integrals into volume integrals,

$$\int_{CS}u_i\left(\rho\boldsymbol{u}\cdot d\boldsymbol{A}\right)=\int_{CV}\boldsymbol{\nabla}\cdot(u_i\rho\boldsymbol{u})\,dV=\int_{CV}\frac{\partial}{\partial x_j}(\rho u_j u_i)\,dV, \tag{5.121}$$

$$\int_{CS}\sigma_{ji}n_j dA=\int_{CV}\frac{\partial \sigma_{ji}}{\partial x_j}dV. \tag{5.122}$$

Substituting these expressions back into the Linear Momentum Equations,

$$\int_{CV}\left[\frac{\partial}{\partial t}(u_i\rho)+\frac{\partial}{\partial x_j}(\rho u_j u_i)-\rho f_{B,i}-\frac{\partial \sigma_{ji}}{\partial x_j}\right]dV=0. \tag{5.123}$$

Since the choice of control volume is arbitrary, the kernel of the integral must be zero, i.e.,

$$\frac{\partial}{\partial t}(u_i\rho) + \frac{\partial}{\partial x_j}(\rho u_j u_i) - \rho f_{B,i} - \frac{\partial \sigma_{ji}}{\partial x_j} = 0. \tag{5.124}$$

This is the same expression as Eq. (5.110) so we see that the final result will be the same,

$$\boxed{\rho\frac{Du_i}{Dt} = \rho f_{B,i} + \frac{\partial \sigma_{ji}}{\partial x_j}}. \tag{5.125}$$

Notes:

(1) In order to be more useful to us, we need to have some way of relating the stresses acting on the fluid element (or control volume) to other properties of the flow, namely the velocities. This connection is accomplished using a constitutive law, which in this case relates the stresses to the strain rates for a particular fluid or class of fluids.
(2) Equation (5.113) is valid for any continuous substance.
(3) Equation (5.110) is the conservative form (i.e., Eulerian form) of the Linear Momentum Equations. Equation (5.113) is the non-conservative form (i.e., Lagrangian form).

Consider the flow of a mixture of liquid water and small water vapor bubbles. The bubble diameters are very small in comparison to the length scales of interest in the flow so that the properties of the mixture can be considered point functions. For example, the density of the mixture at a "point" can be written as:

$$\rho_{\mathrm{M}} = \alpha\rho_{\mathrm{V}} + (1-\alpha)\rho_{\mathrm{L}}$$

where ρ_M is the mixture density, ρ_L is the liquid density, ρ_V is the vapor density, and α is the "void fraction" or the fraction of volume that is vapor in a unit volume of the mixture. Assume that evaporation occurs at the bubble surface so that the liquid water turns to water vapor at a mass flow rate per unit volume denoted by *s*.

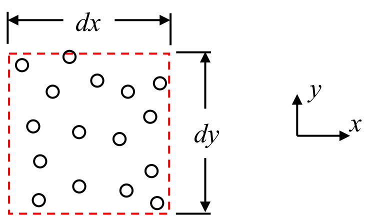
a. What is the continuity equation for the *mixture*?
b. What is the continuity equation for the *liquid water phase*?
c. What are the momentum equations for the *liquid water phase*?

SOLUTION:

The continuity equation for the mixture will be the "normal" continuity equation:

$$\boxed{\frac{\partial \rho_M}{\partial t} + \frac{\partial}{\partial x_i}(\rho_M u_i) = 0} \tag{1}$$

To show that this relation is true, consider the control volume shown below.



The rate of change of mass within the control volume is:

$$\frac{\partial}{\partial t}(\rho_M dxdydz) = \frac{\partial \rho_M}{\partial t} dxdydz \tag{2}$$

The net mass flux <u>into the CV</u> in the *x*-direction is:

$$\dot{m}_{\substack{x,\text{ net}\\ \text{into CV}}} = \rho_M u_x dydz - \left[\rho_M u_x + \frac{\partial}{\partial x}(\rho_M u_x) dx\right] dydz = -\frac{\partial}{\partial x}(\rho_M u_x) dxdydz \tag{3}$$

Following a similar approach in the *y* and *z* directions gives:

$$\dot{m}_{\substack{y,\text{net}\\ \text{into CV}}} = -\frac{\partial}{\partial y}(\rho_M u_y) dxdydz \tag{4}$$

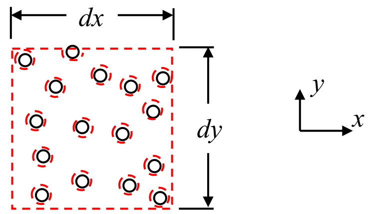
$$\dot{m}_{\substack{z,\text{net}\\ \text{into CV}}} = -\frac{\partial}{\partial z}(\rho_M u_z) dxdydz \tag{5}$$

Thus, from conservation of mass:

$$\frac{\partial \rho_M}{\partial t} dxdydz = -\frac{\partial}{\partial x}(\rho_M u_x) dxdydz - \frac{\partial}{\partial y}(\rho_M u_y) dxdydz - \frac{\partial}{\partial z}(\rho_M u_z) dxdydz \tag{6}$$

$$\frac{\partial \rho_M}{\partial t} + \frac{\partial}{\partial x_i}(\rho_M u_i) = 0 \tag{7}$$

To determine the continuity equation for the liquid water phase, consider the control volume drawn below where the CV surrounds each vapor bubble.



The rate of change of liquid mass within the control volume is:

$$\frac{\partial}{\partial t}\left[\rho_L(1-\alpha)dxdydz\right]=\frac{\partial}{\partial t}\left[(1-\alpha)\rho_L\right]dxdydz \tag{8}$$

The net liquid mass flux into the CV in the x-direction is:

$$\dot{m}_{\substack{x,\text{ net}\\ \text{into CV}}}=(1-\alpha)\rho_L u_x dydz-\left\{(1-\alpha)\rho_L u_x+\frac{\partial}{\partial x}\left[(1-\alpha)\rho_L u_x\right]dx\right\}dydz=-\frac{\partial}{\partial x}\left[(1-\alpha)\rho_L u_x\right]dxdydz \tag{9}$$

Following a similar approach in the y and z directions gives:

$$\dot{m}_{\substack{y,\text{net}\\ \text{into CV}}}=-\frac{\partial}{\partial y}\left[(1-\alpha)\rho_L u_y\right]dxdydz \tag{10}$$

$$\dot{m}_{\substack{z,\text{net}\\ \text{into CV}}}=-\frac{\partial}{\partial z}\left[(1-\alpha)\rho_L u_z\right]dxdydz \tag{11}$$

The rate at which liquid mass is being converted to vapor mass is:

$$\dot{m}_{\substack{\text{out of CV}\\ \text{due to evap.}}}=s(1-\alpha)dxdydz \tag{12}$$

Thus, from conservation of mass:

$$\begin{aligned}&\frac{\partial}{\partial t}(1-\alpha)\rho_L dxdydz=\\&-\frac{\partial}{\partial x}\left[(1-\alpha)\rho_L u_x\right]dxdydz-\frac{\partial}{\partial y}\left[(1-\alpha)\rho_L u_y\right]dxdydz-\frac{\partial}{\partial z}\left[(1-\alpha)\rho_L u_z\right]dxdydz-s(1-\alpha)dxdydz\end{aligned} \tag{13}$$

$$\boxed{\frac{\partial}{\partial t}\left[(1-\alpha)\rho_L\right]+\frac{\partial}{\partial x_i}\left[(1-\alpha)\rho_L u_i\right]=-(1-\alpha)s}\quad\text{(continuity eqn. for liquid phase)} \tag{14}$$

To determine the momentum equations for the liquid phase, apply the momentum equation to the same control volume used to derive the liquid phase continuity equation. The change in momentum of liquid within the CV is:

$$\frac{d}{dt}\int_{\text{CV}}\mathbf{u}\rho dV=\frac{\partial}{\partial t}\left[u_i\rho_L(1-\alpha)dxdydz\right]=\frac{\partial}{\partial t}\left[u_i\rho_L(1-\alpha)\right]dxdydz \tag{15}$$

The net flux of linear momentum out of the CV through the sides of the CV is:

$$\begin{aligned}\int_{\text{CS}}\mathbf{u}\rho(\mathbf{u}_{\text{rel}}\cdot d\mathbf{A})&=\frac{\partial}{\partial x}\left[u_i(1-\alpha)\rho_L u_x\right]dxdydz+\frac{\partial}{\partial y}\left[u_i(1-\alpha)\rho_L u_y\right]dxdydz+\frac{\partial}{\partial z}\left[u_i(1-\alpha)\rho_L u_z\right]dxdydz+u_i s(1-\alpha)dxdydz\\&=\frac{\partial}{\partial x_j}\left[u_i(1-\alpha)\rho_L u_j\right]dxdydz+u_i s(1-\alpha)dxdydz\end{aligned} \tag{16}$$

(Note that the term involving s is the rate at which momentum leaves the liquid phase due to the fact that the liquid is evaporating.)

The surface forces acting on the control surface are:

$$\mathbf{F}_S + \mathbf{F}_B = -\frac{\partial \sigma_{ji}}{\partial x_j} dxdydz + f_{VonL,i}\rho_L\left(1-\alpha\right)dxdydz + g_i\rho_L\left(1-\alpha\right)dxdydz \tag{17}$$

Note that the stress terms are the surfaces forces acting on the sides of the CV. The term $f_{VonL,i}$ is the force per unit mass that the vapor phase exerts on the liquid phase, and the last term in Eqn. (17) is the body force acting on the liquid phase where g_i is the body force per unit mass.

Substituting into the linear momentum equation and simplifying results in:

$$\boxed{\frac{\partial}{\partial t}\left[u_i\rho_L\left(1-\alpha\right)\right] + \frac{\partial}{\partial x_j}\left[u_i\left(1-\alpha\right)\rho_L u_j\right] = -\frac{\partial \sigma_{ji}}{\partial x_j} + f_{VonL,i}\rho_L\left(1-\alpha\right) - u_i s\left(1-\alpha\right) + g_i\rho_L\left(1-\alpha\right)} \tag{18}$$

The continuity equation derived previously for the liquid phase (Eqn. (14)) could be used to further simplify the momentum equation, if desired.

5.5. Fluid Element Deformations

In order to relate the stresses acting on a fluid element to the other variables in the Momentum Equations, we need to determine a constitutive law. The constitutive law should relate the stresses to the rates of strain (or deformation rates) of a fluid element. For a solid, the necessary constitutive law relates the stresses to the strains (or deformations). In order to derive an appropriate constitutive law, we must first discuss the general types of deformations that can occur for a fluid element and then describe, in mathematical terms, the rates at which these deformations occur.

Any general deformation can be decomposed into a combination of translation, dilation (aka dilatation), rigid body rotation, and angular deformation as shown in Figure 5.13.

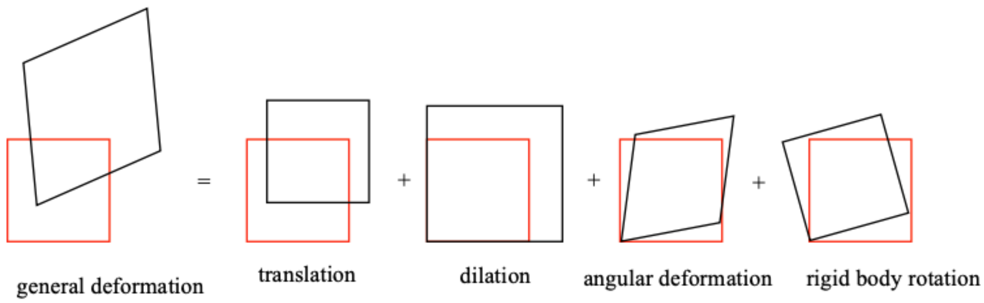


Figure 5.13. Any general deformation can be divided into four fundamental deformations: translation, dilation, angular deformation, and rigid body rotation.

Now let's describe the rate at which each of these deformations occurs.

Translation: The rate of translation is described by the time rate of change of the position of the element (Figure 5.14), i.e., the velocity,

$$\text{rate of translation} = \frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}. \tag{5.126}$$

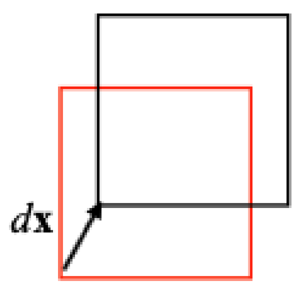


Figure 5.14. Translation of a fluid element.

Dilation (aka Dilatation): The rate of dilation can be described by the rate at which the relative volume of the element increases with time,

$$\text{volumetric dilation rate}, \theta := \frac{1}{V}\frac{dV}{dt}. \tag{5.127}$$

The velocity of point A relative to point O in the x_1 direction is (Figure 5.15),

$$u_1 = \frac{\partial u_1}{\partial x_1} dx_1. \tag{5.128}$$

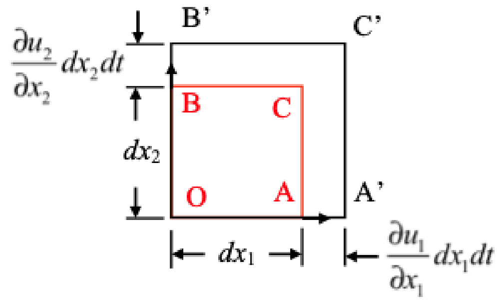


FIGURE 5.15. Dilation of a fluid element.

Thus, point A "stretches" the element in the x_1 direction over time dt a distance of,

$$\frac{\partial u_1}{\partial x_1} dx_1 dt. \tag{5.129}$$

The increase in volume of the element due to the relative movement of point A is,

$$dV_A = \left(\frac{\partial u_1}{\partial x_1} dx_1 dt \right) dx_2 dx_3. \tag{5.130}$$

A similar approach can be followed for stretching in the x_2 and x_3 directions. The total increase in the element volume is,

$$dV = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) dx_1 dx_2 dx_3 dt. \tag{5.131}$$

Note that higher order volume terms have been neglected in deriving the previous result (Figure 5.16).

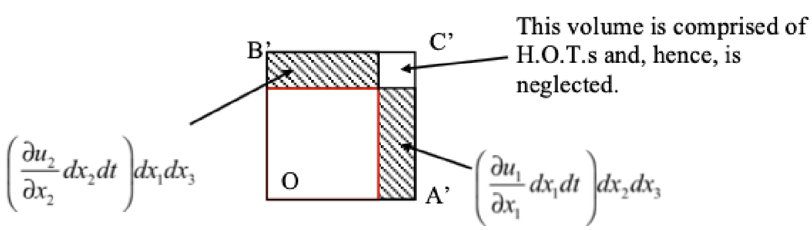


FIGURE 5.16. Dilation of a fluid element showing the regions of increased volume, including the part that is neglected since it's a higher order term volume (H.O.T.).

The volumetric dilation rate is, thus,

$$\theta := \frac{1}{V} \frac{dV}{dt} = \frac{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) dx_1 dx_2 dx_3 dt}{dx_1 dx_2 dx_3 dt}, \tag{5.132}$$

$$\theta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i} = \boldsymbol{\nabla} \cdot \boldsymbol{u}. \tag{5.133}$$

Notes:

(1) For an incompressible fluid, the volumetric dilation rate is zero since, if the volume of the element changes, the density must also change (from Conservation of Mass).

Angular Deformation: The rate of angular deformation in the 1-2 plane (Figure 5.17) can be described as

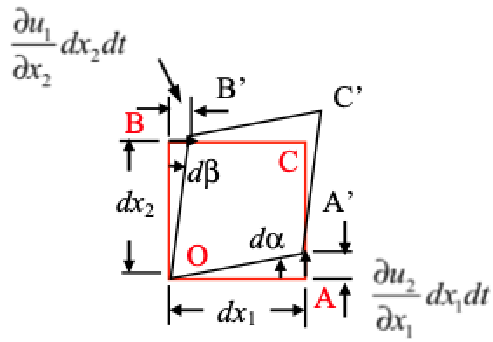


Figure 5.17. Angular deformation of a fluid element.

the average rate at which the sides of the element approach one another, i.e., the average rate at which the angles AOA' and BOB' increase. The angle AOA' ($d\alpha$) is,

$$\tan(d\alpha) = \frac{\frac{\partial u_2}{\partial x_1} dx_1 dt}{dx_1}. \tag{5.134}$$

Since the angle $d\alpha$ is very small, $\tan(d\alpha) \approx d\alpha$ and,

$$d\alpha = \frac{\partial u_2}{\partial x_1} dt. \tag{5.135}$$

Similarly, the angle BOB' ($d\beta$) is,

$$d\beta = \frac{\partial u_1}{\partial x_2} dt. \tag{5.136}$$

Define the rate of angular deformation, S_{12} (aka rate of shearing strain), in the 1-2 plane as the average time rate of change of these two angles,

$$S_{12} \coloneqq \frac{1}{2}\left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right) = \frac{1}{2}\left(\frac{du_2}{dx_1} + \frac{du_1}{dx_2}\right). \tag{5.137}$$

Similarly, we can determine the angular deformation rate in the 1-3 and 2-3 planes,

$$S_{13} = \frac{1}{2}\left(\frac{du_3}{dx_1} + \frac{du_1}{dx_3}\right) \quad \text{and} \quad S_{23} = \frac{1}{2}\left(\frac{du_3}{dx_2} + \frac{du_2}{dx_3}\right). \tag{5.138}$$

Combine the angular deformation rate and the dilation rate into one tensor quantity called the shearing strain tensor, S_{ij},

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{5.139}$$

Notes:

(1) Dilation rate is the trace of S_{ij},

$$\theta = \text{trace}(S_{ij}) = S_{ii} = S_{11} + S_{22} + S_{33} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \tag{5.140}$$

(2) The shearing strain tensor is symmetric, i.e., $S_{ij} = S_{ji}$.

Rigid Body Rotation: The rate at which the fluid element, shown in Figure 5.18, rotates about the 3-axis in rigid body motion can be described as the average rate at which the sides of the element rotate in the same direction. The rotation rate about the 3-axis, Ω_3, is,

$$\Omega_3 \coloneqq \frac{1}{2}\left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right) = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right). \tag{5.141}$$

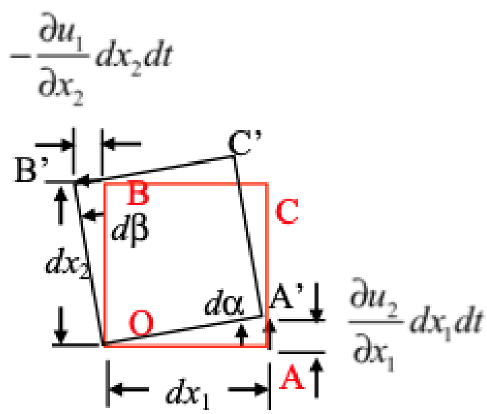


Figure 5.18. A fluid element undergoing rigid body rotation.

Note that the angle $d\beta$ in Figure 5.18 is different than the $d\beta$ in Figure 5.17. Rotations about the 1 and 2 axes can be found in a similar manner,

$$\Omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \quad \text{and} \quad \Omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right). \tag{5.142}$$

The rate of rotation of the element can be summarized using the rotation rate vector, Ω,

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\hat{\boldsymbol{e}}_1 + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\hat{\boldsymbol{e}}_2 + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\hat{\boldsymbol{e}}_3, \tag{5.143}$$

$$= \frac{1}{2}\left(\boldsymbol{\nabla} \times \boldsymbol{u}\right). \tag{5.144}$$

Notes:

(1) The rotation rate vector, $\boldsymbol{\Omega}$, is written in index notation form as,

$$\Omega_i = \frac{1}{2}\epsilon_{ijk}\frac{\partial u_k}{\partial x_j}. \tag{5.145}$$

(2) The rotation rate vector can also be written as an anti-symmetric rotation rate tensor, R_{ij},

$$R_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = -\epsilon_{ijk}\Omega_k \quad (\text{Note: } R_{ij} = -R_{ji}), \tag{5.146}$$

where R_{ij} is the rotation rate in the $i-j$ plane. Note that the diagonal elements of the tensor R_{ij} are zero.

(3) The vorticity, $\boldsymbol{\omega}$, of a fluid element (a vector quantity) is defined to be twice the rotation rate of the element,

$$\boldsymbol{\omega} := 2\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \boldsymbol{u}, \tag{5.147}$$

or in index notation,

$$\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}. \tag{5.148}$$

An irrotational flow is one in which $\boldsymbol{\omega} = \boldsymbol{0}$. A rotational flow is one in which $\boldsymbol{\omega} \neq \boldsymbol{0}$.

Now that we have described the deformation rate components (e.g., dilation, angular deformation, and rigid body rotation; translations are treated separately) of a fluid element, let's combine these into a single tensor quantity known as the deformation rate tensor, e_{ij},

$$e_{ij} := \frac{\partial u_i}{\partial x_j} = S_{ij} + R_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{5.149}$$

We will use the deformation rate tensor when deriving the constitutive relations between stress and strain (deformation) rates in a fluid.

A fluid has a velocity field given by

$$\mathbf{u} = 2x\hat{\mathbf{i}} - 3y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

At the location (x, y, z) = (-2, -1, 2), calculate:

a. the normal and shearing strain rates at the location, and
b. the rotational velocity of the fluid.

SOLUTION:

The strain rate tensor is given by,

$$S_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

so that the normal strain rates are,

$$\left(S_{xx}, S_{yy}, S_{zz}\right) = \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_y}{\partial y}, \frac{\partial u_z}{\partial z}\right)$$

$$\boxed{\therefore \left(S_{xx}, S_{yy}, S_{zz}\right) = \left(2, -3, 1\right)}$$

and the shearing strain rates are,

$$\left(S_{xy} = S_{yx}, S_{xz} = S_{zx}, S_{yz} = S_{zy}\right) = \left[\tfrac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \tfrac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \tfrac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)\right]$$

$$\boxed{\therefore \left(S_{xy} = S_{yx}, S_{xz} = S_{zx}, S_{yz} = S_{zy}\right) = \left(0, 0, 0\right)}$$

The rotational velocity of a fluid element is given by,

$$\Omega_i = \tfrac{1}{2}\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$ (Note that the vorticity is twice the rotation rate, i.e., $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$.)

and, thus, for the given case,

$$\left(\Omega_x, \Omega_y, \Omega_z\right) = \frac{1}{2}\left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right), \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\right]$$

$$\boxed{\therefore \left(\Omega_x, \Omega_y, \Omega_z\right) = \left(0, 0, 0\right)}$$ (Fluid elements are not rotating anywhere! The flow is irrotational.)

5.6. Stress-Strain Rate Relations for a Newtonian Fluid

The following assumptions are based on observation and intuition. The key assumptions in deriving the stress-strain rate constitutive relations for a Newtonian fluid are:

(1) When the fluid is at rest, the pressure exerted on the fluid is the thermodynamic pressure, p.
(2) For a Newtonian fluid, the stress tensor, σ_{ij}, is linearly related to the deformation rate tensor, e_{kl}, and depends only on that tensor.
(3) There are no preferred directions in the fluid so that the fluid properties are point functions. This is the condition of isotropy.

Now let's examine how these assumptions aid us in deriving the appropriate constitutive law.

Assumption 1: When the fluid is at rest, the pressure exerted by the fluid is the thermodynamic pressure, p. This assumption implies the following,

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}, \tag{5.150}$$

where τ_{ij} is referred to as the viscous stress tensor (aka deviatoric stress tensor) and it is only a function of the fluid motion, i.e., $\tau_{ij} = 0$ for a static fluid. Note that the pressure term is negative since compression of the fluid element is indicated by a negative normal stress.

Assumption 2: For a Newtonian fluid, the stress tensor, σ_{ij}, is linearly related to the deformation rate tensor, e_{kl}, and depends only on that tensor. The nine elements of τ_{ij} can be written as a linear combination of the nine elements of e_{kl},

$$\tau_{ij} = A_{ijkl}e_{kl}, \tag{5.151}$$

where A_{ijkl} is a tensor of rank 4 (81 elements) that depends only on the local state of the fluid.

Notes:

(1) Recall that the deformation rate tensor is given as,

$$e_{kl} \coloneqq \frac{\partial u_k}{\partial x_l} = S_{kl} + R_{kl} = \underbrace{\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)}_{\text{symmetric, } S_{kl}=S_{lk}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} - \frac{\partial u_l}{\partial x_k}\right)}_{\text{anti-symmetric, } R_{kl}=-R_{lk}}. \tag{5.152}$$

(2) Air and water are common examples of Newtonian fluids.

Since the stress tensor, σ_{ij}, is symmetric, τ_{ij} must also be symmetric. In addition, since τ_{ij} is symmetric, the components of the A_{ijkl} tensor multiplied by the anti-symmetric part of the deformation rate tensor, e_{kl}, must be zero. Thus,

$$\tau_{ij} = \frac{1}{2}B_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \tag{5.153}$$

where B_{ijkl} is the A_{ijkl} tensor with the A_{ijkl} components multiplied by the components of the anti-symmetric part of e_{kl} set equal to zero.

Assumption 3: There are no preferred directions in the fluid so the fluid properties are point functions. This condition is known as the condition of isotropy. The condition of isotropy means that the fluid properties are the same in all directions. Examples of non-isotropic materials include fluids comprised of long chain molecules or oriented fibrous solids such as wood. It can be shown (out of the scope of these notes) that the most general fourth-order isotropic tensor can be written as,

$$B_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \gamma\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right), \tag{5.154}$$

where λ, μ, and γ are scalar properties.

Substitute Eq. (5.154) into Eq. (5.153) and simplify,

$$\tau_{ij} = \frac{1}{2}\left[\lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \gamma\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)\right]\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \tag{5.155}$$

where,

$$\frac{1}{2}\lambda\delta_{ij}\delta_{kl}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)=\frac{1}{2}\lambda\delta_{ij}\left(2\frac{\partial u_k}{\partial x_k}\right)=\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}, \tag{5.156}$$

$$\frac{1}{2}\mu\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)=\frac{1}{2}\mu\left[\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\right]=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{5.157}$$

$$\frac{1}{2}\gamma\left(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right)\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)=\frac{1}{2}\gamma\left[\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\right]=0, \tag{5.158}$$

so that,

$$\tau_{ij}=\lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{5.159}$$

Substituting Eq. (5.159) into Eq. (5.150) gives,

$$\sigma_{ij}=-p\delta_{ij}+\tau_{ij}, \tag{5.160}$$

$$=-p\delta_{ij}+\lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{5.161}$$

$$\boxed{\therefore\sigma_{ij}=-\left(p+\lambda\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}. \tag{5.162}$$

The boxed equation is the stress-strain rate constitutive relation for a Newtonian fluid.

Notes:

(1) The quantity, μ, is referred to as the dynamic viscosity.
(2) The quantity, λ, is referred to as the second coefficient of viscosity.
(3) How is the thermodynamic pressure related to the normal stresses? Define the mechanical pressure, $\bar{p}$, as the average of the normal stresses,

$$\bar{p}\coloneqq-\frac{1}{3}\text{trace}(\sigma_{ij})=-\frac{1}{3}\sigma_{ii}=-\frac{1}{3}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right). \tag{5.163}$$

For a Newtonian fluid,

$$\bar{p}=-\frac{1}{3}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right), \tag{5.164}$$

$$=-\frac{1}{3}\left[\left(-p+\lambda\frac{\partial u_k}{\partial x_k}+2\mu\frac{\partial u_1}{\partial x_1}\right)+\left(-p+\lambda\frac{\partial u_k}{\partial x_k}+2\mu\frac{\partial u_2}{\partial x_2}\right)+\left(-p+\lambda\frac{\partial u_k}{\partial x_k}+2\mu\frac{\partial u_3}{\partial x_3}\right)\right], \tag{5.165}$$

$$=p-\lambda\frac{\partial u_k}{\partial x_k}-\frac{2}{3}\mu\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right), \tag{5.166}$$

$$\therefore\bar{p}=p-\underbrace{\left(\lambda+\frac{2}{3}\mu\right)}_{=K}\frac{\partial u_k}{\partial x_k}\quad\text{or}\quad p=\bar{p}+\underbrace{\left(\lambda+\frac{2}{3}\mu\right)}_{=K}\frac{\partial u_k}{\partial x_k}, \tag{5.167}$$

where K, bulk viscosity $\coloneqq\lambda+\frac{2}{3}\mu$. In general, the thermodynamic pressure is not the same as the mechanical pressure. What then is the physical significance of the bulk viscosity, K, term? The mechanical pressure is a measure of the translational energy only. The thermodynamic pressure, however, is a measure of the total energy (translational, rotational, vibrational, etc.) The bulk viscosity, K, is a measure of the transfer of energy from the translational mode to the other modes. For example, when fluid flows through a shock wave, there is a considerable transfer of energy between the translational mode and the other modes; hence, the bulk viscosity cannot be neglected for such a flow process.

For typical flows, however, the bulk viscosity is often neglected. For example,

(1) For monatomic gases the only energy mode is the translational mode so that,

$$K=0\implies p=\bar{p}. \tag{5.168}$$

For polyatomic gases and liquids, the bulk viscosity is often small so we usually assume,

$$K \approx 0 \implies p \approx \bar{p}. \tag{5.169}$$

The assumption that the bulk viscosity is zero (or equivalently, $\lambda = -\frac{2}{3}\mu$) is known as Stokes' Relation.

(2) For an incompressible fluid the velocity divergence term is zero (from the Continuity Equation) so the bulk viscosity is irrelevant,

$$\frac{\partial u_k}{\partial x_k} = 0 \implies p = \bar{p}. \tag{5.170}$$

(3) The bulk viscosity term is generally not negligible when there is a rapid expansion or contraction of the fluid such as when fluid passes through a shock wave or when considering acoustic absorption.

(4) The stress tensor given in Eq. (5.162) can be substituted into the Momentum Equations to give the Navier-Stokes Equations,

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i, \tag{5.171}$$

$$\boxed{\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \rho f_i}. \tag{5.172}$$

For an incompressible fluid with constant dynamic viscosity,

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_i}\underbrace{\frac{\partial u_k}{\partial x_k}}_{=0}\right) + \rho f_i, \tag{5.173}$$

$$\boxed{\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_i \partial x_j} + \rho f_i} \quad \text{or} \quad \rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{f}. \tag{5.174}$$

For an inviscid fluid ($\mu = 0$) that follows Stokes' Relation or is incompressible,

$$\boxed{\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho f_i} \quad \text{or} \quad \rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \rho \boldsymbol{f}. \tag{5.175}$$

These relations are known as Euler's Equations. The Navier-Stokes Equations are the Momentum Equations for a Newtonian fluid. Euler's Equations are the Momentum Equations for an inviscid fluid.

Prove that a Newtonian fluid with constant viscosity in incompressible flow obeys the relation:

$$\nabla \cdot \underline{\underline{\boldsymbol{\tau}}} = \mu \nabla^2 \mathbf{u}$$

where $\boldsymbol{\tau}$ is the viscous part of the stress tensor.

SOLUTION:

The viscous part of the stress tensor for an incompressible Newtonian fluid with constant viscosity is:

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{1}$$

Thus:

$$\begin{aligned}
\left(\nabla \cdot \underline{\underline{\boldsymbol{\tau}}} \right)_j &= \frac{\partial \tau_{ij}}{\partial x_i} = \mu \frac{\partial}{\partial x_i} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\
&= \mu \left(\frac{\partial^2 u_i}{\partial x_i \partial x_j} + \frac{\partial^2 u_j}{\partial x_i \partial x_i} \right) \\
&= \mu \frac{\partial}{\partial x_j} \underbrace{\left(\frac{\partial u_i}{\partial x_i} \right)}_{=0 \text{ (continuity)}} + \mu \frac{\partial^2 u_j}{\partial x_i \partial x_i} \\
&= \mu \frac{\partial^2 u_j}{\partial x_i \partial x_i} = \mu \left(\nabla^2 \mathbf{u} \right)_j
\end{aligned}$$

$$\boxed{\therefore \nabla \cdot \underline{\underline{\boldsymbol{\tau}}} = \mu \nabla^2 \mathbf{u}} \tag{2}$$

Consider a 3D steady flow of an incompressible, Newtonian liquid with a velocity field given by:

$$\mathbf{u} = ax\hat{\mathbf{i}} + ay\hat{\mathbf{j}} - 2az\hat{\mathbf{k}}$$

There are no body forces acting on the flow and the pressure at the origin is p_0.

a. Show that the continuity equation is satisfied,
b. Determine the pressure field.
c. Determine the vorticity field.

SOLUTION:

The Continuity Equation is:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = a + a - 2a = 0 \quad \text{Continuity is satisfied!}$$

The pressure field may be found using the Navier-Stokes equations. Note that the body forces are zero.

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{\partial p}{\partial x} = -\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right) + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) \Rightarrow \frac{\partial p}{\partial x} = -\rho a^2 x$$

$$\Rightarrow p(x,y,z) = -\tfrac{1}{2}\rho a^2 x^2 + f(y,z)$$

$$\frac{\partial p}{\partial y} = -\rho\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \right) + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \right) \Rightarrow \frac{\partial p}{\partial y} = -\rho a^2 y$$

$$\Rightarrow p(x,y,z) = -\tfrac{1}{2}\rho a^2 y + g(x,z)$$

$$\frac{\partial p}{\partial z} = -\rho\left(u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \right) + \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) \Rightarrow \frac{\partial p}{\partial z} = -4\rho a^2 z$$

$$\Rightarrow p(x,y,z) = -2\rho a^2 z^2 + h(x,y)$$

Combining the previous expressions and noting that $p(0, 0, 0) = p_0$:

$$\boxed{p(x,y,z) = p_0 - \tfrac{1}{2}\rho a^2 x^2 - \tfrac{1}{2}\rho a^2 y - 2\rho a^2 z^2}$$

The vorticity field is:

$$\omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

$$\omega_x = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} = 0$$

$$\omega_y = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} = 0$$

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 0$$

$$\boxed{\boldsymbol{\omega}(x,y,z) = \mathbf{0}} \quad \text{The flow is irrotational!}$$

Note that the viscous force terms in the Navier-Stokes equation are zero ($\mu u_{i,jj} = 0$).

5.7. Acceleration of a Fluid Particle in Streamline Coordinates

Often it's helpful to use streamline coordinates (s, n) instead of Cartesian coordinates (x, y) when describing the motion of a fluid particle. Let's determine a fluid particle's acceleration parallel (s-direction) and normal (n-direction) to a streamline for a steady, 2D flow. Consider Figure 5.19.

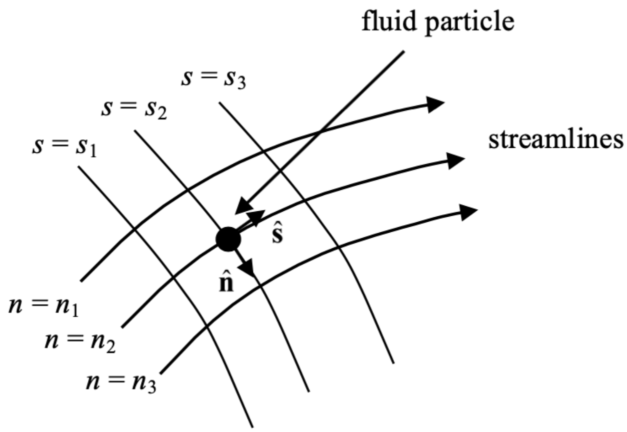


Figure 5.19. An illustration of a streamline coordinate system.

Notes:

(1) The coordinates (s, n) are just like (x, y) coordinates. They specify the location of the fluid particle.
(2) Lines of constant s and n are perpendicular.
(3) The unit vector $\hat{\boldsymbol{s}}$ points in the direction tangent to the streamline.
(4) The unit vector $\hat{\mathbf{n}}$ points toward the center of curvature of the streamline.

The acceleration of the fluid particle is,

$$\boldsymbol{a} = \frac{D\boldsymbol{u}}{Dt}, \tag{5.176}$$

where $\boldsymbol{u} = u\hat{\boldsymbol{s}}$. Substituting and expanding gives,

$$\boldsymbol{a} = \frac{D(u\hat{\boldsymbol{s}})}{Dt} = \hat{\boldsymbol{s}}\frac{Du}{Dt} + u\frac{D\hat{\boldsymbol{s}}}{Dt}. \tag{5.177}$$

Now expand the Lagrangian derivative terms keeping in mind that $u = u(s, n)$,

$$\frac{Du}{Dt} = \underbrace{\frac{\partial u}{\partial t}}_{\substack{=0,\\ \text{(steady)}}} + \underbrace{u_n}_{\substack{=0,\\ \text{(flow tangent}\\ \text{to streamline)}}}\frac{\partial u}{\partial n} + \underbrace{u_s}_{\substack{=\text{u},\\ \text{(flow tangent}\\ \text{to streamline)}}}\frac{\partial u}{\partial s} = u\frac{\partial u}{\partial s}, \tag{5.178}$$

and,

$$\frac{D\hat{\boldsymbol{s}}}{Dt} = \underbrace{\frac{\partial \hat{\boldsymbol{s}}}{\partial t}}_{\substack{=0,\\ \text{(steady)}}} + \underbrace{u_n}_{\substack{=0,\\ \text{(flow tangent}\\ \text{to streamline)}}}\frac{\partial \hat{\boldsymbol{s}}}{\partial n} + \underbrace{u_s}_{\substack{=\text{u},\\ \text{(flow tangent}\\ \text{to streamline)}}}\frac{\partial \hat{\boldsymbol{s}}}{\partial s} = u\frac{\partial \hat{\boldsymbol{s}}}{\partial s}. \tag{5.179}$$

To determine how $\hat{\boldsymbol{s}}$ varies with the s-coordinate, consider Figure 5.20. Note that the triangles AOB and A'O'B' are similar. Hence,

$$\frac{ds}{R} = \frac{|d\hat{\boldsymbol{s}}|}{\underbrace{|\hat{\boldsymbol{s}}|}_{=1}} = |d\hat{\boldsymbol{s}}| \implies \frac{|d\hat{\boldsymbol{s}}|}{ds} = \frac{1}{R}. \tag{5.180}$$

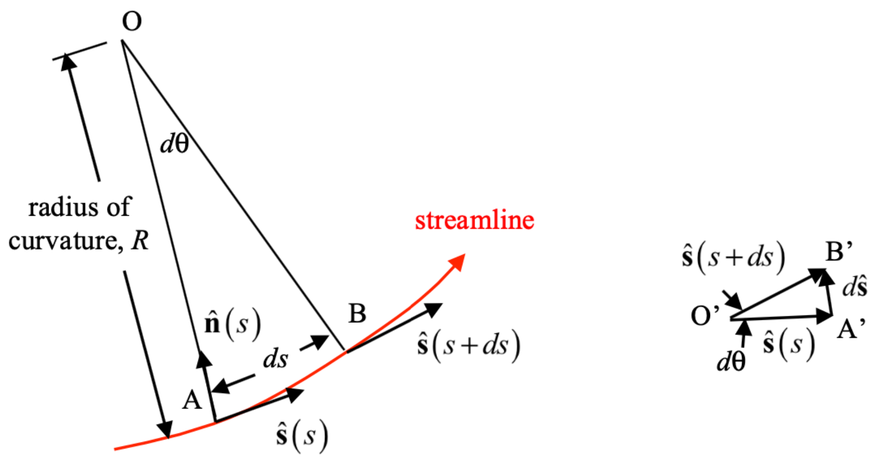


Figure 5.20. Illustration showing how the change in the $\hat{\boldsymbol{s}}$ direction varies with the s coordinate.

Also, as $ds \to 0$, $d\hat{\boldsymbol{s}}$ points in the $\hat{\mathbf{n}}$ direction so,

$$\frac{d\hat{\boldsymbol{s}}}{ds} = \frac{1}{R}\hat{\mathbf{n}}. \tag{5.181}$$

Substituting Eq. (5.181) into Eq. (5.179),

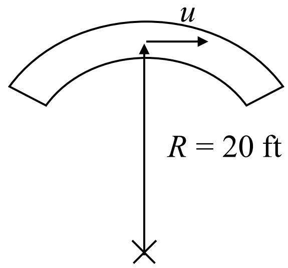
$$\frac{D\hat{\boldsymbol{s}}}{Dt} = \frac{u}{R}\hat{\mathbf{n}}. \tag{5.182}$$

Substituting Eq. (5.182) and Eq. (5.178) into Eq. (5.177) gives the fluid particle acceleration in streamline coordinates,

$$\boldsymbol{a} = \underbrace{\left(u\frac{\partial u}{\partial s}\right)}_{\substack{\text{tangential}\\ \text{acceleration}}} \hat{\boldsymbol{s}} + \underbrace{\left(\frac{u^2}{R}\right)}_{\substack{\text{normal}\\ \text{acceleration}}} \hat{\mathbf{n}}. \tag{5.183}$$

Water flows through the curved hose shown below with an increasing speed of $u = 10t$ ft/s, where t is in seconds. For $t = 2$ s determine:

a. the component of acceleration along the streamline,
b. the component of acceleration normal to the streamline, and
c. the net acceleration (magnitude and direction).



SOLUTION:

The acceleration component in the streamline direction is,

$$a_s = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s}, \tag{1}$$

where,

$$\frac{\partial u}{\partial t} = 10 \ \text{ft/s}^2 \quad \text{(The flow is unsteady.)}$$

$$\frac{\partial u}{\partial s} = 0 \quad \text{(The flow velocity doesn't change with respect to position along the streamline.)}$$

$$\boxed{\therefore a_s = 10 \ \text{ft/s}^2}$$

The acceleration component normal to the streamline is,

$$a_n = \frac{u^2}{R}, \tag{2}$$

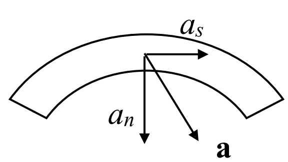
where,

$$\frac{u^2}{R} = \frac{(10*2 \text{ ft/s})^2}{20 \text{ ft}} = 20 \ \text{ft/s}^2 \quad \text{(The velocity is evaluated at } t = 2 \text{ s.)}$$

$$\boxed{\therefore a_n = 20 \ \text{ft/s}^2} \quad \text{(The acceleration is toward the center of curvature.)}$$

The net acceleration is,

$$\mathbf{a} = a_n\hat{\mathbf{n}} + a_s\hat{\mathbf{s}} \tag{3}$$



$$\boxed{\mathbf{a} = (20\hat{\mathbf{n}} + 10\hat{\mathbf{s}}) \ \text{ft/s}^2}$$

$$\boxed{|\mathbf{a}| = 22.4 \ \text{ft/s}^2}$$

5.8. Euler's Equations in Streamline Coordinates

Recall from previous analyses (Section 5.6) that the differential equations of motion for a fluid particle in an inviscid flow in a gravitational field are,

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \rho \boldsymbol{g} \quad \text{Euler's Equations}. \tag{5.184}$$

For simplicity, further assume that we're dealing with a 2D, steady flow. Now write Eq. (5.184) in streamline coordinates (s, n) (Figure 5.21),

$$s - \text{direction:} \quad \rho a_s = -\frac{\partial p}{\partial s} + \rho g_s, \tag{5.185}$$

$$n - \text{direction:} \quad \rho a_n = -\frac{\partial p}{\partial n} + \rho g_n. \tag{5.186}$$

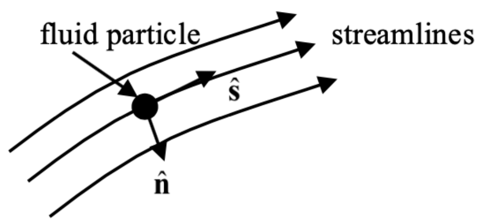


Figure 5.21. A fluid particle in streamline coordinates.

Recall that in streamline coordinates (refer to the previous section),

$$a_s = u\frac{\partial u}{\partial s} \quad \text{and} \quad a_n = \frac{u^2}{R}, \tag{5.187}$$

so that Eqs. (5.185) and (5.186) become,

$$u\frac{\partial u}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + g_s, \tag{5.188}$$

$$\frac{u^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g_n. \tag{5.189}$$

These are the 2D, steady Euler's Equations in streamline coordinates.

We can draw an important and very useful conclusion from Eq. (5.189). For a flow moving in a straight line ($R \to \infty$) and neglecting gravity ($g_n = 0$) we have,

$$\frac{\partial p}{\partial n} = 0, \tag{5.190}$$

i.e., the pressure does not change normal to the direction of the flow! This result is very helpful when considering the pressure in a free jet (Figure 5.22). Since free jets typically have negligible curvature and gravitational effects, the pressure everywhere normal to the free jet will be the same!

Similarly, for a flow with parallel streamlines adjacent to a flat boundary (Figure 5.23), the pressure gradient normal to the flow is,

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g \implies \frac{\partial p}{\partial n} = \rho g. \tag{5.191}$$

Thus, the pressure normal to the flow varies hydrostatically.

Now consider flow in a bend, as shown in Figure 5.24. Here, in the $\hat{\mathbf{n}}$ direction,

$$\frac{u^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} \implies \frac{\partial p}{\partial n} = -\rho\frac{u^2}{R}. \tag{5.192}$$

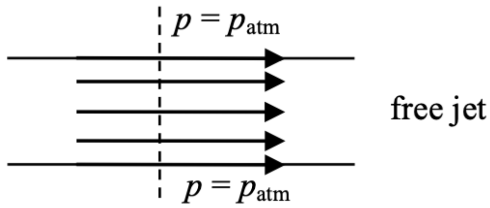


Figure 5.22. Streamlines in a free jet with no gravity.

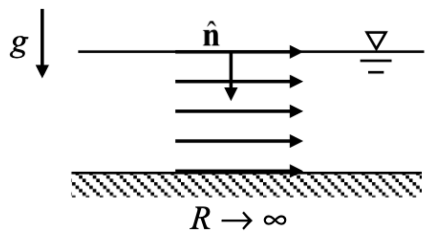


Figure 5.23. Streamlines for a flow parallel to a flat boundary.

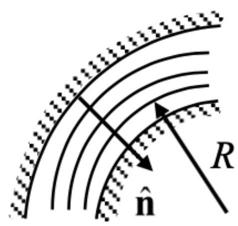


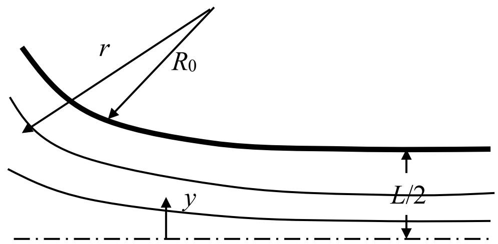
Figure 5.24. Streamlines in a curved bend.

Thus, the pressure increases as one moves in the negative n direction. The largest pressure is on the outside of the bend while the smallest pressure is on the inside part of the bend. If the fluid is a liquid and the inside bend pressure reaches the vapor pressure of the liquid, cavitation will occur.

In the curved inlet section of a wind tunnel the velocity distribution has a streamline radius of curvature given by:

$$r = R_0 \frac{L}{2y}$$

As a first approximation, assume the air speed along each streamline is 20 m/s. Evaluate the pressure change from the center line of the tunnel to the wall (located at $y = L/2$) if $L = 150$ mm and $R_0 = 0.6$ m.



SOLUTION:

Apply Euler's equation across the streamlines.

$$\frac{dp}{dr} = \rho \frac{V^2}{r} \tag{1}$$

Note that in the channel:

$$y = \left(R_0 + L/2\right) - r \quad \Rightarrow \quad dy = -dr \tag{2}$$

Substitute for the curvature radius and solve for the pressure difference.

$$-\frac{dp}{dy} = \rho \frac{V^2}{R_0 \frac{L}{2y}} \quad \Rightarrow \quad dp = -\frac{2\rho V^2}{R_0 L} y dy \tag{3}$$

$$\int_{p=p_{y=0}}^{p=p_{y=L/2}} dp = -\frac{2\rho V^2}{R_0 L} \int_{y=0}^{y=L/2} y dy \tag{4}$$

$$\boxed{p_{y=L/2} - p_{y=0} = -\frac{\rho V^2 L}{4R_0}} \tag{5}$$

Using the given data:

ρ = 1.23 kg/m^3
V = 20 m/s
R_0 = 0.6 m
L = 150e-3 m
$\Rightarrow$ $\boxed{p_{y=L/2} - p_{y=0} = \text{-30.8 Pa}}$

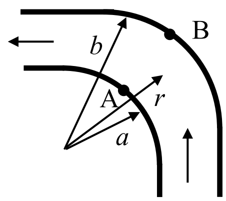
The velocity distribution in a horizontal, two-dimensional bend through which an ideal fluid flows can be approximated:

$$u_\theta = \frac{k}{r}$$

where r is the radius of curvature and k is a constant. Show that the volumetric flow rate through the bend, Q, is related to the pressure difference, $\Delta p = p_B - p_A$, and fluid density, ρ, via:

$$Q = C\sqrt{\frac{\Delta p}{\rho}}$$

where C is a constant that depends upon the bend geometry.



SOLUTION:

Apply Euler's equation across the streamlines:

$$\frac{dp}{dr} = \rho\frac{u_\theta^2}{r} \tag{1}$$

Substitute for the given velocity profile and solve the differential equation.

$$\frac{dp}{dr} = \rho\frac{k^2}{r^3} \tag{2}$$

$$\int_{p=p_A}^{p=p_B} dp = \rho k^2 \int_{r=a}^{r=b} \frac{dr}{r^3} \tag{3}$$

$$\Delta p = p_B - p_A = -\frac{\rho k^2}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = \frac{\rho k^2}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \tag{4}$$

Relate k to the volumetric flow rate using the velocity profile.

$$Q = \int_{r=a}^{r=b} u_\theta dr = k\int_{r=a}^{r=b} \frac{dr}{r} = k\ln\left(\frac{b}{a}\right) \tag{5}$$

$$\therefore k = \frac{Q}{\ln\left(\frac{b}{a}\right)} \tag{6}$$

Substitute Eqn. (6) into Eqn. (4) and simplify.

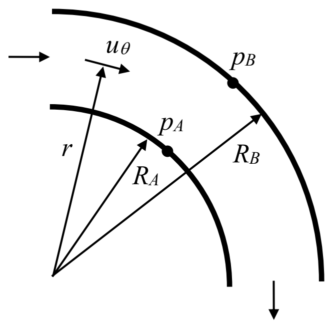
$$\Delta p = \frac{\rho}{2}\left[\frac{Q}{\ln\left(\frac{b}{a}\right)}\right]^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \tag{7}$$

$$\boxed{\therefore Q = \frac{\sqrt{2}\ln\left(\frac{b}{a}\right)}{\sqrt{\left(\frac{1}{a^2} - \frac{1}{b^2}\right)}}\sqrt{\frac{\Delta p}{\rho}} = C\sqrt{\frac{\Delta p}{\rho}}} \tag{8}$$

Consider the steady, inviscid flow through a smooth, constant diameter pipe bend as shown in the figure below. Gravity may be neglected in this problem. The fluid velocity in the bend is inversely proportional to the radius, *i.e.*,

$$u_\theta = \frac{k}{r},$$

where k is a constant.



How does the pressure difference, $p_B - p_A$, change as the radius R_B increases (R_A remains constant)?

A. increases

B. decreases

C. remains the same

D. not enough information is given

E. it's twice the change in the momentum flux

SOLUTION:

Simplify the radial component of Euler's equation.

$$\frac{dp}{dr} = \rho \frac{u_\theta^2}{r} \Rightarrow \frac{dp}{dr} = \rho \frac{(k/r)^2}{r} \tag{1}$$

$$\int_{p=p_A}^{p=p_B} dp = \rho k^2 \int_{r=R_A}^{r=R_B} \frac{dr}{r^3} \Rightarrow p\Big|_{p_A}^{p_B} = \rho k^2 \left(-\frac{1}{2}\frac{1}{r^2}\right)_{R_A}^{R_B} \tag{2}$$

$$p_B - p_A = -\frac{1}{2}\rho k^2 \left(\frac{1}{R_B^2} - \frac{1}{R_A^2}\right) \Rightarrow \boxed{p_B - p_A = \frac{1}{2}\rho k^2 \left(\frac{1}{R_A^2} - \frac{1}{R_B^2}\right)} \tag{3}$$

Thus, as R_B increases, $\boxed{p_B - p_A \text{ increases}}$.

5.9. The Energy Equation (aka The First Law for a Differential Control Volume)

The Energy Equation, which is The First Law of Thermodynamics for a differential fluid element or control volume, can be derived several different ways. Two of these methods are given in this section.

Method 1: Apply the integral approach to a differential control volume, as shown in Figure 5.25. Assume that

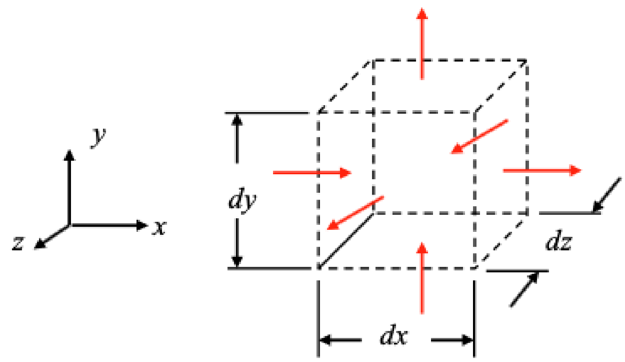


Figure 5.25. The differential control volume used in the derivation of the Energy Equation.

the density, specific total energy, and velocity are ρ, e, and $\boldsymbol{u}$, respectively, at the control volume's center. The total energy flow rates through each of the sides of the control volume are,

$$(\dot{m}_x e)_{\text{in through left}} = (\dot{m}_x e)_{\text{center}} + \frac{\partial (\dot{m}_x e)_{\text{center}}}{\partial x}\left(-\frac{1}{2}dx\right), \tag{5.193}$$

$$= (\rho u_x dydze) + \frac{\partial}{\partial x}(\rho u_x dydze)\left(-\frac{1}{2}dx\right), \tag{5.194}$$

$$= \left[\rho u_x e + \frac{\partial}{\partial x}(\rho u_x e)\left(-\frac{1}{2}dx\right)\right](dydz), \tag{5.195}$$

where $\dot{m}_{x,\text{center}}$ is the mass flow rate in the x direction at the center of the control volume. Similarly through the other faces,

$$(\dot{m}_x e)_{\text{out through right}} = \left[\rho u_x e + \frac{\partial}{\partial x}(\rho u_x e)\left(\frac{1}{2}dx\right)\right](dydz), \tag{5.196}$$

$$(\dot{m}_y e)_{\text{in through bottom}} = \left[\rho u_y e + \frac{\partial}{\partial y}(\rho u_y e)\left(-\frac{1}{2}dy\right)\right](dxdz), \tag{5.197}$$

$$(\dot{m}_y e)_{\text{out through top}} = \left[\rho u_y e + \frac{\partial}{\partial y}(\rho u_y e)\left(\frac{1}{2}dy\right)\right](dxdz), \tag{5.198}$$

$$(\dot{m}_z e)_{\text{in through back}} = \left[\rho u_z e + \frac{\partial}{\partial z}(\rho u_z e)\left(-\frac{1}{2}dz\right)\right](dxdy), \tag{5.199}$$

$$(\dot{m}_z e)_{\text{out through front}} = \left[\rho u_z e + \frac{\partial}{\partial z}(\rho u_z e)\left(\frac{1}{2}dz\right)\right](dxdy). \tag{5.200}$$

Note that in these previous equations the specific total energy (not including the potential energy), e, is given by,

$$e = u + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}, \tag{5.201}$$

where u is the specific internal energy. Thus, the net flow rate of total energy out of the control volume is,

$$(\dot{m}e)_{\text{net, out of CV}} = \left[\frac{\partial}{\partial x}(\rho u_x e) + \frac{\partial}{\partial y}(\rho u_y e) + \frac{\partial}{\partial z}(\rho u_z e)\right](dxdydz). \tag{5.202}$$

The rate at which the total energy increases within the control volume is,

$$\frac{\partial}{\partial t}(me)_{\text{within CV}} = \frac{\partial}{\partial t}(e\rho dxdydz) = \frac{\partial}{\partial t}(e\rho)(dxdydz), \tag{5.203}$$

where ρ and e are the density and specific internal energy, respectively, at the center of the control volume. Note that since these quantities vary linearly within the control volume (from the Taylor Series approximation), the averages within the control volume are ρ and e.

The rate at which heat is added to the control volume is,

$$\dot{Q}_{\text{into CV}} = \delta\dot{q}_{\text{into CV}}(dxdydz), \tag{5.204}$$

where $\delta\dot{q}_{\text{into CV}}$ is the rate of heat transfer into the control volume per unit volume. Note that the mode of heat transfer is not indicated at this point in the derivation.

The rate at which work is done on the control volume due to body forces is,

$$\dot{W}_{B,\text{on CV}} = (\boldsymbol{f}_B \cdot \boldsymbol{u})(\rho dxdydz) = (f_{B,i}u_i)(\rho dxdydz). \tag{5.205}$$

Note that the potential energy is not included in Eq. (5.201) since that term is included in the rate of body force work term in Eq. (5.205).

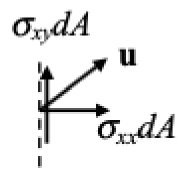


Figure 5.26. A sketch illustrating how the surface forces, in terms of stresses, do work on the control volume.

The rate at which work is done on the control volume due to surface forces is (Figure 5.26),

$$\begin{aligned}
\dot{W}_{S,\text{on CV}} &= d\boldsymbol{F}_S \cdot \boldsymbol{u}, \\
&= \left\{\left[\sigma_{xx}u_x + \frac{\partial(\sigma_{xx}u_x)}{\partial x}\left(\frac{1}{2}dx\right)\right] - \left[\sigma_{xx}u_x + \frac{\partial(\sigma_{xx}u_x)}{\partial x}\left(-\frac{1}{2}dx\right)\right]\right\}(dydz) \\
&+ \left\{\left[\sigma_{xy}u_y + \frac{\partial(\sigma_{xy}u_y)}{\partial x}\left(\frac{1}{2}dx\right)\right] - \left[\sigma_{xy}u_y + \frac{\partial(\sigma_{xy}u_y)}{\partial x}\left(-\frac{1}{2}dx\right)\right]\right\}(dydz) \\
&+ \left\{\left[\sigma_{xz}u_z + \frac{\partial(\sigma_{xz}u_z)}{\partial x}\left(\frac{1}{2}dx\right)\right] - \left[\sigma_{xz}u_z + \frac{\partial(\sigma_{xz}u_z)}{\partial x}\left(-\frac{1}{2}dx\right)\right]\right\}(dydz) \\
&+ \cdots \\
&= \frac{\partial}{\partial x_j}(\sigma_{ji}u_i)(dxdydz).
\end{aligned} \tag{5.206}$$

The First Law of Thermodynamics states that the rate of increase of total energy within the control volume plus the net rate at which total energy leaves the control volume must equal the rate at which heat is added to the control volume plus the rate at which work is done on the control volume,

$$\frac{\partial}{\partial t}(me)_{\text{within CV}} + (\dot{m}e)_{\text{net, out of CV}} = \dot{Q}_{\text{into CV}} + \dot{W}_{B,\text{on CV}} + \dot{W}_{S,\text{on CV}}. \tag{5.207}$$

Substituting Eq. (5.202) – (5.206) into Eq. (5.207) gives,

$$\begin{aligned}\frac{\partial}{\partial t}(e\rho)(dxdydz) + \left[\frac{\partial}{\partial x}(\rho u_x e) + \frac{\partial}{\partial y}(\rho u_y e) + \frac{\partial}{\partial z}(\rho u_z e)\right](dxdydz) = \\ + \delta\dot{q}_{\text{into}}(dxdydz) + f_{B,i}u_i(\rho dxdydz) + \frac{\partial}{\partial x_j}(\sigma_{ji}u_i)(dxdydz),\end{aligned} \tag{5.208}$$

$$\frac{\partial}{\partial t}(e\rho) + \frac{\partial}{\partial x_j}(e\rho u_j) = \delta\dot{q}_{\text{into CV}} + \rho u_i f_{B,i} + \frac{\partial}{\partial x_j}(\sigma_{ji}u_i). \tag{5.209}$$

Expand the left-hand side of the previous equation and utilize the Continuity Equation,

$$\frac{\partial}{\partial t}(e\rho) + \frac{\partial}{\partial x_j}(e\rho u_j) = e\frac{\partial \rho}{\partial t} + \rho\frac{\partial e}{\partial t} + e\frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j\frac{\partial e}{\partial x_j}, \tag{5.210}$$

$$= e\underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right]}_{=0,\text{(Continuity Eq.)}} = \rho\underbrace{\left(\frac{\partial e}{\partial t} + u_j\frac{\partial e}{\partial x_j}\right)}_{=\frac{De}{Dt}}. \tag{5.211}$$

Substituting back into Eq. (5.209) results in the Energy Equation,

$$\boxed{\rho\frac{De}{Dt} = \delta\dot{q}_{\text{into CV}} + \rho u_i f_{B,i} + \frac{\partial}{\partial x_j}(\sigma_{ji}u_i)}. \tag{5.212}$$

Method 2: Recall the integral form of the First Law of Thermodynamics,

$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS} e\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\text{into CV}} + \dot{W}_{B,\text{on CV}} + \dot{W}_{S,\text{on CV}}. \tag{5.213}$$

Consider a fixed control volume so that,

$$\frac{d}{dt}\int_{CV} e\rho dV = \int_{CV}\frac{\partial(e\rho)}{\partial t}dV \quad \text{and} \quad \boldsymbol{u}_{\text{rel}} = \boldsymbol{u}. \tag{5.214}$$

Note that the heat transfer into the control volume can be written as,

$$\dot{Q}_{\text{into CV}} = \int_{CV}\delta\dot{q}_{\text{into CV}}dV. \tag{5.215}$$

The work on the control volume due to body forces is,

$$\dot{W}_{B,\text{on CV}} = \int_{CV}(\boldsymbol{f}_B\cdot\boldsymbol{u})\rho dV = \int_{CV}(f_{B,i}u_i)\rho dV, \tag{5.216}$$

and the work on the control volume due to surface forces is,

$$\dot{W}_{S,\text{on CV}} = \int_{CS}(\boldsymbol{f}_S\cdot\boldsymbol{u})\rho dA = \int_{CS}(f_{S,i}u_i)\rho dA = \int_{CS}\sigma_{ji}n_j u_i dA, \tag{5.217}$$

where the surface forces have been written in terms of the stress tensor.

By utilizing Gauss' Theorem (aka the Divergence Theorem), we can convert the area integrals into volume integrals,

$$\int_{CS} e\rho\left(\boldsymbol{u}\cdot d\boldsymbol{A}\right) = \int_{CV}\boldsymbol{\nabla}\cdot(e\rho\boldsymbol{u})dV = \int_{CV}\frac{\partial}{\partial x_j}(e\rho u_j)dV, \tag{5.218}$$

and,

$$\int_{CS}\sigma_{ji}n_j u_i dA = \int_{CV}\frac{\partial(u_i\sigma_{ji})}{\partial x_j}dV. \tag{5.219}$$

Substitute these expressions into the First Law of Thermodynamics to get,

$$\int_{CV}\left[\frac{\partial}{\partial t}(e\rho) + \frac{\partial}{\partial x_j}(e\rho u_j) - \delta\dot{q}_{\text{into CV}} - \rho f_{B,i}u_i - \frac{\partial}{\partial x_j}(\sigma_{ji}u_i)\right]dV = 0. \tag{5.220}$$

Since the choice of control volume is arbitrary, the kernel of the integral must be zero, i.e.,

$$\frac{\partial}{\partial t}(e\rho) + \frac{\partial}{\partial x_j}(e\rho u_j) - \delta\dot{q}_{\text{into CV}} - \rho f_{B,i} u_i - \frac{\partial}{\partial x_j}(\sigma_{ji} u_i) = 0. \tag{5.221}$$

This equation is the same one as Eq. (5.209) so we see that the final result will be the same,

$$\boxed{\rho \frac{De}{Dt} = \delta\dot{q}_{\text{into CV}} + \rho u_i f_{B,i} + \frac{\partial}{\partial x_j}(\sigma_{ji} u_i)}. \tag{5.222}$$

Notes:

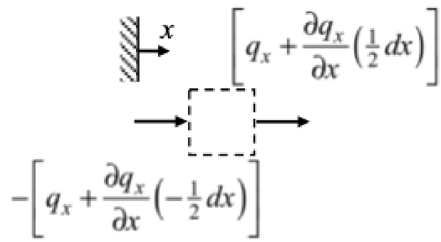


Figure 5.27. A sketch illustrating the heat transfer into the control volume entering through x faces.

(1) The rate of heat transfer may be re-written in terms of the rate of heat transfer per unit area out of the control volume through the control surface, $\boldsymbol{q}$. In terms of the Method 1 approach, the rate of total heat transfer into the control volume may be written as (refer to Figure 5.27),

$$\begin{aligned} \delta\dot{Q}_{\text{into CV}} = &-\left\{-\left[q_x + \frac{\partial q_x}{\partial x}\left(-\frac{1}{2}dx\right)\right] + \left[q_x + \frac{\partial q_x}{\partial x}\left(\frac{1}{2}dx\right)\right]\right\}(dydz) \\ &-\left\{-\left[q_y + \frac{\partial q_y}{\partial y}\left(-\frac{1}{2}dy\right)\right] + \left[q_y + \frac{\partial q_y}{\partial y}\left(\frac{1}{2}dy\right)\right]\right\}(dxdz) \\ &-\left\{-\left[q_z + \frac{\partial q_z}{\partial z}\left(-\frac{1}{2}dz\right)\right] + \left[q_z + \frac{\partial q_z}{\partial z}\left(\frac{1}{2}dz\right)\right]\right\}(dxdy) \end{aligned} \tag{5.223}$$

Simplifying the previous relation gives,

$$\delta\dot{Q}_{\text{into CV}} = -\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right](dxdydz) = -\left(\boldsymbol{\nabla}\cdot\boldsymbol{q}\right)(dxdydz) = -\frac{\partial q_j}{\partial x_j}(dxdydz). \tag{5.224}$$

The rate of total heat transfer in terms of the rate of heat transfer per unit area may also be derived using the Method 2 approach and the Divergence Theorem,

$$\delta\dot{Q}_{\text{into CV}} = -\int_{CS}(\boldsymbol{q}\cdot d\boldsymbol{A}) = -\int_{CS} q_j n_j dA = -\int_{CV}\frac{\partial q_j}{\partial x_j}dV. \tag{5.225}$$

Substituting the previous expressions in to the Energy Equation (Eq. (5.212)),

$$\boxed{\rho\frac{De}{Dt} = -\frac{\partial q_i}{\partial x_i} + \rho u_i f_{B,i} + \frac{\partial}{\partial x_j}(\sigma_{ji}u_i)}. \tag{5.226}$$

This is the <u>Energy Equation in terms of the heat transfer per unit area</u>.

(2) The Energy Equation (either Eq. (5.212) or (5.226)) may be simplified further by noting that,

$$\frac{De}{Dt} = \frac{D}{Dt}\left(u + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) = \frac{\partial u}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{2}u_i u_i\right) + u_j\frac{\partial u}{\partial x_j} + u_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i u_i\right) \tag{5.227}$$

$$= \underbrace{\left(\frac{\partial u}{\partial t} + u_j\frac{\partial u}{\partial x_j}\right)}_{=\frac{Du}{Dt}} + \underbrace{u_i\frac{\partial u_i}{\partial t} + u_j u_i\frac{\partial u_i}{\partial x_j}}_{=u_i\frac{Du_i}{Dt}}, \tag{5.228}$$

and,

$$\frac{\partial}{\partial x_j}\left(\sigma_{ji}u_i\right) = u_i\frac{\partial \sigma_{ji}}{\partial x_j} + \sigma_{ji}\frac{\partial u_i}{\partial x_j}. \tag{5.229}$$

Substituting these expressions into Eq. (5.212) gives,

$$\rho\left(\frac{Du}{Dt} + u_i\frac{Du_i}{Dt}\right) = \delta\dot{q}_{\text{into CV}} + \rho u_i f_{B,i} + u_i\frac{\partial \sigma_{ji}}{\partial x_j} + \sigma_{ji}\frac{\partial u_i}{\partial x_j}, \tag{5.230}$$

$$\rho\frac{Du}{Dt} = \delta\dot{q}_{\text{into CV}} + \sigma_{ji}\frac{\partial u_i}{\partial x_j} + u_i\underbrace{\left(\rho f_{B,i} + \frac{\partial \sigma_{ji}}{\partial x_j} - \rho\frac{Du_i}{Dt}\right)}_{=0}, \tag{5.231}$$

but the terms in the parentheses of this last equation are the Momentum Equations (Eq. (5.113))! The dot product of the Momentum Equations with the velocity is known as the Mechanical Energy Equation,

$$\boxed{\underbrace{u_i\rho\frac{Du_i}{Dt}}_{\substack{\text{rate of increase}\\ \text{of kinetic energy of}\\ \text{fluid element}}} = \underbrace{\rho u_i f_{B,i}}_{\substack{\text{rate at which work is}\\ \text{done on the fluid element}\\ \text{due to body forces}}} + \underbrace{u_i\frac{\partial \sigma_{ji}}{\partial x_j}}_{\substack{\text{rate at which work is}\\ \text{done on the fluid element}\\ \text{due to stress gradients}}}}. \tag{5.232}$$

Note that,

$$\frac{D}{Dt}\left(\frac{1}{2}u_i u_i\right) = u_i\frac{Du_i}{Dt}. \tag{5.233}$$

The Energy Equation without the Mechanical Energy Equation terms is known as the Thermal Energy Equation and is given by,

$$\boxed{\rho\frac{Du}{Dt} = \delta\dot{q}_{\text{into CV}} + \sigma_{ji}\frac{\partial u_i}{\partial x_j}}, \tag{5.234}$$

or, in terms of the heat transfer per unit area,

$$\boxed{\underbrace{\rho\frac{Du}{Dt}}_{\substack{\text{rate of increase}\\ \text{of internal energy}\\ \text{within the fluid element}}} = \underbrace{-\frac{\partial q_i}{\partial x_i}}_{\substack{\text{rate at which}\\ \text{heat is added}\\ \text{to the fluid element}\\ \text{through the surface area}}} + \underbrace{\sigma_{ji}\frac{\partial u_i}{\partial x_j}}_{\substack{\text{rate at which mechanical}\\ \text{energy is converted to thermal}\\ \text{energy due to deformations}\\ \text{of the fluid element}}}} \tag{5.235}$$

(3) The heat flux term, $\boldsymbol{q}$, may be written in terms of a temperature gradient using Fourier's Law of Conduction, assuming that conduction is the dominant mode of heat transfer,

$$\boldsymbol{q} = -k\boldsymbol{\nabla}T \quad \text{or} \quad q_i = -k\frac{\partial T}{\partial x_i}, \tag{5.236}$$

where k is the thermal conductivity (in its most general form, the thermal conductivity is a tensor quantity) of the substance and T is the temperature. Note that the negative sign is included in the equation to account for the fact that heat flows from regions of high temperature to regions of low temperature. Thus, the Thermal Energy Equation using Fourier's Law of Conduction is,

$$\boxed{\rho \frac{Du}{Dt} = \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) + \sigma_{ji}\frac{\partial u_i}{\partial x_j}}. \tag{5.237}$$

(4) The rate of work term in the Thermal Energy Equation (Eq. (5.235)) includes both reversible and irreversible work terms. Consider the rate of work term using the stress tensor for a Newtonian fluid,

$$\sigma_{ji}\frac{\partial u_i}{\partial x_j} = \left[\left(-p + \lambda\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right]\left(\frac{\partial u_i}{\partial x_j}\right), \tag{5.238}$$

$$= \underbrace{-p\left(\frac{\partial u_j}{\partial x_j}\right)}_{\substack{\text{reversible}\\ \text{pressure work}}} + \underbrace{\lambda\left(\frac{\partial u_k}{\partial x_k}\right)^2 + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right)}_{\text{irreversible viscous work}}. \tag{5.239}$$

The irreversible rate of work term (the rate at which mechanical energy is converted into thermal energy) is referred to as the Energy Dissipation Function, Φ,

$$\boxed{\Phi = \lambda\left(\frac{\partial u_k}{\partial x_k}\right)^2 + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right)}. \tag{5.240}$$

Thus, the Thermal Energy Equation can be written as,

$$\rho\frac{Du}{Dt} = \delta\dot{q}_{\text{into CV}} - p\left(\frac{\partial u_j}{\partial x_j}\right) + \Phi, \tag{5.241}$$

or, if conduction is the significant mode of heat transfer,

$$\rho\frac{Du}{Dt} = \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) - p\left(\frac{\partial u_j}{\partial x_j}\right) + \Phi. \tag{5.242}$$

(5) Note that for an incompressible fluid, the Continuity Equation,

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{5.243}$$

can be used to simplify the Thermal Energy Equation to the following form,

$$\boxed{\rho\frac{Du}{Dt} = \delta\dot{q}_{\text{into CV}} + \Phi} \quad \underline{\text{Thermal Energy Eq. for an incompressible fluid}}, \tag{5.244}$$

where the Energy Dissipation Function is,

$$\Phi = \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right). \tag{5.245}$$

(6) The Energy Dissipation Function for a Newtonian fluid is a positive definite quantity, which means that viscosity always acts to convert mechanical energy into thermal energy,

$$\Phi = \lambda \left(\frac{\partial u_k}{\partial x_k}\right)^2 + \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right), \tag{5.246}$$

$$= \lambda \left(\frac{\partial u_k}{\partial x_k}\right)^2 + \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left[\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)\right], \tag{5.247}$$

$$= \lambda \left(\frac{\partial u_k}{\partial x_k}\right)^2 + \frac{1}{2}\mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)^2 + \frac{1}{2}\mu\left(\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right), \tag{5.248}$$

$$= \lambda \left(\frac{\partial u_k}{\partial x_k}\right)^2 + \frac{1}{2}\mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)^2 + \frac{1}{2}\mu\underbrace{\left(-\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)}_{\substack{=0,\,\text{(since } i \text{ and } j\\ \text{are dummy indices)}}}, \tag{5.249}$$

$$\therefore \Phi = \lambda \left(\frac{\partial u_k}{\partial x_k}\right)^2 + \frac{1}{2}\mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)^2. \tag{5.250}$$

Note that for an incompressible fluid,

$$\frac{\partial u_k}{\partial x_k} = 0, \tag{5.251}$$

so that,

$$\Phi = \frac{1}{2}\mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)^2 > 0, \tag{5.252}$$

since $\mu > 0$.

For a compressible, Newtonian fluid, Stokes' Hypothesis states that,

$$\lambda = -\frac{2}{3}\mu. \tag{5.253}$$

After simplify Eq. (5.250), it can also be shown that $\Phi > 0$.

(7) The Thermal Energy Equation may also be written in a form using the specific enthalpy, h, which is defined as,

$$h \coloneqq u + \frac{p}{\rho}. \tag{5.254}$$

Consider the Thermal Energy Equation (Eq. (5.241)) using the definition of the Energy Dissipation Function (Eq. (5.240)),

$$\rho\frac{Du}{Dt} = \delta\dot{q}_{\text{into CV}} - p\left(\frac{\partial u_j}{\partial x_j}\right) + \Phi. \tag{5.255}$$

Re-write the pressure term in the previous equation utilizing the Continuity Equation,

$$p\left(\frac{\partial u_j}{\partial x_j}\right) = p\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right) = \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{Dp}{Dt} + \Phi, \tag{5.256}$$

$$\rho\frac{D}{Dt}\left(u + \frac{p}{\rho}\right) = \delta\dot{q}_{\text{into CV}} + \frac{Dp}{Dt} + \Phi, \tag{5.257}$$

$$\boxed{\therefore \rho\frac{Dh}{Dt} = \delta\dot{q}_{\text{into CV}} + \frac{Dp}{Dt} + \Phi}. \tag{5.258}$$

(8) For convenience, rewrite the Thermal Energy Equation for an incompressible fluid (Eq. (5.244)),

$$\rho\frac{Du}{Dt} = \delta\dot{q}_{\text{into CV}} + \Phi. \tag{5.259}$$

Recall that for an incompressible fluid, the internal energy is a function only of temperature, i.e., $u = \int c(T)dT$ where $c(T)$ is the fluid's specific heat, which, in general, is a function of temperature. Thus, we observe that the Energy Equation is uncoupled from the Continuity and Momentum

Equations for an incompressible flow. In other words, the unknowns of velocity, u_i, and pressure, p, (four unknowns) can be solved using the Continuity and Momentum Equations (four equations). Once these quantities have been determined, the internal energy, which is a function of temperature, can be calculated using the Thermal Energy Equation.

(9) For an adiabatic, inviscid flow, Eq. (5.258) reduces to,

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt}. \tag{5.260}$$

The Mechanical Energy Equation (Eq. (5.232)) can be used to re-write the Lagrangian derivative of the pressure,

$$\rho u_i \frac{Du_i}{Dt} = -u_i \frac{\partial p}{\partial x_i} + \rho u_i f_{B,i}, \tag{5.261}$$

$$\rho u_i \frac{Du_i}{Dt} - \frac{\partial p}{\partial t} = -\underbrace{\left(\frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i}\right)}_{=\frac{Dp}{Dt}} + \rho u_i f_{B,i}, \tag{5.262}$$

$$\therefore \frac{Dp}{Dt} = \frac{\partial p}{\partial t} - \rho u_i \frac{Du_i}{Dt} + \rho u_i f_{B,i}. \tag{5.263}$$

Substituting this last equation into Eq. (5.260),

$$\rho \frac{Dh}{Dt} = \frac{\partial p}{\partial t} - \rho u_i \frac{Du_i}{Dt} + \rho u_i f_{B,i} = \frac{\partial p}{\partial t} - \rho \frac{D}{Dt}\left(\frac{1}{2}u_i u_i\right) + \rho u_i f_{B,i}, \tag{5.264}$$

$$\rho \frac{D}{Dt}\left(h + \frac{1}{2}u_i u_i\right) = \frac{\partial p}{\partial t} + \rho u_i f_{B,i}. \tag{5.265}$$

If the body force is conservative, i.e., $\boldsymbol{f}_B = -\boldsymbol{\nabla} G$, (weight is a conservative body force, for example) then,

$$\rho u_i f_{B,i} = -\rho u_i \frac{\partial G}{\partial x_i} = -\rho \frac{DG}{Dt} + \rho \frac{\partial G}{\partial t}. \tag{5.266}$$

Furthermore, if the body force is also independent of time (obviously a good assumption if gravity is the only body force considered), then Eq. (5.266) may be substituted into Eq. (5.265) and then simplified to give,

$$\rho \frac{D}{Dt}\left(h + \frac{1}{2}u_i u_i\right) = \frac{\partial p}{\partial t} - \rho \frac{DG}{Dt}, \tag{5.267}$$

$$\boxed{\frac{D}{Dt} \underbrace{\left(h + \frac{1}{2}u_i u_i + G\right)}_{=h_T\text{, total specific enthalpy}} = \frac{1}{\rho}\frac{\partial p}{\partial t}}. \tag{5.268}$$

With the additional assumption that the flow is steady, we observe that the total specific enthalpy of a fluid particle will remain constant, i.e., $\frac{Dh_T}{Dt} = 0$. In particular, the total specific enthalpy will remain constant along a streamline.

For a Newtonian fluid show that the viscous dissipation is given by:

$$\underline{\underline{\boldsymbol{\tau}}} : \nabla \mathbf{u} = -\tfrac{2}{3}\mu\left(\nabla \cdot \mathbf{u}\right)^2 + 2\mu \underline{\underline{\mathbf{S}}} : \underline{\underline{\mathbf{S}}}$$

where $\boldsymbol{\tau}$ is the viscous part of the stress tensor and $\mathbf{S}$ is the shearing strain tensor.

SOLUTION:

$$\underline{\underline{\boldsymbol{\tau}}} : \nabla \mathbf{u} = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \left[\lambda \frac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]\frac{\partial u_i}{\partial x_j}$$

$$= \lambda \frac{\partial u_k}{\partial x_k}\frac{\partial u_j}{\partial x_j} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}$$

$$= \lambda \left(\frac{\partial u_k}{\partial x_k}\right)^2 + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}$$

$$\underline{\underline{\boldsymbol{\tau}}} : \nabla \mathbf{u} = -\tfrac{2}{3}\mu\left(\nabla \cdot \mathbf{u}\right)^2 + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} \quad (1)$$

where Stokes' hypothesis has been used ($\lambda + {}^2/_3\mu = 0$).

The last term in Eqn. (1) is most easily put in the required form by working backwards from the desired result.

$$2\mu \underline{\underline{\mathbf{S}}} : \underline{\underline{\mathbf{S}}} = 2\mu S_{ij}S_{ji}$$

$$= 2\mu \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

$$= \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right)$$

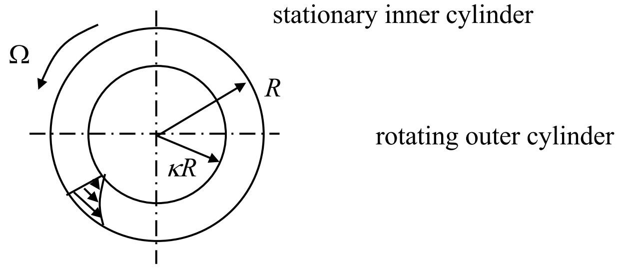
$$= \mu\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)$$

$$\therefore 2\mu \underline{\underline{\mathbf{S}}} : \underline{\underline{\mathbf{S}}} = \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\frac{\partial u_i}{\partial x_j} \quad (2)$$

Thus, combining Eqns. (1) and (2) demonstrates that:

$$\underline{\underline{\boldsymbol{\tau}}} : \nabla \mathbf{u} = -\tfrac{2}{3}\mu\left(\nabla \cdot \mathbf{u}\right)^2 + 2\mu \underline{\underline{\mathbf{S}}} : \underline{\underline{\mathbf{S}}} \quad (3)$$

A scientist is interested in knowing the temperature distribution in a steadily rotating viscometer (shown below).



For this geometry, the laminar velocity profile for a incompressible, constant viscosity fluid can be determined from the Navier-Stokes equations to be:

$$u_\theta = \Omega R\frac{\left(\frac{r}{\kappa R}-\frac{\kappa R}{r}\right)}{\left(\frac{1}{\kappa}-\kappa\right)} \text{ and } u_r = u_z = 0$$

If the temperature at the wall of the rotating outer cylinder is T_O and the temperature at the inner stationary cylinder is T_I, determine the resulting temperature distribution in the fluid. You may assume that the variations in the fluid density, viscosity, and thermal conductivity with respect to temperature are negligible.

SOLUTION:

Substitute the given velocity profile into the thermal energy equation (assuming an incompressible, constant viscosity, constant thermal conductivity, Newtonian fluid and cylindrical coordinates):

$$\rho c\underbrace{\left(\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}+u_z\frac{\partial T}{\partial z}\right)}_{=\frac{DT}{Dt}} = k\underbrace{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right]}_{=\nabla^2 T}$$

$$+\mu\underbrace{\left\{2\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)^2+\left(\frac{\partial u_z}{\partial z}\right)^2\right]+\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right)^2+\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)^2\right\}}_{=\sigma_{ji}\frac{\partial u_i}{\partial x_j}}$$

Noting that $u_r = u_z = 0$, assuming steady conditions ($\partial/\partial t = 0$), and that $T = T(r)$, the previous equation simplifies to:

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\mu\left(\frac{du_\theta}{dr}-\frac{u_\theta}{r}\right)^2$$

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\mu\left[\frac{\Omega R}{\left(\frac{1}{\kappa}-\kappa\right)}\left(\frac{1}{\kappa R}+\frac{\kappa R}{r^2}\right)-\Omega\frac{R}{r}\frac{\left(\frac{r}{\kappa R}-\frac{\kappa R}{r}\right)}{\left(\frac{1}{\kappa}-\kappa\right)}\right]^2 \tag{1}$$

where

$$\left[\frac{\Omega R}{\left(\frac{1}{\kappa}-\kappa\right)}\left(\frac{1}{\kappa R}+\frac{\kappa R}{r^2}\right)-\Omega\frac{R}{r}\frac{\left(\frac{r}{\kappa R}-\frac{\kappa R}{r}\right)}{\left(\frac{1}{\kappa}-\kappa\right)}\right]^2 = \frac{\Omega^2 R^2}{\left(\frac{1}{\kappa}-\kappa\right)^2}\left[\frac{1}{\kappa R}+\frac{\kappa R}{r^2}-\frac{1}{\kappa R}+\frac{\kappa R}{r^2}\right]^2$$

$$= \frac{4\Omega^2\kappa^4}{\left(1-\kappa^2\right)^2}\frac{R^4}{r^4} \tag{2}$$

Substituting Eqn. (2) into Eqn. (1) gives:

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{4\mu\Omega^2\kappa^4}{\left(1-\kappa^2\right)^2}\frac{R^4}{r^4} \tag{3}$$

Solving this differential equation gives:

$$d\left(r\frac{dT}{dr}\right) = -\frac{4\mu\Omega^2\kappa^4}{k\left(1-\kappa^2\right)^2}\frac{R^4}{r^3}dr$$

$$r\frac{dT}{dr} = \frac{2\mu\Omega^2\kappa^4}{k\left(1-\kappa^2\right)^2}\frac{R^4}{r^2} + c_1$$

$$dT = \frac{2\mu\Omega^2\kappa^4}{k\left(1-\kappa^2\right)^2}\frac{R^4}{r^3}dr + \frac{c_1}{r}dr$$

$$T = -\frac{\mu\Omega^2 R^2}{k}\frac{\kappa^4}{\left(1-\kappa^2\right)^2}\frac{R^2}{r^2} + c_1\ln r + c_2 \tag{4}$$

Apply the given boundary conditions to determine the unknown constants c_1 and c_2.

$$T\left(r=\kappa R\right) = T_I \tag{5}$$

$$T\left(r=R\right) = T_O \tag{6}$$

For simplicity, define the following dimensionless parameters:

$$\xi \equiv \frac{r}{R} \quad \text{(dimensionless radius)} \tag{7}$$

$$\Theta \equiv \frac{T-T_I}{T_O-T_I} \quad \text{(dimensionless temperature)} \tag{8}$$

$$\mathrm{N} \equiv \frac{\mu\Omega^2 R^2}{k\left(T_O-T_I\right)}\frac{\kappa^4}{\left(1-\kappa^2\right)^2} = \mathrm{Br}\frac{\kappa^4}{\left(1-\kappa^2\right)^2} \tag{9}$$

(where Br is a Brinkman number which is the ratio of the rate of viscous heat generation to the heat flux by conduction)

(10)

Substituting these dimensionless numbers into Eqn. (4) and boundary conditions (5) and (6) gives:

$$\frac{T-T_I}{\left(T_O-T_I\right)} = -\frac{\mu\Omega^2 R^2}{k\left(T_O-T_I\right)}\frac{\kappa^4}{\left(1-\kappa^2\right)^2}\frac{R^2}{r^2} + d_1\ln\left(\frac{r}{R}\right) + d_2 \quad (d_1 \text{ and } d_2 \text{ are new constants})$$

$$\Theta = -N\frac{1}{\xi^2} + d_1\ln\xi + d_2 \tag{11}$$

$$\Theta\left(\xi=\kappa\right) = 0 \tag{12}$$

$$\Theta\left(\xi=1\right) = 1 \tag{13}$$

Applying the boundary conditions gives:

$$1 = -N + d_2 \Rightarrow d_2 = 1 + N \tag{14}$$

$$0 = -N\frac{1}{\kappa^2} + d_1\ln\kappa + 1 + N \Rightarrow d_1 = \frac{N\left(\frac{1-\kappa^2}{\kappa^2}\right) - 1}{\ln\kappa} \tag{15}$$

Hence, the temperature distribution (in dimensionless form) is:

$$\boxed{\Theta = -N\frac{1}{\xi^2} + \left[N\left(\frac{1-\kappa^2}{\kappa^2}\right) - 1\right]\frac{\ln\xi}{\ln\kappa} + (1+N)} \tag{16}$$

Note that if N is large enough, the maximum temperature may be located at:

$$\frac{d\Theta}{d\xi} = 0 = 2N\frac{1}{\xi^3} + \left[N\left(\frac{1-\kappa^2}{\kappa^2}\right) - 1\right]\frac{1}{\xi\ln\kappa} \tag{17}$$

$$\xi_{\max} = \sqrt{\frac{\ln\kappa}{2N}\left[1 - N\left(\frac{1-\kappa^2}{\kappa^2}\right)\right]^{-1}} \tag{18}$$

with the temperature in the fluid being greater than the boundary temperatures. (The boundary temperatures would also need to be checked to see if this value is the maximum.) This occurs due to the viscous work done on the fluid.

5.10. The Entropy Equation (aka The Second Law for a Differential Control Volume)

The Entropy Equation, which is the Second Law of Thermodynamics for a differential fluid element or control volume, can be derived several different ways. Two of these methods are given in this section.

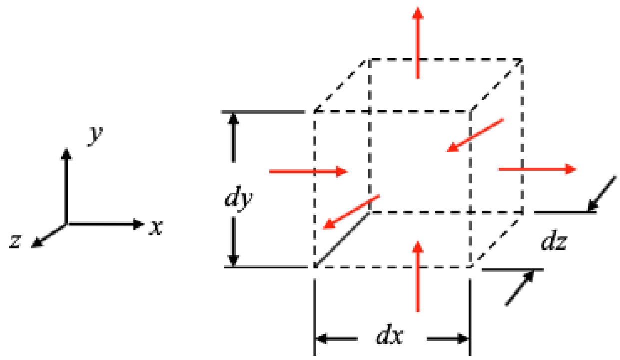


Figure 5.28. The differential control volume used in deriving the Entropy Equation.

Method 1: Apply the integral approach to the differential control volume shown in Figure 5.28. Assume that the density, specific entropy, and velocity are ρ, s, and u, respectively, at the control volume's center. The entropy fluxes through each of the side of the control volume are given by,

$$(\dot{m}_x s)_{\text{in through left}} = (\dot{m}_x s)_{\text{center}} + \frac{\partial(\dot{m}_x s)_{\text{center}}}{\partial x}\left(-\frac{1}{2}dx\right), \tag{5.269}$$

$$= (\rho u_x dydzs) + \frac{\partial}{\partial x}(\rho u_x dydzs)\left(-\frac{1}{2}dx\right), \tag{5.270}$$

$$= \left[\rho u_x s + \frac{\partial}{\partial x}(\rho u_x s)\left(-\frac{1}{2}dx\right)\right](dydz), \tag{5.271}$$

where $\dot{m}_{x,\text{center}}$ is the mass flow rate in the x direction at the center of the control volume. Applying the same approach through the other faces,

$$(\dot{m}_x s)_{\text{out through right}} = \left[\rho u_x s + \frac{\partial}{\partial x}(\rho u_x s)\left(\frac{1}{2}dx\right)\right](dydz), \tag{5.272}$$

$$(\dot{m}_y s)_{\text{in through bottom}} = \left[\rho u_y s + \frac{\partial}{\partial y}(\rho u_y s)\left(-\frac{1}{2}dy\right)\right](dxdz), \tag{5.273}$$

$$(\dot{m}_y s)_{\text{out through top}} = \left[\rho u_y s + \frac{\partial}{\partial y}(\rho u_y s)\left(\frac{1}{2}dy\right)\right](dydz), \tag{5.274}$$

$$(\dot{m}_z s)_{\text{in through back}} = \left[\rho u_z s + \frac{\partial}{\partial z}(\rho u_z s)\left(-\frac{1}{2}dz\right)\right](dxdy), \tag{5.275}$$

$$(\dot{m}_z s)_{\text{out through front}} = \left[\rho u_z s + \frac{\partial}{\partial z}(\rho u_z s)\left(\frac{1}{2}dz\right)\right](dxdy). \tag{5.276}$$

Thus, the net entropy flow rate out of the control volume is,

$$(\dot{m}s)_{\text{net, out of CV}} = \left[\frac{\partial}{\partial x}(\rho u_x s) + \frac{\partial}{\partial y}(\rho u_y s) + \frac{\partial}{\partial z}(\rho u_z s)\right](dxdydz). \tag{5.277}$$

The rate at which the entropy increases within the control volume is,

$$\frac{\partial}{\partial t}(ms)_{\text{within CV}} = \frac{\partial}{\partial t}(s\rho dxdydz) = \frac{\partial}{\partial t}(s\rho)(dxdydz), \tag{5.278}$$

where ρ and s are the density and specific entropy, respectively, at the center of the control volume. Note that since these quantities vary linearly within the control volume (from the Taylor Series approximation), the averages within the control volume are ρ and s.

The Second Law of Thermodynamics (applied to a control volume) states that the rate at which entropy increases within the control volume plus the net rate at which entropy leaves the control volume is equal to the rate of heat transfer into the control volume divided by the absolute temperature where the heat is added plus the rate of entropy generation within the control volume,

$$\frac{\partial}{\partial t}(ms)_{\text{within CV}} + (\dot{m}s)_{\text{net, out of CV}} = \frac{\delta\dot{q}_{\text{into CV}}(dxdydz)}{T} + \dot{\sigma}(dxdydz). \tag{5.279}$$

In this equation, $\delta\dot{q}_{\text{into CV}}$ is the rate of heat transfer into the control volume per unit volume and $\dot{\sigma}$ is the rate of entropy production in the control volume per unit volume ($\dot{\sigma} \geq 0$). Note that if the temperature at the center of the control volume is T, then the temperature at one of the faces where the heat is added would be differentially different. For example, at the right face the temperature would be $T + \frac{\partial T}{\partial x}\left(\frac{1}{2}dx\right)$. However, since the differential part is very small compared to T, the temperature at the face is just equal to T.

Substituting Eqs. (5.277) and (5.278) into Eq. (5.279) and simplifying gives,

$$\frac{\partial}{\partial t}(s\rho)(dxdydz) + \left[\frac{\partial}{\partial x}(\rho u_x s) + \frac{\partial}{\partial y}(\rho u_y s) + \frac{\partial}{\partial z}(\rho u_z s)\right](dxdydz) = \frac{\delta\dot{q}_{\text{into CV}}(dxdydz)}{T} + \dot{\sigma}(dxdydz), \tag{5.280}$$

$$\frac{\partial}{\partial t}(s\rho) + \left[\frac{\partial}{\partial x}(\rho u_x s) + \frac{\partial}{\partial y}(\rho u_y s) + \frac{\partial}{\partial z}(\rho u_z s)\right] = \frac{\delta\dot{q}_{\text{into CV}}}{T} + \dot{\sigma}. \tag{5.281}$$

This equation can be written in the following, compact form,

$$\boxed{\frac{\partial}{\partial t}(s\rho) + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}s) = \frac{\delta\dot{q}_{\text{into CV}}}{T} + \dot{\sigma} \quad \text{or} \quad \frac{\partial}{\partial t}(s\rho) + \frac{\partial}{\partial x_j}(\rho u_j s) = \frac{\delta\dot{q}_{\text{into CV}}}{T} + \dot{\sigma}}. \tag{5.282}$$

Expand the left-hand side of this equation and utilize the Continuity Equation,

$$\frac{\partial}{\partial t}(s\rho) + \frac{\partial}{\partial x_j}(\rho u_j s) = s\frac{\partial\rho}{\partial t} + \rho\frac{\partial s}{\partial t} + s\frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j\frac{\partial s}{\partial x_j}, \tag{5.283}$$

$$= s\underbrace{\left[\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right]}_{=0,\text{ (Continuity Eq.)}} + \rho\underbrace{\left(\frac{\partial s}{\partial t} + u_j\frac{\partial s}{\partial x_j}\right)}_{=\frac{Ds}{Dt}}. \tag{5.284}$$

Substituting back into Eq. (5.282),

$$\boxed{\rho\frac{Ds}{Dt} = \frac{\delta\dot{q}_{\text{into CV}}}{T} + \dot{\sigma}} \quad \underline{\text{The Entropy Equation}} \tag{5.285}$$

Method 2: Apply the Second Law of Thermodynamics directly to a small piece of fluid,

$$\frac{D}{Dt}(s\rho dxdydz) = \frac{\delta\dot{q}_{\text{into CV}}}{T}(dxdydz) + \dot{\sigma}(dxdydz). \tag{5.286}$$

Expand the Lagrangian derivative,

$$\frac{D}{Dt}(s\rho dxdydz) = \frac{Ds}{Dt}(\rho dxdydz) + s\frac{D}{Dt}(\rho dxdydz). \tag{5.287}$$

The second term on the right-hand side of this equation is zero since the mass of the fluid element remains constant. Thus, Eq. (5.286) can be written as,

$$\rho\frac{Ds}{Dt} = \frac{\delta\dot{q}_{\text{into CV}}}{T} + \dot{\sigma}, \tag{5.288}$$

which is the same Entropy Equation found using Method 1.

Notes:

(1) For an internally reversible, adiabatic flow ($\Longrightarrow$ an isentropic flow),

$$\frac{Ds}{Dt} = 0. \tag{5.289}$$

(2) Recall that for a simple, compressible system where the only surface forces are reversible pressure forces (so that $\dot{\sigma} = 0$), the First and Second Laws of Thermodynamics may be combined to give,

$$\frac{De}{Dt} = \frac{1}{\rho}\delta\dot{q}_{\text{into sys}} - p\frac{Dv}{Dt}, \tag{5.290}$$

where v is the specific volume and for a simple, compressible system the total specific energy is equal to the specific internal energy, i.e., $e = u$. Re-writing the specific volume in terms of the density, utilizing Eq. (5.285), and simplifying gives,

$$\frac{Du}{Dt} = T\frac{Ds}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right), \tag{5.291}$$

$$T\frac{Ds}{Dt} = \frac{Du}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}. \tag{5.292}$$

This expression may also be written in terms of the specific enthalpy, h, by re-writing the second term on the right-hand side of the equation,

$$-p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \frac{1}{\rho}\frac{Dp}{Dt} - \frac{D}{Dt}\left(\frac{p}{\rho}\right) \Longrightarrow T\frac{Ds}{Dt} = \underbrace{\frac{Du}{Dt} + \frac{D}{Dt}\left(\frac{p}{\rho}\right)}_{=\frac{Dh}{Dt}} - \frac{1}{\rho}\frac{Dp}{Dt}, \tag{5.293}$$

$$\boxed{T\frac{Ds}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}}. \tag{5.294}$$

5.11. Vorticity Dynamics

Recall that the vorticity $\boldsymbol{\omega}$ of a fluid element is equal to twice the rotation rate of the element,

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} \quad \text{or} \quad \omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}. \tag{5.295}$$

Notes:

(1) A rotational flow is defined as one in which the vorticity is *not* zero.
(2) An irrotational flow is defined as one in which there is no vorticity, i.e.,

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}. \tag{5.296}$$

A useful concept when discussing vorticity is the vortex line. A vortex line is a line that is everywhere tangent to the flow's vorticity vectors.

Notes:

(1) A vortex line is analogous to a streamline.
(2) A vortex tube is a tube made by all the vortex lines passing through a closed curve (Figure 5.29).

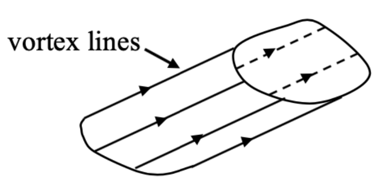


Figure 5.29. A sketch of a vortex tube.

(3) A vortex filament is a vortex tube with infinitesimally-small cross-sectional area.
(4) There are no vortex lines in an irrotational flow.
(5) There can be no sources or sinks of vorticity in a flow. This fact follows from the following vector identity,

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = 0, \tag{5.297}$$

$$\therefore \boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0. \tag{5.298}$$

Zero divergence of vorticity means that there are no sources or sinks of vorticity, which in turn means that vorticity is neither created nor destroyed in a flow. So how then is vorticity generated in a flow? It must be introduced at a fluid or solid boundary. According to Eq. (5.298), vortex lines must either form closed curves or start and end at boundaries.
(6) Another useful quantity for the discussion of vorticity dynamics is the circulation, Γ (Figure 5.30),

$$\Gamma \coloneqq \oint_C \boldsymbol{u} \cdot d\boldsymbol{s}. \tag{5.299}$$

The relationship between the vorticity and the circulation about a curve, C, enclosing an area, A,

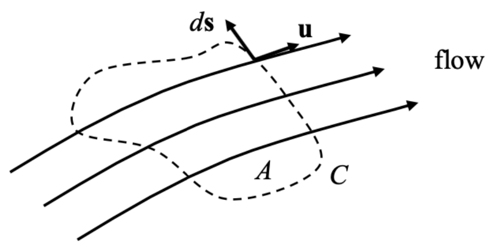


Figure 5.30. A sketch illustrating the concept of circulation.

with unit normal, $\hat{\mathbf{n}}$, is found using Stokes' Theorem,

$$\Gamma = \oint_C \boldsymbol{u} \cdot d\boldsymbol{s} = \int_A (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \hat{\mathbf{n}} dA, \tag{5.300}$$

$$\therefore \Gamma = \int_A \boldsymbol{\omega} \cdot \hat{\mathbf{n}} dA \quad \text{or} \quad \frac{d\Gamma}{dA} = \boldsymbol{\omega} \cdot \hat{\mathbf{n}}. \tag{5.301}$$

(7) The circulation around any cross-section of the same vortex tube remains constant. Recall that,

$$\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0, \tag{5.302}$$

so that,

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{\omega}) dV = 0 \underbrace{\Longrightarrow}_{\text{div thm}} \int_S (\boldsymbol{\omega} \cdot \hat{\mathbf{n}}) dA = 0, \tag{5.303}$$

where V is the volume enclosed within the vortex tube and S is the surface area of this volume.

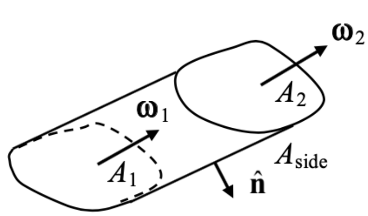


Figure 5.31. The vorticity entering and exiting a vortex tube.

Breaking the total area into the area of the top, bottom, and sides (refer to Figure 5.31),

$$\int_S (\boldsymbol{\omega} \cdot \hat{\mathbf{n}}) dA = 0 = \int_{A_1} (\boldsymbol{\omega} \cdot \hat{\mathbf{n}}) dA + \int_{A_2} (\boldsymbol{\omega} \cdot \hat{\mathbf{n}}) dA + \int_{A_{\text{side}}} (\boldsymbol{\omega} \cdot \hat{\mathbf{n}}) dA. \tag{5.304}$$

On the sides of the vortex tube, the normal vector for the area is perpendicular to the vorticity vectors (from the definition of a vortex tube) so that,

$$(\boldsymbol{\omega} \cdot \hat{\mathbf{n}})_{\text{sides}} = 0. \tag{5.305}$$

Thus,

$$0 = \int_{A_1} (\boldsymbol{\omega} \cdot \hat{\mathbf{n}}) dA + \int_{A_2} (\boldsymbol{\omega} \cdot \hat{\mathbf{n}}) dA, \tag{5.306}$$

$$-\int_{A_1} (\boldsymbol{\omega} \cdot \hat{\mathbf{n}}) dA = \int_{A_2} (\boldsymbol{\omega} \cdot \hat{\mathbf{n}}) dA. \tag{5.307}$$

Using Eq. (5.301) and noting that the outward pointing normal vector on area A_1 points in the opposite direction as the vorticity vector there (Figure 5.31), we have,

$$\Gamma_1 = \Gamma_2. \tag{5.308}$$

Hence, the circulation around any cross-section of the same vortex tube remains constant. This observation is also known as Helmholtz's Third Law.

5.12. Vorticity Transport Equations (aka Helmholtz Equations)

The Vorticity Transport Equations are an alternate expression of the Navier-Stokes Equations. Consider the Navier-Stokes Equations for an incompressible fluid with constant dynamic viscosity,

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{f}. \tag{5.309}$$

Divide through by the density, ρ, (note that it is a constant here since we're considering an incompressible fluid) and also write the body force, $\boldsymbol{f}$, as the gradient of a potential function, G (allowable if $\boldsymbol{f}$ is a conservative body force, i.e., $\boldsymbol{f} = -\boldsymbol{\nabla} G$),

$$\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) + \frac{\mu}{\rho}\nabla^2 \boldsymbol{u} - \boldsymbol{\nabla} G. \tag{5.310}$$

Note that the kinematic viscosity, $\nu = \mu/\rho$, is the ratio of the dynamic viscosity to the density. Now expand the acceleration term on the left-hand side of the equation,

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}. \tag{5.311}$$

The second term in the previous equation can be expanded using the following vector identity,

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times \underbrace{(\boldsymbol{\nabla} \times \boldsymbol{u})}_{=\boldsymbol{\omega}}, \tag{5.312}$$

where $\boldsymbol{\omega}$ is the vorticity. Substituting these relations into Eq. (5.310) gives,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times \boldsymbol{\omega} = -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) + \nu \nabla^2 \boldsymbol{u} - \boldsymbol{\nabla} G. \tag{5.313}$$

Now take the curl of this equation and simplify,

$$\boldsymbol{\nabla} \times \left[\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times \boldsymbol{\omega} = -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) + \nu \nabla^2 \boldsymbol{u} - \boldsymbol{\nabla} G\right], \tag{5.314}$$

where,

$$\nabla \times \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \boldsymbol{u}) = \frac{\partial \boldsymbol{\omega}}{\partial t}, \tag{5.315}$$

$$\nabla \times \frac{1}{2}\nabla(\boldsymbol{u} \cdot \boldsymbol{u}) = \mathbf{0} \quad \text{(From the vector identity } \nabla \times \nabla\phi = \mathbf{0}.\text{)}, \tag{5.316}$$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{u} \underbrace{(\nabla \cdot \boldsymbol{\omega})}_{\substack{=0, \text{ vorticity} \\ \text{is divergence free}}} - \boldsymbol{\omega} \underbrace{(\nabla \cdot \boldsymbol{u})}_{\substack{=0, \text{ Continuity} \\ \text{Equation}}} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} \quad \text{(using a vector identity)}, \tag{5.317}$$

$$\nabla \times \nabla\left(\frac{p}{\rho}\right) = \mathbf{0}, \tag{5.318}$$

$$\nabla \times \nu\nabla^2\boldsymbol{u} = \nu\nabla^2(\nabla \times \boldsymbol{u}) = \nu\nabla^2\boldsymbol{\omega}, \tag{5.319}$$

$$\nabla \times \nabla G = \mathbf{0}. \tag{5.320}$$

Substituting and simplifying,

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}}_{=\frac{D\boldsymbol{\omega}}{Dt}} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \nu\nabla^2\boldsymbol{\omega}, \tag{5.321}$$

$$\boxed{\therefore \frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega}}. \tag{5.322}$$

These are the Vorticity Transport Equations for an incompressible, Newtonian fluid (aka the Helmholtz Equations).

Notes:

(1) Let's interpret what each of the terms in the vorticity transport equation means,

$$\underbrace{\frac{D\boldsymbol{\omega}}{Dt}}_{\substack{\text{rate of change} \\ \text{of fluid element} \\ \text{vorticity}}} = \underbrace{(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}}_{\substack{\text{stretching and} \\ \text{turning of a} \\ \text{vortex line}}} + \underbrace{\nu\nabla^2\boldsymbol{\omega}}_{\substack{\text{diffusion of} \\ \text{vorticity}}}. \tag{5.323}$$

(2) The Vorticity Transport Equations do not contain pressure or body force terms explicitly. Assuming uniform density, the pressure and body forces act through the center of mass of the element and, thus, cannot produce rotation. Only the shear stresses may produce vorticity. Note that in a stratified flow where the density gradient results in a non-coincident geometric center and center of mass, the pressure forces can produce rotation of the fluid element. Hence, Eq. (5.322) should not be used for stratified flows.

(3) The Vorticity Transport Equations are sometimes used in numerical calculations in place of the Navier-Stokes Equations.

(4) For a 2D flow, the Vorticity Transport Equations simplify to,

$$\frac{D\omega}{Dt} = \nu\nabla^2\omega, \tag{5.324}$$

since the vorticity points in a direction perpendicular to the streamlines. Hence, vorticity can only diffuse (and not stretch) in a 2D flow. If the flow is inviscid, then,

$$\frac{D\omega}{Dt} = 0, \tag{5.325}$$

and the vorticity remains constant for each fluid element. Hence, in a 2D inviscid flow, if the flow starts off irrotational, then it must remain irrotational! This result is very important and will be explored more fully when discussing Kelvin's Theorem in Section 5.14.

Consider the momentum equations for the planar flow of an incompressible, inviscid fluid under the action of conservative body forces. By eliminating the pressure, *p*, and using the continuity equation show that:

$$\frac{\partial \omega}{\partial t}+u_x\frac{\partial \omega}{\partial x}+u_y\frac{\partial \omega}{\partial y}=0$$

where ω is the magnitude of the vorticity. What can you conclude about the vorticity of a particular element of fluid in such a flow?

SOLUTION:

The momentum equations for a planar, incompressible, inviscid flow with conservative body forces are:

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho\frac{\partial F}{\partial x} \tag{1}$$

$$\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\rho\frac{\partial F}{\partial y} \tag{2}$$

where F is the potential function describing the force, *i.e.* $\partial F/\partial x_i = f_i$. To eliminate the pressure, take the partial derivative of Eqn. (1) with respect to y and subtract the partial derivative of Eqn. (2) with respect to x.

$$\rho\left(\frac{\partial^2 u_x}{\partial y\partial t}+\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial x}+u_x\frac{\partial^2 u_x}{\partial x\partial y}+\frac{\partial u_y}{\partial y}\frac{\partial u_x}{\partial y}+u_y\frac{\partial^2 u_x}{\partial y^2}\right)-\rho\left(\frac{\partial^2 u_y}{\partial x\partial t}+\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial x}+u_x\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y}+u_y\frac{\partial^2 u_y}{\partial x\partial y}\right)$$

$$=-\frac{\partial^2 p}{\partial x\partial y}+\rho\frac{\partial^2 F}{\partial x\partial y}+\frac{\partial^2 p}{\partial x\partial y}-\rho\frac{\partial^2 F}{\partial x\partial y}$$

$$\rho\left(\frac{\partial^2 u_x}{\partial y\partial t}-\frac{\partial^2 u_y}{\partial x\partial t}+\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial x}-\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial x}+u_x\frac{\partial^2 u_x}{\partial x\partial y}-u_x\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial u_y}{\partial y}\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y}+u_y\frac{\partial^2 u_x}{\partial y^2}-u_y\frac{\partial^2 u_y}{\partial x\partial y}\right)=0$$

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\right)}_{=-\omega}+u_x\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\right)}_{=-\omega}+u_y\frac{\partial}{\partial y}\underbrace{\left(\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\right)}_{=-\omega}+\frac{\partial u_y}{\partial y}\underbrace{\left(\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\right)}_{=-\omega}+\frac{\partial u_x}{\partial x}\underbrace{\left(\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\right)}_{=-\omega}=0$$

$$\frac{\partial \omega}{\partial t}+u_x\frac{\partial \omega}{\partial x}+u_y\frac{\partial \omega}{\partial y}+\omega\underbrace{\left(\frac{\partial u_y}{\partial y}+\frac{\partial u_x}{\partial x}\right)}_{=0\ \text{(from continuity)}}=0$$

$$\boxed{\frac{\partial \omega}{\partial t}+u_x\frac{\partial \omega}{\partial x}+u_y\frac{\partial \omega}{\partial y}=0}\ \text{or}\ \boxed{\frac{D\omega}{Dt}=0} \tag{3}$$

Equation (3) implies that the <u>vorticity of a fluid element in such a flow remains constant</u>.

For a free vortex flow:

a. Calculate the circulation, Γ, for any closed curve not including the origin.
b. Now calculate the circulation for any closed curve that does include the origin.
c. What can you conclude about the vorticity for a free vortex flow?
d. Explain how the vorticity for this flow (at points not including the origin) can be zero yet the flow streamlines can be circles.

SOLUTION:

The stream function, ψ, for a free vortex flow is:

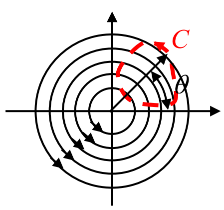
$$\psi = -\frac{\Gamma}{2\pi}\ln r \tag{1}$$

Hence, the velocities are:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \tag{2}$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi}\frac{1}{r} \tag{3}$$

First consider the circulation around a contour, C, that does not include the origin, as shown below.



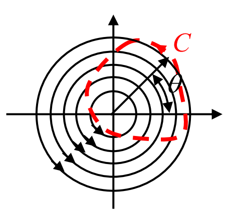
$$\Gamma = \oint_C \mathbf{u}\cdot d\mathbf{s} = \oint_C \left(\frac{\Gamma}{2\pi}\frac{1}{r}\hat{\mathbf{e}}_\theta\right)\cdot\left(dr\hat{\mathbf{e}}_r + rd\theta\hat{\mathbf{e}}_\theta\right)$$

$$\therefore \Gamma = \frac{\Gamma}{2\pi}\oint_C d\theta \tag{4}$$

For a contour that does not include the origin, the starting angle and ending angle will be identical and, hence,

$\boxed{\Gamma = 0}$ (circulation around a contour that does not include the origin) (5)

For a contour that does include the origin, the starting angle and ending angle will be different by 2π.



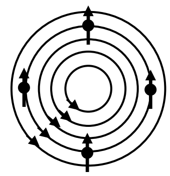
Hence,

$\boxed{\Gamma = \Gamma}$ (circulation around a contour that does include the origin) (6)

A more formal proof of the concepts given above can be found in Churchill, R.V. and Brown, J.W., *Complex Variables and Applications*, McGraw-Hill.

From the results presented above, we can conclude that all of the vorticity in a free vortex is concentrated at the vortex's origin. The free vortex is irrotational everywhere except at the origin.

A flow can have curved streamlines yet be irrotational since the vorticity is twice the rotation rate of a fluid element, not the rotation of the fluid flow as a whole. If a small arrow is attached to a piece of fluid in the free vortex flow (the arrow indicates the fluid element's orientation), the arrow would not rotate as the fluid element moved around the origin.



Since the fluid element doesn't rotate, the flow is irrotational.

Using index notation, prove that the divergence of the vorticity is zero, i.e., $\nabla\cdot\boldsymbol{\omega} = 0$. What is the physical meaning of this statement?

SOLUTION:

$$\nabla\cdot\boldsymbol{\omega} = \nabla\cdot\left(\nabla\times\mathbf{u}\right)$$
$$= \frac{\partial}{\partial x_i}\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$
$$= \varepsilon_{ijk}\frac{\partial^2 u_k}{\partial x_i \partial x_j} = \varepsilon_{ijk}\frac{\partial^2 u_k}{\partial x_j \partial x_i} = \varepsilon_{jik}\frac{\partial^2 u_k}{\partial x_i \partial x_j}$$

However, $\varepsilon_{ijk} = \varepsilon_{jik}$ only when they are equal to zero so that we must have:

$$\nabla\cdot\boldsymbol{\omega} = \varepsilon_{ijk}\frac{\partial^2 u_k}{\partial x_i \partial x_j} = \varepsilon_{jik}\frac{\partial^2 u_k}{\partial x_i \partial x_j} = 0$$

Physically, $\nabla\cdot\boldsymbol{\omega} = 0$ means that there can be no sources or sinks of vorticity in a flow. Vorticity can only be generated or removed at a boundary.

Consider a two-dimensional flow with velocity components $u_x = cx$ and $u_y = -cy$. Find expressions for the vorticity and the deformation rate tensor.

SOLUTION:

The vorticity (in Cartesian coordinates) is given by:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{\mathbf{k}} \tag{1}$$

For the given velocity field:

$$\boxed{\boldsymbol{\omega} = \mathbf{0}} \text{ The flow is irrotational!} \tag{2}$$

The deformation rate tensor is given by:

$$e_{ij} = \underbrace{S_{ij}}_{\text{shearing strain tensor}} + \underbrace{R_{ij}}_{\text{rotation rate tensor}} = \frac{\partial u_i}{\partial x_j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{3}$$

Using the given velocity field:

$$\underline{\underline{\mathbf{e}}} = \begin{bmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Note that since the flow is irrotational, the rotation rate tensor, R_{ij}, is zero and the deformation rate tensor is equal to the shearing strain tensor, S_{ij}.

5.13. Bernoulli's Equation

Euler's Equations, i.e., the Momentum Equations for an inviscid fluid, can be simplified to an expression known as Bernoulli's Equation for the conditions given below.

- Steady flow of an inviscid fluid in a conservative force field along either a streamline or a vortex line,
$$\int \frac{dp}{\rho} + \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}) + G = \text{constant}. \tag{5.326}$$
- Irrotational flow of an inviscid fluid in a conservative force field,
$$\frac{\partial \phi}{\partial t} + \int \frac{dp}{\rho} + \frac{1}{2}\left(\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi\right) + G = F(t), \tag{5.327}$$
where $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ and $F(t)$ is a function only of time.

Derivation of Bernoulli's Equation: To begin, first consider Euler's Equations (recall that Euler's Equations are the Momentum Equations for an inviscid fluid),
$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}G, \tag{5.328}$$
where a conservative body force ($\boldsymbol{f}_B = -\boldsymbol{\nabla}G$) has been assumed. Re-write the convective acceleration term using the following vector identity,
$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) - \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}), \tag{5.329}$$
$$\implies \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) - \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}G. \tag{5.330}$$
Collect gradient terms on the left-hand side of the equation,
$$\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) + \boldsymbol{\nabla}G = -\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}). \tag{5.331}$$
Note that the pressure gradient term can be re-written in a slightly different form,
$$\left(\frac{1}{\rho}\boldsymbol{\nabla}p\right) \cdot d\boldsymbol{x} = \frac{dp}{\rho} = d\left(\int \frac{dp}{\rho}\right) = \boldsymbol{\nabla}\left(\int \frac{dp}{\rho}\right) \cdot d\boldsymbol{x}, \tag{5.332}$$
$$\therefore \frac{1}{\rho}\boldsymbol{\nabla}p = \boldsymbol{\nabla}\left(\int \frac{dp}{\rho}\right). \tag{5.333}$$
Note that in the previous set of equations, the following relationship was used,
$$\boldsymbol{\nabla}a \cdot d\boldsymbol{x} = \left(\frac{\partial a}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial a}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial a}{\partial z}\hat{\mathbf{e}}_z\right), \tag{5.334}$$
$$= \frac{\partial a}{\partial x}dx + \frac{\partial a}{\partial y}dy + \frac{\partial a}{\partial z}dz, \tag{5.335}$$
$$= da. \tag{5.336}$$
Substituting Eq. (5.333) into Eq. (5.331), simplifying, and noting that $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$,
$$\boldsymbol{\nabla}\left(\int \frac{dp}{\rho} + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u} + G\right) = -\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \times \boldsymbol{\omega}. \tag{5.337}$$

Now consider two particular cases.

- *Steady flow along a streamline or vortex line.* A steady flow results in $\partial \boldsymbol{u}/\partial t = \mathbf{0}$. Taking the dot product of Eq. (5.337) with a small length of line $d\boldsymbol{x}$ that is along either streamline or vortex line gives,
$$\boldsymbol{\nabla}\left(\int \frac{dp}{\rho} + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u} + G\right) \cdot d\boldsymbol{x} = (\boldsymbol{u} \times \boldsymbol{\omega}) \cdot d\boldsymbol{x}. \tag{5.338}$$

Since the vector $(\boldsymbol{u}\times\boldsymbol{\omega})$ is perpendicular to both the streamline and vortex line, the dot product with $d\boldsymbol{x}$ will be zero,

$$(\boldsymbol{u}\times\boldsymbol{\omega})\cdot d\boldsymbol{x} = 0. \tag{5.339}$$

Furthermore, the dot product of the gradient on the left-hand side of Eq. (5.338) with $d\boldsymbol{x}$ results in an ordinary differential,

$$\boldsymbol{\nabla}\left(\int\frac{dp}{\rho}+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}+G\right)\cdot d\boldsymbol{x} = d\left(\int\frac{dp}{\rho}+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}+G\right). \tag{5.340}$$

Thus,

$$d\left(\int\frac{dp}{\rho}+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}+G\right) = 0, \tag{5.341}$$

$$\int\frac{dp}{\rho}+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}+G = \text{constant}. \tag{5.342}$$

- *Irrotational Flow.* In an irrotational flow, the velocity can be written as the gradient of a velocity potential function, ϕ, i.e., $\boldsymbol{u}=\boldsymbol{\nabla}\phi$, since in an irrotational flow, $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\boldsymbol{u}=\boldsymbol{0}$, and from the vector identity, $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi=\boldsymbol{0}$. Thus $\boldsymbol{u}$ can be written as $\boldsymbol{u}=\boldsymbol{\nabla}\phi$. Substituting into Eq. (5.337) and noting that the vorticity is zero in an irrotational flow,

$$\boldsymbol{\nabla}\left(\int\frac{dp}{\rho}+\frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi+G\right) = -\frac{\partial(\boldsymbol{\nabla}\phi)}{\partial t} = -\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t}\right). \tag{5.343}$$

Combining gradient terms and simplifying,

$$\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t}+\int\frac{dp}{\rho}+\frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi+G\right) = \boldsymbol{0}. \tag{5.344}$$

Now take the dot product of the previous equation with a short distance in *any* direction, $d\boldsymbol{x}$, and integrate the resulting expression along that path,

$$\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t}+\int\frac{dp}{\rho}+\frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi+G\right)\cdot d\boldsymbol{x} = 0, \tag{5.345}$$

$$\implies d\left(\frac{\partial\phi}{\partial t}+\int\frac{dp}{\rho}+\frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi+G\right) = 0, \tag{5.346}$$

$$\frac{\partial\phi}{\partial t}+\int\frac{dp}{\rho}+\frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi+G = F(t), \tag{5.347}$$

where $F(t)$ is a function only of time. This term is introduced in the integration step since the terms in the equation may vary with both position and time.

Notes:

(1) For a fluid with constant density, i.e., $\rho=\text{constant}$,

$$\int\frac{dp}{\rho} = \frac{1}{\rho}\int dp, \tag{5.348}$$

$$\therefore \int\frac{dp}{\rho} = \frac{p}{\rho}. \tag{5.349}$$

(2) For an ideal gas ($p=\rho RT$):

(a) *Isothermal case:*

$$\int\frac{dp}{\rho} = \int\frac{d(\rho RT_0)}{\rho} = RT_0\int\frac{d\rho}{\rho}, \tag{5.350}$$

$$\therefore \int\frac{dp}{\rho} = RT_0\ln\left(\frac{\rho}{\rho_0}\right), \tag{5.351}$$

where ρ_0 and T_0 are a reference density and temperature, respectively.

(b) *Isentropic case:* Recall that for an ideal gas undergoing an isentropic process,

$$0 = c_p(T)\frac{dT}{T} - R\frac{dp}{p}, \tag{5.352}$$

$$dp = \frac{c_p(T)}{R}(\rho R T)\frac{dT}{T}, \tag{5.353}$$

$$dp = \rho c_p(T) dT. \tag{5.354}$$

Thus,

$$\int \frac{dp}{\rho} = \int c_p(T) dT = \int dh = \Delta h, \tag{5.355}$$

where h is the specific enthalpy. If the ideal gas has constant specific heats, i.e., is a "perfect" gas, then, $\Delta h = c_p \Delta T$ and,

$$\int \frac{dp}{\rho} = c_p \Delta T. \tag{5.356}$$

5.13.1. Another Approach to Deriving Bernoulli's Equation

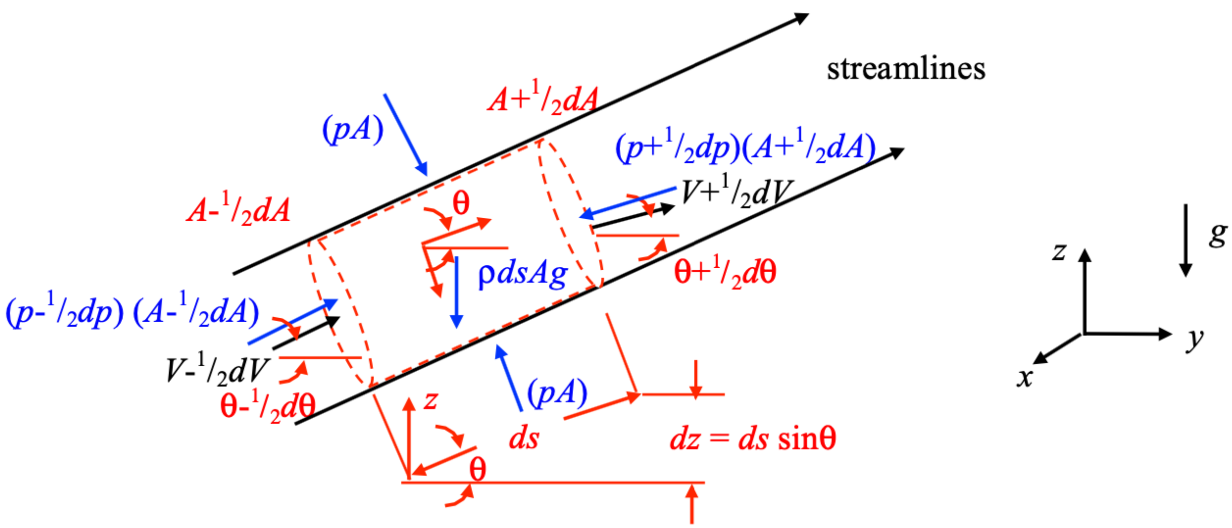


Figure 5.32. The differential control volume used to derive Bernoulli's Equation.

We can also derive Bernoulli's Equation using the Linear Momentum Equations and Conservation of Mass applied to a differential control volume as shown in Figure 5.32. Note that the control volume shown in the figure follows the streamlines. In the following analysis, we'll make the following simplifying assumptions:

(1) steady flow,
(2) inviscid flow, and
(3) incompressible fluid.

First apply Conservation of Mass to the control volume,

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0, \tag{5.357}$$

where,

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}, \tag{5.358}$$

$$\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \rho \left(V + \frac{1}{2}dV\right)\left(A + \frac{1}{2}dA\right) - \rho\left(V - \frac{1}{2}dV\right)\left(A - \frac{1}{2}dA\right), \tag{5.359}$$

$$= \rho V dA + \rho A dV + \text{H.O.T.s} \tag{5.360}$$

Note that there's no flow across the streamlines. Substituting these expressions into Conservation of Mass gives,

$$V dA = -A dV. \tag{5.361}$$

Now apply the Linear Momentum Equation to the same control volume in the streamline direction,

$$\frac{d}{dt}\int_{CV} u_s \rho dV + \int_{CS} u_s (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = F_{B,s} + F_{S,s}, \tag{5.362}$$

where,

$$\frac{d}{dt}\int_{CV} u_s \rho dV = 0 \quad \text{(steady flow)}, \tag{5.363}$$

$$\int_{CS} u_s (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = -\rho\left(V - \frac{1}{2}dV\right)^2\left(A - \frac{1}{2}dA\right) + \rho\left(V + \frac{1}{2}dV\right)^2\left(A + \frac{1}{2}dA\right), \tag{5.364}$$

$$= 2\rho V A dV + \rho V^2 dA + \text{H.O.T.s}, \tag{5.365}$$

$$F_{B,s} = \rho ds A(-g\sin\theta) = -\rho A g \underbrace{ds\sin\theta}_{=dz} = -\rho A g dz, \tag{5.366}$$

$$F_{S,s} = \left(p - \frac{1}{2}dp\right)\left(A - \frac{1}{2}dA\right) - \left(p + \frac{1}{2}dp\right)\left(A + \frac{1}{2}dA\right) + p dA, \tag{5.367}$$

$$= -A dp + \text{H.O.T.s}, \tag{5.368}$$

Combining these terms together into the Linear Momentum Equation,

$$2\rho V A dV + \rho V^2 dA = -\rho A g dz - A dp. \tag{5.369}$$

Now substitute the result from Conservation of Mass into the result from the Linear Momentum Equation and simplify,

$$2\rho V A dV + \underbrace{\rho V^2 dA}_{=-\rho V A dV} = -\rho A g dz - A dp, \tag{5.370}$$

$$\frac{dp}{\rho} + V dV + g dz = 0. \tag{5.371}$$

We can integrate this equation along the streamline to get,

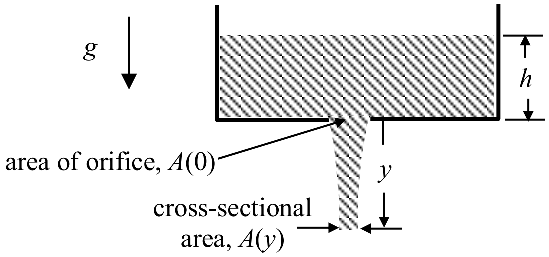
$$\boxed{\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{constant}} \tag{5.372}$$

Again, it's important to review the assumptions builtin to the derivation of Eq. (5.372):

(1) steady flow,
(2) inviscid flow,
(3) incompressible fluid, and
(4) flow along a streamline.

Video solution: https://www.youtube.com/watch?v=dzmDzfNoMdA

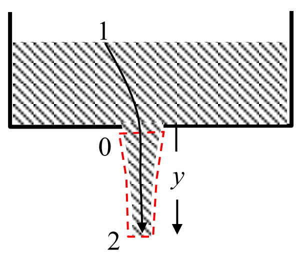
A water tank has an orifice in the bottom of the tank:



The height, h, of water in the tank is kept constant by a supply of water which is not shown. A jet of water emerges from the orifice; the cross-sectional area of the jet, $A(y)$, is a function of the vertical distance, y. Neglecting viscous effects and surface tension, find an expression for $A(y)$ in terms of $A(0)$, h, and y.

SOLUTION:

Apply Conservation of Mass to the following CV:



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0$$

where,

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(The flow is steady.)}$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho V_0 A_0 + \rho V_2 A_2$$

Substitute and simplify,

$$V_2 = V_0 \frac{A_0}{A_2} \tag{1}$$

Now apply Bernoulli's Equation from point 1 to point 0 and from point 1 to point 2,

$$\left(p + \tfrac{1}{2}\rho V^2 - \rho g y\right)_1 = \left(p + \tfrac{1}{2}\rho V^2 - \rho g y\right)_0 = \left(p + \tfrac{1}{2}\rho V^2 - \rho g y\right)_2$$

where,

$p_1 = p_0 = p_2 = p_{\text{atm}}$ (These points are all at free surfaces.)

$V_1 = 0$ and V_0 and V_2 are related through Eq. (1).

$y_1 = -h, y_0 = 0, y_2 = y$

Substitute and simplify,

$$\rho g h = \tfrac{1}{2}\rho V_0^2 = \tfrac{1}{2}\rho V_2^2 - \rho g y$$

$$\rho g h = \tfrac{1}{2}\rho V_0^2 = \tfrac{1}{2}\rho V_0^2 \left(\frac{A_0}{A_2}\right)^2 - \rho g y$$

The first two equations in the previous expression state that,

$$V_0 = \sqrt{2gh} \tag{2}$$

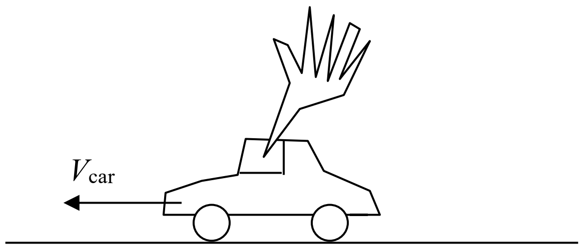
Equation (2) combined with the second two equations gives,

$$\left(\frac{A_0}{A_2}\right)^2 = 1 + \frac{\rho g y}{\tfrac{1}{2}\rho V_0^2}$$

$$\left(\frac{A_0}{A_2}\right)^2 = 1 + \frac{\rho g y}{\rho g h} = 1 + \frac{y}{h}$$

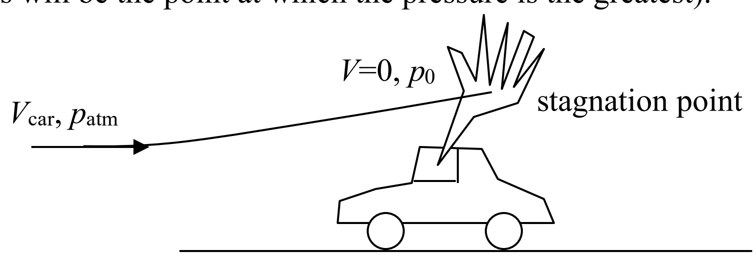
$$\boxed{\frac{A_2}{A_0} = \frac{1}{\sqrt{1 + \dfrac{y}{h}}}}$$

A person holds their hand out of a car window while driving through still air at a speed of V_{car}. What is the maximum pressure on the person's hand?



SOLUTION:

Change the frame of reference so that the car is stationary and the air approaches the car at a velocity, V_{car}. Apply Bernoulli's equation, neglecting elevation differences, along a streamline from a point far upstream of the car to the stagnation point on the person's hand (this will be the point at which the pressure is the greatest).

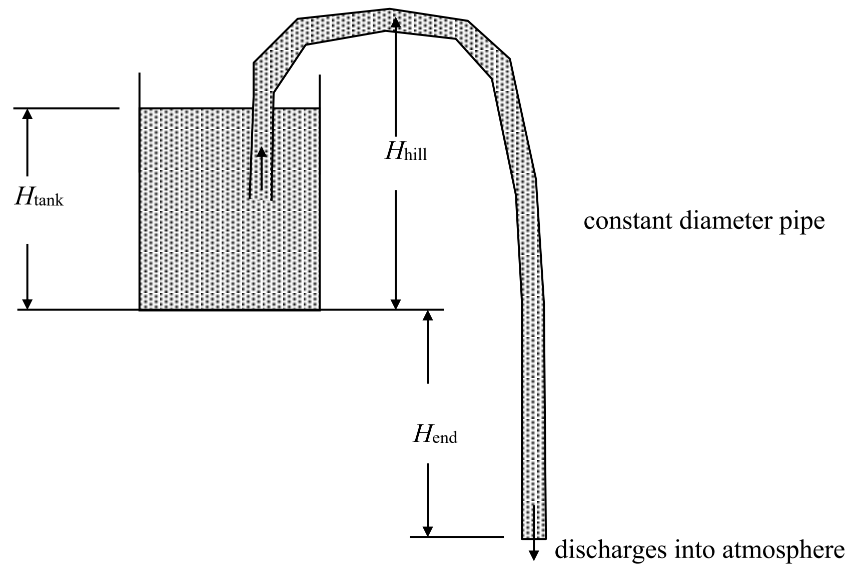


$$p_{\text{atm}} + \tfrac{1}{2}\rho V_{\text{car}}^2 = p_0 + \tfrac{1}{2}\rho \underbrace{V_0^2}_{=0}$$

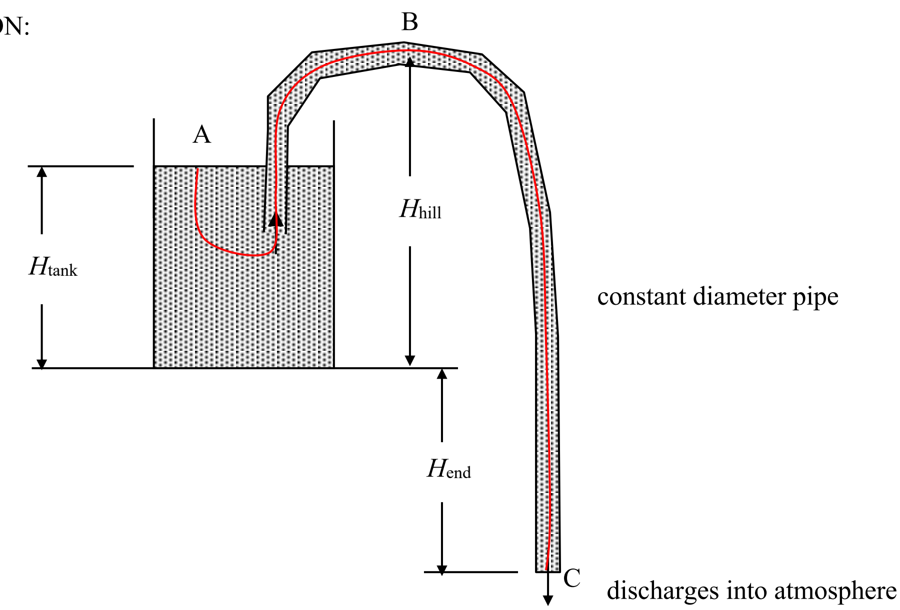
$$\boxed{p_0 = p_{\max} = p_{\text{atm}} + \tfrac{1}{2}\rho V_{\text{car}}^2} \tag{1}$$

Video solution: https://www.youtube.com/watch?v=PDtuZK7C-60

Water is siphoned from a large tank through a constant diameter hose as shown in the figure. Determine the maximum height of the hill, H_{hill}, over which the water can be siphoned without cavitation occurring. Assume that the vapor pressure of the water is p_v, the height of the water free surface in the tank is H_{tank}, and the vertical distance from the end of the hose to the base of the tank is H_{end}.



SOLUTION:



Apply Bernoulli's equation along a streamline from the tank free surface (point A) to the end of the tube (point C).

$$\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_A=\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_C \tag{1}$$

where

$p_A = p_C = p_{\text{atm}}$

$V_A \approx 0$ (free surface of a large tank)

$z_A - z_C = H_{\text{tank}} + H_{\text{end}}$

Solving Eqn. (1) for V_C gives:

$$V_C=\sqrt{2g\left(H_{\text{tank}}+H_{\text{end}}\right)} \tag{2}$$

Now apply Bernoulli's equation along a streamline from the tank free surface (point A) to the top of the tube (point B). Note that the velocity everywhere within the tube will be equal to V_C (from conservation of mass).

$$\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_A=\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_B \tag{3}$$

where

$p_A = p_{\text{atm}}$

$p_B = p_{\text{v}}$

(From Eqn. (3) we see that the pressure at point B will decrease as H_{hill} increases so we should use the smallest allowable pressure at point B to determine the maximum H_{hill}.)

$V_A \approx 0$ (free surface of a large tank)

$V_B = V_C = \sqrt{2g\left(H_{\text{tank}}+H_{\text{end}}\right)}$ (from conservation of mass)

$z_A - z_B = H_{\text{tank}} - H_{\text{hill}}$

Substituting into Eqn. (3) and solving for H_{hill} gives:

$$\frac{p_{\text{atm}}}{\rho g}+H_{\text{tank}}-H_{\text{hill}}=\frac{p_v}{\rho g}+H_{\text{tank}}+H_{\text{end}}$$

$$\boxed{H_{\text{hill}}=\frac{p_{\text{atm}}-p_v}{\rho g}-H_{\text{end}}} \tag{4}$$

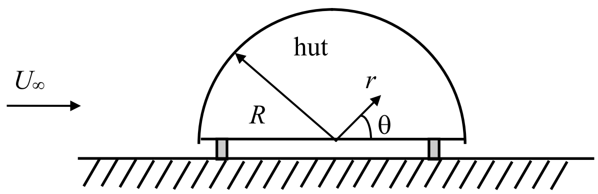
Video solution: https://www.youtube.com/watch?v=2wHc99esfoA

You are to design Quonset huts for a military base. The design wind speed is $U_\infty = 30$ m/s and the free-stream pressure and density are $p_\infty = 101$ kPa and $\rho_\infty = 1.2$ kg/m^3, respectively. The Quonset hut may be considered to be a closed (no leaks) semi-cylinder with a radius of $R = 5$ m which is mounted on tie-down blocks as shown in the figure. The flow is such that the velocity distribution over the top of the hut is approximated by:

$$u_r(r = R) = 0$$

$$u_\theta(r = R) = -2U_\infty \sin\theta$$

The air under the hut is at rest.



a. What is the pressure distribution over the top surface of the Quonset hut?
b. What is the net lift force acting on the Quonset hut due to the air? Don't forget to include the effect of the air under the hut.
c. What is the net drag force acting on the hut? (Hint: A calculation may not be necessary here but some justification is required.)

SOLUTION:

Apply Bernoulli's equation over a streamline adjacent to the upper surface of the hut to determine the pressure distribution. Neglect elevation effects since the fluid is a gas and the elevation differences are small.

$$\left(p + \tfrac{1}{2}\rho V^2\right)_\infty = \left(p + \tfrac{1}{2}\rho V^2\right)_{\text{surface}} \tag{1}$$

where

$$p_\infty = 101 \text{ kPa}$$

$$V_\infty^2 = U_\infty^2 = (30 \text{ m/s})^2 = 900 \text{ m}^2/\text{s}^2$$

$$p_{\text{surface}} = ?$$

$$V_{\text{surface}}^2 = u_\theta^2 = 4U_\infty^2 \sin^2\theta \quad (0 \le \theta \le \pi)$$

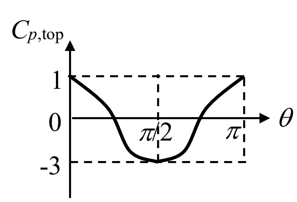
$$\rho = 1.2 \text{ kg/m}^3$$

Substitute and solve for the pressure on the hut's upper surface.

$$p_{\text{surface}} = p_\infty + \tfrac{1}{2}\rho\left(V_\infty^2 - V_{\text{surface}}^2\right)$$

$$p_{\text{surface}} = p_\infty + \tfrac{1}{2}\rho U_\infty^2\left(1 - 4\sin^2\theta\right)$$

$$\boxed{C_{p,\text{top}} = \frac{p_{\text{surface}} - p_\infty}{\tfrac{1}{2}\rho U_\infty^2} = 1 - 4\sin^2\theta} \tag{2}$$

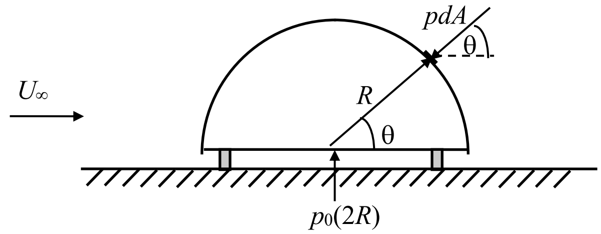


where C_p is known as a "pressure coefficient."

The pressure under the hut will be the stagnation pressure. It can also be found by applying Bernoulli's equation and noting that under the hut the velocity is zero.

$$C_{p,\text{bottom}} = \frac{p_0 - p_\infty}{\tfrac{1}{2}\rho U_\infty^2} = 1 \tag{3}$$

The net lift force is determined by integrating the vertical component of the pressure forces over the entire surface of the hut.



$$L = \underbrace{\int_{\theta=0}^{\theta=\pi} (-p\sin\theta)\underbrace{(Rd\theta)}_{=dA}}_{\text{top}} + \underbrace{p_0(2R)}_{\text{bottom}} \quad \text{(Note that positive lift is directed upwards.)} \tag{4}$$

where p_0 is the stagnation pressure.

$$L = -\int_{\theta=0}^{\theta=\pi}\left[p_\infty + \tfrac{1}{2}\rho U_\infty^2\left(1-4\sin^2\theta\right)\right]\sin\theta R d\theta + \left(p_\infty + \tfrac{1}{2}\rho U_\infty^2\right)(2R)$$

$$C_L = \frac{L}{\tfrac{1}{2}\rho U_\infty^2(2R)} = -\frac{1}{2}\int_{\theta=0}^{\theta=\pi}\left[\frac{p_\infty}{\tfrac{1}{2}\rho U_\infty^2} + \left(1-4\sin^2\theta\right)\right]\sin\theta d\theta + \left(\frac{p_\infty}{\tfrac{1}{2}\rho U_\infty^2}+1\right)$$

where C_L is a "lift coefficient."

$$C_L = -\frac{1}{2}\frac{p_\infty}{\tfrac{1}{2}\rho U_\infty^2}\underbrace{\int_{\theta=0}^{\theta=\pi}\sin\theta\, d\theta}_{=-\cos\theta|_0^\pi=1+1=2} - \frac{1}{2}\int_{\theta=0}^{\theta=\pi}\left(\sin\theta - 4\sin^3\theta\right)d\theta + \frac{p_\infty}{\tfrac{1}{2}\rho U_\infty^2}+1$$

$$= -\frac{p_\infty}{\tfrac{1}{2}\rho U_\infty^2} - \frac{1}{2}\underbrace{\int_{\theta=0}^{\theta=\pi}\sin\theta\, d\theta}_{=-\cos\theta|_0^\pi=1+1=2} + 2\underbrace{\int_{\theta=0}^{\theta=\pi}\sin^3\theta\, d\theta}_{=-\frac{1}{3}\left(2+\sin^2\theta\right)\cos\theta|_0^\pi=\frac{2}{3}+\frac{2}{3}=\frac{4}{3}} + \frac{p_\infty}{\tfrac{1}{2}\rho U_\infty^2}+1$$

$$\boxed{C_L = \frac{L}{\tfrac{1}{2}\rho U_\infty^2(2R)} = \frac{8}{3}} \tag{5}$$

The net drag force is determined by integrating the horizontal component of the pressure forces over the entire surface of the hut.

$$D = \int_{\theta=0}^{\theta=\pi}(-p\cos\theta)(Rd\theta) \tag{6}$$

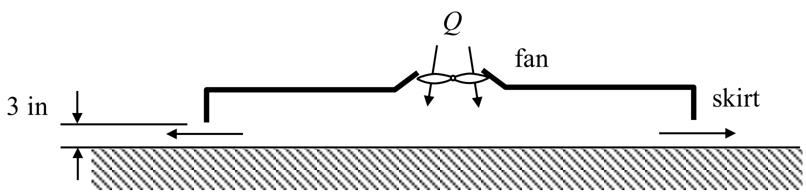
$$D = -\int_{\theta=0}^{\theta=\pi}\left[p_\infty + \tfrac{1}{2}\rho U_\infty^2\left(1-4\sin^2\theta\right)\right]\cos\theta R\, d\theta$$

$$= -Rp_\infty\underbrace{\int_{\theta=0}^{\theta=\pi}\cos\theta\, d\theta}_{=\sin\theta|_0^\pi=0} - \tfrac{1}{2}\rho U_\infty^2 R\left[\underbrace{\int_{\theta=0}^{\theta=\pi}\cos\theta\, d\theta}_{=\sin\theta|_0^\pi=0} - 4\underbrace{\int_{\theta=0}^{\theta=\pi}\sin^2\theta\cos\theta\, d\theta}_{=\frac{1}{3}\sin^3\theta|_0^\pi=0}\right]$$

$$\boxed{\therefore D = 0} \tag{7}$$

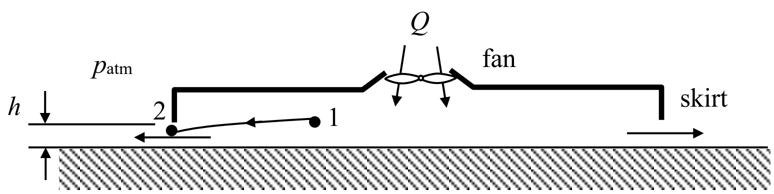
We could have also anticipated that the drag would be zero since the velocity field is symmetric between the upstream and downstream sides of the hut.

An air cushion vehicle is supported by forcing air into the chamber created by a skirt around the periphery of the vehicle as shown. The air escapes through the 3 in. clearance between the lower end of the skirt and the ground (or water). Assume the vehicle weighs 10,000 lb_f and is essentially rectangular in shape, 30 by 50 ft. The volume of the chamber is large enough so that the kinetic energy of the air within the chamber is negligible. Determine the flowrate, Q, needed to support the vehicle.



SOLUTION:

The weight of the vehicle is supported by the increased pressure within the chamber.



A simple force balance gives:

$$W = (p_1 - p_{atm}) A_{projected} \tag{1}$$

Note that we have neglected the downward momentum flux of the air caused by the fan since it will be negligible when compared to the weight of the vehicle.

The pressure within the chamber, p_1, can be found using Bernoulli's equation applied along the streamline shown in the previous figure.

$$\left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_2 = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_1 \tag{2}$$

where

$p_2 = p_{atm}$ $\qquad$ $p_1 = ?$

$V_2 = \dfrac{Q}{A_{skirt}}$ $\qquad$ $V_1 \approx 0$ (large chamber)

$z_2 \approx z_1$ (Elevation differences are negligible, especially since a gas is being considered.)

Substitute and simplify.

$$p_1 - p_{atm} = \tfrac{1}{2}\rho\left(\frac{Q}{A_{skirt}}\right)^2 \tag{3}$$

Substitute Eqn. (3) into Eqn. (1) and solve for the flow rate Q.

$$W = \tfrac{1}{2}\rho\left(\frac{Q}{A_{skirt}}\right)^2 A_{projected}$$

$$\boxed{Q = \sqrt{\frac{2WA_{skirt}^2}{\rho A_{projected}}}} \tag{4}$$

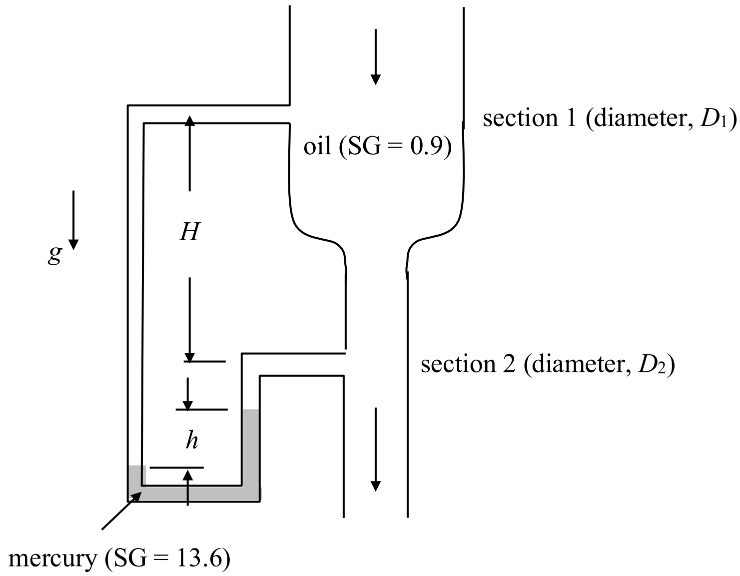
Substitute the given parameters.

$$W = 10000 \text{ lb}_f = 322{,}000 \text{ lb}_m\text{ft/s}^2$$
$$A_{skirt} = (3 \text{ in.})(\text{ft}/12 \text{ in.})[2(30 \text{ ft} + 50 \text{ ft})] = 40 \text{ ft}^2 \quad \text{(rectangular cross-section)}$$
$$\rho = 7.68\text{e-2 lb}_m/\text{ft}^3$$
$$A_{projected} = (30 \text{ ft})(50 \text{ ft}) = 1500 \text{ ft}^2 \quad \text{(rectangular cross-section)}$$
$$\Rightarrow \boxed{Q = 2990 \text{ ft}^3/\text{s}}$$

Video solution: https://www.youtube.com/watch?v=0yg7Vd_mFQc

Oil flows through a contraction with circular cross-section as shown in the figure below. A manometer, using mercury as the gage fluid, is used to measure the pressure difference between sections 1 and 2 of the pipe. Assuming frictionless flow, determine:

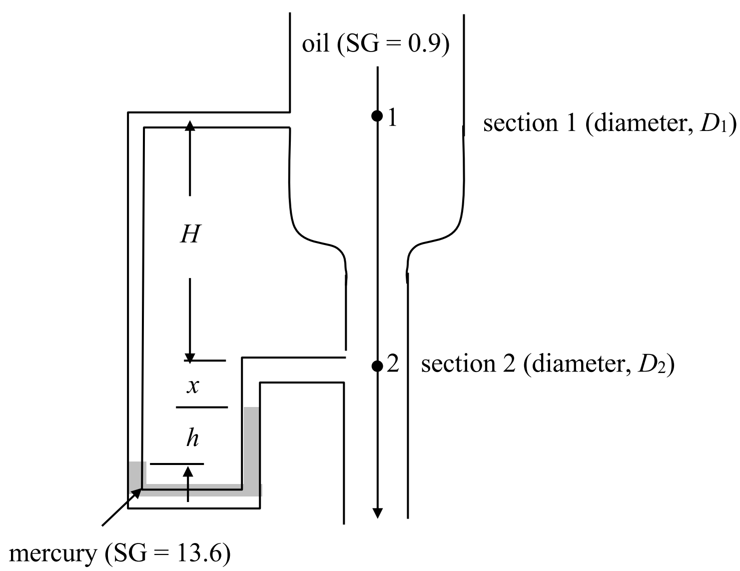
a. the pressure difference, p_1-p_2, between sections 1 and 2, and
b. the mass flow rate through the pipe.



D_1 = 300 mm
D_2 = 100 mm
H = 600 mm
h = 100 mm

SOLUTION:

First determine the pressure difference using the manometer.



$$p_2 = p_1 + \rho_{\text{oil}} g\left(H + x + h\right) - \rho_{\text{Hg}} gh - \rho_{\text{oil}} gx$$
$$p_2 = p_1 + SG_{\text{oil}} \rho_{\text{H20}} g\left(H + h\right) - SG_{\text{Hg}} \rho_{\text{H20}} gh$$
$$\boxed{p_1 - p_2 = \rho_{\text{H20}} g\left[SG_{\text{Hg}} h - SG_{\text{oil}}\left(H + h\right)\right]} \tag{1}$$

Use the given parameters.

ρ_{H20}	=	1000 kg/m^3
g	=	9.81 m/s^2
SG_{Hg}	=	13.6
h	=	100e-3 m
SG_{oil}	=	0.9
H	=	600e-3 m
$\Rightarrow$		$\boxed{p_1 - p_2 = 7.2 \text{ kPa}}$

Now apply Bernoulli's equation along a streamline from 1 to 2 to determine the mass flow rate.

$$\left(\frac{p}{\rho_{\text{oil}}g}+\frac{V^2}{2g}+z\right)_2=\left(\frac{p}{\rho_{\text{oil}}g}+\frac{V^2}{2g}+z\right)_1$$

where

$$p_2 - p_1 = 7200 \text{ N/m}^2 \quad \text{(found previously)}$$

$$V_2=\frac{Q}{\frac{\pi D_2^2}{4}}=\frac{4Q}{\pi D_2^2} \qquad V_1=\frac{Q}{\frac{\pi D_1^2}{4}}=\frac{4Q}{\pi D_1^2}$$

$$z_1 - z_2 = H$$

Substitute and simplify.

$$\frac{p_2-p_1}{\rho_{\text{oil}}g}-H=\frac{8Q^2}{\pi^2 g}\left(\frac{1}{D_1^4}-\frac{1}{D_2^4}\right)$$

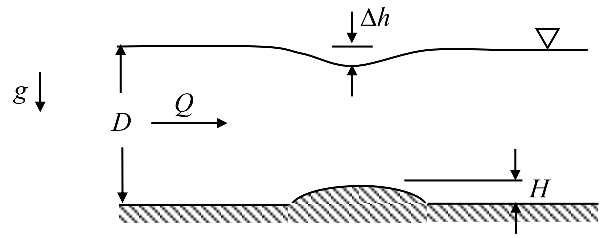
$$\boxed{\dot{m}_{\text{oil}}=\rho_{\text{oil}}Q=\rho_{\text{oil}}\sqrt{\frac{\pi^2 g}{8}\left(\frac{D_1^4D_2^4}{D_2^4-D_1^4}\right)\left(\frac{p_2-p_1}{\rho_{\text{oil}}g}-H\right)}} \tag{2}$$

Use the given parameters.

ρ_{H20}	=	1000 kg/m^3
SG_{oil}	=	0.9
g	=	9.81 m/s^2
H	=	600e-3 m
D_1	=	300e-3 m
D_2	=	100e-3 m
$p_1 - p_2$	=	7200 N/m^2
$\Rightarrow$		$\boxed{\dot{m} = 37.5 \text{ kg/s}}$

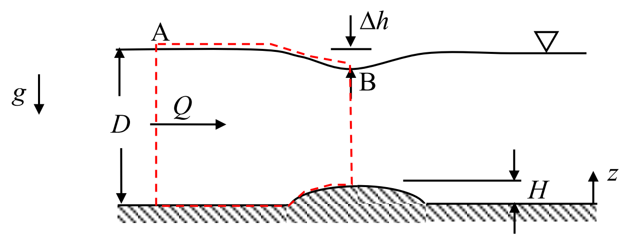
Video solution: https://www.youtube.com/watch?v=P-05A4AWeb4

If the approach velocity is not too large, a hump of height, H, in the bottom of a water channel will cause a dip of magnitude Δh in the water level. This depression in the water can be used to determine the flow rate of the water. Assuming no losses and that the incoming flow has a depth, D, determine the volumetric flow rate, Q, as a function of Δh, H, D, and g (the acceleration due to gravity).



SOLUTION:

Assume steady, incompressible, inviscid flow with uniform velocity profiles at the inlet and outlet of the control volume.



Apply Bernoulli's Equation along a streamline on the free surface from point A to point B.

$$\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_B=\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_A \tag{1}$$

where

$$p_B = p_A = p_{\text{atm}} \tag{2}$$

$$V_B=\frac{Q}{D-H-\Delta h} \quad \text{and} \quad V_A=\frac{Q}{D} \tag{3}$$

$$z_B = D-\Delta h \quad \text{and} \quad z_A = D \tag{4}$$

Substitute and simplify.

$$\frac{p_{\text{atm}}}{\rho g}+\frac{1}{2g}\left(\frac{Q}{D-H-\Delta h}\right)^2+D-\Delta h=\frac{p_{\text{atm}}}{\rho g}+\frac{1}{2g}\left(\frac{Q}{D}\right)^2+D \tag{5}$$

$$\left(\frac{Q}{D-H-\Delta h}\right)^2-2g\Delta h=\left(\frac{Q}{D}\right)^2$$

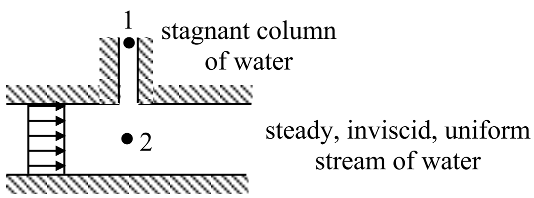
$$\boxed{\therefore Q=\sqrt{\frac{2g\Delta h}{\left(\frac{1}{D-H-\Delta h}\right)^2-\left(\frac{1}{D}\right)^2}}} \tag{6}$$

In which of the following scenarios is applying the following form of Bernoulli's equation:

$$\frac{p}{\rho g}+\frac{V^2}{2g}+z=\text{constant}$$

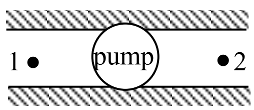
from point 1 to point 2 valid?

a.

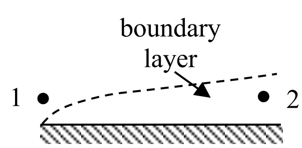


b.

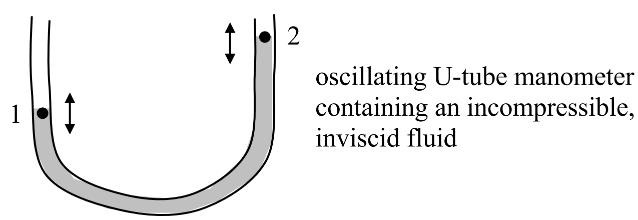
c.



d.



e.



SOLUTION:

Bernoulli's equation, as written in the problem statement, can be used in <u>NONE</u> of the scenarios presented.

a. The flow is rotational at the interface between the vertical and horizontal channels and, hence, Bernoulli's equation cannot be applied across the flow streamlines.

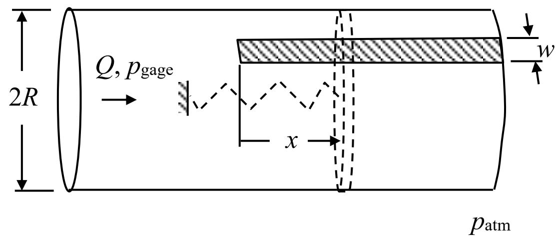
b. Since Ma > 0.3, the flow should be considered compressible. The given form of Bernoulli's equation is valid only for incompressible flows. An alternate form of Bernoulli's equation that takes compressibility effects into account could be used, however.

c. The pump between points 1 and 2 adds energy to the flow and, hence, the constant in Bernoulli's equation changes across the pump. The Extended Bernoulli's Equation (aka energy equation) could be used in this scenario instead of the given form of Bernoulli's equation.

d. Bernoulli's equation assumes inviscid flow. Viscous effects are significant in boundary layers and thus Bernoulli's equation may not be used.

e. The given form of Bernoulli's equation assumes steady flow. The oscillating U-tube is unsteady and the given Bernoulli's equation cannot be used. Note that it is possible to derive an unsteady form of Bernoulli's equation that could be used in the given situation.

The device shown in the figure below is proposed for measuring the exhalation pressure and volume flow rate of a person (the device is known as a "peak flow meter"). A circular tube, with inside radius R, has a slit of width w running down the length of it (a cut-out in the cylinder). Inside the tube is a lightweight, freely moving piston attached to a linear spring (with spring constant k). The equilibrium position of the piston is at $x = 0$ where the slit begins.



Derive equations for:

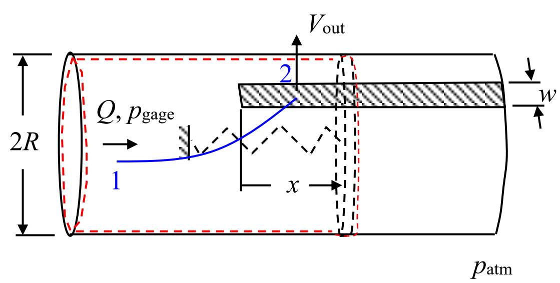
a. the volumetric flow rate, Q, and

b. the gage pressure in the tube, p_{gage},

in terms of (a subset of) the piston displacement, x, as well as the tube radius, R, slit width, w, spring constant, k, and the properties of air. Assume that the slit width, w, is so small that the outflow area is much smaller than the tube's cross-sectional area, πR^2, even at the piston's full extension.

SOLUTION:

Apply conservation of mass to the control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(at steady state)} \tag{2}$$

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho Q + \rho V_{\text{out}} wx \tag{3}$$

Substitute and simply to get:

$$-\rho Q + \rho V_{\text{out}} wx = 0 \tag{4}$$

$$Q = V_{\text{out}} wx \tag{5}$$

where V_{out} is the speed of the air flowing out of the slit. This speed may be found by applying Bernoulli's equation from a point located within the tube (1) and a point just at the slit exit (2).

$$\left(p + \tfrac{1}{2}\rho V^2\right)_1 = \left(p + \tfrac{1}{2}\rho V^2\right)_2 \tag{6}$$

where

$$p_1 = p_{\text{gage}} \tag{7}$$

$$p_2 = 0 \ (p_{\text{atm,gage}} = 0) \tag{8}$$

$$V_1 = Q/(\pi R^2) \tag{9}$$

$$V_2 = V_{\text{out}} \tag{10}$$

Since the slit area is much smaller than the outlet area, $V_1 << V_2$, Eqn. (6) becomes

$$V_{\text{out}} = \sqrt{\frac{2p_{\text{gage}}}{\rho}}\,. \tag{11}$$

Substituting into Eqn. (5) gives:

$$Q = wx\sqrt{\frac{2p_{\text{gage}}}{\rho}} \tag{12}$$

The pressure, p_{gage}, may be found by balancing forces on the piston:

$$p_{\text{gage}}\pi R^2 - kx = 0 \tag{13}$$

$$\boxed{p_{\text{gage}} = \frac{kx}{\pi R^2}} \tag{14}$$

Note that we could have used the linear momentum equation in the x-direction on the same control volume to arrive at this expression (see below).

Combining Eqns. (12) and (14) gives:

$$\boxed{Q = wx^{3/2}\sqrt{\frac{2k}{\rho\pi R^2}}} \tag{15}$$

Thus, by measure the displacement of the piston on the simple device shown in the figure, lung functions such as pressure and volumetric flow rate can be easily determined.

Note that we could have also worked out Eq. (14) of this problem using the linear momentum equation in the x-direction applied to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho\, dV + \int_{\text{CS}} u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{16}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho\, dV = 0 \quad \text{(steady flow)} \tag{17}$$

$$\int_{\text{CS}} u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \frac{Q}{\pi R^2}\left(-\rho\frac{Q}{\pi R^2}\pi R^2\right) = -\rho\frac{Q^2}{\pi R^2} \quad \text{(no } x\text{-momentum flux out through side)} \tag{18}$$

$$F_{B,x} = 0 \tag{19}$$

$$F_{S,x} = p_{\text{gage}}\pi R^2 - kx \tag{20}$$

Substitute and simplify.

$$-\rho\frac{Q^2}{\pi R^2} = p_{\text{gage}}\pi R^2 - kx \tag{21}$$

Substituting from Eq. (12) ,

$$-\rho\frac{(wx)^2\left(2p_{\text{gage}}/\rho\right)}{\pi R^2} = p_{\text{gage}}\pi R^2 - kx \tag{22}$$

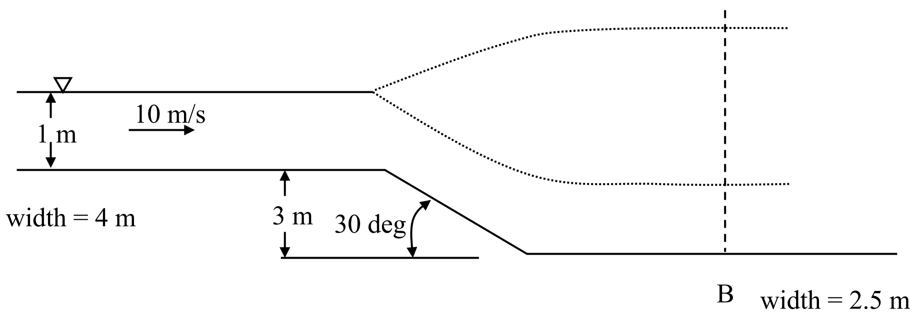
$$-\left[2\left(\frac{wx}{\pi R^2}\right)^2 + 1\right]\pi R^2 p_{\text{gage}} = -kx \tag{23}$$

But since $wx \ll \pi R^2$,

$$p_{\text{gage}} = \frac{kx}{\pi R^2} \quad \text{which is the same as Eq. (14)!} \tag{24}$$

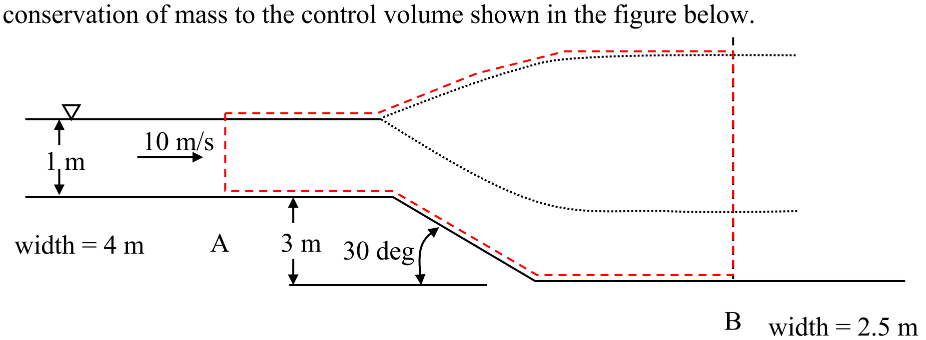
Water 1 m deep is flowing steadily at 10 m/s in a channel 4 m wide. The channel drops 3 m at 30 deg, and simultaneously narrows to 2.5 m as shown in the accompanying sketch.

Determine the two possible water depths at downstream station B. Neglect all losses.



SOLUTION:

Apply conservation of mass to the control volume shown in the figure below.



$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow)} \tag{2}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\left(\rho V z w\right)_A + \left(\rho V z w\right)_B \tag{3}$$

Substitute and simplify noting that the water density remains constant.

$$-\left(\rho V z w\right)_A + \left(\rho V z w\right)_B = 0 \;\Rightarrow\; \left(V z w\right)_B = \left(V z w\right)_A \;\Rightarrow\; V_B = V_A\left(\frac{z_A}{z_B}\right)\left(\frac{w_A}{w_B}\right) \tag{4}$$

Now apply Bernoulli's equation along the free surface of the stream from point A to point B.

$$\left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_B = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_A \tag{5}$$

where

$$p_A = p_B = p_{\text{atm}} \tag{6}$$

$$\frac{V_B^2}{2g} + z_B = \frac{V_A^2}{2g} + z_A \tag{7}$$

Substitute Eq. (4) and solve for z_B.

$$\frac{V_A^2 \left(\frac{z_A}{z_B}\right)^2 \left(\frac{w_A}{w_B}\right)^2}{2g} + z_B = \frac{V_A^2}{2g} + z_A \tag{8}$$

$$V_A^2 \left(\frac{z_A}{z_B}\right)^2 \left(\frac{w_A}{w_B}\right)^2 + 2gz_B = V_A^2 + 2gz_A \tag{9}$$

$$V_A^2 z_A^2 \left(\frac{w_A}{w_B}\right)^2 + 2gz_B^3 = \left(V_A^2 + 2gz_A\right) z_B^2 \tag{10}$$

$$z_B^3 - \left(\frac{V_A^2}{2g} + z_A\right) z_B^2 + \frac{V_A^2}{2g} z_A^2 \left(\frac{w_A}{w_B}\right)^2 = 0 \tag{11}$$

Using the given parameters:

V_A = 10 m/s
g = 9.81 m/s^2
z_A = 1 m
w_A = 4 m
w_B = 2.5 m

Eq. (11) may be written as,

$$z_B^3 - (6.097\text{ m}) z_B^2 + (13.05\text{ m}) = 0 \tag{12}$$

Solve Eq. (12) numerically to get,

$$z_B = 1.728\text{ m, } 5.695\text{ m, } -1.326\text{ m} \tag{13}$$

Thus, the two possible depths at location B are: $\boxed{\text{1.7 m and 5.7 m}}$.

Video solution: https://www.youtube.com/watch?v=_NHZL_Z6xGE

A Venturi pump is used in the design of a carburetor, a device used to create a fuel-air mixture to be fed into the cylinder of an internal combustion engine. Simplified schematics of a carburetor are shown in the following figures.

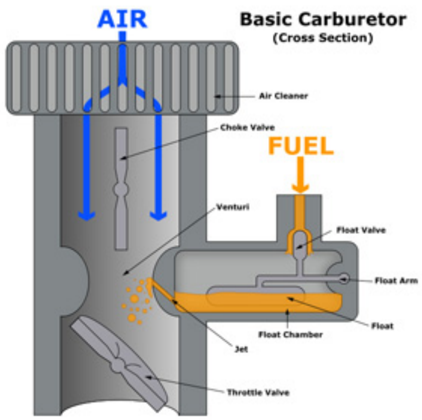
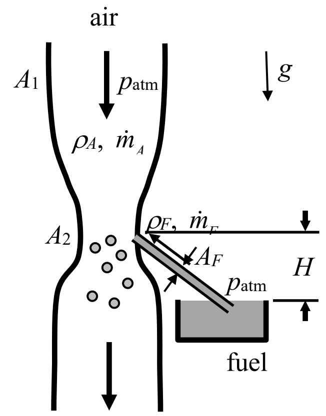


Image from: http://hdabob.com/wp-content/uploads/2009/10/carburetor.jpg

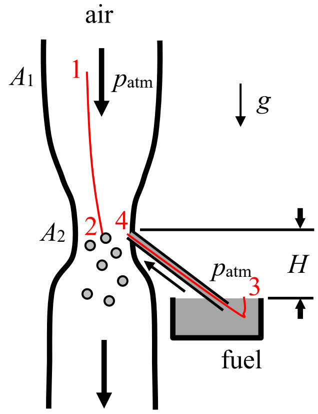


The air, which may reasonably be assumed to be incompressible, has a density ρ_A and the liquid fuel has density ρ_F. The fuel reservoir is located a distance H below the inlet port into the Venturi. The inlet air is at atmospheric pressure as is the free surface of the fuel reservoir. The air inlet cross-sectional area is A_1 and the Venturi throat area is A_2. The fuel line cross-sectional area is A_F.

If the desired air-to-fuel mass flow rate ratio at the outlet of the carburetor is R (= $\dot{m}_A/\dot{m}_F$), determine the required ratio A_1/A_2 in terms of (a subset of) the air-to-fuel ratio R, air density ρ_A, the fuel density ρ_F, the inlet air mass flow rate $\dot{m}_A$, the acceleration due to gravity g, the height from the fuel reservoir to the Venturi throat H, the fuel pipe area A_F, and the air inlet area A_1.

SOLUTION:

Apply Bernoulli's equation from 1 to 2.



$$\left(\frac{p}{\rho_A g}+\frac{V^2}{2g}+z\right)_2=\left(\frac{p}{\rho_A g}+\frac{V^2}{2g}+z\right)_1 \tag{1}$$

where

$$p_1 = p_{atm} \text{ and } p_2 = ? \tag{2}$$

$$V_1=\frac{\dot{m}_A}{\rho_A A_1} \text{ and } V_2=\frac{\dot{m}_A}{\rho_A A_2} \tag{3}$$

Δz is negligible compared to the other terms in B.E. since the fluid is a gas (4)

Substitute and simplify.

$$p_2-p_{atm}=\frac{1}{2}\frac{\dot{m}_A^2}{\rho_A}\left(\frac{1}{A_1^2}-\frac{1}{A_2^2}\right) \tag{5}$$

Apply Bernoulli's equation from 3 to 4.

$$\left(\frac{p}{\rho_F g}+\frac{V^2}{2g}+z\right)_4=\left(\frac{p}{\rho_F g}+\frac{V^2}{2g}+z\right)_3 \tag{6}$$

where

$$p_3 = p_{atm} \text{ and } p_4 = p_2 \tag{7}$$

$$V_3 \approx 0 \text{ and } V_4=\frac{\dot{m}_F}{\rho_F A_F} \tag{8}$$

$$\Delta z = z_4 - z_3 = H \tag{9}$$

Substitute and simplify.

$$\underbrace{p_4}_{=p_2}-p_{atm}=-\rho_F gH-\frac{1}{2}\frac{\dot{m}_F^2}{\rho_F A_F^2} \tag{10}$$

Combine Eqs. (5) and (10) and solve for A_1/A_2.

$$\frac{1}{2}\frac{\dot{m}_A^2}{\rho_A}\left(\frac{1}{A_1^2}-\frac{1}{A_2^2}\right)=-\rho_F gH-\frac{1}{2}\frac{\dot{m}_F^2}{\rho_F A_F^2} \tag{11}$$

$$\left(\frac{1}{A_1^2}-\frac{1}{A_2^2}\right)=-2\rho_A\rho_F\frac{gH}{\dot{m}_A^2}-\frac{1}{A_F^2}\frac{\rho_A}{\rho_F}\frac{\dot{m}_F^2}{\dot{m}_A^2} \tag{12}$$

$$\left(1-\frac{A_1^2}{A_2^2}\right)=-2\rho_A\rho_F\frac{gHA_1^2}{\dot{m}_A^2}-\frac{A_1^2}{A_F^2}\frac{\rho_A}{\rho_F}\frac{\dot{m}_F^2}{\dot{m}_A^2} \tag{13}$$

$$\left(\frac{A_1}{A_2}\right)^2=1+2\rho_A\rho_F\frac{gHA_1^2}{\dot{m}_A^2}+\left(\frac{A_1}{A_F}\right)^2\left(\frac{\rho_A}{\rho_F}\right)\left(\frac{\dot{m}_F}{\dot{m}_A}\right)^2 \tag{14}$$

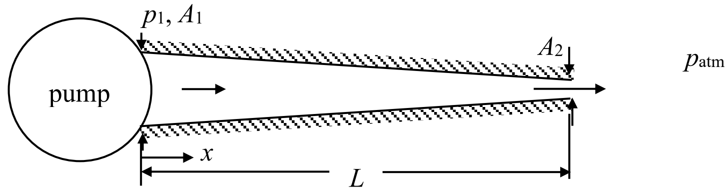
$$\boxed{\frac{A_1}{A_2}=\sqrt{1+2\rho_A\rho_F\frac{gHA_1^2}{\dot{m}_A^2}+\left(\frac{A_1}{A_F}\right)^2\left(\frac{\rho_A}{\rho_F}\right)\left(\frac{\dot{m}_F}{\dot{m}_A}\right)^2}}\text{ or }\boxed{\frac{A_1}{A_2}=\sqrt{1+2\rho_A\rho_F\frac{gHA_1^2}{\dot{m}_A^2}+\left(\frac{A_1}{A_F}\right)^2\left(\frac{\rho_A}{\rho_F}\right)\frac{1}{R^2}}} \tag{15}$$

For a typical carburetor,

ρ_F = 770 kg/m^3 (gasoline)
ρ_A = 1.23 kg/m^3 (air)
A_1 = 1.34*10^{-3} m^2 (D_1 = 4.13cm = 1 5/8 in.)
A_F = 1.70*10^{-6} m^2 (D_F = 1.47 mm = 0.058 in.)
R = 14.7 (ideal fuel to air ratio for gasoline)
g = 9.81 m/s^2
H = 2.00*10^{-2} m (= 2 cm)
$\dot{m}_A$ = 0.290 kg/s (500 cfm @ 1.23 kg/m^3)

$\Rightarrow$ A_1/A_2 = 2.36 $\Rightarrow$ D_2 = 2.69 cm

Consider a pipe of length L with variable cross-sectional area connected to a pump as shown in the figure. The cross-sectional area of the pipe varies linearly with position, x, from an initial area of A_1 to a final area of A_2. Assume that just downstream of the pump the pressure remains constant at p_1 (absolute) and that the flow from the pipe discharges into the atmosphere with pressure p_{atm} (absolute). The pipe is horizontal so that the inlet and exit of the pipe are at the same elevation. At $t = 0$, where t is time, the fluid in the pipe is at rest. Assuming that the flow within the pipe is one-dimensional, incompressible, inviscid, and unsteady, derive an expression for the fluid velocity at the exit of the pipe (station 2) as a function of time.



SOLUTION:

Apply the unsteady form of Bernoulli's equation from point 1 to point 2. Note that a 1D flow

$$\left(\frac{\partial\phi}{\partial t}+\frac{p}{\rho}+\frac{1}{2}\nabla\phi\cdot\nabla\phi-G\right)_1=F(t)=\left(\frac{\partial\phi}{\partial t}+\frac{p}{\rho}+\frac{1}{2}\nabla\phi\cdot\nabla\phi-G\right)_2 \tag{1}$$

where

$$p_1=p_1 \tag{2}$$
$$p_2=p_{\text{atm}}$$
$$\nabla\phi=V \tag{3}$$
$$G_1=G_2 \quad \text{(The points 1 and 2 are at the same elevation.)} \tag{4}$$

Substitute and simplify.

$$\left.\frac{\partial\phi}{\partial t}\right|_1-\left.\frac{\partial\phi}{\partial t}\right|_2=\frac{p_{\text{atm}}-p_1}{\rho}+\frac{1}{2}\left(V_2^2-V_1^2\right) \tag{5}$$

Note that from conservation of mass the velocity, V, at any position, x, at any instant in time, t, is:

$$V=V_2\frac{A_2}{A} \tag{6}$$

where A is the area at position x. Note that V_2 varies with time.

The velocity at any cross-section within the pipe is related to the potential function:

$$\frac{\partial\phi}{\partial x}=V=V_2\frac{A_2}{A} \quad \text{(using Eqn. (6))} \tag{7}$$

The pipe area varies linearly with the position, x:

$$A=A_1+\left(\frac{A_2-A_1}{L}\right)x \tag{8}$$

so that:

$$\phi(x,t)=\int_{x=0}^{x=x}Vdx+f(t)=\int_{x=0}^{x=x}V_2\frac{A_2}{A_1+\frac{A_2-A_1}{L}x}dx+f(t)=V_2A_2\int_{x=0}^{x=x}\frac{dx}{A_1+\frac{A_2-A_1}{L}x}+f(t) \tag{9}$$

$$\phi(x,t)=\left(\frac{V_2L}{1-A_1/A_2}\right)\ln\left[1+\left(A_2/A_1-1\right)\frac{x}{L}\right]+f(t) \qquad \text{(Note: } V_2=V_2(t).\text{)} \tag{10}$$

Taking the partial derivative of ϕ with respect to time gives:

$$\frac{\partial \phi}{\partial t} = \left(\frac{\frac{dV_2}{dt} L}{1 - A_1/A_2} \right) \ln\left[1 + \left(A_2/A_1 - 1 \right) \frac{x}{L} \right] + f'(t) \tag{11}$$

Evaluating Eqn. (11) at points 1 and 2:

$$\left. \frac{\partial \phi}{\partial t} \right|_1 = f'(t) \tag{12}$$

$$\left. \frac{\partial \phi}{\partial t} \right|_2 = \frac{dV_2}{dt} \left[\frac{L \ln\left(A_2/A_1 \right)}{1 - A_1/A_2} \right] + f'(t) \tag{13}$$

Substitute Eqns. (12) and (13) into Eqn. (5) and simplify.

$$-\frac{dV_2}{dt} \left[\frac{L \ln\left(A_2/A_1 \right)}{1 - A_1/A_2} \right] + f'(t) = \frac{p_{\text{atm}} - p_1}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2 \right) + f'(t) \tag{14}$$

$$\frac{dV_2}{dt} \left[\frac{L \ln\left(A_2/A_1 \right)}{1 - A_1/A_2} \right] = \frac{p_1 - p_{\text{atm}}}{\rho} - \frac{1}{2} V_2^2 \left[1 - \left(\frac{A_2}{A_1} \right)^2 \right] \text{ (using Eqn. (6))} \tag{15}$$

Let:

$$\alpha^2 \equiv \left[\frac{1 - A_1/A_2}{L \ln\left(A_2/A_1 \right)} \right] \left(\frac{p_1 - p_{\text{atm}}}{\rho} \right) \text{ (Note: } \alpha^2 > 0 \text{ since } A_1/A_2 > 1 \text{ and } p_1 > p_{\text{atm}}\text{.)} \tag{16}$$

$$\beta^2 \equiv \frac{1}{2} \left[\frac{1 - A_1/A_2}{L \ln\left(A_2/A_1 \right)} \right] \left[1 - \left(\frac{A_2}{A_1} \right)^2 \right] \text{ (Note: } \beta^2 > 0 \text{ since } A_1/A_2 > 1\text{.)} \tag{17}$$

so that Eqn. (15) becomes:

$$\frac{dV_2}{dt} = \alpha^2 - \beta^2 V_2^2 \tag{18}$$

Another approach to solving the problem is to solve Euler's equation in the x-direction.

$$\rho\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)=-\frac{\partial p}{\partial x} \tag{19}$$

where

$$V=\frac{V_2A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}} \tag{20}$$

so that

$$\frac{\partial V}{\partial t}=\frac{\frac{dV_2}{dt}A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}} \tag{21}$$

$$\frac{\partial V}{\partial x}=\frac{-V_2A_2\left[\left(A_2-A_1\right)\frac{1}{L}\right]}{\left[A_1+\left(A_2-A_1\right)\frac{x}{L}\right]^2} \tag{22}$$

Substitute and simplify.

$$\frac{\frac{dV_2}{dt}A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}}+\left[\frac{V_2A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}}\right]\left\{\frac{-V_2A_2\left[\left(A_2-A_1\right)\frac{1}{L}\right]}{\left[A_1+\left(A_2-A_1\right)\frac{x}{L}\right]^2}\right\}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{23}$$

$$\frac{\frac{dV_2}{dt}A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}}-\frac{V_2^2A_2^2\left[\left(A_2-A_1\right)\frac{1}{L}\right]}{\left[A_1+\left(A_2-A_1\right)\frac{x}{L}\right]^3}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{24}$$

$$\int_{x=0}^{x=L}\frac{\frac{dV_2}{dt}A_2dx}{A_1+\left(A_2-A_1\right)\frac{x}{L}}-\int_{x=0}^{x=L}\frac{V_2^2A_2^2\left[\left(A_2-A_1\right)\frac{1}{L}\right]}{\left[A_1+\left(A_2-A_1\right)\frac{x}{L}\right]^3}dx=-\frac{1}{\rho}\int_{p=p_1}^{p=p_2}dp \tag{25}$$

$$\frac{dV_2}{dt}\left[\frac{L\ln\left(A_1/A_2\right)}{1-A_1/A_2}\right]=\frac{p_1-p_{\text{atm}}}{\rho}-\frac{1}{2}V_2^2\left[1-\left(\frac{A_2}{A_1}\right)^2\right] \tag{26}$$

This is the same result as Eqn. (15)!

$$\int_{V_2=0}^{V_2=V_2} \frac{dV_2}{\alpha^2 - \beta^2 V_2^2} = \int_{t=0}^{t=t} dt \tag{27}$$

$$\frac{-1}{2\alpha\beta} \ln\left(\frac{\beta V_2 - \alpha}{\beta V_2 + \alpha}\right) = t \tag{28}$$

$$\frac{V_2 - \alpha/\beta}{V_2 + \alpha/\beta} = \exp(-2\alpha\beta t) \tag{29}$$

$$V_2 - \alpha/\beta = \exp(-2\alpha\beta t)\left(V_2 + \alpha/\beta\right) \tag{30}$$

$$V_2\left[1 - \exp(-2\alpha\beta t)\right] = \alpha/\beta\left[\exp(-2\alpha\beta t) + 1\right]$$

$$\boxed{\therefore V_2 = \left(\alpha/\beta\right)\left[\frac{1 + \exp(-2\alpha\beta t)}{1 - \exp(-2\alpha\beta t)}\right]} \tag{31}$$

Another approach to solving the problem is to solve Euler's equation in the x-direction.

$$\rho\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)=-\frac{\partial p}{\partial x} \tag{32}$$

where

$$V=\frac{V_2 A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}} \tag{33}$$

so that

$$\frac{\partial V}{\partial t}=\frac{\frac{dV_2}{dt}A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}} \tag{34}$$

$$\frac{\partial V}{\partial x}=\frac{-V_2 A_2\left[\left(A_2-A_1\right)\frac{1}{L}\right]}{\left[A_1+\left(A_2-A_1\right)\frac{x}{L}\right]^2} \tag{35}$$

Substitute and simplify.

$$\frac{\frac{dV_2}{dt}A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}}+\left[\frac{V_2 A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}}\right]\left\{\frac{-V_2 A_2\left[\left(A_2-A_1\right)\frac{1}{L}\right]}{\left[A_1+\left(A_2-A_1\right)\frac{x}{L}\right]^2}\right\}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{36}$$

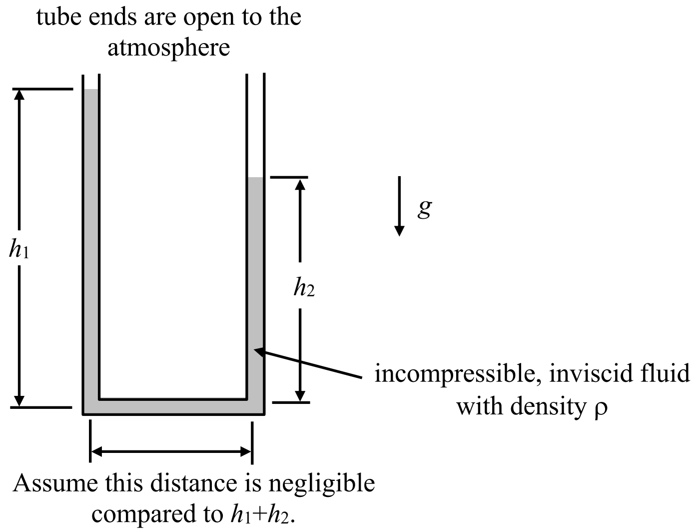
$$\frac{\frac{dV_2}{dt}A_2}{A_1+\left(A_2-A_1\right)\frac{x}{L}}-\frac{V_2^2 A_2^2\left[\left(A_2-A_1\right)\frac{1}{L}\right]}{\left[A_1+\left(A_2-A_1\right)\frac{x}{L}\right]^3}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{37}$$

$$\int_{x=0}^{x=L}\frac{\frac{dV_2}{dt}A_2 dx}{A_1+\left(A_2-A_1\right)\frac{x}{L}}-\int_{x=0}^{x=L}\frac{V_2^2 A_2^2\left[\left(A_2-A_1\right)\frac{1}{L}\right]}{\left[A_1+\left(A_2-A_1\right)\frac{x}{L}\right]^3}dx=-\frac{1}{\rho}\int_{p=p_1}^{p=p_2}dp \tag{38}$$

$$\frac{dV_2}{dt}\left[\frac{L\ln\left(A_1/A_2\right)}{1-A_1/A_2}\right]=\frac{p_1-p_{\text{atm}}}{\rho}-\frac{1}{2}V_2^2\left[1-\left(\frac{A_2}{A_1}\right)^2\right] \tag{39}$$

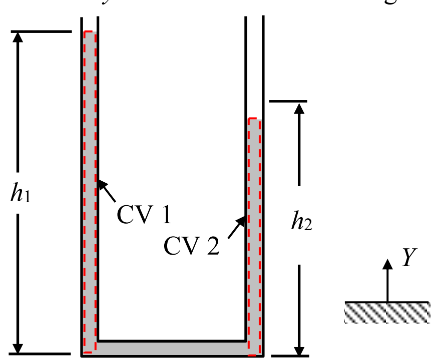
This is the same result as Eqn. (15)!

a. Using an integral approach, write the differential equation governing the motion of an inviscid, incompressible fluid (with density ρ) oscillating within the U-tube manometer shown. The manometer cross-sectional area is A.
b. What is the natural frequency of the fluid motion?
c. What are the implications of this result for making time-varying pressure measurements using a manometer?



SOLUTION:

Apply COLM in the y-direction to the following two CVs.



CV 1:

$$\frac{d}{dt}\int_{\text{CV}} u_Y \rho dV + \int_{\text{CS}} u_Y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{BY} + F_{SY}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_Y \rho dV = \frac{d}{dt}\left(\frac{dh_1}{dt}\rho A h_1\right) = \rho A\left[h_1 \frac{d^2 h_1}{dt^2} + \left(\frac{dh_1}{dt}\right)^2\right]$$

$$\int_{\text{CS}} u_Y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0$$

$$F_{BY} = -\rho A h_1 g$$

$$F_{SY} = p_{\text{bottom}} A$$

Substitute and simplify:

$$\rho A\left[h_1 \frac{d^2 h_1}{dt^2} + \left(\frac{dh_1}{dt}\right)^2\right] = -\rho A h_1 g + p_{\text{bottom}} A$$

$$h_1 \frac{d^2 h_1}{dt^2} = -h_1 g - \left(\frac{dh_1}{dt}\right)^2 + \frac{p_{\text{bottom}}}{\rho} \tag{1}$$

CV 2:

$$\frac{d}{dt}\int_{\text{CV}} u_Y \rho dV + \int_{\text{CS}} u_Y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{BY} + F_{SY}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_Y \rho dV = \frac{d}{dt}\left(\frac{dh_2}{dt}\rho A h_2\right) = -\rho A\left[h_2 \frac{d^2 h_2}{dt^2} + \left(\frac{dh_2}{dt}\right)^2\right]$$

$$\int_{\text{CS}} u_Y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0$$

$$F_{BY} = -\rho A h_2 g$$

$$F_{SY} = p_{\text{bottom}} A$$

Substitute and simplify:

$$\rho A\left[h_2 \frac{d^2 h_2}{dt^2} + \left(\frac{dh_2}{dt}\right)^2\right] = -\rho A h_2 g + p_{\text{bottom}} A$$

$$h_2 \frac{d^2 h_2}{dt^2} = -h_2 g - \left(\frac{dh_2}{dt}\right)^2 + \frac{p_{\text{bottom}}}{\rho} \tag{2}$$

From conservation of mass considering both CVs combined together:

$$\frac{dh_2}{dt} = -\frac{dh_1}{dt} \tag{3}$$

(One side moves down at the same rate that the other side moves up.)

Subtract Eqn. (2) from (1) and make use of Eqn. (3).

$$h_1\frac{d^2h_1}{dt^2} - h_2\frac{d^2h_2}{dt^2} = -h_1g - \left(\frac{dh_1}{dt}\right)^2 + \frac{p_{\text{bottom}}}{\rho} + h_2g + \left(\frac{dh_2}{dt}\right)^2 - \frac{p_{\text{bottom}}}{\rho}$$

$$h_1\frac{d^2h_1}{dt^2} - h_2\frac{d}{dt}\left(-\frac{dh_1}{dt}\right) = \left(h_2 - h_1\right)g - \left(\frac{dh_1}{dt}\right)^2 + \left(\frac{dh_1}{dt}\right)^2$$

$$\left(h_1 + h_2\right)\frac{d^2h_1}{dt^2} + g\left(h_1 - h_2\right) = 0 \tag{4}$$

Let:

$$L = h_1 + h_2$$

$$z = h_1 - h_2$$

$$\frac{dz}{dt} = \frac{dh_1}{dt} - \frac{dh_2}{dt} = \frac{dh_1}{dt} + \frac{dh_1}{dt} = 2\frac{dh_1}{dt}$$

$$\frac{d^2z}{dt^2} = 2\frac{d^2h_1}{dt^2}$$

so that Eqn. (4) becomes:

$$\boxed{\frac{d^2z}{dt^2} + \frac{2g}{L}z = 0} \tag{5}$$

The general solution to this differential equation is:

$$z = A\sin\left(t\sqrt{\frac{2g}{L}}\right) + B\cos\left(t\sqrt{\frac{2g}{L}}\right) \tag{6}$$

Specifying the following initial conditions:

$$z\left(t=0\right) = z_0$$

$$\frac{dz}{dt}\left(t=0\right) = V \tag{7}$$

gives the solution:

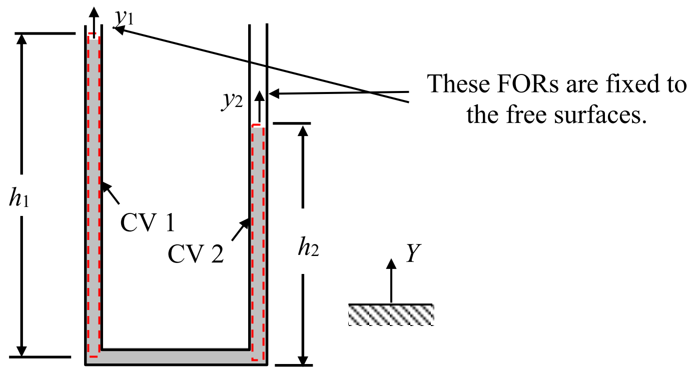
$$\boxed{z = V\sqrt{\frac{L}{2g}}\sin\left(t\sqrt{\frac{2g}{L}}\right) + z_0\cos\left(t\sqrt{\frac{2g}{L}}\right)} \tag{8}$$

The natural (radian) frequency of the manometer, ω, is:

$$\boxed{\omega = \sqrt{\frac{2g}{L}}} \tag{9}$$

The practical implication of this result is that one must make sure that the manometer fluid length, *L*, is sufficiently small so that the manometer's natural frequency is large enough to accurately capture the temporal variations in the pressure measurements. In other words, a manometer with a large *L* will not be able to capture rapid pressure fluctuations.

This problem may also be solved using a accelerating frames of reference.



COLM in the y-direction using the indicated accelerating FORs:

$$\frac{d}{dt}\int_{\text{CV}} u_y \rho dV + \int_{\text{CS}} u_y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{By} + F_{Sy} - \int_{\text{CV}} a_{y/Y} \rho dV$$

where

$$\frac{d}{dt}\int_{\text{CV1}} u_y \rho dV = 0 \qquad \frac{d}{dt}\int_{\text{CV2}} u_y \rho dV = 0$$

$$\int_{\text{CS1}} u_y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0 \qquad \int_{\text{CS2}} u_y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0$$

$$F_{By1} = -\rho h_1 A g \qquad F_{By2} = -\rho h_2 A g$$

$$F_{Sy1} = p_{\text{bottom}} A \qquad F_{Sy2} = p_{\text{bottom}} A$$

$$\int_{\text{CV1}} a_{y/Y} \rho dV = \frac{d^2 h_1}{dt^2} \rho h_1 A \qquad \int_{\text{CV2}} a_{y/Y} \rho dV = \frac{d^2 h_2}{dt^2} \rho h_2 A$$

$$\therefore 0 = -\rho h_1 A g + p_{\text{bottom}} A - \frac{d^2 h_1}{dt^2} \rho h_1 A \qquad \therefore 0 = -\rho h_2 A g + p_{\text{bottom}} A - \frac{d^2 h_2}{dt^2} \rho h_2 A$$

Subtract the two previous equations and simplify:

$$0 = \left(h_2 - h_1\right) g - \frac{d^2 h_1}{dt^2} h_1 + \frac{d^2 h_2}{dt^2} h_2$$

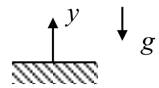
Make use of the Eqn. (3) and re-arrange to get:

$$\left(h_1 + h_2\right) \frac{d^2 h_1}{dt^2} + g\left(h_1 - h_2\right) = 0$$

This is the same equation as Eqn. (4)!

This problem may also be solved using the unsteady Bernoulli equation. Assuming the flow is irrotational, inviscid, and incompressible, Bernoulli's equation may be written at an instant in time as:

$$\left(\frac{\partial \phi}{\partial t}+\frac{p}{\rho}+\tfrac{1}{2}V^2+gy\right)_1=\left(\frac{\partial \phi}{\partial t}+\frac{p}{\rho}+\tfrac{1}{2}V^2+gy\right)_2 \tag{10}$$



where the point 1 is on the free surface of the left leg of the manometer and point 2 is on the free surface of the right leg of the manometer. In addition:

$$V=\frac{\partial \phi}{\partial y}\Rightarrow \begin{array}{l} \left.\dfrac{\partial \phi}{\partial t}\right|_1=\dfrac{\partial}{\partial t}\displaystyle\int_{y=0}^{y=h_1}Vdy=\dfrac{d}{dt}\left(V_1h_1\right)=h_1\dfrac{dV_1}{dt}+V_1\dfrac{dh_1}{dt}=h_1\dfrac{d^2h_1}{dt^2}+\left(\dfrac{dh_1}{dt}\right)^2 \\ \left.\dfrac{\partial \phi}{\partial t}\right|_2=\dfrac{\partial}{\partial t}\displaystyle\int_{y=0}^{y=h_2}Vdy=\dfrac{d}{dt}\left(V_2h_2\right)=h_2\dfrac{dV_1}{dt}+V_1\dfrac{dh_2}{dt}=h_2\dfrac{d^2h_2}{dt^2}+\left(\dfrac{dh_2}{dt}\right)^2 \end{array}$$

$$p_1=p_2=p_{\text{atm}}$$

$$V_1^2=V_2^2$$

$$y_1=h_1$$

$$y_2=h_2$$

Substituting and simplifying gives:

$$h_1\frac{d^2h_1}{dt^2}+\left(\frac{dh_1}{dt}\right)^2+gh_1=h_2\frac{d^2h_2}{dt^2}+\left(\frac{dh_2}{dt}\right)^2+gh_2$$

Making use of Eqn. (3) gives:

$$\left(h_1+h_2\right)\frac{d^2h_1}{dt}+g\left(h_1-h_2\right)=0 \tag{11}$$

Eqn. (11) is identical to Eqn. (4) so the solution will be the same as that derived previously.

5.14. Kelvin's Theorem

In an inviscid flow of a fluid with constant density, or a fluid where the pressure is a function only of the density, where the only body forces are conservative, the vorticity of each fluid element is preserved.

Notes:

(1) A conservative body force is one that can be written as the gradient of a potential function, i.e.,

$$\boldsymbol{f} = -\boldsymbol{\nabla} G. \tag{5.373}$$

The force due to gravity is an example of a conservative body force,

$$\boldsymbol{f} = -\boldsymbol{\nabla}(gz) = -g\hat{\mathbf{e}}_z. \tag{5.374}$$

(2) A fluid in which the pressure is a function only of the density, i.e., $p = p(\rho)$, is called a barotropic fluid. An example of such a fluid would be an ideal gas undergoing an isentropic flow process,

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{5.375}$$

where γ is the specific heat ratio for the ideal gas.

(3) An important result of Kelvin's Theorem is that if a flow starts off irrotational, viscous forces are negligible, and the fluid has either constant density or is a barotropic fluid, then the flow will always remain irrotational.

Proof of Kelvin's Theorem: Consider the flow of an inviscid fluid where only conservative body forces are considered. The time rate of change of the circulation about a specific collection of fluid particles is given by,

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_C \boldsymbol{u}\cdot d\boldsymbol{l} = \frac{D}{Dt}\oint_C u_i dx_i, \tag{5.376}$$

where Γ is the circulation, C is the contour about the fluid particles, $\boldsymbol{u}$ is the fluid velocity, and $d\boldsymbol{l} = dx_i\hat{\boldsymbol{e}}_i$ is a small displacement along the contour. The Lagrangian time derivative can be brought inside the contour integral (refer to Pneuli, D. and Gutfinger, C., *Fluid Mechanics*, pp. 310-312 for the proof) to give,

$$\frac{D\Gamma}{Dt} = \oint_C \left[\frac{Du_i}{Dt}dx_i + u_i\frac{D(dx_i)}{Dt}\right]. \tag{5.377}$$

Note that,

$$\frac{D(dx_i)}{Dt} = d\left(\frac{Dx_i}{Dt}\right) = d\left(\underbrace{\frac{\partial x_i}{\partial t}}_{=0} + u_j\underbrace{\frac{\partial x_i}{\partial x_j}}_{=\delta_{ij}}\right) = du_i. \tag{5.378}$$

We can also substitute in for the fluid particle acceleration using Euler's Equations,

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial G}{\partial x_i}. \tag{5.379}$$

Note that in the previous equation conservative body forces have been assumed where,

$$f_i = -\frac{\partial G}{\partial x_i}. \tag{5.380}$$

Substituting Eqs. (5.379) and (5.378) into Eq. (5.377) gives,

$$\frac{D\Gamma}{Dt} = \oint_C \left[-\frac{1}{\rho}\frac{\partial p}{\partial x_i}dx_i - \frac{\partial G}{\partial x_i}dx_i + u_i du_i\right]. \tag{5.381}$$

Several of the terms in this equation may be simplified since,

$$\frac{\partial p}{\partial x_i} dx_i = dp, \tag{5.382}$$

$$\frac{\partial G}{\partial x_i} dx_i = dG, \tag{5.383}$$

$$u_i du_i = d\left(\frac{1}{2} u_i u_i\right). \tag{5.384}$$

Substituting these equations into Eq. (5.381),

$$\frac{D\Gamma}{Dt} = \oint_C \left[-\frac{dp}{\rho} - dG + d\left(\frac{1}{2} u_i u_i\right)\right] = -\oint_C \frac{dp}{\rho} - \oint_C dG + \oint_C d\left(\frac{1}{2} u_i u_i\right). \tag{5.385}$$

The second and third terms in this equation are zero since these functions are single-valued, i.e., at each location the quantities have a unique value, and the contour C is a closed curve,

$$\oint_C dG = 0, \tag{5.386}$$

$$\oint_C d\left(\frac{1}{2} u_i u_i\right) = 0. \tag{5.387}$$

Thus,

$$\frac{D\Gamma}{Dt} = -\oint_C \frac{dp}{\rho}. \tag{5.388}$$

If the density is a constant, then,

$$\frac{D\Gamma}{Dt} = -\frac{1}{\rho} \oint_C dp, \tag{5.389}$$

and, since the pressure is a single-valued function,

$$\frac{D\Gamma}{Dt} = 0. \tag{5.390}$$

Since the circulation about the fluid particles remains unchanged, the vorticity of the fluid particles will also remain unchanged.

Also, if the pressure is a function only of the density then,

$$p = p(\rho) \implies dp = \frac{dp}{d\rho} d\rho \implies dp = f(\rho) d\rho, \tag{5.391}$$

$$\frac{D\Gamma}{Dt} = -\oint_C \frac{dp}{\rho} = -\oint_C f(\rho) \frac{d\rho}{\rho}, \tag{5.392}$$

$$\therefore \frac{D\Gamma}{Dt} = 0, \tag{5.393}$$

since the density is a single-valued function. Therefore, we see that for the flow of an inviscid fluid in a conservative force field where either the density of the fluid is constant or where the pressure is a function only of the density, the vorticity of a collection of fluid particles will remain unchanged.

Notes:

(1) When the pressure is a function of variables other than the density, the contour integral will not, in general, be zero.
(2) As might be anticipated, the vorticity of a fluid element may be changed through the action of viscosity, non-conservative forces, or density variations that are not a function solely of the pressure variations.
(3) Kelvin's Theorem applies strictly to a simply-connected region, i.e., a contour that does not intersect itself and contains only fluid. A contour that surrounds some object, e.g., an airfoil, is not a simply-connected region and, therefore, Kelvin's Theorem does not hold for such a contour. This fact is

significant when examining the lift on an airfoil since it is possible to have circulation around an airfoil, i.e., around a non-simply connected region, in an otherwise irrotational flow.

5.15. Crocco's Equation

Crocco's Equation relates the vorticity of a flow field to the gradients in the entropy and stagnation enthalpy of the fluid in a flow where viscosity and body forces are negligible. Crocco's Equation is given as,

$$\boldsymbol{u} \times \boldsymbol{\omega} + T\boldsymbol{\nabla} s = \boldsymbol{\nabla}\left(h + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \frac{\partial \boldsymbol{u}}{\partial t}. \tag{5.394}$$

Derivation of Crocco's Equation: Consider the Momentum Equations for an inviscid fluid (Euler's Equations) for a flow in which body forces are negligible,

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p. \tag{5.395}$$

Re-write the convective derivative term using the following vector identity,

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})\,. \tag{5.396}$$

Also make use of the definition of vorticity, $\boldsymbol{\omega}$,

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}, \tag{5.397}$$

and substitute into Eq. (5.395) to get,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) - \boldsymbol{u}\times\boldsymbol{\omega} = -\frac{1}{\rho}\boldsymbol{\nabla} p. \tag{5.398}$$

Now consider the First Law of Thermodynamics for a fluid element (assumed to be a pure substance) where only reversible pressure work is considered,

$$du = \frac{1}{\rho}\delta q - pd\left(\frac{1}{\rho}\right). \tag{5.399}$$

Note that δq is the amount of heat added to the fluid element per unit volume. Substituting the definition of entropy for a reversible (zero viscosity has been assumed) process,

$$Tds = \frac{1}{\rho}\delta q, \tag{5.400}$$

and enthalpy,

$$dh = du + d\left(\frac{p}{\rho}\right) = du + pd\left(\frac{1}{\rho}\right) + \frac{dp}{\rho}, \tag{5.401}$$

into Eq. (5.399) and simplifying gives,

$$dh = Tds + \frac{dp}{\rho}. \tag{5.402}$$

Note that we can write Eq. (5.402) in a slightly different form by utilizing the following,

$$dh = \boldsymbol{\nabla} h \cdot d\boldsymbol{x}, \tag{5.403}$$

$$ds = \boldsymbol{\nabla} s \cdot d\boldsymbol{x}, \tag{5.404}$$

$$dp = \boldsymbol{\nabla} p \cdot d\boldsymbol{x}, \tag{5.405}$$

where $d\boldsymbol{x}$ is a small length in any direction. Substituting these relations into Eq. (5.402) and simplifying,

$$-\frac{\boldsymbol{\nabla} p}{\rho} = T\boldsymbol{\nabla} s - \boldsymbol{\nabla} h. \tag{5.406}$$

Substituting Eq. (5.406) into Eq. (5.398),

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) - \boldsymbol{u}\times\boldsymbol{\omega} = T\boldsymbol{\nabla} s - \boldsymbol{\nabla} h. \tag{5.407}$$

Re-arranging this equation results in Crocco's Equation,

$$\boldsymbol{u} \times \boldsymbol{\omega} + T\boldsymbol{\nabla} s = \boldsymbol{\nabla}\left(h + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \frac{\partial \boldsymbol{u}}{\partial t}. \tag{5.408}$$

Recall that this equation holds for a flow in which viscous and body forces are negligible.

Notes:

(1) Consider a steady, inviscid flow in which body forces are negligible so that Eq. (5.394) becomes,

$$\boldsymbol{u} \times \boldsymbol{\omega} = T\boldsymbol{\nabla} s = \boldsymbol{\nabla} h_0 \quad \text{where} \quad h_0 = h + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}. \tag{5.409}$$

Let's restrict our investigation to flow along a streamline by taking the dot product of the previous equation with $\boldsymbol{u}$,

$$\underbrace{\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{\omega})}_{=0\,\text{(vector identity)}} = T(\boldsymbol{u}\cdot\boldsymbol{\nabla})s = (\boldsymbol{u}\cdot\boldsymbol{\nabla})h_0. \tag{5.410}$$

Note that since we're concerned here with steady flows,

$$\frac{Ds}{Dt} = (\boldsymbol{u}\cdot\boldsymbol{\nabla})s \quad \text{and} \quad \frac{Dh_0}{Dt} = (\boldsymbol{u}\cdot\boldsymbol{\nabla})h_0, \tag{5.411}$$

and Eq. (5.410) becomes,

$$T\frac{Ds}{Dt} = \frac{Dh_0}{Dt}. \tag{5.412}$$

Hence, if the flow remains isentropic along a streamline (i.e., $Ds/Dt = 0$), then the stagnation enthalpy, h_0, must remain constant along the streamline since $Dh_0/Dt = 0$.

(2) Now consider the case where the stagnation enthalpy in a steady flow is uniform along the streamlines so that $\boldsymbol{\nabla} h_0 = 0$. For this case, Eq. (5.394) becomes,

$$\boldsymbol{u} \times \boldsymbol{\omega} = -T\boldsymbol{\nabla} s. \tag{5.413}$$

For such a flow, we can conclude the following important statement: For the steady flow of an inviscid fluid in which body forces are negligible and where the stagnation enthalpy is constant, an irrotational flow will be isentropic and an isentropic flow will be irrotational.

(3) Consider the uniform, supersonic flow in front of a blunt-nosed object. The incoming flow will have a constant stagnation enthalpy and will also be irrotational (and, thus, isentropic). A curved shock wave will stand in front of the object. Across the shock wave the stagnation enthalpy will remain constant but the entropy will change (flow across a shock wave is a non-isentropic process). Since the shock wave is curved, there will be a gradient in the entropy normal to the downstream streamlines. Thus, from Crocco's Equation we observe that vorticity will also be generated downstream of the curved shock wave and the flow will, by definition, be rotational.

Although there's a change in entropy across a normal shock wave and an oblique shock wave, there's no entropy gradient in the normal direction (the entropy change across the shock is uniform along the length of the shock) and, thus, the flow will remain irrotational downstream of normal and oblique shocks if the upstream flow is irrotational. Note that Crocco's Equation does not strictly apply across a shock wave since the large velocity gradient within the shock means that viscous effects are also significant there. However, Crocco's Equation can be applied just upstream and downstream of the shock wave.

5.16. Review Questions

(1) Describe what each term represents in the Lagrangian derivative.
(2) What is the formal definition of an incompressible fluid? Give an example of an incompressible flow where the fluid density is not uniform.
(3) Write the Continuity Equation for an incompressible fluid.
(4) Write the Continuity Equation for a fluid with constant and uniform density.
(5) Describe the naming and sign convention for stresses, σ_{ij}.

(6) Is the stress tensor always symmetric?
(7) Describe the various ways in which a fluid element can deform.
(8) What is the vorticity of a fluid element (in words and in mathematical form)?
(9) What is the deformation rate tensor (in words and in mathematical form)?
(10) What is meant by an "irrotational" flow?
(11) What three key assumptions are made in deriving the stress-strain rate constitutive relation for a Newtonian fluid?
(12) How is the mechanical pressure related to the thermodynamic pressure?
(13) Why doesn't the bulk viscosity enter into incompressible fluid flow problems?
(14) Describe what each term represents in the Navier-Stokes Equations.
(15) What are Euler's Equations?
(16) Describe the various ways in which a fluid element can deform.
(17) What does each term represent in the Energy Equation?
(18) What does each term represent in the Mechanical Energy Equation?
(19) What does each term represent in the Thermal Energy Equation?
(20) What is Fourier's Law of Conduction (in mathematical terms)?
(21) What does the Energy Dissipation Function represent?
(22) Is the Energy Dissipation Function always positive?
(23) Is the Thermal Energy Equation required to solve for the flow velocity and pressure in incompressible flows? How about for compressible flows?
(24) Under what conditions can vorticity be generated within a flow?
(25) How are vorticity and circulation related?
(26) What are the assumptions that go into the following form of Bernoulli's Equation?

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = c \tag{5.414}$$

(27) What are the assumptions that go into the following form of Bernoulli's Equation?

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{V^2}{2} + gz = F(t) \tag{5.415}$$

(28) What does Bernoulli's Equation look like for an ideal gas flowing isentropically?
(29) What is Kelvin's Theorem? What is its significance?

CHAPTER 6

Potential Flows

1. Stream Functions

A stream function is a special scalar function that is useful when analyzing 2D flows. As will be shown, a stream function has the following properties:

1. A stream function satisfies the continuity equation.
2. A stream function is a constant along a streamline.
3. The flow rate between two streamlines is equal to the difference in the streamlines' stream functions.

First, let's define the stream function.

Define a scalar function, ψ, called a **stream function**, such that the continuity equation is automatically satisfied for 2D (planar and axi-symmetric) flows.

For a 2D incompressible flow in rectangular coordinates, define $\psi = \psi(x, y)$ such that:

$$u_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y = -\frac{\partial \psi}{\partial x} \tag{1}$$

If the stream function is defined in this manner, then the continuity equation will automatically be satisfied:

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

For a 2D incompressible flow in polar coordinates, $\psi = \psi(r, \theta)$:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r} \tag{2}$$

The continuity equation for a 2D, incompressible flow in polar coordinates is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta \partial r} = 0$$

Now let's consider some of the additional properties of the stream function.

The Stream Function is Constant Along a Streamline

Let's determine the curve along which the stream function remains constant. We'll only consider an incompressible flow in rectangular coordinates here for simplicity (the same result holds for polar coordinates and compressible flows).

The total change in the stream function, $d\psi$, where $\psi=\psi(x, y)$, over some displacement (dx, dy) is given by:

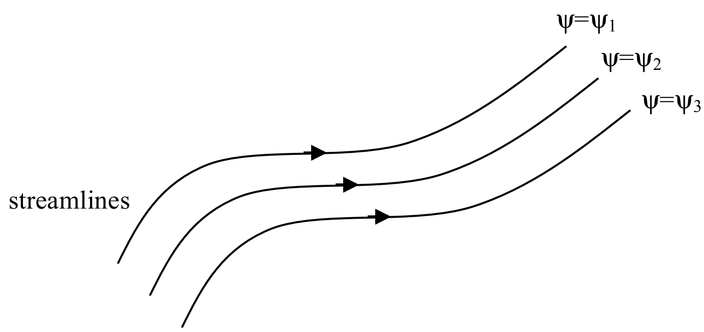
$$\psi = \psi(x, y) \quad \Rightarrow \quad d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -u_y dx + u_x dy$$

where the definition of the stream function has been used to write $d\psi$ in terms of the velocities. To find the slope of the curve along which ψ=constant, we let $d\psi=0$.

$$d\psi = 0 = -u_y dx + u_x dy$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{\psi=\text{constant}} = \frac{u_y}{u_x}$$

Notice that the slope of the curve along which the stream function is constant is exactly the same as the slope of a streamline. Thus, we conclude that **the stream function is constant along a streamline**!



where ψ_1, ψ_2, and ψ_3 are constants.

Note that these constants may vary from streamline to streamline.

Example:

A particular planar, incompressible flow can be described with the following stream function:

$$\psi = Axy$$

where A is a constant.

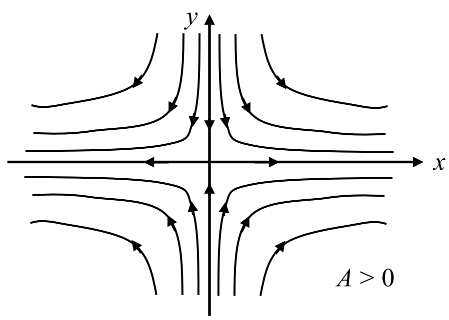
a. Sketch the streamlines for the flow.
b. Determine the velocity components for the flow.

SOLUTION:

The stream function is a constant along a streamline so the equation of the streamlines will be:

$$y = \frac{\psi}{A}\frac{1}{x} \quad \text{(hyperbolas!)}$$

A plot of the streamlines is shown below. Note that A has been assumed to be a positive constant (i.e., $A > 0$) in determining the direction of the flow.

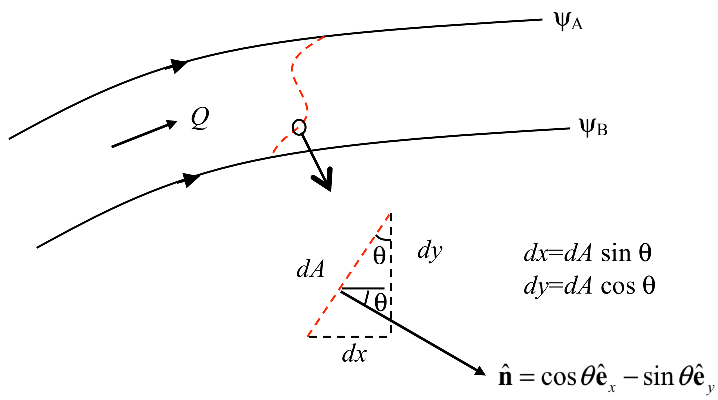


The velocities are determined from the definition of the stream function.

$$\boxed{u_x = \frac{\partial \psi}{\partial y} = Ax} \text{ and } \boxed{u_y = -\frac{\partial \psi}{\partial x} = -Ay}$$

The Flow Rate Between Two Streamlines is Equal to the Difference in their Stream Functions

Now let's examine how the flow rate between two streamlines is related to the stream function. Consider the sketch below.



The volumetric flow rate passing between the two streamlines, and thus crossing through a line drawn between the two streamlines can be found by calculating the volumetric flow rate through a small piece of the line and then integrating from one streamline to the other.

$$dQ = (\mathbf{u}\cdot d\mathbf{A}) = \left(u_x\hat{\mathbf{e}}_x + u_y\hat{\mathbf{e}}_y\right)\cdot\left(\cos\theta\hat{\mathbf{e}}_x - \sin\theta\hat{\mathbf{e}}_y\right)dA = \left(u_x\hat{\mathbf{e}}_x + u_y\hat{\mathbf{e}}_y\right)\cdot\left(\frac{dy}{dA}\hat{\mathbf{e}}_x - \frac{dx}{dA}\hat{\mathbf{e}}_y\right)dA$$

$$= u_x dy - u_y dx = \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx$$

$$\therefore dQ = d\psi$$

Integrating from streamline A to streamline B gives:

$$Q_{AB} = \psi_B - \psi_A$$

The volumetric flow rate between two streamlines is equal to the difference in the streamline stream functions!

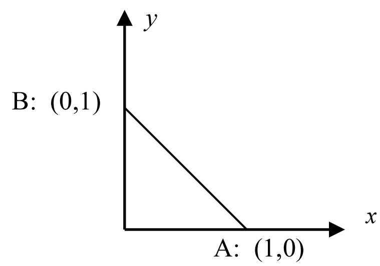
Note that if $\psi_B > \psi_A$ then the flow is from left to right. If $\psi_B < \psi_A$ then the flow is from right to left.

Example:

The velocity field for a planar, incompressible flow is given by:

$$\mathbf{u} = 2(x^2 - y^2)\hat{\mathbf{e}}_x - 4xy\hat{\mathbf{e}}_y$$

a. Determine the stream function for this flow field if ψ(0,0)=0.
b. Determine the volumetric flow rate across the line AB shown in the figure.



SOLUTION:

Recall that the velocities are determined from the stream function in the following manner.

$$u_x = \frac{\partial \psi}{\partial y} = 2\left(x^2 - y^2\right) \quad \Rightarrow \quad \psi = 2\left(x^2 y - \tfrac{1}{3}y^3\right) + f(x) \tag{3}$$

$$u_y = -\frac{\partial \psi}{\partial x} = 4xy \quad \Rightarrow \quad \psi = -2x^2 y + g(y) \tag{4}$$

where f and g are, at this point, unknown functions of x and y, respectively. Comparing Eqs. (3) and (4) indicates that:

$$f(x) = c \quad \text{and} \quad g(y) = -\tfrac{2}{3}y^3 + c$$

where c is a constant. Hence, the stream function is:

$$\psi = 2\left(x^2 y - \tfrac{1}{3}y^3\right) + c \tag{5}$$

Knowing that ψ(0, 0) = 0 we can conclude that c = 0 and:

$$\boxed{\psi = 2\left(x^2 y - \tfrac{1}{3}y^3\right)} \tag{6}$$

Recall that the volumetric flow rate between two streamlines is equal to the difference in the streamline stream functions.

$$Q_{AB} = \psi_B - \psi_A \tag{7}$$

where

$$\psi_B = \psi(0,1) = -\tfrac{2}{3}$$

$$\psi_A = \psi(1,0) = 0$$

so that

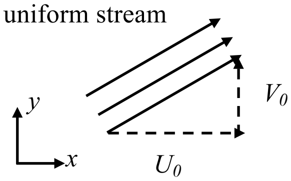
$$\boxed{Q_{AB} = -\tfrac{2}{3}} \tag{8}$$

"Building Block" Stream Functions

The properties of stream functions described previously are enough to justify their use. There are additional reasons to use stream functions, however. Models of real flows can be produced by combining "building block" stream functions. The significance of this topic will be discussed in greater detail when examining the potential function (especially the complex potential); a topic covered later in the notes. For now, however, it is sufficient to present some "building block" stream functions and discuss how they can be combined to produce models of actual flows. First, let's examine a few basic "building block" stream functions.

<table>
<tr><td>

uniform stream



shown for $U_0, V_0 > 0$

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$$\psi = -V_0 x + U_0 y$$
$$u_x = U_0; \quad u_y = V_0$$

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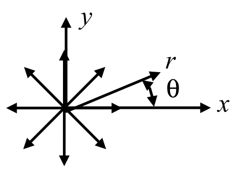
line source (m>0) or sink (m<0)



shown for $m > 0$

m is referred to as the <u>source (sink) strength</u>.

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$$\psi = \frac{m}{2\pi}\theta \quad (0 \le \theta < 2\pi)$$
$$u_r = \frac{m}{2\pi}\frac{1}{r}; \quad u_\theta = 0$$

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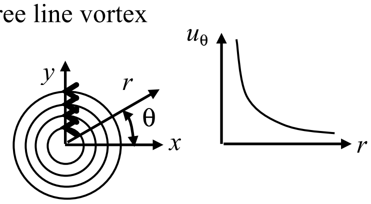
<tr><td>

free line vortex



shown for $\Gamma > 0$

Γ is referred to as the <u>circulation</u>.

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$$\psi = \frac{-\Gamma}{2\pi}\ln r$$
$$u_r = 0; \quad u_\theta = \frac{\Gamma}{2\pi}\frac{1}{r}$$

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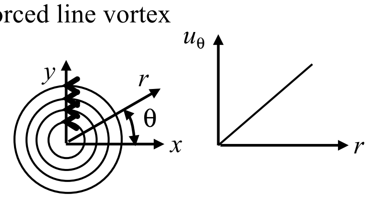
forced line vortex



shown for $K > 0$

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$$\psi = \frac{-Kr^2}{2}$$
$$u_r = 0; \quad u_\theta = Kr$$

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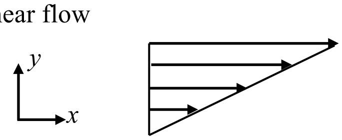
<tr><td>

shear flow



shown for $A > 0$

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$$\psi = Ay^2$$
$$u_x = 2Ay; \quad u_y = 0$$

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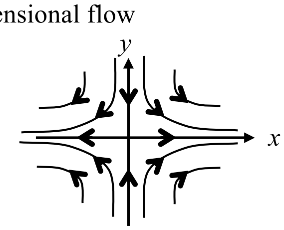
<tr><td>

extensional flow



shown for $A > 0$

Can also be used to model flow in a corner and stagnation point flow.

</td><td>

$$\psi = Axy$$
$$u_x = Ax; \quad u_y = -Ay$$

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</table>

Superposition of Stream Functions

Because the continuity equation is a linear PDE, the "building block" stream functions just presented can be combined together to produce new stream function flows.

Proof:

Let $\psi_T=\psi_1+\psi_2$ where ψ_1 and ψ_2 are stream functions that satisfy the continuity equation. The velocities determined from the new stream function are given by:

$$u_x = \frac{\partial \psi_T}{\partial y} = \frac{\partial(\psi_1+\psi_2)}{\partial y} = \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial y}$$

$$u_y = -\frac{\partial \psi_T}{\partial x} = -\frac{\partial(\psi_1+\psi_2)}{\partial x} = -\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2}{\partial x}$$

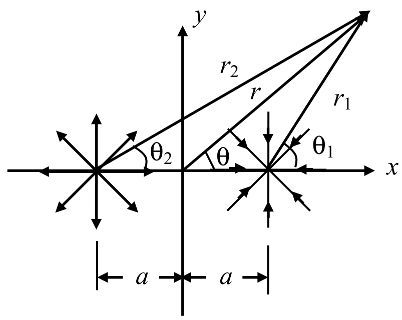
Substitute these velocities into the continuity equation:

$$\begin{aligned}\nabla \cdot \mathbf{u} &= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial x}\right) \\ &= \frac{\partial \psi_1}{\partial x \partial y} + \frac{\partial \psi_2}{\partial x \partial y} - \frac{\partial \psi_1}{\partial y \partial x} - \frac{\partial \psi_2}{\partial y \partial x} \\ &= 0\end{aligned}$$

Thus, the new stream function, ψ_T, formed by the superposition of the original stream functions also satisfies the continuity equation.

Example:

The **doublet** is formed by superposing a source and sink of equal strength separated by an infinitesimal distance.



The stream function for the source/sink pair is given by ($m > 0$):

$$\psi = \frac{m}{2\pi}\theta_2 - \frac{m}{2\pi}\theta_1 = \frac{m}{2\pi}\left(\theta_2 - \theta_1\right)$$

Re-arrange and take the tangent of both sides:

$$\tan\left(\frac{2\pi\psi}{m}\right) = \tan\left(\theta_2 - \theta_1\right) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\theta_1} \tag{9}$$

Note that a trig identity has been used in deriving the previous expression. From the figure we observe that:

$$\tan\theta_1 = \frac{r\sin\theta}{r\cos\theta - a} \quad \text{and} \quad \tan\theta_2 = \frac{r\sin\theta}{r\cos\theta + a}$$

Substitute these expressions into Eq. (9) and simplify:

$$\tan\left(\frac{2\pi\psi}{m}\right) = \frac{\left(\dfrac{r\sin\theta}{r\cos\theta + a}\right) - \left(\dfrac{r\sin\theta}{r\cos\theta - a}\right)}{1 + \left(\dfrac{r\sin\theta}{r\cos\theta + a}\right)\left(\dfrac{r\sin\theta}{r\cos\theta - a}\right)}$$

$$= \frac{\dfrac{r^2\sin\theta\cos\theta - ar\sin\theta - r^2\sin\theta\cos\theta - ar\sin\theta}{r^2\cos^2\theta - a^2}}{1 + \dfrac{r^2\sin^2\theta}{r^2\cos^2\theta - a^2}}$$

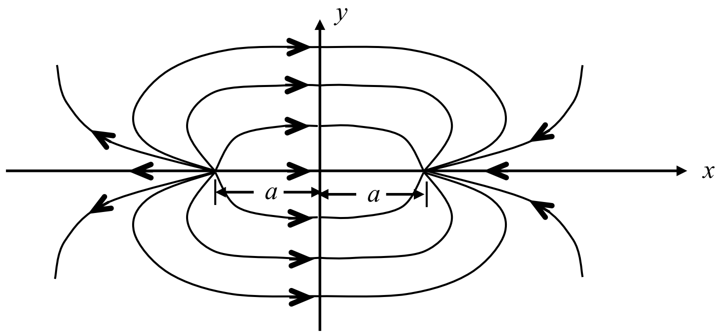
$$= \frac{\dfrac{-2ar\sin\theta}{r^2\cos^2\theta - a^2}}{\dfrac{r^2\cos^2\theta - a^2 + r^2\sin^2\theta}{r^2\cos^2\theta - a^2}}$$

$$= \frac{-2ar\sin\theta}{r^2 - a^2}$$

$$\therefore \psi = \frac{m}{2\pi}\tan^{-1}\left(\frac{-2ar\sin\theta}{r^2 - a^2}\right) \tag{10}$$

Stream function for a source/sink pair of equal strength each located a distance *a* from the origin along the *x*-axis.

The streamlines for the stream function given in Eq. (10) are shown in the following figure.



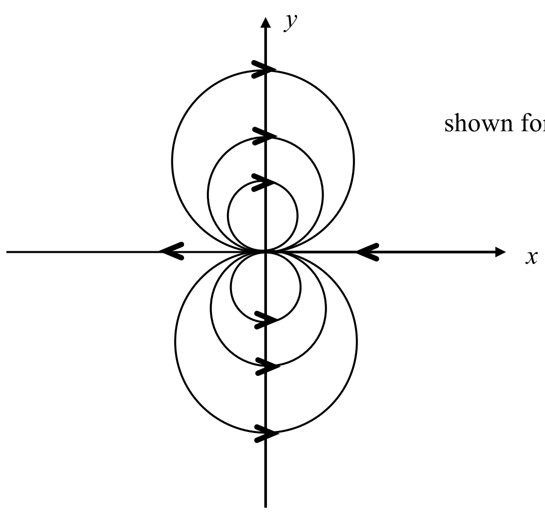
Note that as $a \to 0$, Eq. (10) becomes:

$$\lim_{a \to 0} \psi = \frac{m}{2\pi}\left(\frac{-2ar\sin\theta}{r^2 - a^2}\right)$$

since the tangent of a very small angle approaches the value of the angle. If we let the source and sink approach each other ($a \to 0$) while we let the source/sink strength approach infinity ($m \to \infty$) such that the ratio $ma/\pi = K$=constant, then the stream function becomes:

$$\psi_{\substack{\text{doublet oriented} \\ \text{along } x\text{-axis}}} = \frac{-K\sin\theta}{r} \qquad \text{Note: } 0 \le \theta < 2\pi$$

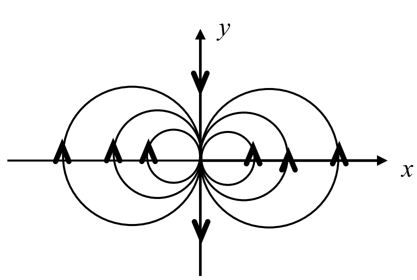
The streamlines for the doublet are circles passing through the origin as shown in the figure below.



Note:

1. Doublets have an orientation. The stream function for a doublet oriented in the y-direction is given by:

$$\psi_{\substack{\text{doublet oriented} \\ \text{along y-axis}}} = \frac{K\cos\theta}{r}$$

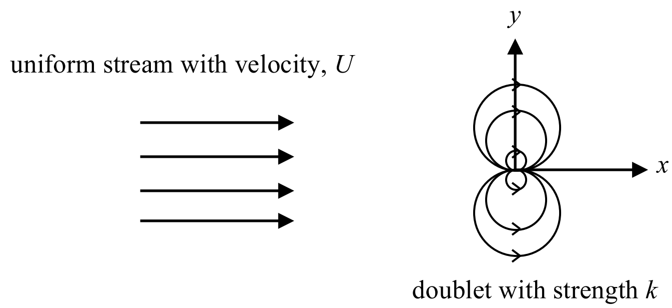


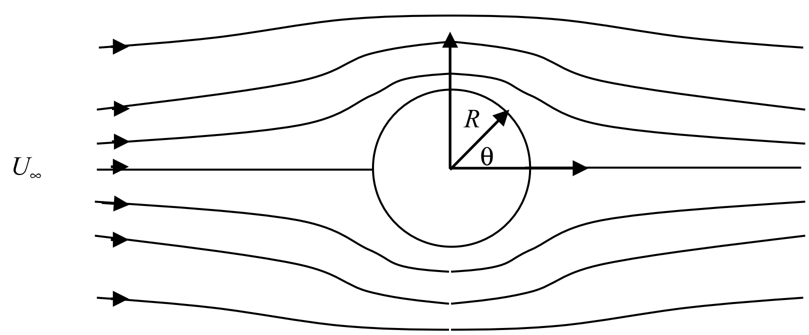
Example:

The flow of a frictionless fluid (there can be slip at solid surfaces) around a non-rotating cylinder can be modeled as a uniform stream superimposed with a doublet:

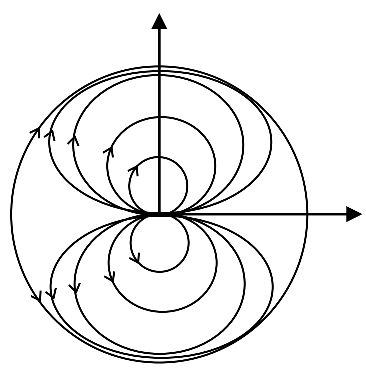
$$\psi_{\substack{\text{flow around}\\ \text{cylinder}}} = \psi_{\substack{\text{uniform}\\ \text{stream}}} + \psi_{\text{doublet}}$$

If the cylinder radius is R, determine the velocity of the fluid on the surface of the cylinder as a function of angular position, θ.





Note that inside the cylinder the streamlines look like:



SOLUTION:

The stream function for flow around a non-rotating cylinder is given by combining the stream function for a uniform stream with the stream function for a (horizontally-oriented) doublet.

$$\psi_{\substack{\text{flow around}\\\text{cylinder}}} = \psi_{\substack{\text{uniform}\\\text{stream}}} + \psi_{\text{doublet}}$$
$$= Uy - \frac{K\sin\theta}{r}$$
$$\therefore \psi_{\substack{\text{flow around}\\\text{cylinder}}} = Ur\sin\theta - \frac{K\sin\theta}{r}$$

At the moment, K is an unknown constant. It can be determined by noting that there is no flow through the cylinder surface. Hence, the radial velocity, u_r, at $r = R$ should be zero.

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta - \frac{K\cos\theta}{r^2}$$
$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta - \frac{K\sin\theta}{r^2}$$

No flow through the cylinder surface at $r = R$:

$$u_r\big|_{r=R} = 0 = U\cos\theta - \frac{K\cos\theta}{R^2}$$
$$\therefore K = UR^2$$

Hence, the stream function and flow velocities for flow around a non-rotating cylinder are:

$$\psi_{\substack{\text{flow around}\\\text{cylinder}}} = Ur\sin\theta\left[1-\left(\frac{R}{r}\right)^2\right]$$
$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = Ur\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right]$$
$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right]$$

On the cylinder surface ($r = R$):

$$\boxed{u_r\big|_{r=R} = 0} \quad \text{and} \quad \boxed{u_\theta\big|_{r=R} = -2U\sin\theta}$$

Note that there are stagnation points at $\theta = 0, \pi$. A maximum speed of $2U$ occurs at $\theta = \pi/2, -\pi/2$.

Notes:

1. Stream functions can also be defined for steady, compressible, 2D flows. For example, in rectangular coordinates:

   $$u_x = \frac{\rho_0}{\rho}\frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y = -\frac{\rho_0}{\rho}\frac{\partial \psi}{\partial x} \quad \text{where } \rho_0 \text{ is a reference density}$$

   The continuity equation for these conditions is:

   $$\nabla \cdot (\rho \mathbf{u}) = \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} = \frac{\partial}{\partial x}\left(\rho \frac{\rho_0}{\rho}\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\rho \frac{\rho_0}{\rho}\frac{\partial \psi}{\partial x}\right) = 0$$

2. Stream functions also exist for axi-symmetric, incompressible flows (referred to as Stokes' stream functions): $\psi = \psi(r, z)$

   $$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

   The continuity equation for these conditions is:

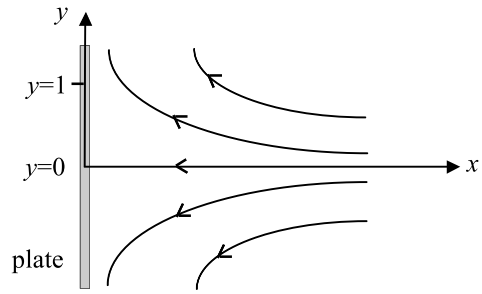
   $$\frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0$$

3. Stream functions <u>cannot</u> be defined for <u>arbitrary</u> 3D flows.

The following stream function is an exact solution to the Navier-Stokes equations and represents the steady, planar flow of a viscous, Newtonian, incompressible fluid of dynamic viscosity, μ, approaching a flat plate:

$$\psi = -Ax^2 y - Bx^3 y$$

where A and B are known constants. The following is a sketch of the flow:



Neglecting gravity, evaluate:

a. the velocity of the flow in terms of A, B, x, and y,
b. the vorticity in the flow in terms of A, B, x, and y,
c. the shear stress on the plate as a function of y (also indicate the direction), and
d. the shear force on that portion of the plate between y=0 and y=1 per unit depth into the page (also indicate the direction).
e. Can we write a potential function for this flow? Explain your answer.
f. Determine the normal force acting on the wall between y = 0 and y = 1 per unit depth into the page. You may assume that the pressure at the origin, p_0, is known.

SOLUTION:

Determine the velocities from the stream function.

$$\boxed{u_x = \frac{\partial \psi}{\partial y} = -Ax^2 - Bx^3} \text{ and } \boxed{u_y = -\frac{\partial \psi}{\partial x} = 2Axy + 3Bx^2 y} \tag{1}$$

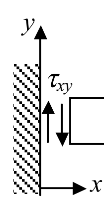
The vorticity is given by:

$$\boldsymbol{\xi} = \nabla \times \mathbf{u}$$

$$\boxed{\xi_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 2Ay + 6Bxy} \tag{2}$$

The shear stress on the fluid element adjacent to the wall is:

$$\tau_{xy}\Big|_{x=0} = \mu \frac{\partial u_y}{\partial x}\Big|_{x=0} = \mu\left(2Ay + 6Bxy\right)\Big|_{x=0}$$



$$\boxed{\therefore \tau_{xy}\Big|_{x=0} = 2\mu A y} \tag{3}$$

Thus, the shear stress acting on the wall is equal in magnitude but in the opposite direction. A positive stress acting on the wall is shown in the figure above.

The shear force acting on the wall is found by integrating the shear stress over the given area.

$$F_S\Big|_{x=0} = \int_{y=0}^{y=1} \tau_{xy}\Big|_{x=0} dy = \int_{y=0}^{y=1} 2\mu A y dy$$

$$\boxed{\therefore F_S\Big|_{x=0} = \mu A} \text{ (The force is directed in the } +y\text{-direction.)} \tag{4}$$

We <u>cannot write a potential function</u> for the given flow since the flow is not irrotational (refer to Eqn. (2)).

The normal force acting on the wall is found by integrating the pressure acting on the wall. The pressure is determined by solving the Navier-Stokes equation in the y-direction.

$$\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)+\rho g_y$$

$$\rho\left[\left(-Ax^2-Bx^3\right)\left(2A+6Bxy\right)+\left(2Axy+3Bx^2y\right)\left(2Ax+3Bx^2\right)\right]=-\frac{\partial p}{\partial y}+\mu\left[\left(6By\right)+\left(0\right)\right]$$

$$\rho\left[-2A^2x^2-6ABx^3y-2ABx^3-6B^2x^4y+4A^2x^2y+6ABx^3y+6ABx^3y+9B^2x^4y\right]=-\frac{\partial p}{\partial y}+6\mu By$$

$$\rho\left[-2A^2x^2-2ABx^3+4A^2x^2y+6ABx^3y+3B^2x^4y\right]=-\frac{\partial p}{\partial y}+6\mu By$$

$$\left.\frac{\partial p}{\partial y}\right|_{x=0}=6\mu By$$

$$\left.p\right|_{x=0}=3\mu By^2+c$$

(Note that in general the pressure will include an unknown function of x upon integrating with respect to y. However, since the pressure is being evaluated at $x = 0$ the function can be, at most, a constant.)

Evaluating the pressure at $y = 0$ gives the constant to be $c = p_0$. Hence,

$$\left.p\right|_{x=0}=3\mu By^2+p_0$$

The force is found by integrating the pressure over the area from $y = 0$ to $y = 1$.

$$F_N=\int_{y=0}^{y=1}-\left.p\right|_{x=0}dy=-\int_{y=0}^{y=1}\left(3\mu By^2+p_0\right)dy$$

$$\boxed{\therefore F_N=-\left(\mu B+p_0\right)}\quad\text{(i.e., The force acts in the }-x\text{-direction.)}\tag{5}$$

The velocity in the y-direction in a 2D, incompressible flow is given by $v=Ay$ where A is a constant.

a. Find the velocity in the x-direction, u, if $u(1,y) = -A$.
b. Determine the stream function for the flow if $\Psi(0,0) = 0$.
c. Determine the location of any stagnation points in the flow.
d. Sketch the streamlines (and their directions) for the flow. Assume $A<0$.
e. What might this flow represent?
f. Determine the pressure gradient in the x-direction at any point along the x-axis. Assume the fluid is inviscid and neglect gravity.

SOLUTION:

Determine the x-velocity using the continuity equation.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}=-A$$

$$u=-Ax+f\left(y\right) \text{ where } f(y) \text{ is an unknown function of } y \tag{2}$$

Since we're given that $u(1,y) = -A$, then $f(y)$ must be identically zero. Hence,

$$\boxed{u=-Ax} \tag{3}$$

The stream function may be determined from the velocities.

$$\frac{\partial \psi}{\partial y}=u=-Ax \quad \Rightarrow \quad \psi=-Axy+g\left(x\right) \text{ where } g(x) \text{ is an unknown function of } x \tag{4}$$

$$-\frac{\partial \psi}{\partial x}=v=Ay \quad \Rightarrow \quad \psi=-Axy+f\left(y\right) \text{ where } f(y) \text{ is an unknown function of } y \tag{5}$$

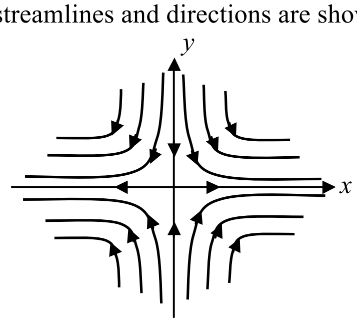
Comparing Eqns. (4) and (5) and noting that $\psi(0,0) = 0$ indicates that the unknown functions are identically zero. Hence:

$$\boxed{\psi=-Axy} \tag{6}$$

Stagnation points occur where the velocity is zero.

$$u=0=-Ax \quad \Rightarrow \quad x=0$$
$$v=0=Ay \quad \Rightarrow \quad y=0$$
$$\boxed{\therefore\left(x,y\right)=\left(0,0\right)} \text{ location of the only stagnation point in the flow} \tag{7}$$

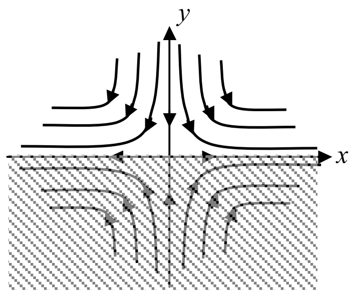
The flow streamlines and directions are shown in the following plot.



$$y=-\frac{\psi}{A}\frac{1}{x} \text{ (hyperbolas!)}$$

shown for $A < 0$

The flow could represent friction flow against a horizontal wall (known as "stagnation point flow"). Or it could represent frictionless flow in a 90° corner.



The pressure gradient can be determined using Bernoulli's equation along the x-axis.

$$p+\tfrac{1}{2}\rho V^2 = \text{constant}$$

$$\frac{\partial p}{\partial x} = -\rho V\frac{\partial V}{\partial x}$$

On the x-axis the velocity component is $V = u = -Ax$ so that:

$$\boxed{\frac{\partial p}{\partial x} = -\rho A^2 x} \qquad (8)$$

A particular planar, incompressible flow can be described with the following stream function:

$$\psi = Axy$$

where A is a constant.

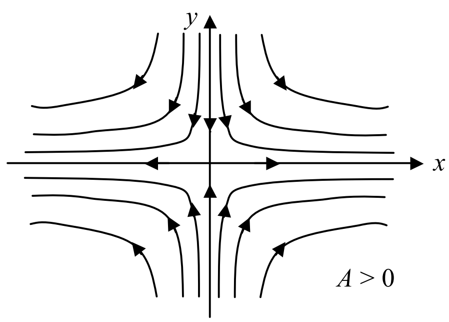
a. Sketch the streamlines for the flow.
b. Determine the velocity components for the flow.

SOLUTION:

The stream function is a constant along a streamline so the equation of the streamlines will be:

$$y = \frac{\psi}{A}\frac{1}{x} \quad \text{(hyperbolas!)}$$

A plot of the streamlines is shown below. Note that A has been assumed to be a positive constant (i.e., $A > 0$) in determining the direction of the flow.



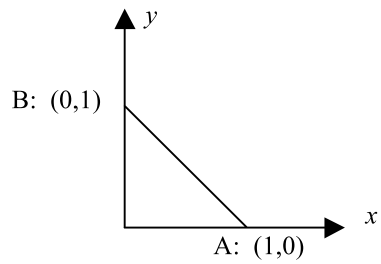
The velocities are determined from the definition of the stream function.

$$\boxed{u_x = \frac{\partial \psi}{\partial y} = Ax} \quad \text{and} \quad \boxed{u_y = -\frac{\partial \psi}{\partial x} = -Ay}$$

The velocity field for a planar, incompressible flow is given by:

$$\mathbf{u} = 2(x^2 - y^2)\hat{\mathbf{e}}_x - 4xy\hat{\mathbf{e}}_y$$

a. Determine the stream function for this flow field if $\psi(0,0)=0$.
b. Determine the volumetric flow rate across the line AB shown in the figure.



SOLUTION:

Recall that the velocities are determined from the stream function in the following manner.

$$u_x = \frac{\partial \psi}{\partial y} = 2\left(x^2 - y^2\right) \quad \Rightarrow \quad \psi = 2\left(x^2 y - \tfrac{1}{3}y^3\right) + f(x) \tag{1}$$

$$u_x = -\frac{\partial \psi}{\partial x} = 4xy \quad \Rightarrow \quad \psi = -2x^2 y + g(y) \tag{2}$$

where f and g are, at this point, unknown functions of x and y, respectively. Comparing Eqs. (1) and (2) indicates that:

$$f(x) = c \quad \text{and} \quad g(y) = -\tfrac{2}{3}y^3 + c$$

where c is a constant. Hence, the stream function is:

$$\psi = 2\left(x^2 y - \tfrac{1}{3}y^3\right) + c \tag{3}$$

Knowing that $\psi(0, 0) = 0$ we can conclude that $c = 0$ and:

$$\boxed{\psi = 2\left(x^2 y - \tfrac{1}{3}y^3\right)} \tag{4}$$

Recall that the volumetric flow rate between two streamlines is equal to the difference in the streamline stream functions.

$$Q_{AB} = \psi_B - \psi_A \tag{5}$$

where

$$\psi_B = \psi(0,1) = -\tfrac{2}{3}$$

$$\psi_A = \psi(1,0) = 0$$

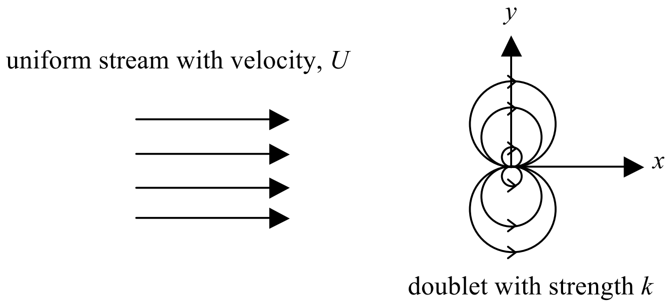
so that

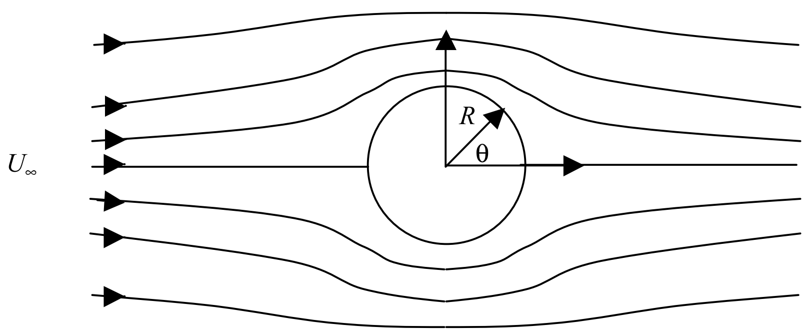
$$\boxed{Q_{AB} = -\tfrac{2}{3}} \tag{6}$$

The flow of a frictionless fluid (there can be slip at solid surfaces) around a non-rotating cylinder can be modeled as a uniform stream superimposed with a doublet:

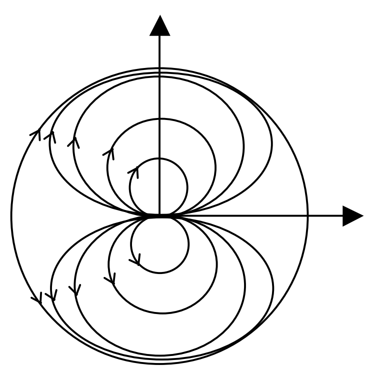
$$\psi_{\substack{\text{flow around}\\\text{cylinder}}} = \psi_{\substack{\text{uniform}\\\text{stream}}} + \psi_{\text{doublet}}$$

If the cylinder radius is R, determine the velocity of the fluid on the surface of the cylinder as a function of angular position, θ.





Note that inside the cylinder the streamlines look like:



SOLUTION:

The stream function for flow around a non-rotating cylinder is given by combining the stream function for a uniform stream with the stream function for a (horizontally-oriented) doublet.

$$\psi_{\substack{\text{flow around}\\ \text{cylinder}}} = \psi_{\substack{\text{uniform}\\ \text{stream}}} + \psi_{\text{doublet}}$$

$$= Uy - \frac{K\sin\theta}{r}$$

$$\therefore \psi_{\substack{\text{flow around}\\ \text{cylinder}}} = Ur\sin\theta - \frac{K\sin\theta}{r}$$

At the moment, K is an unknown constant. It can be determined by noting that there is no flow through the cylinder surface. Hence, the radial velocity, u_r, at $r = R$ should be zero.

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta - \frac{K\cos\theta}{r^2}$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta - \frac{K\sin\theta}{r^2}$$

No flow through the cylinder surface at $r = R$:

$$u_r\big|_{r=R} = 0 = U\cos\theta - \frac{K\cos\theta}{R^2}$$

$$\therefore K = UR^2$$

Hence, the stream function and flow velocities for flow around a non-rotating cylinder is:

$$\psi_{\substack{\text{flow around}\\ \text{cylinder}}} = Ur\sin\theta\left[1-\left(\frac{R}{r}\right)^2\right]$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = Ur\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right]$$

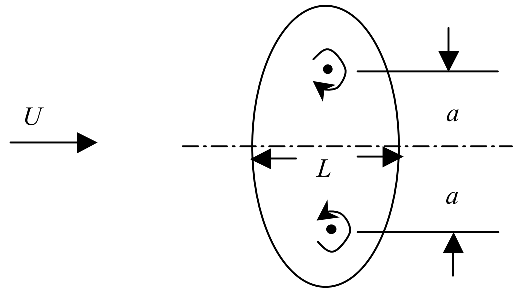
$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right]$$

On the cylinder surface ($r = R$):

$$\boxed{u_r\big|_{r=R} = 0} \quad \text{and} \quad \boxed{u_\theta\big|_{r=R} = -2U\sin\theta}$$

Note that there are stagnation points at $\theta = 0, \pi$. A maximum speed of $2U$ occurs at $\theta = \pi/2, -\pi/2$.

The planar flow of an incompressible fluid around a finite body is simulated by the superposition of two free vortices of opposite rotation and a uniform stream of velocity, U (such a body is called a Kelvin oval). The vortices have the same magnitude of circulation, Γ, and are located a distance $2a$ apart as shown in the figure.



Find the axial length, L, of the body in terms of a, U, and Γ.

SOLUTION:

Form the stream function by superposing a uniform stream with a clockwise free vortex centered at $(0, a)$ and a counter-clockwise free vortex centered at $(0, -a)$.

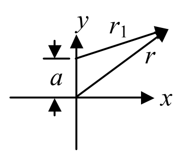
$$\psi = \psi_{\substack{\text{uniform}\\ \text{stream in}\\ x\text{-direction}}} + \psi_{\substack{\text{clockwise}\\ \text{free vortex}\\ \text{centered at }(0,a)}} + \psi_{\substack{\text{counter-clockwise}\\ \text{free vortex}\\ \text{centered at }(0,-a)}}$$

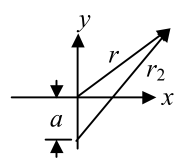
where

$$\psi_{\substack{\text{uniform}\\ \text{stream in}\\ x\text{-direction}}} = Uy$$

$$\psi_{\substack{\text{clockwise}\\ \text{free vortex}\\ \text{centered at }(0,a)}} = \frac{\Gamma}{2\pi}\ln r_1$$

$$\psi_{\substack{\text{counter-clockwise}\\ \text{free vortex}\\ \text{centered at }(0,-a)}} = \frac{-\Gamma}{2\pi}\ln r_2$$





Convert the r_1 and r_2 coordinates into common coordinates:

$$r_1 = \sqrt{x^2 + (y-a)^2}$$

$$r_2 = \sqrt{x^2 + (y+a)^2}$$

Re-write the stream function:

$$\psi = Uy + \frac{\Gamma}{2\pi}\ln\sqrt{x^2 + (y-a)^2} - \frac{\Gamma}{2\pi}\ln\sqrt{x^2 + (y+a)^2}$$

Determine velocities from the stream function:

$$u_x = \frac{\partial \psi}{\partial y} = U + \frac{\Gamma}{2\pi}\frac{(y-a)}{x^2+(y-a)^2} - \frac{\Gamma}{2\pi}\frac{(y+a)}{x^2+(y+a)^2}$$

$$u_y = -\frac{\partial \psi}{\partial x} = \frac{\Gamma}{2\pi}\frac{x}{x^2+(y-a)^2} - \frac{\Gamma}{2\pi}\frac{x}{x^2+(y-a)^2}$$

The axial length, L, is the distance between the leading and trailing stagnation points on the x-axis.

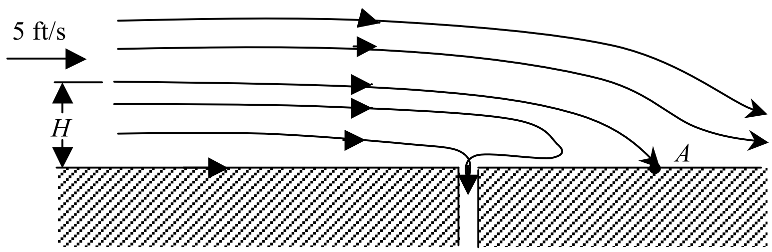
$$u_x\left(x=\tfrac{1}{2}L, y=0\right) = 0 = U - \frac{\Gamma}{2\pi}\frac{a}{\frac{1}{4}L^2+a^2} - \frac{\Gamma}{2\pi}\frac{a}{\frac{1}{4}L^2+a^2}$$

$$\boxed{\therefore L = 2\sqrt{\frac{\Gamma}{\pi}\frac{a}{U} - a^2}}$$ (Note: We must have $\Gamma/(\pi U a) > 1$ for the Kelvin oval to exist.)

Note that we could have used a potential function for this problem instead of a stream function.

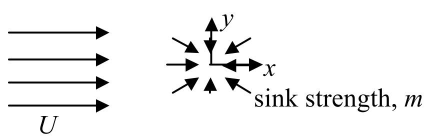
Water flows over a flat surface at 5 ft/s as shown in the figure. A pump draws off water through a narrow slit at a volume flow rate of 0.1 ft^3/s per foot length of the slit. Assume that the fluid is incompressible and inviscid and can be represented by the combination of a uniform flow and a sink.

a. Locate the stagnation point on the wall (point *A*) and determine the equation for the stagnation streamline.
b. How far above the surface, *H*, must the fluid be so that it does not get sucked into the slit?



SOLUTION:

Model the given flow as a uniform flow plus a line sink.



$$\psi = \psi_{\substack{\text{uniform}\\\text{flow}}} + \psi_{\substack{\text{line}\\\text{sink}}} = Uy - \frac{m}{2\pi}\theta = Ur\sin\theta - \frac{m}{2\pi}\theta \quad \text{(where } m > 0\text{)} \tag{1}$$

Determine the velocity field from the stream function.

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta - \frac{m}{2\pi}\frac{1}{r} \tag{2}$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -U\sin\theta \tag{3}$$

The stagnation point is where $\mathbf{u} = \mathbf{0}$.

$$u_\theta = 0 = -U\sin\theta_S \Rightarrow \theta_S = 0 \tag{4}$$

(Neglect $\theta_S = \pi$ since the stagnation point will occur downstream of the sink.)

$$u_r = 0 = U\underbrace{\cos\theta_S}_{=1} - \frac{m}{2\pi}\frac{1}{r_S} \Rightarrow r_S = \frac{m}{2\pi U} \tag{5}$$

(Note that if $\theta_S = \pi$, then $r_S = -m/(2\pi U)$. It's the same as the previously determined point!)

$$\boxed{\therefore (r_S, \theta_S) = \left(\frac{m}{2\pi U}, 0\right)} \tag{6}$$

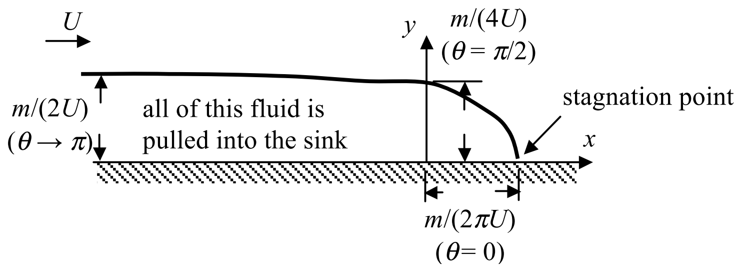
The equation for the stagnation streamline can be determined by evaluating the stream function value at the stagnation point.

$$\psi_S = Ur_S \sin\theta_S - \frac{m}{2\pi}\theta_S = 0 \tag{7}$$

Hence, the stagnation streamline equation is:

$$U\underbrace{r\sin\theta}_{=y} - \frac{m}{2\pi}\theta = 0 \tag{8}$$

$$\boxed{\therefore y = \frac{m}{2\pi U}\theta} \tag{9}$$



In order for the fluid not to be sucked into the slit, it must be on a streamline that is above the stagnation streamline, i.e.,

$$\boxed{H > \frac{m}{2U}} \tag{10}$$

For the given values:

U = 5 ft/s

m = $2(0.1\ \text{ft}^3/(\text{s}\cdot\text{ft})) = 0.2\ \text{ft}^3/(\text{s}\cdot\text{ft})$

(Recall that m is the strength of the <u>full</u> source. Here, we know that <u>one half</u> of the strength, m, is $0.1\ \text{ft}^3/(\text{s}\cdot\text{ft})$.)

$\Rightarrow$ $\boxed{H = 0.02\ \text{ft}}$

We could have also used conservation of mass to determine the height of the stagnation streamline far upstream.

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho UH + \rho Q \quad \text{(where } Q \text{ is the flow rate into the sink)}$$

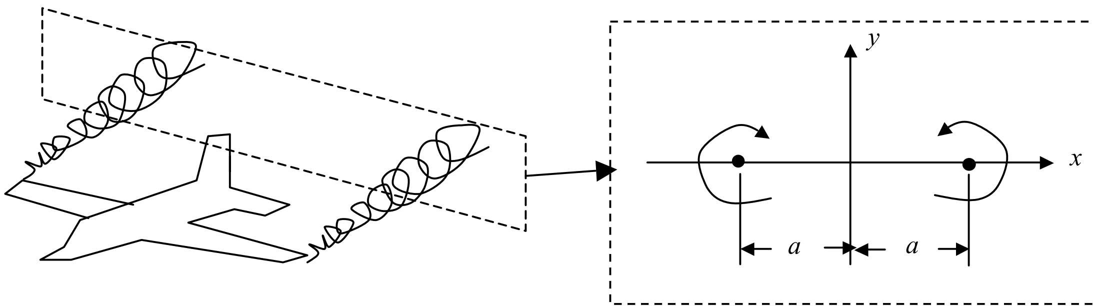
Substitute and simplify.

$$-\rho UH + \rho Q = 0$$

$$\therefore H = \frac{Q}{U} \tag{11}$$

where $Q = 0.1\ \text{ft}^3/(\text{s}\cdot\text{ft})$ and $U = 5$ ft/s $\Rightarrow$ $H = 0.02$ ft. This is the same answer as before!

Wing-tip vortices form on aircraft wings as a result of higher pressure air on the bottom of the wing "wrapping over" to the lower-pressure air on the top of the wing. This flow can be modeled as the superposition of two equal strength, counter-rotating free vortices as shown in the figure.



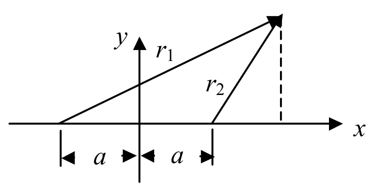
frontal view

a. Write the stream function for the flow in terms of x and y using the coordinate system shown.
b. Calculate the velocity field resulting from these vortices.
c. Are there any stagnant regions in the flow? If so, where are they located? (Careful on this one.)
d. What trajectory will these vortices have over time? Why? What significance would this phenomenon have on the landing of aircraft at airports?

SOLUTION:

The stream function for the system of vortices is:

$$\psi = \underset{\text{vortex}}{\psi_{\text{left}}} + \underset{\text{vortex}}{\psi_{\text{right}}} = \frac{\Gamma}{2\pi}\ln r_1 - \frac{\Gamma}{2\pi}\ln r_2 \tag{1}$$



where

$$r_1^2 = (x+a)^2 + y^2 \tag{2}$$

$$r_2^2 = (x-a)^2 + y^2 \tag{3}$$

$$\boxed{\therefore \psi = \frac{\Gamma}{4\pi}\ln\left[(x+a)^2 + y^2\right] - \frac{\Gamma}{4\pi}\ln\left[(x-a)^2 + y^2\right]} \tag{4}$$

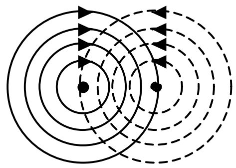
The velocities may be found from the stream function.

$$\boxed{u_x = \frac{\partial \psi}{\partial y} = \frac{\Gamma}{4\pi}\left[\frac{2y}{(x+a)^2 + y^2} - \frac{2y}{(x-a)^2 + y^2}\right]} \tag{5}$$

$$\boxed{u_y = -\frac{\partial \psi}{\partial x} = -\frac{\Gamma}{4\pi}\left[\frac{2(x+a)}{(x+a)^2 + y^2} - \frac{2(x-a)}{(x-a)^2 + y^2}\right]} \tag{6}$$

Stagnation points occur where $\mathbf{u} = \mathbf{0}$. Both velocity components are zero only far from the center of the vortices, i.e., $\boxed{\text{the flow stagnation points (a region actually) occur at } r \to \infty.}$

The vortices will move in the $-y$-direction over time since the "core" of each vortex will move downward due to the influence of the other vortex.

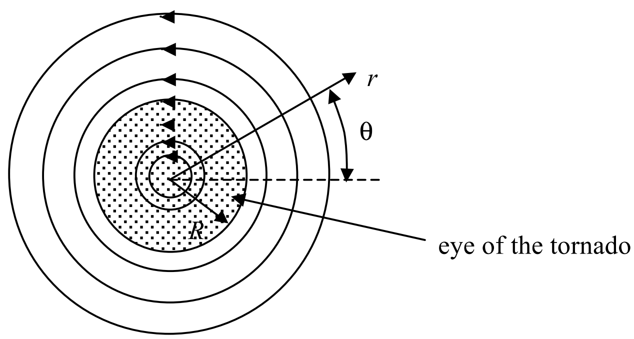


There is downward flow at the center of the right vortex due to the flow created by the left vortex.

Aircraft at airports are spaced out so that the vortices have enough time to dissipate (due to viscous effects) before the next aircraft comes in to land. A number of aircraft crashes have been attributed to this phenomenon (see, for example, the National Transportation Safety Board (NTSB) web site: http://www.ntsb.gov/aviation.htm).

The eye of a tornado has a radius, R. In the eye, the tornado flow field is approximated as solid body rotation while outside the eye the flow is a free vortex.

a. Write the stream function for the tornado.
b. Determine the tangential velocity distribution of the tornado, $u_\theta(r)$, if the maximum wind velocity is U_{max}.
c. Where in the flow field is the flow irrotational? Prove it.
d. Determine the pressure variation, $p(r)$, resulting from the tornado. Note that the pressure far from the tornado is atmospheric pressure, p_{atm}.
e. Where will the minimum pressure occur?
f. Explain why closed windows in a home often blow *outward* when a tornado is nearby.

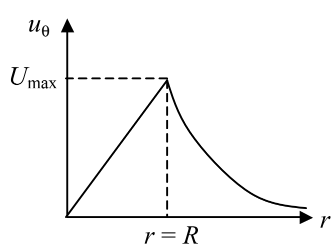


SOLUTION:

The stream function for this flow consists of a free vortex for $r > R$ and a forced vortex for $0 \le r \le R$:

$$\boxed{\psi = \begin{cases} -\frac{1}{2}kr^2 & 0 \le r \le R \\ \dfrac{-\Gamma}{2\pi}\ln r & r > R \end{cases}} \tag{1}$$

The maximum wind speed will occur at the edge of the tornado eye ($r = R$).



$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = \begin{cases} kr & 0 \le r \le R \\ \dfrac{\Gamma}{2\pi}\dfrac{1}{r} & r > R \end{cases}$$

Since U_{max} occurs at $r = R$:

$$u_\theta\big|_{r=R} = U_{max} = \begin{cases} kR \\ \dfrac{\Gamma}{2\pi}\dfrac{1}{R} \end{cases} \Rightarrow \begin{matrix} k = \dfrac{U_{max}}{R} \\ \Gamma = 2\pi R U_{max} \end{matrix}$$

and

$$\boxed{u_\theta = \begin{cases} U_{max}\dfrac{r}{R} & 0 \le r \le R \\ U_{max}\dfrac{R}{r} & r > R \end{cases}} \tag{2}$$

The flow is irrotational if the vorticity, $\boldsymbol{\xi}$, is zero (here we only care about the z-component):

$$\xi_z = (\nabla\times\mathbf{u})_z = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) = \begin{cases} \dfrac{2U_{\max}}{R} & 0\le r\le R \\ 0 & r>R \end{cases}$$

Hence, the flow is irrotational for $r > R$ but rotational for $0 \le r \le R$.

Bernoulli's equation can be used everywhere within the free vortex region ($r > R$) since the flow is irrotational there.

$$\left(p+\tfrac{1}{2}\rho V^2\right)_r = \left(p+\tfrac{1}{2}\rho V^2\right)_\infty$$

where

$$V_r = U_{\max}\frac{R}{r}$$

$$V_\infty = 0$$

Substitute and simplify:

$$p_r - p_\infty = -\tfrac{1}{2}\rho V_r^2 = -\tfrac{1}{2}\rho U_{\max}^2\left(\frac{R}{r}\right)^2$$

$$\boxed{\frac{p_r-p_\infty}{\frac{1}{2}\rho U_{\max}^2} = -\left(\frac{R}{r}\right)^2} \text{ for } r>R \tag{3}$$

Within the forced vortex ($0 \le r \le R$) Bernoulli's equation is restricted to a streamline since the flow is rotational there. Hence, Euler's equations must be used to determine the pressure distribution across the streamlines. Euler's equation in the radial direction, assuming planar flow and no body forces, is:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta^2}{r}\right) = -\frac{\partial p}{\partial r}$$

Noting that $u_r = 0$ and $p = p(r)$ and simplifying gives:

$$\rho\frac{u_\theta^2}{r} = \frac{dp}{dr}$$

Making use of Eq. (2) and simplifying gives:

$$\rho\frac{U_{\max}^2\left(\frac{r}{R}\right)^2}{r} = \frac{dp}{dr}$$

$$\rho U_{\max}^2\frac{r}{R^2} = \frac{dp}{dr}$$

$$p_r - p_R = \tfrac{1}{2}\rho U_{\max}^2\left[\left(\frac{r}{R}\right)^2 - 1\right]$$

$$\frac{p_r-p_R}{\frac{1}{2}\rho U_{\max}^2} = \left(\frac{r}{R}\right)^2 - 1 \tag{4}$$

where p_R is the pressure at $r = R$ which can be found using Eq. (3):

$$\frac{p_R-p_\infty}{\frac{1}{2}\rho U_{\max}^2} = -1 \Rightarrow \frac{p_R}{\frac{1}{2}\rho U_{\max}^2} = -1 + \frac{p_\infty}{\frac{1}{2}\rho U_{\max}^2}$$

Substituting into Eq. (4) and simplifying gives:

$$\boxed{\frac{p_r-p_\infty}{\frac{1}{2}\rho U_{\max}^2} = \left(\frac{r}{R}\right)^2 - 2} \text{ for } 0\le r\le R \tag{5}$$

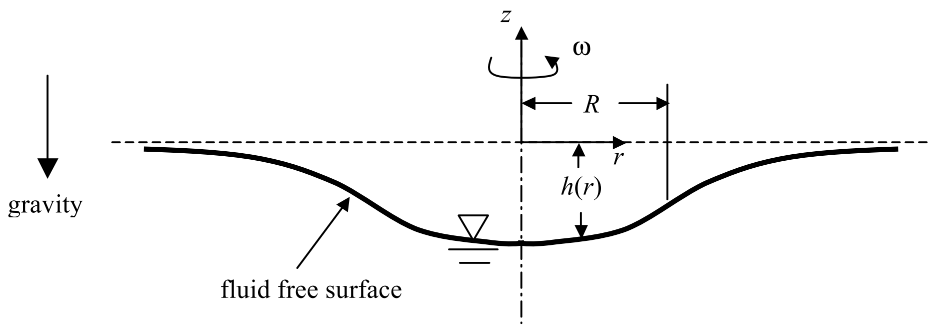
Examining Eqs. (3) and (5) indicates that the minimum pressure for the tornado will occur at the tornado eye ($r = 0$) and will be:

$$\boxed{\frac{p_{r=0} - p_{\infty}}{\frac{1}{2}\rho U_{\max}^2} = -2}$$

Windows in a house tend to blow outward because the pressure inside the house (which is essentially equal to p_{∞} since the air there is stagnant) is larger than the outside pressure resulting from the tornado.

Consider the whirlpool formed by the combination of a forced vortex ($0 < r < R$) and a free vortex ($r > R$). The inner part (forced vortex) of the whirlpool rotates with angular velocity ω.

Determine the profile of the surface, $h(r)$.



SOLUTION:

The forced vortex/free vortex combination can be written in terms of the stream function:

$$\psi = \begin{cases} -\frac{1}{2}kr^2 & 0 \le r \le R \text{ (forced vortex - rotational)} \\ -\frac{\Gamma}{2\pi}\ln r & r > R \text{ (free vortex - irrotational)} \end{cases}$$

The corresponding velocity field is:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = \begin{cases} kr & 0 \le r \le R \\ \frac{\Gamma}{2\pi}\frac{1}{r} & r > R \end{cases}$$

Since the forced vortex rotates with constant angular velocity, ω:

$$u_\theta(r = R) = \omega R = \begin{cases} kR \\ \frac{\Gamma}{2\pi}\frac{1}{R} \end{cases} \Rightarrow \begin{matrix} k = \omega \\ \Gamma = 2\pi\omega R^2 \end{matrix}$$

Hence,

$$u_\theta = \begin{cases} \omega R\left(\frac{r}{R}\right) & 0 \le r \le R \\ \omega R\left(\frac{R}{r}\right) & r > R \end{cases} \tag{1}$$

Use Bernoulli's equation to determine the surface height in the irrotational region ($r > R$). Apply Bernoulli's equation between a point on the free surface located at a distance r from the origin and another point located at a point $r \to \infty$ also on the free surface.

$$\left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_r = \text{constant} = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_\infty$$

where

$$p(r) = p_\infty = p_{\text{atm}}$$
$$V(r) = \omega R\left(\frac{R}{r}\right)$$
$$V_\infty = 0$$
$$z(r) = h(r)$$
$$z_\infty = 0$$

Substitute and solve for $h(r)$.

$$h(r) = -\frac{(\omega R)^2}{2g}\left(\frac{R}{r}\right)^2 \quad \text{for } (r > R) \tag{2}$$

In the rotational zone ($0 \le r \le R$) we must use the Navier-Stokes equations to determine the pressure variation as we cross streamlines. Note that since the flow is in solid body rotation, the viscous stress terms will be zero and the Navier-Stokes equations reduce to Euler's equations. Euler's equation in the r-direction, assuming steady flow with no body forces in the r-direction, is:

$$\rho\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r}$$

Substitute for the velocity and solve for the pressure.

$$\rho\frac{\left[\omega R\left(\frac{r}{R}\right)\right]^2}{r} = \frac{\partial p}{\partial r}$$

$$p(r,z) = \tfrac{1}{2}\rho(\omega R)^2\left(\frac{r}{R}\right)^2 + f(z) \quad \text{where } f(z) \text{ is an unknown function of } z. \tag{3}$$

Euler's equation in the z-direction gives:

$$\frac{\partial p}{\partial z} = -\rho g$$

$$p(r,z) = -\rho g z + g(r) \quad \text{where } g(r) \text{ is an unknown function of } r. \tag{4}$$

Comparing Eqs. (3) and (4) shows that:

$$p(r,z) = \tfrac{1}{2}\rho(\omega R)^2\left(\frac{r}{R}\right)^2 - \rho g z + c \quad \text{where } c \text{ is a constant.} \tag{5}$$

Along the free surface, $p = p_{\text{atm}}$ so that:

$$p_{\text{atm}} = \tfrac{1}{2}\rho(\omega R)^2\left(\frac{r}{R}\right)^2 - \rho g h + c$$

$$h(r) = \frac{(\omega R)^2}{2g}\left(\frac{r}{R}\right)^2 - \frac{p_{\text{atm}}}{\rho g} + c \quad \text{for } (0 \le r \le R)$$

The constant, c, can be determined using Eq. (2) by matching the free surface height at $r = R$.

$$\frac{(\omega R)^2}{2g} - \frac{p_{\text{atm}}}{\rho g} + c = -\frac{(\omega R)^2}{2g}$$

$$\therefore c = \frac{p_{\text{atm}}}{\rho g} - \frac{(\omega R)^2}{g}$$

Thus, the free surface height for the whirlpool is given by:

$$\boxed{h(r) = \begin{cases} \dfrac{(\omega R)^2}{2g}\left[\left(\dfrac{r}{R}\right)^2 - 2\right] & 0 \le r \le R \\ -\dfrac{(\omega R)^2}{2g}\left(\dfrac{R}{r}\right)^2 & r > R \end{cases}} \tag{6}$$

An alternate method for finding the surface slope is to note that at any point on the free surface:

$$dp = 0 = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz = \frac{\rho u_\theta^2}{r}dr - \rho g dz \;\Rightarrow\; \left.\frac{dz}{dr}\right|_{\text{surface}} = \frac{u_\theta^2}{gr}$$

What are three properties of a stream function? Under what general conditions can stream functions <u>not</u> be written?

SOLUTION:

Three properties of a stream function:

1. Velocities determined from the stream function automatically satisfy the continuity equation.
2. The stream function is a constant along a streamline.
3. The volumetric flow rate between two streamlines is equal to the difference in their stream functions.

Stream functions cannot be written for 3D flows, in general.

Consider the plane, incompressible, Cartesian stream function in the region $0 \le y \le \infty$:

$$\psi = ax + by + \frac{b}{c}\exp(-cy)$$

where a, b, and c are positive constants.

a. Does this equation satisfy the continuity equation? Show your work.
b. What relationship must hold between the constants a, b, and c in order for the Navier-Stokes equations to be satisfied if gravity and pressure gradients are neglected?
c. Determine the equation of the streamline passing through the origin.
d. Determine the vorticity field for the flow.
e. Can a potential function be written for the flow? Explain your answer.

SOLUTION:

The given stream function will satisfy the continuity equation based on the definition of the stream function. To demonstrate that this particular stream function satisfies the continuity equation, first find the velocity field from the stream function.

$$u_x = \frac{\partial \psi}{\partial y} = b - b\exp(-cy) = b\left[1-\exp(-cy)\right] \tag{1}$$

$$u_y = -\frac{\partial \psi}{\partial x} = -a \tag{2}$$

The continuity equation for a plane, incompressible flow is:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{3}$$

Substituting Eqs. (1) and (2) into the continuity equation (Eq. (3)) gives:

$$\frac{\partial}{\partial x}\left\{b\left[1-\exp(-cy)\right]\right\} + \frac{\partial}{\partial y}\left\{-a\right\} = 0 + 0 = 0 \tag{4}$$

Thus, we see that $\boxed{\text{this stream function does indeed satisfy the continuity equation}}$.

The Navier-Stokes equations for a planar, incompressible, constant viscosity flow in Cartesian coordinates and neglecting gravity and pressure gradients are:

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) = \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) \tag{5}$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y}\right) = \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) \tag{6}$$

Note that:

$$u_x = b\left[1-\exp(-cy)\right] \Rightarrow \frac{\partial u_x}{\partial t} = \frac{\partial u_x}{\partial x} = \frac{\partial^2 u_x}{\partial x^2} = 0\text{, } \frac{\partial u_x}{\partial y} = bc\exp(-cy)\text{, and } \frac{\partial^2 u_x}{\partial y^2} = -bc^2\exp(-cy) \tag{7}$$

$$u_y = -a \Rightarrow \frac{\partial u_y}{\partial t} = \frac{\partial u_y}{\partial x} = \frac{\partial^2 u_y}{\partial x^2} = \frac{\partial u_y}{\partial y} = \frac{\partial^2 u_y}{\partial y^2} = 0 \tag{8}$$

Substituting Eqs. (7) and (8) into the Navier-Stokes equations (Eqs. (5) and (6)) gives:

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0}+u_x\underbrace{\frac{\partial u_x}{\partial x}}_{=0}+\underbrace{u_y}_{=-a}\underbrace{\frac{\partial u_x}{\partial y}}_{=bc\exp(-cy)}\right)=\mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0}+\underbrace{\frac{\partial^2 u_x}{\partial y^2}}_{=-bc^2\exp(-cy)}\right)\Rightarrow -\rho abc\exp(-cy)=-\mu bc^2\exp(-cy) \tag{9}$$

$\boxed{\therefore a=\nu c}$ in order for the x-component of the N-S equations to be satisfied at all points where ν is the kinematic viscosity. The constant b can have any non-zero value.

$$\rho\left(\underbrace{\frac{\partial u_y}{\partial t}}_{=0}+u_x\underbrace{\frac{\partial u_y}{\partial x}}_{=0}+u_y\underbrace{\frac{\partial u_y}{\partial y}}_{=0}\right)=\mu\left(\underbrace{\frac{\partial^2 u_y}{\partial x^2}}_{=0}+\underbrace{\frac{\partial^2 u_y}{\partial y^2}}_{=0}\right)\Rightarrow 0=0 \tag{10}$$

The y-component of the N-S equations is automatically satisfied.

The streamline passing through the origin will have the stream constant:

$$\psi_{(0,0)}=a\underbrace{x}_{=0}+b\underbrace{y}_{=0}+\frac{b}{c}\exp\left(-c\underbrace{y}_{=0}\right)=\frac{b}{c} \tag{11}$$

Since the stream function is a constant along a streamline, the equation of the streamline passing through the origin is:

$$\boxed{\frac{b}{c}=ax+by+\frac{b}{c}\exp(-cy)} \tag{12}$$

The vorticity may be found from:

$$\boldsymbol{\omega}=\nabla\times\mathbf{u}\Rightarrow\omega_z=\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right) \tag{13}$$

Substituting Eqs. (7) and (8) gives:

$$\omega_z=\left(\underbrace{\frac{\partial u_y}{\partial x}}_{=0}-\underbrace{\frac{\partial u_x}{\partial y}}_{=bc\exp(-cy)}\right)=-bc\exp(-cy) \tag{14}$$

$\boxed{\therefore\omega_z=-bc\exp(-cy)}$ (The other vorticity terms are zero.) (15)

Since the flow is rotational in general, $\boxed{\text{a potential function cannot be written for the flow}}$.

2. Potential Functions

The velocity field for an irrotational flow can be written as the gradient of a potential function, ϕ:

$$\mathbf{u} = \nabla\phi \tag{11}$$

since, from a vector identity:

$$\nabla\times\nabla\phi = \mathbf{0}$$

and because an irrotational flow is defined as one with zero vorticity, i.e.:

$$\nabla\times\boldsymbol{\omega} = \mathbf{0}$$

Now let's ensure that the continuity equation is satisfied for an incompressible fluid (compressible potential flows will be considered in a separate set of notes dedicated specifically to compressible flows):

$$\nabla\cdot\mathbf{u} = 0 \quad \Rightarrow \quad \nabla\cdot\nabla\phi = 0$$

$$\therefore \nabla^2\phi = 0 \tag{12}$$

This is Laplace's Equation!, a well studied, linear, elliptic partial differential equation that appears in many other disciplines such as electromagnetics and conduction heat transfer.

The momentum equations for a potential flow simplify to Bernoulli's equation since the flow is everywhere irrotational (refer to an earlier set of notes concerning Bernoulli's equation):

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \tfrac{1}{2}\left(\nabla\phi\cdot\nabla\phi\right) + G = F(t) \tag{13}$$

where G is a conservative body potential (e.g., for gravity, $G = gz$ where $\mathbf{g}$ and $\hat{\mathbf{e}}_z$ point in opposite directions) and $F(t)$ is a function of time. For a steady potential flow, Eq. (13) simplifies to:

$$\frac{p}{\rho} + \tfrac{1}{2}\left(\nabla\phi\cdot\nabla\phi\right) + G = \text{constant} \tag{14}$$

Note that the momentum equation (Eq. (13) or (14)) need not be solved to determine the fluid kinematics. Solving Eq. (12) subject to appropriate boundary conditions is sufficient to determine the flow velocity field. This occurs because we placed two restrictions on the flow field: the continuity equation and the irrotationality assumption. The momentum equation can be solved to determine the fluid pressure field once the velocity field is known.

The appropriate boundary conditions for Laplace's equation are either Dirichlet (the functions value is specified), Neumann (the functions gradient is specified), or mixed. At solid surfaces the appropriate boundary condition for the flow is that the flow velocity normal to the surface is equal to the surface velocity, i.e.:

$$\mathbf{u}\cdot\hat{\mathbf{n}} = \mathbf{U}\cdot\hat{\mathbf{n}} \tag{15}$$

where $\mathbf{u}$ is the fluid velocity, $\mathbf{U}$ is the boundary velocity, and $\hat{\mathbf{n}}$ is the normal vector to the boundary. This is a Dirichlet or kinematic boundary condition. Note that the no-slip condition is not satisfied for potential flows. This occurs because potential flows have no viscous force contributions since the viscous terms in the Navier-Stokes equations (i.e., momentum equations) drop out due to the irrotationality assumption. As a result, the Navier-Stokes equations, which are normally 2nd order PDEs, simplify to the 1st order Euler's equations (and can be simplified further to Bernoulli's equation). Hence, only a single boundary condition must be specified.

Neumann boundary conditions are specified at free surfaces, i.e., surfaces where the pressure is defined. These are sometimes called dynamic boundary conditions. Bernoulli's Equation (Eqs. (13) or (14)) is used to relate free pressure boundary conditions to the velocity field.

Notes:

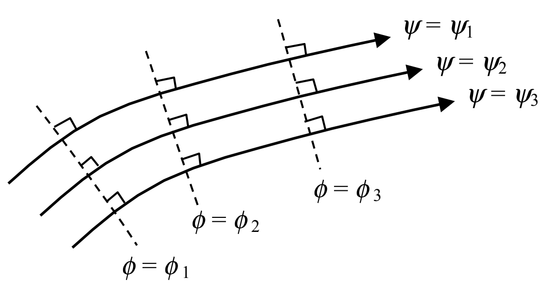
1. Incompressible potential flows are often referred to as **ideal fluid flows** since the fluid is incompressible and viscous forces are negligible.

2. Potential functions can be defined for 3D flows (as long as they're irrotational). Recall that stream functions could only be specified for 2D flows.

3. The governing equation for potential flows (Laplace's equation) is a linear PDE so that the principle of superposition can be used to combine potential flow solutions. The approach is similar to that discussed previously for stream functions.

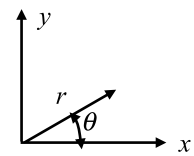
4. Potential functions and stream functions are intimately related. This will become clear in the following section of notes concerning the complex potential function.

5. Streamlines (ψ = constant) and equi-potential lines (ϕ = constant) are perpendicular everywhere in the flow. Consider the curves along which ψ = constant (a streamline) and ϕ= constant:
   $$d\psi = 0 = \nabla\psi \cdot d\mathbf{x}$$
   $$d\phi = 0 = \nabla\phi \cdot d\mathbf{x}$$
   where $d\boldsymbol{x}$ is a small distance along the curves. Re-write these relations in terms of the velocities:
   $$\nabla\psi \cdot d\mathbf{x} = 0 = -u_y dx + u_x dy \Rightarrow \frac{dy}{dx} = \frac{u_y}{u_x}$$
   $$\nabla\phi \cdot d\mathbf{x} = 0 = u_x dx + u_y dy \Rightarrow \frac{dy}{dx} = -\frac{u_x}{u_y}$$
   From analytical geometry, two curves are perpendicular to each other if the slopes of the curves multiplied together equals -1. Hence, we see that the streamlines and equi-potential lines will always be perpendicular to each other. The resulting mesh of streamlines and equi-potential lines is known as the **flow net**.



3. Complex Variable Methods for Investigating Planar, Ideal, Irrotational Flows
A good mathematics reference for this topic is: Churchill, R.V. and Brown, J.W., *Complex Variables and Applications*, McGraw-Hill.

Let's define the complex potential, $f(z)$: $f(z) = \phi + i\psi$



where $z = x+iy = r\exp(i\theta)$ where $0 \leq \theta < 2\pi$ and $\exp(i\theta)=\cos(\theta)+i\sin(\theta)$
ϕ is the velocity potential
ψ is the stream function

Why do this? Because it allows us to present information in a compact manner and because we can use tools from complex variable mathematics to analyze fluid flows.

Notes:

1. A few complex variables preliminaries:
   a. $z = x+iy = r\exp(i\theta)$ where $0 \leq \theta < 2\pi$ and $\exp(i\theta)=\cos(\theta)+i\sin(\theta)$
   b. $|z| = (x^2+y^2)^{1/2} = r$
   c. $\arg(z) = \tan^{-1}(y/x) = \theta$
   d. $z\bar{z} = |z|^2 = x^2 + y^2$
   e. $\log(z) = \ln(r) + i\theta$
   f. A function f of the complex variable z is analytic on an open set if it has a derivative at each point in that set. (counter-example: $f(z) = |z|^2$ is not analytic anywhere since its derivative exists only at $z = 0$.)
   g. A function, h, is harmonic if it has continuous partial derivatives of the first and second order and satisfies Laplace's equation:
   $$\nabla^2 h = 0$$
   h. If a function, $f(z)=a(x,y)+ib(x,y)$, is analytic in D, then the first order partial derivatives of its component functions, a and b, must satisfy the Cauchy-Riemann equations throughout D.
   $$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \quad \text{and} \quad \frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$$
   i. If two functions, a and b, are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equations throughout D, b is said to be a harmonic conjugate of a.

2. If a function, $f(z)=a(x,y)+ib(x,y)$, is analytic in a domain D then its component functions, a and b, are harmonic conjugates in D.

   Proof:
   Since $f(z)$ is analytic in a domain D, then the first order partial derivatives of its component functions, $a(x,y)$ and $b(x,y)$, satisfy the Cauchy-Riemann equations throughout D (Note #1h).

   Differentiating the Cauchy-Riemann equations gives:
   $$\frac{\partial^2 a}{\partial x^2} = \frac{\partial^2 b}{\partial x \partial y} \qquad \frac{\partial^2 a}{\partial x \partial y} = -\frac{\partial^2 b}{\partial x^2}$$
   $$\text{and}$$
   $$\frac{\partial^2 a}{\partial y \partial x} = \frac{\partial^2 b}{\partial y^2} \qquad \frac{\partial^2 a}{\partial y^2} = -\frac{\partial^2 b}{\partial y \partial x}$$
   But from advanced calculus:
   $$\frac{\partial^2 a}{\partial x \partial y} = \frac{\partial^2 a}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 b}{\partial x \partial y} = \frac{\partial^2 b}{\partial y \partial x}$$

   Substituting and simplifying:

$$\frac{\partial^2 a}{\partial x^2} = -\frac{\partial^2 a}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 b}{\partial y^2} = -\frac{\partial^2 b}{\partial x^2}$$

$$\Rightarrow \nabla^2 a = 0 \quad \text{and} \quad \nabla^2 b = 0$$

Thus, a and b are harmonic (Note 1g). Since a and b are harmonic and satisfy the Cauchy-Riemann equations in D, they are harmonic conjugates of each other in D (Note 1h). Therefore, if $f(z)=a(x,y)+ib(x,y)$ is analytic in a domain D then its component functions, a and b, are harmonic conjugates in D.

3. Any analytic function, $f(z)=\phi(x,y)+i\psi(x,y)$, is a valid 2D, incompressible, irrotational flow field.

Proof:
Recall that for an irrotational flow, the velocity may be written as the gradient of a potential function, ϕ:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \mathbf{0}$$

from a vector identity: $\nabla \times \nabla\phi = 0$ for any ϕ

$$\Rightarrow \mathbf{u} = \nabla\phi$$

For the flow to satisfy the continuity equation for an incompressible fluid:

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \nabla \cdot \nabla\phi = \nabla^2\phi = 0$$

$\therefore \phi$ is a harmonic function!

Also recall that the stream function, ψ, is defined for 2D flows such that continuity for an incompressible fluid is automatically satisfied:

$$u_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y = -\frac{\partial \psi}{\partial x}$$

$$\text{so that } \nabla \cdot \mathbf{u} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

$\therefore$ continuity is automatically satisfied!

If the flow is also irrotational, the stream function must also satisfy:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \mathbf{0} \quad \Rightarrow \quad \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 0$$

$$\Rightarrow -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\nabla^2\psi = 0$$

$\therefore \psi$ is a harmonic function!

In addition,

$$\left.\begin{aligned} u_x &= \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ u_y &= \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \text{Cauchy-Riemann equations}$$

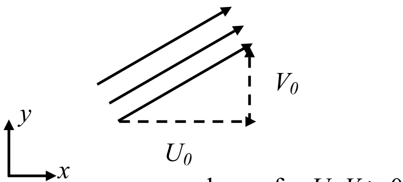
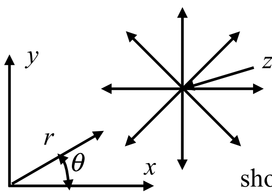
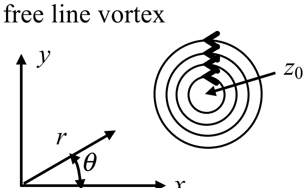
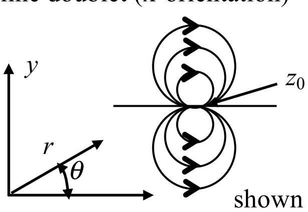
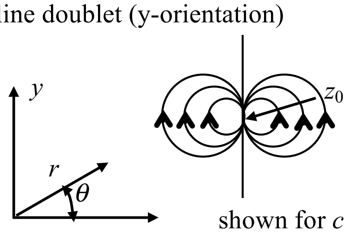
$\therefore$ ϕ and ψ are harmonic conjugates!

From Note 2, the components of any analytic function are harmonic conjugates. Thus, since the governing equations for the fluid are Laplace's equation and since ϕ and ψ are harmonic conjugates, then any analytic $f(z)=\phi+i\psi$ will be a valid flow field.

Thus, by choosing various forms of $f(z)$ that are analytic, we can produce various valid (incompressible and irrotational) flow fields. Whether or not the flow fields are interesting from an engineering perspective is another matter.

Notes…:

4. Some "building block" flows and their complex potentials:

uniform stream  shown for $U_0, V_0 > 0$	$f(z) = (U_0 - iV_0)z$	$\phi = U_0 x + V_0 y$ $\psi = -V_0 x + U_0 y$
line source (m>0) or sink (m<0)  shown for $m > 0$	$f(z) = \dfrac{m}{2\pi}\log(z - z_0)$ $m \in \Re$	$\phi = \dfrac{m}{2\pi}\ln r'$ $\psi = \dfrac{m}{2\pi}\theta'$
free line vortex  shown for $\Gamma > 0$	$f(z) = \dfrac{-i\Gamma}{2\pi}\log(z - z_0)$ $\Gamma \in \Re$	$\phi = \dfrac{\Gamma}{2\pi}\theta'$ $\psi = \dfrac{-\Gamma}{2\pi}\ln r'$
line doublet (x-orientation)  shown for $c > 0$	$f(z) = \dfrac{c}{z - z_0}$ $c \in \Re$	$\phi = \dfrac{c\cos\theta'}{r'}$ $\psi = -\dfrac{c\sin\theta'}{r'}$
line doublet (y-orientation)  shown for $c > 0$	$f(z) = \dfrac{ic}{z - z_0}$ $c \in \Re$	$\phi = \dfrac{c\sin\theta'}{r'}$ $\psi = \dfrac{c\cos\theta'}{r'}$

Note that in the table above:

$z = x + iy$ and $z_0 = x_0 + iy_0$

$0 \le \theta < 2\pi$

$$r' = \left[(x - x_0)^2 + (y - y_0)^2\right]^{1/2} \quad \text{and} \quad \theta' = \tan^{-1}\left(\frac{y - y_0}{x - x_0}\right)$$

Notes…:

5. Fluid velocities are found via differentiation of the complex potential:

$$f'(z) = \frac{df}{dz} = u_x - iu_y$$

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} \quad \text{where } f(z) = \phi(x,y) + i\psi(x,y)$$

$$\text{and } \frac{\partial f}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u_x - iu_y$$

$$\text{and } \frac{\partial z}{\partial x} = 1 \qquad (\text{since } z = x + iy)$$

$$\therefore \frac{df}{dz} = u_x - iu_y \quad \text{(an identical result occurs if we consider } \frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y}\text{)}$$

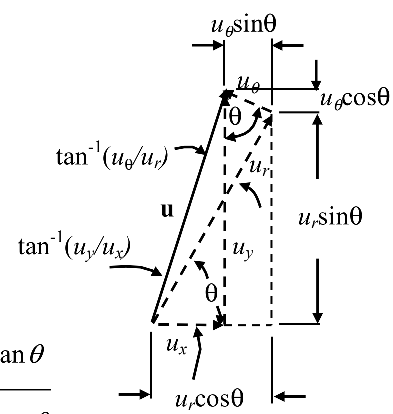
A rectangular coordinates example:

$$f(z) = \frac{c}{z} \quad \text{(line doublet oriented in the } x\text{-direction and centered at the origin)}$$

$$\frac{df}{dz} = -\frac{c}{z^2} = -\frac{c\bar{z}^2}{(z\bar{z})^2} = -\frac{c\left(x^2 - 2xyi - y^2\right)}{\left(x^2+y^2\right)^2} = \frac{-c\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} - i\frac{-2cxy}{\left(x^2+y^2\right)^2}$$

$$\therefore u_x = \frac{-c\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} \text{ and } u_y = \frac{-2cxy}{\left(x^2+y^2\right)^2}$$

A polar coordinate example:



$$f(z) = \frac{m}{2\pi}\log(z) \qquad \text{(source/sink at origin)}$$

$$\frac{df}{dz} = \frac{m}{2\pi}\frac{1}{z} = \frac{m}{2\pi}\frac{1}{r\exp(i\theta)} = \frac{m}{2\pi}\frac{\exp(-i\theta)}{r} = \frac{m}{2\pi r}\left(\cos\theta - i\sin\theta\right)$$

$$\therefore u_x = \frac{m}{2\pi r}\cos\theta \quad \text{and} \quad u_y = \frac{m}{2\pi r}\sin\theta$$

or in polar coordinates using some geometry and trig.:

$$u_r^2 + u_\theta^2 = u_x^2 + u_y^2 \quad \text{and} \quad \tan^{-1}\left(\frac{u_y}{u_x}\right) = \theta + \tan^{-1}\left(u_\theta/u_r\right) \quad \Rightarrow \quad \left(\frac{u_y}{u_x}\right) = \frac{\left(u_\theta/u_r\right) + \tan\theta}{1 - \left(u_\theta/u_r\right)\tan\theta}$$

Substituting in for our values of u_x and u_y and simplifying:

$$u_r^2 + u_\theta^2 = \left(\frac{m}{2\pi r}\right)^2 \quad \text{and} \quad \tan\theta = \frac{\left(u_\theta/u_r\right) + \tan\theta}{1 - \left(u_\theta/u_r\right)\tan\theta} \quad \Rightarrow \quad \left(1 + \tan^2\theta\right)\left(u_\theta/u_r\right) = 0$$

$$\Rightarrow u_\theta = 0 \quad \text{and} \quad u_r = \frac{m}{2\pi r}$$

In general the relation between the velocity components expressed in rectangular and polar coordinates is given by (refer to the figure shown above):

$$u_x = u_r \cos\theta - u_\theta \sin\theta$$

$$u_y = u_r \sin\theta + u_\theta \cos\theta$$

$$\begin{aligned}\frac{df}{dz} &= u_x - iu_y = \left(u_r \cos\theta - u_\theta \sin\theta\right) - i\left(u_r \sin\theta + u_\theta \cos\theta\right) \\ &= u_r\left(\cos\theta - i\sin\theta\right) - iu_\theta\left(\cos\theta - i\sin\theta\right) \\ &= \left(u_r - iu_\theta\right)\left(\cos\theta - i\sin\theta\right)\end{aligned}$$

$$\boxed{\therefore \frac{df}{dz} = u_x - iu_y = \left(u_r - iu_\theta\right)\exp\left(-i\theta\right)} \tag{16}$$

6. We can use the principle of superposition to combine complex potentials and form new complex potentials since if two functions are analytic in a domain *D*, then their sum is also analytic.

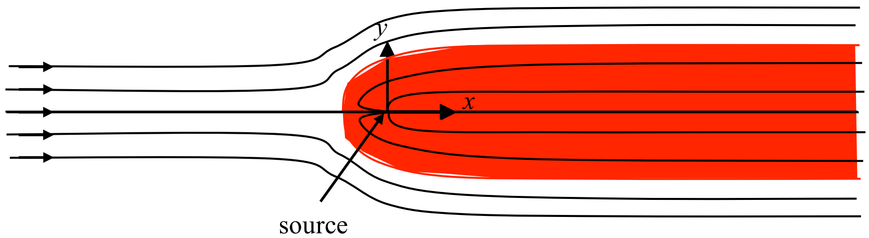
Notes…:

7. A few example flows created by superposition:

Flow over a Rankine half-body:

Combine the complex potentials for a uniform stream and a source (m>0):

$$f(z) = Uz + \frac{m}{2\pi}\log z$$

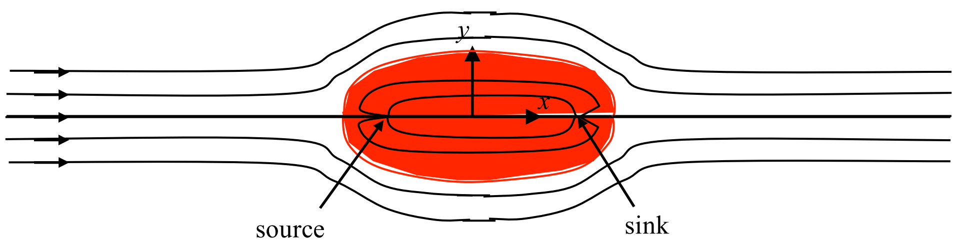


Flow over a Rankine oval:

Combine the complex potentials for a uniform stream, a source, and a sink:

$$f(z) = Uz + \frac{m}{2\pi}\log(z+a) - \frac{m}{2\pi}\log(z-a)$$

where m>0 and $a \in \Re > 0$.



Flow around a non-rotating cylinder of radius R:

Combine the complex potentials for a uniform stream and a doublet:

$$f(z) = Uz + \frac{c}{z}$$

where the constant c is found by not allowing any flow through the cylinder walls, i.e.

$$u_r(r = R) = 0$$

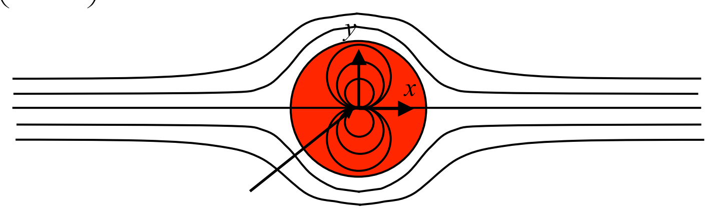
$$f(z) = \underbrace{\left[Ur\cos\theta + \frac{c\cos\theta}{r}\right]}_{\phi} + i\underbrace{\left[Ur\sin\theta - \frac{c\sin\theta}{r}\right]}_{\psi}$$

$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta - \frac{c\cos\theta}{r^2}$$

$$\Rightarrow u_r(r = R) = 0 = U\cos\theta - \frac{c\cos\theta}{R^2} \Rightarrow c = UR^2$$

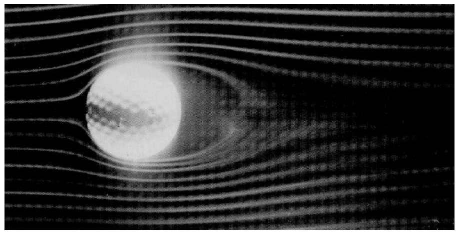
so that the complex potential becomes:

$$f(z) = U\left(z + \frac{R^2}{z}\right) \tag{17}$$



Notes:

1. Real (viscous) flow over a sphere (a golf ball in the figure below) is shown below. The streamlines for flow over a cylinder look much the same.



The streamlines over the front half of a cylinder are similar to those predicted by the potential flow analysis. In fact, the velocity and pressures field on the front half of the cylinder are also accurate (the pressures will be discussed in a moment.) The flow field downstream of the cylinder is not accurately predicted. The discrepancy between the potential flow analysis and real life occurs due to the formation of a viscous boundary layer on the cylinder surface. The boundary layer separates near the top/bottom points of the cylinder and forms a wake. Assuming irrotational flow in the boundary layer and wake are poor assumptions. However, outside the boundary layer and wake, the potential flow assumption is reasonable. We'll discuss boundary layers in a later section of notes.

2. The pressure distribution on the cylinder surface can be predicted using Eq. (17) and Bernoulli's equation:

$$f(z) = U\left(z + \frac{R^2}{z}\right)$$

From Eq. (16), the flow velocity field is:

$$\begin{aligned}
\left(u_r - iu_\theta\right)\exp\left(-i\theta\right) = \frac{df}{dz} = U\left(1 - \frac{R^2}{z^2}\right) &= U\left[1 - \frac{R^2}{r^2\exp\left(2i\theta\right)}\right] \\
&= U\left[1 - \frac{R^2}{r^2}\exp\left(-2i\theta\right)\right] = U\left[\exp\left(i\theta\right) - \frac{R^2}{r^2}\exp\left(-i\theta\right)\right]\exp\left(-i\theta\right) \\
&= U\left[\cos\left(\theta\right) + i\sin\left(\theta\right) - \frac{R^2}{r^2}\left(\cos\left(\theta\right) - i\sin\left(\theta\right)\right)\right]\exp\left(-i\theta\right) \\
&= U\left[\left(1 - \frac{R^2}{r^2}\right)\cos\left(\theta\right) + i\left(1 + \frac{R^2}{r^2}\right)\sin\left(\theta\right)\right]\exp\left(-i\theta\right)
\end{aligned}$$

(Note that for this case it would be easier to determine the velocity using ϕ or ψ directly rather than the complex potential). On the cylinder surface ($r = R$):

$$u_r\big|_{r=R} = 0$$

$$u_\theta\big|_{r=R} = -2U\sin\left(\theta\right)$$

The pressure distribution on the cylinder surface is found via Bernoulli's equation, and expressed in terms of a dimensionless pressure coefficient, c_p:

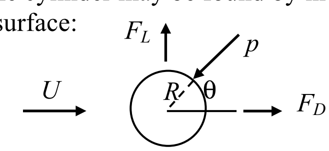
$$c_p = \frac{p_s - p_\infty}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\left(\theta\right) \tag{18}$$

Notes:

a. The total drag (F_D) and lift (F_L) on the cylinder may be found by integrating the pressure distribution over the entire cylinder surface:

$$F_D = \int_{\theta=0}^{\theta=2\pi} -p\cos\theta R\,d\theta$$

$$F_L = \int_{\theta=0}^{\theta=2\pi} -p\sin\theta R\,d\theta \tag{19}$$



Either by actually evaluating Eq. (19) or noting that the velocity field is symmetric over the front and back and upper and lower surfaces, the drag and lift forces on the cylinder are both zero. Of course in real flows we know that the drag is not zero. The fact that the potential flow model predicts zero drag while real flows have non-zero drag is known as **d'Alembert's Paradox**. We, of course, now know that the discrepancies are explained by the formation of a boundary layer and boundary layer separation. d'Alembert's paradox will be discussed again when reviewing Blasius' integral law.

b. Equation (18) is compared to experimental data in the plot below (from Fox, R.W. and McDonald, A.T., *Introduction to Fluid Mechanics*, 5th ed., Wiley.)

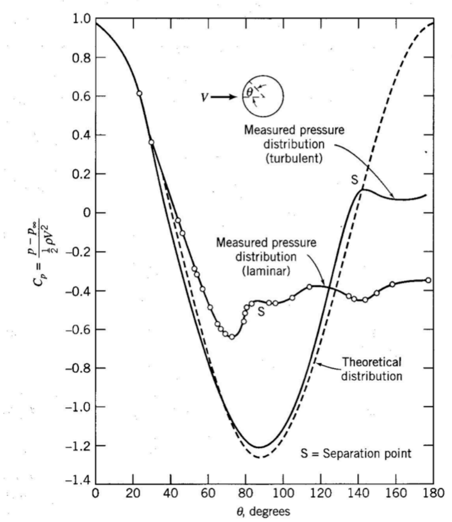


Fig. 9.12 Pressure distribution around a smooth sphere for laminar and turbulent boundary-layer flow, compared with inviscid flow [16].

Again, the potential flow analysis predicts the pressure distribution reasonably well over the upstream part of the cylinder but does a poor job over the back half due to boundary layer separation.

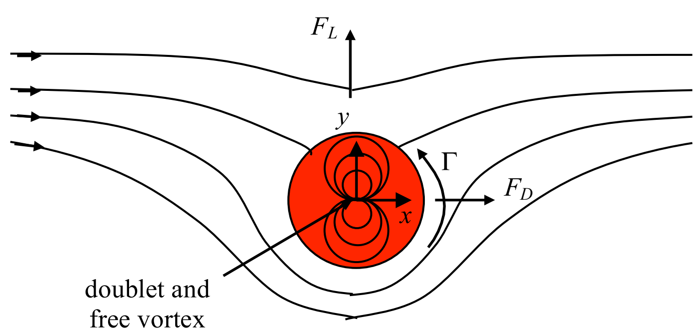
Flow around a rotating cylinder of radius R:

Combine the complex potentials for a uniform stream, a doublet, and a free vortex:

$$f(z) = Uz + \frac{c}{z} - \frac{i\Gamma}{2\pi}\log z$$

where the constant c is found by not allowing any flow through the cylinder walls, just as in the previous example. Note that the addition of a vortex will not change the value of c since a vortex only produces tangential flow and not radial flow. As a result, the complex potential becomes:

$$f(z) = U\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi}\log z$$



Notes:

1. The corresponding velocity field is:

$$\left(u_r - iu_\theta\right)\exp\left(-i\theta\right) = \frac{df}{dz} = U\left(1 - \frac{R^2}{z^2}\right) - \frac{i\Gamma}{2\pi z}$$
$$= U\left[1 - \frac{R^2}{r^2}\exp\left(-2i\theta\right)\right] - \frac{i\Gamma}{2\pi r}\exp\left(-i\theta\right)$$

Using the previous results for a non-rotating cylinder:

$$\begin{aligned} u_r &= U\left(1 - \frac{R^2}{r^2}\right)\cos\left(\theta\right) \\ u_\theta &= -U\left(1 + \frac{R^2}{r^2}\right)\sin\left(\theta\right) + \frac{\Gamma}{2\pi}\frac{1}{r} \end{aligned} \tag{20}$$

On the cylinder surface ($r = R$):

$$\begin{aligned} u_r\big|_{r=R} &= 0 \\ u_\theta\big|_{r=R} &= -2U\sin\left(\theta\right) + \frac{\Gamma}{2\pi}\frac{1}{R} \end{aligned} \tag{21}$$

Using Bernoulli's equation, the pressure coefficient over the surface is:

$$c_p = 1 - 4\sin^2\theta + 4\frac{\Gamma}{2\pi}\frac{1}{UR}\sin\theta - \left(\frac{\Gamma}{2\pi}\frac{1}{UR}\right)^2 \tag{22}$$

The corresponding drag, F_D, and lift, F_L, are:

$$\begin{aligned} F_D &= 0 \\ F_L &= -\rho U\Gamma \end{aligned} \tag{23}$$

Notes:

a. The drag again is zero and is not unexpected due to the fore/aft symmetry of the velocity field.
b. The lift is non-zero and is related to the flow circulation. This type of lift is referred to as **Magnus lift**. Both drag and lift for potential flows will be discussed in detail when reviewing Blasius' integral law and the Kutta-Joukowski theorem.
c. The photo below shows the flow past a rotating golf ball. The flow is from left to right and the golf ball rotates in a clockwise manner ($\Gamma < 0$). From Eq. (23), the lift on the golf ball will be in the positive vertical direction.

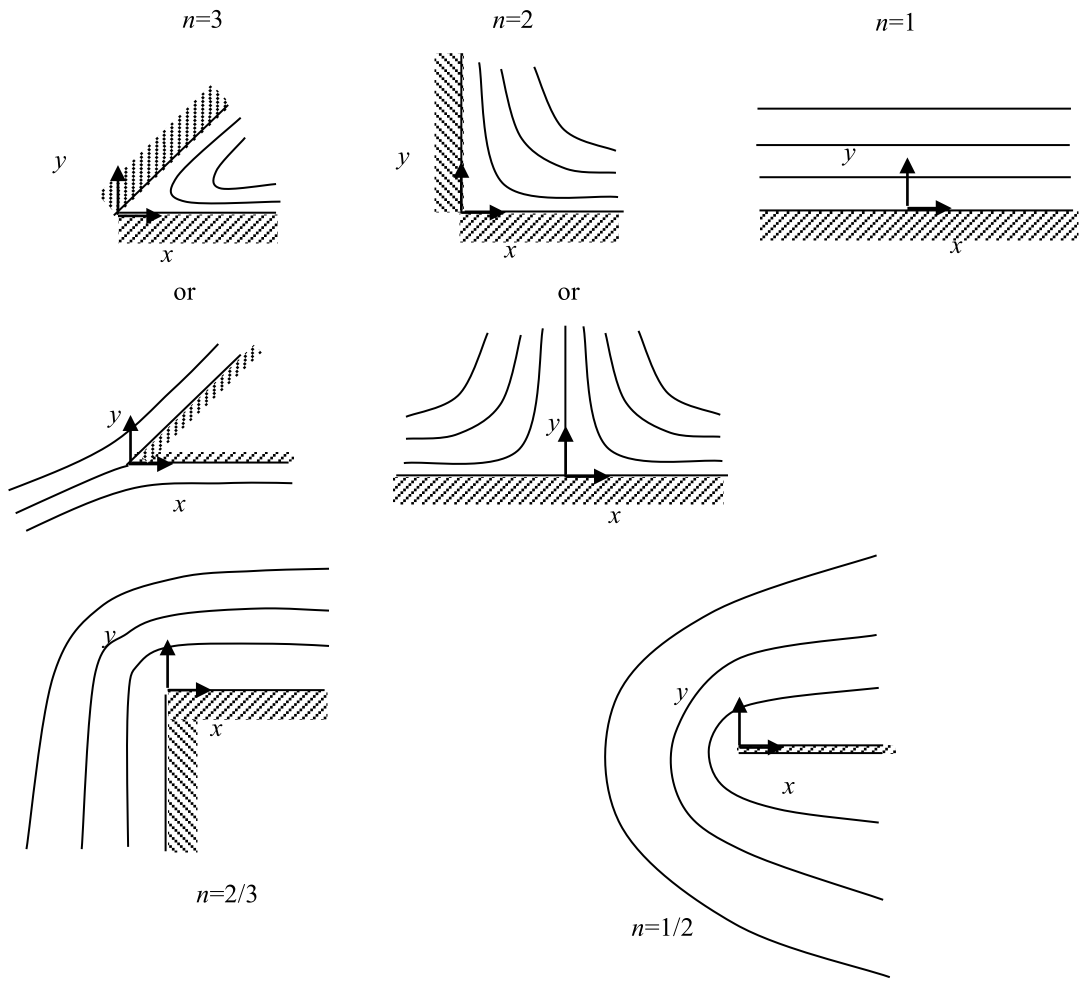
In real (i.e., viscous) flows the lift on a rotating object comes primarily from deflection of the downstream wake (the fluid momentum is directed downward resulting in an upward force on the ball) rather than from the unbalanced pressure distribution on the object. The Magnus effect is often mistakenly referred to as the primary source of the lift force.

Notes…:

7. Flow in and around corners of varying angles can be modeled using the following complex potential:

   $f(z) = Az^n$ where A and n are constants

   a. This produces flows between boundaries intersecting at an angle π/n (only flows with $n \geq 1/2$ are of interest):



   b. The potential and stream functions are given by:

$$f(z) = Az^n = A\left[r\exp\left(i\theta\right)\right]^{\mathrm{n}} = Ar^n\exp\left(in\theta\right) = Ar^n\cos\left(n\theta\right) + iAr^n\sin\left(n\theta\right)$$

$$\therefore \phi = Ar^n\cos\left(n\theta\right) \quad \text{and} \quad \psi = Ar^n\sin\left(n\theta\right)$$

$$u_r = \frac{\partial\varphi}{\partial r} = Anr^{n-1}\cos\left(n\theta\right) \quad \text{and} \quad u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = -Anr^{n-1}\sin\left(n\theta\right)$$

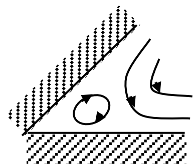
   c. The fluid speed at the origin is:

$$\lim_{r\to 0}\left|\mathbf{u}\right| = \lim_{r\to 0}\left|f'(z)\right| = \lim_{r\to 0}\left|nAr^{n-1}\exp\left(i(n-1)\theta\right)\right| = \lim_{r\to 0}\left|nAr^{n-1}\right|$$

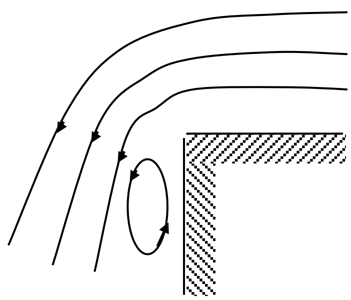
$$\Rightarrow \lim_{r\to 0}\left|\mathbf{u}\right| = \begin{cases} 0 & n > 1 \\ \left|A\right| & n = 1 \\ \infty & n < 1 \end{cases}$$

d. In a real flow (one with viscosity), the flow along the surface streamline would:

for $n>1$: separate <u>before</u> reaching the corner and produce a standing eddy

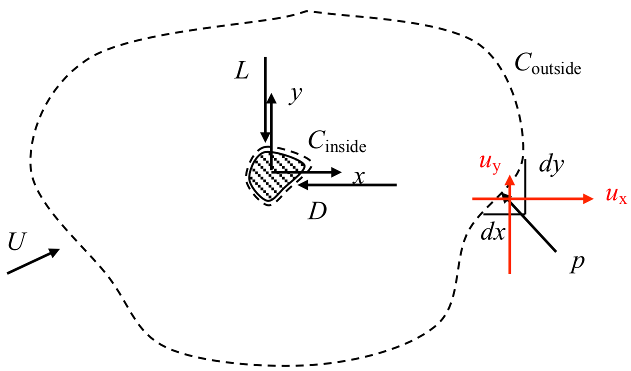


for $n<1$: separate <u>after</u> reaching the corner unless the corner angle is small



4. Blasius Integral Law

Consider the 2D, incompressible, inviscid, steady, irrotational flow around an arbitrary closed body:



Using the LMEs, determine the lift, L, and drag, D, acting on the body:

$$-D - \oint_{C_{\text{outside}}} p\,dy = \oint_{C_{\text{outside}}} u_x \underbrace{\left(\rho u_x dy - \rho u_y dx\right)}_{\rho \mathbf{u}\cdot d\mathbf{A}}$$

$$-L + \oint_{C_{\text{outside}}} p\,dx = \oint_{C_{\text{outside}}} u_y \underbrace{\left(\rho u_x dy - \rho u_y dx\right)}_{\rho \mathbf{u}\cdot d\mathbf{A}}$$

From Bernoulli's equation (neglecting gravity):

$$p + \tfrac{1}{2}\rho\left(u_x^2 + u_y^2\right) = c \quad \Rightarrow \quad p = c - \tfrac{1}{2}\rho\left(u_x^2 + u_y^2\right)$$

where c is a constant. Substituting and re-arranging:

$$D = \oint_{C_{\text{outside}}} \left[-c\,dy + \tfrac{1}{2}\rho\left(u_x^2 + u_y^2\right)dy - \rho\left(u_x^2 dy - u_x u_y dx\right)\right]$$

$$L = \oint_{C_{\text{outside}}} \left[c\,dx - \tfrac{1}{2}\rho\left(u_x^2 + u_y^2\right)dx - \rho\left(u_x u_y dy - u_y^2 dx\right)\right]$$

Noting that:

$$\oint_{C_{\text{outside}}} c\,dy = \oint_{C_{\text{outside}}} c\,dx = 0$$

and simplifying:

$$D = \oint_{C_{\text{outside}}} \left[-\tfrac{1}{2}\rho\left(u_x^2 - u_y^2\right)dy + \rho u_x u_y dx\right]$$

$$L = \oint_{C_{\text{outside}}} \left[-\tfrac{1}{2}\rho\left(u_x^2 - u_y^2\right)dx - \rho u_x u_y dy\right]$$

As shown below, the previous drag and lift relations may be written in terms of the complex potential.

$$
\begin{aligned}
i\frac{\rho}{2}\oint_{C_{\text{outside}}}\left(\frac{df}{dz}\right)^2 dz &= i\frac{\rho}{2}\oint_{C_{\text{outside}}}\left(u_x - iu_y\right)^2\left(dx + idy\right) \\
&= i\frac{\rho}{2}\oint_{C_{\text{outside}}}\left(u_x^2 - 2iu_xu_y - u_y^2\right)\left(dx + idy\right) \\
&= i\frac{\rho}{2}\oint_{C_{\text{outside}}}\left[\left(u_x^2dx - u_y^2dx + 2u_xu_ydy\right) + i\left(u_x^2dy - u_y^2dy - 2u_xu_ydx\right)\right] \\
&= \oint_{C_{\text{outside}}}\left[\underbrace{\left(-\tfrac{1}{2}\rho\left(u_x^2 - u_y^2\right)dy + \rho u_xu_ydx\right)}_{=D} - i\underbrace{\left(-\tfrac{1}{2}\rho\left(u_x^2 - u_y^2\right)dx - \rho u_xu_ydy\right)}_{=L}\right]
\end{aligned}
$$

Substituting the expressions for lift, L, and drag, D, found previously:

$$\boxed{i\frac{\rho}{2}\oint_{C_{\text{outside}}}\left(\frac{df}{dz}\right)^2 dz = D - iL} \qquad \textbf{BLASIUS' INTEGRAL LAW}$$

How is this result used? Typically, it is applied using a theorem from complex variables referred to as the Residue Theorem (Churchill, R.V. and Brown, J.W., *Complex Variables and Applications*, McGraw-Hill, 5th ed., pg. 169) which states:

$$\oint_C w(z)\,dz = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k}\left(w(z)\right)$$

A residue is the coefficient in front of the $1/(z\text{-}z_0)$ term (the b_1 term in the series below) in the Laurent series expansion of an analytic complex function about the point z_0 (Churchill and Brown, pg. 144):

$$w(z) = \sum_{n=0}^{\infty}a_n\left(z - z_0\right)^n + \sum_{n=1}^{\infty}\frac{b_n}{\left(z - z_0\right)^n}$$

where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{2\pi i}\oint_C\frac{w(z)dz}{\left(z - z_0\right)^{n+1}} \quad \text{and} \quad b_n = \frac{1}{2\pi i}\oint_C\frac{w(z)dz}{\left(z - z_0\right)^{-n+1}}$$

The details of the expressions above won't concern us here and are only presented for completeness. Blasius' Integral Law is used in deriving the Kutta-Joukowski theorem given in the following section which relates the lift (and drag) around any arbitrary, closed object to the circulation, Γ, caused by the object.

Example:
Determine, using the Blasius Integral Law, the lift acting on a rotating cylinder.

SOLUTION:

The complex potential function for flow around a rotating cylinder is:

$$f(z) = U\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi}\log z \;\Rightarrow\; \frac{df}{dz} = U\left(1 - \frac{R^2}{z^2}\right) - \frac{i\Gamma}{2\pi}\frac{1}{z} \tag{24}$$

$$\left(\frac{df}{dz}\right)^2 = U^2\left(1 - \frac{R^2}{z^2}\right)^2 - \frac{i\Gamma}{\pi}\frac{1}{z}U\left(1 - \frac{R^2}{z^2}\right) + \frac{\Gamma^2}{4\pi^2}\frac{1}{z^2} \tag{25}$$

Let the contour in the integral law, C, be the circle defined as:

$$z = R'\exp(i\theta) \;\Rightarrow\; dz = iR'\exp(i\theta)d\theta \tag{26}$$

where $0 \le \theta < 2\pi$ and R' is an arbitrary radius greater than R. Thus, the Blasius Integral Law for this problem is:

$$\begin{aligned}
i\frac{\rho}{2}\oint_C \left(\frac{df}{dz}\right)^2 dz &= i\frac{\rho}{2}\int_{\theta=0}^{\theta=2\pi}\left\{U^2\left[1 - \frac{R^2}{R'^2\exp(2i\theta)}\right]^2 - \frac{i\Gamma}{\pi}\frac{1}{R'\exp(i\theta)}U\left[1 - \frac{R^2}{R'^2\exp(2i\theta)}\right] + \frac{\Gamma^2}{4\pi^2}\frac{1}{R'^2\exp(2i\theta)}\right\} iR'\exp(i\theta)d\theta \\
&= -\frac{\rho}{2}R'\int_{\theta=0}^{\theta=2\pi}\left\{U^2\exp(i\theta)\left[1 - 2\frac{R^2}{R'^2}\exp(-2i\theta) + \frac{R^4}{R'^4}\exp(-4i\theta)\right] - \frac{i\Gamma}{\pi R'}U\left[1 - \frac{R^2}{R'^2}\exp(-2i\theta)\right] + \frac{\Gamma^2}{4\pi^2}\frac{\exp(-i\theta)}{R'^2}\right\}d\theta \\
&= -\frac{\rho}{2}R'\left\{-\frac{i\Gamma}{\pi R'}U2\pi\right\}
\end{aligned}$$

$$\therefore D - iL = i\frac{\rho}{2}\oint_C\left(\frac{df}{dz}\right)^2 dz = i\rho U\Gamma \;\Rightarrow\; \boxed{D = 0 \text{ and } L = -\rho U\Gamma} \tag{27}$$

These are the same results that we found previously (Eq. (23))!

5. Kutta-Joukowski Theorem

Now consider the flow around an arbitrary closed body (centered at the origin) in a uniform stream of horizontal velocity, U. Far from the body ($z\to\infty$) the complex potential will be of the following form (a Laurent series expansion):

$$f(z) = Uz + \left(\frac{m - i\Gamma}{2\pi}\right)\log z + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \tag{28}$$

Note that the coefficients, a_n, for the terms involving z^n ($n \geq 2$) in the Laurent series (refer to the previous set of notes on the Blasius integral law) are all zero since we are considering external flows (recall that the velocity field is given by df/dz so that terms involving z^n where $n \geq 2$ will approach ∞ as $z\to\infty$). Furthermore, since we are concerned only with closed bodies, the net source term, m, should also be zero. We, however, will continue to include the source term until the end of this analysis.

Given the complex potential above, let's apply Blasius' Integral Law to determine the lift and drag about an arbitrary object:

$$\frac{df}{dz} = U + \left(\frac{m - i\Gamma}{2\pi}\right)\frac{1}{z} + \sum_{n=1}^{\infty} -nb_n z^{-n+1}$$

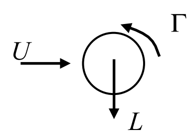
$$\left(\frac{df}{dz}\right)^2 = U^2 + 2\left(\frac{m - i\Gamma}{2\pi}\right)\frac{U}{z} + \left(\frac{m - i\Gamma}{2\pi}\right)^2 \frac{1}{z^2} + O\left(\frac{1}{z^3}\right)$$

Using the Residue Theorem to evaluate Blasius' Integral Law:

$$\begin{aligned} D - iL &= i\frac{\rho}{2}\oint_{C_{\text{outside}}} \left(\frac{df}{dz}\right)^2 dz = i\frac{\rho}{2}\cdot 2\pi \mathrm{i} \operatorname*{Res}_{z=0}\left(\frac{df}{dz}\right)^2 = i\frac{\rho}{2}\cdot 2\pi\mathrm{i}\cdot 2\left(\frac{m - i\Gamma}{2\pi}\right)U \\ &= -\rho mU + i\rho U\Gamma \end{aligned}$$

Thus, we see that for a closed object (m=0):

$$\boxed{D = 0 \ \text{ and } L = -\rho U\Gamma} \qquad \textbf{KUTTA-JOUKOWSKI THEOREM} \tag{29}$$



For an object that is not closed (e.g., a Rankine half-body), we have:

$$D = -\rho mU \ \text{ and } L = -\rho U\Gamma$$

Notes:

1. The result given above indicates that there is no drag around an arbitrary closed object in an a steady, incompressible, irrotational, inviscid flow. In real life of course there is always some drag on an object. The conflict between the derived value of zero drag and the real-life value of non-zero drag is referred to as **d'Alembert's Paradox**. There is no paradox, in fact, if one realizes that it is viscous effects (skin friction drag and form, aka pressure, drag resulting from the formation of a wake (which in turn is a result of boundary layer separation)), which produces drag on an object.

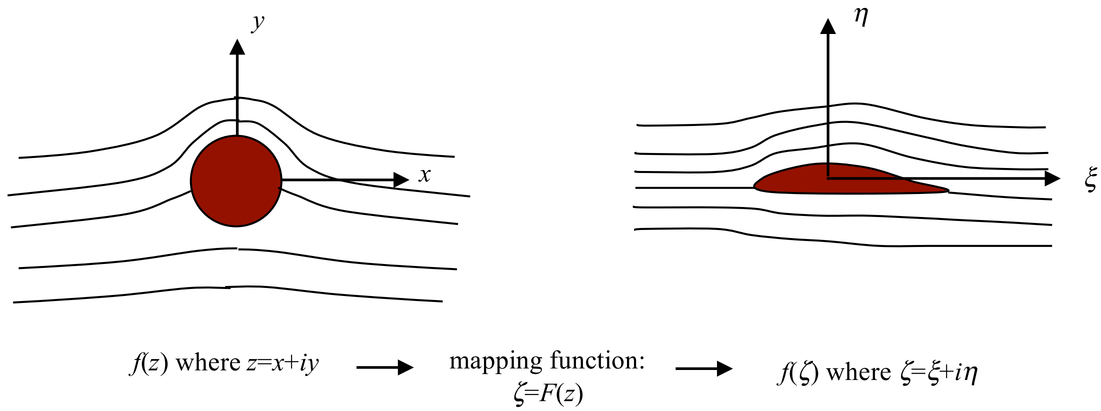
2. Bodies of semi-infinite extent (e.g., a Rankine half-body) do have drag on them due to the fact that the net source term, m, is not zero. The drag is a result of a non-zero flux of horizontal momentum out through the control surface.

3. The Kutta-Joukowski theorem states that the lift on an object is directly proportional to the net circulation, Γ, caused by the object. This is an important observation that is especially useful in aerodynamics when calculating the lift on an airfoil. As will be shown later, the circulation around an airfoil is dependent on the free stream velocity so that the lift turns out to be proportional to the circulation squared.

6. Conformal Mappings

Conformal maps are analytic functions that transform curves (e.g., equi-potential lines and streamlines) in one complex plane, call this the z-plane, to similar curves, but expanded or contracted and rotated at each point, in a different complex plane, call this the ζ-plane.

Conformal maps are useful because they allow us to use the complex potential for a straightforward flow (e.g., flow around a rotating cylinder), after the proper mapping, as the complex potential for a more complex flow (e.g., flow around an airfoil). The complexity comes into play when trying to find the proper mapping that will give us the desired transformation.

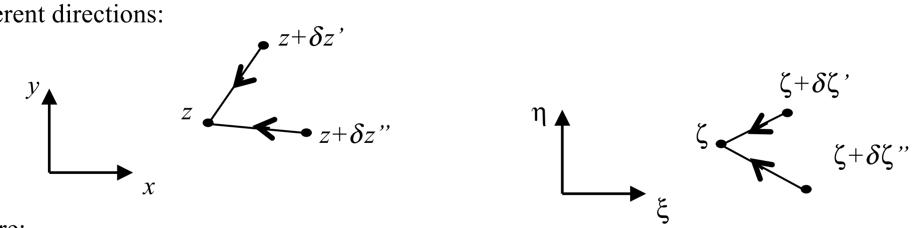


First, let's examine some properties of a conformal map. Let $\zeta=\xi+i\eta$ be an analytic function of $z=x+iy$ given by: $\zeta=F(z)$

Because the transforming function, F, is analytic, there is a connection between curves in the z-plane and corresponding curves in the ζ-plane.

Proof:

Determine the derivative of the function, $\zeta=F(z)$, at a point, z, by approaching the point from two different directions:



where:

$$\zeta+\delta\zeta'=F\left(z+\delta z'\right)$$
$$\zeta+\delta\zeta''=F\left(z+\delta z''\right)$$

The length ratios in the z and ζ planes are:

$$\left|\frac{\delta z'}{\delta z''}\right| \quad \text{and} \quad \left|\frac{\delta\zeta'}{\delta\zeta''}\right|$$

and the angles separating the lines are:

$$\arg(\delta z')-\arg(\delta z'') = \arg\left(\frac{\delta z'}{\delta z''}\right) \quad \text{and} \quad \arg(\delta\zeta')-\arg(\delta\zeta'') = \arg\left(\frac{\delta\zeta'}{\delta\zeta''}\right)$$

Also, because an analytic function has a unique derivative:

$$\delta\zeta'=\frac{d\zeta}{dz}\delta z'+O\left[\left(\delta z'\right)^2\right]$$
$$\delta\zeta''=\frac{d\zeta}{dz}\delta z''+O\left[\left(\delta z''\right)^2\right]$$

So that the length ratios and angles between the lines are the same in each plane:

$$\left|\frac{\delta\zeta'}{\delta\zeta''}\right|=\left|\frac{\delta z'}{\delta z''}\right| \quad \text{and} \quad \arg\left(\frac{\delta\zeta'}{\delta\zeta''}\right)=\arg\left(\frac{\delta z'}{\delta z''}\right)$$

as $\delta z',\delta z''\to 0$

Thus, lengths in the neighborhood of z are stretched by a scale factor, $\left|d\zeta/dz\right|$, and are rotated by an angle, $\arg\left(d\zeta/dz\right)$, into the ζ-plane.

Notes:

1. Curves of <u>small linear dimension</u> in the z-plane are mapped into curves of <u>similar shape</u>, but expanded or contracted and rotated, in the ζ-plane.

2. Note that a large region may be transformed into a region that bears no resemblance to the original one since the scale factor and angle of rotation vary, in general, from point to point.

3. At singular points of the map, *i.e.* points where $\frac{d\zeta}{dz} = 0$ or ∞, the mapping is not conformal.

4. Since lines of constant ϕ and ψ are $\perp$ in the z-plane, they will also be $\perp$ in the ζ-plane, except at singular points. (Refer to Note 1.)

5. Since an analytic function of another analytic function is also analytic (Churchill and Brown, p. 56), we are assured that the function resulting from the conformal map of a complex potential will also be a valid complex potential (it will be a 2D, incompressible, irrotational flow).

6. Velocities in the ζ-plane are proportional to the velocities in the z-plane by the inverse of the scale factor:
$$u_\xi - iu_\eta = \frac{df}{d\zeta} = \frac{df}{dz}\frac{dz}{d\zeta} = \left(u_x - iu_y\right)\frac{dz}{d\zeta}$$

7. Singularities such as vortices and sources/sinks in the z-plane map to identical singularities in the ζ-plane. This can be seen by considering the flow in a neighborhood of the singularity as the neighborhood shrinks to an infinitesimally small radius.
$$m = \oint_C \mathbf{u} \cdot d\mathbf{A} = \oint_C \left(u_x dy - u_y dx\right)$$
$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{s} = \oint_C \left(u_x dx + u_y dy\right)$$
Consider the integral of the complex velocity around the contour C:
$$\oint_C \frac{df}{dz} dz = \oint_C \left(u_x - iu_y\right)\left(dx + idy\right) = \underbrace{\oint_C \left(u_x dx + u_y dy\right)}_{\Gamma} + i\underbrace{\oint_C \left(u_x dy - u_y dx\right)}_{m}$$
$$\therefore \oint_C \frac{df}{dz} dz = \Gamma + im$$
Thus,
$$\Gamma_z + im_z = \oint_{C_z} \frac{df}{dz} dz = \oint_{C_z} \frac{df}{d\zeta}\frac{d\zeta}{dz} dz = \oint_{C_\zeta} \frac{df}{d\zeta} d\zeta = \Gamma_\zeta + im_\zeta$$
Source/sink and free vortex singularities in the z-plane map to similar singularities in the ζ-plane.

8. Doublet singularities in the z-plane map to doublet singularities in the ζ-plane but with the strength changed in magnitude by $\left|d\zeta/dz\right|$ and an orientation rotated by $\arg\left(d\zeta/dz\right)$ (recall that a doublet is formed by bringing a source and sink of equal strength infinitesimally close to each other while keeping the product m/a constant where a is the separation distance between the source and sink).

9. Even though the occurrence of boundary layer separation in the real flow may limit the usefulness of a potential flow model, transformation of the flow field to a different flow field may produce a realistic flow. For example, although in a real flow boundary layer separation occurs for flow over a rotating cylinder, the mapping to an airfoil shape appears realistic.

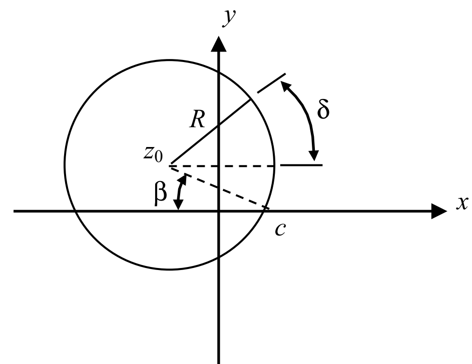
10. Conformal maps are another tool we can use to produce realistic-looking flows using potential functions.

7. Joukowski Transformation

An example of a particular conformal mapping is the Joukowski transformation:

$$\zeta = z + \frac{c^2}{z} \tag{30}$$

where $c \in \Re$. This mapping will transform flow around a rotating cylinder in the z-plane to flow around an airfoil in the ζ-plane. This airfoil is referred to as a Joukowski airfoil.



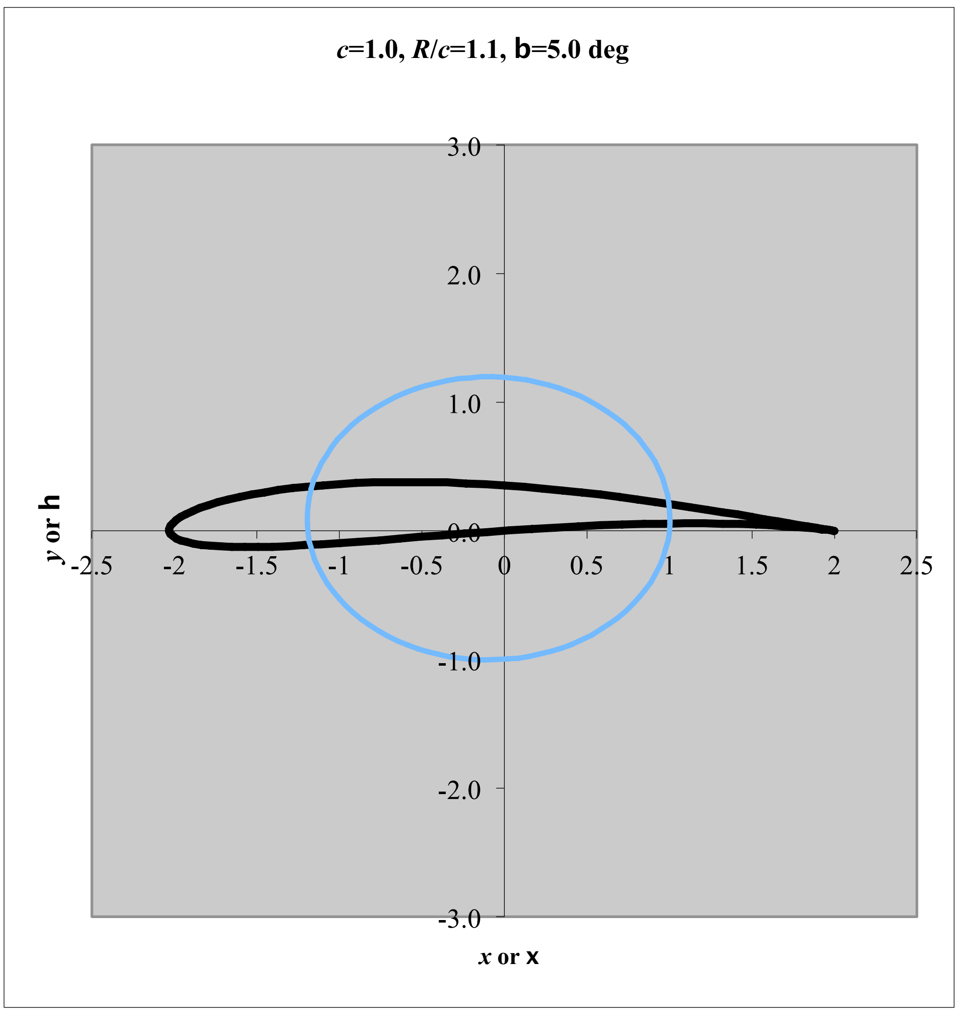
Consider a circle of radius R centered at the point z_0 such that the circumference of the circle passes through the point $z=c$:

$$z_0 = c - R\exp(-i\beta)$$

The points defining the circle are given by:

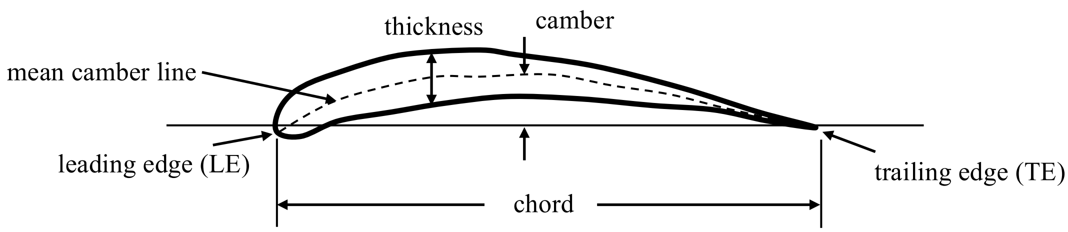
$$z = z_0 + R\exp(i\delta) = c - R\exp(-i\beta) + R\exp(i\delta)$$

If we map the points of the circle in the z-plane to the ζ-plane using the transformation given in Eq. (30), the resulting figure looks like an airfoil:

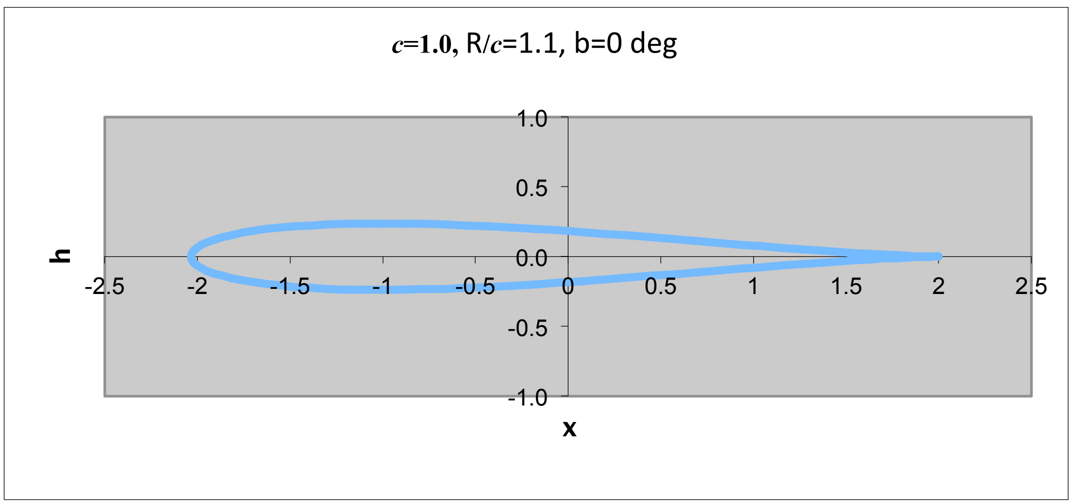


Notes:

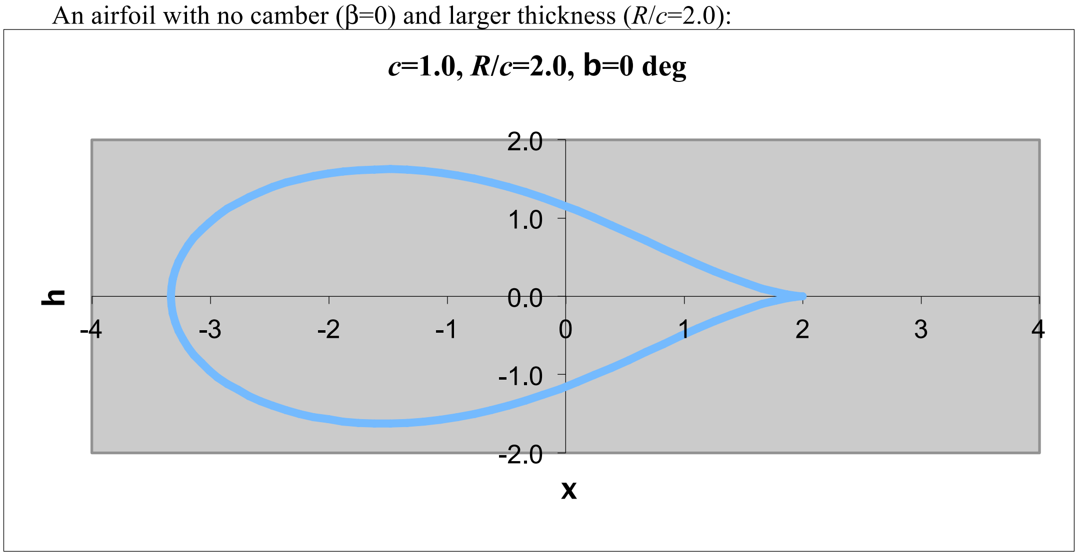
1. The geometry of the Joukowski airfoil is determined by the quantities R/c and β. The **camber** of the airfoil is proportional to β (camber ↑ as β ↑). The **thickness** of the airfoil increases as (R/c) increases. The **chord length** of the airfoil is approximately equal to $4c$ (the exact chord length will also depend on the airfoil thickness).



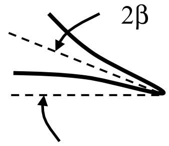
An airfoil with no camber (β=0):

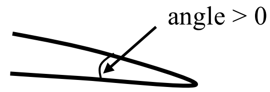


An airfoil with no camber (β=0) and larger thickness (R/c=2.0):



2. The trailing edge of the Joukowski airfoil will be cusp-shaped. Real airfoils typically end in a finite angle.





3. Joukowski airfoils are not commonly used in practice; however, they provide a good model for predicting the general behavior of airfoils at small angles of attack (so that boundary layer separation won't occur in the real-world airfoils).

4. Note that the trailing edge of the airfoil corresponds to the location where the cylinder intersects the location $z=c$. The transformation is not conformal at $z=c$ and thus the angle between intersecting lines in the ζ-plane is not necessarily the same angle between intersecting lines in the z-plane at the point $z=c$.

Now consider, in the z-plane, a uniform flow with velocity U around a rotating cylinder with circulation Γ (an unknown value at this point) and radius R. The complex potential for this flow is given by:

$$f\left(z''\right) = U\left(z'' + \frac{R^2}{z''}\right) - \frac{i\Gamma}{2\pi}\log\left(\frac{z''}{R}\right) \tag{31}$$

Let's rotate the flow so that the incoming stream is at an angle of attack, α, with respect to the horizontal:

$$z' = z''\exp\left(i\alpha\right) \quad \Rightarrow \quad z'' = z'\exp\left(-i\alpha\right)$$

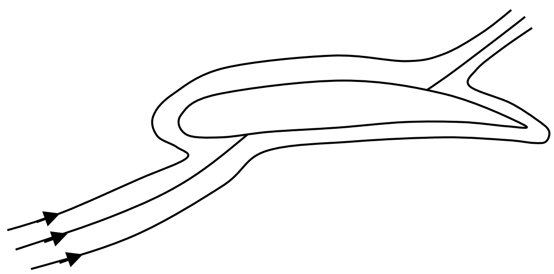
Let's also translate the origin of the cylinder so that it is centered at the position $z_0 = c - R\exp(-i\beta)$:

$$z = z' + z_0 \quad \Rightarrow \quad z' = z - z_0$$

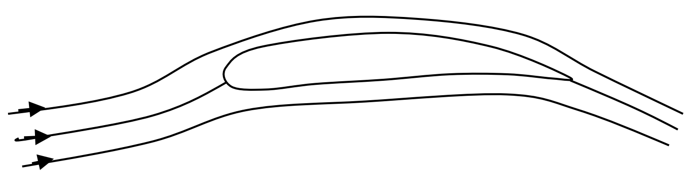
The new complex potential is given by

$$f\left(z\right) = U\left[\left(z - z_0\right)\exp\left(-i\alpha\right) + \frac{R^2}{\left(z - z_0\right)}\exp\left(i\alpha\right)\right] - \frac{i\Gamma}{2\pi}\log\left[\left(\frac{z - z_0}{R}\right)\exp\left(-i\alpha\right)\right] \tag{32}$$

Note that we haven't yet determined the value of the circulation, Γ. This will be found in the section below. First, however, let's plot some streamlines for the case with zero circulation (Γ=0):



Of particular interest in the plot is the condition at the trailing edge of the airfoil. The streamlines at the trailing edge make a very sharp turn (the streamlines are not smooth at the very tip of the airfoil) resulting in infinite fluid accelerations and velocities at the trailing edge. This is not a very realistic flow and does not match what we observe in flows around real airfoils. We can avoid this infinite velocity problem by adjusting the circulation around the airfoil so that the flow leaves smoothly from the trailing edge. This is equivalent to moving the rear stagnation point to the tip of the trailing edge. This adjustment is referred to as the **Kutta Condition**.



To quantitatively determine what the value of the circulation must be to satisfy the Kutta Condition, let's examine the complex velocity of fluid along the airfoil surface:

$$u_\xi - iu_\eta = \frac{df}{d\zeta} = \frac{df}{dz''}\frac{dz''}{dz'}\frac{dz'}{dz}\frac{dz}{d\zeta}$$

where

$$\frac{df}{dz''} = U\left[1-\frac{R^2}{(z'')^2}\right]-\frac{i\Gamma}{2\pi z''}$$

$$\frac{dz''}{dz'} = \exp(-i\alpha)$$

$$\frac{dz'}{dz} = 1$$

$$\frac{dz}{d\zeta} = \left[1-\left(\frac{c}{z}\right)^2\right]^{-1}$$

$$z'' = (z-z_0)\exp(-i\alpha)$$

$$z_0 = c - R\exp(-i\beta)$$

so that the complex velocity in the ζ-plane is given by:

$$u_\xi - iu_\eta = \frac{df}{d\zeta} = \left\{U\left[1-\frac{R^2}{(z'')^2}\right]-\frac{i\Gamma}{2\pi z''}\right\}\exp(-i\alpha)\left[1-\left(\frac{c}{z}\right)^2\right]^{-1} \tag{33}$$

Note that at $z=c$ the magnitude of the complex velocity approaches infinity due to the $(dz/d\zeta)$ term. To prevent infinite velocities from occurring, the term within the curly brackets {} must equal zero at $z=c$:

$$\left.\left\{U\left[1-\frac{R^2}{(z'')^2}\right]-\frac{i\Gamma}{2\pi z''}\right\}\right|_{z=c} = 0$$

$$\Gamma = -2\pi iUz''\left.\left[1-\frac{R^2}{(z'')^2}\right]\right|_{z=c} = -2\pi iU\left[R\exp(-i(\alpha+\beta))-\frac{R^2}{R\exp(-i(\alpha+\beta))}\right]$$

where $z''\big|_{z=c} = (c-z_0)\exp(-i\alpha) = \left[c-c+R\exp(-i\beta)\right]\exp(-i\alpha) = R\exp\left[-i(\alpha+\beta)\right]$

$$\Rightarrow \quad \Gamma = -2\pi iUR\left[\cos(\alpha+\beta)-i\sin(\alpha+\beta)-\cos(\alpha+\beta)-i\sin(\alpha+\beta)\right]$$

$$\therefore \Gamma = -4\pi UR\sin(\alpha+\beta)$$

Thus, to prevent infinite velocities from occurring at the trailing edge of the airfoil, the circulation must be given by:

$$\Gamma = -4\pi UR\sin(\alpha+\beta) \tag{34}$$

Now that we know the circulation about the airfoil, we can use the Kutta-Joukowski Theorem to determine the lift of the airfoil:

$$\begin{aligned} &L = -\rho U\Gamma \\ \Rightarrow \quad &L = 4\pi\rho U^2 R\sin(\alpha+\beta) \end{aligned} \tag{35}$$

The lift is often presented in dimensionless form as the **lift coefficient, c_L**:

$$c_L \equiv \frac{L}{\frac{1}{2}\rho U^2 (4c)} = 2\pi\left(\frac{R}{c}\right)\sin(\alpha+\beta) \tag{36}$$

where $4c$ is the approximate chord length of the airfoil.

We can also determine the pressure distribution on the airfoil surface by using Bernoulli's equation (recall that we're dealing with an incompressible, irrotational flow so the same Bernoulli constant is used everywhere):

$$p_s + \tfrac{1}{2}\rho u_s^2 = p_\infty + \tfrac{1}{2}\rho U^2$$

where p_s and u_s are the pressure and speed on the airfoil surface, and p_∞ and U are the pressure and speed far from the airfoil.

The magnitude of the velocity on the surface of the airfoil can be found using the complex velocity given in Eq. (33) with:

$$z'' = (z - z_0)\exp(-i\alpha) = R\exp(i\delta)\exp(-i\alpha) = R\exp\left[i(\delta-\alpha)\right]$$

where δ defines the location on the airfoil surface. After some algebra, we arrive at:

$$|u_s| = \frac{2U\left[\sin(\delta-\alpha)+\sin(\alpha+\beta)\right]}{\left|1-\left(\frac{c}{z}\right)^2\right|} \tag{37}$$

with $z=c-R\exp(-i\beta)+R\exp(i\delta)$.

The pressure is often expressed non-dimensionally as the **pressure coefficient, c_P**:

$$c_P \equiv \frac{p-p_\infty}{\frac{1}{2}\rho U^2} = 1-\left(\frac{u_s}{U}\right)^2 \tag{38}$$

Notes:

1. The lift coefficient predicted by our potential flow analysis of a Joukowski airfoil is reasonably close to values found experimentally at small angles of attack and small camber (to avoid boundary layer separation).

2. Flow over a flat plate can be found by letting R/c=1 and β=0. The resulting lift coefficient is:

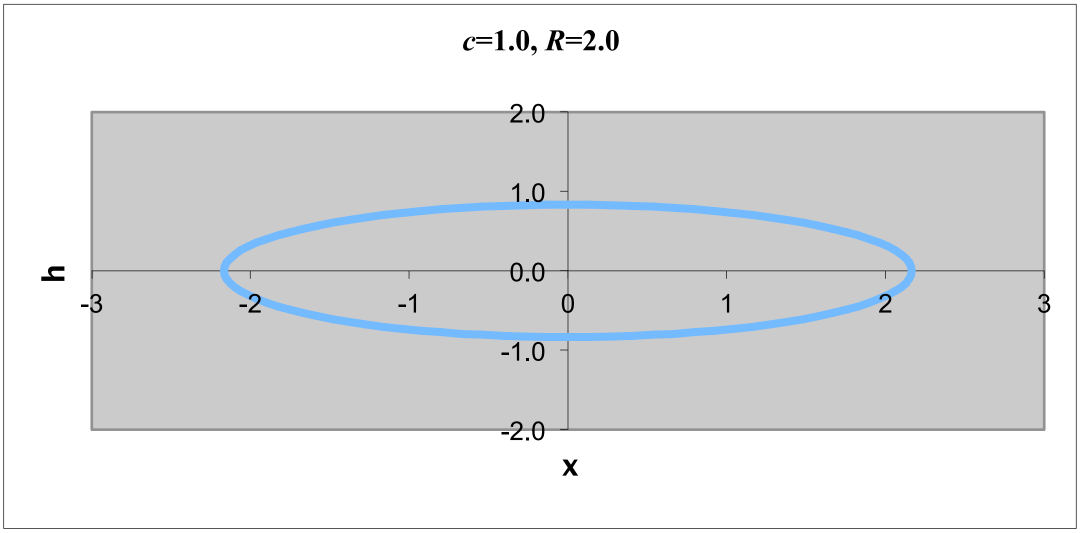
   $$c_L = 2\pi\sin(\alpha)$$

   The Joukowski transformation can also produce flow around curved plates (let R/c=0 and β>0).

3. Note that increasing the angle of attack, α, the camber, β, and the thickness, R/c, all act to increase the lift of an airfoil.

4. The Joukowski transformation can also produce flow around ellipses. To produce this type of flow, we center the cylinder at the origin and choose $R>c$. The points on the cylinder surface are given by

$$z = R\exp(i\delta)$$



where $0 \le \delta < 2\pi$.

8. Method of Images

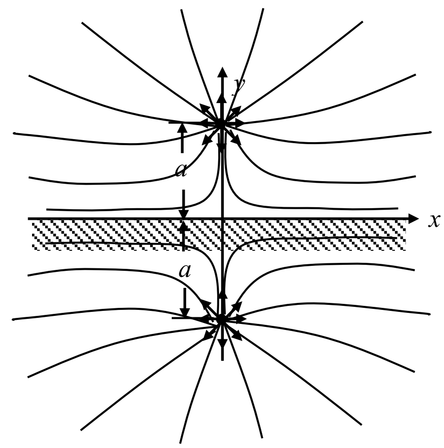
In much of our previous work concerning potential flows, we investigated external flows in an infinite expanse of fluid. Since there are a number of phenomena that are of interest when there is flow near a boundary, we should find a method for modeling flows near walls. The Method of Images is such a method.

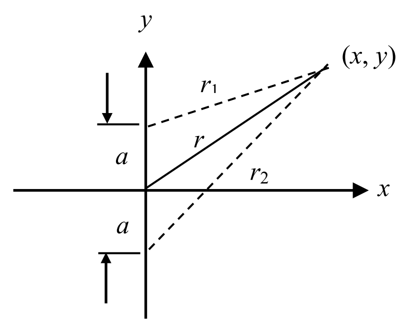
Consider how we can model the flow from a source near a wall. To produce a horizontal streamline representing the wall, we can add to our original source, an "image source" an equal distance away from where we want our wall to be.

$$\phi = \frac{m}{2\pi}\left[\ln r_1 + \ln r_2\right]$$

where $r_1 = \sqrt{x^2 + (y-a)^2}$

$r_2 = \sqrt{x^2 + (y+a)^2}$





Notes:

1. There is a net upward velocity at the location of the original source of:
$$V = \frac{m}{2\pi}\frac{1}{(2a)}$$
where m is the source strength due to the flow induced by the image source.

2. The vertical force acting on the source can be determined by first calculating the pressure force acting on the wall using Bernoulli's equation and then noting that the force acting on the wall is equal, but in the opposite direction, to the force acting on the source. The pressure at the wall is given by:
$$p_w + \tfrac{1}{2}\rho V_w^2 = p_\infty$$
where p_w and V_w are the pressure and velocity magnitude at the wall and p_∞ is the pressure far from the wall (U_∞ approaches zero as we move far from the wall). The velocity along the wall is found from the potential function:
$$\phi = \frac{m}{2\pi}\left[\ln\sqrt{x^2 + (y-a)^2} + \ln\sqrt{x^2 + (y+a)^2}\right]$$
$$u_x\big|_{y=0} = \left.\frac{\partial \phi}{\partial x}\right|_{y=0} = \frac{m}{\pi}\frac{x}{x^2 + a^2}$$
$$u_y\big|_{y=0} = \left.\frac{\partial \phi}{\partial y}\right|_{y=0} = 0$$
$$\Rightarrow p_w = p_\infty - \tfrac{1}{2}\rho\left(\frac{m}{\pi}\frac{x}{x^2 + a^2}\right)^2$$
Note that without the sources, the pressure acting on the wall would be p_∞. Hence, the increase in the force acting on the wall is:

$$F_{\substack{\text{on wall} \\ \text{due to source}}} = \int_{x=-\infty}^{x=\infty} \left[-p_w - (-p_\infty) \right] dx = \int_{x=-\infty}^{x=\infty} \left[\tfrac{1}{2}\rho \left(\frac{m}{\pi} \frac{x}{x^2 + a^2} \right)^2 \right] dx$$

$$= -\frac{1}{4}\rho \left(\frac{m}{\pi} \right)^2 \left[\frac{x}{x^2 + a^2} - \frac{1}{a} \tan^{-1}\left(\frac{x}{a} \right) \right]_{-\infty}^{\infty}$$

$$\therefore F_{\substack{\text{on wall} \\ \text{due to source}}} = \frac{\rho m^2}{4\pi a}$$

Consequently, the force on the source will be:

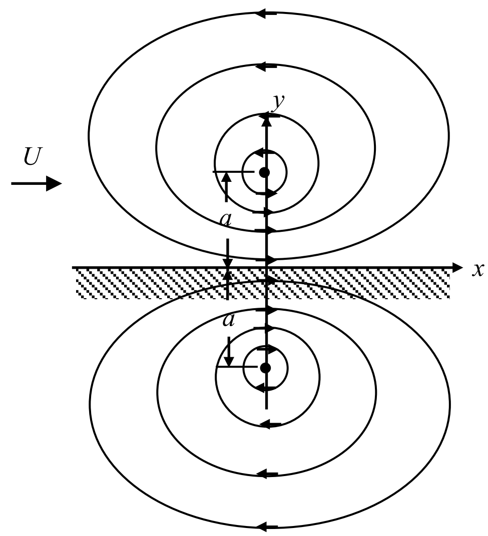
$$F_{\text{on source}} = -\frac{\rho m^2}{4\pi a}$$

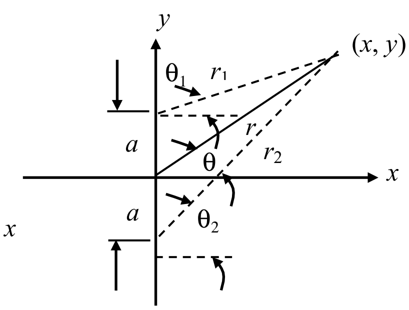
A particularly interesting application of the method of images is investigating the effect that the ground has on the lift of an airfoil. Let's use a crude model consisting of a free vortex combined with a uniform stream to investigate this effect. Recall that in order to satisfy the Kutta condition, an airfoil must have some circulation, which in turn develops lift (from the Kutta-Joukowski Theorem). The potential flow model is given below (drawn for Γ>0).

$$\phi = Ux + \frac{\Gamma}{2\pi}\left[\theta_1 - \theta_2 \right]$$

where $\theta_1 = \tan^{-1}\left(\frac{y-a}{x} \right)$

$$\theta_2 = \tan^{-1}\left(\frac{y+a}{x} \right)$$





As before with the source example, the force acting on the source will have the same magnitude, but with opposite sign, as the force acting on the wall. The force acting on the wall is found by integrating the pressure force over the entire wall:

$$u_x\big|_{y=0} = \frac{\partial \phi}{\partial x}\bigg|_{y=0} = U + \frac{\Gamma}{2\pi}\left[\frac{\frac{a}{x^2}}{1 + \left(\frac{a}{x} \right)^2} \right] = U + \frac{\Gamma}{\pi}\frac{a}{x^2 + a^2}$$

$$u_y\big|_{y=0} = \frac{\partial \phi}{\partial y}\bigg|_{y=0} = 0$$

$$\Rightarrow p_w = p_\infty + \tfrac{1}{2}\rho U^2 - \tfrac{1}{2}\rho \left[U^2 + \frac{2\Gamma}{\pi}\frac{aU}{x^2 + a^2} + \left(\frac{\Gamma}{\pi}\frac{a}{x^2 + a^2} \right)^2 \right]$$

The resulting force acting on the wall due to the vortices (again, subtracting out the pressure when no vortices are present) is then:

$$F_{\substack{\text{on wall}\\ \text{due to vortex}}} = \int_{x=-\infty}^{x=\infty}\left[-p_w-(-p_\infty)\right]dx = \int_{x=-\infty}^{x=\infty}\left\{\tfrac{1}{2}\rho\left[\frac{2\Gamma}{\pi}\frac{aU}{x^2+a^2}+\left(\frac{\Gamma}{\pi}\frac{a}{x^2+a^2}\right)^2\right]\right\}dx$$

$$= \tfrac{1}{2}\rho\left\{\frac{2\Gamma U}{\pi}\tan^{-1}\left(\frac{x}{a}\right)+\left(\frac{\Gamma}{\pi}\right)^2\frac{1}{2}\left[\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]\right\}_{-\infty}^{\infty}$$

$$= \tfrac{1}{2}\rho\left[2\Gamma U+\left(\frac{\Gamma}{\pi}\right)^2\frac{\pi}{2a}\right]$$

$$\therefore F_{\substack{\text{on wall}\\ \text{due to vortex}}} = \rho U\Gamma\left(1+\frac{\Gamma}{4\pi aU}\right)$$

Thus, the force acting on the vortex is:

$$\therefore F_{\text{on vortex}} = -\rho U\Gamma\left(1+\frac{\Gamma}{4\pi aU}\right)$$

Recall that the lift force acting on an object in an infinite expanse of fluid with circulation Γ is given by the Kutta-Joukowski theorem as:

$$\therefore L_{\substack{\text{infinite}\\ \text{expanse}}} = -\rho U\Gamma$$

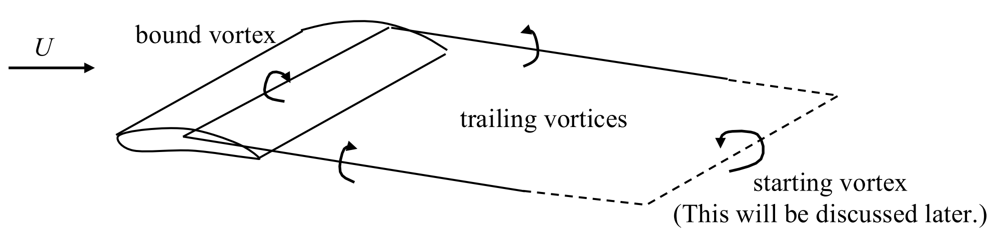
Keep in mind that from the previous discussion regarding Joukowski airfoils, the circulation around an airfoil is negative ($\Gamma < 0$) in order to satisfy the Kutta condition at the trailing edge of the airfoil.

The difference between the lift generated when the wall is nearby versus the lift in an infinite expanse of fluid is:

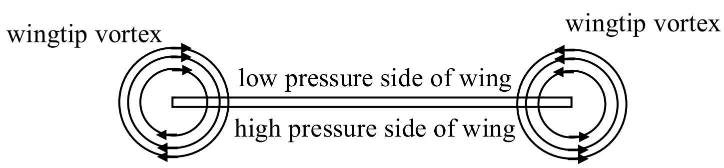
$$\Delta L = L_{\substack{\text{wall}\\ \text{present}}} - L_{\substack{\text{infinite}\\ \text{expanse}}} = -\rho U\Gamma\left(1+\frac{\Gamma}{4\pi aU}\right)-(-\rho U\Gamma)$$

$$= \frac{-\rho\Gamma^2}{4\pi a}$$

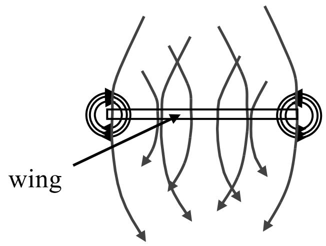
Hence, the wall acts to *decrease* the lift. Experience shows, however, that an airfoil near the ground, aka in **ground effect**, actually has increased lift (and decreased drag) rather than decreased lift. Why do we have this discrepancy? It's because our analysis considers an infinitely long airfoil, i.e., one with no wing tips. At the end of a finite wing, "trailing" vortices (as opposed to the vortex "bound" to the airfoil resulting from the Kutta condition) are generated by the wing tips as shown below. These wingtip vortices occur because air in the high pressure region underneath the airfoil is pushed around the wing tips to the low pressure above the airfoil.

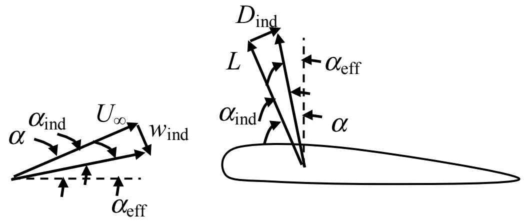


When viewed from behind, the trailing vortices appear as shown below.



The wingtip vortices induce a "downwash" along the wing surface and thus reduce the effective angle of attack that the airfoil sees.





The lift acting on the wing per unit span (i.e., distance into the page), L, will be the lift calculated for the local effective angle of attack, α_{eff}, where:

$$\alpha_{eff} = \alpha - \alpha_{ind} \text{ with } \tan\alpha_{ind} = \frac{w_{ind}}{U_\infty}$$

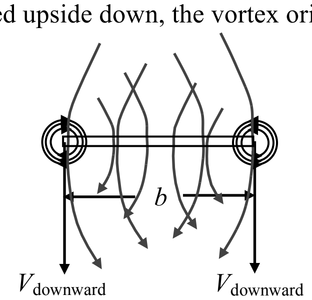
Here, α is the nominal angle of attack and α_{ind} is the induced angle of attack resulting from the wingtip vortices which induce a local downwash velocity of w_{ind}. Since the effective angle of attack is reduced, the lift on the airfoil will also be reduced.

There is also an induced drag on the wing D_{ind} since the local flow is at an angle of α_{ind} from the free stream, which tilts the actual lift vector slightly downstream

$$D_{ind} = L\tan\alpha_{ind}$$

Again, D_{ind} is the induced drag per unit span of the wing. Note that the induced drag on the wing is not due to viscous effects, but is due solely to the induced angle of attack resulting from the induced downwash. Hence, there is a drag on a finite wing, even in an ideal flow, due to the trailing vortices.

The trailing vortices also drift downwards over time due to the flow induced at the center of each vortex by the other vortex. If the airfoil was turned upside down, the vortex orientation would be reversed and the vortices would drift upwards over time!

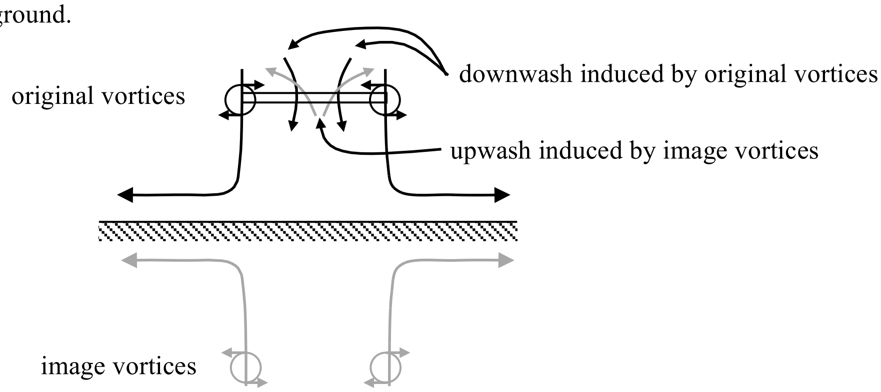


If this vortex has a circulation of Γ, the induced velocity at the center of the other vortex is:

$$V_{downward} = \frac{\Gamma}{2\pi}\frac{1}{b}$$

where b is the distance between the wingtips.

Near the ground, two image vortices must also be included in the analysis in order to make the horizontal ground streamline. As a result of the flow induced by the image vortices, the wingtip vortices drift outward when reaching the ground.

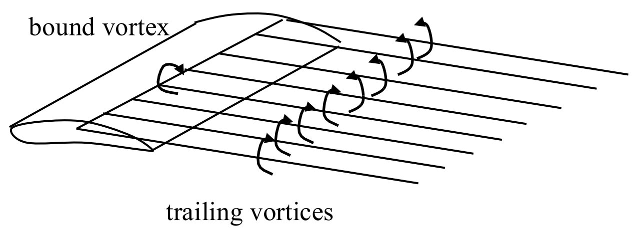


The image vortices also contribute an "upwash" along the wing, which helps to counteract the downwash caused by the original vortices. This reduction in the downwash helps increase the lift and reduce the drag on the finite wing when it is located near the ground. This is the source of the observed "ground effect."

Notes:

1. Recall that from previous discussions regarding vorticity, vortex lines must either form closed loops or terminate on a boundary ($\nabla \cdot \boldsymbol{\omega} = 0$). So how then do the vortex lines corresponding to the trailing vortices terminate? The bound vortex/trailing vortex lines actually form a closed loop through a "starting" vortex line (shown in a previous figure). The starting vortex occurs during the transient when the lift on the airfoil changes (e.g., at start up). Since the starting vortex is typically located far behind the bound vortex, its effects are typically neglected in steady airfoil analyses and the bound vortex/trailing vortex combination is treated as a horseshoe vortex.

2. The trailing vortices are not actually concentrated solely at the wingtips. Instead, there is a distribution of trailing vortices along the wing due to variations in the circulation, which result from changes in airfoil geometry and local flow conditions. These variations must be included a finite wing analyses (see, for example, Kuethe, A.M. and Chow, C-Y., *Foundations of Aerodynamics*, Wiley.)



3. Trailing vortices have been the source of several airline disasters (see, for example, http://www.asy.faa.gov/safety_products/wake.htm and http://aviation-safety.net/events/EFV.shtml). If an aircraft flies behind a preceding aircraft too closely, it can be caught in the trailing vortices and cause the pilot to lose control of the aircraft (this phenomenon is sometimes mistakenly referred to as "wake turbulence"). The strength of the vortices is proportional to the lift generated by the airfoil which in turn is related to the weight of the aircraft. Hence, the spacing between aircraft (near an airport for example) is a function of their relative size.

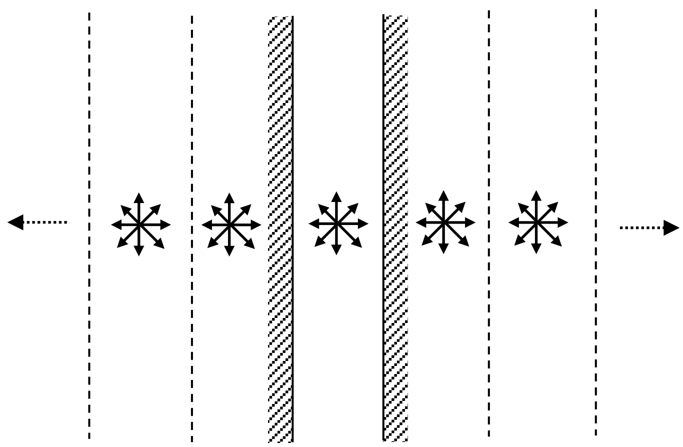
4. Ground effect has been used as a significant component in the design of several aircraft. A search for WIG (wing-in-ground-effect) aircraft on the web will show many different designs.

The "Caspian Sea Monster" developed by the Soviets in the 1960s.

Even pelicans take advantage of ground effect!

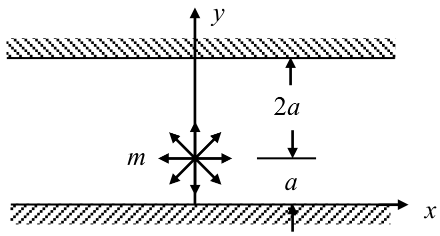
Notes:

1. We may sometimes need an infinite number of reflections to properly model a flow. Consider for example, the flow resulting from a source located midway between two walls. An infinite number of images are required for perfect symmetry.



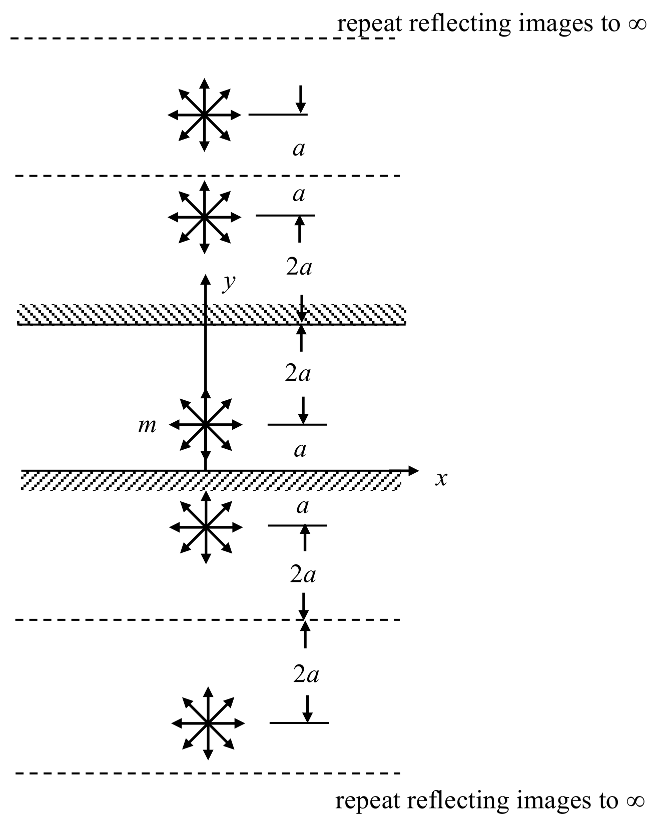
Example:

a. Write the potential function that simulates the flow of a line source placed asymmetrically between two parallel walls as shown in the figure.
b. Compute the dimensionless velocity, $\mathbf{u}' = a\mathbf{u}/(4\pi m)$, on the lower wall at $(x/a, y/a) = (1,0)$ accurate to three significant digits.



SOLUTION:

Use the Method of Images to create the given flow. The sequence of images is shown below.



The potential function for the given flow field is:

$$\phi = \frac{m}{2\pi}\left[\begin{array}{l} \ln\sqrt{x^2+(y-a)^2} + \ln\sqrt{x^2+(y+a)^2} + \ln\sqrt{x^2+(y-5a)^2} + \ln\sqrt{x^2+(y+5a)^2} + \\ \ln\sqrt{x^2+(y-7a)^2} + \ln\sqrt{x^2+(y+7a)^2} + \ln\sqrt{x^2+(y-11a)^2} + \ln\sqrt{x^2+(y+11a)^2} + \cdots \end{array}\right] \tag{39}$$

$$\phi = \frac{m}{4\pi}\sum_{k=0}^{k\to\infty}\left\{\begin{array}{l}\ln\left[x^2+\left[y-(6k+1)a\right]^2\right]+\ln\left[x^2+\left[y+(6k+1)a\right]^2\right]+\\ \ln\left[x^2+\left[y-(6k+5)a\right]^2\right]+\ln\left[x^2+\left[y+(6k+5)a\right]^2\right]\end{array}\right\} \tag{40}$$

or, in dimensionless terms:

$$\boxed{\phi' = \sum_{k=0}^{k\to\infty}\left\{\begin{array}{l}\ln\left[x'^2+\left[y'-(6k+1)\right]^2\right]+\ln\left[x'^2+\left[y'+(6k+1)\right]^2\right]+\\ \ln\left[x'^2+\left[y'-(6k+5)\right]^2\right]+\ln\left[x'^2+\left[y'+(6k+5)\right]^2\right]+4\ln a^2\end{array}\right\}} \tag{41}$$

where the dimensionless potential function is $\phi' = \phi/(4\pi m)$ and the dimensionless positions are $x' = x/a$ and $y' = y/a$.

The dimensionless velocities resulting from this potential function are:

$$u'_x = \frac{\partial\phi'}{\partial x'} = 2x'\sum_{k=0}^{k\to\infty}\left\{\begin{array}{l}\dfrac{1}{x'^2+\left[y'-(6k+1)\right]^2}+\dfrac{1}{x'^2+\left[y'+(6k+1)\right]^2}+\\ \dfrac{1}{x'^2+\left[y'-(6k+5)\right]^2}+\dfrac{1}{x'^2+\left[y'+(6k+5)\right]^2}\end{array}\right\} \tag{42}$$

$$u'_y = \frac{\partial\phi'}{\partial y'} = \sum_{k=0}^{k\to\infty}\left\{\begin{array}{l}\dfrac{y'-(6k+1)}{x'^2+\left[y'-(6k+1)\right]^2}+\dfrac{y'+(6k+1)}{x'^2+\left[y'+(6k+1)\right]^2}+\\ \dfrac{y'-(6k+5)}{x'^2+\left[y'-(6k+5)\right]^2}+\dfrac{y'+(6k+5)}{x'^2+\left[y'+(6k+5)\right]^2}\end{array}\right\} \tag{43}$$

where $u'_x = au_x/(4\pi m)$ and $u'_y = au_y/(4\pi m)$.

The velocity at $(x', y') = (1, 0)$ is:

$$\boxed{u'_x(1,0) = 4\sum_{k=0}^{k\to\infty}\left\{\frac{1}{1+(6k+1)^2}+\frac{1}{1+(6k+5)^2}\right\}} \tag{44}$$

$\boxed{u'_y(1,0)=0}$ (as expected since the point is on a wall) (45)

The value of the horizontal velocity component at $(x', y') = (1,0)$ as a function of k is given in the table below. Note that "%diff from prev" is the percentage change in the value of u'_x from the previous value of u'_x, *i.e.* % diff = $(u'_{x,k+1} - u'_{x,k})/u'_{x,k} * 100\%$.

k	u'$_x$	% diff from prev
0	1.962	
1	2.042	4.08%
2	2.066	1.15%
3	2.077	0.53%
4	2.083	0.31%
5	2.087	0.20%
6	2.090	0.14%
7	2.092	0.10%
8	2.094	0.08%
9	2.095	0.06%
10	2.097	0.05%
11	2.097	0.04%
12	2.098	0.04%
13	2.099	0.03%
14	2.099	0.03%
15	2.100	0.02%
16	2.100	0.02%
17	2.101	0.02%
18	2.101	0.02%
19	2.101	0.01%

Hence, the velocity components at $(x',y') = (1,0)$ are $\boxed{(u'_x, u'_y) = (2.10, 0)}$.

9. Added Mass

Added mass (aka apparent or virtual mass) is the concept whereby we add an extra "mass" to an object when we accelerate the object through a fluid. This added mass term accounts for the force required to accelerate the surrounding fluid to a higher velocity.

To study this concept, let's consider the potential function for a cylinder in a fluid that is stagnant far from the cylinder. To form this potential function, we first form the potential function for a uniform stream with velocity U flowing around a stationary cylinder of radius R.

$$\phi\Big|_{\text{stationary cylinder in uniform stream}} = Ur\cos\theta\left(1+\frac{R^2}{r^2}\right)$$

To change our frame of reference so that the fluid far from the cylinder is stationary, we add in a uniform stream of velocity U in the opposite direction:

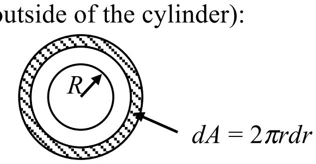
$$\phi\Big|_{\text{cylinder moving through stagnant fluid}} = \phi\Big|_{\text{stationary cylinder in uniform stream of velocity } U} - Ur\cos\theta$$

$$\therefore \phi\Big|_{\text{cylinder moving through stagnant fluid}} = \frac{UR^2\cos\theta}{r}$$

The resulting potential function describes the flow produced by a cylinder of radius R moving at velocity U through an otherwise quiescent fluid.

Let's now determine the total kinetic energy in the fluid (outside of the cylinder):

$$KE_{\text{total}} = \int_{r=R}^{r=\infty} \tfrac{1}{2}\rho\left(2\pi r dr\right)\left(u_r^2+u_\theta^2\right)$$



where the fluid velocity components are given by:

$$u_r = \frac{\partial\phi}{\partial r} = \frac{-UR^2\cos\theta}{r^2} \quad \text{and} \quad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{-UR^2\sin\theta}{r^2}$$

$$\Rightarrow \quad \left(u_r^2+u_\theta^2\right) = \frac{U^2R^4}{r^4}$$

Note that the ρ in the kinetic energy formula is the *fluid* density. Substituting the speed into the expression for the total kinetic energy:

$$KE_{\text{total}} = \int_{r=R}^{r=\infty} \tfrac{1}{2}\rho\left(2\pi r dr\right)\frac{U^2R^4}{r^4} = \left.\frac{-\pi\rho U^2R^4}{2r^2}\right|_{r=R}^{r=\infty} = \frac{\pi R^2}{2}\rho U^2$$

If we apply a force such that it increases the velocity of the cylinder by a small amount δU, the total kinetic energy of the fluid will increase by an amount (neglecting higher order terms):

$$\delta\left(KE_{\text{total}}\right) = \frac{\pi R^2}{2}\rho\left(U+\delta U\right)^2 - \frac{\pi R^2}{2}\rho U^2 = \pi R^2\rho U\left(\delta U\right)$$

The average force we must apply to the cylinder over time δt (the time over which the velocity goes from velocity U to velocity $U+\delta U$) to increase the total kinetic of the fluid is:

$$\underbrace{FU\delta t}_{\text{total work done in time } \delta t} = \delta\left(KE_{\text{total}}\right)$$

$$\Rightarrow \quad F\Big|_{\text{required to increase } KE \text{ of fluid}} = \frac{\pi R^2\rho U\left(\delta U\right)}{U\delta t} = \pi R^2\rho\frac{\delta U}{\delta t}$$

Thus, the total force required to accelerate a cylinder of mass, M, through a quiescent fluid is given by:

$$F = \left(\underbrace{M}_{\substack{\text{cylinder}\\ \text{mass}}} + \underbrace{\rho\pi R^2}_{\substack{\text{added}\\ \text{mass}}}\right)\frac{dU}{dt}$$

The term added to the cylinder mass in the previous equation is referred to as the **added mass** (aka apparent or virtual mass). Thus, the added mass for a cylinder is $M_{\text{added}}=\rho\pi R^2$. Again, the ρ is the fluid density.

Notes:

1. Note that added mass is only a factor for unsteady flows. There is no added mass term for a steady flow.

2. Added mass terms are typically only significant in flows of liquids since the added mass for gases is often small compared to the object's mass (ρ_{gas} is typically very small). Added mass terms in a gas can be significant however if the object is large and has small mass (e.g., a parachute).

3. We could have also found the force on the cylinder by integrating the pressure force around the cylinder surface, which is found using the unsteady Bernoulli equation (neglecting gravitational effects):
$$\left(\rho\frac{\partial\phi}{\partial t}+p+\tfrac{1}{2}\rho V^2\right)_{\text{on surface }(r=R)}=p_\infty$$
$$\Rightarrow\quad p_s=p_\infty-\tfrac{1}{2}\rho U^2-\rho R\cos\theta\frac{dU}{dt}$$
Integrating the pressure force along the surface of the cylinder to find the total force on the cylinder:
$$F=\int_{\theta=0}^{\theta=2\pi}p_s\cos\theta\left(Rd\theta\right)=R\left[p_\infty\int_0^{2\pi}\cos\theta d\theta-\tfrac{1}{2}\rho U^2\int_0^{2\pi}\cos\theta d\theta-\rho R\frac{dU}{dt}\int_0^{2\pi}\cos^2\theta d\theta\right]$$
$$\therefore F=\rho\pi R^2\frac{dU}{dt}\quad\text{(the same answer as before)}$$

4. The added mass is dependent on the shape of the object. It's possible to have different values for the added mass depending on the orientation of the object (e.g., an ellipse will have different added masses depending on its orientation.) We can also have added mass effects due to rotational acceleration of an object.

5. An additional reference concerning added mass is: Yih, C.S., 1969, *Fluid Dynamics*, McGraw-Hill (Now published by West River Press, Ann Arbor, MI).

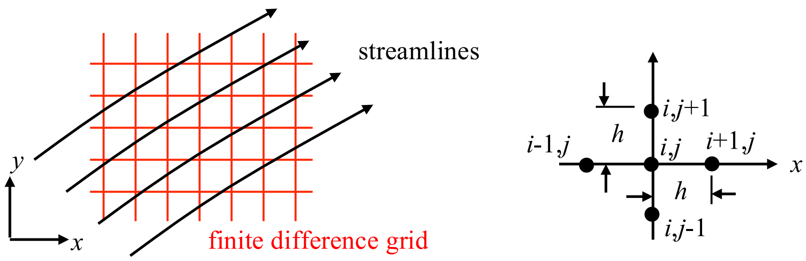
6. The added mass presented here was calculated for an inviscid flow. For unsteady viscous flows an additional term referred to as the **Bassett force** also appears which takes into account unsteady viscous force terms.

10. Finite Difference Methods

Recall that the governing equation for an incompressible, irrotational flow is:

$$\nabla^2 \phi = 0 \tag{46}$$

where ϕ is the velocity potential. Our goal here is to re-write Eq. (46) using a finite difference approximation so that we can solve the equation numerically. We'll assume a 2D flow in Cartesian coordinates to make the following analyses more straightforward; however, the same ideas can be applied to 3D and non-Cartesian (but still orthogonal!) coordinate systems. In particular, we'll solve Eq. (46) at the grid points shown in the figure below. Note that for simplicity, neighboring grid points are assumed to be separated by a distance h in both the x- and y-directions. The derivations given below may also be extended to non-uniform grid spacings.



The values of the second order partial derivatives in Eq. (46), e.g., $\partial^2\phi_{i,j}/\partial x^2$, may be written in terms of the neighboring values of ϕ by using Taylor series expansions about the point i,j. For example, for determining $\partial^2\phi_{i,j}/\partial x^2$, express $\phi_{i+1,j}$ and $\phi_{i-1,j}$ in terms of Taylor series expansions about i,j and then add the two Taylor series together.

$$\phi_{i+1,j} = \phi_{i,j} + \frac{(h)}{1!}\left.\frac{\partial\phi}{\partial x}\right|_{i,j} + \frac{(h)^2}{2!}\left.\frac{\partial^2\phi}{\partial x^2}\right|_{i,j} + \frac{(h)^3}{3!}\left.\frac{\partial^3\phi}{\partial x^3}\right|_{i,j} + \frac{(h)^4}{4!}\left.\frac{\partial^4\phi}{\partial x^4}\right|_{i,j} + \cdots \tag{47}$$

$$\phi_{i-1,j} = \phi_{i,j} + \frac{(-h)}{1!}\left.\frac{\partial\phi}{\partial x}\right|_{i,j} + \frac{(-h)^2}{2!}\left.\frac{\partial^2\phi}{\partial x^2}\right|_{i,j} + \frac{(-h)^3}{3!}\left.\frac{\partial^3\phi}{\partial x^3}\right|_{i,j} + \frac{(-h)^4}{4!}\left.\frac{\partial^4\phi}{\partial x^4}\right|_{i,j} + \cdots \tag{48}$$

$$\phi_{i+1,j} + \phi_{i-1,j} = 2\phi_{i,j} + 2\frac{h^2}{2!}\left.\frac{\partial^2\phi}{\partial x^2}\right|_{i,j} + 2\frac{h^4}{4!}\left.\frac{\partial^4\phi}{\partial x^4}\right|_{i,j} + \cdots \tag{49}$$

The previous expression may be re-arranged to solve for the 2nd order derivative.

$$\left.\frac{\partial^2\phi}{\partial x^2}\right|_{i,j} = \frac{\phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j}}{h^2} + 2\frac{h^2}{4!}\left.\frac{\partial^4\phi}{\partial x^4}\right|_{i,j} + \cdots \tag{50}$$

A similar approach may be used in the y-direction to determine $\partial^2\phi_{i,j}/\partial y^2$.

$$\left.\frac{\partial^2\phi}{\partial y^2}\right|_{i,j} = \frac{\phi_{i,j+1} + \phi_{i,j-1} - 2\phi_{i,j}}{h^2} + 2\frac{h^2}{4!}\left.\frac{\partial^4\phi}{\partial y^4}\right|_{i,j} + \cdots \tag{51}$$

Hence, at point (i,j), the solution to Eq. (46) may be written as:

$$0 = \left.\nabla^2\phi\right|_{i,j} = \left.\frac{\partial^2\phi}{\partial x^2}\right|_{i,j} + \left.\frac{\partial^2\phi}{\partial y^2}\right|_{i,j}$$

$$= \frac{\phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j}}{h^2} + 2\frac{h^2}{4!}\left.\frac{\partial^4\phi}{\partial x^4}\right|_{i,j} + \frac{\phi_{i,j+1} + \phi_{i,j-1} - 2\phi_{i,j}}{h^2} + 2\frac{h^2}{4!}\left.\frac{\partial^4\phi}{\partial y^4}\right|_{i,j} + \cdots$$

$$0 = \phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} + 2\frac{h^4}{4!}\left.\frac{\partial^4\phi}{\partial x^4}\right|_{i,j} + 2\frac{h^4}{4!}\left.\frac{\partial^4\phi}{\partial y^4}\right|_{i,j} + \cdots \tag{52}$$

The previous equation is an *exact* solution to Eq. (46) at the point (i,j) as long as all of the higher order terms are included in Eq. (52). If the value of h is sufficiently small, then we may approximate Eq. (52) by neglecting terms of order h^4 and higher since they will be small in comparison to the remaining terms (as long as the higher order derivatives don't simultaneously become very large). The resulting truncated equation is now only an approximate solution to Eq. (46),

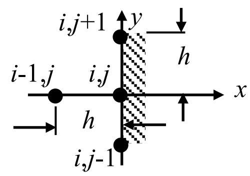
$$0 \approx \phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} \tag{53}$$

where the truncation error in the previous equation is of order h^4.

Notes:

1. Note that at a vertical solid boundary, the horizontal velocity (u_x) is zero. In terms of the velocity potential:

$$\left.\frac{\partial \phi}{\partial x}\right|_{i,j} = u_x\big|_{i,j} = 0$$



Determining the potential at (i-1,j) in terms of the Taylor series expansion about point (i,j) gives:

$$\phi_{i-1,j} = \phi_{i,j} + \frac{(-h)}{1!}\underbrace{\left.\frac{\partial \phi}{\partial x}\right|_{i,j}}_{=0} + \frac{(-h)^2}{2!}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} + \frac{(-h)^3}{3!}\left.\frac{\partial^3 \phi}{\partial x^3}\right|_{i,j} + \frac{(-h)^4}{4!}\left.\frac{\partial^4 \phi}{\partial x^4}\right|_{i,j} + \cdots \tag{54}$$

Re-arranging this equation gives:

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} = \frac{2\left(\phi_{i-1,j} - \phi_{i,j}\right)}{h^2} + O\left(h\left.\frac{\partial^3 \phi}{\partial x^3}\right|_{i,j}\right) \tag{55}$$

Combine Eq. (55) with Eq. (51) to solve for $\nabla^2\phi_{i,j} = 0$.

$$0 = \nabla^2\phi_{i,j} = \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} + \left.\frac{\partial^2 \phi}{\partial y^2}\right|_{i,j}$$

$$= \frac{2\left(\phi_{i-1,j} - \phi_{i,j}\right)}{h^2} + \frac{\phi_{i,j+1} + \phi_{i,j-1} - 2\phi_{i,j}}{h^2} + O\left(h\left.\frac{\partial^3 \phi}{\partial x^3}\right|_{i,j}\right) + O\left(h^2\left.\frac{\partial^4 \phi}{\partial y^4}\right|_{i,j}\right)$$

$$0 = \phi_{i,j+1} + \phi_{i,j-1} + 2\phi_{i-1,j} - 4\phi_{i,j} + O\left(h^3\left.\frac{\partial^3 \phi}{\partial x^3}\right|_{i,j}\right) + O\left(h^4\left.\frac{\partial^4 \phi}{\partial y^4}\right|_{i,j}\right) \tag{56}$$

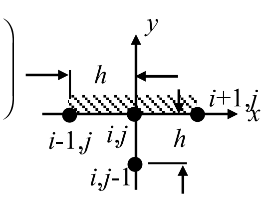
If the previous equation is truncated, then it becomes:

$$\underline{0 \approx \phi_{i,j+1} + \phi_{i,j-1} + 2\phi_{i-1,j} - 4\phi_{i,j}} \tag{57}$$

A similar approach can be used at a horizontal boundary to give:

$$0 = \phi_{i+1,j} + \phi_{i-1,j} + 2\phi_{i,j-1} - 4\phi_{i,j} + O\left(h^3\left.\frac{\partial^3 \phi}{\partial x^3}\right|_{i,j}\right) + O\left(h^4\left.\frac{\partial^4 \phi}{\partial y^4}\right|_{i,j}\right) \tag{58}$$

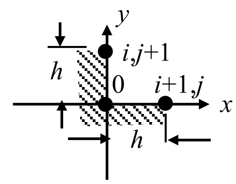
$$\underline{0 \approx \phi_{i+1,j} + \phi_{i-1,j} + 2\phi_{i,j-1} - 4\phi_{i,j}} \tag{59}$$



or at a corner:

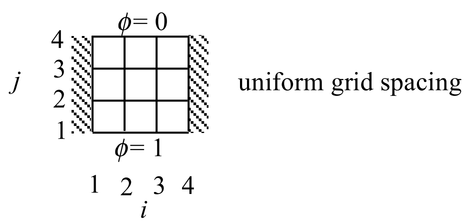
$$0 = 2\phi_{i+1,j} + 2\phi_{i,j+1} - 4\phi_{i,j} + O\left(h^3 \frac{\partial^3 \phi}{\partial x^3}\bigg|_{i,j} \right) + O\left(h^4 \frac{\partial^4 \phi}{\partial y^4}\bigg|_{i,j} \right) \tag{60}$$

$$\underline{0 \approx 2\phi_{i+1,j} + 2\phi_{i,j+1} - 4\phi_{i,j}} \tag{61}$$



2. Equation (46), i.e., Laplace's equation, is an elliptic partial differential equation. In order to have a well-posed problem, i.e., the equation has a unique solution that depends continuously on the boundary and/or initial data, the gridded flow domain must be finite and continuous boundary conditions must be specified along the *entire* boundary. The boundary conditions may be either <u>Dirichlet</u> boundary conditions (where the value of ϕ is specified), <u>Neumann</u> boundary conditions (where the gradient of ϕ is specified), or a combination of both types of boundary conditions (known as <u>mixed</u> boundary conditions).

3. There are two common methods to solving the resulting finite difference approximations to Eq. (46) at every point in the flow domain. Non-iterative, or <u>direct</u>, methods solve the equations directly (in "one step") while <u>iterative</u> methods solve the equations after repeated calculations that (hopefully) converge on the answer. Examples of each of these methods are given in the following discussions using the simple flow field and grid shown below.



Before numerically solving for the values of ϕ at each of the grid points, we can easily observe that for the uniform grid spacing shown, we anticipate that values for ϕ in row 2 ($i,j = 2$) should be $^2/_3$ and the values for ϕ in row 3 ($i, j = 3$) should be $^1/_3$.

For the given example, write the finite difference equations for each point on the grid using the expressions derived previously.

At $(i,j) = (1,1)$: $\phi_{1,1} = 1$ (a given boundary condition)

At $(i,j) = (2,1)$: $\phi_{2,1} = 1$ (a given boundary condition)

At $(i,j) = (3,1)$: $\phi_{3,1} = 1$ (a given boundary condition)

At $(i,j) = (4,1)$: $\phi_{4,1} = 1$ (a given boundary condition)

At $(i,j) = (1,2)$: $\phi_{1,1} + \phi_{1,3} + 2\phi_{2,2} - 4\phi_{1,2} = 0$

At $(i,j) = (2,2)$: $\phi_{1,2} + \phi_{3,2} + \phi_{2,1} + \phi_{2,3} - 4\phi_{2,2} = 0$

At $(i,j) = (3,2)$: $\phi_{2,2} + \phi_{4,2} + \phi_{3,1} + \phi_{3,3} - 4\phi_{3,2} = 0$

At $(i,j) = (4,2)$: $2\phi_{3,2} + \phi_{4,1} + \phi_{4,3} - 4\phi_{4,2} = 0$

At $(i,j) = (1,3)$: $2\phi_{2,3} + \phi_{1,2} + \phi_{1,4} - 4\phi_{1,3} = 0$

At $(i,j) = (2,3)$: $\phi_{1,3} + \phi_{3,3} + \phi_{2,2} + \phi_{2,4} - 4\phi_{2,3} = 0$

At $(i,j) = (3,3)$: $\phi_{2,3} + \phi_{4,3} + \phi_{3,2} + \phi_{3,4} - 4\phi_{3,3} = 0$

At $(i,j) = (3,3)$: $2\phi_{3,3} + \phi_{4,2} + \phi_{4,4} - 4\phi_{4,3} = 0$

At $(i,j) = (1,4)$: $\phi_{1,4} = 0$ (a given boundary condition)

At $(i,j) = (2,4)$: $\phi_{2,4} = 0$ (a given boundary condition)

At $(i,j) = (3,4)$: $\phi_{3,4} = 0$ (a given boundary condition)

At $(i,j) = (4,4)$: $\phi_{4,4} = 0$ (a given boundary condition)

Re-write the previous equations in matrix form.

$$
\underbrace{\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -4 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}}_{=[A]}
\underbrace{\begin{Bmatrix}
\phi_{1,1} \\ \phi_{2,1} \\ \phi_{3,1} \\ \phi_{4,1} \\ \phi_{1,2} \\ \phi_{2,2} \\ \phi_{3,2} \\ \phi_{4,2} \\ \phi_{1,3} \\ \phi_{2,3} \\ \phi_{3,3} \\ \phi_{4,3} \\ \phi_{1,4} \\ \phi_{2,4} \\ \phi_{3,4} \\ \phi_{4,4}
\end{Bmatrix}}_{=\{\phi\}}
=
\underbrace{\begin{Bmatrix}
1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{Bmatrix}}_{=\{b\}}
\tag{62}
$$

A direct method for solving Eq. (62) is Gauss Elimination. The algorithm for Gauss Elimination is not presented here and instead the reader is encouraged to review the method in a numerical methods text (see, for example, Hoffman, J.D., *Numerical Methods for Engineers and Scientists*, 2^{nd} ed., Marcel-Dekker).

Solving Eq. (62) using Gaussian Elimination gives:

$$\begin{Bmatrix} \phi_{1,1} \\ \phi_{2,1} \\ \phi_{3,1} \\ \phi_{4,1} \\ \phi_{1,2} \\ \phi_{2,2} \\ \phi_{3,2} \\ \phi_{4,2} \\ \phi_{1,3} \\ \phi_{2,3} \\ \phi_{3,3} \\ \phi_{4,3} \\ \phi_{1,4} \\ \phi_{2,4} \\ \phi_{3,4} \\ \phi_{4,4} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2/3 \\ 2/3 \\ 2/3 \\ 2/3 \\ 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{These are the results we expected!} \tag{63}$$

Notes:

a. Gaussian elimination is the preferred method for solving systems of linear equations. Modifications to the Gaussian elimination algorithm have been proposed that are optimized for banded matrices (where non-zero entries in the matrix occur in diagonal bands) such as the one in Eq. (62). Thomas' algorithm is one such algorithm that is particularly efficient for tri-diagonal matrices.

b. Direct methods, as opposed to iterative methods, will always converge to a solution (assuming that the given [A] matrix is non-singular, i.e., it has a non-zero determinant).

c. For very large systems, direct methods are generally less efficient than iterative methods.

Another approach for determining the values of $\phi_{i,j}$ in Eq. (62) is to use an iterative scheme. With an iterative method, an initial guess for the solution to $\phi_{i,j}$ is assumed. These initial values for $\phi_{i,j}$ are then used to generate new values for $\phi_{i,j}$ using a scheme that reduces the value between the current values of $\phi_{i,j}$ and the actual solution values. The scheme is repeated using the new values for $\phi_{i,j}$ until the values converge to the solution. Note that convergence of the iterative algorithm is not always guaranteed, which is the major drawback to iterative methods.

One commonly used iterative algorithm is Gauss-Seidel Iteration with Successive Over-Relaxation. In this algorithm, the new value for $\phi_{i,j}^{n+1}$, where the superscript "n+1" indicates the new value, is determined using the previous values, found at iteration step "n", of ϕ at the surrounding points (this is actually known as the Jacobi Iteration Method – Gauss-Seidel Iteration will be discussed in a moment). For example, Eq. (53) may be written in iterative form as:

$$\phi_{i,j}^{n+1} = \tfrac{1}{4}\left(\phi_{i-1,j}^{n} + \phi_{i+1,j}^{n} + \phi_{i,j-1}^{n} + \phi_{i,j+1}^{n}\right) \tag{64}$$

Iteration on all points (i,j) continues until the error between the current iteration value for ϕ and the previous iteration's value for ϕ is less than some tolerance, *i.e.*:

$$\text{Repeat iterations until } \left|\phi_{i,j}^{n+1} - \phi_{i,j}^{n}\right| < \text{tolerance for all } (i, j). \tag{65}$$

The difference between Gauss-Seidel iteration and Jacobi iteration is that Jacobi iteration determines the value for $\phi_{i,j}^{n+1}$ based on all of the previous iteration values whereas Gauss-Seidel iteration makes use of the new values for ϕ as they become available. For example, if we iterate in the previous example moving in the direction of increasing i and increasing j, then the value for $\phi_{3,3}$ will be:

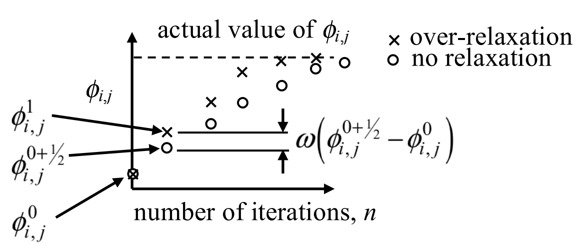
$$\phi_{3,3}^{n+1} = \tfrac{1}{4}\left(\phi_{2,3}^{n+1} + \phi_{4,3}^{n} + \phi_{3,2}^{n+1} + \phi_{3,4}^{n}\right) \tag{66}$$

By using the already updated values at the neighboring grid points, convergence is accelerated.

Often, the rate of convergence of the iterations can be improved by implementing a relaxation scheme. With relaxation the value of $\phi_{i,j}^{n+1}$ is found using a linear combination of Eq. (64) (or rather an equation similar to Eq. (66) depending on the direction of iteration) and the previous value for $\phi_{i,j}^{n}$, *i.e.*:

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \omega\left(\phi_{i,j}^{n+1/2} - \phi_{i,j}^{n}\right) \tag{67}$$

where the superscript $n+{}^{1}/_{2}$ refers to the intermediate value of $\phi_{i,j}$ calculated using Eq. (64) (or Eq. (66)) and ω is referred to as the relaxation parameter. The effect of relaxation can be most easily understood when presented graphically. In many instances the iterative values of $\phi_{i,j}$ approach the actual value of $\phi_{i,j}$ from one direction as shown in the plot below where a particular $\phi_{i,j}$ is shown as a function of the number of iterations, n. We observe from the plot that by using over-relaxation we are extrapolating the value of $\phi_{i,j}^{n+1}$ using $\phi_{i,j}^{n}$ and $\phi_{i,j}^{n+1/2}$ to help reach the converged value more quickly.



Notes:

a. If $\omega > 1$ then the process is known as over-relaxation, $\omega < 1$ is under-relaxation, and when $\omega = 1$ there is no relaxation. Under-relaxation is typically used when the iterations produce oscillatory values for $\phi_{i,j}$.

b. For over-relaxation, the iterative scheme can be shown to diverge if $\omega \geq 2$.

c. Relaxation can reduce the convergence rate considerably, often by one to two orders of magnitude!

d. The optimal choice for the relaxation parameter is not known a priori, in general, and multiple computations using differing values of ω need to be performed to determine ω_{opt}. Despite the additional computations, determining ω_{opt} is still a worthwhile effort, especially if the system of equations must be solved multiple times (if the boundary conditions change for example). As a rule of thumb, larger systems usually have a larger value for ω_{opt}.

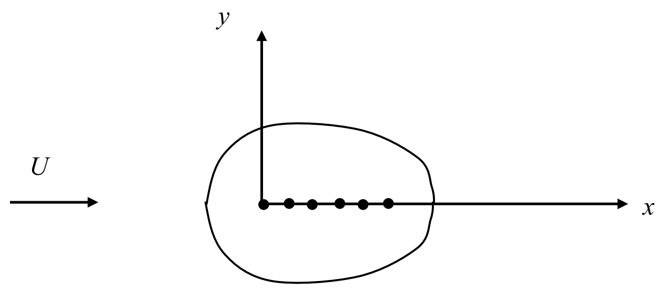
4. Additional issues such as the effects of round-off and truncation errors should be considered in more depth for a better understanding of numerical solutions. The reader is encouraged to study numerical methods texts for more information on these topics.

11. Doublet Distributions

So far we've approached potential flow problems by choosing potential functions and observing what types of flows result. Let's now look at a method of specifying an object shape and determining what the potential function should be.

We'll just examine a simple method here but it should be noted that more sophisticated methods (although based on the same concepts) are addressed in most books on aerodynamics (see, for example, Kuethe, A.M. and Chow, C.-Y., *Foundations of Aerodynamics*, Wiley).

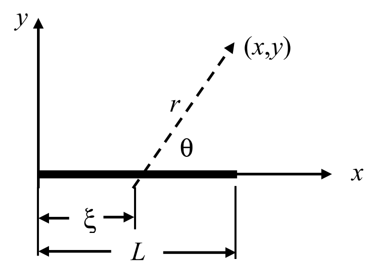
Recall that when we combined a uniform flow with a doublet, flow around a cylinder resulted. Now let's imagine combining on the x-axis a large number of doublets with varying strength.



The stream function evaluated at a point (x, y) for such a flow is given by

$$\psi = Uy - \int_{\xi=0}^{\xi=L} \frac{K(\xi)\, y\, d\xi}{(x-\xi)^2 + y^2}$$

where $K(\xi)d\xi$ is the total strength of the doublets over a very small distance $d\xi$, ξ is the distance from the origin, and L is the total length of the line of doublets.



Since we generally solve these types of problems numerically, re-write the integral as a summation:

$$\psi = Uy - \sum_{j=1}^{j=N} \frac{K_j\, y \Delta\xi}{\left(x-\xi_j\right)^2 + y^2}$$

where $\Delta\xi = L/N$.

We're usually interested in determining what the potential function should be for a specific object. To solve this inverse problem, we note that the object surface is a streamline so the stream function remains constant on the surface. Since we can arbitrarily adjust the value of the stream function (by adding in a constant – remember that only differences or derivatives of the stream function are of interest to us), we can adjust the stream function so that its value is zero on the object surface:

$$\psi_i = 0 = Uy_i - \sum_{j=1}^{j=N} c_{ij} K_j$$

where i is a point on the object surface and c_{ij} is referred to as the "influence coefficient" (the contribution of a doublet of unit density at the location j to the point i):

$$c_{ij} = \frac{y_i \Delta\xi}{\left(x_i - \xi_j\right)^2 + y_i^2} \tag{68}$$

The result is a system of equations we can solve numerically to determine the appropriate values of K_j:

$$\begin{array}{lllll} c_{11}K_1 & +c_{12}K_2 & \ldots & +c_{1N}K_N & =Uy_1 \\ & & \vdots & & \\ c_{N1}K_1 & +c_{N2}K_2 & \ldots & +c_{NN}K_N & =Uy_N \end{array}$$

Here the y_i are known since the geometry is known, U is known, and the c_{ij} are known as discussed previously.

Notes:

1. We could also have used a potential function in the previous analysis but instead of specifying the value of the potential function on the surface of the object, we would instead require that there is no flow through the surface:
   $$u_n = \frac{\partial \phi}{\partial n} = 0$$
   This approach is generally more involved than if we use stream functions.

2. We can extend these ideas to asymmetric objects by distributing doublets along curved paths.

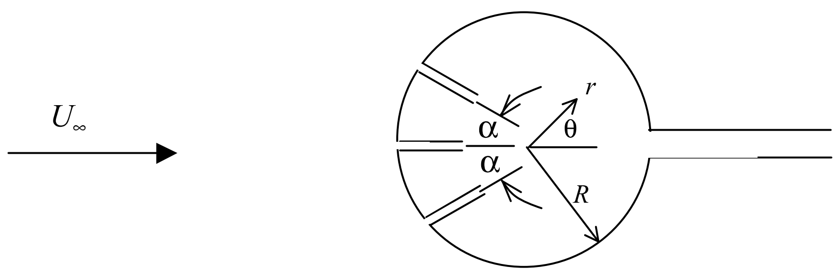
3. Instead of using a line of doublets (aka doublet panel), we could also use lines of sources (aka source panels), or lines of vortices (aka vortex panels).

4. There will be no lift on objects generated with doublets or sources since they produce no net circulation. Only vortex panels will produce circulation and lift.

5. These ideas can be extended to 3D using 3D source/doublet/vortex potentials.

A cylindrical tube with three (very small diameter) radially drilled orifices can be used as a flow-direction indicator. Whenever the pressure on the two side holes is equal, the center hole (located halfway between the side holes) will point in the direction of the flow. The pressure at the center hole is then the stagnation pressure. Such an instrument is called a direction-finding Pitot tube, or a cylindrical yaw probe.

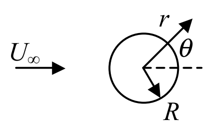
a. If the orifices of a direction-finding Pitot tube were to be used to measure free-stream static pressure, where would they have to be located?
b. For a direction-finding Pitot tube with orifices located as calculated in part (a), what is the device's sensitivity? Let the sensitivity be defined as the pressure change per unit angular change (i.e., $\partial p/\partial\theta$).



SOLUTION:

Model the flow as potential flow around a cylinder. The total potential function is the sum of the potential functions for a uniform stream and a doublet.

$$\phi_{\substack{\text{flow around}\\\text{a non-rotating}\\\text{cylinder}}} = \underbrace{U_\infty x}_{\substack{\text{uniform}\\\text{stream}}} + \underbrace{\frac{K\cos\theta}{r}}_{\text{doublet}} \tag{1}$$



$$\phi = U_\infty r\cos\theta + \frac{K\cos\theta}{r} \tag{2}$$

where the constant K is determined by specifying that there is no flow through the cylinder surface (i.e., $u_r(r = R) = 0$).

$$u_r = \frac{\partial\phi}{\partial r} = U_\infty\cos\theta - \frac{K\cos\theta}{r^2} \tag{3}$$

$$u_r\left(r = R\right) = 0 = U_\infty\cos\theta - \frac{K\cos\theta}{R^2} \Rightarrow K = U_\infty R^2 \tag{4}$$

Thus, Eq. (2) becomes:

$$\underline{\phi = U_\infty r\cos\theta\left(1 + \frac{R^2}{r^2}\right)} \tag{5}$$

The tangential velocity on the surface of the cylinder is found from the potential function.

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U_\infty\sin\theta\left(1 + \frac{R^2}{r^2}\right) \tag{6}$$

On the surface of the cylinder ($r = R$):

$$u_\theta\left(r = R\right) = -2U_\infty\sin\theta \tag{7}$$

The pressure on the surface can be found using Bernoulli's equation.

$$p_s + \tfrac{1}{2}\rho U_s^2 = p_\infty + \tfrac{1}{2}\rho U_\infty^2 \quad (8)$$

where

$$U_s^2 = \left(u_r^2 + u_\theta^2\right)_{r=R} = 4U_\infty^2 \sin^2\theta \quad (9)$$

Substituting Eq. (9) into Eq. (8) gives:

$$\underline{p_s = p_\infty + \tfrac{1}{2}\rho U_\infty^2\left(1 - 4\sin^2\theta\right)} \quad (10)$$

To measure the free stream pressure (i.e., p_∞), we need:

$$1 - 4\sin^2\theta = 0 \quad (11)$$

$$\sin\theta = \tfrac{1}{2}$$

$$\underline{\therefore \theta = \pm 30^\circ, \pm 150^\circ} \quad (12)$$

Note that $\alpha = 180^\circ - \theta$ so that the location of the pressure taps should be at:

$$\boxed{\alpha = \pm 30^\circ} \quad (13)$$

in order to measure the free stream pressure.

The sensitivity of the device is given by $\partial p/\partial\theta$ where the pressure is given in Eq. (10).

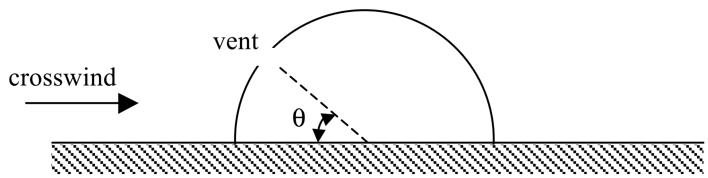
$$\frac{\partial p_s}{\partial\theta} = -4\rho U_\infty^2 \sin\theta\cos\theta \quad (14)$$

For $\alpha = \pm 30^\circ$ ($\theta = \pm 150^\circ$),

$$\frac{\partial p_s}{\partial\theta} = -4\rho U_\infty^2 \left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right)$$

$$\boxed{\frac{\partial p_s}{\partial\theta} = -\sqrt{3}\rho U_\infty^2} \quad (15)$$

A Quonset hut (essentially a cylinder cut in half) is subjected to a crosswind as shown in the figure. The interior of the hut is ventilated to the outside through a small vent at a position, θ, as indicated. Hence, the pressure inside the hut (assumed uniform and constant) is the same as the pressure just outside the vent. Assuming potential flow over the hut, find the angle, θ, at which the net vertical lift on the hut is zero. You may neglect the thickness of the wall of the hut and assume that the vent has no effect on the exterior flow.



SOLUTION:

Model the flow as potential flow over a non-rotating cylinder. The complex potential for this flow is:

$$f(z) = U\left(z + \frac{R^2}{z}\right)$$ (Refer to the course notes for the derivation of this potential function.)

The flow velocities are:

$$\begin{aligned}
(u_r - iu_\theta)\exp(-i\theta) = \frac{df}{dz} &= U\left(1 - \frac{R^2}{z^2}\right) \\
&= U\left[1 - \frac{R^2}{r^2\exp(2i\theta)}\right] \\
&= U\left[1 - \frac{R^2}{r^2}\exp(-2i\theta)\right] \\
&= U\left[\exp(i\theta) - \frac{R^2}{r^2}\exp(-i\theta)\right]\exp(-i\theta) \\
&= U\left[\cos\theta + i\sin\theta - \frac{R^2}{r^2}(\cos\theta - i\sin\theta)\right]\exp(-i\theta) \\
&= U\left[\left(1 - \frac{R^2}{r^2}\right)\cos\theta + i\left(1 + \frac{R^2}{r^2}\right)\sin\theta\right]\exp(-i\theta)
\end{aligned}$$

$$\begin{aligned}
\frac{u_r}{U} &= \left(1 - \frac{R^2}{r^2}\right)\cos\theta \\
\frac{u_\theta}{U} &= -\left(1 + \frac{R^2}{r^2}\right)\sin\theta
\end{aligned} \tag{1}$$

At the cylinder surface ($r = R$):

$$\begin{aligned}
\frac{u_r|_{r=R}}{U} &= 0 \\
\frac{u_\theta|_{r=R}}{U} &= -2\sin\theta
\end{aligned} \tag{2}$$

From Bernoulli's equation, the pressure on the cylinder surface ($r = R$) is:

$$\left(p + \tfrac{1}{2}\rho V^2\right)_{r=R} = \left(p + \tfrac{1}{2}\rho V^2\right)_{r\to\infty}$$

$$\frac{p_R - p_\infty}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta \tag{3}$$

The net vertical force (lift) acting on the hut due to the external flow and internal conditions (assumed constant at the pressure p_{vent}) can be found by integrating the vertical component of the net pressure force.

$$L_{\text{net}} = \int_{\theta=0}^{\theta=\pi} -\left(p_R - p_{\text{vent}}\right)\sin\theta \underbrace{\left(R d\theta\right)}_{=dA}$$

$$L_{\text{net}} = \tfrac{1}{2}\rho U^2 R \int_{\theta=0}^{\theta=\pi} -\left[\left(p_\infty + 1 - 4\sin^2\theta\right) - \left(p_\infty + 1 - 4\sin^2\theta_{\text{vent}}\right)\right]\sin\theta d\theta$$

$$= 2\rho U^2 R\left[\int_0^\pi \sin^3\theta d\theta - \sin^2\theta_{\text{vent}}\int_0^\pi \sin\theta d\theta\right]$$

$$= 2\rho U^2 R\left[\tfrac{4}{3} - 2\sin^2\theta_{\text{vent}}\right]$$

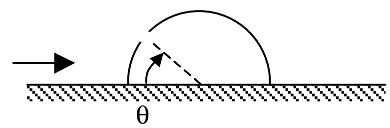
$$= 4\rho U^2 R\left[\tfrac{2}{3} - \sin^2\theta_{\text{vent}}\right]$$

Since we want zero net lift:

$$L_{\text{net}} = 0 = 4\rho U^2 R\left[\tfrac{2}{3} - \sin^2\theta_{\text{vent}}\right]$$

$$\sin^2\theta_{\text{vent}} = \tfrac{2}{3}$$

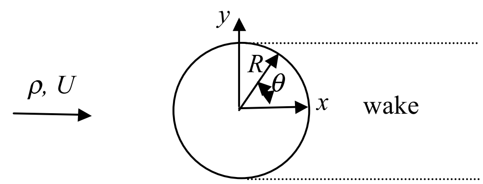
$$\boxed{\therefore \theta_{\text{vent}} = 54.7^\circ} \text{ or } \boxed{\theta_{\text{vent}} = 125.3^\circ}$$



For the purposes of estimating the drag force on a cylinder of radius, R, in a uniform stream of velocity, U, and uniform density, ρ, it is assumed that the pressure distribution over the upstream side (the side facing the oncoming stream) is the same as that in a potential flow, whereas the pressure on the downstream side is constant, simulating the conditions in a wake. Moreover, the pressure in the wake is equal to the pressure at the location θ = π/2. Determine the drag coefficient for the cylinder, C_D, using this model, where the drag coefficient is defined as:

$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho U^2 (2R)}$$

and F_D is the drag force acting on the cylinder.



SOLUTION:

The flow over the upstream side of the cylinder is modeled as a potential flow and combines a uniform stream and a doublet.

$$\phi = Ux + \frac{K\cos\theta}{r} = Ur\cos\theta + \frac{K\cos\theta}{r} \tag{1}$$

where the doublet strength, K, is found by noting that there is no flow through the surface of the cylinder, *i.e.*:

$$u_r\big|_{r=R} = \left.\frac{\partial\phi}{\partial r}\right|_{r=R} = 0 \Rightarrow U\cos\theta - \frac{K\cos\theta}{R^2} = 0 \Rightarrow K = UR^2 \tag{2}$$

Substituting Eq. (2) into Eq. (1) and re-arranging gives:

$$\underline{\phi = Ur\cos\theta\left(1 + \frac{R^2}{r^2}\right)} \tag{3}$$

The velocity components over the surface of the cylinder are found from the potential function.

$$u_r\big|_{r=R} = \left.\frac{\partial\phi}{\partial r}\right|_{r=R} = 0 \quad \text{(Found previously in Eq. (2).)} \tag{4}$$

$$\underline{u_\theta\big|_{r=R} = \left.\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{r=R} = -2U\sin\theta} \tag{5}$$

Bernoulli's equation may be used to find the pressure distribution on the surface of the cylinder.

$$p_\infty + \tfrac{1}{2}\rho U^2 = p_{r=R} + \tfrac{1}{2}\rho\left(u_r^2 + u_\theta^2\right)\Big|_{r=R} \tag{6}$$

$$p_{r=R} = p_\infty + \tfrac{1}{2}\rho\left[U^2 - \left(u_r^2 + u_\theta^2\right)\Big|_{r=R}\right] = p_\infty + \tfrac{1}{2}\rho\left(U^2 - 4U^2\sin^2\theta\right) \tag{7}$$

$$\underline{\therefore p_{r=R} = p_\infty + \tfrac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)} \tag{8}$$

The drag force on the cylinder is found by integrating the pressure force over the surface (assuming unit depth). Note that over the downstream side of the cylinder the pressure force is constant at the value when $\theta = \pi/2$.

$$F_D = \int_{\theta=\pi/2}^{\theta=3\pi/2} -p_{r=R}\cos\theta \underbrace{Rd\theta}_{=dA} - p_{r=R,\,\theta=\frac{\pi}{2}}(2R) \tag{9}$$

$$F_D = \int_{\theta=\pi/2}^{\theta=3\pi/2} -\left[p_\infty + \tfrac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)\right]\cos\theta R d\theta - \left(p_\infty - \tfrac{3}{2}\rho U^2\right)(2R) \tag{10}$$

$$F_D = -p_\infty R \sin\theta\Big|_{\pi/2}^{3\pi/2} - \tfrac{1}{2}\rho U^2 R\left[\sin\theta\Big|_{\pi/2}^{3\pi/2} - \tfrac{4}{3}\sin^3\theta\Big|_{\pi/2}^{3\pi/2}\right] - \left(p_\infty - \tfrac{3}{2}\rho U^2\right)(2R) \tag{11}$$

$$F_D = 2p_\infty R - \tfrac{1}{2}\rho U^2 R\left[-2+\tfrac{8}{3}\right] - \left(p_\infty - \tfrac{3}{2}\rho U^2\right)(2R) \tag{12}$$

$$F_D = -\tfrac{1}{3}\rho U^2 R + 3\rho U^2 R \tag{13}$$

$$\boxed{\therefore C_D \equiv \frac{F_D}{\tfrac{1}{2}\rho U^2 (2R)} = \tfrac{8}{3}} \tag{14}$$

Note that experiments have shown that at large Reynolds numbers (the regime in which potential flow is most applicable), the drag coefficient for a cylinder is approximately 1.5.

Consider the complex potential given by:

$$f(z) = V_0\left[z + y_0 \exp(i2\pi z/\lambda)\right]$$

where V_0, y_0, and λ are constants. Assume that $y_0 \ll \lambda$.

a. Sketch the streamlines for this flow.
b. What might this flow represent? What do the constants y_0 and λ represent?
c. Determine the pressure distribution along the streamline corresponding to $\psi=0$ in terms of V_0, ρ, y_0, λ, and p_∞ where p_∞ is the pressure far from this streamline ($y\to\infty$).

SOLUTION:

Expand the complex potential to determine the potential and stream functions, ϕ and ψ, respectively.

$$\begin{aligned} f(z) = \phi + i\psi &= V_0\left[z + y_0 \exp\left(i2\pi z/\lambda\right)\right] \\ &= V_0\left[x + iy + y_0 \exp\left(i2\pi\left(x+iy\right)/\lambda\right)\right] \\ &= V_0\left[x + iy + y_0 \exp\left(2\pi\left(ix-y\right)/\lambda\right)\right] \\ &= V_0\left[x + iy + y_0 \exp\left(2\pi ix/\lambda\right)\exp\left(-2\pi y/\lambda\right)\right] \\ &= V_0\left[x + iy + y_0 \left(\cos\left(2\pi x/\lambda\right) + i\sin\left(2\pi x/\lambda\right)\right)\exp\left(-2\pi y/\lambda\right)\right] \end{aligned}$$

$$\phi + i\psi = V_0\left[x + y_0 \exp\left(\frac{-2\pi y}{\lambda}\right)\cos\left(\frac{2\pi x}{\lambda}\right)\right] + iV_0\left[y + y_0 \exp\left(\frac{-2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right)\right]$$

$$\phi = V_0\left[x + y_0 \exp\left(\frac{-2\pi y}{\lambda}\right)\cos\left(\frac{2\pi x}{\lambda}\right)\right] \tag{1}$$

$$\psi = V_0\left[y + y_0 \exp\left(\frac{-2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right)\right] \tag{2}$$

The flow velocities are:

$$u_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = V_0\left[1 - y_0\frac{2\pi}{\lambda}\exp\left(\frac{-2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right)\right] \tag{3}$$

$$u_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} = -V_0 y_0\frac{2\pi}{\lambda}\exp\left(\frac{-2\pi y}{\lambda}\right)\cos\left(\frac{2\pi x}{\lambda}\right) \tag{4}$$

Note that along a streamline, the stream function is a constant (call it ψ_0).

$$\frac{\psi_0}{V_0} = y + y_0 \exp\left(\frac{-2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right) \tag{5}$$

This function may be plotted numerically by choosing particular values for ψ_0, V_0, y_0, λ, and x then solving for y (using a Newton-Raphson root finding scheme, for example). A *C* program for this procedure is given at the end of this solution.

We can also determine the general streamline shape without the aid of a computer program by considering conditions at extreme cases. For example, when y is large:

$$\frac{\psi_0}{V_0} = \lim_{y\to\infty}\left[y + y_0 \exp\left(\frac{-2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right)\right] = y \Rightarrow \underline{y = \text{constant}} \quad (6)$$

(i.e., The stream line is a horizontal line when y is very large).

Note that the exponential term may be written as a power series:

$$\exp\left(\frac{-2\pi y}{\lambda}\right) = 1 - \frac{2\pi}{\lambda}y + \frac{1}{2!}\left(\frac{2\pi}{\lambda}y\right)^2 - \frac{1}{3!}\left(\frac{2\pi}{\lambda}y\right)^3 + \cdots$$

If $y \ll \lambda$ then:

$$\exp\left(\frac{-2\pi y}{\lambda}\right) \approx 1 \quad (7)$$

Substituting Eq. (7) into Eq. (5) gives:

$$\frac{\psi_0}{V_0} = y + y_0 \sin\left(\frac{2\pi x}{\lambda}\right) \quad (8)$$

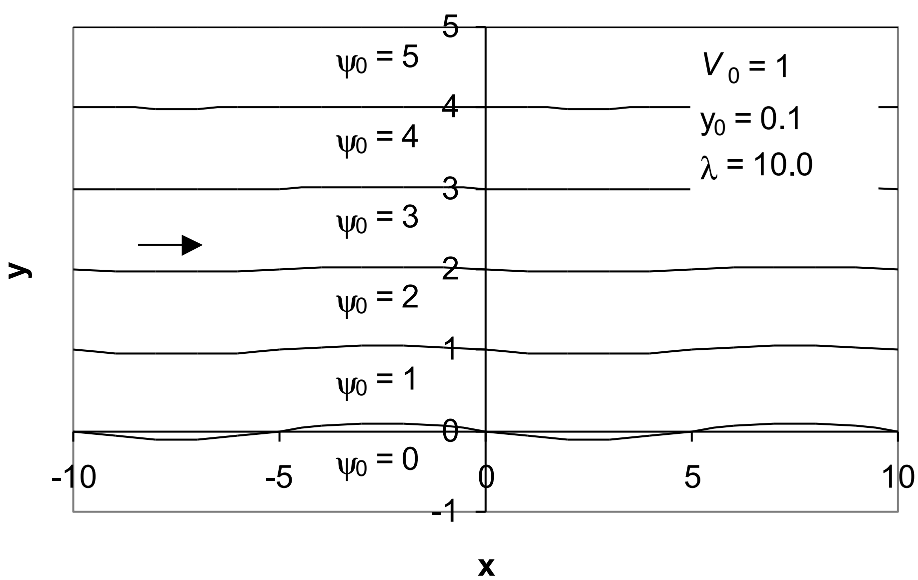
At $\psi_0 = 0$:

$$0 = y + y_0 \sin\left(\frac{2\pi x}{\lambda}\right)$$

$$\underline{y = -y_0 \sin\left(\frac{2\pi x}{\lambda}\right)} \text{ (A sine wave!)} \quad (9)$$

Hence, the stream function represents the flow over a wavy wall where y_0 represents the amplitude of the wall and λ represents the wall's wavelength. The velocity fluctuations decay as the distance from the wavy wall increases (i.e., as y increases).

The following plot presents streamlines (lines of constant ψ) for $V_0 = 1$, $y_0 = 0.1$, and $\lambda = 10.0$ calculated from the attached *C* program. The fluid is moving from left to right in the plot.



The pressure along the y = 0 streamline can be determined using Bernoulli's equation.

$$\left(p+\tfrac{1}{2}\rho V^2\right)_{\psi=0}=\left(p+\tfrac{1}{2}\rho V^2\right)_{y\to\infty}$$

where the flow velocities are given in Eqs. (3) and (4).

$$\begin{aligned}V^2=u_x^2+u_y^2&=V_0^2\left\{\left[1-y_0\frac{2\pi}{\lambda}\exp\left(\frac{-2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right)\right]^2+\left[y_0\frac{2\pi}{\lambda}\exp\left(\frac{-2\pi y}{\lambda}\right)\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2\right\}\\&=V_0^2\left\{1-\frac{4\pi y_0}{\lambda}\exp\left(\frac{-2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right)+\left(\frac{2\pi y_0}{\lambda}\right)^2\exp\left(\frac{-4\pi y}{\lambda}\right)\sin^2\left(\frac{2\pi x}{\lambda}\right)+\left(\frac{2\pi y_0}{\lambda}\right)^2\exp\left(\frac{-4\pi y}{\lambda}\right)\cos^2\left(\frac{2\pi x}{\lambda}\right)\right\}\\&=V_0^2\left\{1-\frac{4\pi y_0}{\lambda}\exp\left(\frac{-2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right)+\left(\frac{2\pi y_0}{\lambda}\right)^2\exp\left(\frac{-4\pi y}{\lambda}\right)\right\}\end{aligned}$$

$$V_{y\to\infty}^2=V_0^2$$

Substituting into Bernoulli's equation gives:

$$\begin{aligned}\frac{p-p_\infty}{\frac{1}{2}\rho V_0^2}&=1-\left(\frac{V}{V_0}\right)^2\\&=\frac{4\pi y_0}{\lambda}\exp\left(\frac{-2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right)+\left(\frac{2\pi y_0}{\lambda}\right)^2\exp\left(\frac{-4\pi y}{\lambda}\right)\end{aligned}\tag{10}$$

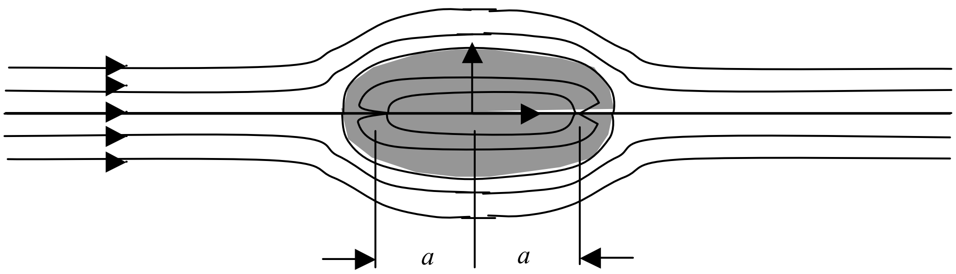
We solve exactly for Eq. (10) along the $\psi = 0$ streamline using a numerical procedure or, assuming that $y_0 << \lambda$ and Eq. (7), we can get the approximate solution:

$$\boxed{\frac{p_{\psi=0}-p_\infty}{\frac{1}{2}\rho V_0^2}=\frac{4\pi y_0}{\lambda}\sin\left(\frac{2\pi x}{\lambda}\right)}\tag{11}$$

```
/*
  A numerical routine for determining the y-values corresponding to
  a particular x-value and a particular stream function.
*/

# include <stdio.h>
# include <stdlib.h>
# include <math.h>
# define PI (4.0*atan(1.0))
# define TWOPI (2.0*PI)

/* The following are constants in the stream function. */
double V0 = 1.0;
double yo = 0.1;
double lambda = 10.0;

/* ***** */

void main(void) {
  double Newton_Raphson(double, double);
  double x, y, psi, x_min, x_max, delta_x, psi_min, psi_max, delta_psi;
  FILE *outfile;

  /* Open an output file to which the data will be written. */
  if ((outfile = fopen("soln_pot_06.out", "wb")) == NULL) {
    printf("Cannot open output file\n");
    exit(1);
  }

  /* Specify the range over which y values will be determined. */
  x_min = -10.0;
  x_max = 10.0;
  delta_x = 1.0;
  psi_min = 0.0;
  psi_max = 5.0;
  delta_psi = 1.0;

  /* Loop through the range of psi and x values. */
  for (psi=psi_min; psi<=psi_max; psi+=delta_psi) {
    for (x=x_min; x<=x_max; x+=delta_x) {
      /* Solve for the y value corresponding to the given psi and x
         values. */
      y = Newton_Raphson(psi, x);
      /* Print the data to the output file. */
      fprintf(outfile, "%.2f\t%.2f\t%.2f\n", psi, x, y);
    }
  }
  /* Close the output file. */
  fclose(outfile);
  printf("Done.\n");
}
```

```
/* ***** */

double Newton_Raphson(double psi, double x) {
  /* This routine solves for the y value corresponding to a
     particular psi and x value using a Newton-Raphson root finding
     algorithm. */
  int num_iterations, max_iterations=100;
  double y, y_old, tol=1.0e-8, f, df_dy;
  extern double V0, yo, lambda;

  num_iterations = 0;
  y = 1.0;  /* An initial guess for y. */
  do {
    num_iterations++;
    y_old = y;
    f = y+yo*exp(-TWOPI*y/lambda)*sin(TWOPI*x/lambda)-psi/V0;
    df_dy = 1-TWOPI*yo/lambda*exp(-TWOPI*y/lambda)*sin(TWOPI*x/lambda);
    y = y_old - f/df_dy;
  } while ((fabs((y-y_old)/y_old) > tol) &&
         (num_iterations < max_iterations)) ;
  if (num_iterations == max_iterations)  {
    printf("A converged y value could not be determined within %d
iterations.\n", max_iterations);
    exit(1);
  }

  return y;
}
```

A Rankine oval is formed by combining a uniform stream of velocity, U_∞, with a source of strength m located at $(x, y) = (-a, 0)$ and a sink of strength m located at $(x, y) = (a, 0)$.



For this problem, let: $U_\infty = 10$ m/s, $m = 1$ m^3/(s·m), and $a = 1$ m.

a. Determine the overall length of the oval.
b. Determine the overall width of the oval.
c. Plot the shape of the oval. (Do not sketch the oval, but actually plot the points corresponding to the oval surface.)
d. Plot the surface velocity and the surface pressure profiles.
e. Calculate the drag coefficient acting on the Rankine oval by integrating the pressure force over the entire surface.
f. Calculate the drag coefficient acting on the oval using the Blasius' Integral Law.

SOLUTION:

First, form the complex potential describing the Rankine oval.

$$f(z) = \underbrace{U_\infty z}_{\substack{\text{uniform}\\ \text{stream}}} + \underbrace{\frac{m}{2\pi}\log(z+a)}_{\substack{\text{source at } z=-a\\ (m>0)}} - \underbrace{\frac{m}{2\pi}\log(z-a)}_{\substack{\text{sink at } z=a\\ (m>0)}} \tag{1}$$

Determine the velocity field from the complex potential.

$$\begin{aligned}
u_x - iu_y = \frac{df}{dz} &= U_\infty + \frac{m}{2\pi}\frac{1}{z+a} - \frac{m}{2\pi}\frac{1}{z-a} \\
&= U_\infty + \frac{m}{2\pi}\left(\frac{-2a}{z^2-a^2}\right) \\
&= U_\infty + \frac{m}{2\pi}\left(\frac{-2a}{x^2+i2xy-y^2-a^2}\right) \\
&= U_\infty + \frac{m}{2\pi}\left(\frac{-2a}{x^2-y^2-a^2+i2xy}\right)\left(\frac{x^2-y^2-a^2-i2xy}{x^2-y^2-a^2-i2xy}\right) \\
&= U_\infty + \frac{m}{2\pi}\left[\frac{-2a\left(x^2-y^2-a^2-i2xy\right)}{\left(x^2-y^2-a^2\right)^2+(2xy)^2}\right]
\end{aligned}$$

$$\therefore u_x - iu_y = U_\infty + \frac{m}{2\pi}\left[\frac{-2a\left(x^2-y^2-a^2\right)}{\left(x^2-y^2-a^2\right)^2+(2xy)^2}\right] - i\frac{m}{2\pi}\left[\frac{-2a(2xy)}{\left(x^2-y^2-a^2\right)^2+(2xy)^2}\right] \tag{2}$$

Hence:

$$u_x = U_\infty + \frac{m}{2\pi}\left[\frac{-2a\left(x^2-y^2-a^2\right)}{\left(x^2-y^2-a^2\right)^2+\left(2xy\right)^2}\right] \tag{3}$$

$$u_y = \frac{m}{2\pi}\left[\frac{-2a\left(2xy\right)}{\left(x^2-y^2-a^2\right)^2+\left(2xy\right)^2}\right] \tag{4}$$

In dimensionless terms, the velocities are:

$$u'_x = 1+\alpha\left[\frac{-2\left(x'^2-y'^2-1\right)}{\left(x'^2-y'^2-1\right)^2+\left(2x'y'\right)^2}\right] \tag{5}$$

$$u'_y = \alpha\left[\frac{-2\left(2x'y'\right)}{\left(x'^2-y'^2-1\right)^2+\left(2x'y'\right)^2}\right] \tag{6}$$

where the dimensionless source strength is $\alpha = m/(2\pi a U_\infty)$, the dimensionless velocities u'_x and u'_y are u_x/U_∞ and u_y/U_∞, respectively, and the dimensionless positions x' and y' are x/a and y/a, respectively.

The equation for the surface of the Rankine oval is the same as the stream function evaluated at the stagnation point on the leading (or trailing) edge of the oval. The stream function can be determined from the complex potential.

$$\begin{aligned}
f(z) &= U_\infty z + \frac{m}{2\pi}\log(z+a) - \frac{m}{2\pi}\log(z-a) \\
&= U_\infty z + \frac{m}{2\pi}\left[\ln|z+a| + i\arg(z+a)\right] - \frac{m}{2\pi}\left[\ln|z-a| + i\arg(z-a)\right] \\
&= U_\infty(x+iy) + \frac{m}{2\pi}\left\{\ln\left[(x+a)^2+y^2\right] + i\tan^{-1}\left(\frac{y}{x+a}\right)\right\} - \frac{m}{2\pi}\left\{\ln\left[(x-a)^2+y^2\right] + i\tan^{-1}\left(\frac{y}{x-a}\right)\right\} \\
&= \underbrace{\left\{U_\infty x + \frac{m}{2\pi}\ln\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]\right\}}_{=\phi} + i\underbrace{\left\{U_\infty y + \frac{m}{2\pi}\left[\tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right)\right]\right\}}_{=\psi}
\end{aligned}$$

$$\therefore \psi = U_\infty y + \frac{m}{2\pi}\left[\tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right)\right] \tag{7}$$

or, in dimensionless terms:

$$\therefore \psi' = y' + \alpha\left[\tan^{-1}\left(\frac{y'}{x'+1}\right) - \tan^{-1}\left(\frac{y'}{x'-1}\right)\right] \tag{8}$$

where $\psi' = \psi/(aU_\infty)$.

The dimensionless location of the leading edge stagnation point (which will occur along the x-axis due to symmetry) is found by setting the dimensionless velocity components equal to zero at (x'_0, $y'_0 = 0$) where the subscript "$_0$" indicates a stagnation point.

$$u'_x\big|_0 = 0 = 1+\alpha\left[\frac{-2}{x_0'^2-1}\right] \Rightarrow x'_0 = -\sqrt{2\alpha+1} \tag{9}$$

$$u'_y\big|_0 = 0 \tag{10}$$

Thus, the leading edge stagnation point is located at:

$$\left(x_0', y_0'\right) = \left(-\sqrt{2\alpha+1}, 0\right) \tag{11}$$

Note that the overall length of the oval is $|2x'_0|$, *i.e.*:

$$\boxed{\text{oval length} = \left|2x_0'\right| = 2\sqrt{2\alpha+1}} \tag{12}$$

Evaluating ψ' at the leading edge stagnation point (x'_0, y'_0) gives:

$$\psi_0' = 0 \tag{13}$$

Thus, the equation for the surface of the Rankine oval is:

$$0 = y_s' + \alpha\left[\tan^{-1}\left(\frac{y_s'}{x_s'+1}\right) - \tan^{-1}\left(\frac{y_s'}{x_s'-1}\right)\right] \tag{14}$$

where the subscript "$_s$" indicates coordinates on the surface of the object. In addition, to remain on the surface of the oval, $-|x'_0| \leq x'_s \leq |x'_0|$.

The maximum width of the oval occurs when $x'_s = 0$ (from symmetry).

$$0 = y_{s,\max}' + \alpha\left[\tan^{-1}\left(\frac{y_{s,\max}'}{1}\right) - \tan^{-1}\left(\frac{y_{s,\max}'}{-1}\right)\right] \tag{15}$$

where

$$\boxed{\text{oval width} = \left|2y_{s,\max}'\right|} \tag{16}$$

To determine the velocity and pressure on the surface of the oval, use the following procedure:

1. Choose a value for x_s where $-|x'_0| \le x'_s \le |x'_0|$ and x'_0 is given by Eqn. (11).
2. Determine the maximum width of the oval (Eqn. (16)) by solving Eqn. (15) numerically.
4. Determine y'_s from Eqn. (14) (solve this equation numerically since it's implicit in y'_s).
5. Evaluate $u'_x|_s$ and $u'_y|_s$ using (x'_s, y'_s) and Eqns. (5) and (6).
6. Evaluate the speed squared at this point on the oval surface using $V'^2_s = (u'_x|_s^2 + u'_y|_s^2)$.
7. Evaluate the pressure at this point on the oval surface using Bernoulli's equation:

$$p_s + \tfrac{1}{2}\rho V_s^2 = p_\infty + \tfrac{1}{2}\rho U_\infty^2 \tag{17}$$

$$c_{p,s} \equiv \frac{p_s - p_\infty}{\tfrac{1}{2}\rho U_\infty^2} = 1 - V_s'^2 \tag{18}$$

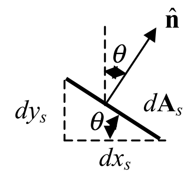
where $c_{p,s}$ is a dimensionless pressure coefficient evaluated on the oval surface and p_∞ is the pressure far upstream of the oval.

8. The force acting on the oval surface can be found by integrating the pressure force.

$$\mathbf{F}_s = \int_s d\mathbf{F}_s = \int_s -p_s d\mathbf{A}_s \tag{19}$$

where $d\mathbf{A}_s$ is a differential area element on the oval's surface.

$$d\mathbf{A}'_s = dy'_s\hat{\mathbf{e}}_x + dx'_s\hat{\mathbf{e}}_y \tag{20}$$



(21)

In dimensionless terms, the drag coefficient, C_D, acting on the oval is:

$$C_D \equiv \frac{\mathbf{F}_s \cdot \hat{\mathbf{e}}_x}{\tfrac{1}{2}\rho U^2 \underbrace{\left|2y_{s,\max}\right|}_{\text{frontal projected area}}} = \int_s -\frac{p_s - p_\infty}{\tfrac{1}{2}\rho U^2}\frac{d\mathbf{A}_s \cdot \hat{\mathbf{e}}_x}{\left|2y_{s,\max}\right|} = \int_s -c_p \frac{dy'_s}{\left|2y_{s,\max}\right|} \tag{22}$$

where the frontal projected area of the oval is used in making the drag force dimensionless. Note that the atmospheric pressure term will not contribute anything to the integral if the integral encompasses the entire oval surface. The lift coefficient may be found in a similar manner.

A C program was created to perform the previously presented calculations. A printout of the source file is attached at the end of this problem solution.

Using the given data, the dimensionless source strength, α, is:

$$\alpha = \frac{m}{2\pi a U_\infty} = \frac{\left(1\text{ m}^3/(\text{s}\cdot\text{m})\right)}{2\pi(1\text{ m})(10\text{ m/s})} = 1.592*10^{-2} \tag{23}$$

$\boxed{\text{The dimensionless overall oval length is 2.032 and the dimensionless overall width is } 9.691*10^{-2}.}$

A plot of the oval surface streamline is shown in Figure 1. The dimensionless surface velocities and pressure coefficient are shown in Figures 2 and 3, respectively.

$\boxed{\text{The drag coefficient determined by integrating the pressure forces is approximately zero.}}$ Note that the C program calculates a drag coefficient of $6.0*10^{-5}$, but there is error expected in this result due to numerical integration and finite machine precision. Decreasing the step size (delta_x) in the numerical integration reduces the drag coefficient.

$\boxed{\text{The drag acting on the entire oval using the Blasius Integral Law is zero since there are no net sources or sinks (the oval is a closed body).}}$

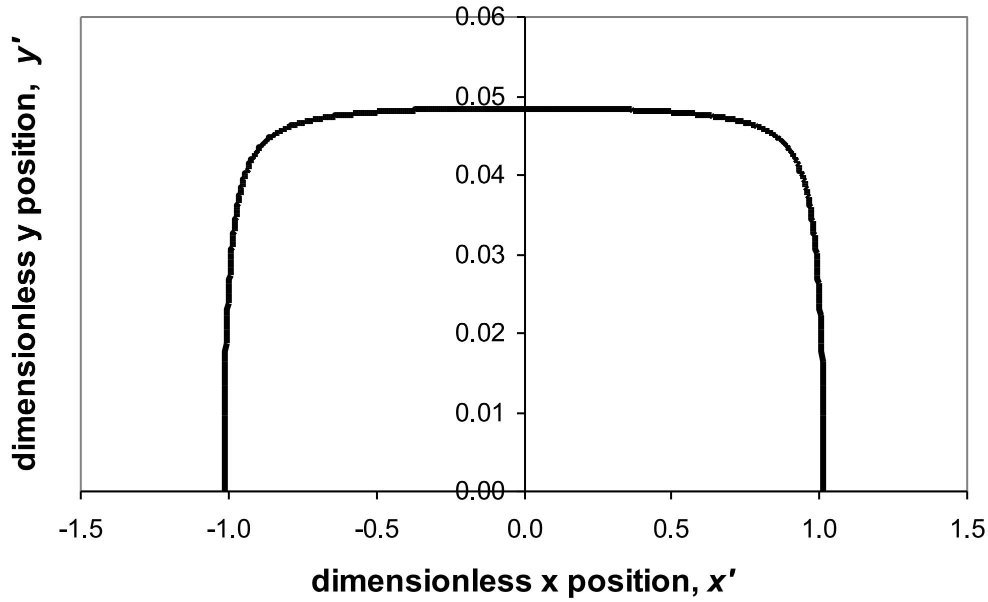
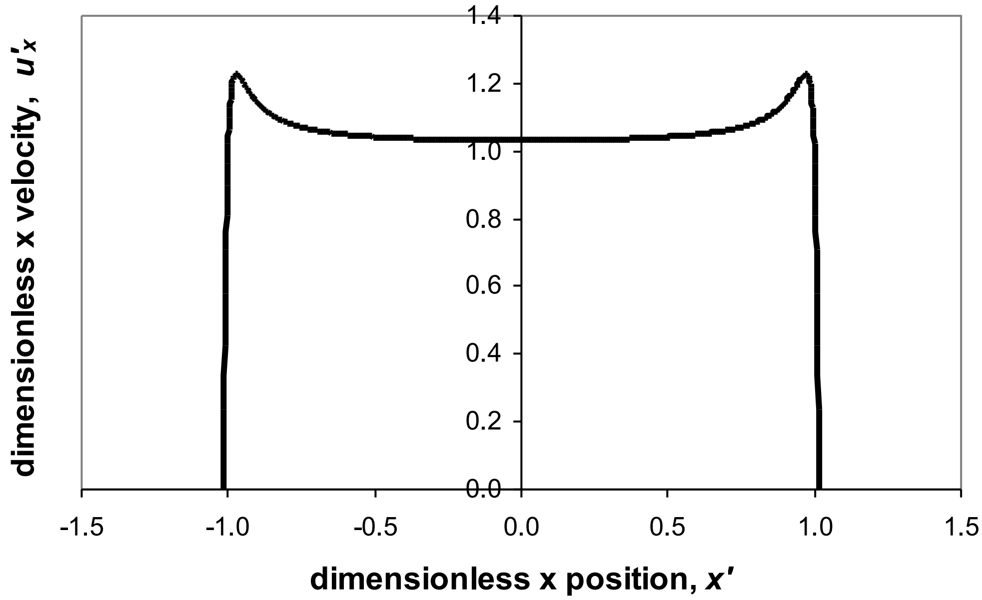
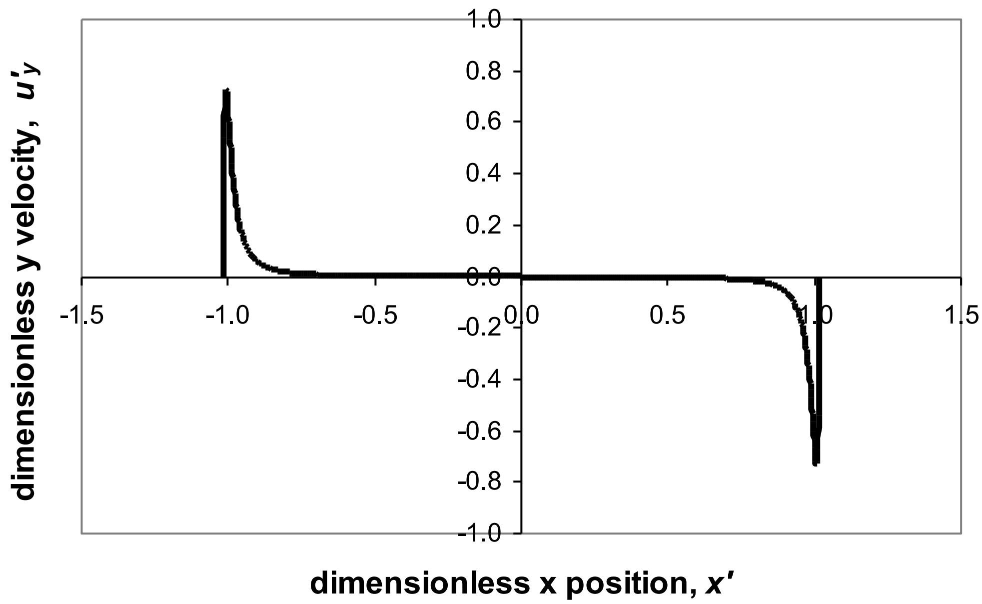


Figure 1. A plot of the Rankine oval surface streamline for α= 1.592*10^{-2}. Only the top half of the oval is shown since the oval is symmetric.



(a)



(b)

Figure 2. (a) A plot of the dimensionless x-velocity on the oval upper surface. (b) A plot of the dimensionless y-velocity on the oval upper surface. The dimensionless source strength is $\alpha = 1.592*10^{-2}$.

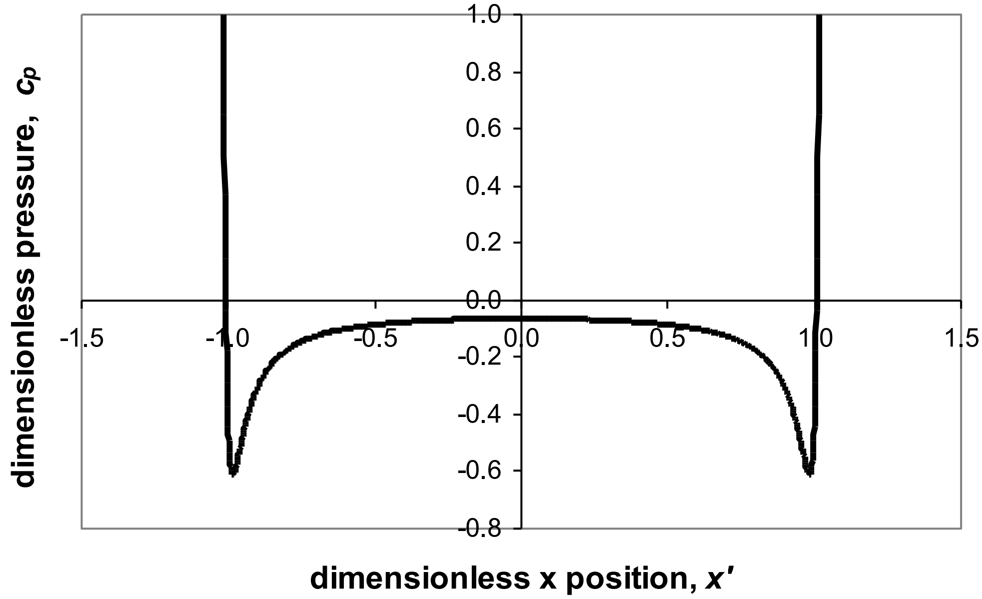


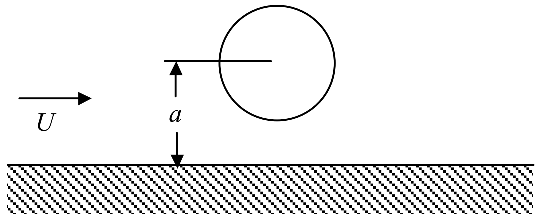
Figure 3. A plot of the pressure coefficient on the oval upper surface for a dimensionless source strength of $\alpha = 1.592*10^{-2}$.

```
/*
  pot_10.c
*/

# include <stdio.h>
# include <stdlib.h>
# include <math.h>
# define PI (4.0*atan(1.0))
# define TWOPI (2.0*PI)

/* ***** */

int main(int argc, char **argv) {
  double Bisection(const double, const double);
  double alpha, x, y, x_min, x_max, delta_x, ux, uy, Vsquared, cp,
y_old,
    delta_y, CD, y_max, y_width;
  FILE *outfile;

  // Make sure there are enough data in the command line.
  if (argc < 2) {
    printf("Usage:  pot_10 <alpha>\n");
    exit(1);
  }

  // Read in the value for alpha.
  alpha = atof(argv[1]);
  if (alpha <= 0.0) {
    printf("alpha must be greater than zero.\n");
    exit(1);
  }

  // Open an output file to which the data will be written.
  if ((outfile = fopen("data.out", "wb")) == NULL) {
    printf("Cannot open output file\n");
    exit(1);
  }

  // Determine the maximum width of the oval.
  y_max = Bisection(alpha,0.0);
  y_width = 2.0*y_max;

  // Specify the range over which y values will be determined.
  x_min = -sqrt(2.0*alpha+1.0);
  x_max = -x_min;
  delta_x = (x_max-x_min)/1.0e6;
  y_old = 0.0;
  CD = 0.0;

  // Loop through the range of x values.
  for (x=x_min; x<=x_max; x+=delta_x) {
    // Solve for the y value corresponding to the given x value.
    y = Bisection(alpha, x);

    // Determine the surface velocity components.
    ux = 1.0+alpha*((-2.0*(x*x-y*y-1.0))/((x*x-y*y-1.0)*(x*x-y*y-1.0) +
(2.0*x*y)*(2.0*x*y)));
```

```
    uy = alpha*((-4.0*x*y)/((x*x-y*y-1.0)*(x*x-y*y-1.0) +
(2.0*x*y)*(2.0*x*y)));

    Vsquared = ux*ux + uy*uy;  // Determine the surface speed.
    cp = 1.0 - Vsquared;  // Determine the surface pressure coefficient.

    delta_y = y-y_old;
    y_old = y;
    CD += -cp*delta_y/y_width;

    // Print the data to the output file.
    fprintf(outfile, "%.6e\t%.6e\t%.6e\t%.6e\t%.6e\t%.6e\n",
          x, y, ux, uy, cp, -cp*delta_y/y_width);
  }

  // Close the output file.
  fclose(outfile);

  printf("alpha = %.4f\n", alpha);
  printf("Leading edge stagnation point dimensionless location:  (%.3e,
0)\n", x_min);
  printf("Rankine oval dimensionless width:  %.3e\n", y_width);
  printf("Rankine oval dimensionless length:  %.3e\n", -2.0*x_min);
  printf("Drag coefficient:  %.3e\n", CD);

  printf("Done.\n");
  return(0);
}

/* ***** */

double Bisection(const double alpha, const double x) {
  /* This routine solves for the y value corresponding to a
     particular alpha and x value using a bisection root finding
     algorithm. */
  int num_iterations, max_iterations=10000, flag;
  double y_left, y_middle, y_right, f_left, f_middle, f_right, tol=1.0e-
9;

  num_iterations = 0;
  flag = 0;

  // Specify some initial bounds.
  y_left = 2.0*tol;
  y_right = 10.0;

  do {
    num_iterations++;

    y_middle = 0.5*(y_left+y_right);
    f_left = y_left + alpha*(atan2(y_left,x+1.0)-atan2(y_left,x-1.0));
    f_right = y_right + alpha*(atan2(y_right,x+1.0)-atan2(y_right,x-
1.0));
    f_middle = y_middle + alpha*(atan2(y_middle,x+1.0)-atan2(y_middle,x-
1.0));

    if (f_left*f_middle < 0.0) {
```

```
      y_left = y_left;
      y_right = y_middle;
    } else if (f_left*f_middle > 0.0) {
      y_left = y_middle;
      y_right = y_right;
    } else {
      y_left = y_right = y_middle;
    }

    if ((y_left == 0.0) || (fabs((y_right-y_left)/y_left) < tol)) {
      flag = 1;
    }
  } while ((num_iterations < max_iterations) && (flag == 0));
  if (num_iterations == max_iterations)  {
    printf("A converged y value could not be determined within %d
iterations.\n", max_iterations);
    exit(1);
  }

  return y_left;
}
```

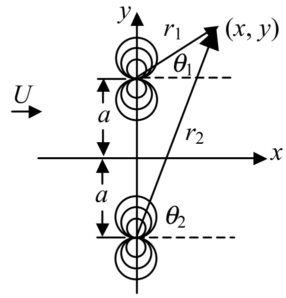
a. Determine the potential function, ϕ, of a cylinder without circulation in a uniform stream of velocity, *U*, centered a vertical distance, *a,* from a horizontal wall. You may leave your answer in terms of the doublet strength, *K*.



b. What is the vertical force acting on the cylinder? Hint: The vertical force acting on the cylinder will be equal but opposite to the vertical force acting on the wall.
c. Discuss how the shape of the cylinder is affected as *a* approaches the cylinder radius.

SOLUTION:

The potential function for the flow is the sum of the potential functions for a uniform stream and two doublets (one plus a reflection).



$$\phi = Ux + \frac{K\cos\theta_1}{r_1} + \frac{K\cos\theta_2}{r_2} \tag{1}$$

where

$$r_1 = \sqrt{x^2 + (y-a)^2} \tag{2}$$

$$r_2 = \sqrt{x^2 + (y+a)^2} \tag{3}$$

$$\cos\theta_1 = \frac{x}{r_1} \tag{4}$$

$$\cos\theta_2 = \frac{x}{r_2} \tag{5}$$

Substitute and simplify.

$$\boxed{\phi = Ux + \frac{Kx}{x^2 + (y-a)^2} + \frac{Kx}{x^2 + (y+a)^2}} \tag{6}$$

The velocities are found from the gradient of the potential function.

$$u_x = \frac{\partial\phi}{\partial x} = U + \frac{K}{x^2 + (y-a)^2} + \frac{Kx(-2x)}{\left[x^2 + (y-a)^2\right]^2} + \frac{K}{x^2 + (y+a)^2} + \frac{Kx(-2x)}{\left[x^2 + (y+a)^2\right]^2} \tag{7}$$

$$u_y = \frac{\partial\phi}{\partial y} = \frac{Kx\left[-2(y-a)\right]}{\left[x^2 + (y-a)^2\right]^2} + \frac{Kx\left[-2(y+a)\right]}{\left[x^2 + (y+a)^2\right]^2} \tag{8}$$

Evaluate u_x and u_y at $y = 0$.

$$u_x\big|_{y=0} = U + \frac{2K}{x^2 + a^2} - \frac{4Kx^2}{\left(x^2 + a^2\right)^2} \tag{9}$$

$$u_y\big|_{y=0} = 0 \quad \text{(as expected)} \tag{10}$$

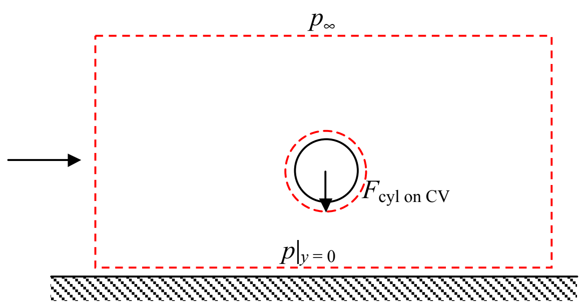
The pressure on the horizontal wall may be found from Bernoulli's equation.

$$p\big|_{y=0} + \tfrac{1}{2}\rho\left(u_x^2 + u_y^2\right)\Big|_{y=0} = p_\infty + \tfrac{1}{2}\rho U^2 \tag{11}$$

Substitute in for the velocity components along the wall (Eqs. (9) and (10)).

$$p\big|_{y=0} = p_\infty + \tfrac{1}{2}\rho\left\{U^2 - \left[U + \frac{2K}{x^2+a^2} - \frac{4Kx^2}{\left(x^2+a^2\right)^2}\right]^2\right\} \tag{12}$$

The force on the cylinder may be found by considering a control volume surrounding the cylinder, bordering the wall, and extending to distance far from the wall in the vertical direction (refer to the figure below).



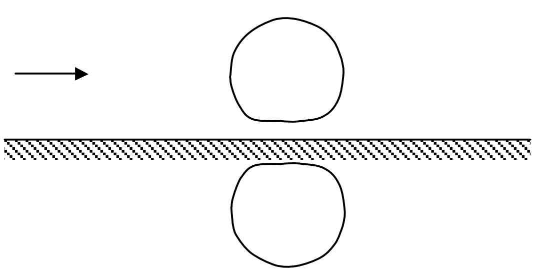
$$\int_{x\to-\infty}^{x\to\infty}\left[p(x)\big|_{y=0} - p_\infty\right]dx - F_{\text{cyl on CV}} = 0 \quad \text{(Assuming unit depth into page.)} \tag{13}$$

$$\therefore F_{\text{on cyl}} = -F_{\text{cyl on CV}} = -\tfrac{1}{2}\rho\int_{x=0}^{x\to\infty}\left\{U^2 - \left[U + \frac{2K}{x^2+a^2} - \frac{4Kx^2}{\left(x^2+a^2\right)^2}\right]^2\right\}dx \tag{14}$$

Solving the integral (using a computational calculus package such as MAPLE for example):

$$\boxed{F_{\substack{\text{on cyl,}\\ \text{per unit depth}}} = \pi\rho\frac{K^2}{a^3}} \quad \text{(Note: As } a \to \infty \text{ or } K \to 0,\ F_{\text{on cyl}} \to 0.\text{)} \tag{15}$$

The cylinder shape will be distorted if we use the potential function in Eq. (1) due to the influence of the image doublet.



Write a computer program (using any language or software package) that will compute the coordinates of a series of points (~100) on the surface of Joukowski airfoils.

a. Plot the shape of two Joukowski airfoils. One should have c=1.0, R/c=1.2, and β=3 deg. The other should have c=1.0, R/c=1.5, and β=5 deg.
b. Plot the pressure distribution, given in terms of the pressure coefficient, as a function of distance, s, from the leading edge stagnation point to the trailing stagnation point on both the pressure and suction sides of the airfoil. Use the following airfoil parameters: c=1.0, R/c=1.1, β=5 deg, and an angle of attack, α=5 deg.

SOLUTION:

The generating circle is given by:

$$z = c - R\exp(-i\beta) + R\exp(i\delta) \tag{1}$$

The Joukowski transformation is given by:

$$\zeta = z + \frac{c^2}{z} \tag{2}$$

A computer program, written in C, that calculates the (x, y) and (ξ, η) coordinates for the cylinder and airfoil, respectively is attached. The generating cylinders and corresponding Joukowski airfoils are shown in Figures 1 and 2.

The lift coefficient, c_L, for a Joukowski airfoil is given by (refer to the course notes for this derivation):

$$c_L = 2\pi\left(\frac{R}{c}\right)\sin(\alpha+\beta) \tag{3}$$

Note that c_L is usually defined in terms of the chord length. Here I've assumed the chord length is approximately equal to $4c$. In fact, the chord length will vary with R/c.

The pressure coefficient on the airfoil surface is given by (refer to the course notes for this derivation):

$$c_p = 1 - \left(\frac{u_s}{U}\right)^2 \tag{4}$$

where

$$\left(\frac{u_s}{U}\right)^2 = \left\{\frac{2\left[\sin(\delta-\alpha)+\sin(\alpha+\beta)\right]}{\left|1-\frac{c^2}{z^2}\right|}\right\}^2 \tag{5}$$

The location on the airfoil surface in terms of the displacement along the surface, s, is:

$$\Delta s = \sqrt{(\Delta\xi)^2 + (\Delta\eta)^2} \tag{6}$$

where

$$\Delta\xi = \mathrm{Re}\left(\zeta^n - \zeta^{n-1}\right)$$

$$\Delta\eta = \mathrm{Im}\left(\zeta^n - \zeta^{n-1}\right)$$

where (n-1) indicates the previous value and (n) indicates the current value. The value of ζ is determined from the transformation given in Eqn. (2). Note that the stagnation points on the airfoil can be determined by setting the left-hand side of Eqn. (5) to zero. The stagnation points occur when δ = -π+2α+β (leading edge) and -β (trailing edge). The pressure coefficient along the airfoil surface is plotted in Figure 3.

A few additional relations are used in the computer program for handling complex variables. These are given below.

$$\begin{aligned}
z &= x+iy = c-R\exp(-i\beta)+R\exp(i\delta) \\
&= c-R\cos\beta+iR\sin\beta+R\cos\delta+i\sin\delta \\
&= (c-R\cos\beta+R\cos\delta)+i(R\sin\beta+\sin\delta) \\
x &= \mathrm{Re}(z) = c-R\cos\beta+R\cos\delta \\
y &= \mathrm{Re}(z) = R\sin\beta+\sin\delta
\end{aligned} \tag{7}$$

$$\zeta = z+\frac{c^2}{z} = z+\frac{c^2\bar{z}}{z\bar{z}} = z+\frac{c^2\bar{z}}{|z|^2}$$

$$\xi = \mathrm{Re}(\zeta) = \mathrm{Re}\left(z+\frac{c^2\bar{z}}{|z|^2}\right) = \mathrm{Re}(z)+\frac{c^2}{|z|^2}\mathrm{Re}(\bar{z})$$

$$\therefore \xi = x\left(1+\frac{c^2}{|z|^2}\right) \tag{8}$$

$$\eta = \mathrm{Im}(\zeta) = \mathrm{Im}\left(z+\frac{c^2\bar{z}}{|z|^2}\right) = \mathrm{Im}(z)+\frac{c^2}{|z|^2}\mathrm{Im}(\bar{z})$$

$$\therefore \eta = y\left(1-\frac{c^2}{|z|^2}\right) \tag{9}$$

$$\begin{aligned}
\left|1-\frac{c^2}{z^2}\right|^2 &= \left|1-\frac{c^2\bar{z}^2}{z^2\bar{z}^2}\right|^2 = \left|1-\frac{c^2\bar{z}^2}{|z|^4}\right|^2 \\
&= \left|1-\frac{c^2(x-iy)^2}{|z|^4}\right|^2 = \left|1-\frac{c^2(x^2-2ixy-y^2)}{|z|^4}\right|^2 \\
&= \left|1-\frac{c^2(x^2-y^2)}{|z|^4}+i\frac{2xyc^2}{|z|^4}\right|^2
\end{aligned}$$

$$\therefore \left|1-\frac{c^2}{z^2}\right|^2 = \left[1-\frac{c^2(x^2-y^2)}{|z|^4}\right]^2+\left[\frac{2xyc^2}{|z|^4}\right]^2 \tag{10}$$

```
/*
  jouk.c
  Program to generate data for Joukowski airfoils.

  To compile with the GNU C compiler:
    gcc -o jouk jouk.c -lm
  To execute:
    ./jouk
  No input files.
  Output files:
    jouk.dat
*/

# include <stdio.h>
# include <stdlib.h>
# include <math.h>
# define PI (4.0*atan(1.0))
# define TWOPI (2.0*PI)

/* ***** */

int main(int argc, char **argv) {
  int max_points=100, flag;
  double c, R, beta, alpha, delta, delta_xi, delta_eta,
    s, delta_s, cp, magz2, V_U, temp1, temp2, num, den;
  FILE *output;
  struct {
    double r, i;
  } z, zeta, zeta_old;

  /* Open the output file. */
  if ((output = fopen("jouk.dat", "wb")) == NULL) {
    printf("Error opening jouk.dat\n");
    exit(1);
  }

  /* Define parameters. */
  c = 1.0;
  R = 1.1;
  beta = 5.0*(PI/180.0);
  alpha = 5.0*(PI/180.0);

  /* Print headers to the output file. */
  fprintf(output, "z.r\tz.i\tzeta.r\tzeta.i\ts\tcp\n");

  /* Evaluate points around the circle in the z-plane starting at the
     leading stagnation point and ending at the trailing stagnation
     point.  First look at the suction side of the airfoil. */
  s = 0.0;
  flag = 1;
  for (delta=PI+2.0*alpha+beta; delta>=-beta; delta-=TWOPI/max_points) {
    /* Real and imaginary parts in the z-plane. */
    z.r = c-R*cos(beta)+R*cos(delta);
    z.i = R*sin(beta)+R*sin(delta);

    /* The magnitude squared of z:  |z|^2 */
    magz2 = z.r*z.r + z.i*z.i;
```

```
    /* The real and imaginary parts in the zeta-plane after
       transformation. */
    zeta.r = (1.0+c*c/magz2)*z.r;
    zeta.i = (1.0-c*c/magz2)*z.i;

    /* Calculate quantities used in determining the velocity
       magnitude. */
    temp1 = (1.0-c*c/(magz2*magz2)*(z.r*z.r-z.i*z.i))*
      (1.0-c*c/(magz2*magz2)*(z.r*z.r-z.i*z.i));
    temp2 = (c*c/(magz2*magz2)*(2.0*z.r*z.i))*
      (c*c/(magz2*magz2)*(2.0*z.r*z.i));
    num = 2.0*(sin(delta-alpha)+sin(alpha+beta));
    den = sqrt(temp1+temp2);

    /* The velocity magnitude divided by the free stream velocity. */
    V_U = num/den;

    /* The pressure coefficient. */
    cp = 1.0-(V_U*V_U);

    /* Print the data to an output file. */
    fprintf(output, "%.3f\t%.3f\t%.3f\t%.3f\t%.3f\t%.3f\n",
          z.r, z.i, zeta.r, zeta.i, s, cp);

    /* The delta displacement in the zeta-plane from the previous zeta
       value. */
    if (flag == 1) {
      zeta_old.r = zeta.r;
      zeta_old.i = zeta.i;
      flag = 0;
    }
    delta_xi = zeta.r-zeta_old.r;
    delta_eta = zeta.i-zeta_old.i;
    delta_s = sqrt(delta_xi*delta_xi + delta_eta*delta_eta);
    s += delta_s;
    zeta_old.r = zeta.r;
    zeta_old.i = zeta.i;
  }

  /* Now look at the pressure side of the airfoil starting from the
     leading edge stagnation point. */
  s = 0.0;
  flag = 1;
  for (delta=-PI+2.0*alpha+beta; delta<=-beta; delta+=TWOPI/max_points)
{
    /* Real and imaginary parts in the z-plane. */
    z.r = c-R*cos(beta)+R*cos(delta);
    z.i = R*sin(beta)+R*sin(delta);

    /* The magnitude squared of z:  |z|^2 */
    magz2 = z.r*z.r + z.i*z.i;

    /* The real and imaginary parts in the zeta-plane after
       transformation. */
    zeta.r = (1.0+c*c/magz2)*z.r;
    zeta.i = (1.0-c*c/magz2)*z.i;
```

```
    /* Calculate quantities used in determining the velocity
       magnitude. */
    temp1 = (1.0-c*c/(magz2*magz2)*(z.r*z.r-z.i*z.i))*
      (1.0-c*c/(magz2*magz2)*(z.r*z.r-z.i*z.i));
    temp2 = (c*c/(magz2*magz2)*(2.0*z.r*z.i))*
      (c*c/(magz2*magz2)*(2.0*z.r*z.i));
    num = 2.0*(sin(delta-alpha)+sin(alpha+beta));
    den = sqrt(temp1+temp2);

    /* The velocity magnitude divided by the free stream velocity. */
    V_U = num/den;

    /* The pressure coefficient. */
    cp = 1.0-(V_U*V_U);

    /* Print the data to an output file. */
    fprintf(output, "%.3f\t%.3f\t%.3f\t%.3f\t%.3f\t%.3f\n",
          z.r, z.i, zeta.r, zeta.i, s, cp);

    /* The delta displacement in the zeta-plane from the previous zeta
       value. */
    if (flag == 1) {
      zeta_old.r = zeta.r;
      zeta_old.i = zeta.i;
      flag = 0;
    }
    delta_xi = zeta.r-zeta_old.r;
    delta_eta = zeta.i-zeta_old.i;
    delta_s = sqrt(delta_xi*delta_xi + delta_eta*delta_eta);
    s += delta_s;
    zeta_old.r = zeta.r;
    zeta_old.i = zeta.i;
  }

  /* Close the output file and quit. */
  fclose(output);
  printf("Done.\n");
  return(0);
}
```

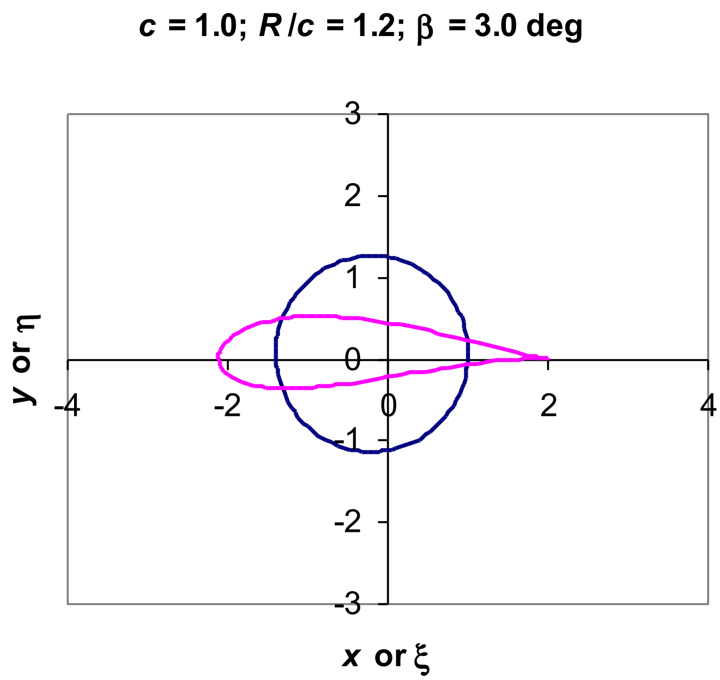


Figure 1. The generating cylinder and corresponding Joukowski airfoil for $c = 1.0$, $R/c = 1.2$, and $\beta = 3.0°$

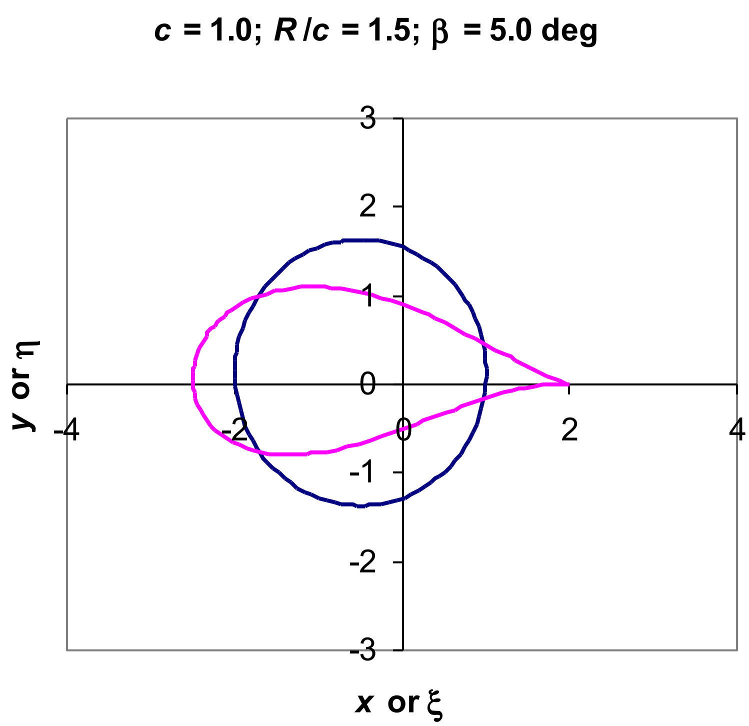


Figure 2. The generating cylinder and corresponding Joukowski airfoil for $c = 1.0$, $R/c = 1.5$, and $\beta = 5.0°$.

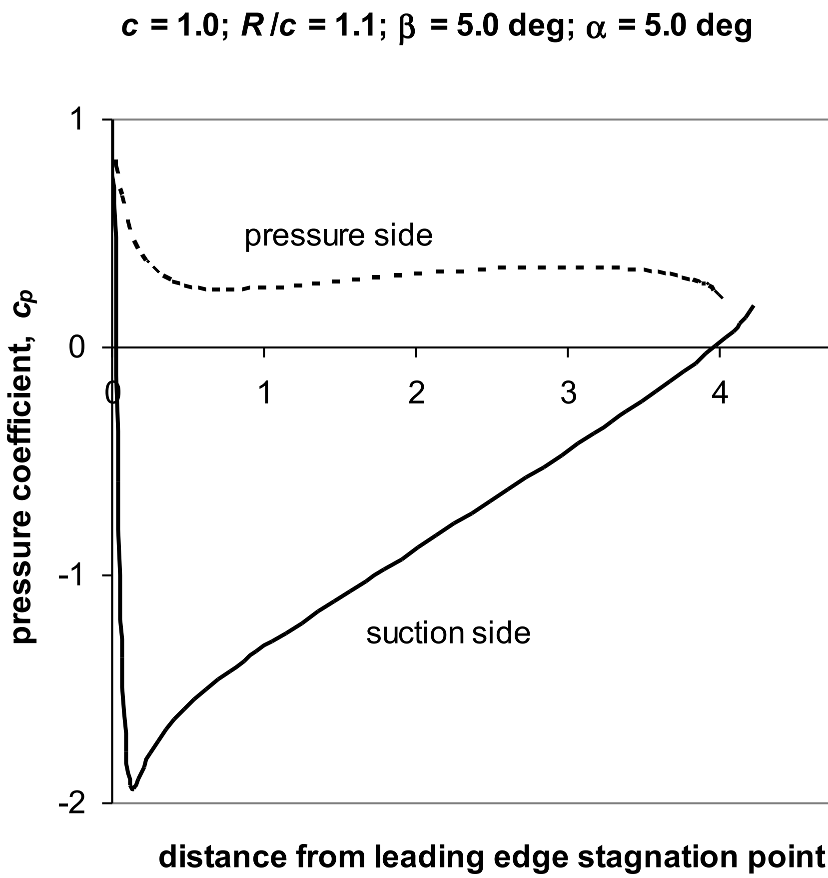
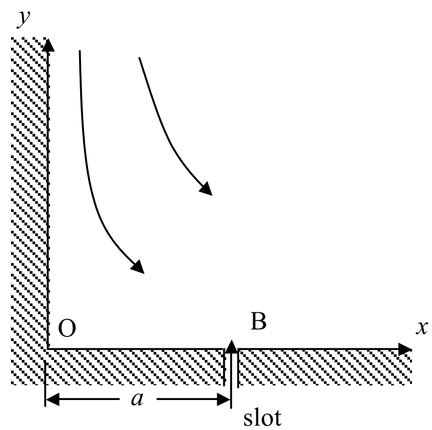


Figure 3. The pressure coefficient, c_p, plotted as a function of position from the leading edge stagnation point *along the airfoil surface* (not along the chord length) to the trailing stagnation point. The Joukowski airfoil parameters are c = 1.0, R/c = 1.1, and β = 5.0° and the angle of attack is α = 5.0°. Note that the distance along the airfoil on the suction surface is longer than the distance along the pressure surface.

Consider the irrotational, planar flow of an incompressible, inviscid fluid in a right-angle corner:



The basic corner flow ($f(z)=Az^2$) is modified by fluid being injected into the flow through a slot at the location, B, shown above where the distance OB is denoted by a. The volume rate of injection of fluid per unit depth normal to the figure is denoted by Q.

a. Determine the location between O and B where the velocity is zero (in terms of A, a, and Q).
b. Determine the normal force acting on the vertical wall between y=0 and y=1. Use the stagnation pressure as a reference pressure. You need not evaluate any integrals that result.

SOLUTION:

Write the potential function for the corner flow including a line source located at $x = a$. An image of the line source will also be required at $x = -a$.

$$f(z) = Az^2 + \frac{m}{2\pi}\log(z-a) + \frac{m}{2\pi}\log(z+a) \tag{1}$$

where $m = 2Q$ since the volume flow rate considered here consists of only half of the source (the contribution in the upper plane – we don't care about the lower plane contribution). The flow velocities may be determined from the complex potential.

$$\begin{aligned}
u_x - iu_y = \frac{df}{dz} &= 2Az + \frac{m}{2\pi}\frac{1}{z-a} + \frac{m}{2\pi}\frac{1}{z+a} \\
&= 2Az + \frac{m}{2\pi}\left(\frac{1}{z-a} + \frac{1}{z+a}\right) \\
&= 2Az + \frac{m}{2\pi}\left[\frac{\bar{z}-a}{(z-a)(\bar{z}-a)} + \frac{\bar{z}+a}{(z+a)(\bar{z}+a)}\right] \\
&= 2Az + \frac{m}{2\pi}\left[\frac{\bar{z}-a}{|z|^2 - az - a\bar{z} + a^2} + \frac{\bar{z}+a}{|z|^2 + az + a\bar{z} + a^2}\right] \\
&= 2Az + \frac{m}{2\pi}\left[\frac{\bar{z}-a}{|z|^2 - 2ax + a^2} + \frac{\bar{z}+a}{|z|^2 + 2ax + a^2}\right] \\
&= 2Az + \frac{m}{2\pi}\left[\frac{\bar{z}-a}{(|z|-a)^2} + \frac{\bar{z}+a}{(|z|+a)^2}\right]
\end{aligned}$$

$$u_x - iu_y = 2Az + \frac{m}{2\pi}\left[\frac{\bar{z}-a}{\left(|z|-a\right)^2} + \frac{\bar{z}+a}{\left(|z|+a\right)^2}\right]$$

$$u_x = 2Ax + \frac{m}{2\pi}\left[\frac{x-a}{\left(\sqrt{x^2+y^2}-a\right)^2} + \frac{x+a}{\left(\sqrt{x^2+y^2}+a\right)^2}\right] \tag{2}$$

$$u_y = -2Ay + \frac{m}{2\pi}\left[\frac{y}{\left(\sqrt{x^2+y^2}-a\right)^2} + \frac{y}{\left(\sqrt{x^2+y^2}+a\right)^2}\right] \tag{3}$$

The stagnation point along the x axis ($y = 0$) can be found by setting $u_x = 0$ (u_y is already zero along the y-axis).

$$0 = 2Ax_s + \frac{m}{2\pi}\left[\frac{x_s-a}{\left(x_s-a\right)^2} + \frac{x_s+a}{\left(x_s+a\right)^2}\right]$$

$$= 2Ax_s + \frac{m}{2\pi}\left[\frac{1}{x_s-a} + \frac{1}{x_s+a}\right]$$

$$= 2Ax_s + \frac{m}{2\pi}\left[\frac{2x_s}{x_s^2-a^2}\right]$$

$$x_s^2 = a^2 - \frac{m}{2\pi A}$$

$$\boxed{\therefore \left(x_s, y_s\right) = \left(\sqrt{a^2 - \frac{Q}{\pi A}}, 0\right)} \tag{4}$$

where $m = 2Q$.

The pressure force acting on the wall ($x = 0$) can be found using Bernoulli's equation and integrating. Choose the stagnation point, s, for the reference pressure.

$$\left(p + \tfrac{1}{2}\rho V^2\right)_s = p_s = \left(p + \tfrac{1}{2}\rho V^2\right) \Rightarrow p = p_s - \tfrac{1}{2}\rho V^2 \tag{5}$$

where, along $x = 0$:

$$V^2\Big|_{x=0} = u_y^2\Big|_{x=0} = \left\{-2Ay + \frac{m}{2\pi}\left[\frac{1}{y-a} + \frac{1}{y+a}\right]\right\}^2$$

$$= \left\{-2Ay + \frac{m}{2\pi}\left[\frac{2y}{y^2-a^2}\right]\right\}^2$$

The pressure force (which acts in the negative x-direction) between $y = 0$ and $y = 1$ is:

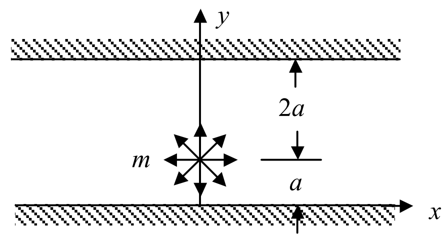
$$F_x = \int_{y=0}^{y=1} -p\,dy = \int_{y=0}^{y=1} -\left(p_s - \tfrac{1}{2}\rho V\big|_{x=0}^{2}\right)dy$$

$$\boxed{\therefore F_x = -p_s\left(1\right) + \tfrac{1}{2}\rho \int_{y=0}^{y=1}\left\{-2Ay + \frac{Q}{\pi}\left[\frac{2y}{y^2-a^2}\right]\right\}^2 dy} \tag{6}$$

Evaluating this integral is tedious and will not be performed here. It can, however, be easily solved using a symbolic/numeric package such as MAPLE.

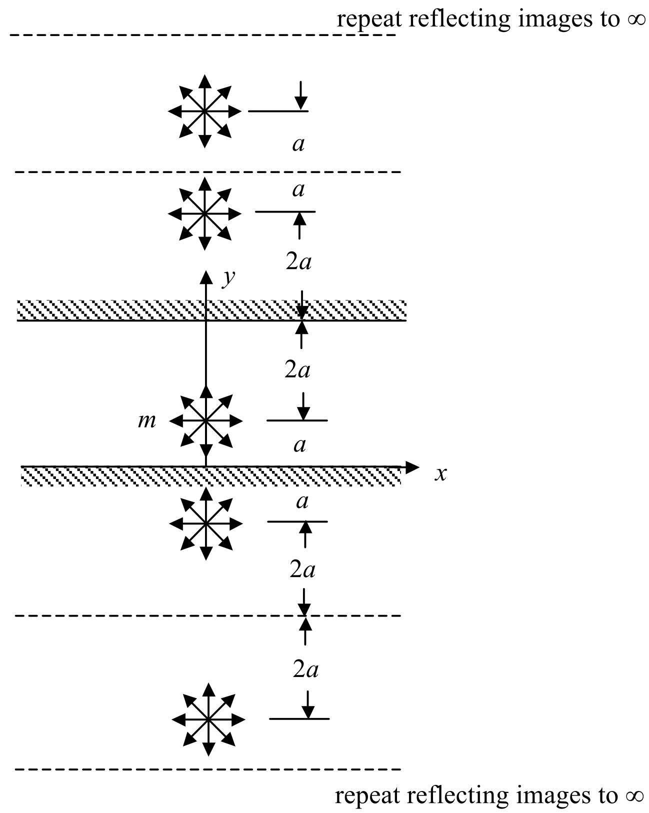
a. Write the potential function that simulates the flow of a line source placed asymmetrically between two parallel walls as shown in the figure.

b. Compute the dimensionless velocity, $\mathbf{u}' = a\mathbf{u}/(4\pi m)$, on the lower wall at $(x/a, y/a) = (1,0)$ accurate to three significant digits.



SOLUTION:

Use the Method of Images to create the given flow. The sequence of images is shown below.



The potential function for the given flow field is:

$$\phi = \frac{m}{2\pi}\left[\begin{array}{l} \ln\sqrt{x^2+\left(y-a\right)^2} + \ln\sqrt{x^2+\left(y+a\right)^2} + \ln\sqrt{x^2+\left(y-5a\right)^2} + \ln\sqrt{x^2+\left(y+5a\right)^2} + \\ \ln\sqrt{x^2+\left(y-7a\right)^2} + \ln\sqrt{x^2+\left(y+7a\right)^2} + \ln\sqrt{x^2+\left(y-11a\right)^2} + \ln\sqrt{x^2+\left(y+11a\right)^2} + \cdots \end{array}\right] \quad (1)$$

$$\phi = \frac{m}{4\pi}\sum_{k=0}^{k\to\infty}\left\{\begin{array}{l}\ln\left[x^2+\left[y-(6k+1)a\right]^2\right]+\ln\left[x^2+\left[y+(6k+1)a\right]^2\right]+\\ \ln\left[x^2+\left[y-(6k+5)a\right]^2\right]+\ln\left[x^2+\left[y+(6k+5)a\right]^2\right]\end{array}\right\} \tag{2}$$

or, in dimensionless terms:

$$\boxed{\phi' = \sum_{k=0}^{k\to\infty}\left\{\begin{array}{l}\ln\left[x'^2+\left[y'-(6k+1)\right]^2\right]+\ln\left[x'^2+\left[y'+(6k+1)\right]^2\right]+\\ \ln\left[x'^2+\left[y'-(6k+5)\right]^2\right]+\ln\left[x'^2+\left[y'+(6k+5)\right]^2\right]+4\ln a^2\end{array}\right\}} \tag{3}$$

where the dimensionless potential function is $\phi' = \phi/(4\pi m)$ and the dimensionless positions are $x' = x/a$ and $y' = y/a$.

The dimensionless velocities resulting from this potential function are:

$$u'_x = \frac{\partial\phi'}{\partial x'} = 2x'\sum_{k=0}^{k\to\infty}\left\{\begin{array}{l}\frac{1}{x'^2+\left[y'-(6k+1)\right]^2}+\frac{1}{x'^2+\left[y'+(6k+1)\right]^2}+\\ \frac{1}{x'^2+\left[y'-(6k+5)\right]^2}+\frac{1}{x'^2+\left[y'+(6k+5)\right]^2}\end{array}\right\} \tag{4}$$

$$u'_y = \frac{\partial\phi'}{\partial y'} = \sum_{k=0}^{k\to\infty}\left\{\begin{array}{l}\frac{y'-(6k+1)}{x'^2+\left[y'-(6k+1)\right]^2}+\frac{y'+(6k+1)}{x'^2+\left[y'+(6k+1)\right]^2}+\\ \frac{y'-(6k+5)}{x'^2+\left[y'-(6k+5)\right]^2}+\frac{y'+(6k+5)}{x'^2+\left[y'+(6k+5)\right]^2}\end{array}\right\} \tag{5}$$

where $u'_x = au_x/(4\pi m)$ and $u'_y = au_y/(4\pi m)$.

The velocity at $(x', y') = (1, 0)$ is:

$$\boxed{u'_x(1,0) = 4\sum_{k=0}^{k\to\infty}\left\{\frac{1}{1+(6k+1)^2}+\frac{1}{1+(6k+5)^2}\right\}} \tag{6}$$

$\boxed{u'_y(1,0)=0}$ (as expected since the point is on a wall) (7)

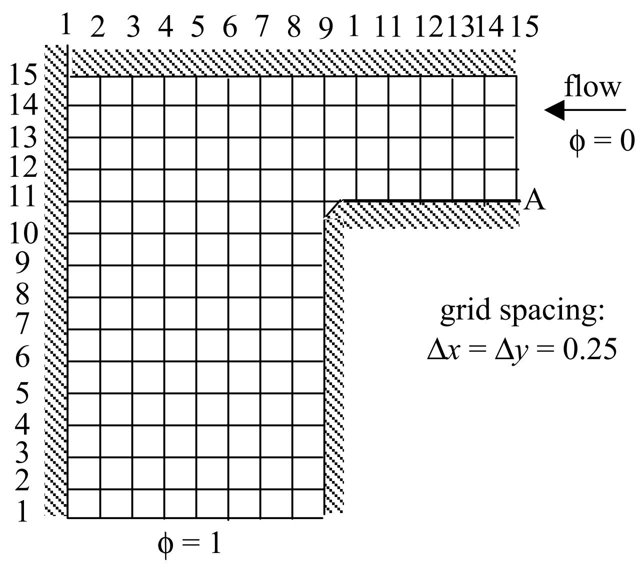
The value of the horizontal velocity component at $(x', y') = (1,0)$ as a function of k is given in the table below. Note that "%diff from prev" is the percentage change in the value of u'_x from the previous value of u'_x, i.e., % diff = $(u'_{x,k+1} - u'_{x,k})/u'_{x,k}$ * 100%.

k	**u'$_x$**	**% diff from prev**
0	1.962	
1	2.042	4.08%
2	2.066	1.15%
3	2.077	0.53%
4	2.083	0.31%
5	2.087	0.20%
6	2.090	0.14%
7	2.092	0.10%
8	2.094	0.08%
9	2.095	0.06%
10	2.097	0.05%
11	2.097	0.04%
12	2.098	0.04%
13	2.099	0.03%
14	2.099	0.03%
15	2.100	0.02%
16	2.100	0.02%
17	2.101	0.02%
18	2.101	0.02%
19	2.101	0.01%

Hence, the velocity components at $(x',y') = (1,0)$ are $\boxed{(u'_x, u'_y) = (2.10, 0)}$.

Obtain a numerical solution to the planar, potential flow around a bend using the grid sketched below. The upstream and downstream boundaries to be used are $\phi = 0$ and $\phi = 1$, respectively. You are not required to use any special procedures to treat the singular behavior near the projecting corner. You are advised to use an iterative method with over-relaxation and, by trial and error, to find an effective value of the over-relaxation factor, which improves the convergence of the iterative process.

a. Determine the velocity potential at all of the nodes.
b. Determine the velocity components at all of the nodes.
c. Determine the volumetric flow rates at the inlet and outlet. What could be done to reduce the error between these flow rates?
d. If the pressure coefficient is defined by $(p-p_A)/(^1/_2\rho U^2)$ where p is the pressure, p_A is the pressure at the point A, and ρ is the fluid density, find the pressure coefficient at all of the nodes.



SOLUTION:

The attached *C* computer program computes the potential function, velocity components, and pressure coefficients at all of the grid points.

The volumetric flow rates at the entrance and exit are found using:

$$Q_{\text{inlet}} = \sum_{\text{inlet nodes}} \tfrac{1}{2}\left(u_{i,j+1} - u_{i,j}\right)\Delta y$$

$$Q_{\text{outlet}} = \sum_{\text{outlet nodes}} \tfrac{1}{2}\left(u_{i+1,j} - u_{i,j}\right)\Delta x$$

For the given conditions, I calculate that $\boxed{Q_{\text{inlet}} = -1.328 \text{ and } Q_{\text{outlet}} = 1.282}$ giving a relative error of 3.5%. The error can be improved if additional nodes are used in the calculations.

The pressure coefficient at any node point can be determined using Bernoulli's equation with the point A serving as a reference.

$$p_{i,j} + \tfrac{1}{2}\rho\underbrace{\left(u_x^2\Big|_{i,j} + u_y^2\Big|_{i,j}\right)}_{=V_{i,j}^2} = p_A + \tfrac{1}{2}\rho V_A^2$$

$$c_p\Big|_{i,j} \equiv \frac{p_{i,j} - p_A}{\tfrac{1}{2}\rho V_A^2} = 1 - \left(\frac{V_{i,j}}{V_A}\right)^2$$

Values of the potential function, horizontal velocity, vertical velocity, and pressure coefficients are given in Tables 1-4, respectively. Figure 1 plots the number of iterations to reach convergence as a function of the relaxation parameter, α.

```
/*
  bend.c

  A finite difference based routine for determining potential flow in
  a pipe bend.

  To compile using the GNU C compiler.
    gcc -o bend bend.c -lm
  To execute:
    bend <relaxation parameter, alpha>
  No input files:
  Output files:
    bend.out
*/

# include <stdio.h>
# include <stdlib.h>
# include <math.h>
# define PI (4.0*atan(1.0))
# define TWOPI (2.0*PI)

int rmax, cmax, r_corner, c_corner;
struct parameters{
  /* node type:
     -1 = ignore this node,
     0 = a Dirichlet node point (a specified phi value),
     1 = an interior node point,
     2 = a downward-facing horizontal wall boundary point,
     3 = an upward-facing horizontal wall boundary point,
     4 = a rightward-facing vertical wall boundary point,
     5 = a leftward-facing vertical wall boundary point,
     6 = a downward/rightward-facing interior corner point,
     7 = a downward/leftward-facing interior corner point,
     8 = an upward/rightward-facing interior corner point,
     9 = an upward/leftward-facing interior corner point
  */
  int type;
  double phi, phi_old, ux, uy, cp;
} **node;

/* ***** */

int main(int argc, char **argv) {
  void InitializePhiValues(void);
  void DeterminePhiValues(double);
  int CheckConvergence(void);
  void PrintNodeValues(void);
  void DetermineVelocities(double, double);
  void DeterminePressureCoefficients(double);

  int num_iterations, max_iterations=5000, r, c;
  double alpha, deltax, deltay, VA;

  /* Check to make sure that alpha is specified in the command line. */
  if (argc < 2) {
    printf("Usage:  bend <alpha>\n");
```

```
    exit(1);
  }

  /* Set the over-relaxation parameter. */
  alpha = atof(argv[1]);
  if ((alpha < -1.0) || (alpha > 1.0)) {
    printf("alpha must be between -1 and +1.\n");
    exit(1);
  }

  /* Set the grid spacing. */
  deltax = 0.25;
  deltay = deltax;

  /* Initialize the phi values. */
  InitializePhiValues();

  /* Initialize the number of iterations required for convergence. */
  num_iterations = 0;

  do {
    /* Increment the number of iterations. */
    num_iterations++;

    /* Determine the new phi values. */
    DeterminePhiValues(alpha);

    /* Check for convergence of the phi values. */
  } while ((CheckConvergence() == 0) && (num_iterations <
max_iterations));

  if (num_iterations < max_iterations) {
    printf("%d iterations required for convergence with an over-
relaxation parameter of alpha = %.2f.\n", num_iterations, alpha);

    /* Determine velocities. */
    DetermineVelocities(deltax, deltay);

    /* Determine pressures coefficients. Use the reference velocity at
       point A. */
    r = r_corner;
    c = cmax;
    VA = sqrt(node[r][c].ux*node[r][c].ux +
node[r][c].uy*node[r][c].uy);
    DeterminePressureCoefficients(VA);

    /* Print the values at the node points. */
    PrintNodeValues();
  } else {
    printf("Could not get a converged solution within %d iterations with
an over-relaxation parameter of %.2f.\n",
          max_iterations, alpha);
  }
  return(0);
}
```

```
/* ***** */

void InitializePhiValues() {
  /* Initialize the boundary potential function values. */
  int r, c;
  extern int rmax, cmax, r_corner, c_corner;
  extern struct parameters **node;

  rmax = 15;  /* 15 */
  cmax = 15;  /* 15 */
  r_corner = 11;  /* 11 */
  c_corner = 9;   /* 9 */

  /* Make space for the node array. */
  if ( (node = (struct parameters **)
      calloc(rmax+1, sizeof(struct parameters *))) == NULL) {
    printf("Not enough storage for **node.\n");
    exit(1);
  }
  for (r=1; r<=rmax; r++) {
    if ( (node[r] = (struct parameters *)
        calloc(cmax+1, sizeof(struct parameters))) == NULL) {
      printf("Not enough storage for *node[%d].\n", r);
      exit(1);
    }
  }

  /* Initialize all points to the "ignore" type. */
  for (r=1; r<=rmax; r++) {
    for (c=1; c<=cmax; c++) {
      node[r][c].type = -1;
    }
  }


  /* At the inlet. */
  c = cmax;
  for (r=r_corner; r<=rmax; r++) {
    node[r][c].type = 0;  /* Dirichlet node. */
    node[r][c].phi = 0.0;
  }

  /* At the outlet. */
  r = 1;
  for (c=1; c<=c_corner; c++) {
    node[r][c].type = 0;  /* Dirichlet node. */
    node[r][c].phi = 1.0;
  }


  /* At the upper wall boundary. */
  r = rmax;
  for (c=2; c<cmax; c++) {
    node[r][c].type = 2;  /* downward-facing horz. wall */
    node[r][c].phi = 0.0;
  }
```

```
  /* At the lower wall inlet boundary. */
  r = r_corner;
  for (c=c_corner+1; c<cmax; c++) {
    node[r][c].type = 3;  /* upward-facing horz. wall */
    node[r][c].phi = 0.0;
  }

  /* At the left wall boundary. */
  c = 1;
  for (r=2; r<rmax; r++) {
    node[r][c].type = 4;  /* rightward-facing vert. wall */
    node[r][c].phi = 0.0;
  }

  /* At the right wall boundary. */
  c = c_corner;
  for (r=2; r<r_corner; r++) {
    node[r][c].type = 5;  /* leftward-facing vert. wall */
    node[r][c].phi = 0.0;
  }

  /* At the upper left corner. */
  r = rmax;
  c = 1;
  node[r][c].type = 6;  /* downward/rightward-facing corner point */
  node[r][c].phi = 0.0;

  /* At the interior points. */
  for (r=r_corner+1; r<rmax; r++) {
    for (c=2; c<cmax; c++) {
      node[r][c].type = 1;  /* interior point */
      node[r][c].phi = 0.0;
    }
  }

  for (r=2; r<=r_corner; r++) {
    for (c=2; c<c_corner; c++) {
      node[r][c].type = 1;  /* interior point */
      node[r][c].phi = 0.0;
    }
  }

  r = r_corner;
  c = c_corner;
  node[r][c].type = 1;  /* interior point */
  node[r][c].phi = 0.0;
}


/* ***** */

void DeterminePhiValues(double alpha) {
  /* Determine the "new" node phi values based on the conditions at
     the neighboring nodes. */
```

```
int r, c;
extern int rmax, cmax;
extern struct parameters **node;

for (r=1; r<=rmax; r++) {
  for (c=1; c<=cmax; c++) {

    /* Set the old phi values. */
    if ((node[r][c].type != -1) || (node[r][c].type != 0)) {
    node[r][c].phi_old = node[r][c].phi;
    }

    /* Update the phi values. */
    if (node[r][c].type == 1) {
    /* This is an interior point. */
    node[r][c].phi = 0.25*(node[r+1][c].phi + node[r-1][c].phi +
                       node[r][c+1].phi + node[r][c-1].phi);
    } else if (node[r][c].type == 2) {
    /* This is a downward-facing horizontal boundary point. */
    node[r][c].phi = 0.25*(2.0*node[r-1][c].phi +
                       node[r][c+1].phi + node[r][c-1].phi);
    } else if (node[r][c].type == 3) {
    /* This is an upward-facing horizontal boundary point. */
    node[r][c].phi = 0.25*(2.0*node[r+1][c].phi +
                       node[r][c+1].phi + node[r][c-1].phi);
    } else if (node[r][c].type == 4) {
    /* This is a rightward-facing vertical boundary point. */
    node[r][c].phi = 0.25*(node[r+1][c].phi + node[r-1][c].phi +
                       2.0*node[r][c+1].phi);
    } else if (node[r][c].type == 5) {
    /* This is a leftward-facing vertical boundary point. */
    node[r][c].phi = 0.25*(node[r+1][c].phi + node[r-1][c].phi +
                       2.0*node[r][c-1].phi);
    } else if (node[r][c].type == 6) {
    /* This is a downward/rightward-facing interior corner point. */
    node[r][c].phi = 0.25*(2.0*node[r-1][c].phi +
                       2.0*node[r][c+1].phi);
    } else if (node[r][c].type == 7) {
    /* This is a downward/leftward-facing interior corner point. */
    node[r][c].phi = 0.25*(2.0*node[r-1][c].phi +
                       2.0*node[r][c-1].phi);
    } else if (node[r][c].type == 8) {
    /* This is a upward/rightward-facing interior corner point. */
    node[r][c].phi = 0.25*(2.0*node[r+1][c].phi +
                       2.0*node[r][c+1].phi);
    } else if (node[r][c].type == 9) {
    /* This is a upward/leftward-facing interior corner point. */
    node[r][c].phi = 0.25*(2.0*node[r+1][c].phi +
                       2.0*node[r][c-1].phi);
    }

    /* Apply over-relaxation to this point. */
    if ((node[r][c].type != -1) || (node[r][c].type != 0)) {
    node[r][c].phi = alpha*node[r][c].phi_old +
      (1.0-alpha)*node[r][c].phi;
    }
```

```
    }
  }
}

/* ***** */

int CheckConvergence() {
  /* Check for convergence of the phi values at all points.  Returning a
     "1" value means that all of the phi values have converged.
     A "0" value means that at least one of the phi values has not
     converged. */
  int r, c;
  double tol=1.0e-8;
  extern int rmax, cmax;
  extern struct parameters **node;

  /* Check to see if all of the points have converged.  If any point
     has not converged, then exit the loop and return a "0" value. */
  for (c=1; c<=cmax; c++) {
    for (r=1; r<=rmax; r++) {
      if ((node[r][c].type != -1) && (node[r][c].type != 0)) {
      if (fabs((node[r][c].phi-node[r][c].phi_old)/
             node[r][c].phi_old) > tol) {
        return 0;
      }
      }
    }
  }

  return 1;
}

/* ***** */

void PrintNodeValues() {
  int r, c;
  extern int rmax, cmax;
  extern struct parameters **node;
  FILE *outfile;

  /* Open an output file to which the data will be written. */
  if ((outfile = fopen("bend.out", "wb")) == NULL) {
    printf("Cannot open output file\n");
    exit(1);
  }

  fprintf(outfile, "node type values:\n");
  for (r=rmax; r>=1; r--) {
    for (c=1; c<=cmax; c++) {
      if (node[r][c].type != -1) {
      fprintf(outfile, "%d\t", node[r][c].type);
      }
    }
    fprintf(outfile, "\n");
  }
```

```
  fprintf(outfile, "\nphi values:\n");
  for (r=rmax; r>=1; r--) {
    for (c=1; c<=cmax; c++) {
      if (node[r][c].type != -1) {
      fprintf(outfile, "%.3f\t", node[r][c].phi);
      }
    }
    fprintf(outfile, "\n");
  }

  fprintf(outfile, "\nux values:\n");
  for (r=rmax; r>=1; r--) {
    for (c=1; c<=cmax; c++) {
      if (node[r][c].type != -1) {
      fprintf(outfile, "%.3f\t", node[r][c].ux);
      }
    }
    fprintf(outfile, "\n");
  }

  fprintf(outfile, "\nuy values:\n");
  for (r=rmax; r>=1; r--) {
    for (c=1; c<=cmax; c++) {
      if (node[r][c].type != -1) {
      fprintf(outfile, "%.3f\t", node[r][c].uy);
      }
    }
    fprintf(outfile, "\n");
  }

  fprintf(outfile, "\ncp values:\n");
  for (r=rmax; r>=1; r--) {
    for (c=1; c<=cmax; c++) {
      if (node[r][c].type != -1) {
      fprintf(outfile, "%.3f\t", node[r][c].cp);
      }
    }
    fprintf(outfile, "\n");
  }

  /* Close the output file. */
  fclose(outfile);
}

/* ***** */

void DetermineVelocities(double deltax, double deltay) {
  /* Determine the velocity at all of the nodes. */
  int r, c;
  double Q_inlet, Q_outlet;
  extern int rmax, cmax;
  extern struct parameters **node;

  for (r=1; r<=rmax; r++) {
    for (c=1; c<=cmax; c++) {
      if (node[r][c].type == 1) {
```

```
      /* interior points */
      node[r][c].ux = (node[r][c+1].phi - node[r][c-
1].phi)/(2.0*deltax);
      node[r][c].uy = (node[r+1][c].phi - node[r-
1][c].phi)/(2.0*deltay);
      } else if ((node[r][c].type == 2) || (node[r][c].type == 3)) {
      /* horizontal wall points */
      node[r][c].ux = (node[r][c+1].phi - node[r][c-
1].phi)/(2.0*deltax);
      node[r][c].uy = 0.0;
      } else if ((node[r][c].type == 4) || (node[r][c].type == 5)){
      /* vertical wall points */
      node[r][c].ux = 0.0;
      node[r][c].uy = (node[r+1][c].phi - node[r-
1][c].phi)/(2.0*deltay);
      } else if ((node[r][c].type == 6) || (node[r][c].type == 7) ||
             (node[r][c].type == 8) || (node[r][c].type == 9)) {
      /* interior corner points */
      node[r][c].ux = 0.0;
      node[r][c].uy = 0.0;
      }
    }
  }

  /* At the inlet. */
  c = cmax;
  for (r=1; r<=rmax; r++) {
    if (node[r][c].type == 0) {
      /* Use a backward differencing scheme. */
      node[r][c].ux = (node[r][c].phi-node[r][c-1].phi)/deltax;
      node[r][c].uy = 0.0;
    }
  }

  /* At the outlet. */
  r = 1;
  for (c=1; c<=cmax; c++) {
    if (node[r][c].type == 0) {
      node[r][c].ux = 0.0;
      /* Use a forward differencing scheme. */
      node[r][c].uy = (node[r+1][c].phi-node[r][c].phi)/deltay;
    }
  }

  /* Determine volumetric flow rates at the inlet and outlet. */
  /* At the inlet. */
  c = cmax;
  Q_inlet = 0.0;
  for (r=1; r<rmax; r++) {
    if ((node[r][c].type == 0) || (node[r+1][c].type == 0)) {
      Q_inlet += 0.5*(node[r][c].ux + node[r+1][c].ux)*deltay;
    }
  }
  printf("Q_inlet = %.3f\n", Q_inlet);

  /* At the outlet. */
```

```
  r = 1;
  Q_outlet = 0.0;
  for (c=1; c<cmax; c++) {
    if ((node[r][c].type == 0) || (node[r][c+1].type == 0)) {
      Q_outlet += 0.5*(node[r][c].uy + node[r][c+1].uy)*deltax;
    }
  }
  printf("Q_outlet = %.3f\n", Q_outlet);
  printf("Q_error = %.2f percent\n",
         fabs((Q_outlet-Q_inlet)/Q_inlet)*100.0);
}

/* ***** */

void DeterminePressureCoefficients(double VA) {
  int r, c;
  double Vsquared;
  extern int rmax, cmax;
  extern struct parameters **node;

  for (r=1; r<=rmax; r++) {
    for (c=1; c<=cmax; c++) {
      if (node[r][c].type != -1) {
      Vsquared = node[r][c].ux*node[r][c].ux +
node[r][c].uy*node[r][c].uy;
      node[r][c].cp = 1.0-Vsquared/(VA*VA);
      }
    }
  }
}
```

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Table 1. The velocity potential at each of the node points.

0.631	0.628	0.620	0.605	0.583	0.554	0.517	0.472	0.417	0.355	0.288	0.218	0.146	0.073	0.000
0.634	0.631	0.623	0.608	0.587	0.558	0.522	0.476	0.421	0.358	0.290	0.219	0.146	0.073	0.000
0.643	0.640	0.632	0.618	0.598	0.570	0.535	0.490	0.433	0.366	0.294	0.221	0.147	0.074	0.000
0.657	0.654	0.647	0.634	0.616	0.591	0.558	0.515	0.456	0.378	0.300	0.224	0.149	0.074	0.000
0.676	0.674	0.667	0.656	0.640	0.619	0.592	0.555	0.499	0.390	0.304	0.225	0.149	0.074	0.000
0.699	0.697	0.692	0.683	0.670	0.654	0.634	0.613	0.597						
0.726	0.724	0.720	0.713	0.703	0.692	0.679	0.667	0.661						
0.755	0.754	0.751	0.746	0.739	0.731	0.723	0.716	0.713						
0.787	0.786	0.784	0.780	0.775	0.770	0.765	0.761	0.759						
0.821	0.820	0.818	0.816	0.812	0.809	0.805	0.803	0.802						
0.855	0.855	0.854	0.852	0.850	0.847	0.845	0.844	0.843						
0.891	0.891	0.890	0.889	0.887	0.886	0.884	0.883	0.883						
0.927	0.927	0.926	0.926	0.925	0.924	0.923	0.923	0.922						
0.963	0.963	0.963	0.963	0.962	0.962	0.962	0.961	0.961						
1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000						

Table 2. The horizontal velocity at each of the node points.

0.000	-0.023	-0.047	-0.073	-0.101	-0.132	-0.165	-0.200	-0.233	-0.258	-0.275	-0.285	-0.290	-0.292	-0.293
0.000	-0.023	-0.047	-0.072	-0.100	-0.130	-0.164	-0.201	-0.236	-0.263	-0.279	-0.287	-0.291	-0.293	-0.293
0.000	-0.022	-0.044	-0.068	-0.095	-0.126	-0.161	-0.203	-0.248	-0.278	-0.290	-0.293	-0.294	-0.295	-0.295
0.000	-0.020	-0.040	-0.062	-0.087	-0.115	-0.152	-0.203	-0.274	-0.313	-0.309	-0.302	-0.299	-0.297	-0.297
0.000	-0.017	-0.035	-0.054	-0.074	-0.097	-0.129	-0.184	-0.330	-0.392	-0.330	-0.309	-0.301	-0.298	-0.297
0.000	-0.014	-0.029	-0.043	-0.058	-0.071	-0.081	-0.075	0.000						
0.000	-0.011	-0.023	-0.033	-0.042	-0.049	-0.049	-0.036	0.000						
0.000	-0.009	-0.017	-0.024	-0.030	-0.032	-0.030	-0.019	0.000						
0.000	-0.006	-0.012	-0.017	-0.021	-0.021	-0.019	-0.011	0.000						
0.000	-0.005	-0.009	-0.012	-0.014	-0.014	-0.012	-0.007	0.000						
0.000	-0.003	-0.006	-0.008	-0.009	-0.009	-0.007	-0.004	0.000						
0.000	-0.002	-0.004	-0.005	-0.006	-0.006	-0.005	-0.003	0.000						
0.000	-0.001	-0.002	-0.003	-0.004	-0.003	-0.003	-0.001	0.000						
0.000	-0.001	-0.001	-0.002	-0.002	-0.002	-0.001	-0.001	0.000						
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000						

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Table 3. The vertical velocity at each of the node points.

0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-0.023	-0.023	-0.024	-0.026	-0.029	-0.032	-0.035	-0.036	-0.032	-0.021	-0.012	-0.006	-0.003	-0.001	0.000
-0.045	-0.046	-0.048	-0.052	-0.058	-0.065	-0.073	-0.077	-0.070	-0.040	-0.020	-0.010	-0.004	-0.002	0.000
-0.066	-0.067	-0.070	-0.076	-0.085	-0.097	-0.113	-0.130	-0.132	-0.048	-0.019	-0.008	-0.003	-0.001	0.000
-0.084	-0.086	-0.090	-0.097	-0.109	-0.126	-0.153	-0.197	-0.281	0.000	0.000	0.000	0.000	0.000	0.000
-0.100	-0.101	-0.106	-0.114	-0.126	-0.145	-0.175	-0.225	-0.323						
-0.113	-0.114	-0.119	-0.126	-0.137	-0.154	-0.176	-0.205	-0.232						
-0.123	-0.124	-0.128	-0.135	-0.144	-0.156	-0.171	-0.187	-0.196						
-0.131	-0.132	-0.135	-0.140	-0.147	-0.156	-0.166	-0.174	-0.178						
-0.136	-0.137	-0.140	-0.144	-0.149	-0.155	-0.161	-0.166	-0.168						
-0.140	-0.141	-0.143	-0.146	-0.150	-0.154	-0.158	-0.161	-0.162						
-0.143	-0.144	-0.145	-0.147	-0.150	-0.153	-0.156	-0.158	-0.159						
-0.145	-0.145	-0.146	-0.148	-0.150	-0.153	-0.155	-0.156	-0.156						
-0.146	-0.146	-0.147	-0.149	-0.150	-0.152	-0.154	-0.155	-0.155						
-0.146	-0.147	-0.147	-0.149	-0.150	-0.152	-0.154	-0.155	-0.155						

Table 4. The pressure coefficient at each of the node points.

1.000	0.994	0.975	0.940	0.885	0.804	0.692	0.548	0.388	0.245	0.144	0.084	0.052	0.037	0.033
0.994	0.988	0.969	0.934	0.878	0.796	0.680	0.529	0.358	0.214	0.120	0.069	0.043	0.031	0.028
0.977	0.971	0.952	0.917	0.860	0.774	0.646	0.465	0.249	0.106	0.046	0.026	0.019	0.018	0.017
0.951	0.945	0.926	0.891	0.834	0.742	0.593	0.341	-0.044	-0.134	-0.082	-0.035	-0.010	0.002	0.005
0.920	0.914	0.895	0.861	0.805	0.714	0.549	0.176	-1.125	-0.735	-0.228	-0.079	-0.026	-0.005	-0.000
0.887	0.881	0.864	0.832	0.782	0.704	0.580	0.363	-0.183						
0.856	0.851	0.835	0.808	0.766	0.706	0.622	0.510	0.390						
0.829	0.824	0.811	0.789	0.756	0.712	0.659	0.603	0.566						
0.806	0.803	0.792	0.775	0.750	0.720	0.686	0.656	0.642						
0.789	0.787	0.778	0.765	0.747	0.726	0.705	0.688	0.680						
0.777	0.775	0.768	0.758	0.745	0.731	0.717	0.706	0.702						
0.768	0.767	0.762	0.754	0.745	0.734	0.725	0.718	0.716						
0.762	0.761	0.757	0.752	0.744	0.737	0.730	0.725	0.723						
0.759	0.758	0.755	0.750	0.744	0.738	0.732	0.729	0.727						
0.758	0.757	0.754	0.750	0.744	0.738	0.733	0.730	0.728						

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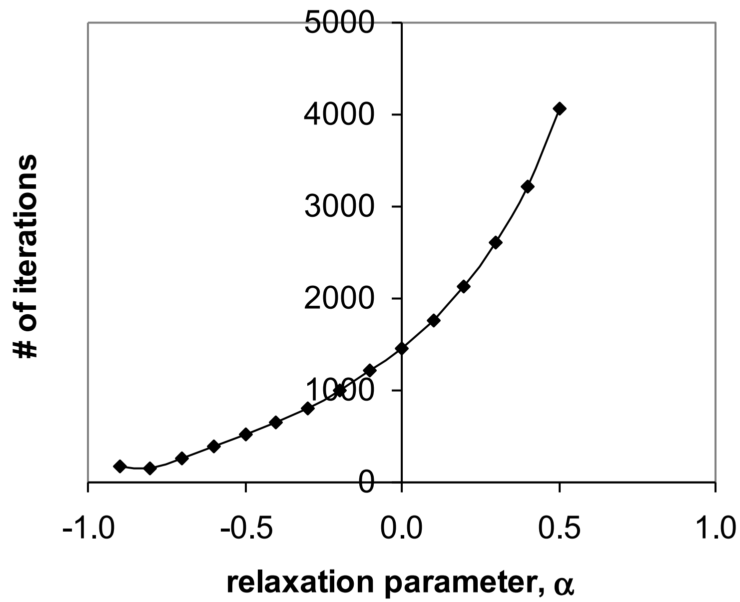


Figure 1. The number of iterations to convergence as a function of the relaxation parameter, α.

Prove that a potential function, ϕ, that satisfies Laplace's equation, $\nabla^2\phi=0$, also satisfies the continuity equation. Show that the incompressible, constant viscosity, Newtonian Navier-Stokes equations with conservative body forces simplifies to the unsteady Bernoulli equation.

SOLUTION:

Laplace's equation may be expanded as:

$$\frac{\partial^2 \phi}{\partial x_j \partial x_j} = 0 \tag{1}$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_j}\right) = 0 \tag{2}$$

Since the velocity potential is defined as $u_j = \partial\phi/\partial x_j$, we can write Eq. (2) as:

$$\frac{\partial}{\partial x_j}\left(u_j\right) = \frac{\partial u_j}{\partial x_j} = 0 \tag{3}$$

which is the continuity equation for an incompressible flow.

The incompressible Navier-Stokes equations for a Newtonian fluid with constant viscosity are:

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho g_i \tag{4}$$

$$\rho\left[\frac{\partial}{\partial t}\frac{\partial \phi}{\partial x_i} + \frac{\partial \phi}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial \phi}{\partial x_i}\right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2}{\partial x_j \partial x_j}\frac{\partial \phi}{\partial x_i} + \rho g_i \tag{5}$$

Re-ordering the time and spatial derivatives in the first term on the left-hand side, and the spatial derivatives in the last term on the right hand side gives:

$$\rho\left[\frac{\partial}{\partial x_i}\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial \phi}{\partial x_i}\right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_i}\frac{\partial^2 \phi}{\partial x_j \partial x_j} + \rho g_i \tag{6}$$

Noting that:

$$\frac{\partial \phi}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial x_j}\frac{\partial}{\partial x_i}\frac{\partial \phi}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right) \tag{7}$$

Substituting Eq. (7) back into Eq. (6) gives:

$$\rho\left[\frac{\partial}{\partial x_i}\frac{\partial \phi}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right)\right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_i}\underbrace{\frac{\partial^2 \phi}{\partial x_j \partial x_j}}_{=0} + \rho g_i \tag{8}$$

Noting that the second term on the RHS is zero (it's Laplace's equation!), re-writing the gravitational acceleration as the gradient of a potential function G, i.e., $g_i = -\partial G/\partial x_i$, and pulling the gradient $\partial/\partial x_i$ outside of each term gives:

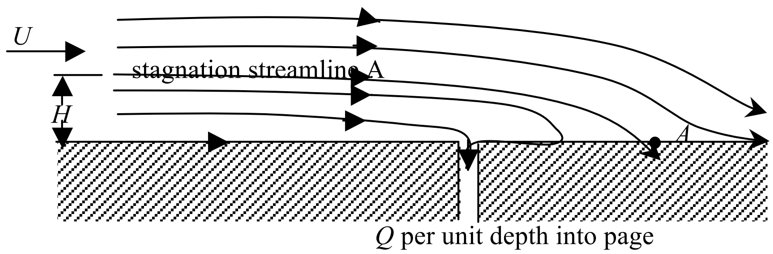
$$\rho \frac{\partial}{\partial x_i}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right)\right] = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial G}{\partial x_i} \tag{9}$$

$$\frac{\partial}{\partial x_i}\left\{\rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right)\right] = -p - \rho G\right\} \tag{10}$$

$$\boxed{\rho \frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho\left(\frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right) + \rho G = F(t)} \text{ The unsteady Bernoulli equation!} \tag{11}$$

Water flows over a flat surface at a velocity of U as shown in the figure. A pump draws off water through a narrow slit at a volume flow rate of Q per unit length of the slit. Assuming that the flow is incompressible and inviscid:

a. locate the stagnation point on the wall (point A),
b. determine the equation for the stagnation streamline A, and
c. determine how far above the surface, H, the fluid must be so that it does not get sucked into the slit.



SOLUTION:

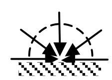
Model the flow using the complex potential function for a uniform stream added to the complex potential function for a line sink centered at the origin.

$$f(z) = Uz - \frac{m}{2\pi}\log z \quad (m > 0) \tag{1}$$

Note that the sink strength m is related to the slit's volume flow rate, Q. Consider the flow rate entering half a line sink (as shown in the diagram below).

$$f(z) = -\frac{m}{2\pi}\log z \Rightarrow \frac{df}{dz} = (u_r - iu_\theta)\exp(-i\theta) = -\frac{m}{2\pi}\frac{1}{z} = -\frac{m}{2\pi}\frac{1}{r}\exp(-i\theta)$$

$$u_r = -\frac{m}{2\pi}\frac{1}{r}$$



$$u_\theta = 0$$

The volumetric flow rate into the half sink, Q, is equal to the fluid velocity at a radius r multiplied by the distance over which the fluid enters, which in this case is half a circle with perimeter, πr.

$$Q = u_r \pi r = \frac{m}{2\pi}\frac{1}{r}\pi r$$

$$\therefore m = 2Q \tag{2}$$

Determine the flow velocities from Eq. (1).

$$u_x - iu_y = \frac{df}{dz} = U - \frac{Q}{\pi}\frac{1}{z} = U - \frac{Q}{\pi}\frac{\bar{z}}{z\bar{z}} = U - \frac{Q}{\pi}\frac{\bar{z}}{|z|} \tag{3}$$

$$u_x - iu_y = U - \frac{Q}{\pi}\frac{x - iy}{x^2 + y^2}$$

$$u_x = U - \frac{Q}{\pi}\frac{x}{x^2 + y^2} \tag{4}$$

$$u_y = -\frac{Q}{\pi}\frac{y}{x^2 + y^2} \tag{5}$$

The stagnation point along the wall ($y = 0$) occurs at:

$$u_x = 0 = U - \frac{Q}{\pi}\frac{1}{x_{\text{stag}}} \Rightarrow \boxed{x_{\text{stag}} = \frac{Q}{\pi U}} \quad \text{(Note that } Q > 0.\text{)}$$

Another way to find the stagnation point is to go directly from Eq. (3):

$$u_x - iu_y = \frac{df}{dz} = U - \frac{Q}{\pi}\frac{1}{z} = U - \frac{Q}{\pi}\frac{1}{x + iy}$$

At the stagnation point, which occurs at $y = 0$, $u_x = u_y = 0$ so that:

$$0 = U - \frac{Q}{\pi}\frac{1}{x_{\text{stag}}} \Rightarrow x_{\text{stag}} = \frac{Q}{\pi U} \quad \text{(The same answer as before!)}$$

The stream function corresponding to the stagnation streamline can be determined by expanding the complex potential (Eq. (1)).

$$f(z) = U(x + iy) - \frac{Q}{\pi}(\ln r + i\theta)$$

$$= \underbrace{\left(Ux - \frac{Q}{\pi}\ln r\right)}_{=\phi} + i\underbrace{\left(Uy - \frac{Q}{\pi}\theta\right)}_{=\psi}$$

$$\psi = Uy - \frac{Q}{\pi}\theta \tag{6}$$

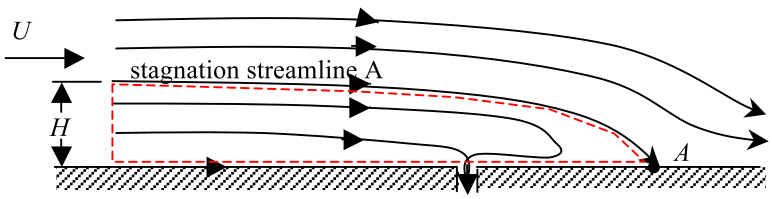
The stream function evaluated at the stagnation point ($y = 0$, $\theta = 0$) is $\psi = 0$ so that the stagnation streamline can be written as:

$$\boxed{y = \frac{Q}{\pi U}\theta = \frac{Q}{\pi U}\tan^{-1}\left(\frac{y}{x}\right)} \quad \text{(Note: } Q > 0.\text{)} \tag{7}$$

Far upstream ($\theta \to \pi$), the stagnation streamline is at a height, H:

$$\boxed{H = \frac{Q}{U}} \quad \text{(Note: } Q > 0.\text{)} \tag{8}$$

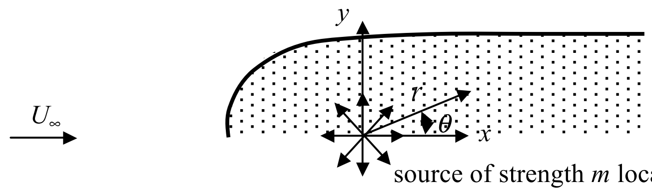
Another way to determine H is to use conservation of mass. The volumetric flow rate entering the slit is Q. Far upstream of the slit the fluid velocity is in the horizontal direction with a uniform velocity U.



Hence, from conservation of mass:

$$HU = Q \Rightarrow H = \frac{Q}{U} \text{ (The same answer as before!)}$$

The upper surface of a half-body is sketched below. This pattern is formed in an ideal flow by combining a source of strength, m, at the origin and a uniform flow from left to right with velocity, U_∞.



a. Determine the potential function for the given flow field in polar coordinates.
b. Determine the equation of the streamline passing through the stagnation point.
c. Obtain an expression for the pressure coefficient, C_p, defined as:

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2}$$

where p is the pressure, p_∞ is the pressure far upstream, and ρ is the fluid density. Express your answer in terms of U_∞, m, r, and θ.

d. Determine the acceleration of a fluid particle traveling along the stagnation streamline upstream of the body.

SOLUTION:

The potential function for the flow field is:

$$\boxed{\phi = U_\infty x + \frac{m}{2\pi}\ln r = U_\infty r\cos\theta + \frac{m}{2\pi}\ln r}$$

and the corresponding stream function for the flow is:

$$\psi = U_\infty y + \frac{m}{2\pi}\theta = U_\infty r\sin\theta + \frac{m}{2\pi}\theta$$

To determine the equation of the streamline through the stagnation point, we must first determine the value of the stream function at the stagnation point (where the flow velocity is zero). The velocity components for the flow are:

$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U_\infty\cos\theta + \frac{m}{2\pi}\frac{1}{r}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = -U_\infty\sin\theta$$

The θ-velocity component is zero when $\theta = 0$ and π. The upstream stagnation point obviously occurs when $\theta = \pi$. The radial velocity component is zero at $\theta = \pi$ when:

$$u_r = 0 = -U_\infty + \frac{m}{2\pi}\frac{1}{r} \;\Rightarrow\; r = \frac{m}{2\pi U_\infty}$$

Thus, the stagnation point occurs when:

$$(r,\theta)_{\text{stag}} = \left(\frac{m}{2\pi U_\infty}, \pi\right)$$

The value of the stream function at this point is:

$$\psi_{\text{stag}} = \tfrac{1}{2}m$$

Hence, the equation of the streamline passing through the stagnation point is:

$$\boxed{\tfrac{1}{2}m = U_\infty r\sin\theta + \frac{m}{2\pi}\theta} \text{ or, after some re-arrangement: } \boxed{r = \frac{m}{2U_\infty\sin\theta}\left(1 - \frac{\theta}{\pi}\right)}$$

The pressure at any point in the flow field may be found using Bernoulli's equation:

$$p+\tfrac{1}{2}\rho U^2 = p_\infty + \tfrac{1}{2}\rho U_\infty^2$$

where the subscript "∞" indicates conditions far upstream. Hence, the pressure coefficient is given by:

$$C_p \equiv \frac{p-p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1-\frac{U^2}{U_\infty^2}$$

where

$$\begin{aligned}
U^2 = u_r^2 + u_\theta^2 &= \left(U_\infty \cos\theta + \frac{m}{2\pi}\frac{1}{r}\right)^2 + \left(-U_\infty \sin\theta\right)^2 \\
&= U_\infty^2 \cos^2\theta + \frac{m}{\pi}\frac{1}{r}U_\infty \cos\theta + \left(\frac{m}{2\pi}\frac{1}{r}\right)^2 + U_\infty^2 \sin^2\theta \\
&= U_\infty^2 + \frac{m}{\pi}\frac{1}{r}U_\infty \cos\theta + \left(\frac{m}{2\pi}\frac{1}{r}\right)^2
\end{aligned}$$

Therefore:

$$\begin{aligned}
C_p &= 1-\frac{1}{U_\infty^2}\left[U_\infty^2 + \frac{m}{\pi}\frac{1}{r}U_\infty \cos\theta + \left(\frac{m}{2\pi}\frac{1}{r}\right)^2\right] \\
&= 1-\left[1+\frac{m}{\pi U_\infty}\frac{1}{r}\cos\theta + \left(\frac{m}{2\pi U_\infty}\frac{1}{r}\right)^2\right]
\end{aligned}$$

$$\boxed{\therefore C_p = -\frac{m}{\pi U_\infty}\frac{1}{r}\cos\theta - \left(\frac{m}{2\pi U_\infty}\frac{1}{r}\right)^2}$$

The acceleration of a fluid particle in polar coordinates is:

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + u_r \frac{\partial \mathbf{u}}{\partial r} + \frac{u_\theta}{r}\frac{\partial \mathbf{u}}{\partial \theta} + u_z \frac{\partial \mathbf{u}}{\partial z}$$

where

$$u_r = U_\infty \cos\theta + \frac{m}{2\pi}\frac{1}{r} \quad \Rightarrow \quad \frac{\partial u_r}{\partial r} = -\frac{m}{2\pi}\frac{1}{r^2} \quad \text{and} \quad \frac{\partial u_r}{\partial \theta} = U_\infty \sin\theta$$

$$u_\theta = -U_\infty \sin\theta \quad \Rightarrow \quad \frac{\partial u_\theta}{\partial r} = 0 \quad \text{and} \quad \frac{\partial u_\theta}{\partial \theta} = -U_\infty \cos\theta$$

Note that the present flow is steady and planar. The stagnation streamline upstream of the body occurs along the line:

$$r > r_{\text{stag}} = \frac{m}{2\pi U_\infty} \quad \text{and} \quad \theta = \pi$$

so that:

$$u_r\big|_{\theta=\pi} = -U_\infty + \frac{m}{2\pi}\frac{1}{r} \quad \text{and} \quad \left.\frac{\partial u_r}{\partial r}\right|_{\theta=\pi} = -\frac{m}{2\pi}\frac{1}{r^2} \quad \text{and} \quad \left.\frac{\partial u_r}{\partial \theta}\right|_{\theta=\pi} = 0$$

$$u_\theta\big|_{\theta=\pi} = 0 \quad \text{and} \quad \left.\frac{\partial u_\theta}{\partial r}\right|_{\theta=\pi} = 0 \quad \text{and} \quad \left.\frac{\partial u_\theta}{\partial \theta}\right|_{\theta=\pi} = U_\infty$$

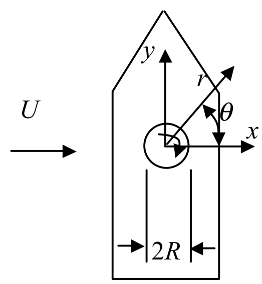
Hence, the acceleration of a fluid particle is:

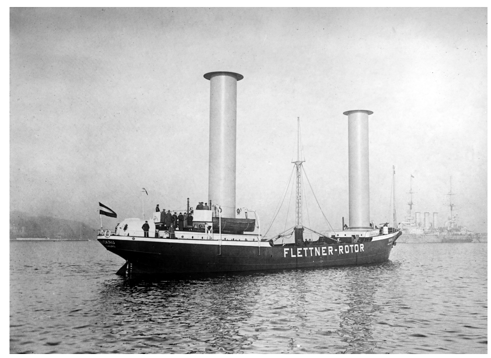
$$\boxed{\begin{aligned}\frac{Du_r}{Dt} &= \left(-U_\infty + \frac{m}{2\pi}\frac{1}{r}\right)\left(-\frac{m}{2\pi}\frac{1}{r^2}\right) \\ \frac{Du_\theta}{Dt} &= 0\end{aligned}} \quad \text{for } r > r_{\text{stag}} = \frac{m}{2\pi U_\infty}$$

A Flettner rotor ship is powered by the wind, but rather than using sails, lift is generated using rotating cylinders. Assume the air speed relative to the boat is U in the y direction as shown in the figure below. Also assume the boat has a single cylinder with a length L and a radius R. The cylinder rotates in a clockwise manner (when viewed from above) with a circulation magnitude of Γ per unit length of the cylinder.

a. Write the velocity components (u_r, u_θ) of the air flow around the cylinder. Express your answer in terms of (a subset of) U, R, Γ, r, and θ.
b. At what angle(s), θ, with respect to the x axis shown in the figure below do the stagnation points occur on the cylinder? Express your answer(s) in terms of (a subset of) U, R, and Γ.
c. Determine the force acting to move the boat forward. Express your answer in terms of (a subset of) ρ, U, R, L, and Γ, where ρ is the air density.

Hint: Be careful in evaluating the sign of the circulation.





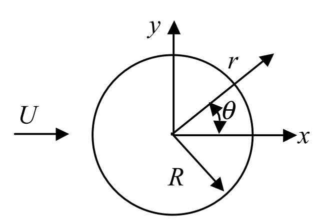
SOLUTION:

Model the flow around the cylinder as a 2D potential flow. Potential flow around a rotating cylinder may be modeled as the sum of the potential flows for a uniform stream, a doublet, and a free vortex. Here the circulation is in the negative direction

$$\phi = Ur\cos\theta + \frac{k\cos\theta}{r} + \frac{\Gamma}{2\pi}\theta \tag{1}$$

where $\Gamma < 0$. The corresponding stream function is:

$$\psi = Ur\sin\theta - \frac{k\sin\theta}{r} - \frac{\Gamma}{2\pi}\ln r \tag{2}$$



Determine the flow velocity components from the potential function.

$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta - \frac{k\cos\theta}{r^2} \tag{3}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = -U\sin\theta - \frac{k\sin\theta}{r^2} + \frac{\Gamma}{2\pi r} \tag{4}$$

The constant k may be found by stipulating that there is no flow through the cylinder, i.e., $u_r(r = R) = 0$.

$$u_r\big|_{r=R} = 0 = U\cos\theta - \frac{k\cos\theta}{R^2} \quad\Rightarrow\quad k = UR^2 \tag{5}$$

Thus, the velocity components are:

$$\boxed{u_r(r,\theta) = U\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right]} \tag{6}$$

$$\boxed{u_\theta(r,\theta) = -U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right] + \frac{\Gamma}{2\pi r}} \tag{7}$$

The location of the stagnation points on the cylinder surface may be found be letting (u_r, u_θ) = (0, 0) when $r = R$.

$$u_r\left(R,\theta\right)=U\cos\theta\left[1-\left(\frac{R}{R}\right)^2\right]=0 \quad \text{(Automatically satisfied! No flow through the cylinder surface.)} \tag{8}$$

$$u_\theta\left(R,\theta_{\text{stag}}\right)=-U\sin\theta_{\text{stag}}\left[1+\left(\frac{R}{R}\right)^2\right]+\frac{\Gamma}{2\pi R}=0 \Rightarrow 0=-2U\sin\theta_{\text{stag}}+\frac{\Gamma}{2\pi R} \Rightarrow \boxed{\sin\theta_{\text{stag}}=\frac{\Gamma}{4\pi RU}} \tag{9}$$

The force acting on the cylinder may be found from the Kutta-Joukowski Theorem:

$$\mathbf{F}=-\rho U\Gamma L\hat{\mathbf{j}} \Rightarrow \boxed{\mathbf{F}=-\rho U\Gamma L\hat{\mathbf{j}}} \tag{10}$$

Note that the lift acts perpendicular to the incoming velocity, and since $\Gamma < 0$, the lift acts in the positive y direction. The drag acting on the cylinder is zero.

Alternately, one could integrate the pressure distribution around the surface of the cylinder.

$$\mathbf{F}=-\int_{\theta=0}^{\theta=2\pi} p_{r=R}\sin\theta R d\theta L\hat{\mathbf{j}} \tag{11}$$

where

$$p_{r=R}+\tfrac{1}{2}\rho U_{r=R}^2=p_\infty+\tfrac{1}{2}\rho U^2 \tag{12}$$

$$p_{r=R}=p_\infty+\tfrac{1}{2}\rho U^2\left[1-\left(\frac{U_{r=R}}{U}\right)^2\right] \tag{13}$$

and

$$U_{r=R}^2=u_{r,r=R}^2+u_{\theta,r=R}^2=0+\left(-2U\sin\theta+\frac{\Gamma}{2\pi R}\right)^2=4U^2\sin^2\theta-4U\sin\theta\frac{\Gamma}{2\pi R}+\frac{\Gamma^2}{4\pi^2R^2} \tag{14}$$

Thus,

$$p_{r=R}=p_\infty+\tfrac{1}{2}\rho U^2\left(1-4\sin^2\theta+\frac{2\Gamma}{\pi RU}\sin\theta-\frac{\Gamma^2}{4\pi^2R^2U^2}\right) \tag{15}$$

and

$$\mathbf{F}=-\int_{\theta=0}^{\theta=2\pi}\left[p_\infty+\tfrac{1}{2}\rho U^2\left(1-4\sin^2\theta+\frac{2\Gamma}{\pi RU}\sin\theta-\frac{\Gamma^2}{4\pi^2R^2U^2}\right)\right]\sin\theta R d\theta L\hat{\mathbf{j}} \tag{16}$$

$$\mathbf{F}=\underbrace{-\int_{\theta=0}^{\theta=2\pi} p_\infty\sin\theta R\,d\theta L\hat{\mathbf{j}}}_{=0}-\int_{\theta=0}^{\theta=2\pi}\tfrac{1}{2}\rho U^2\left(1-4\sin^2\theta+\frac{2\Gamma}{\pi RU}\sin\theta-\frac{\Gamma^2}{4\pi^2R^2U^2}\right)\sin\theta R\,d\theta L\hat{\mathbf{j}} \tag{17}$$

$$\mathbf{F}=-\tfrac{1}{2}\rho U^2RL\hat{\mathbf{j}}\int_{\theta=0}^{\theta=2\pi}\left(1-4\sin^2\theta+\frac{2\Gamma}{\pi RU}\sin\theta-\frac{\Gamma^2}{4\pi^2R^2U^2}\right)\sin\theta d\theta \tag{18}$$

$$\mathbf{F}=-\tfrac{1}{2}\rho U^2RL\hat{\mathbf{j}}\left(\underbrace{\int_{\theta=0}^{\theta=2\pi}\sin\theta\,d\theta}_{=0}-4\underbrace{\int_{\theta=0}^{\theta=2\pi}\sin^3\theta\,d\theta}_{=0}+\frac{2\Gamma}{\pi RU}\underbrace{\int_{\theta=0}^{\theta=2\pi}\sin^2\theta\,d\theta}_{=\pi}-\frac{\Gamma^2}{4\pi^2R^2U^2}\underbrace{\int_{\theta=0}^{\theta=2\pi}\sin\theta\,d\theta}_{=0}\right) \tag{19}$$

$$\mathbf{F}=-\tfrac{1}{2}\rho U^2RL\left(\frac{2\Gamma}{RU}\right)\hat{\mathbf{j}} \tag{20}$$

$$\mathbf{F}=-\rho U\Gamma L\hat{\mathbf{j}} \quad \text{(Note that } \Gamma < 0 \text{ so that the force is in the positive } y \text{ direction. Same as previous result!)} \tag{21}$$

6.13. Review Questions

(1) Describe three properties of stream functions.
(2) Can stream functions be used for rotational flows? How about irrotational flows?
(3) What restrictions are there when using stream functions?
(4) Can stream functions be superposed?
(5) What is the governing equation for an incompressible potential flow?
(6) What are the requirements for modeling a flow as a potential flow?
(7) What are the appropriate boundary conditions for a potential flow?
(8) Can potential functions be written for 3D flows? How about stream functions?
(9) Can stream functions be written for rotational flows?
(10) How are streamlines related to equipotential lines?
(11) Under what conditions can one write a complex potential function to describe a flow?
(12) How is a fluid velocity field determined from a complex potential function?
(13) Describe the potential flow model for ideal fluid flow around a non-rotating cylinder.
(14) Describe the potential flow model for ideal fluid flow around a rotating cylinder.
(15) What is d'Alembert's paradox?
(16) What causes Magnus lift? Is this what causes baseballs or golf balls to curve?
(17) Why does potential flow modeling fail to capture the behavior of real flows downstream of a cylinder?
(18) What is the Blasius integral law?
(19) What is the Kutta-Joukowski theorem?
item Describe the "method of images".
(20) What is meant by "ground effect"?
(21) What is meant by "added mass"? Under what conditions will the added mass on an object be significant?

CHAPTER 7

Dimensional Analysis

7.1. What is Dimensional Analysis?

Dimensional analysis is a method for reducing the number and complexity of variables used to describe a physical system. It's a technique that can be applied to all fields, not just fluid mechanics. The mechanics of dimensional analysis are simple to learn and apply, and the benefits from using it are significant.

Dimensional analysis can be used to present data in an efficient manner, reduce the number of experiments or simulations one needs to perform to investigate the relationship between variables, and scale results. However, dimensional analysis cannot tell us what the functional relationship is between variables. Additional experiments or theoretical analyses are required to determine this information.

7.1.1. Motivating Example 1: A ball falling under gravity

To motivate the use of dimensional analysis, let's consider a simple example involving a ball falling under the action of gravity in a vacuum (Figure 7.1). From basic physics, we know that the vertical position of the ball, y, is given by:

$$y = -\frac{1}{2}gt^2 + \dot{y}_0 t + y_0, \tag{7.1}$$

where g is the acceleration due to gravity, t is the time from when the ball was released, $\dot{y}_0$ is the initial speed of the ball, and y_0 is the initial position of the ball. Note that Eq. (7.1) is dimensional. In other words, each term in the equation has dimensions of length L. For example, the dimension of the first term on the right-hand side is length, $[1/2gt^2] = L$, where the square brackets indicate "dimensions of". If we were to plot the position, y, as a function of time, t, for varying g, $\dot{y}_0$, and y_0, we would have plots that look like Figure 7.2.

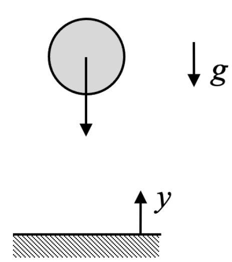


Figure 7.1. A schematic of the ball drop example.

Now let's present the same information, but in dimensionless form. Starting with Eq. (7.1), divide all terms by y_0 (a length), to make each term dimensionless,

$$\frac{y}{y_0} = -\frac{1}{2}t^2\frac{g}{y_0} + \frac{\dot{y}_0}{\sqrt{gy_0}}t\sqrt{\frac{g}{y_0}} + 1, \tag{7.2}$$

or, in a slightly more compact form,

$$y' = -\frac{1}{2}t'^2 + \dot{y}_0' t' + 1, \tag{7.3}$$

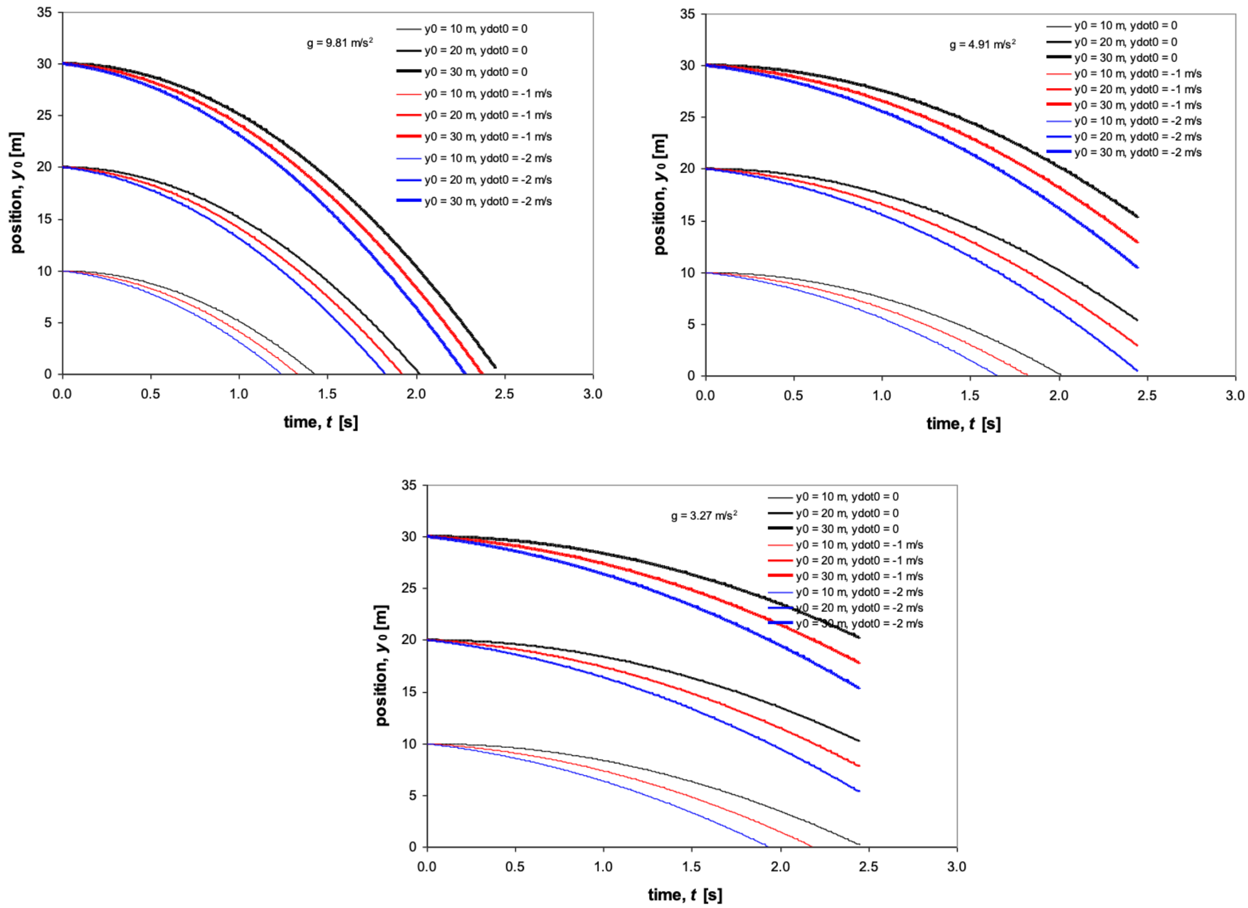


Figure 7.2. Plots of the ball position (vertical axes) as a function of time (horizontal axes) for different initial positions (different intercepts on the vertical axes), different initial speeds (different colors), and different gravitational accelerations (different plots).

where,

$$y' \coloneqq \frac{y}{y_0}, \quad \dot{y}_0' \coloneqq \frac{\dot{y}_0}{\sqrt{g y_0}}, \quad \text{and} \quad t' \coloneqq t\sqrt{\frac{g}{y_0}}, \tag{7.4}$$

are the dimensionless position, initial speed, and time, respectively. Note that Eqs. (7.3) and (7.1) are identical; they're just written in dimensionless or dimensional form. Now if we were to plot Eq. (7.3) for all of the various combinations of variables, we would have the plot shown in Figure 7.3. This dimensionless plot contains all of the information that was contained in the previous dimensional plots. As you can see, presenting data in dimensionless form is very efficient!

Now let's assume we didn't know that Eq (7.1) existed and we had to perform a series of experiments to try to find the functional relationship between the variables,

$$y = f_1(g, t, \dot{y}_0, y_0), \tag{7.5}$$

where f_1 is the unknown function we're trying to determine. Let's say that we perform a series of experiments where we vary each of the variables independently five times. Since we have four independent variables $(g, t, \dot{y}_0, y_0)$, this means we have a total of $5^4 = 625$ experiments to perform! Not only is this a lot of experiments, but some of these experiments are likely to be difficult and expensive to carry out, e.g., varying g isn't trivial.

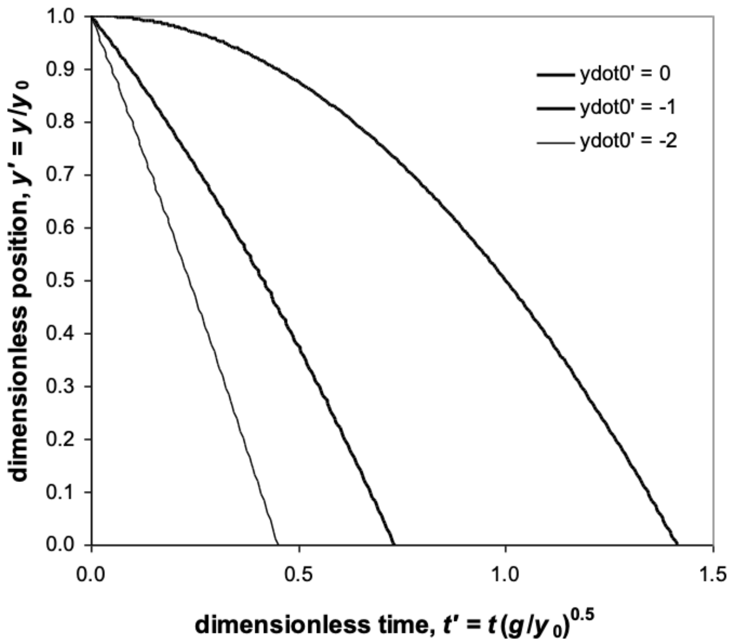


Figure 7.3. The dimensionless ball position (vertical axis) plotted as a function of dimensionless time (horizontal axis) for different dimensionless initial speeds (different lines).

Now if we instead performed a dimensional analysis on Eq. (7.5), which you'll learn how to do later in this set of notes, we could show that Eq. (7.5) can be written in dimensionless form as,

$$\frac{y}{y_0} = f_2\left(t\sqrt{\frac{g}{y_0}}, \frac{\dot{y}_0}{\sqrt{gy_0}}\right). \tag{7.6}$$

(Compare Eqs. (7.6) and (7.3) to verify.) Equation (7.6) contains only two independent variables; hence, varying each parameter five times gives a total of $5^2 = 25$ total experiments. Clearly, <u>performing a dimensional analysis can reduce the number of experiments one needs to perform</u>! Not only are the number of experiments reduced, but the experiments can be much easier to perform. For example, varying the two independent parameters in Eq. (7.6) can be achieved by simply letting time vary, and varying the initial drop speed. We needn't worry about varying gravity, g, independently since g is contained within the term $t\sqrt{g/y_0}$, for example.

Finally, now let's say that we're interested in launching an object on the Moon (indicated by the subscript "M") from a specified height, $y_{0,M} = 1\,\mathrm{m}$, with a specified speed, $\dot{y}_{0,M} = 1\,\mathrm{m\,s^{-1}}$, and want to know how long it will take for the object to impact the ground, $y_M(t_M = ?) = 0$. We know that the acceleration due to gravity on the Moon is $g_M = 1.62\,\mathrm{m/s^2}$. Again, assuming we don't know that Eq. (7.1) exists, we can still determine this time by performing a similar experiment on Earth, and then scale the result. If we're to perform this similar experiment on Earth, where $g_E = 9.81\,\mathrm{m/s^2}$, we need to first determine the initial drop height, $y_{E,0}$, and speed, $\dot{y}_{E,0}$, for the Earth experiment. Since the same physical process holds for both the Moon and Earth, the dimensionless terms describing the process will be identical, i.e., Eq. (7.6) will be the same for the Earth and Moon. Thus, we can equate dimensionless terms to determine the values that should be used on the Moon,

$$\left(\frac{y}{y_0}\right)_E = \left(\frac{y}{y_0}\right)_M \implies y_{0,E} = y_{0,M}\left(\frac{y_E}{y_M}\right) = (1\,\mathrm{m})\left(\frac{0\,\mathrm{m}}{0\,\mathrm{m}}\right) \underbrace{\implies}_{\lim_{y_E, y_M \to 0}} y_{0,E} = 1\,\mathrm{m}, \tag{7.7}$$

$$\left(\frac{\dot{y}_0}{\sqrt{gy_0}}\right)_E = \left(\frac{\dot{y}_0}{\sqrt{gy_0}}\right)_M \implies \dot{y}_{0,E} = \dot{y}_{0,M}\sqrt{\frac{y_{0,E}}{y_{0,M}}}\sqrt{\frac{g_E}{g_M}} = (1\,\mathrm{m\,s^{-1}})\sqrt{\frac{1\,\mathrm{m}}{1\,\mathrm{m}}}\sqrt{\frac{9.81\,\mathrm{m/s^2}}{1.62\,\mathrm{m/s^2}}} \implies \dot{y}_{0,E} = 2.46\,\mathrm{m\,s^{-1}}. \tag{7.8}$$

When performing the drop test on Earth with the given initial conditions, the time required for the ball to hit the ground is $t_E = 0.77\,\mathrm{s}$ (which can be verified using Eq. (7.1)). To determine the corresponding time for the Moon, we equate the last dimensionless term in Eq. (7.8), $t_M = 1.89\,\mathrm{s}$.

$$\left(t\sqrt{\frac{g}{y_0}}\right)_E = \left(t\sqrt{\frac{g}{y_0}}\right)_M \implies t_M = t_E\sqrt{\frac{g_e}{g_M}}\sqrt{\frac{y_{0,M}}{y_{0,E}}} = (0.77\,\mathrm{s})\sqrt{\frac{9.81\,\mathrm{m/s^2}}{1.62\,\mathrm{m/s^2}}}\sqrt{\frac{1\,\mathrm{m}}{1\,\mathrm{m}}} \implies t_M = 1.89\,\mathrm{s}. \quad (7.9)$$

This time is exactly what one would calculate from Eq. (7.1) using $y_{0,M} = 1\,\mathrm{m}$, $\dot{y}_{0,M} = 1\,\mathrm{m\,s^{-1}}$, and $g_M = 1.62\,\mathrm{m/s^2}$. Thus, we see that dimensional analysis can be used for scaling!

Hopefully, you're convinced that dimensional analysis is a worthwhile topic to study and apply. The remainder of this chapter presents the mechanics of performing a dimensional analysis along with examples. In addition, similarity and scaling issues are discussed.

7.1.2. Motivating Example 2: Pressure drop in a pipe

Most fluids engineering problems are too complex to be amenable to analytic, closed-form solutions. As a result, experiments are used to determine relationships between the variables of interest, e.g., pressure and velocity. Let's consider the following example. Say we want to measure the pressure difference, $\Delta p = p_2 - p_1$, between two points separated by a distance, L, in a pipe (Figure 7.4).

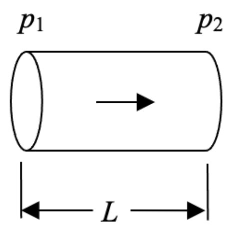


Figure 7.4. A schematic of the pipe pressure drop example.

On what variables do we expect the average pressure gradient, $\Delta p/L$, to depend? From experience and intuition we might expect the following parameters to be important:

$$V := \text{average flow velocity} \quad (7.10)$$
$$D := \text{pipe diameter} \quad (7.11)$$
$$\rho := \text{fluid density} \quad (7.12)$$
$$\mu := \text{fluid dynamic viscosity} \quad (7.13)$$

We can write this relationship in the following, more mathematical form,

$$\Delta p/L = f_1(V, D, \rho, \mu). \quad (7.14)$$

In order to determine the form of this function, it would be logical to design experiments where we vary just one of the parameters while holding the others constant and observe how $\Delta p/L$ varies. Figure 7.5 shows an illustration of typical experimental data one might obtain. This procedure, although logical, can be very time consuming, expensive, and difficult (if not impossible) to perform. For example, can you find fluids that have the same viscosity but varying density? As with the first motivating example, using dimensional analysis will greatly simplify our experimental procedure. This example will be used while presenting the various steps of performing a dimensional analysis in the following sections.

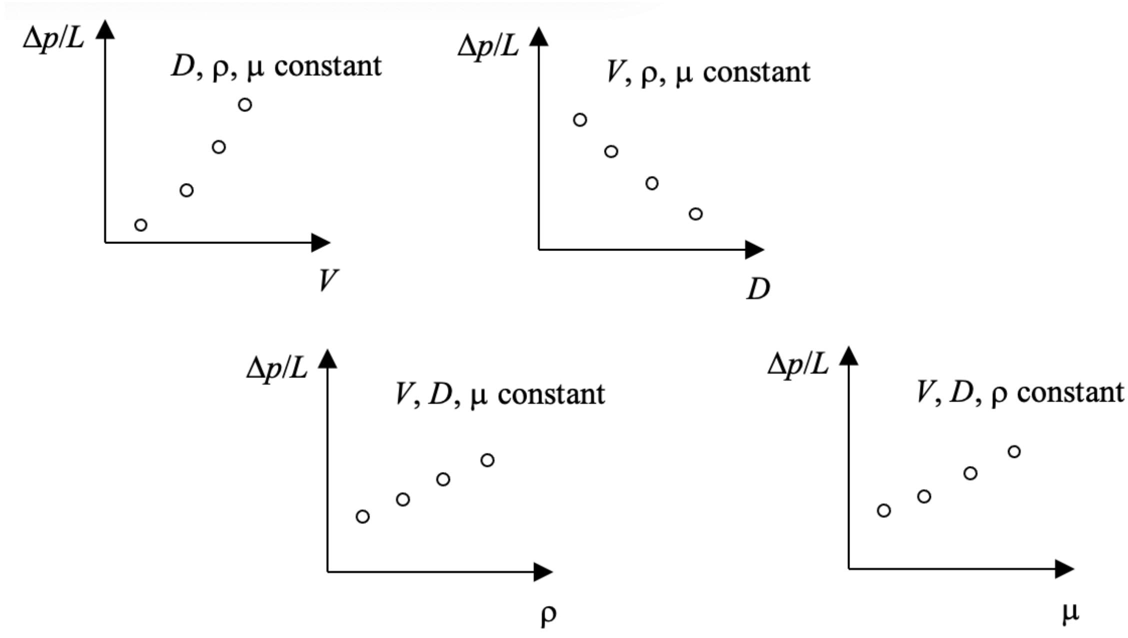


Figure 7.5. An illustration of the measured pressure gradient plotted as a function of the various independent variables.

7.2. Buckingham-Pi Theorem

The key component to dimensional analysis is the Buckingham-Pi Theorem: If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among $k-r$ independent dimensionless products (referred to as Π terms), where r is the minimum number of reference dimensions required to describe the variables, i.e.,

$$(\#\text{ of }\Pi\text{ terms}) = \underbrace{(\#\text{ of variables})}_{=k} - \underbrace{(\#\text{ of reference dimensions})}_{=r}. \tag{7.15}$$

The proof to this theorem will not be presented here.

Notes:

(1) Dimensionally homogeneous means that each term in the equation has the same units. For example, the following form of Bernoulli's equation,

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{constant}, \tag{7.16}$$

is dimensionally homogeneous since each term has units of length (L).

(2) A dimensionless product, also commonly referred to as a Pi (Π) term, is a term that has no dimensions. For example,

$$\frac{p}{\rho V^2}, \tag{7.17}$$

is a dimensionless product since both the numerator and denominator have the same dimensions.

(3) Reference dimensions are usually basic dimensions such as mass (M), length (L), and time (T) or force (F), length (L), and time (T). We'll discuss the "usually" modifier a little later when discussing the Method of Repeating Variables.

7.3. Method of Repeating Variables

The Buckingham-Pi Theorem merely states that a relationship among dimensional variables may be written, perhaps in a more compact form, in terms of dimensionless variables (Π terms). The Pi Theorem does not, however, tell us what these dimensionless variables are. The Method of Repeating Variables is an algorithm that can be used to determine these dimensionless variables.

The Method of Repeating Variables algorithm is as follows:

(1) *List all variables involved in the problem.*
   (a) This is the most difficult step since it requires experience and insight.
   (b) Variables are things like pressure, velocity, gravitational acceleration, viscosity, etc.
   (c) List only independent variables. For example, you can list ρ (density) and g (gravitational acceleration), or ρ and γ (specific weight), or g and γ, but you should not list ρ, g, and γ since one of the variables is dependent on the others.
   (d) If you include variables that are unimportant to the system, then you'll form Π terms that won't have an impact in practice. This situation is the same one you'd have if dimensional variables were used.
   (e) If you leave out an important variable, then you'll find that your relationship between dimensionless terms won't fully describe the system behavior. Again, this situation is the same one you'd have if you used dimensional terms.

(2) *Express each variable in terms of basic dimensions.*
   (a) For fluid mechanics problems we typically use mass (M), length (L), and time (T) or force (F), length (L), and time (T) as basic dimensions. We may occasionally need other basic dimensions such as temperature (θ).
   (b) For example, the dimensions of density can be written as,

$$[\rho] = \frac{M}{L^3} = \frac{FT^2}{L^4}. \tag{7.18}$$

   Note that the square brackets are used to indicate "dimensions of".

(3) *Determine the number of* Π *terms using the Buckingham-Pi Theorem.*
   (a) (# of Π terms) = (# of variables) – (# of reference dimensions)
   (b) Usually the number of reference dimensions will be the same as the number of basic dimensions found in the previous step. There are (rare) cases where some of the basic dimensions always appear in particular combinations so that the number of reference dimensions is less than the number of basic dimensions. For example, say that the variables in the problem are A, B, and C, and their corresponding basic dimensions, are,

$$[A] = \frac{M}{L^3} \quad [B] = \frac{M}{L^3T^2} \quad [C] = \frac{MT}{L^3}. \tag{7.19}$$

   The basic dimensions are M, L, and T (there are three basic dimensions). Notice, however, that the dimensions M and L always appear in the combination M/L^3. Thus, we really only need two reference dimensions, M/L^3 and T, to describe all of the variable dimensions.

(4) *Select repeating variables where the number of repeating variables is equal to the number of reference dimensions.*
   (a) The repeating variables should come from the list of independent variables. In our previous example, the list of independent variables is V, D, ρ, and μ.
   (b) Each repeating variable must have units independent of the other repeating variables.
   (c) Don't make the dependent variable one of the repeating variables. In our previous example, $\Delta p/L$ is the dependent variable. If we do that, then the resulting Π terms may have the dependent variable embedded in them.
   (d) All of the reference dimensions must be included in the group of repeating variables.

(5) *Form a* Π *term by multiplying one of the non-repeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.*
   (a) This step is most clearly illustrated in an example and will not be discussed here.

   (b) Repeat this step for all non-repeating variables.
(6) *Check that all* Π *terms are dimensionless.*
   (a) This is an important, but often overlooked, step to verify that your Π terms are, in fact, dimensionless.
(7) *Express the final form of the dimensional analysis as a relationship among the* Π *terms.*
   (a) For example, $\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$.

Let's use our pipe flow experiment to demonstrate the procedure.

(1) *List all variables involved in the problem.*
The variables that are important in this problem are,

$$\Delta p/L \coloneqq \text{average pressure gradient over length } L \tag{7.20}$$
$$V \coloneqq \text{average flow velocity,} \tag{7.21}$$
$$D \coloneqq \text{pipe diameter,} \tag{7.22}$$
$$\rho \coloneqq \text{fluid density,} \tag{7.23}$$
$$\mu \coloneqq \text{fluid dynamic viscosity.} \tag{7.24}$$

Thus,

$$\Delta p/L = f_1(V, D, \rho, \mu). \tag{7.25}$$

(2) *Express each variable in terms of basic dimensions.*
The basic dimensions of each variable are,

$$\left[\frac{\Delta p}{L}\right] = \frac{F}{L^3} = \frac{M}{L^2T^2}, \tag{7.26}$$
$$[V] = \frac{L}{T}, \tag{7.27}$$
$$[D] = L, \tag{7.28}$$
$$[\rho] = \frac{M}{L^3}, \tag{7.29}$$
$$[\mu] = \frac{FT}{L^2} = \frac{M}{LT}. \tag{7.30}$$

(3) *Determine the number of* Π *terms using the Buckingham-Pi Theorem.*
   - (# of variables) = 5 ($\Delta p/L$, V, D, ρ, μ)
   - (# of reference dimensions) = 3 (F, L, T or M, L, T)
   - (# of Π terms) = (# of variables) − (# of reference dimensions) = 2

Thus, instead of having a relation involving five terms, we actually have a relationship involving just two terms!
(4) *Select repeating variables where the number of repeating variables is equal to the number of reference dimensions.*
Select the following three repeating variables (three since the number of reference dimensions is three): ρ, V, D.
   (a) These three repeating variables have independent dimensions.
   (b) The dependent variable ($\Delta p/L$) is not one of the repeating variables.
   (c) We could have also selected (V, μ, D) or (V, ρ, μ) or (μ, D, ρ) as repeating variables. The choice of repeating variables is somewhat arbitrary as long as they have independent reference dimensions and do not include the dependent variable.

(5) *Form a* Π *term by multiplying one of the non-repeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.*

$$\Pi_1 = \left(\frac{\Delta p}{L}\right)\rho^a V^b D^c \tag{7.31}$$

$$(MLT)^0 = \left(\frac{M}{L^2T^2}\right)\left(\frac{M}{L^3}\right)^a\left(\frac{L}{T}\right)^b\left(\frac{L}{1}\right)^c \tag{7.32}$$

Examining the M, L, and T terms individually,

$$M^0 = M^1M^a \implies 0 = 1 + a \tag{7.33}$$

$$L^0 = L^{-2}L^{-3a}L^bL^c \implies 0 = -2 - 3a + b + c \tag{7.34}$$

$$T^0 = T^{-2}T^{-b} \implies 0 = -2 - b \tag{7.35}$$

Solving this system of equations gives: $a = -1, b = -2, c = 1$,

$$\therefore \Pi_1 = \frac{(\Delta p/L)D}{\rho V^2}. \tag{7.36}$$

Now consider the second Π term.

$$\Pi_2 = \mu\rho^a V^b D^c \tag{7.37}$$

$$(MLT)^0 = \left(\frac{M}{LT}\right)\left(\frac{M}{L^3}\right)^a\left(\frac{L}{T}\right)^b\left(\frac{L}{1}\right)^c \tag{7.38}$$

Examining the M, L, and T terms individually,

$$M^0 = M^1M^a \implies 0 = 1 + a \tag{7.39}$$

$$L^0 = L^{-1}L^{-3a}L^bL^c \implies 0 = -1 - 3a + b + c \tag{7.40}$$

$$T^0 = T^{-1}T^{-b} \implies 0 = -1 - b \tag{7.41}$$

Solving this system of equations gives: $a = -1, b = -1, c = -1$,

$$\therefore \Pi_2 = \frac{\mu}{\rho V D}. \tag{7.42}$$

(6) *Check that all* Π *terms are dimensionless.*

$$[\Pi_1] = \left[\frac{(\Delta p/L)D}{\rho V^2}\right] = \frac{M}{L^2T^2}\frac{L}{1}\frac{L^3}{M}\frac{T^2}{L^2} = M^0L^0T^0 \quad \text{OK!} \tag{7.43}$$

$$[\Pi_2] = \left[\frac{\mu}{\rho V D}\right] = \frac{M}{LT}\frac{L^3}{M}\frac{T}{L}\frac{1}{L} = M^0L^0T^0 \quad \text{OK!} \tag{7.44}$$

(7) *Express the final form of the dimensional analysis as a relationship among the* Π *terms.* Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{(\Delta p/L)D}{\rho V^2} = f_2\left(\frac{\mu}{\rho V D}\right)}. \tag{7.45}$$

Notes:

(1) Instead of having to run four different sets of experiments as was discussed at the beginning of this chapter, we only really need to run one set of experiments where we vary,

$$\Pi_2 = \frac{\mu}{\rho V D}, \tag{7.46}$$

and measure,

$$\Pi_1 = \frac{(\Delta p/L)D}{\rho V^2}. \tag{7.47}$$

All of the information contained in Figure 7.5 is contained within Figure 7.6! This reduces the complexity, cost, and time required to determine the relationship between the average pressure gradient and the other variables.

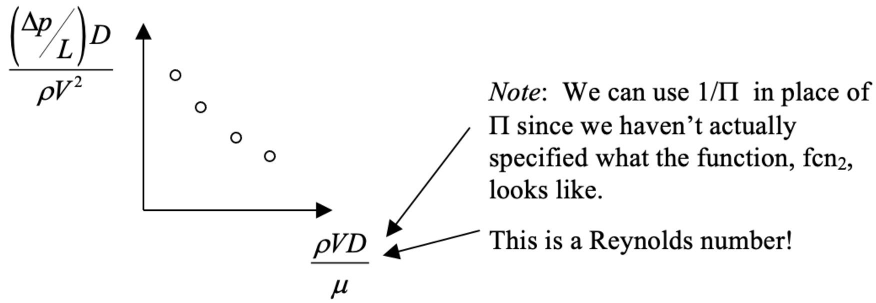


Figure 7.6. An illustration of the dimensionless pressure gradient plotted as a function of the Reynolds number. Compare this plot to the ones in Figure 7.5.

(2) Dimensional analysis is a very powerful tool because it tells us what terms really are important in an equation. For example, we started with the relation,

$$\frac{\Delta p}{L} = f_1(V, D, \rho, \mu), \tag{7.48}$$

leading us to believe that V, D, ρ, and μ are all important terms by themselves. However, dimensional analysis shows us that instead of the terms by themselves, it is the following grouping of terms,

$$\frac{(\Delta p/L)D}{\rho V^2} = f_2\left(\frac{\mu}{\rho V D}\right), \tag{7.49}$$

that is important in the relationship. This is a subtle but very important point.

(3) Dimensional analysis tells us how many dimensionless terms are important in a relation. It does not tell us what the functional relationship is. We need to rely on other analyses or experiments to determine the functional relationship.

(4) The dimensionless Π terms found via dimensionless analysis are not necessarily unique. Had we chosen different repeating variables in the previous example, we would have ended up with different Π terms. One can multiply, divide, or raise their set of Π terms to form the Π terms found by another. The number of Π terms, however, is unique.

(5) After a bit of practice, one can quickly form Π terms by inspection rather than having to go through the method of repeating variables.

An open cylindrical tank having a diameter D is supported around its bottom circumference and is filled to a depth h with a liquid having a specific weight γ. The vertical deflection, δ, of the center of the bottom is a function of D, h, d, γ, and E where d is the thickness of the bottom and E is the modulus of elasticity of the bottom material. Form the dimensionless groups describing this relationship.

SOLUTION:

1. Write the dimensional functional relationship.

$$\delta = f_1\left(D, h, d, \gamma, E\right)$$

2. Determine the basic dimensions of each parameter.

$$[\delta] = L$$
$$[h] = L$$
$$[D] = L$$
$$[d] = L$$
$$[\gamma] = \frac{M}{L^2T^2} = \frac{F}{L^3}$$
$$[E] = \frac{M}{LT^2} = \frac{F}{L^2}$$

3. Determine the number of Π terms required to describe the functional relationship.

   \# of variables = 6 (δ, D, h, d, γ, E)
   \# of reference dimensions = 2 (L, F/L^2 or L, M/T^2)
   (Note that the number of reference dimensions and the number of basic dimensions are not the same for this problem!)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 6 – 2 = 4

4. Choose two repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

   D, γ (Note that the dimensions for D and γ are independent.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = \delta D^a \gamma^b$$
$$\Rightarrow F^0 L^0 = \left(\tfrac{L}{1}\right)\left(\tfrac{L}{1}\right)^a \left(\tfrac{F}{L^3}\right)^b$$
$$F:\ 0 = b$$
$$L:\ 0 = 1 + a - 3b \ \Rightarrow\ a = -1$$
$$\therefore \Pi_1 = \delta/D$$

$$\Pi_2 = hD^a \gamma^b$$
$$\Rightarrow F^0 L^0 = \left(\tfrac{L}{1}\right)\left(\tfrac{L}{1}\right)^a \left(\tfrac{F}{L^3}\right)^b$$
$$F:\ 0 = b$$
$$L:\ 0 = 1 + a - 3b \ \Rightarrow\ a = -1$$
$$\therefore \Pi_2 = h/D$$

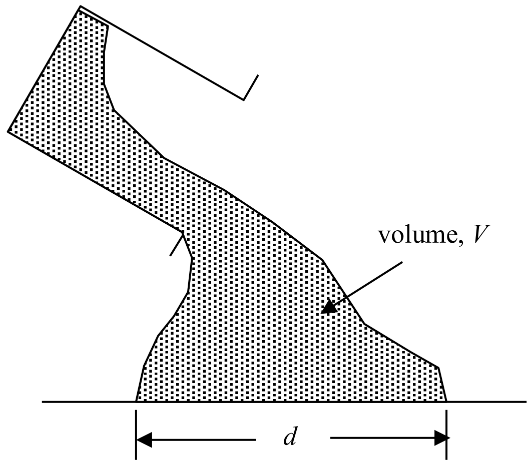
$$\Pi_3 = dD^a\gamma^b$$
$$\Rightarrow \; F^0L^0 = \left(L/1\right)\left(L/1\right)^a\left(F/L^3\right)^b$$
F: $0 = b$
L: $0 = 1 + a - 3b \;\Rightarrow\; a = -1$
$$\therefore \Pi_3 = d/D$$

$$\Pi_4 = ED^a\gamma^b$$
$$\Rightarrow \; F^0L^0 = \left(F/L^2\right)\left(L/1\right)^a\left(F/L^3\right)^b$$
F: $0 = 1 + b \;\Rightarrow\; b = -1$
L: $0 = -2 + a - 3b \;\Rightarrow\; a = -1$
$$\therefore \Pi_4 = E/(D\gamma)$$

6. Verify that each Π term is, in fact, dimensionless.
$$[\Pi_1] = [\delta/D] = L/L = 1 \;\text{ OK!}$$
$$[\Pi_2] = [h/D] = L/L = 1 \;\text{ OK!}$$
$$[\Pi_3] = [d/D] = L/L = 1 \;\text{ OK!}$$
$$[\Pi_4] = \left[E/(D\gamma)\right] = \left(F/L^2\right)\left(1/L\right)\left(L^3/F\right) = 1 \;\text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.
$$\boxed{\frac{\delta}{D} = f_2\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{D\gamma}\right)}$$

A viscous fluid is poured onto a horizontal plate as shown in the figure. Assume that the time, t, required for the fluid to flow a certain distance, d, along the plate is a function of the volume of fluid poured, V, acceleration due to gravity, g, fluid density, ρ, and fluid dynamic viscosity, μ. Determine an appropriate set of dimensionless terms to describe this process.



SOLUTION:

1. Write the dimensional functional relationship.
$$t = f_1(d, V, g, \rho, \mu)$$

2. Determine the basic dimensions of each parameter.
$$[t] = T$$
$$[d] = L$$
$$[V] = L^3$$
$$[g] = L/T^2$$
$$[\rho] = M/L^3$$
$$[\mu] = M/(LT)$$

3. Determine the number of Π terms required to describe the functional relationship.
   \# of variables = 6 (t, d, V, g, ρ, μ)
   \# of reference dimensions = 3 (T, L, M)
   (Note that the number of reference dimensions and the number of basic dimensions are equal for this problem.)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 6 – 3 = 3

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
   d, g, ρ (Note that the dimensions for these variables are independent.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = t d^a g^b \rho^c$$

$$\Rightarrow M^0 L^0 T^0 = \left(\frac{T}{1}\right)\left(\frac{L}{1}\right)^a \left(\frac{L}{T^2}\right)^b \left(\frac{M}{L^3}\right)^c$$

$$\begin{array}{llll} M: & 0 = c & & a = -\tfrac{1}{2} \\ L: & 0 = a + b - 3c & \Rightarrow & b = \tfrac{1}{2} \\ T: & 0 = 1 - 2b & & c = 0 \end{array}$$

$$\therefore \Pi_1 = t\sqrt{\frac{g}{d}}$$

$$\Pi_2 = V d^a g^b \rho^c$$

$$\Rightarrow M^0 L^0 T^0 = \left(\frac{L^3}{1}\right)\left(\frac{L}{1}\right)^a \left(\frac{L}{T^2}\right)^b \left(\frac{M}{L^3}\right)^c$$

$$\begin{array}{llll} M: & 0 = c & & a = -3 \\ L: & 0 = 3 + a + b - 3c & \Rightarrow & b = 0 \\ T: & 0 = -2b & & c = 0 \end{array}$$

$$\therefore \Pi_2 = \frac{V}{d^3}$$

$$\Pi_3 = \mu d^a g^b \rho^c$$

$$\Rightarrow M^0 L^0 T^0 = \left(\frac{M}{LT}\right)\left(\frac{L}{1}\right)^a \left(\frac{L}{T^2}\right)^b \left(\frac{M}{L^3}\right)^c$$

$$\begin{array}{llll} M: & 0 = 1 + c & & a = -\tfrac{3}{2} \\ L: & 0 = -1 + a + b - 3c & \Rightarrow & b = -\tfrac{1}{2} \\ T: & 0 = -1 - 2b & & c = -1 \end{array}$$

$$\therefore \Pi_3 = \frac{\mu}{\rho d\sqrt{gd}}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[t\sqrt{\frac{g}{d}}\right] = \frac{T}{1}\frac{L^{1/2}}{T}\frac{1}{L^{1/2}} = 1 \text{ OK!}$$

$$[\Pi_2] = \left[\frac{V}{d^3}\right] = \frac{L^3}{1}\frac{1}{L^3} = 1 \text{ OK!}$$

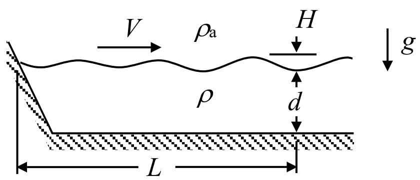
$$[\Pi_3] = \left[\frac{\mu}{\rho d\sqrt{gd}}\right] = \frac{M}{LT}\frac{L^3}{M}\frac{1}{L}\frac{T}{L^{1/2}}\frac{1}{L^{1/2}} = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{t\sqrt{\frac{g}{d}} = f_2\left(\frac{V}{d^3}, \frac{\mu}{\rho d\sqrt{gd}}\right)}$$

It is desired to determine the wave height when wind blows across a lake. The wave height, H, is assumed to be a function of the wind speed, V, the water density, ρ, the air density, ρ_a, the water depth, d, the distance from the shore, L, and the acceleration of gravity, g. Use d, V, and ρ as repeating variables to determine a suitable set of pi terms that could be used to describe this problem.

SOLUTION:



1. Write the dimensional functional relationship.
$$H = f_1(V, \rho, \rho_a, d, L, g)$$

2. Determine the basic dimensions of each parameter.
$$[H] = L$$
$$[V] = L/T$$
$$[\rho] = M/L^3$$
$$[\rho_a] = M/L^3$$
$$[d] = L$$
$$[L] = L$$
$$[g] = L/T^2$$

3. Determine the number of Π terms required to describe the functional relationship.
# of variables = 7 (H, V, ρ, ρ_a, d, L, g)
# of reference dimensions = 3 (L, T, M)

(# Π terms) = (# of variables) – (# of reference dimensions) = 7 – 3 = 4

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
d, V, ρ

5. Make the remaining non-repeating variables dimensionless using the repeating variables.
$$\Pi_1 = \frac{H}{d}$$ (by inspection)
$$\Pi_2 = \frac{\rho_a}{\rho}$$ (by inspection)
$$\Pi_3 = \frac{L}{d}$$ (by inspection)
$$\Pi_4 = \frac{V}{\sqrt{gd}}$$ (by inspection, This is a Froude number!)

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[{}^{H}\!/_{d}\right] = {}^{L}\!/_{1}\,{}^{1}\!/_{L} = 1 \quad \text{OK!}$$

$$[\Pi_2] = \left[{}^{\rho_a}\!/_{\rho}\right] = {}^{M}\!/_{L^3}\,{}^{L^3}\!/_{M} = 1 \quad \text{OK!}$$

$$[\Pi_3] = \left[{}^{L}\!/_{d}\right] = {}^{L}\!/_{1}\,{}^{1}\!/_{L} = 1 \quad \text{OK!}$$

$$[\Pi_4] = \left[{}^{V}\!/_{\sqrt{gd}}\right] = {}^{L}\!/_{T}\,{}^{T}\!/_{L^{1/2}}\,{}^{1}\!/_{L^{1/2}} = 1 \quad \text{OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{H}{d} = f_2\left(\frac{\rho_a}{\rho}, \frac{L}{d}, \frac{V}{\sqrt{gd}}\right)}$$

Small droplets of liquid are formed when a liquid jet breaks up in spray and fuel injection processes. The resulting droplet diameter, d, is thought to depend on liquid density, ρ, viscosity, μ, and surface tension, σ, as well as jet speed, V, and diameter, D. How many dimensionless ratios are required to characterize this process? Determine these ratios.

SOLUTION:

1. Write the dimensional functional relationship.

$$d = f_1\left(\rho, \mu, \sigma, V, D\right)$$

2. Determine the basic dimensions of each parameter.

$$\left[d\right] = L$$
$$\left[\rho\right] = M/L^3$$
$$\left[\mu\right] = M/LT$$
$$\left[\sigma\right] = F/L = M/T^2$$
$$\left[V\right] = L/T$$
$$\left[D\right] = L$$

3. Determine the number of Π terms required to describe the functional relationship.

\# of variables = 6 (d, ρ, μ, σ, V, D)
\# of reference dimensions = 3 (M, L, T)

$$\boxed{(\#\ \Pi \text{ terms}) = (\# \text{ of variables}) - (\# \text{ of reference dimensions}) = 6 - 3 = 3}$$

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

ρ, V, D (Note that these repeating variables have independent dimensions.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = d\rho^a V^b D^c$$
$$\Rightarrow M^0 L^0 T^0 = \left(L/1\right)\left(M/L^3\right)^a \left(L/T\right)^b \left(L/1\right)^c$$

M: $0 = a$ $\Rightarrow$ $a = 0$
T: $0 = -b$ $\Rightarrow$ $b = 0$
L: $0 = 1 - 3a + b + c$ $\Rightarrow$ $c = -1$

$$\therefore \Pi_1 = \frac{d}{D}$$

$$\Pi_2 = \mu\rho^a V^b D^c$$
$$\Rightarrow \; M^0 L^0 T^0 = \left(\frac{M}{LT}\right)\left(M/L^3\right)^a \left(L/T\right)^b \left(L/1\right)^c$$

M: $\quad 0 = 1 + a \quad \Rightarrow \quad a = -1$

T: $\quad 0 = -1 - b \quad \Rightarrow \quad b = -1$

L: $\quad 0 = -1 - 3a + b + c \quad \Rightarrow \quad c = -1$

$$\therefore \Pi_2 = \frac{\mu}{\rho V D} \text{ or } \Pi_2 = \frac{\rho V D}{\mu} \text{ (a Reynolds number!)}$$

$$\Pi_3 = \sigma\rho^a V^b D^c$$
$$\Rightarrow \; M^0 L^0 T^0 = \left(\frac{M}{T^2}\right)\left(M/L^3\right)^a \left(L/T\right)^b \left(L/1\right)^c$$

M: $\quad 0 = 1 + a \quad \Rightarrow \quad a = -1$

T: $\quad 0 = -2 - b \quad \Rightarrow \quad b = -2$

L: $\quad 0 = -3a + b + c \quad \Rightarrow \quad c = -1$

$$\therefore \Pi_3 = \frac{\sigma}{\rho V^2 D} \text{ or } \Pi_3 = \frac{\rho V^2 D}{\sigma} \text{ (a Weber number!)}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{d}{D}\right] = L/1 \; 1/L = 1 \text{ OK!}$$
$$[\Pi_2] = \left[\frac{\rho V D}{\mu}\right] = M/L^3 \; L/T \; L/1 \; LT/M = 1 \text{ OK!}$$
$$[\Pi_3] = \left[\frac{\rho V^2 D}{\sigma}\right] = M/L^3 \; L^2/T^2 \; L/1 \; T^2/M = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{d}{D} = f_2\left(\underbrace{\frac{\rho V D}{\mu}}_{\text{Reynolds \#}}, \underbrace{\frac{\rho V^2 D}{\sigma}}_{\text{Weber \#}}\right)} \tag{1}$$

Spin plays an important role in the flight trajectory of golf, Ping-Pong, and tennis balls. Therefore, it is important to know the rate at which spin decreases for a ball in flight. The aerodynamic torque, T, acting on a ball in flight, is thought to depend on flight speed, V, air density, ρ, air viscosity, μ, ball diameter, D, spin rate (angular speed), ω, and diameter of the dimples on the ball, d. Determine the dimensionless parameters that result.

SOLUTION:

1. Write the dimensional functional relationship.

$$T = f_1\left(V, \rho, \mu, D, \omega, d\right)$$

2. Determine the basic dimensions of each parameter.

$$[T] = F \cdot L = ML^2/T^2$$
$$[V] = L/T$$
$$[\rho] = M/L^3$$
$$[\mu] = M/LT$$
$$[D] = L$$
$$[\omega] = 1/T$$
$$[d] = L$$

3. Determine the number of Π terms required to describe the functional relationship.

   \# of variables = 7 (T, V, ρ, μ, D, ω, d)
   \# of reference dimensions = 3 (M, L, T)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 7 – 3 = 4

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

   ρ, V, D (Note that these repeating variables have independent dimensions.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = T\rho^a V^b D^c$$
$$\Rightarrow\ M^0 L^0 T^0 = \left(ML^2/T^2\right)\left(M/L^3\right)^a \left(L/T\right)^b \left(L/1\right)^c$$

M: $0 = 1 + a \quad \Rightarrow \quad a = -1$

T: $0 = -2 - b \quad \Rightarrow \quad b = -2$

L: $0 = 2 - 3a + b + c \quad \Rightarrow \quad c = -3$

$$\therefore \Pi_1 = \frac{T}{\rho V^2 D^3}$$

$$\Pi_2 = \mu\rho^a V^b D^c$$
$$\Rightarrow\ M^0 L^0 T^0 = \left(\frac{M}{LT}\right)\left(M/L^3\right)^a \left(L/T\right)^b \left(L/1\right)^c$$
M: $0 = 1 + a \Rightarrow a = -1$
T: $0 = -1 - b \Rightarrow b = -1$
L: $0 = -1 - 3a + b + c \Rightarrow c = -1$
$$\therefore \Pi_2 = \frac{\mu}{\rho VD} \text{ or } \Pi_2 = \frac{\rho VD}{\mu} \text{ (a Reynolds number!)}$$

$$\Pi_3 = \omega\rho^a V^b D^c$$
$$\Rightarrow\ M^0 L^0 T^0 = \left(\frac{1}{T}\right)\left(M/L^3\right)^a \left(L/T\right)^b \left(L/1\right)^c$$
M: $0 = a \Rightarrow a = 0$
T: $0 = -1 - b \Rightarrow b = -1$
L: $0 = -3a + b + c \Rightarrow c = 1$
$$\therefore \Pi_3 = \frac{\omega D}{V}$$

$$\Pi_4 = d\rho^a V^b D^c$$
$$\Rightarrow\ M^0 L^0 T^0 = \left(L/1\right)\left(M/L^3\right)^a \left(L/T\right)^b \left(L/1\right)^c$$
M: $0 = a \Rightarrow a = 0$
T: $0 = -b \Rightarrow b = 0$
L: $0 = 1 + c \Rightarrow c = -1$
$$\therefore \Pi_4 = \frac{d}{D}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{T}{\rho V^2 D^3}\right] = ML^2/T^2\ L^3/M\ T^2/L^2\ 1/L^3 = 1 \text{ OK!}$$
$$[\Pi_2] = \left[\frac{\rho VD}{\mu}\right] = M/L^3\ L/T\ L/1\ LT/M = 1 \text{ OK!}$$
$$[\Pi_3] = \left[\frac{\omega D}{V}\right] = 1/T\ L/1\ T/L = 1 \text{ OK!}$$
$$[\Pi_4] = \left[\frac{d}{D}\right] = L/1\ 1/L = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{T}{\rho V^2 D^3} = f_2\left(\underbrace{\frac{\rho VD}{\mu}}_{\text{Reynolds \#}}, \frac{\omega D}{V}, \frac{d}{D}\right)} \tag{1}$$

7.4. Dimensionless Form of the Governing Equations

Consider the dimensional form of the governing equations for an incompressible, Newtonian fluid with constant viscosity in a gravity field:

$$\text{Continuity Equation:} \quad \frac{\partial u_j}{\partial x_j} = 0 \tag{7.50}$$

$$\text{Navier-Stokes Equations:} \quad \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho g_i \tag{7.51}$$

$$\text{Thermal Energy Equation:} \rho c\left(\frac{\partial T}{\partial t} + u_j\frac{\partial T}{\partial x_j}\right) = k\frac{\partial^2 T}{\partial x_j \partial x_j} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right) \tag{7.52}$$

Note that in the thermal energy equation the internal energy has been written as the specific heat (assumed constant, since for an incompressible fluid $c_v = c_p = c$) multiplied by the temperature and the heat transfer has been assumed to be due solely to conduction (Fourier's Law with a constant conduction coefficient, k). Let's re-write these equations in dimensionless form using some characteristic quantities. The variables in the equations are made dimensionless in the following manner:

$$x_i^* := \frac{x_i}{L} \quad \Longrightarrow x_i = Lx_i^*, \tag{7.53}$$

$$\frac{\partial}{\partial x_i^*} := L\frac{\partial}{\partial x_i} \quad \Longrightarrow \frac{\partial}{\partial x_i} = \frac{1}{L}\frac{\partial}{\partial x_i^*}, \tag{7.54}$$

$$u_i^* := \frac{u_i}{U} \quad \Longrightarrow u_i = Uu_i^*, \tag{7.55}$$

$$t^* := \frac{t}{\tau} \quad \Longrightarrow t = \tau t^*, \tag{7.56}$$

$$p^* := \frac{p}{p_0} \quad \Longrightarrow p = p_0 p^*, \tag{7.57}$$

$$T^* := \frac{T}{T_0} \quad \Longrightarrow T = T_0 T^*, \tag{7.58}$$

where the superscript "*" refers to a dimensionless quantity. The quantity L represents a characteristic length for the flow of interest (e.g., a pipe diameter or the diameter of a sphere), U is a characteristic velocity (e.g., the free stream velocity or the average velocity in a pipe), τ is a characteristic time scale (e.g., the period of an oscillating boundary), p_0 is a characteristic pressure (e.g., the free stream pressure or the vapor pressure), and T_0 is a characteristic temperature (e.g., the free stream temperature). These characteristic quantities give us an estimate of the typical magnitude of the various terms in the equations. They give us insight into how a parameter might scale in a flow, e.g., we might expect the fluid velocities in a flow to scale with the incoming free stream velocity.

Now let's rewrite the governing equations using these dimensionless parameters. First start with the Continuity Equation,

$$\frac{\partial(Uu_j^*)}{\partial(Lx_j^*)} = 0, \tag{7.59}$$

$$\left(\frac{U}{L}\right)\frac{\partial u_j^*}{\partial x_j^*} = 0, \tag{7.60}$$

$$\boxed{\frac{\partial u_j^*}{\partial x_j^*} = 0}. \tag{7.61}$$

Thus, the dimensionless Continuity Equation looks identical to the dimensional Continuity Equation. Now examine the Navier-Stokes Equations,

$$\rho\left[\frac{\partial(Uu_i^*)}{\partial(\tau t^*)}+(Uu_j^*)\frac{\partial(Uu_i^*)}{\partial(Lx_j^*)}\right]=-\frac{\partial(p_0p^*)}{\partial(Lx_i^*)}+\mu\frac{\partial^2(Uu_i^*)}{\partial(Lx_j^*)\partial(Lx_j^*)}+\rho g_i, \tag{7.62}$$

$$\left(\frac{\rho U}{\tau}\right)\frac{\partial u_i^*}{\partial t^*}+\left(\frac{\rho U^2}{L}\right)u_j^*\frac{\partial u_i^*}{\partial x_j^*}=-\left(\frac{p_0}{L}\right)\frac{\partial p^*}{\partial x_i^*}+\left(\frac{\mu U}{L^2}\right)\frac{\partial^2 u_i^*}{\partial x_j^*\partial x_j^*}+\rho g_i. \tag{7.63}$$

A dimensional quantity in front of a term represents a particular characteristic force per unit volume, i.e.,

$$\frac{\rho U}{\tau} := \text{characteristic unsteady inertial force per unit volume,} \tag{7.64}$$

$$\frac{\rho U^2}{L} := \text{characteristic convective inertial force per unit volume,} \tag{7.65}$$

$$\frac{p_0}{L} := \text{characteristic pressure force per unit volume,} \tag{7.66}$$

$$\frac{\mu U}{L^2} := \text{characteristic viscous force per unit volume,} \tag{7.67}$$

$$\rho g := \text{characteristic weight per unit volume.} \tag{7.68}$$

In order to make the Navier-Stokes equation dimensionless, the convention is to divide through by the characteristic convective inertial force per unit volume term ($\rho U^2/L$),

$$\underbrace{\left(\frac{L}{\tau U}\right)}_{\text{St}}\frac{\partial u_i^*}{\partial t^*}+u_j^*\frac{\partial u_i^*}{\partial x_j^*}=-\underbrace{\left(\frac{p_0}{\rho U^2}\right)}_{\text{Eu}}\frac{\partial p^*}{\partial x_i^*}+\underbrace{\left(\frac{\mu}{\rho UL}\right)}_{=\frac{1}{\text{Re}}}\frac{\partial^2 u_i^*}{\partial x_j^*\partial x_j^*}+\underbrace{\left(\frac{gL}{U^2}\right)}_{=\frac{1}{\text{Fr}^2}}\hat{g}_i. \tag{7.69}$$

where $\hat{g}_i$ is a unit vector pointing in the direction of the gravitational acceleration. This equation is now dimensionless. Furthermore, the quantities in parentheses in front of each term are characteristic force ratios, which are given special names:

- Strouhal number, $\text{St} = \frac{L}{\tau U}$. Represents the ratio of characteristic (local or Eulerian) inertial forces to characteristic convective inertial forces. The Strouhal number is often significant in unsteady, periodic flows.
- Euler number, $\text{Eu} = \frac{p_0}{\rho U^2}$. Represents the ratio of a characteristic pressure forces to characteristic convective inertial forces. The Euler number is typically significant in flows where large changes in pressure occur. The Euler number is also often written as a pressure coefficient, c_P,

$$c_P := \frac{p-p_0}{\frac{1}{2}\rho U^2}. \tag{7.70}$$

or in flows where cavitation occurs, as the cavitation number, Ca,

$$\text{Ca} := \frac{p-p_v}{\frac{1}{2}\rho U^2}. \tag{7.71}$$

where p_v is the vapor pressure of the liquid.
- Reynolds number, $\text{Re} = \frac{\rho UL}{\mu}$. Represents the ratio of characteristic convective inertial forces to characteristic viscous forces. The Reynolds number is significant in virtually all fluid flows.
- Froude number, $\text{Fr} = \frac{U}{\sqrt{gL}}$. Represents the ratio of characteristic convective inertial forces to characteristic gravitational forces. The Froude (pronounced "'früd") number is typically significant in flows involving a free surface.

Finally, let's make the Thermal Energy Equation dimensionless following the same procedure,

$$\rho c\left[\frac{\partial(T_0T^*)}{\partial(\tau t^*)}+(Uu_j^*)\frac{\partial(T_0T^*)}{\partial(Lx_j^*)}\right)=k\frac{\partial^2(T_0T^*)}{\partial(Lx_j^*)\partial(Lx_j^*)}+\mu\left[\frac{\partial(Uu_j^*)}{\partial(Lx_i^*)}+\frac{\partial(Uu_i^*)}{\partial(Lx_j^*)}\right]\left[\frac{\partial(Uu_i^*)}{\partial(Lx_j^*)}\right], \tag{7.72}$$

$$\left(\frac{\rho cT_0}{\tau}\right)\frac{\partial T^*}{\partial t^*}+\left(\frac{\rho cUT_0}{L}\right)u_j\frac{\partial T^*}{\partial x_j^*}=\left(\frac{kT_0}{L^2}\right)\frac{\partial^2T^*}{\partial x_j^*\partial x_j^*}+\left(\frac{\mu U^2}{L^2}\right)\left(\frac{\partial u_j^*}{\partial x_i^*}+\frac{\partial u_i^*}{\partial x_j^*}\right)\left(\frac{\partial u_i^*}{\partial x_j^*}\right). \tag{7.73}$$

The expressions in parentheses in front of each term has dimensions of $M/(LT^3)$. Now make this equation dimensionless by dividing through by the quantity in front of the convective acceleration term,

$$\underbrace{\left(\frac{L}{\tau U}\right)}_{=\mathrm{St}}\frac{\partial T^*}{\partial t^*}+u_j\frac{\partial T^*}{\partial x_j^*}=\underbrace{\left(\frac{k}{c\mu}\right)}_{=\frac{1}{\mathrm{Pr}}}\underbrace{\left(\frac{\mu}{\rho UL}\right)}_{=\frac{1}{\mathrm{Re}}}\frac{\partial^2T^*}{\partial x_j^*\partial x_j^*}+\underbrace{\left(\frac{U^2}{cT_0}\right)}_{=\mathrm{Ec}}\underbrace{\left(\frac{\mu}{\rho UL}\right)}_{=\frac{1}{\mathrm{Re}}}\left(\frac{\partial u_j^*}{\partial x_i^*}+\frac{\partial u_i^*}{\partial x_j^*}\right)\left(\frac{\partial u_i^*}{\partial x_j^*}\right). \tag{7.74}$$

- Prandtl number, $\mathrm{Pr}=\frac{c\mu}{k}$. Represents the ratio of the characteristic momentum diffusivity, $\nu=\mu/\rho$, to the characteristic thermal diffusivity, $k/(\rho c)$. The Prandtl number gives a measure of how rapidly momentum diffuses through a fluid compared to the diffusion of heat. Most gases have a Prandtl number near one (heat and momentum diffuse at nearly the same rate) while water has a Prandtl number near ten (momentum diffuses faster than heat).
- Eckart number, $\mathrm{Ec}=\frac{U^2}{cT_0}$. Represents the ratio of the characteristic specific macroscopic kinetic energy, U^2, to the characteristic specific internal energy, cT_0. When the Eckart number divided by the Reynolds number is small, i.e., $\mathrm{Ec}/\mathrm{Re}\ll 1$, then the change in the fluid energy due to viscous dissipation can be neglected and the thermal energy equation becomes a balance between advection and conduction.

Additional dimensionless quantities occur when dealing with other equations of significance, e.g., the equations for a compressible fluid, and with the boundary conditions, e.g., surface tension effects or surface roughness.

The differential equation for small-amplitude vibrations of a simple beam is given by:

$$\rho A\frac{\partial^2 y}{\partial t^2}+EI\frac{\partial^4 y}{\partial x^4}=0$$

where

y ≡ vertical displacement of beam
x ≡ horizontal position
t ≡ time
ρ ≡ beam material density
A ≡ cross-sectional area
I ≡ area moment of inertia
E ≡ Young's modulus

Rewrite the differential equation in dimensionless form. Discuss the physical significance of any dimensionless terms in the resulting equation.

SOLUTION:

Re-write the variables y, x, and t in dimensionless form using other variables in the equation where $[y] = L$, $[x] = L$, and $[t] = T$. Use ρ, A, and E as repeating variables where $[\rho] = M/L^3$, $[A] = L^2$, and $[E] = F/L^2 = M/(LT^2)$.

$$y^* \equiv \frac{y}{\sqrt{A}} \qquad \left[\frac{y}{\sqrt{A}}\right]=\frac{L}{\sqrt{L^2}}=1 \text{ OK!} \tag{1}$$

$$x^* \equiv \frac{x}{\sqrt{A}} \qquad \left[\frac{x}{\sqrt{A}}\right]=\frac{L}{\sqrt{L^2}}=1 \text{ OK!} \tag{2}$$

$$t^* \equiv t\sqrt{\frac{E}{\rho A}} \qquad \left[t\sqrt{\frac{E}{\rho A}}\right]=T\sqrt{\frac{M}{LT^2}\frac{L^3}{M}\frac{1}{L^2}}=1 \text{ OK!} \tag{3}$$

Substitute into the original PDE.

$$\rho A\frac{\partial^2\left(y^*\sqrt{A}\right)}{\partial\left(t^*\sqrt{\frac{\rho A}{E}}\right)^2}+EI\frac{\partial^4\left(y^*\sqrt{A}\right)}{\partial\left(x^*\sqrt{A}\right)^4}=0$$

$$\frac{\rho A\sqrt{A}}{\frac{\rho A}{E}}\frac{\partial^2 y^*}{\partial t^{*2}}+\frac{EI\sqrt{A}}{A^2}\frac{\partial^4 y^*}{\partial x^{*4}}=0$$

$$E\sqrt{A}\frac{\partial^2 y^*}{\partial t^{*2}}+\frac{EI}{A^{3/2}}\frac{\partial^4 y^*}{\partial x^{*4}}=0$$

$$\boxed{\frac{\partial^2 y^*}{\partial t^{*2}}+\frac{I}{A^2}\frac{\partial^4 y^*}{\partial x^{*4}}=0} \tag{4}$$

The term I/A^2 is a dimensionless geometric parameter.

Note that if we let:

$$x^* \equiv \frac{x}{I^{1/4}} \tag{5}$$

$$y^* \equiv \frac{y}{\sqrt{A}} \tag{6}$$

$$t^* \equiv t\sqrt{\frac{E}{\rho A}} \tag{7}$$

then:

$$\rho A \frac{\partial^2\left(y^*\sqrt{A}\right)}{\partial\left(t^*\sqrt{\frac{\rho A}{E}}\right)^2} + EI\frac{\partial^4\left(y^*\sqrt{A}\right)}{\partial\left(x^* I^{1/4}\right)^4} = 0$$

$$\frac{\rho A\sqrt{A}}{\frac{\rho A}{E}}\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{EI\sqrt{A}}{I}\frac{\partial^4 y^*}{\partial x^{*4}} = 0$$

$$E\sqrt{A}\frac{\partial^2 y^*}{\partial t^{*2}} + E\sqrt{A}\frac{\partial^4 y^*}{\partial x^{*4}} = 0$$

$$\boxed{\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{\partial^4 y^*}{\partial x^{*4}} = 0} \tag{8}$$

7.5. Modeling and Similarity

Models are often used in fluid mechanics to predict the kinematics and dynamics of full-scale (often referred to as prototype) flows. From previous discussions of dimensional analysis, we observe that we can write the governing equations and boundary conditions of our flow in dimensionless terms (Π terms). Thus, if we have two different flows, e.g., a large-scale, prototype flow and a small scale, model flow, that have identical dimensionless parameters, then the same solution, also in terms of dimensionless parameters, will hold for both. This is extremely helpful when modeling fluid systems.

When a model and the prototype have the same dimensionless parameters, we say that they are similar. We typically discuss similarity in three categories: geometric, dynamic, and kinematic.

- Geometric similarity occurs when the model is an exact geometric replica of the prototype. In other words, all of the lengths in the model are scaled by exactly the same amount as in the prototype, as shown in Figure 7.7. Note that surface roughness may even need to be scaled if it is a significant

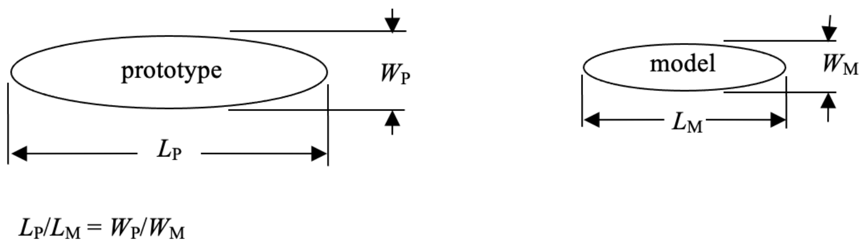


Figure 7.7. An illustration of geometric similarity between a model and prototype. All of the lengths are scaled by the same amount.

factor in the flow.
- Dynamic similarity occurs when the ratio of forces in the model is the same as the ratio of significant forces in the prototype. For example,

$$\text{(ratio of unsteady to conv. inertial forces)}_P = \text{(ratio of unsteady to conv. forces)}_M \implies \mathrm{St}_P = \mathrm{St}_M, \tag{7.75}$$

$$\text{(ratio of inertial to viscous forces)}_P = \text{(ratio of inertial to viscous forces)}_M \implies \mathrm{Re}_P = \mathrm{Re}_M, \tag{7.76}$$

$$\text{(ratio of pressure to inertial forces)}_P = \text{(ratio of pressure to inertial forces)}_M \implies \mathrm{Eu}_P = \mathrm{Eu}_M, \tag{7.77}$$

$$\text{(ratio of inertial to grav. forces)}_P = \text{(ratio of inertial to grav. forces)}_M \implies \mathrm{Fr}_P = \mathrm{Fr}_M. \tag{7.78}$$

- Kinematic similarity occurs when the prototype and model fluid velocity fields have identical streamlines (but scaled speeds). Since the forces affect the fluid motion, geometric similarity and dynamic similarity will automatically ensure kinematic similarity.

Notes:

(1) When modeling, we need to maintain similarity between all of the dimensionless parameters that are *important to the physics of the flow.* This means that we do not necessarily need to have similarity between all Π terms, just the ones that significantly affect the flow physics. Knowing a priori what dimensionless terms are important can be difficult, but with experience the task becomes easier.

(2) It is not uncommon to have the important physics of a system change at different scales. For example, surface tension forces become more pronounced at smaller geometric scales. If one was scaling up a small system in which surface tension was an important effect, but didn't consider the dynamic similarity of the surface tension force at the larger scale, then the scaling experiments would result in incorrect results.

7.5.1. Partial Similarity

True similarity may be difficult to achieve in practice. In such cases, one must either: (a) acknowledge that model testing may not be possible, or (b) relax one or more similarity requirements and use a combination of experimentation and analysis to scale the measurements.

For example, in modeling the flow around ships, both Reynolds number and Froude number similarity are important; however, both are difficult to achieve simultaneously. In such cases, one of the similarity requirements is relaxed (in boat modeling it's the Reynolds number similarity) and a combination of experiments and analysis is utilized to scale the measurements.

The two primary components of drag on a ship's hull are viscous drag, i.e., the friction of the water against the hull's surface, and wave drag, i.e., the force required to create the waves generated by the hull. The two significant dimensionless parameters corresponding to these phenomena are the,

$$\text{Reynolds number: ratio of inertial to viscous forces} \quad \mathrm{Re} = \frac{VL}{\nu}, \tag{7.79}$$

$$\text{Froude number: ratio of inertial to gravitational forces} \quad \mathrm{Fr} = \frac{V}{\sqrt{gL}}. \tag{7.80}$$

Maintaining both Reynolds number and Froude number similarity is difficult to achieve in practice,

$$\mathrm{Fr}_M = \mathrm{Fr}_P \implies \left(\frac{V}{\sqrt{gL}}\right)_M = \left(\frac{V}{\sqrt{gL}}\right)_P \implies V_M = V_P\sqrt{\frac{L_M}{L_P}}\sqrt{\frac{g_M}{g_P}} \implies V_M = V_P\sqrt{\frac{L_M}{L_P}}, \tag{7.81}$$

$$\mathrm{Re}_M = \mathrm{Re}_P \implies \left(\frac{VL}{\nu}\right)_M = \left(\frac{VL}{\nu}\right)_P \implies \nu_M = \nu_P\left(\frac{V_M}{V_P}\right)\left(\frac{L_M}{L_P}\right) \implies \nu_M = \nu_p\left(\frac{L_M}{L_P}\right)^{3/2}, \tag{7.82}$$

where the gravitational acceleration is assumed constant across scales ($g_M = g_P$). As an example, consider a scale model that has $L_P = 100L_M$, $\nu_P = \nu_{H2O} = 1\,\mathrm{cSt} \implies \nu_M = 0.001\,\mathrm{cSt}$. There is no such common model fluid available! Thus, we cannot easily maintain both Froude number and Reynolds number similarity.

How do we resolve this difficulty? In practice, Froude number similarity is maintained with water as both the prototype and model fluid (i.e., Eq. (7.81) holds). Reynolds number similarity is neglected in the experiment and instead analysis or computation is used to estimate the viscous drag contribution. The procedure is as follows:

(1) The total drag acting on the model is measured in the experiment. This drag force is usually expressed in terms of a dimensionless resistance coefficient.
(2) The viscous drag contribution to the total drag is calculated using analysis, e.g., boundary layer analysis, or computation, e.g., computational fluid dynamics.
(3) The difference between the total drag and the viscous drag is the wave drag.
(4) The viscous drag contribution to the total drag is calculated using analysis (e.g., boundary layer analysis) or computation (e.g., computational fluid dynamics).
(5) Estimate the viscous drag contribution for the prototype using analysis or computation.
(6) Sum the predicted viscous drag force (step 5) with the scaled wave drag force (step 4) to get the total prototype drag force.

As a demonstration of how well the procedure works, consider the resistance coefficient data from a 1:80 scale model test of the U.S. Navy guided missile frigate Oliver Hazard Perry (FFG-7) as shown in Figure 7.8. The error between the scaled and actual total drag force measurements is approximately ±5%.

Notes:

(1) Experimental observations have shown that in many (but not all!) cases, Reynolds number similarity may be neglected for sufficiently large Reynolds numbers. For example, consider the Moody plot in Figure 7.9, which plots the dimensionless wall friction coefficient (aka the friction factor) as a function of Reynolds number for varying dimensionless wall roughnesses. At sufficiently large Reynolds numbers, known as the "fully rough zone", the friction factor no longer is a function of the Reynolds number.

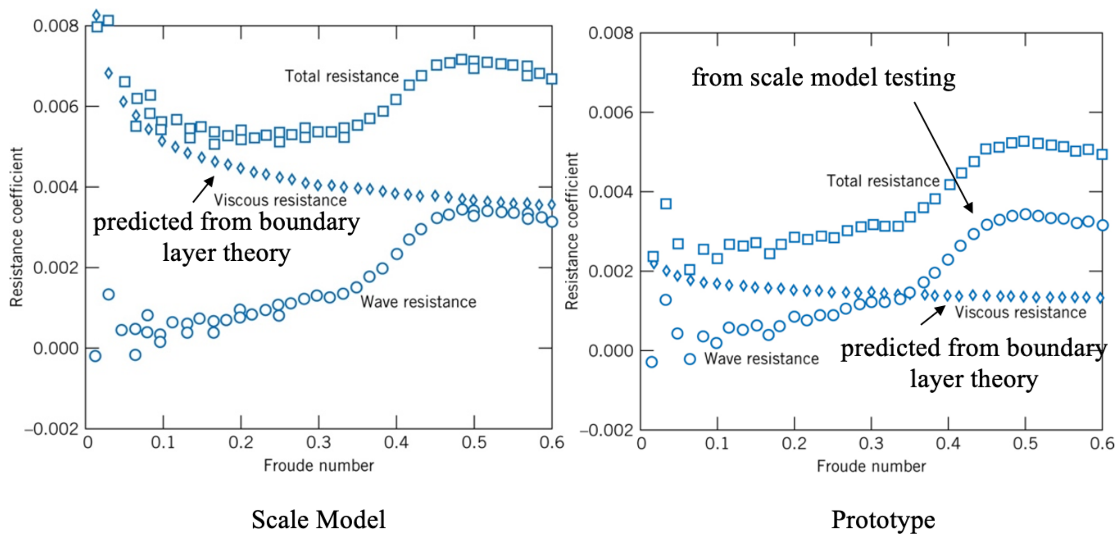


FIGURE 7.8. The resistance coefficient plotted as a function of Froude number for a scale model (left) and prototype (right). These plots are from Figs. 7.2 and 7.3 in Fox, R.W., Pritchard, P.J., and McDonald, A.T., 2008, *Introduction to Fluid Mechanics*, 7th ed., Wiley.

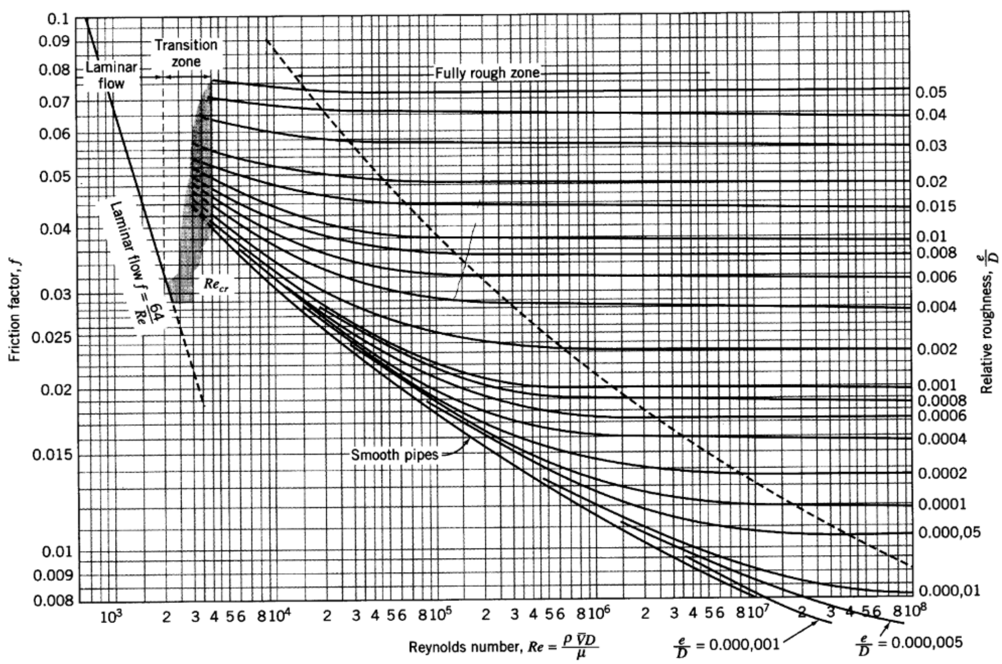


FIGURE 7.9. The Moody plot, which plots the friction coefficient as a function of Reynolds number for varying relative roughness. Note that in the fully rough zone, the friction factor is independent of the Reynolds number. This plot is from Fox, R.W., Pritchard, P.J., and McDonald, A.T., 2008, *Introduction to Fluid Mechanics*, 7th ed., Wiley.

(2) The drag coefficients for flow around sphere and circular disk are insensitive to the Reynolds number over a wide range of Reynolds numbers, as shown in Figure 7.10.

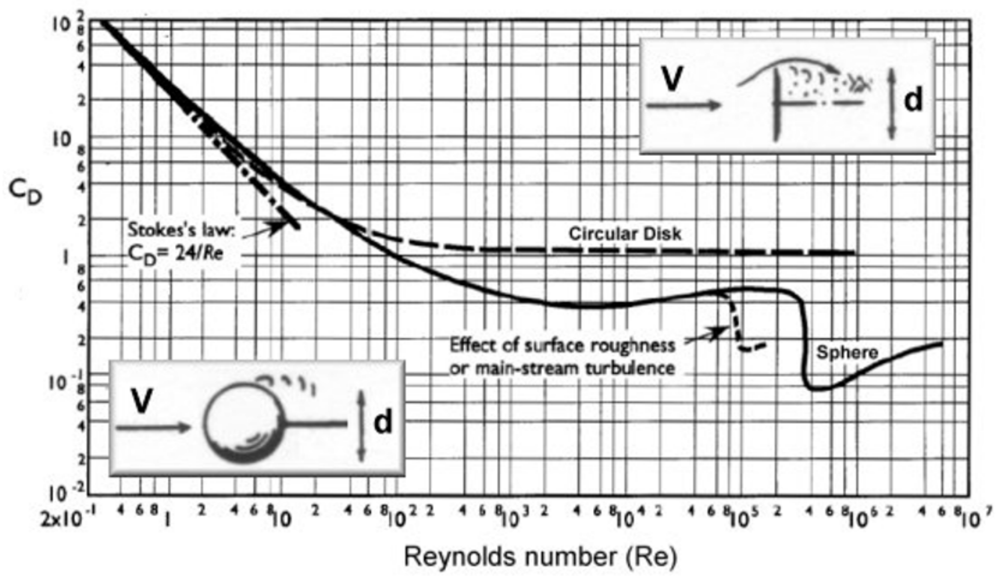


Figure 7.10. The drag coefficients for a sphere and circular disk plotted as a function of the Reynolds number. The drag coefficient is nearly independent of the Reynolds number over a wide range of Reynolds numbers. This figure is from Fox, R.W., Pritchard, P.J., and McDonald, A.T., 2008, *Introduction to Fluid Mechanics*, 7th ed., Wiley.

7.6. The Stokes Number (St) for Small Particles in a Flow

The Stokes number, St, is defined as the ratio of the particle response time, τ_p, to the fluid response time, τ_f,

$$\mathrm{St} \coloneqq \frac{\tau_p}{\tau_f}. \tag{7.83}$$

A response time is a measure of how rapidly a quantity responds to rapid changes. The Stokes number for a particle is essentially a measure of how well the particle follows fluid streamlines. If St $\ll 1$ then the particle will be able to follow the fluid streamlines whereas if St $\gg 1$ then the particle will not be able to follow sudden changes in the fluid velocity. For example, consider driving down a country road late at night during the summer when a lot of bugs are out. If the Stokes number for a bug is small, then it will follow the fluid streamlines as you drive past it and it won't impact your car (Figure 7.11). However, if the Stokes number for the bug is large, it will end up hitting your windshield since it won't be able to follow the fluid streamlines that contour around your car.

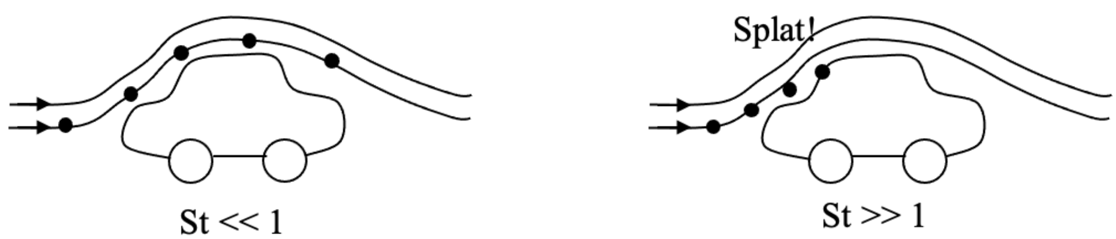


Figure 7.11. An illustration of a bug following the streamlines over a car when St $\ll 1$ (yeah!) or not following the streamlines when St $\gg 1$ (oh no!).

The particle response time can be found by considering the particle equation of motion (assuming spherical particles) for a particle with a speed slower than the surrounding fluid (so the particle accelerates),

$$\left(\rho\frac{\pi}{6}d_p^3\right)\frac{du_p}{dt} = C_D\frac{1}{2}\rho_f\left(u_f - u_p\right)^2\left(\frac{\pi d_p^2}{4}\right), \tag{7.84}$$

where ρ_p an ρ_f are the particle and fluid densities, d_p is the particle diameter, u_p and u_f are the particle and fluid velocities, t is time, and C_D is the particle drag coefficient. Define the Reynolds number for the particle using the local relative velocity,

$$\mathrm{Re}_d = \frac{\rho(u_f - u_p)d_p}{\mu_f} \quad \text{(Note that } u_f > u_p \text{ is assumed.)} \tag{7.85}$$

Substitute Eq. (7.85) into Eq. (7.84) and simplify,

$$\frac{du_p}{dt} = C_D\mathrm{Re}_d\left[\frac{\mu_f}{\rho_f(u_f - u_p)d_p}\right]\rho_f(u_f - u_p)^2\left(\frac{3}{4\rho_p d_p}\right), \tag{7.86}$$

$$= C_D\mathrm{Re}_d\left(\frac{3\mu_f}{4\rho_p d_p^2}\right)(u_f - u_p). \tag{7.87}$$

For small Reynolds numbers the drag coefficient approaches the Stokes drag,

$$C_D = \frac{24}{\mathrm{Re}_d} \quad \text{(We're now assuming that we're dealing with small particles.)}, \tag{7.88}$$

so that the particle equation of motion becomes,

$$\frac{du_p}{dt} = \frac{18\mu_f}{\rho_p d_p^2}(u_f - u_p). \tag{7.89}$$

The solution to this equation, assuming a constant fluid velocity and a particle released from rest ($u_p(t = 0) = 0$), is,

$$u_p = u_f\left[1 - \exp\left(-\frac{t}{\tau_p}\right)\right], \tag{7.90}$$

where τ_p is the particle response time,

$$\boxed{\tau_p = \frac{\rho_p d_p^2}{18\mu_f}}. \tag{7.91}$$

Let the fluid response time, τ_f, for the flow geometry be,

$$\boxed{\tau_f = \frac{L}{u_f}}, \tag{7.92}$$

where L is a typical flow dimension, e.g., the effective frontal diameter of the car in the car/bug example discussed previously. Therefore, the Stokes number for the particle is,

$$\mathrm{St} = \frac{\tau_p}{\tau_f} = \frac{\rho_p d_p^2 u_f}{18\mu_f L}. \tag{7.93}$$

Re-writing in terms of the Reynolds number, Re_L, based on the typical flow dimension, L,

$$\mathrm{Re}_L = \frac{\rho_f u_f L}{\mu_f}, \tag{7.94}$$

the Stokes number is,

$$\mathrm{St} = \frac{\rho_p d_p^2 u_f}{18\mu_f L}\left(\mathrm{Re}_L\frac{\mu_f}{\rho_f u_f L}\right), \tag{7.95}$$

$$\boxed{\mathrm{St} = \frac{\mathrm{Re}_L}{18}\left(\frac{\rho_p}{\rho_f}\right)\left(\frac{d_p}{L}\right)^2}. \tag{7.96}$$

For very small particles compared to the flow dimension, i.e., ($d_p/L \ll 1$), and moderate flow Reynolds numbers and density ratios, we observe that $\mathrm{St} \ll 1$ and the particle should follow the fluid streamlines.

The power, P, to drive an axial flow pump depends on the following variables:

density of the fluid, ρ
angular speed of the rotor, Ω
diameter of the rotor, D
head rise across the pump, ΔH ($= \Delta p/\rho g$)
volumetric flow through the pump, Q

a. Rewrite the functional relationship in dimensionless form.
b. A model scaled to one-third the size of the prototype has the following characteristics:

$\Omega_m = 900$ rpm
$D_m = 5$ in
$\Delta H_m = 10$ ft
$Q_m = 3$ ft^3/s
$P_m = 2$ hp

If the full-size pump is to run at 300 rpm, what is the power required for this pump? What head will the pump maintain? What will the volumetric flow rate be in the prototype?

SOLUTION:

1. Write the dimensional functional relationship.

$$P = f_1\left(\rho, \Omega, D, \Delta H, Q\right)$$

2. Determine the basic dimensions of each parameter.

$$[P] = \frac{FL}{T} = \frac{ML^2}{T^3}$$
$$[\rho] = \frac{M}{L^3}$$
$$[\Omega] = \frac{1}{T}$$
$$[D] = L$$
$$[\Delta H] = L$$
$$[Q] = \frac{L^3}{T}$$

3. Determine the number of Π terms required to describe the functional relationship.

\# of variables = 6 (P, ρ, Ω, D, ΔH, Q)
\# of reference dimensions = 3 (M, L, T)

(# Π terms) = (# of variables) – (# of reference dimensions) = 6 – 3 = 3

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

ρ, D, Ω (Note that these repeating variables have independent dimensions.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = P\rho^a D^b \Omega^c$$

$$\Rightarrow \; M^0 L^0 T^0 = \left(\frac{ML^2}{T^3}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{1}\right)^b \left(\frac{1}{T}\right)^c$$

$$\begin{aligned} &M: && 0 = 1 + a && \Rightarrow \; a = -1 \\ &L: && 0 = 2 - 3a + b && \Rightarrow \; b = -5 \\ &T: && 0 = -3 - c && \Rightarrow \; c = -3 \end{aligned}$$

$$\therefore \Pi_1 = \frac{P}{\rho \Omega^3 D^5}$$

$$\Pi_2 = \Delta H \rho^a D^b \Omega^c$$

$$\Rightarrow \; M^0 L^0 T^0 = \left(\frac{L}{1}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{1}\right)^b \left(\frac{1}{T}\right)^c$$

$$\begin{aligned} &M: && 0 = a && \Rightarrow \; a = 0 \\ &L: && 0 = 1 - 3a + b && \Rightarrow \; b = -1 \\ &T: && 0 = -c && \Rightarrow \; c = 0 \end{aligned}$$

$$\therefore \Pi_2 = \frac{\Delta H}{D}$$

$$\Pi_3 = Q\rho^a D^b \Omega^c$$

$$M^0 L^0 T^0 = \left(\frac{L^3}{T}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{1}\right)^b \left(\frac{1}{T}\right)^c$$

$$\begin{aligned} &M: && 0 = a && \Rightarrow \; a = 0 \\ &L: && 0 = 3 - 3a + b && \Rightarrow \; b = -3 \\ &T: && 0 = -1 - c && \Rightarrow \; c = -1 \end{aligned}$$

$$\therefore \Pi_3 = \frac{Q}{\Omega D^3}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{P}{\rho \Omega^3 D^5}\right] = \frac{ML^2}{T^3}\frac{L^3}{M}\frac{T^3}{1}\frac{1}{L^5} = 1 \quad \text{OK!}$$

$$[\Pi_2] = \left[\frac{\Delta H}{D}\right] = \frac{L}{1}\frac{1}{L} = 1 \quad \text{OK!}$$

$$[\Pi_3] = \left[\frac{Q}{\Omega D^3}\right] = \frac{L^3}{T}\frac{T}{1}\frac{1}{L^3} = 1 \quad \text{OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{P}{\rho \Omega^3 D^5} = f_2\left(\frac{\Delta H}{D}, \frac{Q}{\Omega D^3}\right)}$$

Now perform a scaling analysis assuming that the same fluid is used in the model and prototype, i.e., $\rho_M = \rho_P$. Note that since a one-third scale model is being used, $D_P/D_M = 3/1$.

$$\left(\frac{P}{\rho\Omega^3 D^5}\right)_M = \left(\frac{P}{\rho\Omega^3 D^5}\right)_P$$

$$P_P = P_M\left(\frac{\Omega_P}{\Omega_M}\right)^3\left(\frac{D_P}{D_M}\right)^5$$

$$P_P = (2\text{ hp})\left(\frac{300\text{ rpm}}{900\text{ rpm}}\right)^3\left(\frac{3}{1}\right)^5$$

$$\boxed{P_P = 18\text{ hp}}$$

$$\left(\frac{\Delta H}{D}\right)_M = \left(\frac{\Delta H}{D}\right)_P$$

$$\Delta H_P = \Delta H_M\left(\frac{D_P}{D_M}\right)$$

$$\Delta H_P = (10\text{ ft})\left(\frac{3}{1}\right)$$

$$\boxed{\Delta H_P = 30\text{ ft}}$$

$$\left(\frac{Q}{\Omega D^3}\right)_M = \left(\frac{Q}{\Omega D^3}\right)_P$$

$$Q_P = Q_M\left(\frac{\Omega_P}{\Omega_M}\right)\left(\frac{D_P}{D_M}\right)^3$$

$$\boxed{Q_P = 27\text{ ft}^3/\text{s}}$$

The drag characteristics of a blimp 5 m in diameter and 60 m long are to be studied in a wind tunnel. If the speed of the blimp through still air is 10 m/s, and if a 1/10 scale model is to be tested, what airspeed in the wind tunnel is needed for dynamic similarity? Assume the same air temperature and pressure for both the prototype and model.

SOLUTION:

For dynamic similarity, equate the model and prototype Reynolds numbers.

$$\mathrm{Re}_P = \mathrm{Re}_M$$

$$\Rightarrow \left(\frac{VD}{\nu}\right)_P = \left(\frac{VD}{\nu}\right)_M$$

Since both the model and prototype use air at the same temperature and pressure as the working fluid, $\nu_P = \nu_M$.

$$\Rightarrow V_M = V_P\left(\frac{D_P}{D_M}\right) = \left(10 \ \mathrm{m/s}\right)\left(\frac{10}{1}\right)$$

$$\boxed{\therefore V_M = 100 \ \mathrm{m/s}}$$

Note that the model speed is still low enough that Mach number effects (i.e., compressibility effects) do not come into play.

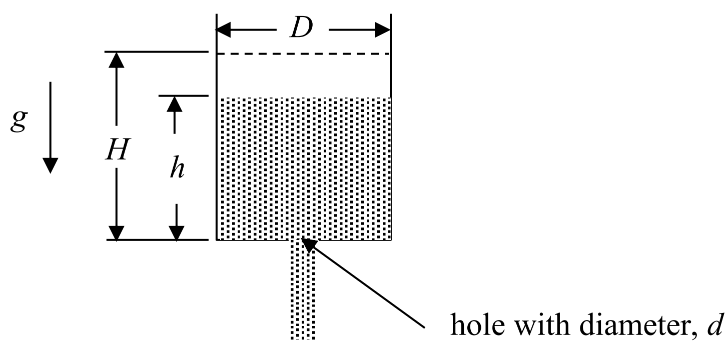
The height of the free surface, h, in a tank of diameter, D, that is draining fluid through a small hole at the bottom with diameter, d, decreases with time, t. This change in free surface height is studied experimentally with a half-scale model. For the prototype tank:

H = 16 in. (the initial height of the free surface)
D = 4.0 in.
d = 0.25 in.

Experimental data is obtained from the prototype and half-scale model and is given below:

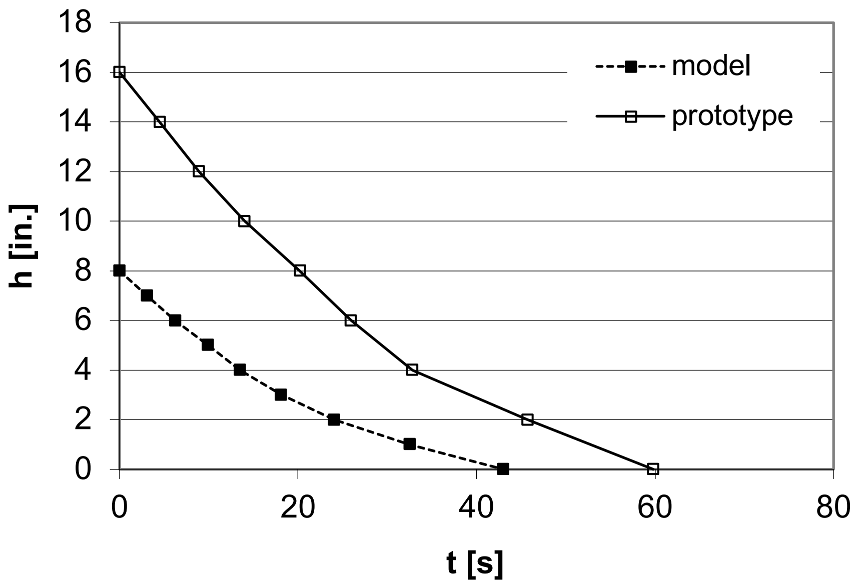
Model Data		Prototype Data	
h [in.]	t [s]	h [in.]	t [s]
8.0	0.0	16.0	0.0
7.0	3.1	14.0	4.5
6.0	6.2	12.0	8.9
5.0	9.9	10.0	14.0
4.0	13.5	8.0	20.2
3.0	18.1	6.0	25.9
2.0	24.0	4.0	32.8
1.0	32.5	2.0	45.7
0.0	43.0	0.0	59.8

1. Plot, on the same graph, the height data as a function of time for both the model and the prototype.
2. Develop a set of dimensionless parameters for this problem assuming that: $h = f(H, D, d, g, t)$
3. Re-plot, on the same graph, the height data as a function of time in non-dimensional form for both the model and prototype.



SOLUTION:

First plot the model and prototype dimensional data.



Now perform a dimensional analysis to determine the dimensionless terms describing the relationship.

1. Write the dimensional functional relationship.

$$h = f_1(H, D, d, g, t)$$

2. Determine the basic dimensions of each parameter.

$$[h] = L$$
$$[H] = L$$
$$[D] = L$$
$$[d] = L$$
$$[g] = L/T^2$$
$$[t] = T$$

3. Determine the number of Π terms required to describe the functional relationship.
   \# of variables = 6 (h, H, D, d, g, t)
   \# of reference dimensions = 2 (L, T)
   (Note that the number of reference dimensions and the number of basic dimensions are equal for this problem.)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 6 – 2 = 4

4. Choose two repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
   H, g (Note that the dimensions for these variables are independent.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = \frac{h}{H} \quad \text{(Found via inspection.)}$$

$$\Pi_2 = \frac{D}{H} \quad \text{(Found via inspection.)}$$

$$\Pi_3 = \frac{d}{H} \quad \text{(Found via inspection.)}$$

$$\Pi_4 = tH^a g^b$$

$$\Rightarrow \; L^0T^0 = \left(\frac{T}{1}\right)\left(\frac{L}{1}\right)^a\left(\frac{L}{T^2}\right)^b$$

$$\begin{aligned} L:&\quad 0 = a+b \\ T:&\quad 0 = 1-2b \end{aligned} \Rightarrow \begin{aligned} a &= -\tfrac{1}{2} \\ b &= \tfrac{1}{2} \end{aligned}$$

$$\therefore \Pi_4 = t\sqrt{\frac{g}{H}}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{h}{H}\right] = \frac{L}{1}\frac{1}{L} = 1 \quad \text{OK!}$$

$$[\Pi_2] = \left[\frac{D}{H}\right] = \frac{L}{1}\frac{1}{L} = 1 \quad \text{OK!}$$

$$[\Pi_3] = \left[\frac{d}{H}\right] = \frac{L}{1}\frac{1}{L} = 1 \quad \text{OK!}$$

$$[\Pi_4] = \left[t\sqrt{\frac{g}{H}}\right] = \frac{T}{1}\frac{L^{1/2}}{T}\frac{1}{L^{1/2}} = 1 \quad \text{OK!}$$

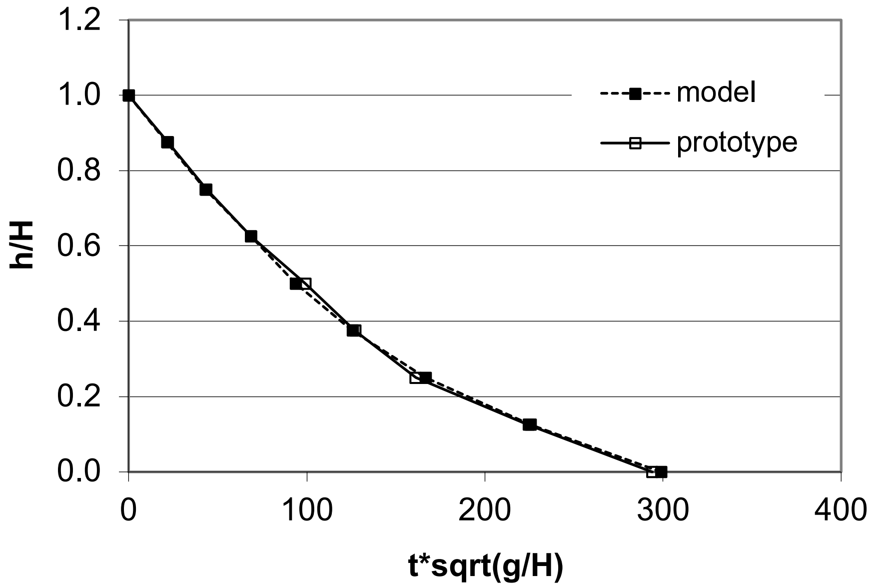
7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{h}{H} = f_2\left(\frac{D}{H}, \frac{d}{H}, t\sqrt{\frac{g}{H}}\right)}$$

Now plot the model and prototype data in dimensionless form. Note that since there is geometric similarity (the model is one-half the size of the prototype):

$$\left(\frac{d}{H}\right)_M = \left(\frac{d}{H}\right)_P \quad \text{and} \quad \left(\frac{D}{H}\right)_M = \left(\frac{D}{H}\right)_P$$

Video solution: https://www.youtube.com/watch?v=Vb-Hjv35DfU



Notice that the data collapse to a single curve when plotted in dimensionless terms.

A model test of a tractor-trailer rig is performed in a wind tunnel. The drag force, F_D, is found to depend on the frontal area, A, wind speed, V, air density, ρ, and air viscosity, μ. The model scale is 1:4 (e.g., 1 m in the model is equivalent to 4 m in the prototype), frontal area of the model is $A = 0.625\text{ m}^2$.

a. Obtain a set of dimensionless parameters suitable to characterize the model test results.
b. If the drag force on the full-scale vehicle traveling at 22.4 m/s is to be predicted from model testing, what should be the wind tunnel air speed? Assume that the air conditions are the same for the model and prototype.
c. When tested at the wind speed found in part (b), the measured drag force on the model was $F_D = 2.46$ kN. Estimate the aerodynamic drag force on the full-scale vehicle.
d. Calculate the power needed to overcome the full-scale drag force.

SOLUTION:

1. Write the dimensional functional relationship.

$$F_D = f_1(A, V, \rho, \mu)$$

2. Determine the basic dimensions of each parameter.

$$[F_D] = \frac{ML}{T^2}$$
$$[A] = L^2$$
$$[V] = \frac{L}{T}$$
$$[\rho] = \frac{M}{L^3}$$
$$[\mu] = \frac{M}{LT}$$

3. Determine the number of Π terms required to describe the functional relationship.

# of variables = 5 (F_D, A, V, ρ, μ)
# of reference dimensions = 3 (L, T, M)

(# Π terms) = (# of variables) – (# of reference dimensions) = 5 – 3 = 2

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

A, V, ρ

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = F_D A^a V^b \rho^c$$

$$\Rightarrow M^0L^0T^0 = \left(\frac{ML}{T^2}\right)\left(\frac{L^2}{1}\right)^a\left(\frac{L}{T}\right)^b\left(\frac{M}{L^3}\right)^c$$

$$\begin{array}{llcl} M: & 0 = 1 + c & & a = -1 \\ L: & 0 = 1 + 2a + b - 3c & \Rightarrow & b = -2 \\ T: & 0 = -2 - b & & c = -1 \end{array}$$

$\therefore \Pi_1 = \dfrac{F_D}{\rho V^2 A}$ (This is a drag coefficient!)

$$\Pi_2 = \mu A^a V^b \rho^c$$

$$\Rightarrow M^0L^0T^0 = \left(\frac{M}{LT}\right)\left(\frac{L^2}{1}\right)^a\left(\frac{L}{T}\right)^b\left(\frac{M}{L^3}\right)^c$$

$$\begin{array}{llcl} M: & 0 = 1 + c & & a = -\tfrac{1}{2} \\ L: & 0 = -1 + 2a + b - 3c & \Rightarrow & b = -1 \\ T: & 0 = -1 - b & & c = -1 \end{array}$$

$\therefore \Pi_2 = \dfrac{\mu}{\rho V\sqrt{A}}$ or $\Pi_2 = \dfrac{\rho V\sqrt{A}}{\mu}$ (This is a Reynolds number!)

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{F_D}{\rho V^2 A}\right] = \frac{ML}{T^2}\frac{L^3}{M}\frac{T^2}{L^2}\frac{1}{L^2} = 1 \text{ OK!}$$

$$[\Pi_2] = \left[\frac{\rho V\sqrt{A}}{\mu}\right] = \frac{M}{L^3}\frac{L}{T}\frac{L}{1}\frac{LT}{M} = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{F_D}{\rho V^2 A} = f_2\left(\frac{\rho V\sqrt{A}}{\mu}\right)} \tag{1}$$

To maintain similarity, the dimensionless terms must be the same between the model and prototype,

$$\boxed{\left(\frac{\rho V\sqrt{A}}{\mu}\right)_M = \left(\frac{\rho V\sqrt{A}}{\mu}\right)_P} \Rightarrow \boxed{\left(\frac{F_D}{\rho V^2 A}\right)_M = \left(\frac{F_D}{\rho V^2 A}\right)_P} \tag{2}$$

To determine the model testing wind speed, keep the Reynolds numbers the same between scales (the Pi term on the right hand side of Eq. (1)),

$$\left(\frac{\rho V\sqrt{A}}{\mu}\right)_M = \left(\frac{\rho V\sqrt{A}}{\mu}\right)_P, \tag{3}$$

$$V_M = V_P \underbrace{\left(\frac{\rho_P}{\rho_M}\right)}_{=1}\underbrace{\left(\frac{\mu_M}{\mu_P}\right)}_{=1}\underbrace{\left(\frac{A_P}{A_M}\right)^{1/2}}_{=(16/1)^{1/2}} \tag{4}$$

(same air properties; $L_P/L_M = 4/1 \Rightarrow A_P/A_M = (4/1)^2 = 16/1$)

$$\Rightarrow \boxed{V_M = 4V_P = 89.6 \text{ m/s}}. \tag{5}$$

The force on the prototype is found using the other Pi term,

$$\left(\frac{F_D}{\rho V^2 A}\right)_M = \left(\frac{F_D}{\rho V^2 A}\right)_P$$

$$F_D\big|_P = \underbrace{F_D\big|_M}_{=2.46\text{ kN}} \underbrace{\left(\frac{\rho_P}{\rho_M}\right)}_{=1} \underbrace{\left(\frac{V_P}{V_M}\right)^2}_{=\left(\frac{22.4\text{ m/s}}{89.6\text{ m/s}}\right)^2} \underbrace{\left(\frac{A_P}{A_M}\right)}_{=\left(\frac{16}{1}\right)} \quad \text{(Note the same air is used in both the model and prototype.)} \tag{6}$$

$$\boxed{\therefore F_D\big|_P = 2.46\text{ kN}}$$

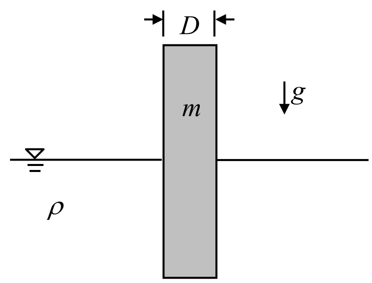
The power required to overcome the prototype drag force is:

$$P_P = \underbrace{F_{D_P}}_{=2.46\text{ kN}} \cdot \underbrace{V_P}_{=22.4\text{ m/s}} \tag{7}$$

$$\boxed{\therefore P_P = 55.1\text{ kW}}$$

A cylinder with a diameter, D, floats upright in a liquid as shown in the figure. When the cylinder is displaced slightly along its vertical axis it will oscillate about its equilibrium position with a frequency, ω. Assume that this frequency is a function of the diameter, D, the mass of the cylinder, m, the liquid density, ρ, and the acceleration due to gravity, g.

If the mass of the cylinder were doubled (assuming the same cylinder material density), by how much would the oscillation frequency change?



SOLUTION:

1. Write the dimensional functional relationship.

$$\omega = f_1\left(D, m, \rho, g\right) \tag{1}$$

2. Determine the basic dimensions of each parameter.

$$[\omega] = 1/T$$
$$[D] = L$$
$$[m] = M$$
$$[\rho] = M/L^3$$
$$[g] = L/T^2$$

3. Determine the number of Π terms required to describe the functional relationship.

\# of variables = 5 (ω, D, m, ρ, g)
\# of reference dimensions = 3 (M, L, T)

(# Π terms) = (# of variables) – (# of reference dimensions) = 5 – 3 = 2

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).

ρ, D, g (Note that these repeating variables have independent dimensions.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = \omega \rho^a D^b g^c$$
$$\Rightarrow M^0 L^0 T^0 = \left(1/T\right)\left(M/L^3\right)^a \left(L/1\right)^b \left(L/T^2\right)^c$$

M: $0 = a \quad \Rightarrow \quad a = 0$

T: $0 = -1 - 2c \quad \Rightarrow \quad c = -\frac{1}{2}$

L: $0 = -3a + b + c \quad \Rightarrow \quad b = \frac{1}{2}$

$$\therefore \Pi_1 = \omega \sqrt{\frac{D}{g}}$$

$$\Pi_2 = m\rho^a D^b g^c$$
$$\Rightarrow \; M^0 L^0 T^0 = \left(\tfrac{M}{1}\right)\left(\tfrac{M}{L^3}\right)^a \left(\tfrac{L}{1}\right)^b \left(\tfrac{L}{T^2}\right)^c$$
$$\begin{array}{lll} M: & 0 = 1 + a & \Rightarrow \; a = -1 \\ T: & 0 = -2c & \Rightarrow \; c = 0 \\ L: & 0 = -3a + b + c & \Rightarrow \; b = -3 \end{array}$$
$$\therefore \Pi_2 = \frac{m}{\rho D^3}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\omega\sqrt{\frac{D}{g}}\right] = \tfrac{1}{T}\,\tfrac{L^{1/2}}{1}\,\tfrac{T}{L^{1/2}} = 1 \;\; \text{OK!}$$
$$[\Pi_2] = \left[\frac{m}{\rho D^3}\right] = \tfrac{M}{1}\,\tfrac{L^3}{M}\,\tfrac{1}{L^3} = 1 \;\; \text{OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\omega\sqrt{\frac{D}{g}} = f_2\left(\frac{m}{\rho D^3}\right) \tag{2}$$

For similarity:

$$\left(\omega\sqrt{\frac{D}{g}}\right)_1 = \left(\omega\sqrt{\frac{D}{g}}\right)_2 \tag{3}$$
$$\left(\frac{m}{\rho D^3}\right)_1 = \left(\frac{m}{\rho D^3}\right)_2 \tag{4}$$

Assuming the same liquid (*i.e.* $\rho_1 = \rho_2$), Eq. (4) indicates:

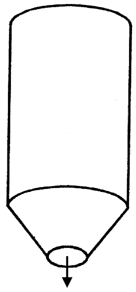
$$\frac{D_2}{D_1} = \left(\frac{m_2}{m_1}\right)^{1/3} \tag{5}$$

Using Eq. (5) with Eq. (3), assuming the same gravitational acceleration (i.e., $g_1 = g_2$), gives:

$$\frac{\omega_2}{\omega_1} = \left(\frac{D_1}{D_2}\right)^{1/2} = \left(\frac{m_1}{m_2}\right)^{1/6} \tag{6}$$

Hence, doubling the mass (i.e., $m_2 = 2m_1$) will result in a smaller frequency with $\omega_2 = 2^{-1/6}\,\omega_1$.

Hoppers are a commonly used device in the handling and storage of particulate materials. A hopper design typically consists of a bin section located above a converging section with a hole located in the bottom through which the particulate material flows (refer to the figures below).



One interesting observation with hopper flows is that the mass flow rate from the hopper exit is independent of the height of the material above the exit and the bin diameter (except when the hopper is nearly empty). The parameters that do affect the discharge rate (assuming cohesionless particles) include the hopper exit diameter, the acceleration due to gravity, the angle of the hopper walls, the friction coefficient between the particulate material and the walls and between the particles themselves, and the bulk density of the material at the discharge plane.

a. Perform a dimensional analysis to determine the dimensionless quantities that govern flow from a hopper.
b. If the same hopper and particulate material are used (i.e., the wall angle and friction properties remain the same), how will the mass flow rate from the hopper change if the hopper exit diameter is doubled?
c. Compare the discharge rate found in part (a) with the mass discharge rate expected for a liquid.

SOLUTION:

1. Write the dimensional functional relationship.

$$\dot{m} = f_1\left(D_E, g, \theta, \mu_{pp}, \mu_{pw}, \rho_b\right) \tag{1}$$

where $\dot{m}$ is the mass discharge rate from the hopper, D_E is the hopper exit diameter, g is the acceleration due to gravity, θ is the hopper wall angle, μ_{pp} and μ_{pw} are the friction coefficients between particles and between particles and the hopper walls, respectively, and ρ_b is the bulk density of the material at the hopper exit (the bulk density is the density of the particulate material including the void space between particles).

2. Determine the basic dimensions of each parameter.

$$[\dot{m}] = M/T$$
$$[D_E] = L$$
$$[g] = L/T^2$$
$$[\theta] = -$$
$$[\mu_{pp}] = [\mu_{pw}] = -$$
$$[\rho_b] = M/L^3$$

3. Determine the number of Π terms required to describe the functional relationship.

# of variables = 7 ($\dot{m}$, D_E, g, θ, μ_{pp}, μ_{pw}, ρ_b)
# of reference dimensions = 3 (L, T, M)

$$(\#\,\Pi \text{ terms}) = (\#\text{ of variables}) - (\#\text{ of reference dimensions}) = 7 - 3 = 4 \tag{2}$$

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
   D_E, g, ρ_b

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = \dot{m} D_E^a g^b \rho_b^c \tag{3}$$

$$\Rightarrow\ M^0 L^0 T^0 = \left(\frac{M}{T}\right)\left(\frac{L}{1}\right)^a\left(\frac{L}{T^2}\right)^b\left(\frac{M}{L^3}\right)^c \tag{4}$$

$$\begin{aligned} M:&\quad 0 = 1 + c & & a = -\tfrac{5}{2} \\ L:&\quad 0 = a + b - 3c & \Rightarrow\ & b = \tfrac{1}{2} \\ T:&\quad 0 = -1 - 2b & & c = -1 \end{aligned} \tag{5}$$

$$\therefore \Pi_1 = \frac{\dot{m}}{\rho_b g^{1/2} D_E^{5/2}} \tag{6}$$

$\Pi_2 = \theta$ (angles are dimensionless)

$\Pi_3 = \mu_{pp}$ (friction coefficients are dimensionless)

$\Pi_4 = \mu_{pw}$ (friction coefficients are dimensionless)

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{\dot{m}}{\rho_b g^{1/2} D_E^{5/2}}\right] = \frac{M/T}{\left(M/L^3\right)\left(L^{1/2}/T\right)\left(L^{5/2}\right)} = 1 \quad \text{OK!}$$

$[\Pi_2] = [\theta] = 1$ OK!

$[\Pi_3] = [\mu_{pp}] = 1$ OK!

$[\Pi_4] = [\mu_{pw}] = 1$ OK!

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{\dot{m}}{\rho_b g^{1/2} D_E^{5/2}} = f_2\left(\theta, \mu_{pp}, \mu_{pw}\right)} \tag{7}$$

If the wall angle and frictional properties remain constant, then doubling the exit diameter increases the mass flow rate by a factor of $2^{5/2} \approx 5.66$.

The mass discharge rate for a liquid discharging from the hopper is given by:

$$\dot{m} = \rho V_E \frac{\pi D_E^2}{4} \tag{8}$$

where ρ is the liquid density and V_E is the liquid speed at the hopper exit. The liquid speed may be found using Bernoulli's equation applied along a streamline from the hopper free surface, located a height, H, above the hopper exit, to the hopper exit. On both surfaces the fluid pressure is atmospheric and, hence:

$$V_E = \sqrt{2gH} \tag{9}$$

assuming that the kinetic energy of the upper free surface is negligible (i.e., it moves at a small velocity). Comparing Eqs. (7) and (8) shows that the mass discharge rate for a liquid depends upon the height of liquid above it while for a particulate material the discharge rate is independent of material height. In addition, the discharge rate for a particulate material is more sensitive to the hopper exit diameter (varying with $D_E^{5/2}$) than it is for a liquid (varying with D_E^2).

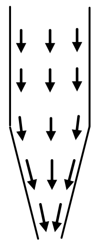
Notes:

1. Beverloo et al. (1961) observed that the experimental data for mass discharge rate from a flat-bottomed hopper is better fit using the following relation:

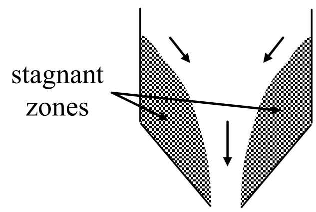
$$W = c\rho_b g^{1/2}\left(D_E - kd\right)^{5/2} \quad \textbf{Beverloo Mass Flow Rate Correlation} \tag{10}$$

where c is a constant incorporating the hopper wall angle and frictional properties (function f_2 in Eq. (7)), k is a constant that depends on the geometry of the exit and particle shape, and d is the effective diameter of the particles. The factor kd accounts for the fact that there is an annular zone at the periphery of the exit within which there are few particles. Hence, the effective exit diameter is reduced. The parameter k typically varies between 1.3 – 2.9 with a value of $k \approx 1.5$ for spherical particles. Angular particles have somewhat larger values for k. A value of $k \approx 1.4$ is a good general estimate if no discharge rate test data is available.

Beverloo et al. also observed that for funnel flow hoppers the parameter c is nearly independent of the friction coefficients, μ_{pp} and μ_{pw}, and the hopper wall angle, θ, and remains at a constant value of $c \approx 0.58$. A funnel flow hopper is one in which material remains stagnant adjacent to the hopper walls. A mass flow hopper is one in which all of the material flows simultaneously within the hopper.



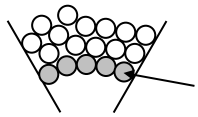
A mass-flow hopper.



A funnel-flow hopper.

3. The bulk density, ρ_b, in Eqs. (7) and (10) is <u>not</u> the bulk density of the material within the hopper. Studies have shown that the discharge rate from a hopper is independent of how the material is originally filled into the hopper. Instead, ρ_b is the bulk density of the *flowing* material. Since we often don't know the flowing bulk density of the material a priori, one can use the bulk density measured by loosely filling a container. The resulting predicted mass flow rate is typically within 5% of the measured value.

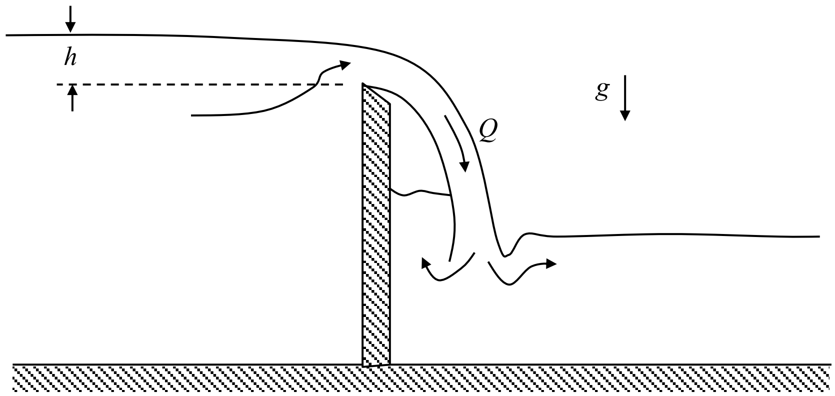
4. Blocking of the hopper exit can occur when the exit diameter is less than about six times the particle diameter. When the exit is smaller than this value, particles can form a mechanical arch that can support the force exerted by the material above it.



References:

Beverloo, W.A., Leniger, H.A., and Van de Velde, J., 1961, "The flow of granular solids through orifices," *Chemical Engineering Science*, Vol. 15, p. 260.

A 1/16th-scale model of a weir has a measured flow rate of $Q = 2.1$ ft^3/s when the upstream water height is $h = 6.3$ in. The flow rate is known to be a function of the acceleration due to gravity, g, the weir width (into the page), b, and the upstream water height, h. Furthermore, the flow rate is found to be directly proportional to the weir width, b. What is the flow rate over the prototype weir when the upstream water height is $h = 3.2$ ft.



SOLUTION:

1. Write the dimensional functional relationship.

$$Q = f_1(g, h, b)$$

2. Determine the basic dimensions of each parameter.

$$[Q] = L^3/T$$
$$[g] = L/T^2$$
$$[h] = L$$
$$[b] = L$$

3. Determine the number of Π terms required to describe the functional relationship.
   # of variables = 4 (Q, g, h, b)
   # of reference dimensions = 2 (L, T)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 4 – 2 = 2

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
   g, h

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = \frac{Q}{\sqrt{gh^5}} \qquad \text{(by inspection)}$$

$$\Pi_2 = \frac{b}{h} \qquad \text{(by inspection)}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{Q}{\sqrt{gh^5}}\right] = \frac{L^3/T}{\sqrt{L/T^2 \cdot L^5}} = 1 \text{ OK!}$$

$$[\Pi_2] = \left[\frac{b}{h}\right] = \frac{L}{L} = 1$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{Q}{\sqrt{gh^5}} = f_2\left(\frac{b}{h}\right)} \tag{1}$$

We are also told that $Q \propto b$ so that Eqn. (1) becomes:

$$\frac{Q}{\sqrt{gh^5}} = c\left(\frac{b}{h}\right) \tag{2}$$

$$\therefore \frac{Q}{b\sqrt{gh^3}} = c \tag{3}$$

where c is a constant of proportionality.

Since the right-hand side of Eq. (1) is a constant, then:

$$\left(\frac{Q}{b\sqrt{gh^3}}\right)_{\text{prototype}} = \left(\frac{Q}{b\sqrt{gh^3}}\right)_{\text{model}}$$

$$Q_{\text{prototype}} = Q_{\text{model}} \frac{\left(b\sqrt{gh^3}\right)_{\text{prototype}}}{\left(b\sqrt{gh^3}\right)_{\text{model}}}$$

The gravitational acceleration is the same for the model and prototype (i.e., $g_1 = g_2$):

$$\boxed{\therefore Q_{\text{prototype}} = Q_{\text{model}}\left(\frac{b_{\text{prototype}}}{b_{\text{model}}}\right)\left(\frac{h_{\text{prototype}}}{h_{\text{model}}}\right)^{3/2}} \tag{4}$$

Use the given data to determine Q_2.

Q_{model} = 2.1 ft^3/s
h_{model} = 6.3 in. = 0.525 ft.
$b_{\text{model}}/b_{\text{prototype}}$ = 1/16
$h_{\text{prototype}}$ = 3.2 ft
$\Rightarrow \boxed{Q_{\text{prototype}} = 506 \text{ ft}^3\text{/s}}$

In the late 1940s, much of the science concerning nuclear bombs was highly classified. In particular, information regarding the energy released in a nuclear explosion, *e.g.* the number of equivalent kilotons of TNT (nowadays the energy is measured in megatons), was top secret. G.I. Taylor, a famous fluid mechanics professor, was asked in 1941 by the British Civil Defence Research Committee of the Ministry of Home Security to predict the dynamics of a blast caused by a nuclear explosion. In his analysis, Taylor assumed that a finite amount of energy, E, is suddenly released in an infinitely concentrated form. The resulting blast wave, with a radius R, then propagates into the surrounding atmosphere, with density ρ_0 and specific heat ratio $\gamma = c_p/c_v$, as a function of time, t. Taylor's analysis resulted in a simple relationship between the blast radius as a function of the time, air density, blast energy, and specific heat ratio. Using declassified photographs of the first nuclear explosion, which occurred at the Trinity test site in New Mexico in 1945, Taylor was able to estimate the energy release to within remarkable accuracy.

Perform a dimensional analysis to determine an expression involving the blast radius as a function of the other significant parameters in the problem.

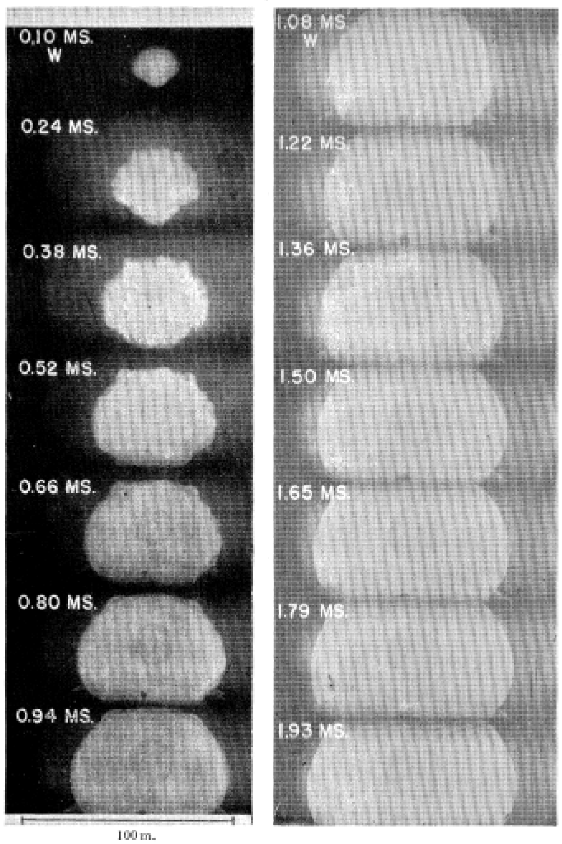


FIGURE 6. Succession of photographs of the 'ball of fire' from $t = 0{\cdot}10$ msec. to $1{\cdot}93$ msec.

References: Taylor, G., 1950, "The formation of a blast wave by a very intense explosion. I. Theoretical analyses," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 201, No. 1065, pp. 159 – 174. Taylor, G., 1950, "The formation of a blast wave by a very intense explosion. II. The atomic explosion of 1945," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 201, No. 1065, pp. 175 – 186.

SOLUTION:

1. Write the dimensional functional relationship.

$$R = f_1\left(t, E, \rho_0, \gamma\right) \tag{1}$$

2. Determine the basic dimensions of each parameter.

$[R]$ = L
$[t]$ = T
$[E]$ = $FL = ML^2/T^2$
$[\rho_0]$ = M/L^3
$[\gamma]$ = -

3. Determine the number of Π terms required to describe the functional relationship.

# of variables = 5
# of reference dimensions = 3 (L, T, M)

(# Π terms) = (# of variables) – (# of reference dimensions) = 5 – 3 = 2

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions): t, E, ρ_0.

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = R t^a E^b \rho_0^c \tag{2}$$

$$\Rightarrow \; M^0 L^0 T^0 = \left(\frac{L}{1}\right)\left(\frac{T}{1}\right)^a \left(\frac{ML^2}{T^2}\right)^b \left(\frac{M}{L^3}\right)^c$$

$$\begin{aligned} M:&\quad 0 = b + c & & a = -\tfrac{2}{5} \\ L:&\quad 0 = 1 + 2b - 3c & \Rightarrow\ & b = -\tfrac{1}{5} \\ T:&\quad 0 = a - 2b & & c = \tfrac{1}{5} \end{aligned}$$

$$\therefore \Pi_1 = \frac{R\rho_0^{1/5}}{t^{2/5} E^{1/5}} \tag{3}$$

$$\Pi_2 = \gamma \quad \text{(the specific heat ratio is already a dimensionless quantity)} \tag{4}$$

6. Verify that each Π term is, in fact, dimensionless.

$$\left[\Pi_1\right] = \left[\frac{R\rho_0^{1/5}}{t^{2/5} E^{1/5}}\right] = \frac{L}{1}\frac{M^{1/5}}{L^{3/5}}\frac{1}{T^{2/5}}\frac{T^{2/5}}{M^{1/5} L^{2/5}} = 1 \quad \text{OK!} \tag{5}$$

$$\left[\Pi_2\right] = \left[\gamma\right] = - \quad \text{OK!} \tag{6}$$

7. Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{R\rho_0^{1/5}}{t^{2/5} E^{1/5}} = f_2\left(\gamma\right)} \tag{7}$$

Note that the specific heat ratio and density of the atmosphere are well known (and assumed constant) so we could write Eq. (7) as:

$$R = c_1 t^{2/5} E^{1/5} \tag{8}$$

where c_1 is a constant (involving ρ_0 and γ). Taking the base 10 logarithm of both sides and re-arranging:

$$\tfrac{5}{2}\log_{10} R = \log_{10} t + \tfrac{5}{2}\left(\log_{10} c_1 + \tfrac{1}{5}\log_{10} E\right) \tag{9}$$

Thus, for a given explosion, the blast radius should follow a straight line when $(5/2)\log_{10}R$ is plotted as a function of $\log_{10}t$. The intercept of the line will be related to the atmospheric conditions (recall that c_1 is a function of ρ_0 and γ) and the blast energy, E. The following plots shows the measurements made from the video of the 1945 nuclear test. It's remarkable that the predictions performed by Taylor four years before the actual test are so accurate. In addition, simple radius vs. time measurements from a movie of the explosion could also easily give an estimate of the energy released. Taylor estimated the energy release to be between 23.7 kilotons of TNT. The actual energy release was estimated to be 20 kilotons.

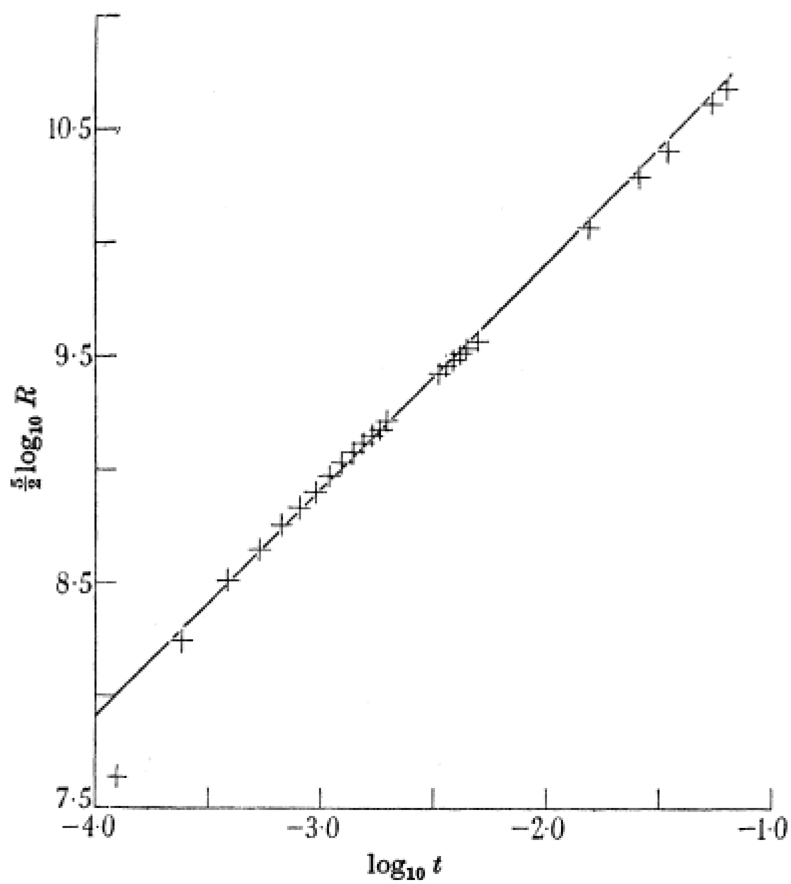


FIGURE 1. Logarithmic plot showing that $R^{\frac{5}{2}}$ is proportional to t.

The aerodynamic behavior of a flying insect is to be investigated in a wind tunnel using a ten-times scale model. If the insect flaps its wings 50 times a second when flying at 4 ft/s, determine the wind tunnel air speed and wing oscillation frequency required for dynamic similarity.

SOLUTION:

Maintain Reynolds number similarity,

$$\mathrm{Re}_M = \mathrm{Re}_P \Rightarrow \left.\frac{VL}{\nu}\right|_M = \left.\frac{VL}{\nu}\right|_P \Rightarrow V_M = V_P\left(\frac{L_P}{L_M}\right)\left(\frac{\nu_M}{\nu_P}\right) \tag{1}$$

where $L_P/L_M = 1/10$ and $\nu_M/\nu_P = 1$ (air used in both cases). Hence,

$$\boxed{V_M = (1/10)V_P = 0.4\text{ ft/s}} \tag{2}$$

Also maintain Strouhal number similarity,

$$\mathrm{St}_M = \mathrm{St}_P \Rightarrow \left.\frac{\omega L}{V}\right|_M = \left.\frac{\omega L}{V}\right|_P \Rightarrow \omega_M = \omega_P\left(\frac{L_P}{L_M}\right)\left(\frac{V_M}{V_P}\right) \tag{3}$$

where $L_P/L_M = 1/10$ and $V_M/V_P = 1/10$. Hence,

$$\boxed{\omega_M = (1/100)\omega_P = 0.5\text{ Hz}} \tag{4}$$

7.7. Review Questions

(1) Describe some of the benefits to performing a dimensional analysis of a problem.
(2) What does the Buckingham-Pi theorem state? Are the dimensionless terms resulting from the theorem unique?
(3) Describe the Method of Repeating Variables. Must this method always be followed to determine dimensionless terms?
(4) What is the difference between "basic dimensions" and "reference dimensions"?
(5) Describe the three types of similarity.
(6) Must there be exact similarity between a model and prototype in order to perform engineering modeling?
(7) In words, define the Reynolds, Froude, Strouhal, and Euler numbers.

CHAPTER 8

Solutions to the Navier-Stokes Equations

8.1. Introduction

Because there is no general method for solving a system of non-linear, partial differential equations, there are only a small number of exact solutions to the governing equations of fluid mechanics. For an incompressible fluid with constant viscosity in a gravity field, the equations governing the fluid motion are the Continuity and Navier-Stokes equations,

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{8.1}$$

$$\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u} \right] = -\boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{g}. \tag{8.2}$$

In general, we must make a number of assumptions to simplify the governing equations so they become manageable analytically. In particular, we often simplify the equations so the non-linear convective term in the Navier-Stokes equations, $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}$, is zero. Although we need to make many assumptions in determining exact solutions, the resulting solutions are still of great engineering value. They are often good models for real-world flows and they are commonly used to validate numerical codes and experimental methods.

One assumption we'll make in all of the solutions is that the flow is laminar as opposed to being turbulent or transitional. A laminar flow means that the fluid moves in smooth layers (or lamina). A turbulent flow is one in which the fluid flows in a chaotic manner with vortices of different size and nearly random spatial and temporal variations in the fluid velocity. A transitional flow is one between the laminar and turbulent states where the flow is mostly laminar, but with occasional turbulent fluctuations.

8.2. Boundary Conditions

When solving the governing equations of fluid dynamics, we'll need to apply boundary conditions (BCs) for specific flow geometries. Two common types of BCs include kinematic and dynamic boundary conditions. Kinematic boundary conditions specify the fluid velocity. One example is the no-slip boundary condition, which states that at either a solid boundary or fluid interface, the fluid velocity must be continuous,

$$\boldsymbol{u}_{\text{fluid}} = \boldsymbol{u}_{\text{boundary}}. \tag{8.3}$$

Another common kinematic boundary condition is that fluid velocities must remain finite. Dynamic boundary conditions specify that stresses must be continuous across solid or fluid interfaces,

$$\sigma_{nn,\text{fluid}} = \sigma_{nn,\text{boundary}}, \tag{8.4}$$

$$\sigma_{ns,\text{fluid}} = \sigma_{ns,\text{boundary}}, \tag{8.5}$$

where the subscripts "nn" and "ns" refer to the normal and shear stresses at the boundary.

Note that there are many graduate-level texts that review more exact solutions than will be presented in these notes. Several good references include:

- White, F.M., *Viscous Fluid Flow*, McGraw-Hill.
- Panton, R.L., *Incompressible Flow*, Wiley.
- Currie, I.G., *Fundamental Mechanics of Fluids*, McGraw-Hill.

8.3. Planar Couette-Poiseuille Flow

Consider the steady flow of an incompressible, constant viscosity Newtonian fluid between two infinitely long, parallel plates separated by a distance, h, as shown in Figure 8.1. Assume the bottom plate is fixed, but the top plate moves at a constant speed U to the right.

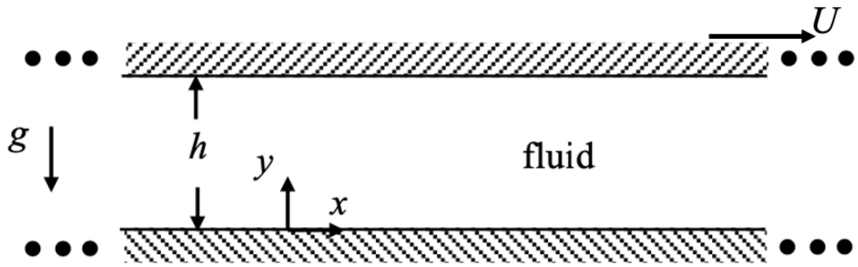


Figure 8.1. The geometry for a planar Couette-Poiseuille flow.

For this flow, we'll make the following assumptions.

(1) The flow is planar. $\implies u_z = \text{constant}$ and $\frac{\partial}{\partial z}(\dots) = 0$,
(2) The flow is at steady-state. $\implies \frac{\partial}{\partial t}(\dots) = 0$,
(3) The flow is fully-developed in the x-direction. $\implies \frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial x} = \frac{\partial u_z}{\partial x} = 0$,
(4) The only body force is due to gravity in the $-y$-direction. $\implies g_x = g_z = 0, g_y = -g$.

First examine the Continuity Equation,

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0\,(\#3)} + \frac{\partial u_y}{\partial y} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0\,(\#1)} = 0, \tag{8.6}$$

$$\frac{\partial u_y}{\partial y} = 0, \tag{8.7}$$

$$\frac{du_y}{dy} = 0, \tag{8.8}$$

$$u_y = \text{constant}. \tag{8.9}$$

The first term in the Continuity Equation is zero due to assumption #3 (the flow is fully-developed in the x-direction) and the third term is zero due to assumption #1 (the flow is planar). We go from a partial derivative in Eq. (8.7) to an ordinary derivative in Eq. (8.8) because we recognize that u_y is not a function of time (assumption #2), it's not a function of x (assumption #3), and it's not a function of z (assumption #1). Thus, at most u_y is only a function of y. Integrating Eq. (8.8) shows that u_y is at most a constant. We can determine the constant by noting that there is no flow through the boundary surfaces, i.e., $u_y(x, y = 0, h) = 0$. Thus,

$$u_y(x, y) = 0 \quad \text{(call this condition \#5)}. \tag{8.10}$$

We could have also stated that the vertical component of the velocity is zero since we're assuming a laminar flow. However, since we didn't make that assumption explicitly here, we instead had to derive the u_y component using the Continuity Equation.

Now let's examine the Navier-Stokes equation in the y-direction,

$$\rho\left(\underbrace{\frac{\partial u_y}{\partial t}}_{=0(\#2,\#5)} + u_x \underbrace{\frac{\partial u_y}{\partial x}}_{=0(\#3,\#5)} + \underbrace{u_y\frac{\partial u_y}{\partial y}}_{=0(\#5)} + u_z \underbrace{\frac{\partial u_y}{\partial z}}_{=0(\#1,\#5)}\right) = -\frac{\partial p}{\partial y} + \mu\left(\underbrace{\frac{\partial^2 u_y}{\partial x^2}}_{=0(\#3,\#5)} + \underbrace{\frac{\partial^2 u_y}{\partial y^2}}_{=0(\#5)} + \underbrace{\frac{\partial^2 u_y}{\partial z^2}}_{=0(\#1,\#5)}\right) + \rho \underbrace{g_y}_{=-g(\#4)} \tag{8.11}$$

$$\therefore \frac{\partial p}{\partial y} = -\rho g. \tag{8.12}$$

Thus, there's a hydrostatic pressure change in the y-direction. If we integrate this equation with respect to y, then,

$$p = -\rho g y + f(x, z), \tag{8.13}$$

where $f(x, z)$ is an unknown function of x and z. This unknown function can also include a constant.

Now let's examine the Navier-Stokes equation in the z-direction,

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(\#2)} + u_x \underbrace{\frac{\partial u_z}{\partial x}}_{=0(\#3)} + \underbrace{u_y}_{=0(\#5)} \frac{\partial u_z}{\partial y} + u_z \underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#1)}\right) = -\frac{\partial p}{\partial z} + \mu\left(\underbrace{\frac{\partial^2 u_z}{\partial x^2}}_{=0(\#3)} + \underbrace{\frac{\partial^2 u_z}{\partial y^2}}_{=0(\#1)} + \underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(\#1)}\right) + \rho \underbrace{g_z}_{0(\#4)} \tag{8.14}$$

$$\therefore \frac{\partial p}{\partial z} = 0. \tag{8.15}$$

Thus, there is no change in pressure in the z-direction. With this in mind, Eq. (8.13) becomes,

$$p = -\rho g y + f(x). \tag{8.16}$$

Note that if we differentiate this equation with respect to x we obtain,

$$\frac{\partial p}{\partial x} = f'(x). \tag{8.17}$$

In other words, $\partial p/\partial x$ is not a function of y. This fact will be useful in the following derivation.

Finally, examine the Navier-Stokes equation in the x-direction,

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#2)} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \underbrace{u_y}_{=0(\#5)} \frac{\partial u_x}{\partial y} + u_z \underbrace{\frac{\partial u_x}{\partial z}}_{=0(\#1)}\right) = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#3)} + \frac{\partial^2 u_x}{\partial y^2} + \underbrace{\frac{\partial^2 u_x}{\partial z^2}}_{=0(\#1)}\right) + \rho \underbrace{g_x}_{0(\#4)} \tag{8.18}$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}, \tag{8.19}$$

$$\frac{d^2 u_x}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}, \tag{8.20}$$

$$\frac{du_x}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + c_1, \tag{8.21}$$

$$u_x = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + c_1 y + c_2, \tag{8.22}$$

where c_1 and c_2 are constants of integration. In going from Eq. (8.19) to Eq. (8.20) we make use of the fact that u_x is not a function of time (assumption #2), it's not a function of x (assumption #3), and it's not a function of z (assumption #1). Thus, u_x is at most a function only of y. Another item to note in going from Eq. (8.20) to (8.22) is that we made use of Eq. (8.17), i.e., the pressure gradient in the x direction is not a function of y.

We need to apply boundary conditions to find the constants of integration in Eq. (8.22),

$$\text{no slip at } y = 0 \implies u_x(y = 0) = 0, \tag{8.23}$$

$$\text{no slip at } y = h \implies u_x(y = h) = U. \tag{8.24}$$

Using these two boundary conditions to solve for the two unknown constants in Eq. (8.22) results in the following velocity profile,

$$\boxed{u_x = U\left(\frac{y}{h}\right) + \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{y}{h}\right)\left(1 - \frac{y}{h}\right)}. \tag{8.25}$$

This type of flow is referred to as a <u>planar Couette-Poiseuille flow</u> (pronounced "'pwäz I").

Notes:

(1) The stress acting on the fluid at any point can be found from the stress-strain rate constitutive relations for a Newtonian fluid,

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{8.26}$$

(2) If we remove the pressure gradient, i.e., $\partial p/\partial x = 0$, and move the fluid using just the moving upper boundary, the velocity profile becomes linear,

$$\boxed{u_x = U\left(\frac{y}{h}\right)}. \tag{8.27}$$

This type of flow is known as a planar Couette flow.

(3) If we fix both boundaries and move the fluid using only a pressure gradient (note that flow in the positive x-direction occurs for $\partial p/\partial x < 0$), the velocity profile becomes,

$$\boxed{u_x = \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right)}. \tag{8.28}$$

This type of flow is called a planar Poiseuille flow.

(4) The average flow velocity may be found by setting the volumetric flow rate using an average velocity profile equal to the volumetric flow rate using the real velocity profile. For example, for planar Poiseuille flow the average velocity is,

$$Q = \bar{u}h = \int_{y=0}^{y=h} \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right)dy = \frac{h^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right), \tag{8.29}$$

$$\bar{u} = \frac{h^2}{12\mu}\left(-\frac{\partial p}{\partial x}\right) = \frac{2}{3}u_{\max}, \tag{8.30}$$

where,

$$u_{\max} = \frac{h^2}{8\mu}\left(-\frac{\partial p}{\partial x}\right). \tag{8.31}$$

(5) Recall that we assumed that these solutions only hold for laminar flows (resulting in $u_y = 0$). Experimentally we observe that planar Couette-Poiseuille flow remains laminar for,

$$\mathrm{Re}_h = \frac{\rho\bar{u}h}{\mu} < 1500, \tag{8.32}$$

where Re_h is the Reynolds number based on the channel height h and $\bar{u}$ is the average flow speed. The value of 1500 is only approximate and can vary considerably depending on how carefully the experiment is performed. Its value is given only as an engineering rule-of-thumb.

(6) Velocity profiles for various conditions are sketched in Figure 8.2.

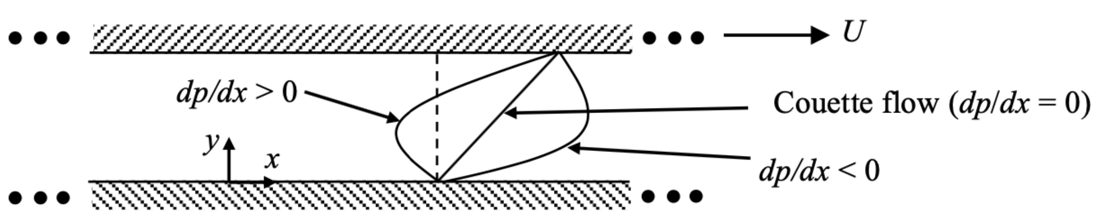


Figure 8.2. Velocity profiles for planar Couette-Poiseuille flow.

(7) There are three different ways to move fluid: (i) via a moving boundary and the no slip condition, e.g., a Couette flow, (ii) via a pressure gradient that pushes the fluid, e.g., a Poiseuille flow, and (iii) via a body force such as gravity, e.g., flow down an inclined plane.

(8) The assumption of fully-developed flow in the x direction means that the flow velocity profile doesn't change in the x direction. This assumption is reasonable for infinitely long plates. There's no reason the profile should look different in the x direction since every x location looks identical.

8.4. Poiseuille Flow

Consider the steady-state flow of an incompressible, constant viscosity, Newtonian fluid within an infinitely long, circular pipe of radius, R, as shown in Figure 8.3.

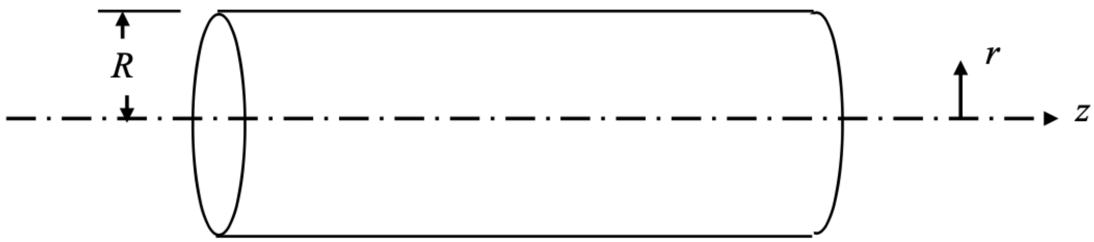


Figure 8.3. A sketch of the Poiseuille flow geometry in a circular pipe.

For this flow, we'll make the following assumptions.

(1) The flow is axi-symmetric and there is no "swirl" velocity component. $\implies \frac{\partial}{\partial \theta}(\dots) = 0 \quad \text{and} \quad u_\theta = 0$,
(2) The flow is at steady-state. $\implies \frac{\partial}{\partial t}(\dots) = 0$,
(3) The flow is fully-developed in the z-direction. $\implies \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = \frac{\partial u_z}{\partial z} = 0$,
(4) There are no body forces $\implies g_r = g_\theta = g_z = 0$.

Regarding this last assumption, we could have also assumed that gravity acts in the z-direction, e.g., a tilted pipe. Simplification of the Navier-Stokes equation in the z-direction will show that the gravitational acceleration can be combined with the pressure gradient in the z direction to form an effective pressure gradient. For the present derivation, however, we'll assume that there are no body forces.

First examine the Continuity Equation,

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#1)} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0\,(\#3)} = 0, \tag{8.33}$$

$$\frac{\partial(ru_r)}{\partial r} = 0, \tag{8.34}$$

$$\frac{d(ru_r)}{dr} = 0, \tag{8.35}$$

$$u_r = \frac{c}{r}. \tag{8.36}$$

The second term in the Continuity Equation is zero due to assumption #1 (the flow is axi-symmetric with no "swirl" velocity component) and the third term is zero due to assumption #3 (the flow is fully-developed in the z-direction). We go from a partial derivative in Eq. (8.34) to an ordinary derivative in Eq. (8.35) because we recognize that u_r is not a function of time (assumption #2), it's not a function of θ (assumption #1), and it's not a function of z (assumption #3). Thus, at most u_r is only a function of r. Integrating Eq. (8.35) gives that (ru_r) is at most a constant.

We can determine the constant by noting that there is no flow through the pipe wall, i.e., $u_r(r = R) = 0$. Thus,

$$u_r = 0 \quad \text{(call this condition \#5)}. \tag{8.37}$$

We could have also concluded that the radial component of the velocity is zero since we're assuming a laminar flow. However, since we didn't make that assumption explicitly here, we instead had to derive $u_r = 0$ using the Continuity Equation.

Now let's examine the Navier-Stokes equation in the z-direction,

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(\#2)} + \underbrace{u_r}_{=0(\#5)}\frac{\partial u_z}{\partial r} + \underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}_{=0(\#1)} + u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#3)}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\underbrace{\frac{\partial^2 u_z}{\partial \theta^2}}_{=0(\#1)} + \underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(\#3)}\right] + \rho\underbrace{g_z}_{=0(\#4)} \tag{8.38}$$

As discussed previously, if there is a body force in the z-direction, the ρg_z term could be combined with the pressure gradient to form an effective pressure gradient, i.e.,

$$\left.\frac{\partial p}{\partial z}\right|_{\text{eff}} = \frac{\partial p}{\partial z} - \rho g_z. \tag{8.39}$$

Simplifying the Navier-Stokes equation in the z-direction,

$$\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{r}{\mu}\frac{dp}{dz}, \tag{8.40}$$

$$r\frac{du_z}{dr} = \frac{r^2}{2\mu}\frac{dp}{dz} + c_1, \tag{8.41}$$

$$u_z = \frac{r^2}{4\mu}\frac{dp}{dz} + c_1\ln r + c_2, \tag{8.42}$$

where c_1 and c_2 are constants of integration. In writing Eq. (8.40) we make use of the fact that since u_r is not a function of time (assumption #2), it's not a function of θ (assumption #1), and it's not a function of z (assumption #3), u_r is at most a function only of r. Furthermore, an analysis of the Navier-Stokes equations in the θ and r directions would show that the pressure gradients in those directions are zero, i.e., $\partial p/\partial\theta = \partial p/\partial r = 0$. Thus, since the pressure doesn't vary with time (steady-state flow), then at most the pressure would be a function only of z and $\partial p/\partial z$ may be written as dp/dz.

Now let's apply boundary conditions to determine the unknown constants c_1 and c_2. First, note that the fluid velocity in a pipe must remain finite as $r \to 0$ so the constant c_1 must be zero (this is a type of kinematic boundary condition). Also, the pipe wall is fixed so we have $u_z(r = R) = 0$ (the no-slip condition). After applying these boundary conditions, Eq. (8.42) is,

$$\boxed{u_z = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1 - \left(\frac{r}{R}\right)^2\right]}. \tag{8.43}$$

This result is the velocity profile for Poiseuille Flow in a Circular Pipe, or pressure-driven flow in a circular pipe.

Notes:

(1) In the previous derivation we used the fact that the flow velocity must remain finite to set $c_1 = 0$. We could have also used a symmetry argument for this boundary condition. Because the pipe is circular, the velocity at the centerline must be a maximum or minimum, which means that the velocity gradient there is zero, i.e., $du_r/dz(r = 0) = 0$. From Eq. (8.41) we see then that $c_1 = 0$.

(2) The velocity profile is a paraboloid with the maximum velocity occurring along the centerline. The average velocity in the pipe is found by equating the volumetric flow rate using an average velocity profile to the volumetric flow rate using the actual velocity profile,

$$\bar{u}\pi R^2 = \int_{r=0}^{r=R} u_z(2\pi r dr), \tag{8.44}$$

$$\bar{u} = \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{1}{2}u_{\max}, \tag{8.45}$$

where,

$$u_{\max} = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right). \tag{8.46}$$

(3) As with planar Couette-Poiseuille flow, we can determine stresses using the constitutive relations for a Newtonian fluid. The shear stress that the pipe walls apply to the fluid, τ_w, is,

$$\tau_w = \frac{R}{2}\left(\frac{dp}{dz}\right) = \frac{-4\mu\bar{u}}{R}. \tag{8.47}$$

An alternate method for determining the average wall shear stress, which in this case is equal to the exact wall shear stress, is to balance shear forces and pressure forces on a small slice of the flow as shown in Figure 8.4.

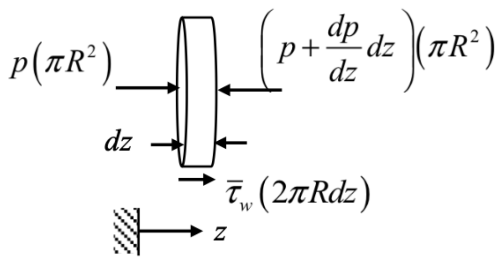


Figure 8.4. A free-body diagram of the forces on a differentially-thick slice of fluid in a circular pipe flow.

$$\sum F_z = 0 = p\pi R^2 - \left(p + \frac{dp}{dz}dz\right)\pi R^2 + \overline{\tau_w}2\pi R dz, \tag{8.48}$$

$$\overline{\tau_w} = \frac{R}{2}\frac{dp}{dz} \quad \text{The same answer as before!} \tag{8.49}$$

In engineering applications it's common to express the average shear stress in dimensionless form by dividing by the dynamic pressure. This quantity is known as the <u>(Darcy) friction factor</u>, f_D,

$$f_D \coloneqq \left|\frac{4\overline{\tau_w}}{\frac{1}{2}\rho\bar{u}^2}\right| = 64\left(\frac{\mu}{\rho\bar{u}D}\right) = \frac{64}{\mathrm{Re}_D}, \tag{8.50}$$

where $D = 2R$ is the pipe diameter and Re_D is the Reynolds number based on the pipe diameter. The Darcy friction factor commonly appears in the Moody plot for incompressible, viscous pipe flow. Note again that this solution is only valid only for a laminar flow. The condition for the flow to remain laminar is found experimentally to be,

$$\mathrm{Re}_D = \frac{\rho\bar{u}D}{\mu} < 2300. \tag{8.51}$$

(4) We can also use the general solution (before applying boundary conditions) to determine the flow between two concentric cylinders by applying different boundary conditions. For example, two fixed cylinders will have the boundary conditions: $u_z(r = R_I) = 0$ and $u_z(r = R_O) = 0$ where R_I and R_O are the inner and outer cylinder radii.

(5) Laminar flow in an elliptical cross-section pipe can be determined by considering the simplified Navier-Stokes equation in the z-direction but using Cartesian coordinates (assuming $u_x = u_y = 0$),

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dz} \quad \text{(Poisson's equation!)}, \tag{8.52}$$

where z is the coordinate along the centerline of the pipe. Note that the pipe wall boundary is the ellipse (Figure 8.5) given by,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \tag{8.53}$$

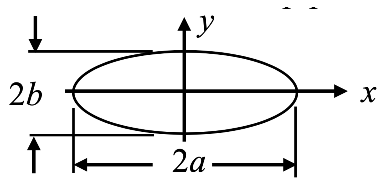


FIGURE 8.5. A pipe with an elliptical cross-section. The major axis length is $2a$ and the minor axis length is $2b$.

where $2a$ and $2b$ are the lengths of the major and minor axes. Since we must satisfy the no-slip boundary at the pipe walls, let's guess that the solution has the form,

$$u_z = \alpha\left[\left(\frac{x}{a}\right)^2 + \left(\frac{x}{b}\right)^2 - 1\right], \tag{8.54}$$

since this profile automatically satisfies the boundary condition. The quantity α is an unknown constant. To determine if this is indeed a valid solution to the fluid equations, we first note that it satisfies the Continuity Equation ($u_x = u_y = 0$ and u_z is not a function of z). If we substitute into the z-component of the Navier-Stokes equations (the Poisson's equation written previously) we find that our guess for the velocity distribution is valid if the constant α is given by,

$$\alpha\left(\frac{2}{a^2} + \frac{2}{b^2}\right) = \frac{1}{\mu}\frac{dp}{dz}, \tag{8.55}$$

$$\alpha = \frac{a^2 b^2}{2\mu(a^2+b^2)}\frac{dp}{dz}, \tag{8.56}$$

which means that the <u>velocity profile for an elliptical pipe</u> is given by,

$$\boxed{u_z = \frac{a^2 b^2}{2\mu(a^2+b^2)}\frac{dp}{dz}\left[\left(\frac{x}{a}\right)^2 + \left(\frac{x}{b}\right)^2 - 1\right]}. \tag{8.57}$$

For very complex cross-sections, we can determine the velocity profile by solving Poisson's equation numerically; however, we must keep in mind that the flow must remain laminar for the solution to be valid.

8.5. Starting Flow Between Two Parallel Plates

Consider a flow starting from rest between two parallel flat plates (Figure 8.6). The bottom plate is fixed while the top plate moves impulsively at $t > 0$ with constant velocity, U. There are no pressure gradients in the flow.

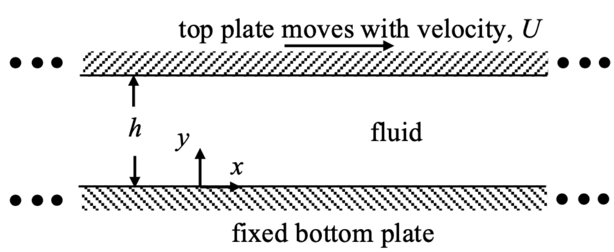


FIGURE 8.6. An illustration for the geometry used to analyze the starting flow between two parallel plates.

For this flow, we'll make the following assumptions.

(1) The flow is planar. $\implies u_z = \text{constant} \quad \text{and} \quad \frac{\partial}{\partial z}(\dots) = 0$,
(2) The flow is fully-developed in the x-direction. $\implies \frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial x} = \frac{\partial u_z}{\partial x} = 0$,
(3) There are no body forces. $\implies g_x = g_y = g_z = 0$,
(4) There is no pressure gradient in the x direction. $\implies \frac{\partial p}{\partial x} = 0$.

First examine the Continuity Equation,

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0\,(\#2)} + \frac{\partial u_y}{\partial y} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0\,(\#1)} = 0, \tag{8.58}$$

$$\frac{\partial u_y}{\partial y} = 0, \tag{8.59}$$

$$u_y = f(t). \tag{8.60}$$

In going from Eq. (8.59) to Eq. (8.60), we note that because of assumptions #1 and #2, the y component of the velocity can be at most a function of time. Since there is no flow through the walls at any time, the y-velocity must be zero,

$$u_y = 0 \quad \text{(Call this condition \#5.)} \tag{8.61}$$

Now simplify the Navier-Stokes equation in the x-direction,

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#2)} + \underbrace{u_y}_{=0(\#5)} \frac{\partial u_x}{\partial y} + u_z \underbrace{\frac{\partial u_x}{\partial z}}_{=0(\#1)}\right) = -\underbrace{\frac{\partial p}{\partial x}}_{=0(\#4)} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#2)} + \frac{\partial^2 u_x}{\partial y^2} + \underbrace{\frac{\partial^2 u_x}{\partial z^2}}_{=0(\#1)}\right) + \rho \underbrace{g_x}_{0(\#3)}, \tag{8.62}$$

$$\implies \frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2} \quad \text{where} \quad \nu = \frac{\mu}{\rho}. \tag{8.63}$$

The initial and boundary conditions for the flow are,

$$\text{no flow initially} \implies u_x(y, t=0) = 0, \tag{8.64}$$

$$\text{no slip at } y=0 \implies u_x(y=0, t>0) = 0, \tag{8.65}$$

$$\text{no slip at } y=h \implies u_x(y=h, t>0) = U. \tag{8.66}$$

Note that as $t \to \infty$, the flow profile should approach the Couette flow profile derived previously, i.e.,

$$u_x(y, t \to \infty) = U\left(\frac{y}{h}\right). \tag{8.67}$$

Hence, let's investigate a solution of the form,

$$u_x' = u_x - U\left(\frac{y}{h}\right). \tag{8.68}$$

Substituting back into Eq. (8.63) and the boundary and initial conditions,

$$\frac{\partial u_x'}{\partial t} = \nu \frac{\partial^2 u_x'}{\partial y^2}, \tag{8.69}$$

$$u_x'(y, t=0) = -U\left(\frac{y}{h}\right), \tag{8.70}$$

$$u_x'(y=0, t>0) = 0, \tag{8.71}$$

$$u_x'(y=h, t>0) = 0. \tag{8.72}$$

To solve Eq. (8.69), try a separation of variables approach,

$$u_x'(y, t) = Y(y)T(t), \tag{8.73}$$

so that, upon substitution into Eq. (8.69),

$$YT' = \nu Y'' T, \tag{8.74}$$

$$\frac{T'}{T} = \nu \frac{Y''}{Y} = -\lambda^2, \tag{8.75}$$

where λ is a constant since the only way the T and Y sides of the equation can be equal for any t and y is if both sides are equal to a constant. Solving for each part of the equation gives,

$$\frac{T'}{T} = -\lambda^2 \quad \Longrightarrow T(t) = c_1 \exp(-\lambda^2 t), \tag{8.76}$$

$$\frac{Y''}{T} = -\frac{\lambda^2}{\nu} \quad \Longrightarrow Y(y) = c_2 \sin\left(\frac{\lambda}{\sqrt{\nu}} y\right) + c_3 \cos\left(\frac{\lambda}{\sqrt{\nu}} y\right). \tag{8.77}$$

Thus, the solution has the form,

$$u_x'(y,t) = \left[c_2 \sin\left(\frac{\lambda}{\sqrt{\nu}} y\right) + c_3 \cos\left(\frac{\lambda}{\sqrt{\nu}} y\right)\right] \left[c_1 \exp(-\lambda^2 t)\right], \tag{8.78}$$

$$u_x'(y,t) = \exp(-\lambda^2 t) \left[c_4 \sin\left(\frac{\lambda}{\sqrt{\nu}} y\right) + c_5 \cos\left(\frac{\lambda}{\sqrt{\nu}} y\right)\right]. \tag{8.79}$$

In order to satisfy the boundary condition at $y = 0$ the constant c_5 must equal zero. The equation now becomes,

$$u_x'(y,t) = c \exp(-\lambda^2 t) \sin\left(\frac{\lambda}{\sqrt{\nu}} y\right). \tag{8.80}$$

In order to satisfy the boundary condition at $y = h$ without having $c = 0$, we must have,

$$\lambda = \frac{n\pi\sqrt{\nu}}{h} \quad \text{where } n \text{ is an integer.} \tag{8.81}$$

Thus,

$$u_{x,n}'(y,t) = c \exp\left(-n^2\pi^2 \frac{\nu t}{h^2}\right) \sin\left(n\pi \frac{y}{h}\right). \tag{8.82}$$

Since Eq. (8.69) and the boundary and initial conditions are linear, we can add the together the solutions in Eq. (8.82) so they satisfy the given initial condition (Eq. (8.70)). Note that we can add together the constants for the negative values of n with the positive values of n since the magnitude of the exponential and sine terms are identical, i.e.,

$$c_{(|n|)} \exp\left[-(|n|)^2\pi^2 \frac{\nu t}{h^2}\right] \sin\left[(|n|)\pi \frac{y}{h}\right] + c_{(-|n|)} \exp\left[-(-|n|)^2\pi^2 \frac{\nu t}{h^2}\right] \sin\left[(-|n|)\pi \frac{y}{h}\right], \tag{8.83}$$

$$= \left[c_{(|n|)} - c_{(-|n|)}\right] \exp\left[-(|n|)^2\pi^2 \frac{\nu t}{h^2}\right] \sin\left[(|n|)\pi \frac{y}{h}\right], \tag{8.84}$$

$$= d_{(|n|)} \exp\left[-(|n|)^2\pi^2 \frac{\nu t}{h^2}\right] \sin\left[(|n|)\pi \frac{y}{h}\right]. \tag{8.85}$$

Furthermore, we needn't include $n = 0$ since it will give $u_{x,n=0}' = 0$ which doesn't contribute to the summation. Hence, the solution to Eq. (8.69) subject to the given boundary conditions (Eqs. (8.71) and (8.72)) is,

$$u_x'(y,t) = \sum_{n=1}^{\infty} d_n \exp\left(-n^2\pi^2 \frac{\nu t}{h^2}\right) \sin\left(n\pi \frac{y}{h}\right), \tag{8.86}$$

where the constants d_n are found by forcing Eq. (8.86) to satisfy the given initial condition (Eq. (8.70)). A Fourier sine series analysis at $t = 0$ gives the constants as,

$$d_n = \frac{2}{h} \int_{y=0}^{y=h} \left(-U \frac{y}{h}\right) \sin\left(n\pi \frac{y}{h}\right) dy = \frac{2U}{n\pi} \cos(n\pi) = \frac{2U}{n\pi} (-1)^n. \tag{8.87}$$

Combining Eqs. (8.68), (8.86), and (8.87) gives,

$$\boxed{\frac{u_x(y,t)}{U} = \frac{y}{h} + \frac{2}{\pi}\sum_{n-1}^{\infty}\frac{(-1)^n}{n}\exp\left(-n^2\pi^2\frac{\nu t}{h^2}\right)\sin\left(n\pi\frac{y}{h}\right)}. \tag{8.88}$$

A plot of the dimensionless velocity profile for various dimensionless times is shown in Figure 8.7.

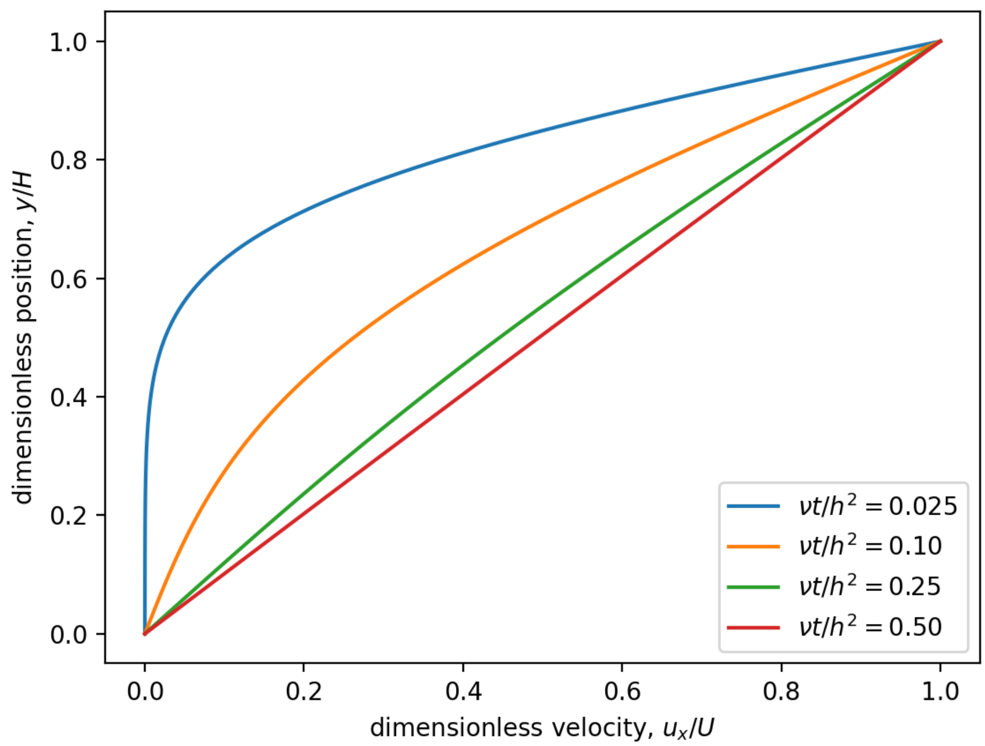


Figure 8.7. The dimensionless velocity plotted as a function of dimensionless position for different dimensionless times.

8.6. Starting Flow in a Circular Pipe

Consider the unsteady, pressure-driven flow of an incompressible, constant viscosity, Newtonian fluid within an infinitely long, circular pipe of radius, R, as shown in Figure 8.8.

For this flow, we'll make the following assumptions.

(1) The flow is axi-symmetric and there is no "swirl" velocity component. $\implies \frac{\partial}{\partial\theta}(\dots) = 0$ and $u_\theta = 0$,
(2) The flow is fully-developed in the z-direction. $\implies \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = \frac{\partial u_z}{\partial z} = 0$,
(3) There are no body forces $\implies g_r = g_\theta = g_z = 0$.

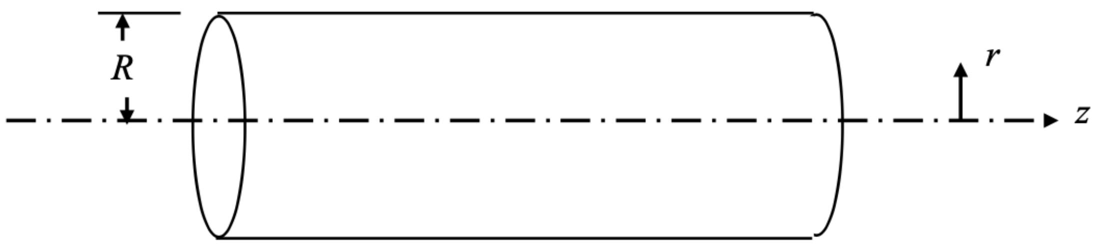


Figure 8.8. A sketch of the starting flow Poiseuille geometry in a circular pipe.

First examine the Continuity Equation,

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#1)}+\underbrace{\frac{\partial u_z}{\partial z}}_{=0\,(\#2)}=0, \tag{8.89}$$

$$\frac{\partial(ru_r)}{\partial r}=0, \tag{8.90}$$

$$u_r=\frac{f(t)}{r}. \tag{8.91}$$

Since there is no flow through the wall regardless of the time, i.e.,

$$u_r(r=R,t)=\frac{f(t)}{R}=0, \tag{8.92}$$

we must have,

$$u_r(r,t)=0 \quad \text{(Call this condition \# 4.)} \tag{8.93}$$

Now examine the Navier-Stokes equation in the z-direction,

$$\rho\left(\frac{\partial u_z}{\partial t}+\underbrace{u_r}_{=0(\#4)}\frac{\partial u_z}{\partial r}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}_{=0(\#1)}+u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#2)}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_z}{\partial \theta^2}}_{=0(\#1)}+\underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(\#2)}\right]+\rho\underbrace{g_z}_{=0(\#3)} \tag{8.94}$$

$$\rho\frac{\partial u_z}{\partial t}=-\frac{dp}{dz}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right). \tag{8.95}$$

Examining the Navier-Stokes equations in the r and θ directions shows that the pressure p is a function only of z and, thus, an ordinary derivative can be used when differentiating the pressure with respect to z.

The initial and boundary conditions for the flow are,

$$\text{no flow initially} \implies u_z(r,t=0)=0, \tag{8.96}$$

$$\text{no-slip at wall} \implies u_z(r=R,t)=0. \tag{8.97}$$

We know that as $t\to\infty$ the flow should approach the Poiseuille flow solution found in Section 8.4,

$$u_z(r,t\to\infty)=\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1-\left(\frac{r}{R}\right)^2\right]. \tag{8.98}$$

Hence, investigate a solution with the following form,

$$u_z(r,t)=\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1-\left(\frac{r}{R}\right)^2\right]-u_z'(r,t). \tag{8.99}$$

Substitute Eq. (8.99) into Eq. (8.95),

$$-\rho\frac{\partial u_z'}{\partial t} = -\frac{dp}{dz} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(-\frac{2r}{R^2}\right) - \frac{\partial u_z'}{\partial r}\right]\right\}, \tag{8.100}$$

$$-\rho\frac{\partial u_z'}{\partial t} = -\frac{dp}{dz} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(-\frac{2r^2}{R^2}\right) - r\frac{\partial u_z'}{\partial r}\right], \tag{8.101}$$

$$-\rho\frac{\partial u_z'}{\partial t} = -\frac{dp}{dz} + \frac{dp}{dz} - \mu\frac{\partial^2 u_z'}{\partial r^2} - \mu\frac{1}{r}\frac{\partial u_z'}{\partial r}, \tag{8.102}$$

$$\frac{\partial u_z'}{\partial t} = \nu\frac{\partial^2 u_z'}{\partial r^2} + \nu\frac{1}{r}\frac{\partial u_z'}{\partial r}, \tag{8.103}$$

$$\frac{\partial u_z'}{\partial t} = \nu\left(\frac{\partial^2 u_z'}{\partial r^2} + \frac{1}{r}\frac{\partial u_z'}{\partial r}\right). \tag{8.104}$$

The corresponding boundary and initial conditions are,

$$\text{no flow initially} \implies u_z'(r, t=0) = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1 - \left(\frac{r}{R}\right)^2\right], \tag{8.105}$$

$$\text{no-slip at wall} \implies u_z'(r=R, t) = 0. \tag{8.106}$$

Try a separation of variables approach to solving Eq. (8.104),

$$u_z' = f(r)g(t). \tag{8.107}$$

Substitute into Eq. (8.104) and re-arrange to put the functions of t on one side and the functions of r on the other,

$$g'f = \nu\left(gf'' + \frac{1}{r}gf'\right), \tag{8.108}$$

$$\frac{1}{\nu}\frac{g'}{g} = \frac{f''}{f} + \frac{1}{r}\frac{f'}{f} = C, \tag{8.109}$$

where C is a constant. Equation (8.109) can be written as two ODEs,

$$\frac{g'}{g} = -\nu C \implies \ln g = -\nu Ct + c_1 \implies g = c_2\exp(-\nu Ct), \tag{8.110}$$

$$f'' + \frac{1}{r}f' + Cf = 0. \tag{8.111}$$

Since we anticipate that u_z' decreases with increasing time, we can assume that $C > 0$. In order to simplify the solution to Eq. (8.111), make the change of variable, $z = \lambda(r/R)$ where $C = \lambda^2/R^2 > 0$ so that Eq. (8.111) becomes,

$$\frac{d}{dr}\frac{df}{dr} + \frac{1}{r}\frac{df}{dr} + Cf = 0, \tag{8.112}$$

$$\frac{\lambda^2}{R^2}\frac{d}{dz}\frac{df}{dz} + \frac{\lambda^2}{R^2 z}\frac{df}{dz} + \frac{\lambda^2}{R^2}f = 0, \tag{8.113}$$

$$\frac{d^2 f}{dz^2} + \frac{1}{z}\frac{df}{dz} + f = 0, \tag{8.114}$$

which can also be written as,

$$\frac{d^2 f}{dz^2} + \frac{1}{z}\frac{df}{dz} + \left(1 - \frac{k^2}{z^2}\right)f = 0, \tag{8.115}$$

where $k = 0$. The reason for writing the differential equation in the form of Eq. (8.115) is because it is the canonical form of the Bessel equation, which has the solution,

$$f = c_3 J_k(z) + c_4 Y_k(z), \tag{8.116}$$

which, for $k = 0$ and our definition for z, becomes,

$$f = c_3 J_0\left(\frac{\lambda r}{R}\right) + c_4 Y_0\left(\frac{\lambda r}{R}\right). \tag{8.117}$$

Note that J_k and Y_k are called Bessel functions of the first and second kind. The Bessel function of the second kind approaches negative infinity as r approaches zero, i.e.,

$$\lim_{r\to 0}\left[Y_0\left(\frac{\lambda r}{R}\right)\right] \to -\infty, \tag{8.118}$$

which implies that $c_4 = 0$ in order to keep the velocity finite at the centerline of the pipe. Thus, the form of u_z' is,

$$u_z' = c_3 J_0\left(\frac{\lambda r}{R}\right) c_2 \exp(-\nu C t) = c_5 J_0\left(\frac{\lambda r}{R}\right)\exp\left(-\lambda^2\frac{\nu t}{R^2}\right). \tag{8.119}$$

To find the constants c_5 and λ we must use the initial and boundary conditions (Eqs. (8.105) and (8.106)). Note also that Eq. (8.119) approaches zero as t approaches infinity, which was one of the conditions specified previously. In order to satisfy Eq. (8.106), we must have,

$$J_0(\lambda_n) = 0, \tag{8.120}$$

where λ_n are the zeros of J_0 (recall that the Bessel function of the first kind is oscillatory). Thus, Eq. (8.119) now becomes,

$$u_z' = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\lambda_n r}{R}\right)\exp\left(-\lambda_n^2\frac{\nu t}{R^2}\right). \tag{8.121}$$

The constant c_5 is now written as c_n since each zero of the Bessel function may contribute a different amount to the total constant. These constants c_n can be found by using the initial condition (Eq. (8.105)),

$$\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1-\left(\frac{r}{R}\right)^2\right] = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\lambda_n r}{R}\right). \tag{8.122}$$

The solution approach to finding these c_n is complex and is not detailed in these notes. Instead, the reader is referred to Langlois and Deville, *Slow Viscous Flow* (2014) for details.

The final solution for the starting flow is,

$$\boxed{\frac{u_z(r,t)}{\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)} = \left[1-\left(\frac{r}{R}\right)^2\right] - 8\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}\frac{J_0(\lambda_n r/R)}{J_1(\lambda_n)}\exp\left(-\lambda_n^2\frac{\nu t}{R^2}\right)}, \tag{8.123}$$

where λ_n are the zeros of J_0 (refer to Eq. (8.120)).

8.7. Impulsively Started Flat Plate (aka Stokes' First Problem, aka the Rayleigh Problem)

Consider the incompressible, constant viscosity, Newtonian fluid flow resulting from the sudden movement of an infinitely long flat plate. The geometry of the problem is shown in Figure 8.9.

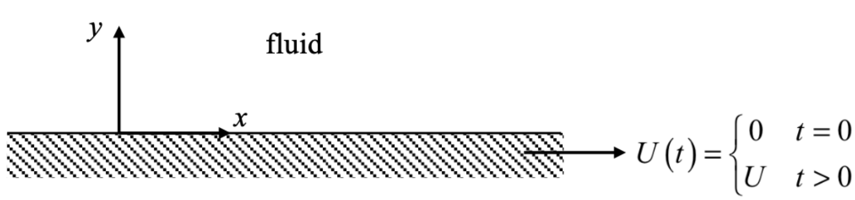


Figure 8.9. An illustration for the geometry used to analyze the impulsively started flow above a flat plate.

For this unsteady flow, we'll make the following assumptions.

(1) The flow is planar. $\implies u_z = \text{constant}$ and $\frac{\partial}{\partial z}(\dots) = 0$,
(2) The flow is fully-developed in the x-direction. $\implies \frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial x} = \frac{\partial u_z}{\partial x} = 0$,
(3) There are no body forces. $\implies g_x = g_y = g_z = 0$,
(4) There is no pressure gradient in the x-direction. $\implies \frac{\partial p}{\partial x} = 0$.

Simplifying the Continuity Equation using these assumptions gives $u_y = 0$. Simplifying the Navier-Stokes equation in the x-direction gives,

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}. \tag{8.124}$$

The boundary and initial conditions for this flow are,

$$\begin{aligned} &\text{no slip at the plate} && \implies u_x(y=0, t>0) = U, && (8.125)\\ &\text{finite velocity far from plate} && \implies u_x(y \to, t) \text{ remains finite}, && (8.126)\\ &\text{the fluid is initially at rest} && \implies u_x(y, t=0) = 0. && (8.127)\end{aligned}$$

Note that there is no geometric length scale in the problem, which suggests that we can use a similarity variable, η, to reduce the number of independent variables from two (t and y) to one ($\eta = \eta(y,t)$), i.e., we can convert the PDE into an ODE. We may anticipate this reduction in the number of variables by considering where and when the fluid velocity reaches some value, e.g., $u_x(y,t) = 0.4U$. It is reasonable to expect that the location, y, where the velocity reaches $0.4U$ will vary depending on t, e.g., the location y gets farther from the plate as t gets larger. Thus, u_x will not depend on the parameters y and t independently but will instead depend on some combination of y and t. To determine this combination, let's first re-write the velocity in dimensionless form using the plate velocity, U,

$$u^* := \frac{u_x}{U}, \tag{8.128}$$

so that the original PDE becomes,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \implies \frac{\partial u^*}{\partial t} = \nu \frac{\partial^2 u^*}{\partial y^2}, \tag{8.129}$$

with the boundary and initial conditions,

$$u^*(y=0, t>0) = 1, \tag{8.130}$$

$$u^*(y \to, t) \text{ remains finite}, \tag{8.131}$$

$$u^*(y, t=0) = 0. \tag{8.132}$$

Since the velocity is dimensionless, it must depend only on dimensionless quantities. The only dimensional quantities in the PDE are t, y, and ν. We can form only one dimensionless variable from these quantities, call it η, the similarity variable,

$$\eta := \frac{y}{\sqrt{4\nu t}}. \tag{8.133}$$

The "4" is added to the similarity variable for convenience in solving the resulting differential equation. The dimensionless velocity will be a function only of the similarity variable, i.e.,

$$u^* = f(\eta). \tag{8.134}$$

Re-writing the original PDE in terms of this similarity variable gives,

$$f' \frac{\partial \eta}{\partial t} = \nu f'' \left(\frac{\partial \eta}{\partial y}\right)^2, \tag{8.135}$$

where,

$$f' = \frac{df}{d\eta} \quad \text{and} \quad f'' = \frac{d^2 f}{d\eta^2}, \tag{8.136}$$

$$\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t} \quad \text{and} \quad \frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}, \tag{8.137}$$

so that the final equation becomes,

$$f'\left(-\frac{\eta}{2t}\right) = f''\frac{1}{4t}, \tag{8.138}$$
$$\implies f'' + 2\eta f' = 0, \tag{8.139}$$

subject to the boundary conditions,

$$f(\eta) = \begin{cases} 0 & \text{for} \quad \eta \to \infty \\ 1 & \text{for} \quad \eta = 0 \end{cases} \tag{8.140}$$

Note that the initial condition is subsumed into the $\eta \to \infty$ condition.

Thus, we see that by using a similarity variable (justified based on dimensional arguments), the PDE with two independent variables is reduced into a (linear) ODE. Now we must solve the ODE. Fortunately, we can solve the resulting ODE without much difficulty,

$$f'' + 2\eta f' = 0 \implies \frac{d}{d\eta}(\ln f') = -2\eta, \tag{8.141}$$
$$\implies \frac{df}{d\eta} = c_1 \exp(-\eta^2), \tag{8.142}$$
$$\implies f(\eta) = c_1 \int_0^\eta \exp(-\xi^2) d\xi + c_2, \tag{8.143}$$

where ξ is a dummy variable of integration. Applying the boundary conditions to determine the constants c_1 and c_2,

$$f(0) = 1 = c_2, \tag{8.144}$$
$$f(\infty) = 0 = c_1 \int_0^\infty \exp(-\xi^2) d\xi + 1 \implies c_1 = -\frac{2}{\sqrt{\pi}}, \tag{8.145}$$

where the indefinite integral has been evaluated. Thus, the velocity distribution for an impulsively started flat plate flow is,

$$\boxed{\frac{u_x}{U} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{y}{\sqrt{4\nu t}}} \exp(-\xi^2) d\xi = 1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)}, \tag{8.146}$$

where the integral is known as the error function (erf).

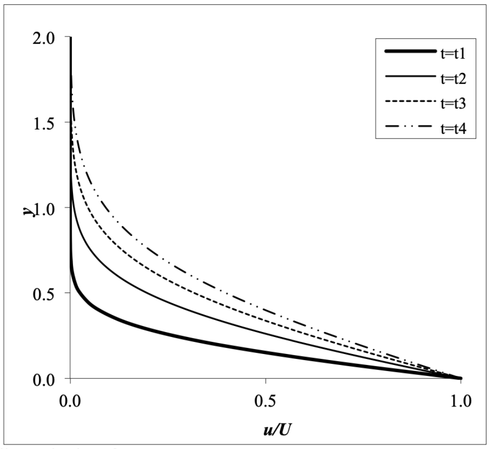
Notes:

(1) A plot of the flow profile in terms of dimensional and dimensionless quantities is shown in Figure 8.10.

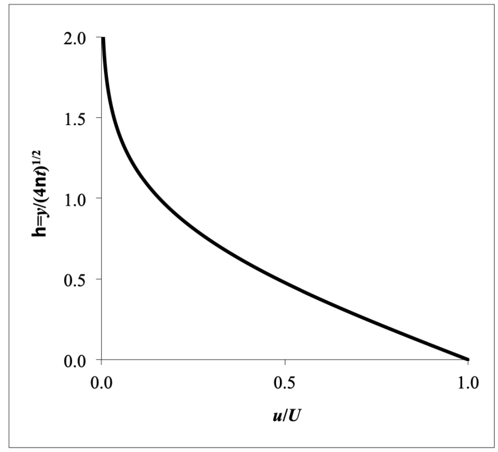
(2) As we can see from the plots shown in Note #1, the effect of the plate diffuses into the remainder of the fluid. An estimate of the depth of fluid that is affected by the movement of the plate may be found by determining the distance from the plate, δ, where the velocity is 1% that of the plate velocity, i.e., $u/U = 0.01$. This distance is,

$$\frac{u_x}{U}(\eta = 1.8) = 0.01 \implies \delta = 3.6\sqrt{\nu t}. \tag{8.147}$$

Thus, we see that the thickness of the affected layer is proportional to the square root of the kinematic viscosity and to the square root of the time (note that it is not a function of the plate velocity, U, or the absolute viscosity, μ). The distance, δ, which is also referred to as the shear layer thickness, is an important parameter that gives us an estimate of how far into the flow the effects of the boundary are felt. We will come across this parameter again, in terms of a boundary layer thickness, in Chapter 9.



(A)



(B)

Figure 8.10. Velocity profiles for an impulsively-started flat plate flow. (A) Using dimensional variables. (B) Using the dimensionless similarity variable.

The shear layer thickness after one minute in:

$$\text{air is:} \quad \delta = 10.8\,\text{cm} \quad (\nu_{\text{air}} = 0.150\,\text{cm}^2/\text{s}), \tag{8.148}$$

$$\text{water is:} \quad \delta = 2.8\,\text{cm} \quad (\nu_{\text{water}} = 0.010\,\text{cm}^2/\text{s}). \tag{8.149}$$

$$\tag{8.150}$$

Thus, we see that the effects of a boundary are felt further into a flow of air than into a flow of water in a specified amount of time!

(3) We can also use this solution to examine the flow resulting from a fluid with a uniform velocity U over a plate that has come to a sudden stop. To produce the resulting velocity profile, we note that this flow can be produced via a Galilean transformation of the problem we just investigated, $(u/U)_{\text{stopped plate}} = 1 - (u/U)_{\text{moving plate}}$. The resulting flow profile is,

$$\frac{u_x}{U} = \text{erf}\left(\frac{y}{\sqrt{4\nu t}}\right). \tag{8.151}$$

(4) The vorticity in the flow is found via,

$$\omega_z = -\frac{\partial u_x}{\partial y} = -\frac{\partial}{\partial y}\left\{U\left[1 - \text{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]\right\} = \frac{U}{\sqrt{\pi\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right). \tag{8.152}$$

A plot of the vorticity as a function of time is shown in Figure 8.11. Vorticity is created at the wall through the no-slip condition and diffuses through the rest of the fluid through the action of viscosity.

8.8. Oscillating Flat Plate Flow (aka Stokes' Second Problem, aka the Rayleigh Problem)

Consider the incompressible, constant viscosity, Newtonian fluid flow resulting from the sinusoidal oscillation of an infinitely long flat plate. The geometry of the problem is shown in Figure 8.12.

For this unsteady flow, we'll make the following assumptions.

(1) The flow is planar. $\implies u_z = \text{constant}$ and $\frac{\partial}{\partial z}(\dots) = 0$,
(2) The flow is fully-developed in the x-direction. $\implies \frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial x} = \frac{\partial u_z}{\partial x} = 0$,
(3) There are no body forces. $\implies g_x = g_y = g_z = 0$,
(4) There is no pressure gradient in the x-direction. $\implies \frac{\partial p}{\partial x} = 0$

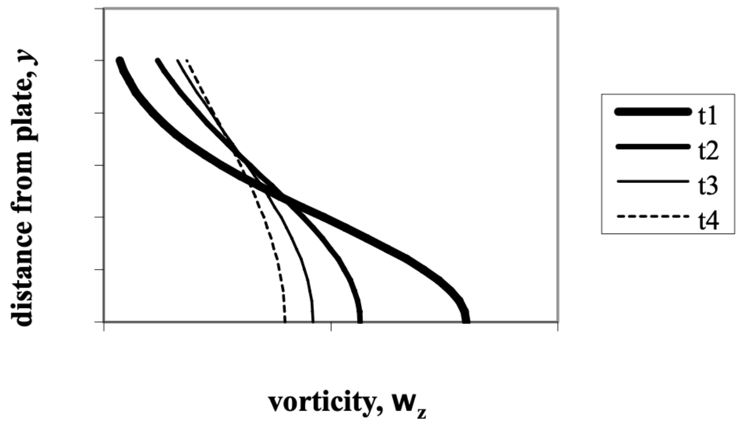


Figure 8.11. The vorticity plotted as a function of distance from the plate at different times for flow generated by an impulsively started flat plate.

Figure 8.12. An illustration for the geometry used to analyze flow above an oscillating flat plate.

Simplifying the Continuity Equation using these assumptions gives $u_y = 0$. Simplifying the Navier-Stokes equation in the x-direction gives,

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}. \tag{8.153}$$

The boundary and initial conditions for this flow are,

$$\begin{aligned} &\text{no slip at the plate} && \Longrightarrow u_x(y=0, t>0) = U\cos(\omega t), && (8.154)\\ &\text{finite velocity far from plate} && \Longrightarrow u_x(y \to, t)\text{remains finite}, && (8.155)\\ &\text{the fluid is initially at rest} && \Longrightarrow u_x(y, t=0) = 0. && (8.156) \end{aligned}$$

Since the boundary condition is time dependent and oscillatory, we might expect that the fluid velocity will have the following (separation of variables) form,

$$u_x(y,t) = f(y)\exp(i\omega t), \tag{8.157}$$

where only the real part of the velocity is relevant to the solution. Substituting into the PDE and simplifying,

$$\begin{aligned} i\omega f\exp(i\omega t) &= \nu f''\exp(i\omega t), && (8.158)\\ \Longrightarrow\ \nu f''\exp(i\omega t) &- i\omega f\exp(i\omega t) = 0, && (8.159)\\ \Longrightarrow\ f'' &- \frac{i\omega}{\nu} f = 0. && (8.160) \end{aligned}$$

Solving for f,

$$f(y) = A \exp\left(-y\sqrt{\frac{i\omega}{\nu}}\right), \tag{8.161}$$

$$\implies u_x(y,t) = A \exp\left(-y\sqrt{\frac{i\omega}{\nu}}\right)\exp(i\omega t) = A \exp\left(-y\sqrt{\frac{i\omega}{\nu}} + i\omega t\right), \tag{8.162}$$

which can be simplified to,

$$u_x(y,t) = A \exp\left(-\frac{1+i}{\sqrt{2}}y\sqrt{\frac{\omega}{\nu}} + i\omega t\right) = A \exp\left(-y\sqrt{\frac{\omega}{2\nu}}\right)\exp\left[i\left(-y\sqrt{\frac{\omega}{2\nu}} + \omega t\right)\right], \tag{8.163}$$

$$u_x(y,t) = A \exp\left(-y\sqrt{\frac{\omega}{2\nu}}\right)\left[\cos\left(-y\sqrt{\frac{\omega}{2\nu}} + \omega t\right) + i \sin\left(-y\sqrt{\frac{\omega}{2\nu}} + \omega t\right)\right], \tag{8.164}$$

$$u_x(y,t) = A \exp\left(-y\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(-y\sqrt{\frac{\omega}{2\nu}} + \omega t\right). \tag{8.165}$$

$$\tag{8.166}$$

In the last step, only the real part of the velocity component is relevant to the solution. Applying the boundary conditions gives the velocity profile for flow over an oscillating plate,

$$\boxed{u_x(y,t) = U \exp\left(-y\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right)}. \tag{8.167}$$

Notes:

(1) A plot of the velocity profile is shown in Figure 8.13.

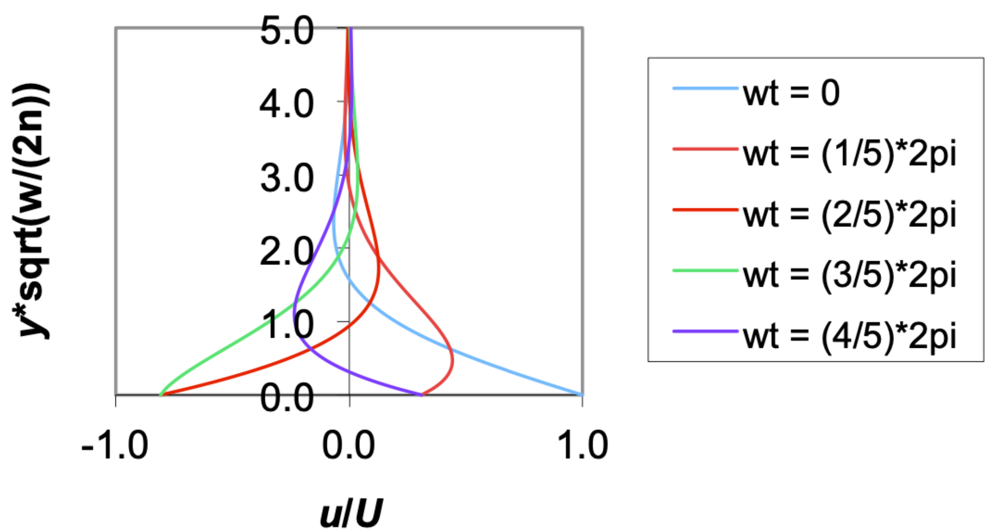


Figure 8.13. The dimensionless velocity plotted as a function of the dimensionless distance above an oscillating plate at different dimensionless times.

(2) The velocity amplitude decreases exponentially with the distance from the plate. There is a phase lag in the velocity profile compared to the plate, which is a function of distance from the plate.

(3) The region of fluid affected by the plate can be estimated by determining the y location ($y = \delta$) at which $u/U = 0.01$. We'll also assume the maximum value for the cosine function,

$$\frac{u_x}{U} = 0.01 = \exp\left(-\delta\sqrt{\frac{\omega}{2\nu}}\right)\underbrace{\cos\left(\omega t - \delta\sqrt{\frac{\omega}{2\nu}}\right)}_{=1}, \tag{8.168}$$

$$\implies \delta = -\ln(0.01)\sqrt{\frac{2\nu}{\omega}} \approx 6.51\sqrt{\frac{\nu}{\omega}}. \tag{8.169}$$

Notice that $\delta \propto \sqrt{\nu}$, just like in the impulsively started flat plate system.

(4) We can also use this solution to investigate flow oscillating far from the plate and having a fixed plate by performing a Galilean transformation on the velocity profile.

8.9. Planar Stagnation Point Flow (aka Hiemenz Flow)

Consider the flow in the vicinity of a stagnation point as shown in Figure 8.14.

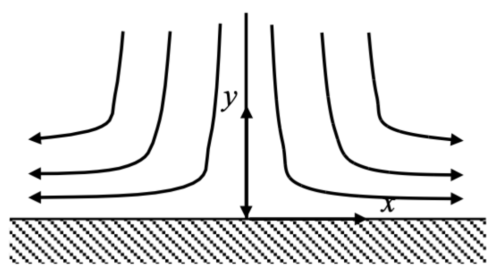


Figure 8.14. An illustration for the geometry used to analyze planar stagnation point flow.

For this flow, we'll make the following assumptions.

(1) The flow is planar. $\implies u_z = \text{constant}$ and $\frac{\partial}{\partial z}(\dots) = 0$,
(2) The flow is steady. $\implies \frac{\partial}{\partial t}(\dots) = 0$
(3) There are no body forces. $\implies g_x = g_y = g_z = 0$,

Recall from the discussion of potential flows in a different chapter that the complex potential model for this type of flow is,

$$f(z) = Az^2 = \underbrace{A(x^2 - y^2)}_{=\phi} + i\underbrace{(2Axy)}_{=\psi}, \tag{8.170}$$

where A is a constant that is proportional to the velocity far from the body, U_∞, divided by a characteristic length of the body, L,

$$A \propto \frac{U_\infty}{L}. \tag{8.171}$$

The constant of proportionality depends on the exact shape of the body. The velocity components for the flow are,

$$f'(z) = u_x - iu_y = 2Az = 2Ax + i2Ay, \tag{8.172}$$

$$\implies u_x = 2Ax \quad \text{and} \quad u_y = -2Ay. \tag{8.173}$$

The pressure distribution at any point (x, y) in the flow is,

$$p(x, y) = p_0 - \frac{1}{2}\rho(u_x^2 + u_y^2) = p_0 - 2\rho A^2(x^2 + y^2), \tag{8.174}$$

where p_0 is the pressure at the stagnation point.

Note that this potential flow solution satisfies our governing equations of fluid dynamics (Continuity and Navier-Stokes) and it satisfies part of the no-slip condition (no flow through the surface). It does not, however, satisfy the tangential component of the no-slip condition. Thus, the potential flow solution is of limited use since it won't be a good model close to the plate surface.

To determine a valid solution close to the plate surface, let's try modifying the potential flow model (we'll work with the stream function since close to the surface the flow will be rotational) so that it does satisfy the no-slip boundary conditions. Let's try the following stream function,

$$\psi = 2Axf, \tag{8.175}$$

$$\implies u_x = 2Axf' \quad \text{and} \quad u_y = -2Af, \tag{8.176}$$

where $f = f(y)$ and $f' = df/dy$. The function, f, is unknown at this point. We'll place several constraints on the function, f, so that it satisfies the governing fluid equations (Continuity and Navier-Stokes) and we'll also make sure that far from the plate the flow has the same form as the original potential flow solution.

We know that since we're using a stream function, the Continuity Equation is automatically satisfied. To make sure we satisfy the momentum equations we substitute the velocity components into the Navier- Stokes equations (simplified using our assumptions),

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right), \tag{8.177}$$

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right), \tag{8.178}$$

$$4A^2x(f')^2 - 4A^2xf'' = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2A\nu x f''', \tag{8.179}$$

$$4A^2 ff' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2A\nu x f''. \tag{8.180}$$

$$\tag{8.181}$$

We need to say something about the pressure distribution before proceeding further. Let's integrate the y-component of the Navier-Stokes equations with respect to y,

$$4A^2 ff' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2A\nu f'', \tag{8.182}$$

$$\frac{\partial p}{\partial y} = -4\rho A^2 ff' - 2A\rho\nu f'', \tag{8.183}$$

$$p(x,y) = -2\rho A^2 f^2 - 2A\rho\nu f' + g(x), \tag{8.184}$$

where $g(x)$ is an unknown function of x (since p is a function of both x and y). To determine the form of $g(x)$ we recall that far from the plate the current solution should approach the potential flow solution where the pressure distribution is given by,

$$p(x,y) = p_0 - 2\rho A^2(x^2 + y^2), \tag{8.185}$$

and the function $f(y \to \infty) \to y$, which gives the original potential flow function. Thus, the unknown function of x should be (as y becomes very large),

$$p(x,y) = -2\rho A^2 y^2 - 2A\rho\nu + g(x) = p_0 - 2\rho A^2(x^2 + y^2), \tag{8.186}$$

$$\implies g(x) = p_0 - 2\rho A^2 x^2 + 2\rho\nu A, \tag{8.187}$$

and the pressure distribution becomes,

$$p(x,y) = p_0 - 2\rho A^2 f^2 + 2A\rho\nu(1 - f') - 2\rho A^2 x^2. \tag{8.188}$$

Substituting this pressure distribution into the x-component of the Navier-Stokes equations gives,

$$\frac{\partial p}{\partial x} = -4\rho A^2 x, \tag{8.189}$$
$$\implies 4A^2x(f')^2 - 4A^2 x f f'' = 4A^2 x + 2A\nu x f''', \tag{8.190}$$
$$\implies \frac{\nu}{2A} f''' + f f'' - (f')^2 + 1 = 0. \tag{8.191}$$

Thus, the function f must satisfy this non-linear, third order ODE to satisfy the x-momentum equation (note that we've already established that the original stream function satisfies the Continuity Equation and y-momentum equation). The boundary conditions for the ODE are,

$$\text{no slip at surface in } x\text{-direction}: \quad u_x(x, y=0) = 0 \implies f'(y=0) = 0, \tag{8.192}$$
$$\text{no slip at surface in } y\text{-direction}: \quad u_y(x, y=0) = 0 \implies f(y=0) = 0, \tag{8.193}$$
$$\text{potential flow far from plate}: \quad f(y\to\infty) \to y \implies f'(y\to\infty) = 1. \tag{8.194}$$
$$\tag{8.195}$$

Currently the ODE and boundary conditions are in dimensional form. To make the solution to the ODE general, let's re-write it in terms of dimensionless parameters,

$$f(y) = \sqrt{\frac{\nu}{2A}} F(\eta) \quad \text{where} \quad y = \eta\sqrt{\frac{\nu}{2A}}, \tag{8.196}$$
$$\implies f' = \frac{df}{dy} = \frac{df}{d\eta}\left(\frac{d\eta}{dy}\right) = \sqrt{\frac{\nu}{2A}} F' \left(\sqrt{\frac{2A}{\nu}}\right) = F', \tag{8.197}$$
$$\implies f'' = \frac{d^2 f}{d\eta^2}\left(\frac{d\eta}{dy}\right)^2 = \sqrt{\frac{\nu}{2A}} F'' \left(\frac{2A}{\nu}\right) = \sqrt{\frac{2A}{\nu}} F'', \tag{8.198}$$
$$\implies f''' = \frac{d^3 f}{d\eta^3}\left(\frac{d\eta}{dy}\right)^3 = \sqrt{\frac{\nu}{2A}} F''' \left(\frac{2A}{\nu}\right)^{3/2} = \frac{2A}{\nu} F''', \tag{8.199}$$
$$\tag{8.200}$$

so that the original dimensional ODE is now in dimensionless form,

$$\boxed{F''' + FF'' - (F')^2 + 1 = 0}, \tag{8.201}$$

subject to the boundary conditions,

$$F'(\eta = 0) = 0, \tag{8.202}$$
$$F(\eta = 0) = 0, \tag{8.203}$$
$$F'(\eta \to \infty) = 1. \tag{8.204}$$

An exact analytical solution to this ODE has not been found so we solve it numerically (using, for example, a Runge-Kutta numerical scheme). Even though we solve the equation numerically, we still consider the result an "exact" solution since we can find the solution numerically to any precision.

The velocity components and pressure distribution are found using the original assumed stream function,

$$\psi = \sqrt{2A\nu}\, x F, \tag{8.205}$$
$$u_x = 2AxF' \quad \text{and} \quad u_y = -2\sqrt{2A\nu} F, \tag{8.206}$$
$$p(x, y) = p_0 - \rho A \nu F^2 + 2\rho A\nu(1 - F') - 2\rho A^2 x^2. \tag{8.207}$$

Notes:

(1) The functions F, F', and F'' are plotted as functions of η in Figure 8.15 .
(2) For this flow the non-linear convective terms in the Navier-Stokes equations, $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$, do not drop out as they have in the previous exact solutions.

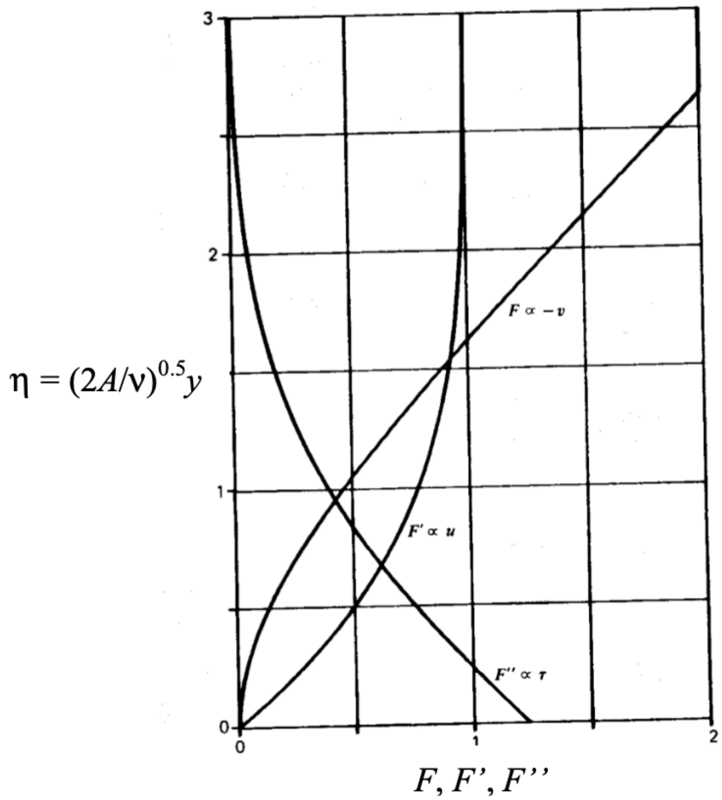


Figure 8.15. The functions F, F', and F'' plotted as functions of η. This plot is from Panton, R.L., *Incompressible Flow*, Wiley.

(3) In flows around objects with surface curvature (e.g., a torpedo-shaped object), this solution will still be valid in some vicinity of the stagnation point. As we zoom in very close to the stagnation point, the local object surface will be approximately flat.

(4) Recall that far from the plate, the viscous flow solution should approach the potential flow solution. We can estimate this distance by determining the location, $y = \delta$, at which the x-velocity is 99% that of the velocity far from the plate at the same x-location, U_∞ (for $y > \delta$ the vorticity will be very small since the velocity gradients are small),

$$u_x = 2AxF' \quad \text{and} \quad u_x(\eta \to \infty) = U_\infty = 2Ax, \tag{8.208}$$

$$\implies \frac{u_x}{U_\infty} = F' = 0.99 \quad \text{when} \quad \eta = \delta\sqrt{\frac{2A}{\nu}} \approx 2.4, \tag{8.209}$$

$$\boxed{\delta \approx 2.4\sqrt{\frac{\nu}{2A}}}. \tag{8.210}$$

The distance, δ, is referred to as the (99%) boundary layer thickness. Note that here the boundary layer thickness is a constant value and proportional to the square root of the kinematic viscosity. Since the boundary layer thickness is constant, we can interpret that the shear layer displaces the outer potential flow a constant distance from the surface. From the plot we note that as $\eta \to \infty$, $F \approx (\eta - 0.65)$ (recall that in the potential flow region "far" from the boundary, F is linear) so that this displacement thickness, δ_D, is given by,

$$\eta = \delta_D\sqrt{\frac{2A}{\nu}} \approx 0.65 \implies \delta_D \approx 0.65\sqrt{\frac{\nu}{2A}}. \tag{8.211}$$

We'll address the concept of a displacement thickness again when discussing boundary layer flow.

(5) The pressure gradients for the flow are given by,

$$\frac{\partial p}{\partial x} = -4\rho A^2 x = -\rho U_\infty \frac{dU_\infty}{dx} \quad \text{(using the } U_\infty \text{ defined in the previous note)}, \tag{8.212}$$

$$\frac{\partial p}{\partial x} = -2\rho A\nu(FF' + F''). \tag{8.213}$$

The pressure gradient in the x-direction is the same as that given by Bernoulli's equation using the outer potential flow velocity while the pressure gradient in the y-direction will be small if the kinematic viscosity is small (F, F', and F'' are all of order one near the surface). Thus, the pressure in the shear layer is nearly constant in the y-direction and it has the same magnitude as the pressure in the outer potential flow. This is an important result that will be discussed again when investigating boundary layer flows in Chapter 9.

(6) An exact solution for axisymmetric stagnation point flow can also be found. Its solution was first presented by Homann (1936). The approach for the axisymmetric problem is very similar to what was presented here for planar flow except a different stream function is used. Refer to White, F.M., *Viscous Fluid Flow*, McGraw-Hill for more details. The resulting velocity profiles, pressure, and shear stress distributions for the axisymmetric case are similar to those found for the planar case, but with slightly different magnitudes.

8.10. Very Low Reynolds Number Flows (aka Creeping Flows, aka Stokes Flows)

Consider the governing equations for an incompressible fluid, neglecting body forces, in dimensional form,

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{8.214}$$

$$\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\right] = -\boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u}. \tag{8.215}$$

Recall that when the Reynolds number is very small, viscous forces dominate the inertial forces. Let's re-write these equations in dimensionless form keeping in mind that we'll be investigating flows where viscous forces dominate (or where the fluid inertia is negligible). The variables in the equations are normalized in the following manner,

$$x^* := \frac{x}{L} \implies x = Lx^*, \quad u^* := \frac{u}{U} \implies u = Uu^*, \tag{8.216}$$

$$t^* := \frac{tU}{L} \implies t = \frac{Lt^*}{U}, \quad p^* := \frac{p}{\mu U/L} \implies p = \frac{\mu U}{L} p^*, \tag{8.217}$$

where the superscript "*" refers to a dimensionless quantity and L and U represent, respectively, a characteristic length and velocity for the flow of interest. Note that the pressure has been made dimensionless using a characteristic viscous stress, $\mu U/L$, rather than the usual dynamic pressure term, i.e., $1/2\rho U^2$. This difference is because here we're investigating flows where viscous forces dominate (or fluid inertia is negligible). Also note that we've assumed that the time scale is set by the flow velocity and length scale. This assumption is fine unless there is some other well-defined time scale in the problem, such as an oscillation period (for acoustic applications, for example). Now let's rewrite the governing equations using these dimensionless parameters,

$$\boldsymbol{\nabla}^* \cdot \boldsymbol{u}^* = 0, \tag{8.218}$$

$$\frac{\rho U^2}{L} \left[\frac{\partial \boldsymbol{u}^*}{\partial t^*} + (\boldsymbol{u}^* \cdot \boldsymbol{\nabla}^*)\boldsymbol{u}^*\right] = -\frac{\mu U}{L^2} \boldsymbol{\nabla}^* p^* + \frac{\mu U}{L^2} \nabla^{*2} \boldsymbol{u}^*. \tag{8.219}$$

Dividing through by the characteristic viscous force term gives,

$$\frac{\rho U L}{\mu}\left[\frac{\partial \boldsymbol{u}^*}{\partial t^*}+(\boldsymbol{u}^*\cdot\boldsymbol{\nabla}^*)\boldsymbol{u}^*\right]=-\boldsymbol{\nabla}^*p^*+\nabla^{*2}\boldsymbol{u}^*, \tag{8.220}$$

$$\mathrm{Re}_L\left[\frac{\partial \boldsymbol{u}^*}{\partial t^*}+(\boldsymbol{u}^*\cdot\boldsymbol{\nabla}^*)\boldsymbol{u}^*\right]=-\boldsymbol{\nabla}^*p^*+\nabla^{*2}\boldsymbol{u}^*, \tag{8.221}$$

where Re_L is the Reynolds number, which is a ratio of typical fluid inertial forces to viscous forces in a flow. If the viscous forces dominate, then the Reynolds number should be small. For creeping flows we investigate the limit when $\mathrm{Re}_L \to 0$, i.e., the fluid has negligible inertia. Thus, for creeping flows the governing fluid equations simplify to,

$$\boldsymbol{\nabla}^*\cdot\boldsymbol{u}^*=0 \tag{8.222}$$

$$,\boldsymbol{\nabla}^*p^*=\nabla^{*2}\boldsymbol{u}^*, \tag{8.223}$$

or, in dimensional form,

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{u}=0,} \tag{8.224}$$

$$\boxed{\boldsymbol{\nabla}p=\mu\nabla^2\boldsymbol{u},} \tag{8.225}$$

These equations are known as Stokes' Equations. Note that ρ doesn't appear in these equations, indicating that Stokes flows behave the same regardless of the surrounding fluid density.

Two additional useful relations can be found if we take the curl of both sides of the momentum equation,

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}p=\mu\nabla^2(\boldsymbol{\nabla}\times\boldsymbol{u}), \tag{8.226}$$

$$\boxed{\nabla^2\boldsymbol{\omega}=0}. \tag{8.227}$$

If we take the divergence of both sides of the momentum equation (and use the Continuity Equation),

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}p=\mu\nabla^2(\boldsymbol{\nabla}\cdot\boldsymbol{u}), \tag{8.228}$$

$$\boxed{\nabla^2 p=0}. \tag{8.229}$$

In a creeping flow, the vorticity and pressure fields both satisfy Laplace's equation.

Notes:

(1) Examples of where creeping flow might occur ($\mathrm{Re}_L \ll 1$):
- small length dimensions (flow in small pipes or channels, around small particles, flow through small pores),
- very viscous fluids,
- small velocities.

A good reference for this topic is Happel and Brenner (1965).

(2) Since Laplace's equation is a linear PDE, we can add together solutions to form new solutions (the principle of superposition). This approach is very similar to what we did with potential flows where we could add together valid velocity fields (in the form of a potential function) to form new velocity fields. The difference, however, is that here we can also add together pressure and vorticity fields. Note that in potential flows we couldn't add together pressure fields since the pressure was found using the non-linear Bernoulli's equation.

(3) If we consider a two-dimensional flow and use a stream function to describe the velocity field, we find that the vorticity can be written in terms of the stream function as,

$$\omega_z=\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}=-\nabla^2\psi. \tag{8.230}$$

Substituting this equation into Eq. (8.227),

$$\boxed{\nabla^2(\nabla^2\psi)=\nabla^4\psi=0}. \tag{8.231}$$

This is the governing equation for a two-dimensional Stokes flow. Note that in 2D Cartesian coordinates,

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{8.232}$$

Equation (8.231) is a *biharmonic equation* and is a common equation found in other fields of study, such as solid mechanics where the Airy stress function is used to solve two-dimensional problems in elasticity.

(4) The pressure increases proportionally with the dynamic viscosity of the fluid assuming that the viscosity is independent of pressure (Eq. (8.225)). When the pressures become very large, such as in lubrication flows, the viscosity becomes a function of pressure (recall that the viscosity is also a function of temperature).

(5) There are several approaches to finding solutions to creeping flow problems. These include:
   (a) forming "building block" solutions that we can add together to form new solutions. We used a similar approach with potential flow problems. Note that we can add together pressure fields since Eq. (8.229) is linear. We can't do this for potential flows since Bernoulli's equation is non-linear in terms of the velocities.
   (b) solving the boundary value problem for the given geometry and boundary conditions,
   (c) borrowing solutions from other disciplines that have the same governing equations, e.g., the Airy stress function in solid mechanics, and
   (d) using computational methods to solve the governing equations.

(6) Note that if $\boldsymbol{u}$ is a Stokes flow solution, then $\boldsymbol{u}' = -\boldsymbol{u}$ is also a solution since,

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}' = 0 \quad \text{and} \quad \boldsymbol{\nabla} p' = \mu \nabla^2 \boldsymbol{u}' \implies \boldsymbol{\nabla} p' = -\boldsymbol{\nabla} p. \tag{8.233}$$

In addition, $\nabla^2 \boldsymbol{\omega}' = 0$ and $\nabla^2 p' = 0$ where $\boldsymbol{\omega}' = \boldsymbol{\nabla} \times \boldsymbol{u}' = -\boldsymbol{\nabla} \times \boldsymbol{u}$. Hence, Stokes flows are kinematically reversible and flow around symmetric objects will produce symmetric streamlines.

8.11. Stokes Flow Around a Sphere

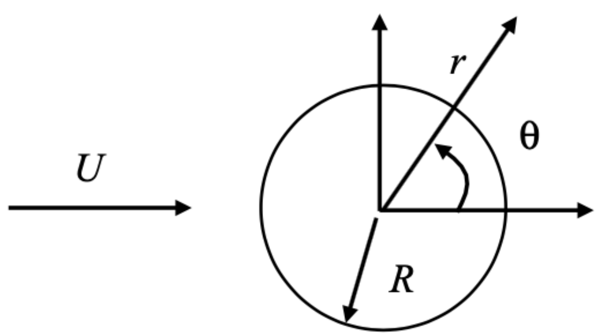


Figure 8.16. The geometry for Stokes flow around a sphere.

Now let's examine the creeping flow around a sphere of radius, R, in a uniform stream of velocity, U (Figure 8.16). For axisymmetric creeping flows it is convenient to use a stream function in spherical polar coordinates, (r, θ, ϕ), to describe the fluid velocity. The angle ϕ is zero when aligned with the incoming free stream. Since the flow is axisymmetric, the stream function will be a function only of r and θ. After substituting the stream function into the biharmonic equation (in spherical coordinates and noting that for an axisymmetric problem there is no variation in the ϕ-direction),

$$\nabla^4 \psi = 0, \tag{8.234}$$

$$\implies \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}\right]^2 \psi = 0. \tag{8.235}$$

The velocity components are related to the spherical stream function by,

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}. \tag{8.236}$$

These forms of the velocity components in terms of the stream function can be verified by substituting them into the (incompressible) Continuity Equation in spherical polar coordinates. Recall that the stream function is defined such that it automatically satisfies the Continuity Equation,

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) = 0, \tag{8.237}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}}_{=u_r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\underbrace{-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}}_{=u_\theta}\sin\theta\right) = 0, \tag{8.238}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = 0, \tag{8.239}$$

$$\frac{1}{r^2\sin\theta}\left(\frac{\partial^2\psi}{\partial r\partial\theta}\right) - \frac{1}{r^2\sin\theta}\left(\frac{\partial^2\psi}{\partial\theta\partial r}\right) = 0 \quad \text{(Continuity Equation satisfied!)} \tag{8.240}$$

The no-slip condition at the surface means that,

$$u_r(r=R) = u_\theta(r=R) = 0, \tag{8.241}$$

$$\implies \frac{\partial\psi}{\partial\theta}(r=R) = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial r}(r=R) = 0, \tag{8.242}$$

and far from the sphere (as $r \to \infty$), the stream function approaches the stream function for a uniform stream,

$$\psi(r\to\infty) = \frac{r^2}{2}U\sin^2\theta + \text{constant}, \tag{8.243}$$

$$\implies u_r \to U\cos\theta \quad \text{and} \quad u_\theta \to -U\sin\theta. \tag{8.244}$$

Solve the differential equation with the given boundary conditions using separation of variables. Based on the form of the stream function far from the origin, let's assume that the solution has the form $\psi = f(r)\sin^2\theta$,

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\right]^2\left[f(r)\sin^2\theta\right] = 0. \tag{8.245}$$

After simplifying,

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 f(r) = 0. \tag{8.246}$$

In trying to solve this ODE, let's try a solution of the form $f(r) = r^n$,

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 r^n = \left[(n-2)(n-3)-2\right]\left[(n)(n-1)-2\right]r^{n-4} = 0, \tag{8.247}$$

$$\implies n = -1, 1, 2, 3, \tag{8.248}$$

$$\therefore f(r) = \frac{A}{r} + Br + Cr^2 + Dr^3. \tag{8.249}$$

The corresponding stream function and velocity components are,

$$\psi(r,\theta) = \left(\frac{A}{r} + Br + Cr^2 + Dr^3\right)\sin^2\theta, \tag{8.250}$$

$$u_r = 2\left(\frac{A}{r^3} + \frac{B}{r} + C + Dr\right)\cos\theta, \tag{8.251}$$

$$u_\theta = -\left(\frac{A}{r^3} + \frac{B}{r} + 2C + 3Dr\right)\sin\theta. \tag{8.252}$$

Applying the boundary conditions we find that,

$$A = \frac{UR^3}{4} \quad B = \frac{-3UR}{4} \quad C = \frac{U}{2} \quad D = 0. \tag{8.253}$$

Consequently,

$$\boxed{\begin{aligned} \psi(r,\theta) &= \frac{R^2U}{4}\left(\frac{R}{r} - \frac{3r}{R} + \frac{2r^2}{R^2}\right)\sin^2\theta, \quad (8.254)\\ u_r &= U\cos\theta\left(1 + \frac{R^3}{2r^3} - \frac{3R}{2r}\right), \quad (8.255)\\ u_\theta &= U\sin\theta\left(-1 + \frac{R^3}{4r^3} + \frac{3R}{4r}\right). \quad (8.256)\end{aligned}}$$

The pressure, found using the momentum equation ($\boldsymbol{\nabla} p = \mu\nabla^2\boldsymbol{u}$) is,

$$p = p_\infty - \frac{3}{2}\frac{\mu U}{R}\left(\frac{R}{r}\right)^2\cos\theta. \tag{8.257}$$

The viscous stresses are found using the constitutive relations for a Newtonian fluid,

$$\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r}, \tag{8.258}$$

$$\sigma_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \tag{8.259}$$

$$\sigma_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right), \tag{8.260}$$

$$\sigma_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right], \tag{8.261}$$

$$\sigma_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right], \tag{8.262}$$

$$\sigma_{\phi\theta} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right]. \tag{8.263}$$

Evaluating at the sphere's surface ($r = R$) gives,

$$\sigma_{rr}|_{r=R} = \sigma_{\phi\phi}|_{r=R} = \sigma_{\theta\theta}|_{r=R} = \sigma_{r\phi}|_{r=R} = \sigma_{\phi\theta}|_{r=R} = 0, \tag{8.264}$$

$$\sigma_{r\theta}|_{r=R} = -\frac{3}{2}\frac{\mu U}{R}\sin\theta. \tag{8.265}$$

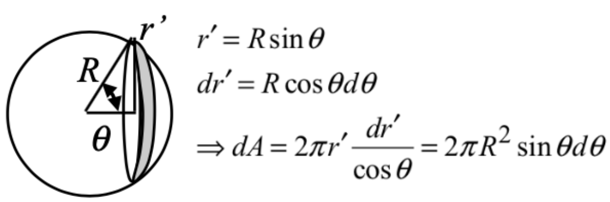


Figure 8.17. The geometry for calculating the Stokes flow drag on a sphere.

The drag force acting on the sphere surface ($r = R$) is found by integrating the pressure and viscous forces in the horizontal direction over the entire sphere's surface (Figure 8.17),

$$F = -\int_0^\pi \sigma_{r\theta}|_{r=R}\sin\theta dA - \int_0^\pi p|_{r=R}\cos\theta dA \quad \text{where} \quad dA = 2\pi R^2\sin\theta d\theta, \tag{8.266}$$

$$\implies F = 4\pi\mu UR + 2\pi\mu UR. \tag{8.267}$$

Thus, the total force acting on the sphere consists of 2/3 viscous force (the first term in the previous equation) and 1/3 pressure force (the second term in the previous equation) giving the total Stokes drag force on a sphere,

$$\boxed{F = 6\pi\mu U R}. \tag{8.268}$$

Notes:

(1) The drag is independent of the fluid density and is proportional to the velocity (and not velocity squared).

(2) Stokes drag is usually presented in dimensionless form as a drag coefficient, c_D. The usual form is,

$$c_D := \frac{F}{\frac{1}{2}\rho U^2(\pi R^2)} = \frac{6\pi\mu U R}{\frac{1}{2}\rho U^2(\pi R^2)} = \frac{12\mu}{\rho U R} = \frac{24\mu}{\rho U D}, \tag{8.269}$$

$$\boxed{\therefore c_D = \frac{24}{\mathrm{Re}_D}}, \tag{8.270}$$

where $\mathrm{Re}_D = \rho U D/\mu$ is the Reynolds number based on the sphere diameter ($D = 2R$).

(3) Stokes drag is strictly valid only when $\mathrm{Re}_D \to 0$, but it is found experimentally to be a reasonable estimate up to $\mathrm{Re}_D = 1$.

(4) Oseen (1910) included first-order inertial effects in the drag analysis and found a drag coefficient of,

$$c_D = \frac{24}{\mathrm{Re}_D}\left(1 + \frac{3}{16}\mathrm{Re}_D\right). \tag{8.271}$$

This drag coefficient is found to give good results up to $\mathrm{Re}_D \approx 5$.

(5) Although the streamlines for flow around a sphere look similar for a potential flow and a Stokes flow (in the sphere's frame of reference, FOR), there are some important differences. The streamlines for potential flows are grouped closer together near the sphere than they are for a Stokes flow. More strikingly, if we plot the streamlines using a frame of reference in which the fluid is at rest far from the sphere (and the sphere moves with a velocity $-U$), we find that in a potential flow the fluid is "pushed" out of the way while in a Stokes flow the fluid is "dragged" along with the sphere. These phenomena are illustrated in Figure 8.18.

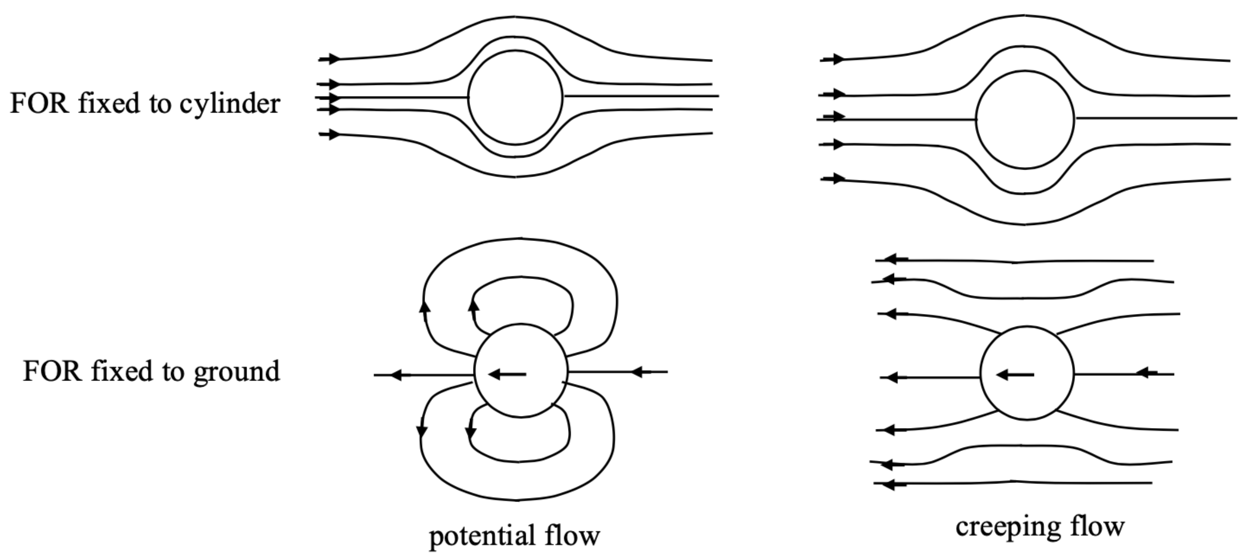


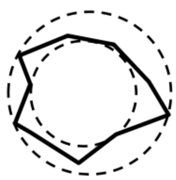
Figure 8.18. Streamlines for potential flow around a sphere and Stokes flow around a sphere.

(6) We can also use the solution approach presented here to determine the drag on a spherical droplet of a fluid (with dynamic viscosity μ_i) in a different fluid (of dynamic viscosity μ_o). The general differential equation is the same, but the boundary conditions are different. For the spherical droplet problem, the boundary conditions at the sphere radius consist of continuous velocity components (no-slip, but the tangential velocity is not zero) and continuous stresses between the droplet fluid and the outer fluid. The resulting drag force acting on the droplet becomes,

$$F = 6\pi R\mu_o U \frac{1+\frac{2}{3}\frac{\mu_o}{\mu_i}}{1+\frac{\mu_o}{\mu_i}}. \tag{8.272}$$

For $\mu_i \gg \mu_o$, e.g., a solid droplet in a gas or liquid, we get the original Stokes drag equation, $F = 6\pi\mu_o UR$. For $\mu_i \ll \mu_o$, e.g., a gas bubble in a liquid, then we get a smaller drag force since the outer fluid can slip at the boundary surface: $F = 4\pi\mu_o RU$.

(7) It can be shown that the drag on an irregular object in a Stoke's flow is bounded by the drag on a sphere that inscribes the object and the drag on a sphere that circumscribes the object (refer to Hill and Power, 1956) (Figure 8.19). This result is useful for practical applications.



$$D_{\text{inscribed sphere}} < D_{\text{irregular object}} < D_{\text{circumscribed sphere}}$$

Figure 8.19. A sphere that inscribes an irregular particle and a sphere that circumscribes the particle.

(8) An interesting observation, referred to as the Stokes Paradox, can be made using dimensional analysis. Assuming that inertia is negligible for a Stokes flow, the force, F, on an object is a function of the dynamic viscosity, the flow speed, and the object size, i.e.,

$$F = f(\mu, U, L). \tag{8.273}$$

For a 2D flow, the dimensions of the force will be F/L (force per unit depth) while for a 3D flow the dimension of the force will simply be F. Thus, from dimensional analysis,

$$F'_{2D} = \frac{F_{2D}}{\mu U} = \text{constant}, \tag{8.274}$$

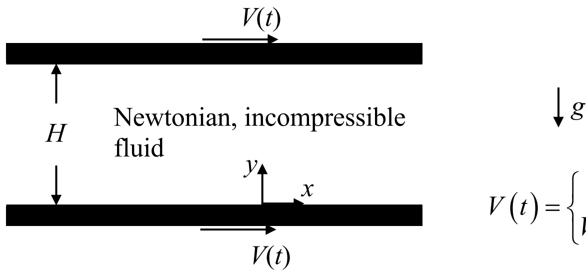
$$F'_{3D} = \frac{F_{3D}}{\mu UL} = \text{constant}, \tag{8.275}$$

where F' is the dimensionless force. The first equation indicates that in a 2D Stokes flow the force on an object is independent of the object size. This result contradicts what we observe in reality. Hence, our initial assumption that the fluid inertia is negligible must be incorrect. For 2D flows, the density must be a factor in determining the force on an object, i.e.,

$$F_{2D} = f(\rho, \mu, U, L), \tag{8.276}$$

$$\implies F'_{2D} = \frac{F_{2D}}{\mu U} = f\left(\frac{\rho UL}{\mu}\right). \tag{8.277}$$

Consider two infinitely long parallel plates separated by a constant distance H as shown in the figure below. Between the plates is a Newtonian, incompressible fluid with density ρ and viscosity μ. There are no pressure gradients in the direction of the flow and gravity acts in the negative y-direction.



$$V(t) = \begin{cases} 0 & t < 0 \\ V_0 & t \geq 0 \end{cases}$$

Assume at time $t < 0$ the entire system is at rest. For $t \geq 0$, the walls are impulsively started and move at a constant speed V_0 in the $+x$-direction.

For these conditions, which of the following simplifications to the Navier-Stokes equations provides the governing equation for determining the velocity profile in the x-direction?

A. $\rho \dfrac{\partial u_x}{\partial t} = \mu \dfrac{\partial^2 u_x}{\partial y^2}$

B. $\rho \left(\dfrac{\partial u_x}{\partial t} + u_y \dfrac{\partial u_x}{\partial y} \right) = \mu \dfrac{\partial^2 u_x}{\partial y^2}$

C. $0 = \mu \dfrac{\partial^2 u_x}{\partial y^2}$

D. $0 = \mu \dfrac{\partial^2 u_x}{\partial x^2} + \rho g$

E. $\rho \dfrac{\partial u_x}{\partial t} = -\dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u_x}{\partial y^2}$

F. $\rho \left(\dfrac{\partial u_y}{\partial t} + u_y \dfrac{\partial u_y}{\partial y} \right) = \mu \dfrac{\partial^2 u_y}{\partial y^2} - \rho g$

SOLUTION:

Make the following assumption about the flow.

1. fully developed flow in the x-direction $\Rightarrow$ $\dfrac{\partial u_x}{\partial x} = \dfrac{\partial u_y}{\partial x} = \dfrac{\partial u_z}{\partial x} = 0$ (1)
2. planar flow $\Rightarrow$ $u_z = \text{const}; \dfrac{\partial}{\partial z}(\cdots) = 0$ (2)
3. no pressure gradient in the x-direction $\Rightarrow$ $\dfrac{\partial p}{\partial x} = 0$ (3)
4. no gravity in the x-direction $\Rightarrow$ $g_x = 0$ (4)

Note that the flow velocity chanes with time. Hence, the flow is unsteady.

Consider the continuity equation for an incompressible fluid.

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0\,(\#1)} + \frac{\partial u_y}{\partial y} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0\,(\#2)} = 0 \;\Rightarrow\; \frac{\partial u_y}{\partial y} = 0 \tag{5}$$

Since $u_y \neq fcn(x, y, z)$ (assumption #1, Eq. (5), and assumption #2), at most we can have:

$$u_y = fcn(t) \tag{6}$$

However, at the boundaries $u_y(t) = 0$, thus, $\underline{u_y = 0}$ everywhere and for all times. Call this condition #5.

Now consider the Navier-Stokes equation for an incompressible, Newtonian fluid in the x-direction.

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0\,(\#1)} + \underbrace{u_y}_{=0\,(\#5)} \frac{\partial u_x}{\partial y} + u_z \underbrace{\frac{\partial u_x}{\partial z}}_{=0\,(\#2)}\right) = -\underbrace{\frac{\partial p}{\partial x}}_{=0\,(\#3)} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0\,(\#1)} + \frac{\partial^2 u_x}{\partial y^2} + \underbrace{\frac{\partial^2 u_x}{\partial z^2}}_{=0\,(\#2)}\right) + \rho \underbrace{g_x}_{=0\,(\#4)} \tag{7}$$

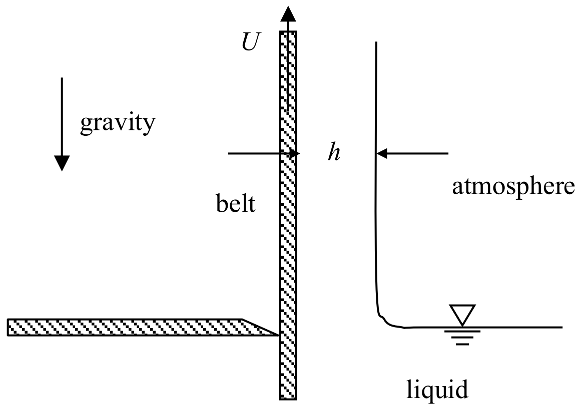
Thus,

$$\boxed{\rho \frac{\partial u_x}{\partial t} = \mu \frac{\partial^2 u_x}{\partial y^2}} \tag{8}$$

The boundary conditions for the flow are,

no-slip at the top and bottom boundaries $\Rightarrow$ $u_x(t; y = 0, H) = \begin{cases} 0 & t < 0 \\ V_0 & t \geq 0 \end{cases}$ (9)

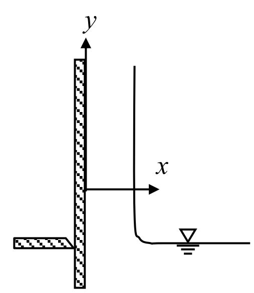
A wide flat belt moves vertically upward at constant speed, U, through a large bath of viscous liquid as shown in the figure. The belt carries with it a layer of liquid of constant thickness, h. The motion is steady and fully-developed after a small distance above the liquid surface level. The external pressure is atmospheric (constant) everywhere.



a. Simplify the governing equations to a form applicable for this particular problem.
b. State the appropriate boundary conditions
c. Determine the velocity profile in the liquid.
d. Determine the volumetric flow rate per unit depth.

SOLUTION:

Make the following assumptions.



1. steady flow $\Rightarrow \partial/\partial t(\cdots)=0$
2. planar flow $\Rightarrow u_z = \partial/\partial z(\cdots)=0$
3. fully-developed flow in the y-direction $\Rightarrow \partial u_x/\partial y = \partial u_y/\partial y = 0$
4. gravity acts only in the $-y$-direction $\Rightarrow g_x = g_z = 0, g_y = -g$

Consider the continuity equation.

$$\frac{\partial u_x}{\partial x}+\underbrace{\frac{\partial u_y}{\partial y}}_{=0(\#3)}=0 \Rightarrow \frac{\partial u_x}{\partial x}=0 \Rightarrow u_x = \text{constant}$$ (Note that u_x does not vary with either y or z either.)

Since there is no flow through the belt,

$\boxed{u_x = 0}$ (Call this condition #5.) (1)

Consider the Navier-Stokes equation in the x-direction.

$$\rho\left[\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#1,\#5)}+\underbrace{u_x\frac{\partial u_x}{\partial x}}_{=0(\#5)}+u_y\underbrace{\frac{\partial u_x}{\partial y}}_{=0(\#3,\#5)}\right]=-\frac{\partial p}{\partial x}+\mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#5)}+\underbrace{\frac{\partial^2 u_x}{\partial y^2}}_{=0(\#3,\#5)}\right)+\rho\underbrace{g_x}_{=0(\#4)}$$

$$\therefore \frac{\partial p}{\partial x}=0 \quad (2)$$

Note that along the free surface of the liquid film the pressure remains constant (= p_{atm}). Hence, from Eqn. (2) the pressure everywhere in the film will be the same, *i.e.* $p(x,y) = p_{atm}$ (call this condition #6.).

Now consider the Navier-Stokes equation in the y-direction.

$$\rho\left[\underbrace{\frac{\partial u_y}{\partial t}}_{=0(\#1)} + \underbrace{u_x}_{=0(\#5)}\frac{\partial u_y}{\partial x} + u_y\underbrace{\frac{\partial u_y}{\partial y}}_{=0(\#3)}\right] = -\underbrace{\frac{\partial p}{\partial y}}_{=0(\#6)} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \underbrace{\frac{\partial^2 u_y}{\partial y^2}}_{=0(\#3)}\right) + \rho\underbrace{g_y}_{=-g}$$

Note that since u_y is neither a function of y or z, we can replace the partial derivative with an ordinary derivative.

$$\boxed{0 = \mu\frac{d^2 u_y}{dx^2} - \rho g} \tag{3}$$

Solve the differential equation given in Eqn. (3).

$$\frac{du_y}{dx} = \frac{\rho g}{\mu}x + c_1$$

$$u_y = \frac{1}{2}\frac{\rho g}{\mu}x^2 + c_1 x + c_2 \tag{4}$$

Apply the following boundary conditions.

$\boxed{\text{no-slip at } x = 0}$ $\quad\Rightarrow\quad u_y(x=0) = U \quad\Rightarrow\quad c_2 = U$

At the free surface, the air will provide a negligible resisting shear stress so:

$\boxed{\text{no shear at } x = h}$ $\quad\Rightarrow\quad \mu\frac{du_y}{dx}(x=h) = 0 \quad\Rightarrow\quad c_1 = -\frac{\rho g}{\mu}h$

Hence,

$$\boxed{u_y = \frac{1}{2}\frac{g}{\nu}x^2 - \frac{gh}{\nu}x + U} \quad (0 \le x \le h) \tag{5}$$

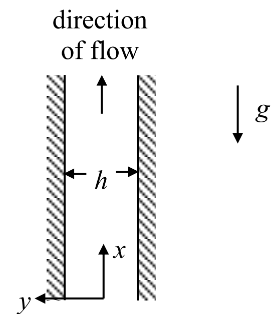
The volumetric flow rate in the film (per unit depth), Q, is given by:

$$Q = \int_{x=0}^{x=h} u_y\,dx = \int_{x=0}^{x=h}\left(\frac{1}{2}\frac{g}{\nu}x^2 - \frac{gh}{\nu}x + U\right)dx$$

$$= \left[\frac{1}{6}\frac{g}{\nu}x^3 - \frac{gh}{2\nu}x^2 + Ux\right]_0^h = \frac{1}{6}\frac{g}{\nu}h^3 - \frac{gh}{2\nu}h^2 + Uh$$

$$\boxed{\therefore Q = -\frac{gh^3}{3\nu} + Uh} \tag{6}$$

A viscous, incompressible fluid flows between the two infinite, vertical, parallel plates shown in the figure. Determine, by use of the Navier-Stokes equations, an expression for the pressure gradient in the direction of flow. Express your answer in terms of the mean velocity. Assume that the flow is steady and fully developed in the *x* direction.



SOLUTION:

Make the following assumptions about the flow:

1. The flow is planar. $\Rightarrow \partial/\partial z(\cdots) = 0, u_z = \text{constant}$
2. The flow is steady. $\Rightarrow \partial/\partial t(\cdots) = 0$
3. The flow is fully developed in the *x*-direction. $\Rightarrow \partial u_x/\partial x = \partial u_y/\partial x = 0$
4. Gravity acts in the –*x* direction. $\Rightarrow g_x = -g; g_y = g_z = 0$

The continuity equation for an incompressible, planar flow is:

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \frac{\partial u_y}{\partial y} = 0 \;\Rightarrow\; \frac{\partial u_y}{\partial y} = 0 \tag{1}$$

Since the flow is also steady (#2), fully developed (#3), and planar (#1), the *y*-velocity can be at most a constant. Since $u_y = 0$ at the wall, then u_y everywhere is:

$$u_y = 0 \quad \text{(Call this condition \#5.)} \tag{2}$$

Now examine the *x*-momentum equation:

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#2)} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \underbrace{u_y}_{=0(\#5)} \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#3)} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho \underbrace{g_x}_{=-g(\#4)}$$

$$0 = -\frac{dp}{dx} + \mu \frac{d^2 u_x}{dy^2} - \rho g \tag{3}$$

where the partial derivatives have been replaced by ordinary derivatives since u_x is not a function of *x* (#3), *t* (#2), or *z* (#1). In addition, consideration of the *y* and *z*-momentum equations will show that *p* is not a function of either *x* or *y* and since the flow is fully developed, dp/dx = constant.

Now solve Eq. (3) for the velocity profile,

$$\frac{d^2 u_x}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} + \frac{\rho}{\mu} g = \text{constant} = \alpha \tag{4}$$

$$\frac{du_x}{dx} = \alpha y + c_1 \tag{5}$$

$$u_x = \tfrac{1}{2}\alpha y^2 + c_1 y + c_2 \tag{6}$$

Apply boundary conditions to determine the unknown constant c_1 and c_2.

no-slip at $y = -1/2h$ $\Rightarrow$ $u_x\left(y=-\tfrac{1}{2}h\right)=0$ $\Rightarrow$ $0=\tfrac{1}{8}\alpha h^2-\tfrac{1}{2}c_1 h+c_2$ (7)

no-slip at $y = 1/2h$ $\Rightarrow$ $u_x\left(y=\tfrac{1}{2}h\right)=0$ $\Rightarrow$ $0=\tfrac{1}{8}\alpha h^2+\tfrac{1}{2}c_1 h+c_2$ (8)

Substract Eq. (8) from Eq. (7) to determine c_1.

$c_1 = 0$ (Note that we could have also determined this from symmetry and Eq. (5).) (9)

The other constant, c_2, is thus:

$$c_2=-\tfrac{1}{8}\alpha h^2 \tag{10}$$

and the velocity profile is:

$$u_x=\frac{1}{8}\alpha h^2\left[\left(\frac{2y}{h}\right)^2-1\right] \tag{11}$$

where α is given in Eq. (4)

$$\boxed{u_x=\frac{1}{8}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^2\left[\left(\frac{2y}{h}\right)^2-1\right]} \tag{12}$$

The average velocity is found from the volumetric flow rate, Q.

$$Q=\int_{y=-\frac{1}{2}h}^{y=\frac{1}{2}h}u_x dy=\int_{y=-\frac{1}{2}h}^{y=\frac{1}{2}h}\frac{1}{8}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^2\left[\left(\frac{2y}{h}\right)^2-1\right]dy \tag{13}$$

$$Q=\frac{1}{8}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^2\left[\frac{4}{3}\frac{y^3}{h^2}-y\right]_{y=-\frac{1}{2}h}^{y=\frac{1}{2}h} \tag{14}$$

$$\therefore Q=-\frac{1}{12}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^3 \tag{15}$$

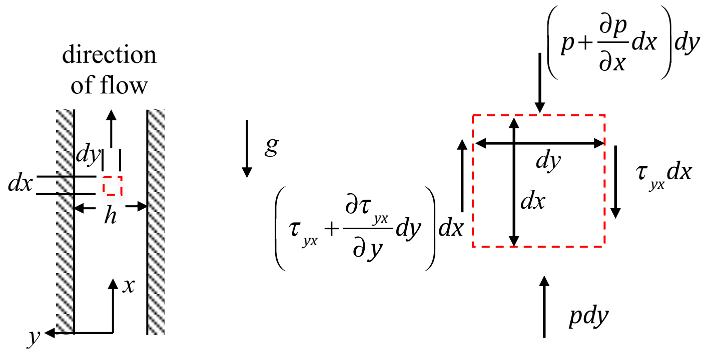
$$\bar{u}_x h=Q \tag{16}$$

$$\therefore \bar{u}_x=-\frac{1}{12}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^2 \tag{17}$$

Re-arrange to solve for the pressure gradient in terms of the average velocity.

$$\boxed{\frac{dp}{dx}=-\left(\frac{12\mu\bar{u}_x}{h^2}+\rho g\right)} \tag{18}$$

Now choose a differential control volume and apply conservation of mass and the linear momentum equation to solve the problem.



$$\frac{d}{dt}\int_{CV}\rho dV+\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=0\,, \tag{19}$$

where,

$$\frac{d}{dt}\int_{CV}\rho dV=0 \quad \text{(steady flow)}, \tag{20}$$

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=-\rho u_x dy+\rho\left(u_x+\frac{\partial u_x}{\partial x}dx\right)dy-\rho u_y dx+\rho\left(u_y+\frac{\partial u_y}{\partial y}dy\right)dx\,, \tag{21}$$

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=\rho\frac{\partial u_x}{\partial x}dxdy+\rho\frac{\partial u_y}{\partial y}dydx\,,$$

assuming unit depth and planar flow.

Substitute and simplify,

$$\rho\frac{\partial u_x}{\partial x}dxdy+\rho\frac{\partial u_y}{\partial y}dydx=0\,, \tag{22}$$

$$\underbrace{\frac{\partial u_x}{\partial x}}_{\substack{=0 \text{ since} \\ \text{the flow is FD} \\ \text{in the } x \text{ direction}}}+\frac{\partial u_y}{\partial y}=0\,, \tag{23}$$

which is the same as Eq. (1)

Now consider the linear momentum equation in the x direction using the same differential control volume,

$$\frac{d}{dt}\int_{CV}u_x\rho dV+\int_{CS}u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=F_{B,x}+F_{S,x}\,, \tag{24}$$

where,

$$\frac{d}{dt}\int_{CV}u_x\rho dV=0 \quad \text{(steady flow)}, \tag{25}$$

$$\int_{CS}u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=0 \quad \text{(the flow is fully developed in the } x \text{ direction, planar, and } u_y=0\text{)}, \tag{26}$$

$$F_{B,x}=-\rho g dxdy\,, \tag{27}$$

$$F_{S,x}=pdy-\left(p+\frac{\partial p}{\partial x}dx\right)dy-\tau_{yx}dx+\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}dy\right)dx\,. \tag{28}$$

Substitute and simplify.

$$0 = -\rho g dx dy + p dy - \left(p + \frac{\partial p}{\partial x}dx\right)dy - \tau_{yx}dx + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right)dx\ , \tag{29}$$

$$0 = -\rho g - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\ . \tag{30}$$

Note that for a Newtonian fluid,

$$\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u_x}{\partial y} + \underbrace{\frac{\partial u_y}{\partial x}}_{\substack{=0\text{ since}\\ u_y=0}}\right)\right] = \mu\frac{\partial^2 u_x}{\partial y^2}\ . \tag{31}$$

Substitute this expression into Eq. (30),

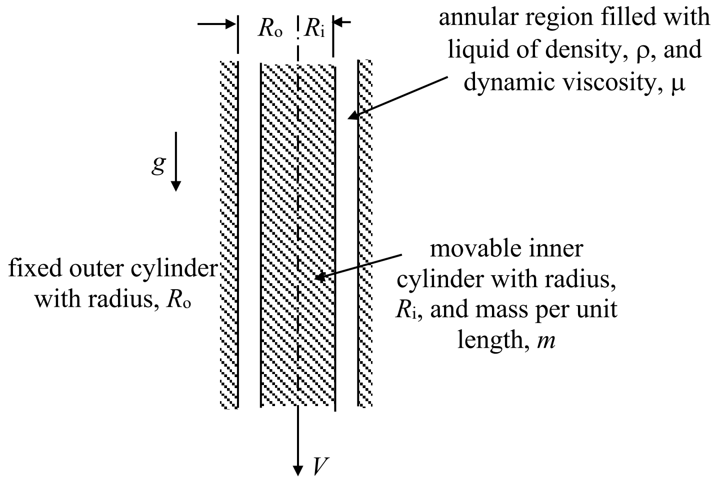
$$0 = -\rho g - \frac{\partial p}{\partial x} + \mu\frac{\partial^2 u_x}{\partial y^2}\ . \tag{32}$$

Since the flow is fully developed, steady, and planar, the last term in Eq. (31) may be written in terms of ordinary derivatives. In addition, apply the linear momentum equation in the *y* and *z* directions would show that the pressure gradient in both of those directions is zero. Hence, Eq. (32) becomes,

$$0 = -\rho g - \frac{dp}{dx} + \mu\frac{d^2 u_x}{dy^2}\ . \tag{33}$$

This equation is exactly the same as Eq. (4)

Consider two concentric cylinders with a Newtonian liquid of constant density, ρ, and constant dynamic viscosity, μ, contained between them. The outer pipe, with radius, R_o, is fixed while the inner pipe, with radius, R_i, and mass per unit length, m, falls under the action of gravity at a constant speed. There is no pressure gradient within flow and no swirl velocity component. Determine the vertical speed, V, of the inner cylinder as a function of the following (subset of) parameters: g, R_o, R_i, m, ρ, and μ.



SOLUTION:

First determine the velocity profile of the fluid within the annulus. Make the following assumptions:

1. steady flow $\Rightarrow \dfrac{\partial u_r}{\partial t} = \dfrac{\partial u_\theta}{\partial t} = \dfrac{\partial u_z}{\partial t} = 0$
2. gravity acts in the z-direction $\Rightarrow f_z = g, f_r = 0, f_\theta = 0$
3. fully-developed flow in the z-direction $\Rightarrow \dfrac{\partial u_r}{\partial z} = \dfrac{\partial u_\theta}{\partial z} = \dfrac{\partial u_z}{\partial z} = 0$
4. the flow is axi-symmetric and there is no swirl velocity $\Rightarrow \dfrac{\partial u_r}{\partial \theta} = \dfrac{\partial u_\theta}{\partial \theta} = \dfrac{\partial u_z}{\partial \theta} = 0, u_\theta = 0$
5. no pressure gradients in the z direction $\Rightarrow \dfrac{\partial p}{\partial z} = 0$

Consider the continuity equation.

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(4)} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0(3)} = 0 \Rightarrow \frac{\partial(ru_r)}{\partial r} = 0 \Rightarrow ru_r = \text{constant} \tag{1}$$

Note that from assumptions 3 and 4, u_r is not a function of either θ or z. Since there is no radial flow at r = R_i or $r = R_o$, the constant in the previous equation must be zero. Thus,

$$u_r = 0 \quad \text{(condition 6)} \tag{2}$$

Now consider the Navier-Stokes equations.

$$\rho\left(\underbrace{\frac{\partial u_r}{\partial t}}_{=0(\#1,\#6)}+u_r\underbrace{\frac{\partial u_r}{\partial r}}_{=0(\#6)}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}_{=0(\#4,\#6)}-\underbrace{\frac{u_\theta^2}{r}}_{=0(\#4)}+u_z\underbrace{\frac{\partial u_r}{\partial z}}_{=0(\#3,\#6)}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_r}_{=0(\#6)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_r}{\partial \theta^2}}_{=0(\#4,\#6)}+\underbrace{\frac{\partial^2 u_r}{\partial z^2}}_{=0(\#3,\#6)}-\frac{2}{r^2}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}\right]+\rho\underbrace{f_r}_{=0(\#2)} \tag{3}$$

$$\Rightarrow \frac{\partial p}{\partial r}=0 \tag{4}$$

$$\rho\left(\underbrace{\frac{\partial u_\theta}{\partial t}}_{=0(\#1,\#4)}+u_r\underbrace{\frac{\partial u_\theta}{\partial r}}_{=0(\#6)}+\frac{u_\theta}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}+\underbrace{\frac{u_r u_\theta}{r}}_{=0(\#6)}+u_z\underbrace{\frac{\partial u_\theta}{\partial z}}_{=0(\#3)}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_\theta}_{=0(\#4)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_\theta}{\partial \theta^2}}_{=0(\#4)}+\underbrace{\frac{\partial^2 u_\theta}{\partial z^2}}_{=0(\#3,\#4)}+\frac{2}{r^2}\underbrace{\frac{\partial u_r}{\partial \theta}}_{=0(\#4,\#6)}\right]+\rho\underbrace{f_\theta}_{=0(\#2)} \tag{5}$$

$$\Rightarrow \frac{\partial p}{\partial \theta}=0 \tag{6}$$

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(\#1)}+\underbrace{u_r}_{=0(\#6)}\frac{\partial u_z}{\partial r}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}_{=0(\#4)}+u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#3)}\right)=-\underbrace{\frac{\partial p}{\partial z}}_{=0(\#5)}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_z}{\partial \theta^2}}_{=0(\#4)}+\underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(\#3)}\right]+\rho\underbrace{f_z}_{=g(\#2)} \tag{7}$$

$$\Rightarrow 0=\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\rho g \tag{8}$$

Note that since u_z is not a function of θ (assumption #4) or z (assumption #3), then $u_z = u_z(r)$ and the partial differentials in Eqn. (8) may be written as ordinary differentials. Solve the ODE given in Eqn. (8).

$$\frac{d}{dr}\left(r\frac{du_z}{dr}\right)=-\frac{\rho g}{\mu}r$$

$$r\frac{du_z}{dr}=-\frac{\rho g}{2\mu}r^2+c_1$$

$$\frac{du_z}{dr}=-\frac{\rho g}{2\mu}r+\frac{c_1}{r}$$

$$u_z=-\frac{\rho g}{4\mu}r^2+c_1\ln r+c_2 \tag{9}$$

Apply the following boundary conditions to determine the constants c_1 and c_2.

no-slip at $r = R_0$: $$u_z(r=R_o)=0 \Rightarrow 0=-\frac{\rho g}{4\mu}R_o^2+c_1\ln R_o+c_2 \tag{10}$$

no-slip at $r = R_i$: $$u_z(r=R_i)=V \Rightarrow V=-\frac{\rho g}{4\mu}R_i^2+c_1\ln R_i+c_2 \tag{11}$$

First determine c_1 by subtracting Eqn. (10) from Eqn. (11).

$$V=-\frac{\rho g}{4\mu}\left(R_i^2-R_o^2\right)+c_1\ln\frac{R_i}{R_o}$$

$$c_1=\frac{\left[V+\frac{\rho g}{4\mu}\left(R_i^2-R_o^2\right)\right]}{\ln\frac{R_i}{R_o}} \tag{12}$$

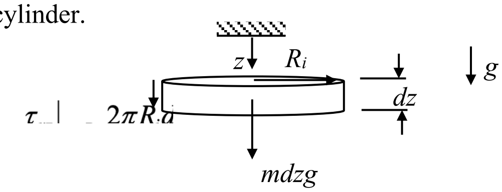
Find c_2 by applying the no-slip condition at $r = R_o$:

$$0 = -\frac{\rho g}{4\mu} R_o^2 + \frac{\left[V + \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)\right]}{\ln \frac{R_i}{R_o}} \ln R_o + c_2 \tag{13}$$

$$c_2 = \frac{\rho g}{4\mu} R_o^2 - \frac{\left[V + \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)\right]}{\ln \frac{R_i}{R_o}} \ln R_o \tag{14}$$

Perform a force balance on a small length of the cylinder.

$$\sum F_z = 0 = mdzg + \tau_{rz}\big|_{r=R_i} 2\pi R_i dz \tag{15}$$



$$\tau_{rz}\big|_{r=R_i} = -\frac{mg}{2\pi R_i} \tag{16}$$

Since the fluid is Newtonian:

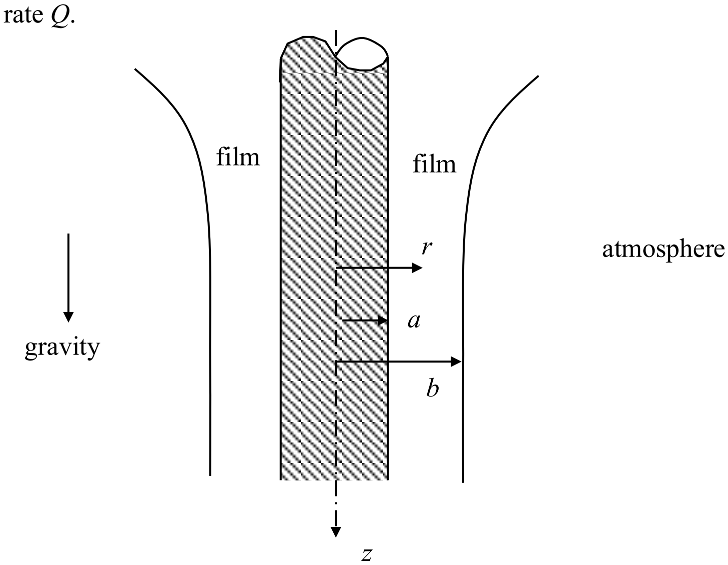
$$\tau_{rz} = \mu \frac{du_z}{dr} \Rightarrow \tau_{rz}\big|_{r=R_i} = \mu\left[-\frac{\rho g}{2\mu} r + \frac{c_1}{r}\right]_{r=R_i} = \mu\left\{-\frac{\rho g}{2\mu} R_i + \frac{\left[V + \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)\right]}{R_i \ln \frac{R_i}{R_o}}\right\} \tag{17}$$

Substitute Eqn. (17) into Eqn. (16) and solve for V.

$$\mu\left\{-\frac{\rho g}{2\mu} R_i + \frac{\left[V + \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)\right]}{R_i \ln \frac{R_i}{R_o}}\right\} = -\frac{mg}{2\pi R_i} \tag{18}$$

$$\boxed{V = \left(R_i \ln \frac{R_i}{R_o}\right)\left(\frac{\rho g}{2\mu} R_i - \frac{mg}{2\pi R_i \mu}\right) - \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)} \tag{19}$$

Consider a film of Newtonian liquid draining at volume flow rate Q down the outside of a vertical rod of radius, a, as shown in the figure. Some distance down the rod, a fully developed region is reached where fluid shear balances gravity and the film thickness remains constant. Assuming incompressible laminar flow and negligible shear interaction with the atmosphere, find an expression for $u_z(r)$ and a relation for the volumetric flow rate Q.



SOLUTION:

The continuity and momentum equations in cylindrical coordinates for an <u>incompressible</u>, <u>Newtonian</u> fluid with <u>constant viscosity</u> are:

$$\frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0$$

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]+\rho f_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_\theta\right)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]+\rho f_\theta$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho f_z$$

Make the following additional assumptions:

1. steady flow $\Rightarrow \frac{\partial}{\partial t}\left(\cdots\right)=0$
2. gravity acts in the z-direction $\Rightarrow f_z=g, f_r=0, f_\theta=0$
3. fully-developed flow in the z-direction $\Rightarrow \frac{\partial u_r}{\partial z}=\frac{\partial u_\theta}{\partial z}=\frac{\partial u_z}{\partial z}=0$
4. the flow is axi-symmetric and there is no swirl velocity $\Rightarrow \frac{\partial u_r}{\partial \theta}=\frac{\partial u_\theta}{\partial \theta}=\frac{\partial u_z}{\partial \theta}=0, u_\theta=0$
5. no pressure gradients in the z direction (due to free surface) $\Rightarrow \frac{\partial p}{\partial z}=0$

Simplify the continuity equation using the given assumptions:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial\theta}}_{=0(4)}+\underbrace{\frac{\partial u_z}{\partial z}}_{=0(3)}=0 \Rightarrow \frac{\partial(ru_r)}{\partial r}=0 \Rightarrow ru_r=\text{constant}$$

Note that from assumptions 3 and 4, u_r is not a function of either θ or z. Since there is no radial flow at this inner boundary ($r = a$), the constant in the previous equation must be zero. Thus,

$u_r = 0$ (condition 6)

Now simplify the momentum equations using our assumptions and condition 6:

$$\rho\left(\underbrace{\frac{\partial u_r}{\partial t}}_{=0(1,6)}+\underbrace{u_r\frac{\partial u_r}{\partial r}}_{=0(6)}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}}_{=0(4,6)}-\underbrace{\frac{u_\theta^2}{r}}_{=0(4)}+\underbrace{u_z\frac{\partial u_r}{\partial z}}_{=0(3,6)}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_r}_{=0(6)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_r}{\partial\theta^2}}_{=0(4,6)}+\underbrace{\frac{\partial^2 u_r}{\partial z^2}}_{=0(3,6)}-\frac{2}{r^2}\underbrace{\frac{\partial u_\theta}{\partial\theta}}_{=0(4)}\right]+\rho\underbrace{f_r}_{=0(2)}$$

$$\rho\left(\underbrace{\frac{\partial u_\theta}{\partial t}}_{=0(1,4)}+\underbrace{u_r\frac{\partial u_\theta}{\partial r}}_{=0(4,6)}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}}_{=0(4)}+\underbrace{\frac{u_r u_\theta}{r}}_{=0(4,6)}+\underbrace{u_z\frac{\partial u_\theta}{\partial z}}_{=0(3,4)}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_\theta}_{=0(4)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_\theta}{\partial\theta^2}}_{=0(4)}+\underbrace{\frac{\partial^2 u_\theta}{\partial z^2}}_{=0(3,4)}+\frac{2}{r^2}\underbrace{\frac{\partial u_r}{\partial\theta}}_{=0(4,6)}\right]+\rho\underbrace{f_\theta}_{=0(2)}$$

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(1)}+\underbrace{u_r}_{=0(6)}\frac{\partial u_z}{\partial r}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}}_{=0(4)}+u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0(3)}\right)=-\underbrace{\frac{\partial p}{\partial z}}_{=0(5)}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_z}{\partial\theta^2}}_{=0(4)}+\underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(3)}\right]+\rho\underbrace{f_z}_{=g(2)}$$

$$\frac{\partial p}{\partial r}=0$$

$$\frac{\partial p}{\partial\theta}=0$$

(Note that since the pressure on the free surface remains constant at atmospheric pressure and the pressure not a function of either r or θ, the pressure everywhere in the z direction will remain constant. Hence, assumption #5 is a reasonable one.)

$$\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)=-\rho g$$

Note that since u_z is not a function of θ or z (assumptions 3 and 4), the partial derivatives with respect to r in the last equation can be written as ordinary derivatives.

Simplifying the last equation gives:

$$\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)=-\rho g \Rightarrow r\frac{du_z}{dr}=-\frac{\rho g}{2\mu}r^2+c_1$$

$$\frac{du_z}{dr}=-\frac{\rho g}{2\mu}r+\frac{c_1}{r}$$

$$u_z=-\frac{\rho g}{4\mu}r^2+c_1\ln r+c_2$$

Apply the following boundary conditions to determine the unknown constants:

no-shear stress at $r = b$:

$$\sigma_{rz}(r=b)=0=\mu\frac{du_z}{dr}(r=b)=0=-\frac{\rho g}{2\mu}b+\frac{c_1}{b}\Rightarrow c_1=\frac{\rho g b^2}{2\mu}$$

no-slip at $r = a$:

$$u_z(r=a)=0=-\frac{\rho g}{4\mu}a^2+\frac{\rho g b^2}{2\mu}\ln a+c_2\Rightarrow c_2=\frac{\rho g}{4\mu}a^2-\frac{\rho g b^2}{2\mu}\ln a$$

Hence, the velocity profile in the z-direction is:

$$\boxed{u_z = -\frac{\rho g}{4\mu}\left(r^2 - a^2\right) + \frac{\rho g b^2}{2\mu}\ln\left(\frac{r}{a}\right)}$$

and the shear stress on an r-face in the z-direction is:

$$\sigma_{rz} = \mu\frac{du_z}{dr} = \frac{-\rho g\left(r^2 - b^2\right)}{2r}$$

The volumetric flow rate, Q, is given by:

$$Q = \int_{r=a}^{r=b} u_z\left(2\pi r dr\right)$$

$$= \int_a^b\left[\frac{-\rho g}{4\mu}\left(r^2 - a^2\right) + \frac{\rho g b^2}{2\mu}\ln\left(\frac{r}{a}\right)\right]\left(2\pi r dr\right)$$

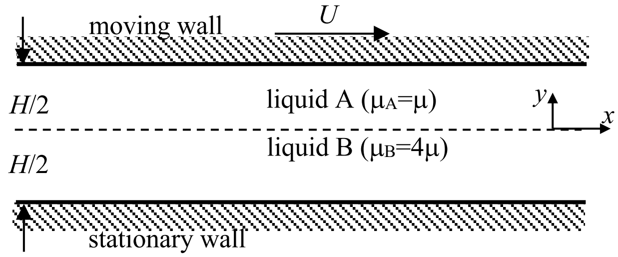
$$= -\frac{\pi\rho g}{2\mu}\int_a^b\left(r^2 - a^2\right)rdr + \frac{\pi\rho g b^2}{\mu}\int_a^b\ln\left(\frac{r}{a}\right)rdr$$

$$= -\frac{\pi\rho g}{2\mu}\left[\frac{b^4 - a^4}{4} - \frac{a^2b^2 - a^4}{2}\right] + \frac{\pi\rho g b^2}{\mu}\left[\frac{b^2}{2}\ln\left(b/a\right) - \frac{b^2 - a^2}{4}\right]$$

$$= -\frac{\pi\rho g}{2\mu}\left[\frac{b^4 - a^4}{4} - \frac{2a^2b^2 - 2a^4}{4} - b^4\ln\left(b/a\right) + \frac{2b^4 - 2a^2b^2}{4}\right]$$

$$\boxed{\therefore Q = -\frac{\pi\rho g}{2\mu}\left[\frac{3b^4 - 4a^2b^2 + a^4}{4} - b^4\ln\left(b/a\right)\right]}$$

Two immiscible viscous liquids are introduced into a Couette flow device so that they form two layers of equal height as shown:



The dynamic viscosity, μ, of liquid A is one quarter that of liquid B. The upper plate is moved at a constant velocity, U, while the bottom plate remains stationary.

a. Determine the velocity of the interface between the two liquids.
b. Determine the "apparent viscosity" of the mixture as seen by an experimenter who believes that only one liquid is in the device.

SOLUTION:

Make several assumptions regarding the flow in each region:

1. planar flow $\Rightarrow$ $\partial/\partial z(\cdots)=0, u_z=0$
2. steady flow $\Rightarrow$ $\partial/\partial t(\cdots)=0$
3. fully-developed flow in the x-direction $\Rightarrow$ $\partial u_x/\partial x = \partial u_y/\partial x = 0$
4. no pressure gradient in the x-direction $\Rightarrow$ $\partial p/\partial x = 0$
5. no body forces $\Rightarrow$ $g_x = g_y = g_z = 0$

First examine the continuity equation.

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \frac{\partial u_y}{\partial y} = 0 \;\Rightarrow\; \frac{\partial u_y}{\partial y} = 0$$

This result combined with assumptions 1 and 3 indicate that u_y = constant. Since there is no flow through the walls, $u_y = 0$ everywhere (call this condition #6).

Simplify the Navier-Stokes equation in the x-direction.

$$\rho\left[\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#2)} + u_x\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \underbrace{u_y}_{=0(\#6)}\frac{\partial u_x}{\partial y}\right] = -\underbrace{\frac{\partial p}{\partial x}}_{=0(\#4)} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#3)} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho\underbrace{g_x}_{=0(\#5)}$$

$$\frac{d^2 u_x}{dy^2} = 0 \quad \text{(Note that } u_x = u_x(y) \text{ from assumptions 1, 2, and 3.)} \tag{1}$$

Integrating twice with respect to y gives:

$$\frac{du_x}{dy} = c_1 \tag{2}$$

$$u_x = c_1 y + c_2 \tag{3}$$

Note that Eqns. (1) - (3) are valid for both fluids (A and B).

Now apply boundary conditions for each fluid.

Fluid B:

no-slip at $y = 0$ $\Rightarrow$ $u_{xB}(y=0) = V_i \Rightarrow c_{2B} = V_i$ (where V_i is the interface velocity)

no-slip at $y = -H/2$ $\Rightarrow$ $u_{xB}\left(y=-\frac{H}{2}\right) = 0 \Rightarrow c_{1B} = \dfrac{2V_i}{H}$

$$\therefore u_{xB} = V_i\left[2\left(\frac{y}{H}\right)+1\right] \tag{4}$$

Fluid A:

no-slip at $y = 0$ $\Rightarrow$ $u_{xA}(y=0) = V_i \Rightarrow c_{2A} = V_i$ (where V_i is the interface velocity)

no-slip at $y = H/2$ $\Rightarrow$ $u_{xA}\left(y=\frac{H}{2}\right) = U \Rightarrow c_{1A} = \dfrac{2(U-V_i)}{H}$

$$\therefore u_{xA} = 2(U-V_i)\left(\frac{y}{H}\right)+V_i \tag{5}$$

Also note that the shear stress is continuous at the interface.

$$\mu_A \left.\frac{du_{xA}}{dy}\right|_{y=0} = \mu_B \left.\frac{du_{xB}}{dy}\right|_{y=0}$$

$$\mu\frac{2(U-V_i)}{H} = 4\mu\frac{2V_i}{H}$$

$$\boxed{\therefore V_i = \tfrac{1}{5}U} \tag{6}$$

To determine the "apparent viscosity", μ^*, note that if there was only a single fluid between the two plates with viscosity, μ^*, the shear stress exerted by the upper plate would be:

$$\tau = \mu^*\left(\frac{U}{H}\right) \tag{7}$$

With the two fluids, the shear stress exerted by the upper plate on fluid A is:

$$\tau = \mu\frac{2(U-V_i)}{H} = \mu\frac{2\left(U-\frac{1}{5}U\right)}{H} = \tfrac{8}{5}\mu\left(\frac{U}{H}\right) \tag{8}$$

Equating Eqns. (7) and (8) shows that the apparent viscosity is:

$$\boxed{\mu^* = \tfrac{8}{5}\mu} \tag{9}$$

In cylindrical coordinates, the momentum equations for an inviscid fluid (Euler's equations) become:

$$\rho\left(\frac{Du_r}{Dt}-\frac{u_\theta^2}{r}\right)=-\frac{\partial p}{\partial r}+\rho f_r$$

$$\rho\left(\frac{Du_\theta}{Dt}-\frac{u_\theta u_r}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho f_\theta$$

$$\rho\frac{Du_z}{Dt}=-\frac{\partial p}{\partial z}+\rho f_z$$

where u_r, u_θ, and u_z are the velocities in the r, θ, and z directions, p is the pressure, ρ is the fluid density, and f_r, f_θ, and f_z are the body force components. The Lagrangian derivative is:

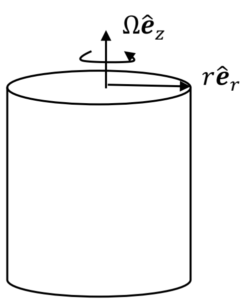
$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial \theta}+u_z\frac{\partial}{\partial z}$$

A cylinder is rotated at a constant angular velocity denoted by Ω. The cylinder contains a compressible fluid which rotates with the cylinder so that the fluid velocity at any point is $u_\theta=\Omega r$ ($u_r=u_z=0$). If the density of the fluid, ρ, is related to the pressure, p, by the polytropic relation:

$$p = A\rho^k$$

where A and k are known constants, find the pressure distribution $p(r)$ assuming that the pressure, p_0, at the center (r=0) is known. Neglect all body forces.

SOLUTION:



Make the following assumptions.

1. steady flow $\Rightarrow \frac{\partial}{\partial t}(\cdots)=0$
2. no body forces $\Rightarrow f_r = f_\theta = f_z = 0$
3. no flow in the z-direction and no variation in the z-direction $\Rightarrow u_z = 0; \frac{\partial}{\partial z}(\cdots)=0$
4. no flow in the r-direction $\Rightarrow u_r = 0$
5. solid body rotation $\Rightarrow u_\theta = \Omega r$

Simplify Euler's equations using the given assumptions.
r-direction:

$$\rho\left(\underbrace{\frac{\partial u_r}{\partial t}}_{=0(\#1)}+\underbrace{u_r\frac{\partial u_r}{\partial r}}_{=0(\#4)}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}_{=0(\#4)}+\underbrace{u_z\frac{\partial u_r}{\partial z}}_{=0(\#3,\#4)}-\underbrace{\frac{u_\theta^2}{r}}_{=\Omega^2 r}\right)=-\frac{\partial p}{\partial r}+\underbrace{\rho f_r}_{=0(\#2)}$$

$$\frac{\partial p}{\partial r}=\rho\Omega^2 r \tag{1}$$

θ-direction:

$$\rho\left(\underbrace{\frac{\partial u_\theta}{\partial t}}_{=0(\#1)}+\underbrace{u_r}_{=0(\#4)}\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#5)}+\underbrace{u_z\frac{\partial u_\theta}{\partial z}}_{=0(\#3)}-\underbrace{\frac{u_\theta u_r}{r}}_{=0(\#4)}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\underbrace{\rho f_\theta}_{=0(\#2)}$$

$$\frac{\partial p}{\partial \theta}=0 \tag{2}$$

z-direction:

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(\#1)}+\underbrace{u_r\frac{\partial u_z}{\partial r}}_{=0(\#3,\#4)}+\frac{u_\theta}{r}\underbrace{\frac{\partial u_z}{\partial \theta}}_{=0(\#3)}+\underbrace{u_z\frac{\partial u_z}{\partial z}}_{=0(\#3)}\right)=-\frac{\partial p}{\partial z}+\underbrace{\rho f_z}_{=0(\#2)}$$

$$\frac{\partial p}{\partial z}=0 \tag{3}$$

Note that Eqns. (2) and (3) imply that:

$$p=p(r)\Rightarrow\frac{\partial p}{\partial r}=\frac{dp}{dr} \tag{4}$$

so that Eqn. (1) becomes:

$$\frac{dp}{dr}=\rho\Omega^2 r \tag{5}$$

Substituting the given relation between the pressure and density gives:

$$\frac{dp}{dr}=\left(\frac{p}{A}\right)^{1/k}\Omega^2 r$$

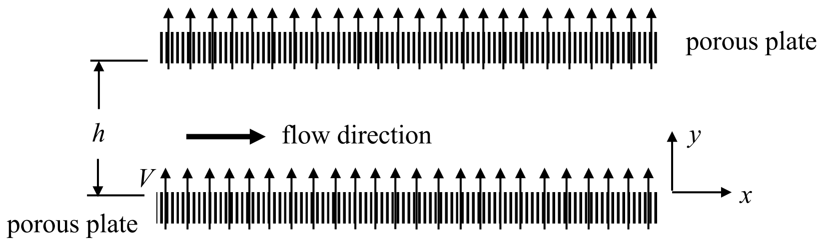
$$\int_{p=p_0}^{p=p} p^{-1/k}dp=A^{-1/k}\Omega^2\int_{r=0}^{r=r} r\,dr$$

$$\frac{1}{1-1/k}\left(p^{1-1/k}-p_0^{1-1/k}\right)=\tfrac{1}{2}A^{-1/k}\Omega^2 r^2$$

$$p^{1-1/k}=p_0^{1-1/k}+\left(1-1/k\right)\tfrac{1}{2}A^{-1/k}\Omega^2 r^2$$

$$\boxed{\therefore\ p(r)=\left[p_0^{\frac{k-1}{k}}+\tfrac{k-1}{2k}A^{-1/k}\Omega^2 r^2\right]^{\frac{k}{k-1}}} \tag{6}$$

An incompressible fluid flows between two porous, parallel flat plates as shown:



An identical fluid is injected at a constant speed V through the bottom plate and simultaneously extracted from the upper plate at the same velocity. Assume the flow to be steady, fully-developed, the pressure gradient in the x-direction is a constant, and neglect body forces. Determine appropriate expressions for the x and y velocity components.

SOLUTION:

First, make several assumptions regarding the flow.

1. The flow is steady. $\Rightarrow$ $\partial/\partial t(\cdots) = 0$
2. The flow is fully developed in the x-direction. $\Rightarrow$ $\partial u_x/\partial x = \partial u_y/\partial x = \partial u_z/\partial x = 0$
3. The flow is planar. $\Rightarrow$ $u_z = \text{constant}, \partial/\partial z(\cdots) = 0$
4. The pressure gradient in the x-direction is constant. $\Rightarrow$ $\partial p/\partial x = \text{constant}$
5. Body forces can be neglected. $\Rightarrow$ $g_x = g_y = g_z = 0$
6. The fluid is incompressible and Newtonian.

First examine the continuity equation.

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#2)} + \frac{\partial u_y}{\partial y} = 0 \;\Rightarrow\; \frac{\partial u_y}{\partial y} = 0$$

Since the y-velocity doesn't vary in the x or z-directions either (assumptions #2 and #3, respectively), the y-velocity must be a constant, *i.e.* u_y = constant. Since the y-velocity at the lower plate is $u_y = V$, we must have everywhere:

$$\boxed{u_y = V} \quad \text{(Call this condition \#7.)} \tag{1}$$

Now examine the Navier-Stokes equation in the x-direction.

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#1)} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#2)} + \underbrace{u_y}_{=V(\#7)} \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#2)} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho \underbrace{g_x}_{=0(\#5)}$$

$$\rho V \frac{du_x}{dy} = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u_x}{dy^2} \tag{2}$$

Let $z = du_x/dy$ so that Eqn. (2) becomes:

$$\rho V z = -\frac{\partial p}{\partial x} + \mu \frac{dz}{dy}$$

$$\int \frac{\mu dz}{\frac{\partial p}{\partial x} + \rho V z} = \int dy$$

$$\frac{\mu}{\rho V} \ln\left(\frac{\partial p}{\partial x} + \rho V z\right) = y + c \quad \text{(where } c \text{ is a constant)}$$

$$\frac{\partial p}{\partial x} + \rho V \frac{du_x}{dy} = c \exp\left(\frac{\rho V y}{\mu}\right)$$

$$\frac{du_x}{dy} = c \exp\left(\frac{\rho V y}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x}$$

$$\int du_x = c \int \exp\left(\frac{\rho V y}{\mu}\right) dy - \frac{1}{\rho V}\frac{\partial p}{\partial x}\int dy$$

$$u_x = c_1 \exp\left(\frac{\rho V y}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x} y + c_2 \quad \text{(where } c_1 \text{ and } c_2 \text{ are constants)} \tag{3}$$

Apply boundary conditions.

no-slip at $y = 0$ $\Rightarrow$ $u_x(y=0) = 0$ $\Rightarrow$ $c_2 = -c_1$

no-slip at $y = h$ $\Rightarrow$ $u_x(y=h) = 0$

$$\Rightarrow \quad 0 = c_1 \exp\left(\frac{\rho V h}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x} h + c_2$$

$$0 = c_1 \exp\left(\frac{\rho V h}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x} h - c_1$$

$$c_1\left[\exp\left(\frac{\rho V h}{\mu}\right) - 1\right] = \frac{1}{\rho V}\frac{\partial p}{\partial x} h$$

$$c_1 = \frac{\frac{h}{\rho V}\left(-\frac{\partial p}{\partial x}\right)}{1 - \exp\left(\frac{\rho V h}{\mu}\right)}$$

Hence,

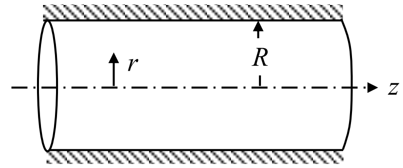
$$u_x = c_1 \exp\left(\frac{\rho V y}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x} y - c_1$$

$$= c_1\left[\exp\left(\frac{\rho V y}{\mu}\right) - 1\right] - \frac{y}{\rho V}\frac{\partial p}{\partial x}$$

$$\boxed{u_x = \frac{h}{\rho V}\left(\frac{\partial p}{\partial x}\right)\left\{\left[\frac{1 - \exp\left(\frac{\rho V y}{\mu}\right)}{1 - \exp\left(\frac{\rho V h}{\mu}\right)}\right] - \frac{y}{h}\right\}}$$

Consider the fully-developed, steady, laminar circular pipe flow of an incompressible, *non-Newtonian* fluid due to a constant pressure gradient $dp/dz < 0$. Gravitational effects may be neglected. The normal stress in this fluid in the z-direction, *i.e.* σ_{zz}, is equal to $-p$ where p is the pressure. The shear stress, σ_{rz}, is related to the velocity gradient by:

$$\sigma_{rz} = C\left(-\frac{du_z}{dr}\right)^2$$



where C is a known constant.

Find:
1. the velocity profile, $u_z(r)$, and
2. the friction factor, f, (*i.e.* the wall shear stress made dimensionless using the dynamic pressure based on the average velocity in the pipe)

for this pipe flow in terms of C, ρ (the fluid density), dp/dz, r, and R (the radius of the pipe), or a subset of these parameters.

SOLUTION:
Make the following assumptions regarding the flow.

1. steady flow $\Rightarrow$ $\frac{\partial}{\partial t}(\cdots) = 0$ (1)
2. fully developed flow in z-direction $\Rightarrow$ $\frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = \frac{\partial u_z}{\partial z} = 0$ (2)
3. axisymmetric flow with no swirl velocity $\Rightarrow$ $\frac{\partial}{\partial \theta}(\cdots) = 0, u_\theta = 0$ (3)
4. negligible gravitational forces in z-dir. $\Rightarrow$ $f_z = 0$ (4)

First consider the continuity equation.

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#3)} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#2)} = 0 \quad \Rightarrow \quad u_r = \frac{\text{constant}}{r} \tag{5}$$

Since there is no flow through the pipe wall, we must have $\underline{u_r = 0}$ (call this condition #6).

Now consider the momentum equation in the z-direction. Note that the Navier-Stokes equation should not be used since the flow is non-Newtonian.

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(\#1)} + \underbrace{u_r}_{=0(\#5)}\frac{\partial u_z}{\partial r} + \underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}_{=0(\#2)} + u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#2)}\right) = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{1}{r}\underbrace{\frac{\partial \sigma_{\theta z}}{\partial \theta}}_{=0(\#3)} + \frac{\partial \sigma_{zz}}{\partial z} + \rho\underbrace{f_z}_{=0(\#4)} \tag{6}$$

$$\frac{\partial}{\partial r}(r\sigma_{rz}) + r\frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{7}$$

We're given the expressions for the flow stresses.

$$\sigma_{zz} = -p \quad \text{and} \quad \sigma_{rz} = C\left(-\frac{du_z}{dr}\right)^2 \tag{8}$$

Substitute and simplify.

$$\frac{d}{dr}\left[rC\left(-\frac{du_z}{dr}\right)^2\right]-r\frac{dp}{dz}=0 \tag{9}$$

(Note: $u_z = u_z(r)$ so we may replace the partial derivative in the first term with an ordinary derivative.)

$$d\left[rC\left(-\frac{du_z}{dr}\right)^2\right]=r\frac{dp}{dz}dr$$

$$rC\left(-\frac{du_z}{dr}\right)^2=\frac{r^2}{2}\frac{dp}{dz}+c_1 \tag{10}$$

The velocity profile will be symmetric about the centerline of the pipe. Hence:

$$\frac{du_z}{dr}(r=0)=0 \;\Rightarrow\; c_1=0 \tag{11}$$

Simplify Eqn. (10) further.

$$\frac{du_z}{dr}=-\left(\frac{r}{2C}\frac{dp}{dz}\right)^{1/2} \tag{12}$$

$$u_z=-\frac{2}{3}\left(\frac{r^3}{2C}\frac{dp}{dz}\right)^{1/2}+c_2 \tag{13}$$

Apply the no-slip boundary condition, $u_z = 0$, at $r = R$ to determine the unknown constant.

$$u_z(r=R)=0=-\frac{2}{3}\left(\frac{R^3}{2C}\frac{dp}{dz}\right)^{1/2}+c_2 \quad\Rightarrow\quad c_2=\frac{2}{3}\left(\frac{R^3}{2C}\frac{dp}{dz}\right)^{1/2} \tag{14}$$

Hence, the velocity profile for this flow is:

$$\boxed{u_z=\frac{2}{3}\left(\frac{1}{2C}\frac{dp}{dz}\right)^{1/2}\left(R^{3/2}-r^{3/2}\right)} \tag{15}$$

or, in dimensionless form:

$$\boxed{\frac{u_z}{\frac{2}{3}\left(\frac{R^3}{2C}\frac{dp}{dz}\right)^{1/2}}=1-\left(\frac{r}{R}\right)^{3/2}} \tag{16}$$

Note that $dp/dz < 0 \Rightarrow C < 0$.

The wall shear stress may be found from the given stress relation and velocity profile:

$$\sigma_{rz}=C\left(-\frac{du_z}{dr}\right)^2=C\left(\frac{r}{2C}\frac{dp}{dz}\right)=\frac{r}{2}\frac{dp}{dz} \tag{17}$$

$$f\equiv\frac{\sigma_{rz}\big|_{r=R}}{\frac{1}{2}\rho\bar{u}_z^2}=\frac{R}{\rho\bar{u}_z^2}\frac{dp}{dz} \tag{18}$$

where

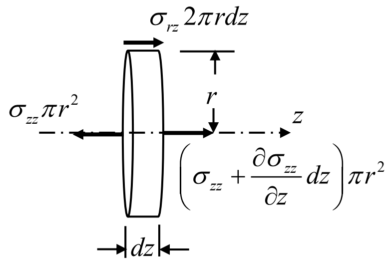
$$\bar{u}_z=\frac{1}{\pi R^2}\int_{r=0}^{r=R}u_z(2\pi r\,dr)=\frac{1}{R^2}\frac{4}{3}\left(\frac{1}{2C}\frac{dp}{dz}\right)^{1/2}\int_{r=0}^{r=R}\left(R^{3/2}-r^{3/2}\right)r\,dr=\frac{1}{R^2}\frac{4}{3}\left(\frac{1}{2C}\frac{dp}{dz}\right)^{1/2}\left(\tfrac{1}{2}R^{7/2}-\tfrac{2}{7}R^{7/2}\right) \tag{19}$$

$$\bar{u}_z=\frac{2}{7}\left(\frac{R^3}{2C}\frac{dp}{dz}\right)^{1/2} \tag{20}$$

Thus,

$$f = \frac{R \, {}^{dp}\!/_{dz}}{\rho \left(\frac{2}{7}\right)^2 \left(\frac{R^3}{2C}\frac{dp}{dz}\right)} \Rightarrow \boxed{f = \frac{49}{2}\frac{C}{\rho R^2}} \tag{21}$$

Note that we could have also arrived at Eqn. (12) by performing a force balance on the differential fluid element shown below.



Since the flow is steady and fully developed in the z-direction, the forces must balance in the z-direction.

$$\sum F_z = 0 = -\sigma_{zz}\pi r^2 + \left(\sigma_{zz} + \frac{\partial \sigma_{zz}}{\partial z} dz\right)\pi r^2 + \sigma_{rz} 2\pi r dz = 0 \tag{22}$$

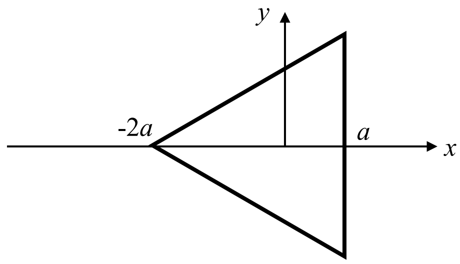
$$-\frac{\partial \sigma_{zz}}{\partial z} r = 2\sigma_{rz}$$

Substituting the given stresses gives:

$$\frac{dp}{dz} r = 2C\left(-\frac{du_z}{dr}\right)^2$$

$$\therefore \frac{du_z}{dr} = -\left(\frac{r}{2C}\frac{dp}{dz}\right)^{1/2} \quad \text{(This is the same as Eqn. (12)!)} \tag{23}$$

Consider a viscous flow through a pipe with an equilateral triangle cross-section:



a. Determine the velocity distribution in the pipe assuming a constant pressure gradient, $dp/dz<0$. Hint: Find a function that is zero on the boundary and check if it's a solution.
b. Determine the magnitude and location of the maximum shear stress on the pipe wall.
c. Determine the average shear stress that the walls exert on the fluid. Note: There is an easy way to calculate this quantity.

SOLUTION:

First, make several assumptions regarding this flow:

1. constant property fluid $\Rightarrow$ ρ, μ are constants
2. fully-developed flow in the z-direction $\Rightarrow$ $\partial u_x/\partial z = \partial u_y/\partial z = \partial u_z/\partial z = 0$
3. steady flow $\Rightarrow$ $\partial/\partial t(\cdots) = 0$
4. negligible body forces $\Rightarrow$ $g_x = g_y = g_z = 0$
5. laminar, unidirectional flow in z-direction $\Rightarrow$ $u_x = u_y = 0$

Simplify the Navier-Stokes equation in the z-direction:

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(\#3)} + \underbrace{u_x}_{=0(\#5)}\frac{\partial u_z}{\partial x} + \underbrace{u_y}_{=0(\#5)}\frac{\partial u_z}{\partial y} + u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#2)}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(\#2)}\right) + \rho\underbrace{g_z}_{=0(\#4)}$$

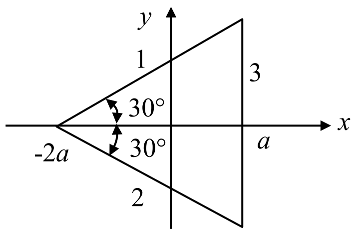
$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dz} \quad \text{(Poisson's Equation!)} \tag{1}$$

Notes:
1. Using our assumptions the continuity equation is automatically satisfied.
2. The x and y components of the Navier-Stokes equations indicate that:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \;\Rightarrow\; p = p(z)$$

The boundary conditions for this flow are that there is no slip at the walls and that the flow velocities remain finite everywhere.

If we can find a velocity profile that satisfies the PDE and the boundary conditions, then it will be the solution for the flow. Let's try to find a function that will automatically satisfy the no-slip condition at the pipe walls. If we multiply together the equations describing each boundary, then the resulting equation will satisfy the boundary conditions at any boundary.



Wall 1: $y = \tan 30^\circ (x+2a)$

Wall 2: $y = -\tan 30^\circ (x+2a)$

Wall 3: $x = a$

Let's assume:

$$u_z = c\left[y - \tfrac{\sqrt{3}}{3}(x+2a)\right]\left[y + \tfrac{\sqrt{3}}{3}(x+2a)\right]\left[x-a\right] \tag{2}$$

which, after some algebra, simplifies to:

$$u_z = c\left[xy^2 - y^2 a - \tfrac{1}{3}x^3 - x^2 a + \tfrac{4}{3}a^3\right] \tag{3}$$

since this function will automatically satisfy the boundary conditions.

The function must also satisfy the governing equations to be a valid fluid velocity profile. It clearly satisfies the continuity equation since $u_z = u_z(x, y)$ and $u_x = u_y = 0$. To see if it satisfies the Navier-Stokes equation, substitute Eqn. (3) into Eqn. (1).

$$\frac{1}{\mu}\frac{dp}{dz} = \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = c\left[-2x-2a\right] + c\left[2x-2a\right] = -4ac$$

$$\therefore c = -\frac{1}{4a\mu}\frac{dp}{dz} \quad \text{If } c \text{ has this value, then the } z\text{-momentum equation is satisfied.} \tag{4}$$

Hence, the velocity profile is:

$$\boxed{u_z = \frac{1}{4a\mu}\left(-\frac{dp}{dz}\right)\left[y - \tfrac{\sqrt{3}}{3}(x+2a)\right]\left[y + \tfrac{\sqrt{3}}{3}(x+2a)\right]\left[x-a\right]} \tag{5}$$

or

$$u_z = \frac{1}{4a\mu}\left(-\frac{dp}{dz}\right)\left[xy^2 - y^2 a - \tfrac{1}{3}x^3 - x^2 a + \tfrac{4}{3}a^3\right] \tag{6}$$

The shear stress acting on the fluid is found using the constitutive stress-strain rate relation for a Newtonian fluid:

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

For this flow, $u_x = u_z = 0$ and $u_z = u_z(x, y)$ so that we have as the only shear stresses:

$$\sigma_{xz} = \sigma_{zx} = \mu\frac{\partial u_z}{\partial x} = \frac{1}{4a}\left(-\frac{dp}{dz}\right)\left[y^2 - x^2 - 2xa\right] \quad (7)$$

$$\sigma_{yz} = \sigma_{zy} = \mu\frac{\partial u_z}{\partial y} = \frac{1}{4a}\left(-\frac{dp}{dz}\right)\left[2xy - 2ya\right] \quad (8)$$

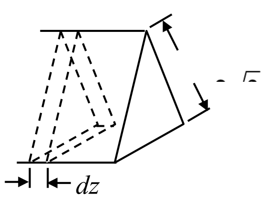
Since the flow has symmetry, the shear stress distribution on each of the walls will be the same. Let's find the maximum shear stress on the $x = a$ wall (note that $\sigma_{yz} = 0$ along this wall):

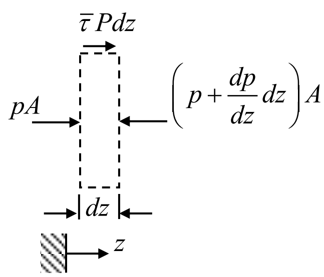
$$\max\left(\sigma_{xz}\big|_{x=a}\right) = \max\left\{\frac{1}{4a}\left(-\frac{dp}{dz}\right)\left[y^2 - 3a^2\right]\right\}$$

$$\boxed{\therefore \max\left(\sigma_{xz}\big|_{x=a}\right) = \left|\frac{3a}{4}\left(\frac{dp}{dz}\right)\right|} \text{ which occurs at } y = 0 \text{ along the } x = a \text{ wall} \quad (9)$$

Note that from symmetry, the other two walls will also have the same maximum shear stress at their mid-lines.

The average shear stress can be found by considering a force balance on a differential CV:





where:
$\bar{\tau} \equiv$ average wall shear stress
$P \equiv$ pipe perimeter = $3\left(2\sqrt{3}a\right) = 6\sqrt{3}a$
$A \equiv$ pipe cross-sectional area = $\frac{1}{2}\left(2\sqrt{3}a\right)\left(3a\right) = 3\sqrt{3}a^2$

$$\sum F_z = 0 = \bar{\tau}Pdz + pA - \left(p + \frac{dp}{dz}dz\right)A$$

$$\bar{\tau} = \frac{dp}{dz}\frac{A}{P} = \frac{dp}{dz}\frac{3\sqrt{3}a^2}{6\sqrt{3}a}$$

$$\boxed{\therefore \bar{\tau} = \frac{a}{2}\frac{dp}{dz}}$$

(This is the shear stress the wall applies to the fluid. Note that $dp/dz < 0$ for flow in the positive z-direction.)

We could have also found the average wall stress by averaging the stress distribution found earlier for the $x = a$ wall.

$$\bar{\tau}\big|_{x=a} = \frac{1}{2a\sqrt{3}}\int_{y=-a\sqrt{3}}^{y=+a\sqrt{3}}\frac{1}{4a}\left(-\frac{dp}{dz}\right)\left[y^2 - 3a^2\right]dy = \frac{1}{8a^2\sqrt{3}}\left(-\frac{dp}{dz}\right)\left[\tfrac{1}{3}y^3 - 3a^2y\right]\Big|_{y=-a\sqrt{3}}^{y=+a\sqrt{3}}$$

$$= \frac{a^3}{8a^2\sqrt{3}}\left(-\frac{dp}{dz}\right)\left[2\sqrt{3} - 6\sqrt{3}\right]$$

$$\bar{\tau}\big|_{x=a} = \frac{a}{2}\left(\frac{dp}{dz}\right) \text{ (This is the same answer as before!)}$$

To find the stress distribution on the walls other than the $x = a$ wall, we need to make use of Cauchy's formula concerning stresses. Consider for example the wall described by the equation:

$$y = \tfrac{\sqrt{3}}{3}\left(x + 2a\right) \quad \text{(This is wall \#1.)} \tag{10}$$

The normal vector for this surface is:

$$\hat{\mathbf{n}} = -\sin 30^\circ \hat{\mathbf{e}}_x + \cos 30^\circ \hat{\mathbf{e}}_y = \underbrace{-\tfrac{1}{2}}_{n_x}\hat{\mathbf{e}}_x + \underbrace{\tfrac{\sqrt{3}}{2}}_{=n_y}\hat{\mathbf{e}}_y$$

The stress on this surface (acting in the z-direction) is:

$$\sigma_{z\hat{\mathbf{n}}} = \sigma_{ji} n_j = \sigma_{zx} n_x + \sigma_{zy} n_y \tag{11}$$

where σ_{zx} and σ_{zy} are given by Eqns. (7) and (8), respectively. Along the wall surface (given by Eqn. (10)) we have:

$$\sigma_{zx} = \frac{1}{4a}\left(-\frac{dp}{dz}\right)\left[\tfrac{1}{3}\left(x^2 + 4xa + 4a^2\right) - x^2 - 2xa\right] \tag{12}$$

$$\sigma_{zy} = \frac{1}{4a}\left(-\frac{dp}{dz}\right)\left[\tfrac{2\sqrt{3}}{3}x^2 + \tfrac{4\sqrt{3}}{3}xa - \tfrac{2\sqrt{3}}{3}xa - \tfrac{4\sqrt{3}}{3}a^2\right] \tag{13}$$

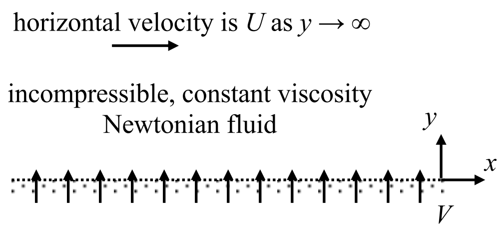
Substituting Eqns. (12) and (13) into Eqn. (11) gives:

$$\sigma_{z\hat{\mathbf{n}}} = \sigma_{ji} n_j = \frac{1}{4a}\left(-\frac{dp}{dz}\right)\left[-\tfrac{1}{2}\left(-\tfrac{2}{3}x^2 - \tfrac{2}{3}xa + \tfrac{4}{3}a^2\right) + \tfrac{\sqrt{3}}{2}\left(\tfrac{2\sqrt{3}}{3}x^2 + \tfrac{2\sqrt{3}}{3}xa - \tfrac{4\sqrt{3}}{3}a^2\right)\right]$$

$$\boxed{\therefore \sigma_{z\hat{\mathbf{n}}} = \frac{1}{4a}\left(-\frac{dp}{dz}\right)\left[\tfrac{4}{3}x^2 + \tfrac{4}{3}xa - \tfrac{8}{3}a^2\right]} \tag{14}$$

$\left|\max\limits_{x=[-2a,a]}\left(\sigma_{z\hat{\mathbf{n}}}\right)\right| = \left|\frac{3}{4a}\left(\frac{dp}{dz}\right)\right|$ occurring at $x = -a/2$ (at the middle of the wall).

Consider steady flow at horizontal velocity U (at $y \to \infty$) past an infinitely long and wide plate. The plate is porous and there is uniform flow normal to the surface at a constant velocity, V. Assume there are no pressure gradients and that gravity is negligible.



a. Determine the y-velocity at all points in the flow field.
b. Determine the x-velocity at all points in the flow field.
c. What restriction is there on the velocity V?
d. Quantify how far into the flow the wall effects are felt. Clearly indicate what criterion you are using.

SOLUTION:

Make the following assumptions about the flow:

1. The flow is planar. $\Rightarrow$ $\partial/\partial z(\cdots) = 0, u_z = 0$
2. The flow is steady. $\Rightarrow$ $\partial/\partial t(\cdots) = 0$
3. The flow is fully-developed in the x-direction. $\Rightarrow$ $\partial u_x/\partial x = \partial u_y/\partial x = 0$
4. Neglect gravity. $\Rightarrow$ $g_x = g_y = g_z = 0$
5. No pressure gradients. $\Rightarrow$ $\partial p/\partial x = \partial p/\partial y = \partial p/\partial z = 0$

The continuity equation for an incompressible, planar flow is:

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \frac{\partial u_y}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial u_y}{\partial y} = 0 \quad \text{(call this condition \#6)}$$

Since u_y is not a function of x (#3), z (#1), or y (#6), then u_y = constant. Since the flow at the wall has vertical velocity, V:

$$\boxed{u_y = V} \tag{1}$$

Now examine the x-momentum equation:

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#2)}+u_x\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)}+\underbrace{u_y}_{=V(\#6)}\frac{\partial u_x}{\partial y}\right)=-\underbrace{\frac{\partial p}{\partial x}}_{=0(\#5)}+\mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#3)}+\frac{\partial^2 u_x}{\partial y^2}\right)+\rho\underbrace{g_x}_{=0(\#4)}$$

$$V\frac{du_x}{dy}=\nu\frac{d^2u_x}{dy^2} \tag{2}$$

where the partial derivatives have been replaced by ordinary derivatives since u_x is not a function of x (#3) or z (#1). Not also that $\nu = \mu/\rho$.

Solving Eqn. (2):

$$\frac{V}{\nu}=\frac{d}{dy}\left(\ln\frac{du_x}{dy}\right)$$

$$\frac{V}{\nu}y+c_1=\ln\frac{du_x}{dy}$$

$$\exp\left(\frac{V}{\nu}y+c_1\right)=c_2\exp\left(\frac{V}{\nu}y\right)=\frac{du_x}{dy}$$

$$u_x=c_3\exp\left(\frac{V}{\nu}y\right)+c_4 \tag{3}$$

Apply the following boundary conditions:

no-slip at $y = 0$ $\Rightarrow$ $u_x(y=0)=0$ (4)

horz. velocity is U as $y \to \infty$ $\Rightarrow$ $u_x(y\to\infty)=U$ (5)

$$u_x(y=0)=c_3+c_4=0 \tag{6}$$

$$u_x(y\to\infty)=c_3\lim_{y\to\infty}\left[\exp\left(\frac{V}{\nu}y\right)\right]+c_4=U \tag{7}$$

Note that in order to have u_x remain finite as $y \to \infty$, we must have $\boxed{V<0}$. Hence, Eqn. (7) implies that $c_4 = U$. Substituting and simplifying gives:

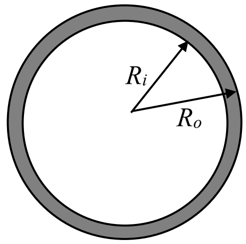
$$\boxed{u_x=U\left[1-\exp\left(\frac{V}{\nu}y\right)\right]} \tag{8}$$

To quantify the distance into the flow that the wall effects are felt, use the 99% boundary layer thickness, δ, *i.e.*:

$$\frac{u_x}{U}=0.99=1-\exp\left(\frac{V}{\nu}\delta\right)$$

$$\boxed{\delta=\frac{\nu}{V}\ln 0.01} \tag{9}$$

An incompressible, Newtonian liquid of density ρ and dynamic viscosity μ is sheared between concentric cylinders as shown in the sketch below. The inner cylinder radius is R_i and the outer cylinder radius is R_o.



a. Determine the velocity profile for the liquid in the gap assuming that the inner cylinder rotates with constant angular speed, ω. Do not assume that $(R_o - R_i) << R_o$.
b. Determine the torque (per unit depth into the page) acting on the outer wall of the cylinder.

SOLUTION:

The continuity and momentum equations in cylindrical coordinates for an incompressible, Newtonian fluid with constant viscosity are:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0$$

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]+\rho f_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]+\rho f_\theta$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho f_z$$

Make the following additional assumptions:

1. steady flow $\Rightarrow \frac{\partial}{\partial t}(\cdots)=0$
2. neglect gravity $\Rightarrow f_{B,z}=0, f_{B,r}=0, f_{B,\theta}=0$
3. planar flow $\Rightarrow u_z=\text{constant}, \frac{\partial}{\partial z}(\cdots)=0$
4. axi-symmetric flow $\Rightarrow \frac{\partial u_r}{\partial \theta}=\frac{\partial u_\theta}{\partial \theta}=\frac{\partial u_z}{\partial \theta}=0$
5. no pressure gradients in the θ direction $\Rightarrow \frac{\partial p}{\partial \theta}=0$ (due to the axi-symmetric flow assumption)

Simplify the continuity equation using the given assumptions:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}+\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#3)}=0 \;\Rightarrow\; \frac{\partial(ru_r)}{\partial r}=0 \Rightarrow ru_r=\text{constant}$$

Note that from assumptions #3 and #4, u_r is not a function of either z or θ. Since there is no radial flow at the inner boundary ($r = R_i$), the constant in the previous equation must be zero. Thus,

$u_r = 0$ (condition #6)

Now simplify the momentum equations using our assumptions and condition #6:

$$\rho\left(\underbrace{\frac{\partial u_r}{\partial t}}_{=0(\#1,\#6)}+\underbrace{u_r\frac{\partial u_r}{\partial r}}_{=0(\#6)}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}_{=0(\#4,\#6)}-\frac{u_\theta^2}{r}+\underbrace{u_z\frac{\partial u_r}{\partial z}}_{=0(\#3,\#6)}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_r}_{=0(\#6)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_r}{\partial \theta^2}}_{=0(\#4,\#6)}+\underbrace{\frac{\partial^2 u_r}{\partial z^2}}_{=0(\#3,\#6)}-\frac{2}{r^2}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}\right]+\rho\underbrace{f_r}_{=0(\#2)}$$

$$\Rightarrow\quad \frac{\partial p}{\partial r}=\rho\frac{u_\theta^2}{r}$$

$$\rho\left(\underbrace{\frac{\partial u_\theta}{\partial t}}_{=0(\#1)}+\underbrace{u_r\frac{\partial u_\theta}{\partial r}}_{=0(\#6)}+\frac{u_\theta}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(\#4)}+\underbrace{\frac{u_r u_\theta}{r}}_{=0(\#6)}+u_z\underbrace{\frac{\partial u_\theta}{\partial z}}_{=0(\#3)}\right)=-\frac{1}{r}\underbrace{\frac{\partial p}{\partial \theta}}_{=0(\#5)}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_\theta}{\partial \theta^2}}_{=0(\#4)}+\underbrace{\frac{\partial^2 u_\theta}{\partial z^2}}_{=0(\#3,\#4)}+\frac{2}{r^2}\underbrace{\frac{\partial u_r}{\partial \theta}}_{=0(\#4,\#6)}\right]+\rho\underbrace{f_\theta}_{=0(\#2)}$$

$$\Rightarrow\quad \mu\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru_\theta)\right]=0$$

Note that since u_θ is not a function of z or θ (assumptions #3 and #4), the partial derivatives with respect to r in the last equation can be written as ordinary derivatives. Integrating the second equation with respect to r twice gives:

$$u_\theta(r)=c_1 r+\frac{c_2}{r}$$

where c_1 and c_2 are constants.

The boundary conditions for the flow are:

no-slip at $r = R_i$: $\quad u_\theta(r=R_i)=\omega R_i$

no-slip at $r = R_o$: $\quad u_\theta(r=R_o)=0$

The boundary conditions are used to determine the unknown constants.

$$u_\theta(r=R_o)=0=c_1R_o+\frac{c_2}{R_o}\quad\Rightarrow\quad c_1=-\frac{c_2}{R_0^2}$$

$$u_\theta(r=R_i)=\omega R_i=c_1R_i+\frac{c_2}{R_i}=-\frac{c_2}{R_o^2}R_i+\frac{c_2}{R_i}=c_2\left(\frac{R_o^2-R_i^2}{R_iR_o^2}\right)\quad\Rightarrow\quad c_2=\omega\left(\frac{R_i^2R_o^2}{R_o^2-R_i^2}\right)$$

Hence:

$$u_\theta(r)=\omega\left(\frac{R_i^2R_o^2}{R_o^2-R_i^2}\right)\left(-\frac{r}{R_0^2}+\frac{1}{r}\right)$$

$$\boxed{\therefore u_\theta(r)=\omega r\left(\frac{R_i^2R_o^2}{R_o^2-R_i^2}\right)\left(\frac{R_0^2-r^2}{r^2R_0^2}\right)}$$

The torque on the outer wall (assuming unit depth) is given by:

$$T = R_o\left(2\pi R_o \sigma_{r\theta}\big|_{r=R_o}\right) = 2\pi R_o^2\left\{\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\underbrace{\frac{\partial u_r}{\partial\theta}}_{=0}\right]\right\}_{r=R_o}$$

$$= 2\pi R_o^2\left\{\mu r\frac{\partial}{\partial r}\left[\omega\left(\frac{R_i^2 R_o^2}{R_o^2-R_i^2}\right)\left(\frac{R_0^2-r^2}{r^2 R_0^2}\right)\right]\right\}_{r=R_o}$$

$$= 2\pi\mu\omega R_o^2\left(\frac{R_i^2 R_o^2}{R_o^2-R_i^2}\right)\left\{r\frac{\partial}{\partial r}\left[\left(\frac{1}{r^2}-\frac{1}{R_0^2}\right)\right]\right\}_{r=R_o}$$

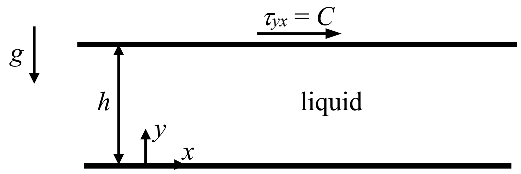
$$= 2\pi\mu\omega R_o^2\left(\frac{R_i^2 R_o^2}{R_o^2-R_i^2}\right)\left\{\frac{-2}{r^2}\right\}_{r=R_o}$$

$$T = -4\pi\mu\omega\left(\frac{R_i^2 R_o^2}{R_o^2-R_i^2}\right)$$

Note that the stress $\sigma_{r\theta}$ is the stress acting on the fluid. The stress acting on the cylinder will be in the opposite direction. Thus, the torque on the cylinder is:

$$\boxed{\therefore T = 4\pi\mu\omega\left(\frac{R_i^2 R_o^2}{R_o^2-R_i^2}\right)}$$

Consider a Newtonian liquid film that is driven by a constant shear stress, $\tau_{yx}(y = h) = C$, applied by a plate to the top surface. Assume that the liquid film is flat, fully developed, and has a constant pressure gradient in the x-direction such that there is zero net flow rate ($Q = 0$).



flow has width b into the page

Determine the velocity profile $u(y)$ and the pressure gradient dp/dx.

a. Use the continuity and Navier-Stokes equations to solve this problem.
b. Use a differential control volume and apply conservation of mass and the linear momentum equation to solve this problem.

SOLUTION:
Make the following assumptions about the flow:

1. The flow is planar. $\Rightarrow \partial/\partial z(\cdots) = 0, u_z = \text{constant}$
2. The flow is steady. $\Rightarrow \partial/\partial t(\cdots) = 0$
3. The flow is fully developed in the x-direction. $\Rightarrow \partial u_x/\partial x = \partial u_y/\partial x = 0$
4. Gravity acts in the $-y$ direction. $\Rightarrow g_y = -g; g_x = g_z = 0$

The continuity equation for an incompressible, planar flow is:

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \frac{\partial u_y}{\partial y} = 0 \Rightarrow \frac{\partial u_y}{\partial y} = 0. \tag{1}$$

Since the flow is also steady (#2), fully developed (#3), and planar (#1), the y-velocity can be at most a constant. Since $u_y = 0$ at the wall, then u_y everywhere is:

$$u_y = 0 \quad \text{(Call this condition \#5.)} \tag{2}$$

Now examine the x-momentum equation:

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#2)} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \underbrace{u_y}_{=0(\#5)} \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#3)} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho \underbrace{g_x}_{=0(\#4)}$$

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u_x}{dy^2} \tag{3}$$

where the partial derivatives have been replaced by ordinary derivatives since u_x is not a function of x (#3), t (#1), or z (#1).

Now consider the y-momentum equation,

$$\rho\left(\underbrace{\frac{\partial u_y}{\partial t}}_{=0(\#2)} + u_x \underbrace{\frac{\partial u_y}{\partial x}}_{=0(\#3)} + \underbrace{u_y}_{=0(\#5)} \underbrace{\frac{\partial u_y}{\partial y}}_{=0(\#5)}\right) = -\frac{\partial p}{\partial y} + \mu\left(\underbrace{\frac{\partial^2 u_y}{\partial x^2}}_{=0(\#5)} + \underbrace{\frac{\partial^2 u_y}{\partial y^2}}_{=0(\#5)}\right) + \rho \underbrace{g_y}_{=-g(\#4)}, \tag{4}$$

$$0=-\frac{\partial p}{\partial y}-\rho g\ , \tag{5}$$

$$\frac{\partial p}{\partial y}=-\rho g\ , \tag{6}$$

$$p(x,y)=\rho g(h-y)+f(x)\ . \tag{7}$$

Note that the pressure is not a function of the z direction since the flow is planar.

Now solve Eqn. (3) for the velocity profile.

$$\frac{d^2u_x}{dy^2}=\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{1}{\mu}\frac{df}{dx}\ , \tag{8}$$

$$\frac{du_x}{dy}=\frac{1}{\mu}\frac{df}{dx}y+c_1 \tag{9}$$

$$u_x=\frac{1}{2\mu}\frac{df}{dx}y^2+c_1y+c_2 \tag{10}$$

Apply boundary conditions to determine the unknown constant c_1 and c_2.

no-slip at $y = 0$ $\quad\Rightarrow\ u_x(y=0)=0 \quad\Rightarrow\ 0=c_2$ (11)

constant stress at $y = h$ $\quad\Rightarrow\ \tau_{yx}(y=h)=\mu\frac{du_x}{dy}(y=h)=C\ \Rightarrow\ \mu\left(\frac{1}{\mu}\frac{df}{dx}h+c_1\right)=C$ (12)

$$c_1=\frac{1}{\mu}\left(C-\frac{df}{dx}h\right) \tag{13}$$

Re-write the velocity profile,

$$u_x=\frac{1}{2\mu}\frac{df}{dx}y^2+\frac{1}{\mu}\left(C-\frac{df}{dx}h\right)y\ , \tag{14}$$

Since the flow has zero volumetric flow rate,

$$Q=\int_{y=0}^{y=h}u_xb\,dy=0\ , \tag{15}$$

$$\int_{y=0}^{y=h}\left[\frac{1}{2\mu}\frac{df}{dx}y^2+\frac{1}{\mu}\left(C-\frac{df}{dx}h\right)y\right]dy=0\ , \tag{16}$$

$$\frac{1}{6\mu}\frac{df}{dx}h^3+\frac{1}{2\mu}\left(C-\frac{df}{dx}h\right)h^2=0\ , \tag{17}$$

$$\frac{1}{6\mu}\frac{df}{dx}h^3+\frac{Ch^2}{2\mu}-\frac{1}{2\mu}\frac{df}{dx}h^3=0\ , \tag{18}$$

$$\frac{-1}{3\mu}\frac{df}{dx}h^3+\frac{Ch^2}{2\mu}=0\ , \tag{19}$$

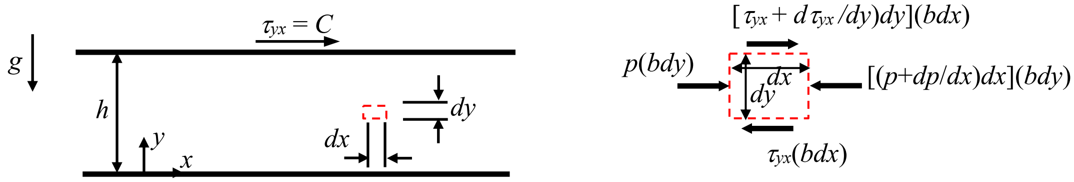
$\boxed{\frac{df}{dx}=\frac{3C}{2h}}$. Note that $\frac{\partial p}{\partial x}=\frac{df}{dx}$ (refer to Eq. (7)). (20)

Substitute back into Eq. (14) to get,

$$u_x = \frac{3}{4\mu}\frac{C}{h}y^2 + \frac{1}{\mu}\left(C - \frac{3}{2}C\right)y\,, \tag{21}$$

$$\boxed{u_x = \frac{3}{4\mu}\frac{C}{h}y^2 - \frac{1}{2\mu}Cy}. \tag{22}$$

Now solve the same problem, but using the fixed differential control volume shown in the following figure.



Apply conservation of mass,

$$\frac{d}{dt}\int_{CV}\rho\,dV + \int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0\,, \tag{23}$$

where,

$$\frac{d}{dt}\int_{CV}\rho\,dV = 0 \text{ (steady)}, \tag{24}$$

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\rho u_x(bdy) + \rho\left(u_x + \frac{\partial u_x}{\partial x}dx\right)(bdy) - \rho u_y(bdx) + \rho\left(u_y + \frac{\partial u_y}{\partial y}dy\right)(bdx), \tag{25}$$

(Note: Assuming planar flow so the z component isn't shown.)

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = \rho\frac{\partial u_x}{\partial x}(bdxdy) + \rho\frac{\partial u_y}{\partial y}(bdxdy)\,. \tag{26}$$

Note that the flow is fully developed in the x direction, planar, and steady,

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = \rho\frac{du_y}{dy}(bdxdy)\,. \tag{27}$$

Substitute and simplify,

$$\rho\frac{du_y}{dy}(bdxdy) = 0 \quad => \quad \frac{du_y}{dy} = 0\,. \tag{28}$$

Using the same logic used to derive Eq. (2) gives $u_y = 0$.

Now apply the linear momentum equation in the x direction,

$$\frac{d}{dt}\int_{CV}u_x\rho\,dV + \int_{CS}u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}\,, \tag{29}$$

where,

$$\frac{d}{dt}\int_{CV}u_x\rho\,dV = 0 \text{ (steady)}, \tag{30}$$

$$\int_{CS}u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = 0 \text{ (fully developed in the } x\text{-direction; } u_y = 0), \tag{31}$$

$$F_{B,x} = 0\,, \tag{32}$$

$$F_{S,x} = pbdy - \left(p + \frac{\partial p}{\partial x}dx\right)bdy - \tau_{yx}bdx + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right)bdx \ . \tag{33}$$

Substitute and simplify,

$$0 = pbdy - \left(p + \frac{\partial p}{\partial x}dx\right)bdy - \tau_{yx}bdx + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right)bdx \ , \tag{34}$$

$$0 = -\frac{\partial p}{\partial x}dxdy + \frac{\partial \tau_{yx}}{\partial y}dxdy \ , \tag{35}$$

$$\frac{d\tau_{yx}}{dy} = \frac{dp}{dx} \ . \tag{36}$$

Note that,

$$\tau_{yx} = \mu\left(\frac{\partial u_x}{\partial y} + \underbrace{\frac{\partial u_y}{\partial x}}_{\substack{=0 \text{ since} \\ u_y=0}}\right) \Rightarrow \frac{\partial \tau_{yx}}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u_x}{\partial y}\right) = \mu\frac{\partial^2 u_x}{\partial y^2} \ . \tag{37}$$

Since the flow is steady, full developed in the *x* direction, and planar, Eq. (37) can be written in terms of an ordinary derivative,

$$\frac{\partial \tau_{yx}}{\partial y} = \mu\frac{d^2 u_x}{dy^2}$$

Thus, Eq. (36) becomes,

$$\mu\frac{d^2 u_x}{dy^2} = \frac{\partial p}{\partial x} \ , \tag{38}$$

$$\frac{d^2 u_x}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \ . \tag{39}$$

This equation is the same as that found previously (Eq. (8)).

The pressure gradient in the *y* direction can be found by applying the linear momentum equation in the *y* direction to the same control volume,

$$\frac{d}{dt}\int_{\text{CV}} u_y \rho \, dV + \int_{\text{CS}} u_y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,y} + F_{S,y} \ , \tag{40}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} u_y \rho \, dV = 0 \ \ \text{(steady; } u_y = 0\text{)}, \tag{41}$$

$$\int_{\text{CS}} u_y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0 \ \ (u_y = 0), \tag{42}$$

$$F_{B,y} = -\rho g b dx dy \ , \tag{43}$$

$$F_{S,y} = pbdx - \left(p + \frac{\partial p}{\partial y}dy\right)bdx - \tau_{xy}bdy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}dx\right)bdy \ . \tag{44}$$

Substitute and simplify,

$$0 = -\rho g b dx dy + pbdx - \left(p + \frac{\partial p}{\partial y}dy\right)bdx - \tau_{xy}bdy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}dx\right)bdy \ , \tag{45}$$

$$-\frac{\partial p}{\partial y}dxdy+\frac{\partial \tau_{xy}}{\partial x}dxdy=\rho g dxdy \ , \tag{46}$$

Note that,

$$\tau_{xy}=\mu\left(\underbrace{\frac{\partial u_y}{\partial x}}_{\substack{=0\text{ since}\\ u_y=0}}+\frac{\partial u_x}{\partial y}\right)\Rightarrow\frac{\partial \tau_{yx}}{\partial x}=\frac{\partial}{\partial x}\left(\mu\frac{\partial u_x}{\partial y}\right)=\mu\frac{\partial}{\partial y}\left(\underbrace{\frac{\partial u_x}{\partial x}}_{\substack{=0\text{ since}\\ \text{fully-developed}\\ \text{in } x \text{ direction}}}\right)=0\ , \tag{47}$$

where the order of the partial derivative have been flipped near the end of the equation.

Equation (46) now becomes,

$$-\frac{\partial p}{\partial y}dxdy=\rho g dxdy \ , \tag{48}$$

$$\frac{\partial p}{\partial y}=-\rho g\ , \tag{49}$$

which is precisely the same as Eq. (6)

8.12. Lubrication Flow

One very important application of creeping flows is in the study of lubrication problems. Let's consider the example of a simple, stationary, planar slipper pad bearing as shown in Figure 8.20.

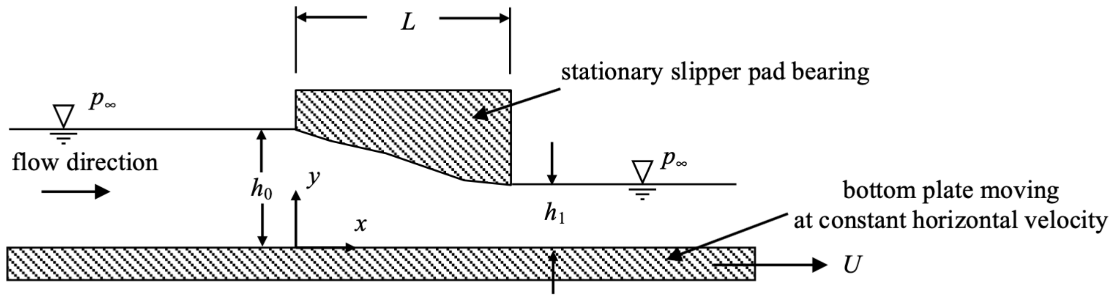


Figure 8.20. The slipper pad bearing lubrication flow geometry.

To analyze this flow, let's examine the typical magnitudes of various terms in the Navier-Stokes equations. First let's consider the x-component of the Navier-Stokes equations for the steady flow of an incompressible fluid with negligible body forces (the gravitational body force term in lubrication problems is typically very small in comparison to the other term and so is neglected),

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{8.278}$$

The characteristic magnitudes of the velocity, length, and vertical direction are,

$$u\sim U \quad x\sim L \quad y\sim h_0. \tag{8.279}$$

The characteristic y-velocity can be determine from the Continuity Equation,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \implies \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}\sim\frac{U}{L} \implies v\sim\frac{Uh_0}{L}. \tag{8.280}$$

Now examine the magnitudes of the convective inertial forces,

$$\rho u\frac{\partial u}{\partial x}\sim\frac{\rho U^2}{L} \quad \text{and} \quad \rho v\frac{\partial u}{\partial y}\sim\frac{\rho U^2}{L}. \tag{8.281}$$

Both convective terms are of similar magnitude. The magnitudes of the viscous forces are,

$$\mu\frac{\partial^2 u}{\partial x^2}\sim\frac{\mu U}{L^2} \quad \text{and} \quad \mu\frac{\partial^2 u}{\partial y^2}\sim\frac{\mu U}{h_0^2}. \tag{8.282}$$

Since we're investigating flows where $h_0/L \ll 1$, the second term will dominate the magnitude of the viscous forces.

Let's examine the case where convective inertial terms can be neglected in comparison to the viscous terms (a creeping flow),

$$\frac{\rho U^2}{L}\ll\frac{\mu U}{h_0^2} \quad \implies \quad \frac{\rho UL}{\mu}\left(\frac{h_0}{L}\right)^2\ll 1, \tag{8.283}$$

$$\boxed{\therefore \mathrm{Re}_L\left(\frac{h_0}{L}\right)^2\ll 1}. \tag{8.284}$$

To check the value of this ratio for a typical lubrication problem, consider the following parameters

$$U=10\,\mathrm{m\,s^{-1}}, L=4\,\mathrm{cm}, h_0=0.1\,\mathrm{mm}, \nu=5\times10^{-4}\,\mathrm{m^2/s}\ \text{(SAE 30 oil)},$$

which gives,

$$\mathrm{Re}_L = 800, \quad \text{but} \quad \mathrm{Re}_L \left(\frac{h_0}{L}\right)^2 = 0.005.$$

Thus, this flow can be considered a creeping flow.

Using the simplifications just discussed, the Navier-Stokes equation in the x-direction reduces to,

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}. \tag{8.285}$$

Note that we expect the magnitude of the pressure gradient in the x-direction to be of the order,

$$\frac{\partial p}{\partial x} \sim \frac{\mu U}{h_0^2}, \tag{8.286}$$

based on the previous scaling arguments.

Considering the Navier-Stokes equation in the y-direction we find that the pressure gradient in the y-direction should be of the order,

$$\left(\frac{\partial p}{\partial y} \sim \frac{\mu U}{h_0 L}\right) \ll \left(\frac{\partial p}{\partial x} \sim \frac{\mu U}{h_0^2}\right), \tag{8.287}$$

where $h_0/L \ll 1$ has been assumed. Thus, it's reasonable to assume that the pressure remains essentially constant in the y-direction in comparison to how the pressure changes in the x-direction.

Solving the differential equation given in Eq. (8.285) gives,

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + c_1 y + c_2. \tag{8.288}$$

Applying no-slip boundary conditions at the top and bottom walls,

$$u(y=0) = U \quad \text{and} \quad u(y=h) = 0, \tag{8.289}$$

results in the following velocity profile,

$$u = \underbrace{\frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{y}{h}\right)\left(1 - \frac{y}{h}\right)}_{\text{Poiseuille flow}} + \underbrace{U\left(1 - \frac{y}{h}\right)}_{\text{Couette flow}}. \tag{8.290}$$

We see that the velocity profile is a combination of a Poiseuille flow and a Couette flow.

We now have a relation relating two unknowns, the velocity profile and the pressure distribution. We must use another relation solve for the pressure distribution (or velocity distribution) in terms of known quantities. So far we've used the momentum equations to derive Eq. (8.290), now let's consider the Continuity Equation. Specifically, the mass flow rate at any cross-section must remain the same,

$$\frac{\partial \dot{m}}{\partial x} = \frac{\partial}{\partial x}\int_{y=0}^{y=h(x)} u dy = 0. \tag{8.291}$$

Note that the fluid was assumed incompressible when writing this equation. Substituting in for the velocity using Eq. (8.290),

$$\frac{\partial}{\partial x}\int_{y=0}^{y=h(x)} u dy = \frac{\partial}{\partial x}\int_{y=0}^{y=h(x)} \left[\frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{y}{h}\right)\left(1 - \frac{y}{h}\right) + U\left(1 - \frac{y}{h}\right)\right] = 0, \tag{8.292}$$

$$\implies \frac{\partial}{\partial x}\left[\frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{h^2}{2h} - \frac{h^3}{3h^2}\right) + U\left(h - \frac{h^2}{2h}\right)\right] = 0, \tag{8.293}$$

$$\implies \frac{\partial}{\partial x}\left[\frac{h^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right) + U\frac{h}{2}\right] = 0, \tag{8.294}$$

$$\boxed{\therefore \frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) = 6\mu U \frac{\partial h}{\partial x}}. \tag{8.295}$$

This last equation is known as Reynolds' Equation for Lubrication in a Planar Channel.

Notes:

(1) We can use Reynolds' Equation to solve for the pressure distribution $p(x)$ assuming we know the bearing geometry $h(x)$. Let's consider the simple example using the slipper pad bearing shown in Figure 8.20. The bearing geometry, which is a straight line, is given by,

$$h(x) = h_0 + (h_1 - h_0)\frac{x}{L}. \tag{8.296}$$

Substituting this expression for $h(x)$ into Eq. (8.295) and solving for the pressure gradient using the boundary conditions $p(x=0) = p(x=L) = p_\infty$ gives,

$$\frac{p - p_\infty}{\frac{\mu U L}{h_0^2}} = \frac{6\left(\frac{x}{L}\right)\left(1 - \frac{x}{L}\right)\left(1 - \frac{h_1}{h_0}\right)}{\left(1 + \frac{h_1}{h_0}\right)\left[1 - \left(1 - \frac{h_1}{h_0}\right)\left(\frac{x}{L}\right)\right]^2}. \tag{8.297}$$

A plot of this dimensionless pressure distribution is shown in Figure 8.21. The magnitude of the

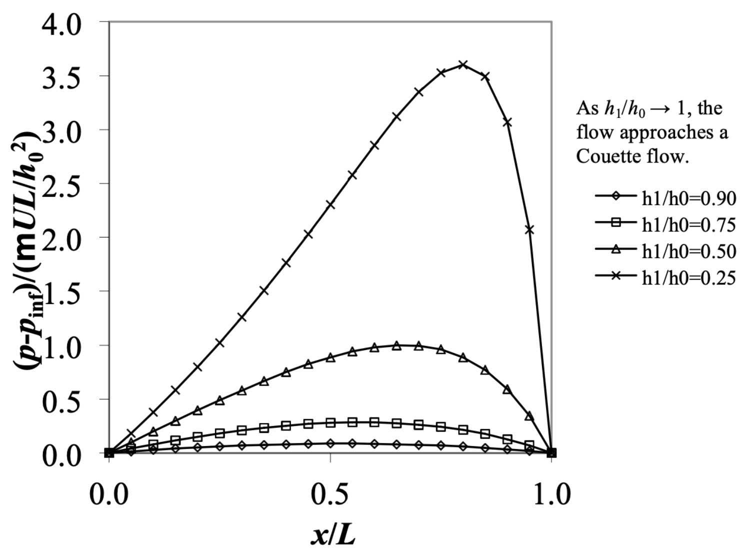


Figure 8.21. The dimensionless pressure plotted as a function of dimensionless position in a slipper pad bearing.

maximum pressure can be quite large. Consider, for example, the following typical parameters: $U = 10\,\mathrm{m\,s^{-1}}$, $L = 4\,\mathrm{cm}$, $h_0 = 0.1\,\mathrm{mm}$, $\mu = 0.4\,\mathrm{Pa\,s}$ (SAE 30 oil) $\implies$ $\mu U L/h_0^2 \approx 1.6\,\mathrm{MPa} \approx 160\,\mathrm{atm}$! A more accurate analysis of the flow would also include variations in the fluid viscosity due to the large pressure variations.

(2) A truly symmetric bearing and flow would result in zero lift on the bearing since the pressure increase on the upstream side of the bearing would be offset by an equivalent pressure reduction on the downstream side (the solid line in Figure 8.22). The reason real bearings can support a load is because the pressure distribution is, in fact, not symmetric. When a liquid flows past the centerline of the bearing the pressure begins to decrease below zero gage pressure as shown in the figure. However, the minimum pressure is limited by the vapor pressure of the liquid (the dashed line in the figure). Hence, there is a net positive gage pressure force acting on the bearing surface, which acts to produce a lift force on the bearing.

8.13. Review Questions

(1) Describe several common assumptions used to simplify the Navier-Stokes equations.
(2) Describe several common boundary conditions used when solving the Navier-Stokes equations.

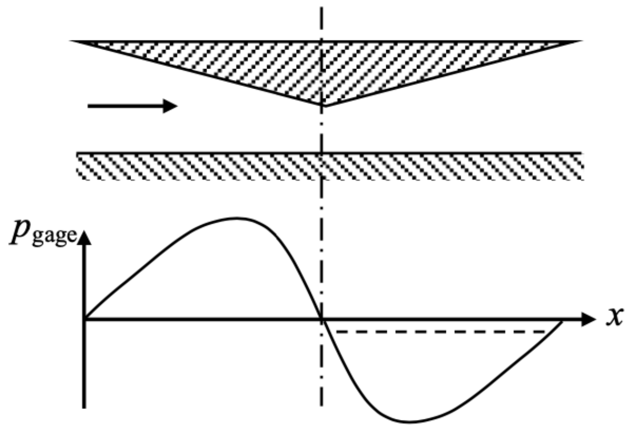
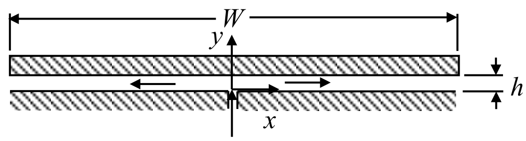


Figure 8.22. A symmetric planar bearing geometry. The symmetric solid line is the pressure distribution that would occur if cavitation didn't occur, e.g., a gas as the lubricating fluid. In this situation there would be no lift force. If cavitation does occur, which would be the case for liquid as the lubricating fluid, then the pressure on the downstream side of the bearing would follow the dashed line. For this case there would be a net lift force on the bearing.

(3) At what (rule of thumb) Reynolds number does transition from laminar to turbulent flow occur for planar Couette flow?
(4) At what (rule of thumb) Reynolds number does transition from laminar to turbulent flow occur for Poiseuille flow?
(5) Sketch the velocity profiles for a planar Couette-Poiseuille flow with different pressure gradients.
(6) What is meant by the "shear layer thickness"?
(7) How does the shear layer thickness typically depend on the kinematic viscosity for laminar flows?

A hydrostatic bearing is to support a load of 3600 lb_f per foot of length perpendicular to the diagram. The bearing is supplied with SAE 30 oil at 100 ºF and 100 psig through the central slit. Since the oil is viscous and the gap is small, the flow may be considered fully developed. Calculate:

a. the required width of the bearing pad,
b. the resulting pressure gradient, dp/dx, and
c. the gap height if $Q = 0.0006$ ft^3/min per foot of length.



SOLUTION:

For a fully-developed flow, the pressure gradient remains constant, *i.e.* $dp/dx = c$, where the constant c is:

$$c = \frac{dp}{dx} = \frac{\Delta p}{\Delta x} = \frac{-p_0}{\frac{1}{2}W} \tag{1}$$

where p_0 is the gage pressure at the origin.

The pressure in the gap is found by integrating the pressure gradient.

$$\int_{p=p_0}^{p=p} dp = c\int_{x=0}^{x=x} dx \tag{2}$$

$$\therefore p = p_0 + cx \tag{3}$$

The force (per unit depth into the page) acting on the bearing pad is found by integrating the pressure force. Since the geometry is symmetric, the total force is twice the force acting on one side of the bearing pad. Also note that gage pressures are being used.

$$F = 2\int_{x=0}^{x=\frac{1}{2}W} p\,dx = 2\int_{x=0}^{x=\frac{1}{2}W} \left(p_0 + cx\right)dx \tag{4}$$

$$\therefore F = 2\left[p_0 \tfrac{1}{2}W + \tfrac{1}{2}c\left(\tfrac{1}{2}W\right)^2\right] = p_0 W + \tfrac{1}{4}cW^2 \tag{5}$$

Substitute Eqn. (1) into Eqn. (5) and solve for the width, W, in terms of F.

$$F = p_0 W + \tfrac{1}{4}\left(\frac{-p_0}{\frac{1}{2}W}\right)W^2 = p_0 W - \tfrac{1}{2}p_0 W = \tfrac{1}{2}p_0 W \tag{6}$$

$$\boxed{\therefore W = \frac{2F}{p_0}} \tag{7}$$

Using the given data:

F = 3600 lb_f/ft
p_0 = 100 psi = 14400 lb_f/ft^2
$\Rightarrow$ $\boxed{W = 0.5 \text{ ft}}$

The pressure gradient is found using Eqn. (1) and the previous result.

$$\boxed{\frac{dp}{dx} = \frac{-p_0}{\frac{1}{2}W}} \tag{8}$$

$\Rightarrow$ $\boxed{dp/dx = -57600 \text{ lb}_f/\text{ft}^3 = -400 \text{ psi/ft}}$

The gap height is found using the velocity profile within the gap.

$$Q = \int_{y=0}^{y=h} u_x dx \tag{9}$$

where, for a planar Poiseuille flow (refer to the course notes or course textbook), the velocity profile is:

$$u_x = \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right) \tag{10}$$

Substituting Eqn. (10) into Eqn. (9) and simplifying gives:

$$\begin{aligned} Q &= 2\int_{y=0}^{y=h} \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right)dy \\ &= \frac{h^2}{\mu}\left(-\frac{dp}{dx}\right)\left[\frac{h^2}{2h}-\frac{h^3}{3h^2}\right] \end{aligned} \tag{11}$$

$$\therefore Q = \frac{h^3}{6\mu}\left(-\frac{dp}{dx}\right) \tag{12}$$

Thus:

$$\boxed{h = \left[\frac{6\mu Q}{\left(-{dp}/{dx}\right)}\right]^{1/3}} \tag{13}$$

Using the given data:

μ = 2.30e-3 $lb_f \cdot s/ft^2$
Q = 0.0006 ft^3/min/ft
dp/dx = -400 psi/ft
$\Rightarrow$ $\boxed{h = 1.6\text{e-3 in.}}$

Double-check the laminar flow assumption by calculating the Reynolds number.

$$\mathrm{Re} = \frac{\bar{u}h}{\nu} = \frac{h}{\nu}\underbrace{\left[\frac{h^2}{6\mu}\left(-\frac{\partial p}{\partial x}\right)\right]}_{=\bar{u}} \tag{14}$$

where the average velocity, $\bar{u}$, is found via:

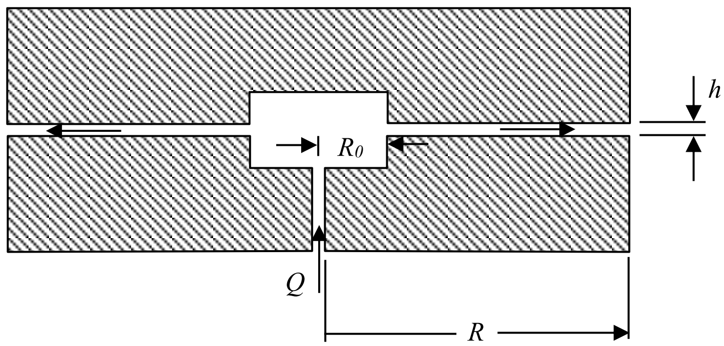
$$\bar{u}h = Q = \frac{h^3}{6\mu}\left(-\frac{dp}{dx}\right) \tag{15}$$

Using the given data:

ν = 1.29e-3 ft^2/s
$\Rightarrow$ Re = 8.04e-4 < 1500 $\Rightarrow$ The laminar flow assumption is valid.

Viscous oil is supplied at a low volumetric flow rate, Q, through a central opening to fill the space between two parallel disks as shown in the figure.

a. Assuming that the flow is axi-symmetric and that the gap is narrow ($h << R$), simplify the Navier-Stokes equations to the form that describes the flow field in the narrow gap.
b. Solve the equation in part (a) for p-p_{atm} and the velocity field in terms of Q for the narrow gap.



SOLUTION:

Simplify the continuity and Navier-Stokes equations (consider only the r-direction) using the following assumptions:

1. steady flow $\Rightarrow$ $\partial/\partial t(\cdots)=0$
2. axi-symmetric flow $\Rightarrow$ $\partial/\partial\theta(\cdots)=0$
3. no "swirl" velocity $\Rightarrow$ $u_\theta = 0$
4. neglect body forces $\Rightarrow$ $f_r = f_\theta = f_z = 0$
5. laminar flow $\Rightarrow$ $u_z = 0$

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial\theta}}_{=0(\#2,3)}+\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#5)}=0$$

$$\therefore \frac{1}{r}\frac{\partial}{\partial r}(ru_r)=0 \quad \text{(Call this condition \#6.)} \tag{1}$$

Navier-Stokes in the r-direction:

$$\rho\left(\underbrace{\frac{\partial u_r}{\partial t}}_{=0(\#1)}+u_r\frac{\partial u_r}{\partial r}+\underbrace{\frac{u_\theta}{r}}_{=0(\#3)}\frac{\partial u_r}{\partial r}-\underbrace{\frac{u_\theta^2}{r}}_{=0(\#3)}+\underbrace{u_z}_{=0(\#5)}\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+$$

$$\mu\left\{\frac{\partial}{\partial r}\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)}_{=0(\#6)}\right]+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_r}{\partial\theta^2}}_{=0(\#2)}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\underbrace{\frac{\partial u_\theta}{\partial\theta}}_{=0(\#2,3)}\right\}+\rho\underbrace{f_r}_{=0(\#4)}$$

$$\rho u_r\frac{\partial u_r}{\partial r}=-\frac{\partial p}{\partial r}+\mu\frac{\partial^2 u_r}{\partial z^2} \tag{2}$$

Since the gap size is small compared to the channel length, *i.e.* $h << R$, it is possible to reduce Eqn. (2) further using scaling arguments. The velocity and lengths in the flow are expected to scale with the following parameters:

$$\begin{aligned} u_r &\sim u \sim \frac{Q}{rh} \\ r &\sim R \\ z &\sim h \end{aligned} \tag{3}$$

Hence, the ratio of the convective inertial term to the viscous term is of the order:

$$\frac{\rho u_r \dfrac{\partial u_r}{\partial r}}{\mu \dfrac{\partial^2 u_r}{\partial z^2}} = \frac{\rho u h}{\mu}\frac{h}{R} = \mathrm{Re}_h\left(\frac{h}{R}\right) \tag{4}$$

Thus, as long as the Reynolds number is not very large (it won't be here since the flow rate is small, the oil is viscous, and r does not approach zero (so that u remains finite)) and $h << R$, then the convective inertial term may be neglected in comparison to the viscous term. As a result, Eqn. (2) simplifies to:

$$\boxed{\frac{\partial^2 u_r}{\partial z^2} = \frac{1}{\mu}\frac{dp}{dr}} \tag{5}$$

Note that the Navier-Stokes equations in the θ and z directions indicate that $\partial p/\partial\theta = 0$ and $\partial p/\partial z = 0$, respectively, so that $p = p(r)$.

Solving the ODE given in Eqn. (5) gives:

$$\begin{aligned} \frac{\partial u_r}{\partial z} &= \frac{1}{\mu}\frac{dp}{dr}z + f_1(r) \\ u_r(r,z) &= \frac{1}{2\mu}\frac{dp}{dr}z^2 + zf_1(r) + f_2(r) \end{aligned} \tag{6}$$

where $f_1(r)$ and $f_2(r)$ are unknown functions of r.

The boundary conditions for the flow are:

no-slip on the bottom surface $\Rightarrow\ u_r(r, z=0) = 0 \ \Rightarrow\ f_2(r) = 0$ (7)

no-slip on the top surface $\Rightarrow\ u_r(r, z=h) = 0 \ \Rightarrow\ f_1(r) = -\dfrac{1}{2\mu}\dfrac{dp}{dr}h$ (8)

Using the given boundary conditions, Eqn. (6) becomes:

$$u_r = \frac{h^2}{2\mu}\left(-\frac{dp}{dr}\right)\left(\frac{z}{h}\right)\left[1-\left(\frac{z}{h}\right)\right] \tag{9}$$

From conservation of mass, the flow rate at any radius, r, is:

$$\begin{aligned} Q &= 2\pi r\int_{z=0}^{z=h} u_r dz \\ &= 2\pi r\frac{h^2}{2\mu}\left(-\frac{dp}{dr}\right)h\int_{z/h=0}^{z/h=1}\left(\frac{z}{h}\right)\left[1-\left(\frac{z}{h}\right)\right]d\left(\frac{z}{h}\right) \\ \therefore Q &= \frac{\pi h^3 r}{6\mu}\left(-\frac{dp}{dr}\right) \end{aligned} \tag{10}$$

Substituting into Eqn. (9) gives:

$$u_r = \frac{3Q}{\pi r h}\left(\frac{z}{h}\right)\left[1-\left(\frac{z}{h}\right)\right]$$

$$\boxed{\therefore \frac{u_r}{\left(Q/rh\right)} = \frac{3}{\pi}\left(\frac{z}{h}\right)\left[1-\left(\frac{z}{h}\right)\right]} \text{ where } R_0 \le r \le R \tag{11}$$

Re-arranging Eqn. (10) and solving for the pressure as a function of r:

$$\int_{p=p_{\text{atm}}}^{p=p} dp = -\frac{6\mu Q}{\pi h^3}\int_{r=R}^{r=r}\frac{dr}{r}$$

$$p - p_{\text{atm}} = -\frac{6\mu Q}{\pi h^3}\ln\left(\frac{r}{R}\right)$$

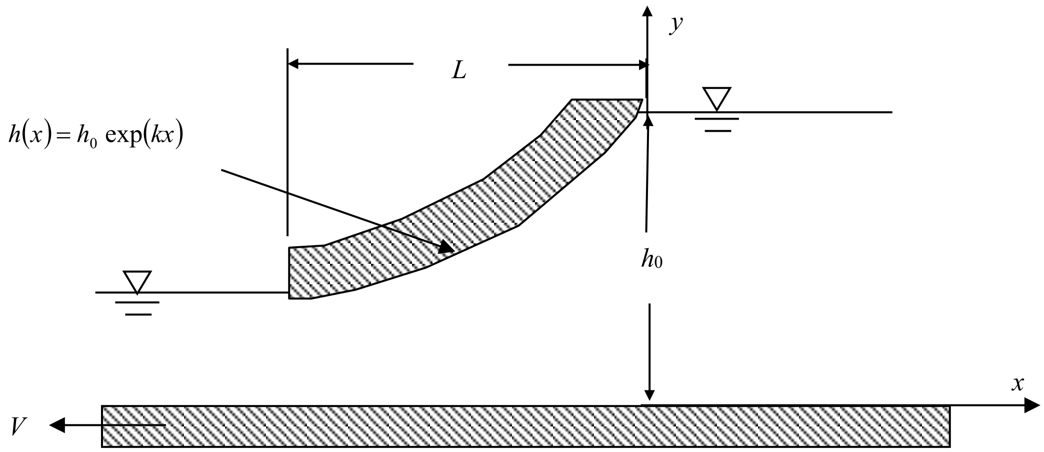
$$\boxed{\therefore \frac{p - p_{\text{atm}}}{\left(\mu Q/h^3\right)} = \frac{6}{\pi}\ln\left(\frac{R}{r}\right)} \text{ where } R_0 \le r \le R \tag{12}$$

A slider bearing has an exponentially varying gap so that:

$$h(x) = h_0 \exp(kx)$$

where k is a positive constant and the coordinate system is shown in the illustration. If the pressure is zero (gage pressure) at each end of the slider, determine:

a. and plot the pressure profile, $p(x)$,
b. the total lift on the stationary part, and
c. the total drag force on the stationary part.



SOLUTION:
From the Reynolds lubrication equation,

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}h^3\right) = -6\mu V\frac{\partial h}{\partial x} \quad \text{(Note that the plate velocity is in the -}x\text{ direction.)} \tag{1}$$

where,

$$h = h_0 \exp(kx) \ \ (-L \le x \le 0), \tag{2}$$

$$h^3 = h_0^3 \exp(3kx), \tag{3}$$

$$\frac{\partial h}{\partial x} = kh_0 \exp(kx). \tag{4}$$

Substitute and integrate,

$$\frac{\partial}{\partial x}\left[\frac{\partial p}{\partial x}h_0^3 \exp(3kx)\right] = -6\mu V k h_0 \exp(kx), \tag{5}$$

$$\frac{\partial p}{\partial x}h_0^3 \exp(3kx) = -6\mu V h_0 \exp(kx) + c_1, \tag{6}$$

$$\frac{\partial p}{\partial x} = -\frac{6\mu V}{h_0^2}\exp(-2kx) + c_2 \exp(-3kx), \tag{7}$$

$$p = \frac{3\mu V}{kh_0^2}\exp(-2kx) - c_3 \exp(-3kx) + c_4, \tag{8}$$

$$p' \coloneqq \frac{p}{\frac{3\mu V}{kh_0^2}} = \exp(-2kLx') - c_5 \exp(-3kLx') + c_6. \tag{9}$$

where $x' = x/L$.

Apply boundary conditions (note that the following are gage pressures),

$$p'(x' = 0) = 0 \ \Rightarrow \ 0 = 1 - c_5 + c_6 \ \Rightarrow \ c_6 = c_5 - 1, \tag{10}$$

$$p'(x' = -1) = 0 \ \Rightarrow \ 0 = \exp(2kL) - c_5 \exp(3kL) + (c_5 - 1), \tag{11}$$

Solve for c_5 and c_6,

$$c_5[\exp(3kL) - 1] = \exp(2kL) - 1 \ \Rightarrow \ c_5 = \frac{\exp(2kL)-1}{\exp(3kL)-1}, \tag{12}$$

$$c_6 = \frac{\exp(2kL)-1}{\exp(3kL)-1} - 1\ . \tag{13}$$

Thus,

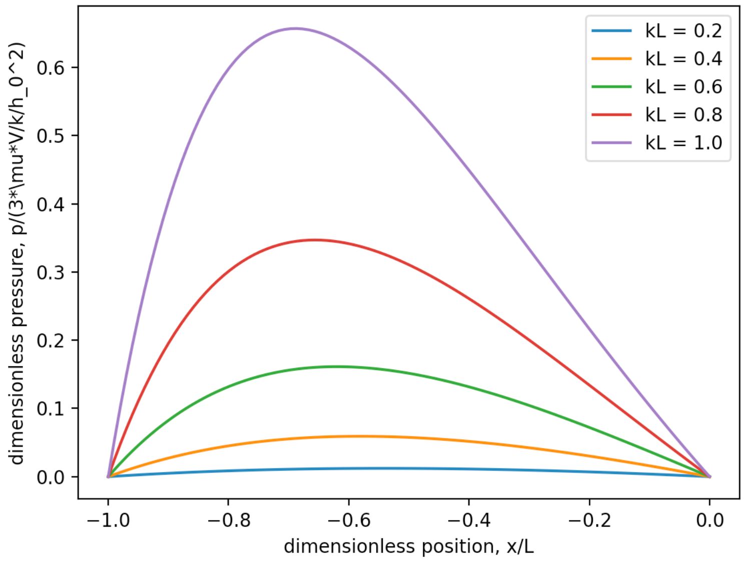
$$p' = \exp(-2kLx') - \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right]\exp(-3kLx') + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right] - 1, \tag{14}$$

$$\boxed{p' = \exp(-2kLx') - 1 + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right][1 - \exp(-3kLx')]}. \tag{15}$$

or, using dimensional variables,

$$\boxed{\frac{p}{\frac{3\mu V}{kh_0^2}} = \exp(-2kx) - 1 + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right][1 - \exp(-3kx)]}. \tag{16}$$

The dimensionless pressure is plotted below for various kL values.



Since the pressure on the pad is much larger than the shear stress on the part, the lift will be due primarily to the pressure,

$$p \sim \frac{\mu V L}{h_0^2} \text{ and } \tau \sim \frac{\mu V}{h_0} \text{ so } \frac{p}{\tau} \sim \left(\frac{\mu V L}{h_0^2}\right)\left(\frac{h_0}{\mu V}\right) = \frac{L}{h_0} \gg 1. \tag{17}$$

The lift on the pad is,

$$\text{lift} = \int_{x=-L}^{x=0} p\,dx, \tag{18}$$

(Pressure multiplied by the projected area in the horizontal direction, assuming unit depth into the page.) In dimensionless form,

$$\text{lift}' = \frac{\text{lift}}{\frac{3\mu V L}{kh_0^2}} = \int_{-1}^{0} p'\,dx', \tag{19}$$

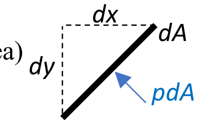
$$\frac{\text{lift}}{\frac{3\mu V L}{kh_0^2}} = \int_{-1}^{0}\left\{\exp(-2kLx') - 1 + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right][1 - \exp(-3kLx')]\right\}dx', \tag{20}$$

$$\frac{\text{lift}}{\frac{3\mu V L}{kh_0^2}} = \frac{1}{-2kL}[1 - \exp(2kL)] - 1 + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right]\left\{1 - \frac{1}{-3kL}[1 - \exp(3kL)]\right\}, \tag{21}$$

$$\boxed{\frac{\text{lift}}{\frac{3\mu V}{kL}\left(\frac{L}{h_0}\right)^2} = \frac{1}{2kL}[\exp(2kL) - 1] - 1 + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right]\left\{1 + \frac{1}{3kL}[1 - \exp(3kL)]\right\}}. \tag{22}$$

The drag on the pad is,

$$\text{drag} = -\int_{y=h_0\exp(-kL)}^{y=h_0} p\,dy, \quad \text{(pressure acting on the vertical projected area)} \tag{23}$$



where,

$$y = h_0\exp(kx) \;\; \Rightarrow \;\; dy = kh_0\exp(kx)\,dx \tag{24}$$

and,

$$y = h_0 \;\; \Rightarrow \;\; x = 0 \;\;\text{and}\;\; y = h_0\exp(-kL) \;\; \Rightarrow \;\; x = -L\,. \tag{25}$$

Thus,

$$\text{drag} = -kh_0\int_{x=-L}^{x=0} p\exp(kx)\,dx, \tag{26}$$

$$\text{drag} = -kh_0\frac{3\mu V}{kh_0^2}\int_{-L}^{0}\left\{\exp(-2kx) - 1 + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right][1-\exp(-3kx)]\right\}\exp(kx)\,dx, \tag{27}$$

$$\frac{\text{drag}}{\frac{3\mu V}{h_0}} = \int_0^{-L}\left\{\exp(-kx) - \exp(kx) + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right][\exp(kx)-\exp(-2kx)]\right\}dx, \tag{28}$$

$$\frac{\text{drag}}{\frac{3\mu V}{h_0}} = \frac{1}{-k}\exp(-kx)\Big|_0^{-L} - \frac{1}{k}\exp(kx)\Big|_0^{-L} + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right]\left[\frac{1}{k}\exp(kx)\Big|_0^{-L} - \frac{1}{-2k}\exp(-2kx)\Big|_0^{-L}\right], \tag{29}$$

$$\frac{\text{drag}}{\frac{3\mu V}{h_0}} = \frac{1}{k}[1-\exp(kL)] - \frac{1}{k}[\exp(-kL)-1] + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right]\left\{\frac{1}{k}[\exp(-kL)-1] + \frac{1}{2k}[\exp(2kL)-1]\right\}, \tag{30}$$

$$\frac{\text{drag}}{\frac{3\mu V}{kh_0}} = 1 - \exp(kL) + 1 - \exp(-kL) + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right]\left[\exp(-kL) - 1 + \frac{1}{2}\exp(2kL) - \frac{1}{2}\right], \tag{31}$$

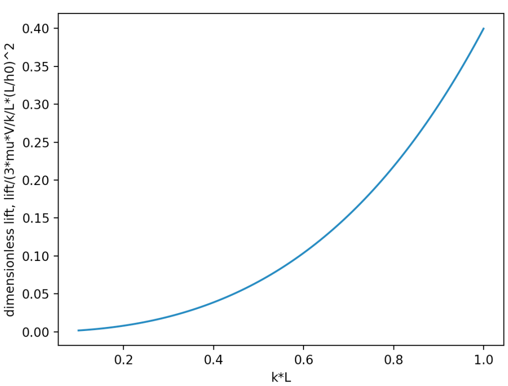
$$\frac{\text{drag}}{\frac{3\mu V}{kh_0}} = 2 - \exp(kL) - \exp(-kL) + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right]\left[\exp(-kL) + \frac{1}{2}\exp(2kL) - \frac{3}{2}\right]. \tag{32}$$

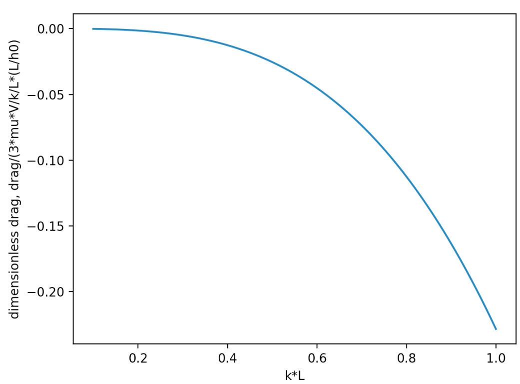
$$\boxed{\frac{\text{drag}}{\frac{3\mu V}{kL}\frac{L}{h_0}} = 2 - \exp(kL) - \exp(-kL) + \left[\frac{\exp(2kL)-1}{\exp(3kL)-1}\right]\left[\exp(-kL) + \frac{1}{2}\exp(2kL) - \frac{3}{2}\right].} \tag{33}$$

Note that the scaling on the drag force is different than the scaling on the lift,

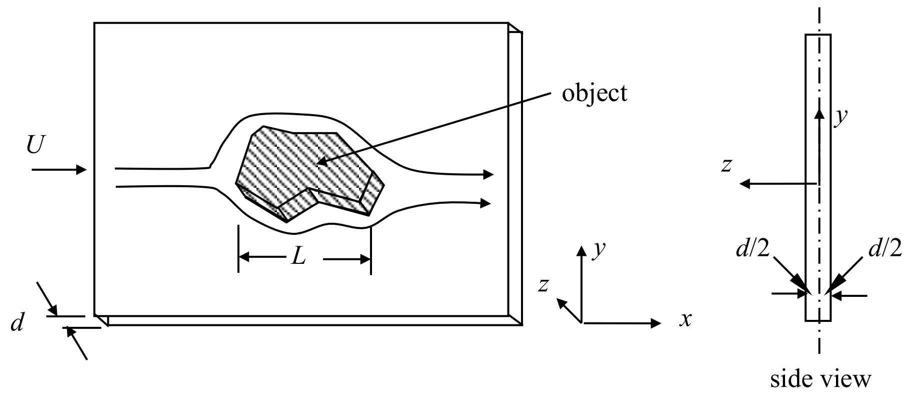
$$\text{lift} \sim \frac{3\mu V}{kL}\left(\frac{L}{h_0}\right)^2 \;\;\text{and}\;\; \text{drag} \sim \frac{3\mu V}{kL}\frac{L}{h_0} \;\; \Rightarrow \;\; \text{lift} \gg \text{drag} \;\;\text{since}\;\; h_0/L \ll 1 \tag{34}$$

The dimensionless lift and drag are plotted below as functions of kL. The drag is negative since it points in the $-x$ direction.





Consider the slow flow (with velocity, U) of a viscous, incompressible fluid between two parallel plates separated by a small width, d, as shown in the figure. Between the plates is an object of characteristic length, L, where $L >> d$.



a. Simplify the governing equations using order of magnitude arguments to a form applicable to this geometry. You may assume a steady flow and neglect body forces.
b. Using the simplified equations from part (a), determine the fluid velocity components in the x and y directions in terms of the local pressure gradients, the fluid dynamic viscosity, the gap width, and the z-position.
c. Show that the flow between these plates will produce the same streamlines as those found in a potential flow. Hint: Show that the same governing equations are satisfied. Discuss the relation between the pressure in this flow and the velocity potential in a potential flow.
d. Discuss the limitations of using this device to model potential flows.

This type of device is called a **Hele-Shaw cell** and is often used to visualize potential flows.

SOLUTION:

First consider the characteristic dimensions of various parameters in the system:

$$u_x \sim u_y \sim U$$
$$x \sim y \sim L$$
$$z \sim d$$

From the continuity equation:

$$\frac{\partial u_z}{\partial z} = -\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}$$

$$\Rightarrow u_z \sim \frac{Ud}{L} \tag{1}$$

Now consider the Navier-Stokes equation in the x-direction assuming steady flow and no body forces.

$$\underbrace{u_x \frac{\partial u_x}{\partial x}}_{\sim U^2/L} + \underbrace{u_y \frac{\partial u_x}{\partial y}}_{\sim U^2/L} + \underbrace{u_z \frac{\partial u_x}{\partial z}}_{\sim U^2/L} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \underbrace{\nu \frac{\partial^2 u_x}{\partial x^2}}_{\sim \nu U/L^2} + \underbrace{\nu \frac{\partial^2 u_x}{\partial y^2}}_{\sim \nu U/L^2} + \underbrace{\nu \frac{\partial^2 u_x}{\partial z^2}}_{\sim \nu U/d^2} \tag{2}$$

Note that the first two terms in the viscous forces are much smaller than the last viscous force term since $d << L$, *i.e.*:

$$\frac{\nu U}{L^2} << \frac{\nu U}{d^2} \tag{3}$$

For a slow flow of a viscous fluid with $L >> d$, inertial terms are expected to be negligible in comparison to viscous terms. Another way of stating this is:

$$\underbrace{\frac{U^2}{L}}_{\text{inertial}} << \underbrace{\frac{\nu U}{d^2}}_{\text{viscous}} \Rightarrow \frac{UL}{\nu}\left(\frac{d}{L}\right)^2 << 1 \Rightarrow \mathrm{Re}_L\left(\frac{d}{L}\right) << 1 \quad \text{(to have creeping flow)} \tag{4}$$

Note that the pressure gradient will be of the same order of size as the dominant viscous force since it is the pressure gradient that drives the flow.

Examining the Navier-Stokes equation in the y-direction will give similar results. Examining the Navier-Stokes equation in the z-direction indicates that:

$$\frac{\partial p}{\partial z} << \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \tag{5}$$

which implies that the pressure does not vary much in the z-direction (as compared to the x- and y-directions) and hence is considered to be independent of z, *i.e.* $p = p(x, y)$.

The Navier-Stokes equations simplify to the following relations using the scaling arguments discussed previously.

$$\frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{\partial^2 u_x}{\partial z^2} \tag{6}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial y} = \frac{\partial^2 u_y}{\partial z^2} \tag{7}$$

No-slip boundary conditions occur at the top and bottom plate surfaces.

$$\begin{aligned} u_x\left(z = \pm\tfrac{1}{2}d\right) &= 0 \\ u_y\left(z = \pm\tfrac{1}{2}d\right) &= 0 \end{aligned} \tag{8}$$

Combining Eqns. (6) and (7) into one equation gives:

$$\nabla p = \mu \frac{\partial^2 \mathbf{u}}{\partial z^2} \tag{9}$$

Taking the divergence of both sides of the equation results in:

$$\nabla \cdot \nabla p = \nabla \cdot \mu \frac{\partial^2 \mathbf{u}}{\partial z^2} = \mu \frac{\partial^2 \left(\nabla \cdot \mathbf{u}\right)}{\partial z^2} = 0 \quad \text{(where the continuity equation has been used in the last step)}$$

$$\boxed{\therefore \nabla^2 p = 0} \tag{10}$$

Equation (10) is the governing equation for this viscous-dominated flow. Note that it's Laplace's equation, which is exactly the same equation for incompressible potential flow ($\nabla^2\phi = 0$).

Integrate Eqns. (6) and (7) with respect to z (taking into account Eqn. (5)).

$$u_x = \frac{1}{2\mu}\frac{\partial p}{\partial x} z^2 + c_1 z + c_2 \tag{11}$$

$$u_y = \frac{1}{2\mu}\frac{\partial p}{\partial y} z^2 + c_3 z + c_4 \tag{12}$$

Using the boundary conditions in Eqn. (8) and substituting into Eqns. (11) and (12) gives:

$$u_x = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(z^2 - \tfrac{1}{4}d^2\right) \tag{13}$$

$$u_y = \frac{1}{2\mu}\frac{\partial p}{\partial y}\left(z^2 - \tfrac{1}{4}d^2\right) \tag{14}$$

or, in a more compact form:

$$\boxed{\mathbf{u} = \frac{1}{2\mu}\nabla p\left(z^2 - \tfrac{1}{4}d^2\right)} \tag{15}$$

Note that just like with potential flows, the velocity in this flow is found from the gradient of a function (recall that $\mathbf{u} = \nabla\phi$ for a potential flow). In this case, the function is a "constant" multiplied by the pressure:

$$\mathbf{u} = \nabla(cp) = c\nabla p \quad \text{(the gradient here operates only in the } x \text{ and } y \text{ directions)} \tag{16}$$

where

$$c = \frac{1}{2\mu}\left(z^2 - \tfrac{1}{4}d^2\right) \quad \text{(At any given elevation, } z\text{, the value of } c \text{ is constant.)} \tag{17}$$

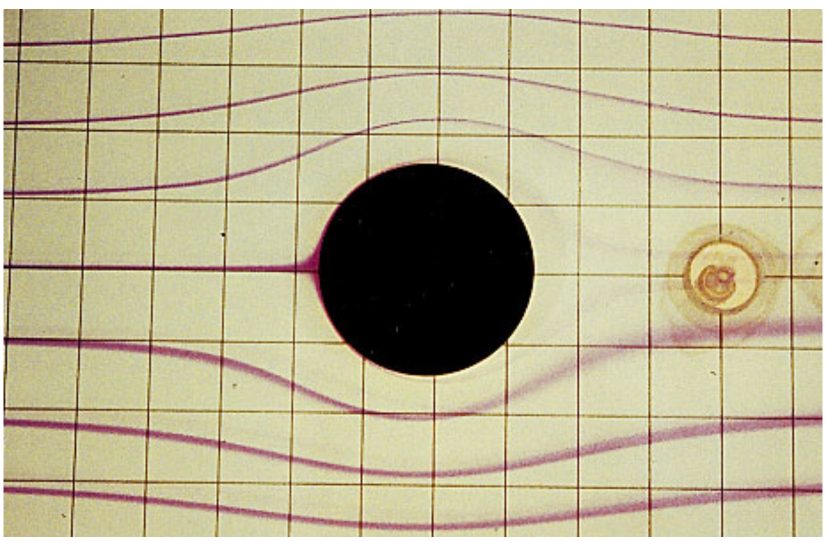
Lastly, there will be no flow through the object's surface:

$$\mathbf{u}\cdot\hat{\mathbf{n}}\Big|_{\substack{\text{at object's}\\ \text{surface}}} = 0 \tag{18}$$

just like the case with a potential flow; however, the viscous-dominated flow will satisfy the no-slip condition whereas the potential flow will not. This region is localized very near to the object's surface since $d/L \ll 1$, and it is here where the assumption that $\partial u/\partial x$, $\partial u/\partial y \ll \partial u/\partial z$ breaks down. A small distance away from the surface, *i.e.* outside the boundary layer in the x, y direction, the move essentially parallel to the object's surface and appears visually to not violate the no-slip condition.

We observe that the pressure for this viscous-dominated flow is analogous to the potential function in an incompressible potential flow. The pressure must satisfy Laplace's equation, just like the potential function, the velocity is found from the pressure gradient, just like with a potential flow, and there is no flow through the object's surface with slip along the object's surface (or at least very near to the surface), just like with a potential flow. As a result, the viscous-flow streamlines look identical to what one would expect in a potential flow. Recall that the limitation for the analogy to be valid is that $\mathrm{Re}_L(d/L)^2 \ll 1$.

The following image shows the flow pattern around a cylinder in a Hele-Shaw cell. Note that the streamlines look very similar to what would be expected from a potential flow solution.



CHAPTER 9

Boundary Layers

9.1. Boundary Layer Structure

Boundary layers are the regions near a boundary in which rotational and viscous effects are significant. The various flow field regions are shown in Figure 9.1.

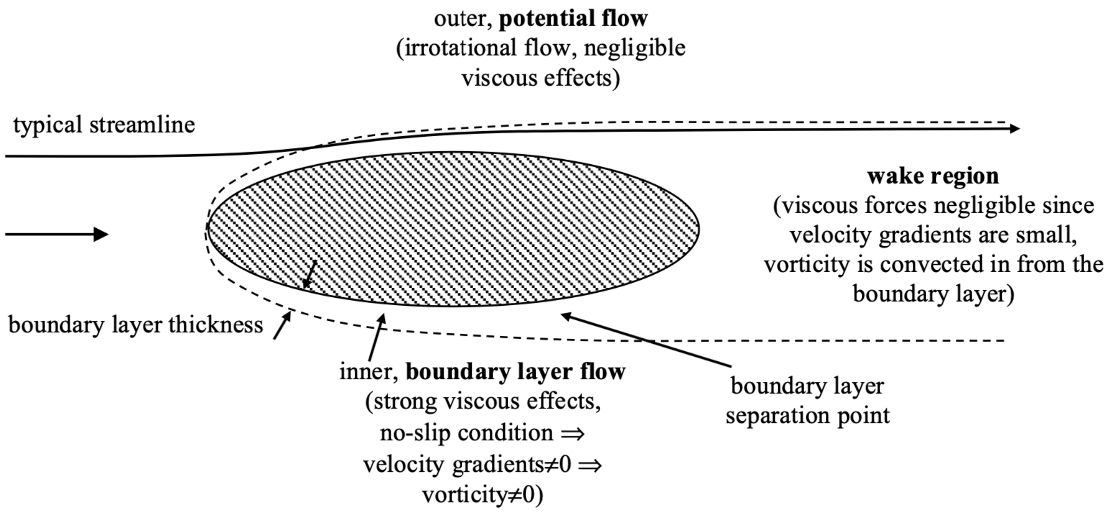


Figure 9.1. A schematic illustrating different regions in the external flow around an object.

9.2. Boundary Layer Thickness Definitions

Before continuing further, we should define what we mean by the "thickness" of a boundary layer. There are three commonly used definitions.

(1) *99% boundary layer thickness, δ or $\delta_{99\%}$.* This thickness definition is the most commonly used definition. The boundary layer thickness, δ, is defined as the distance from the boundary at which the fluid velocity, u, is 99% that of the outer velocity, U (Figure 9.2),

$$\boxed{u(y=\delta)=0.99U}. \tag{9.1}$$

(2) *displacement thickness, δ_D or δ^*.* The displacement thickness, δ_D, is the distance at which the undisturbed outer flow is displaced from the boundary by a stagnant layer of fluid that removes the same mass flow as the actual boundary layer profile (Figure 9.3),

$$\int_0^\infty \rho(U-u)dy=\rho U\delta_D, \tag{9.2}$$

$$\boxed{\delta_D=\int_0^\infty\left(1-\frac{u}{U}\right)dy\approx\int_0^\delta\left(1-\frac{u}{U}\right)dy.} \tag{9.3}$$

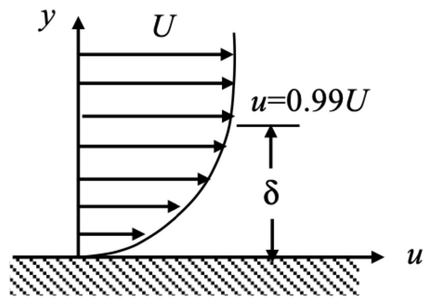


Figure 9.2. A schematic illustrating the 99% boundary layer thickness definition.

The approximate sign in Eq. (9.3) is because in going from $y = \delta$ to $y \to \infty$, the velocity u is less than 1% different from the free stream value U.

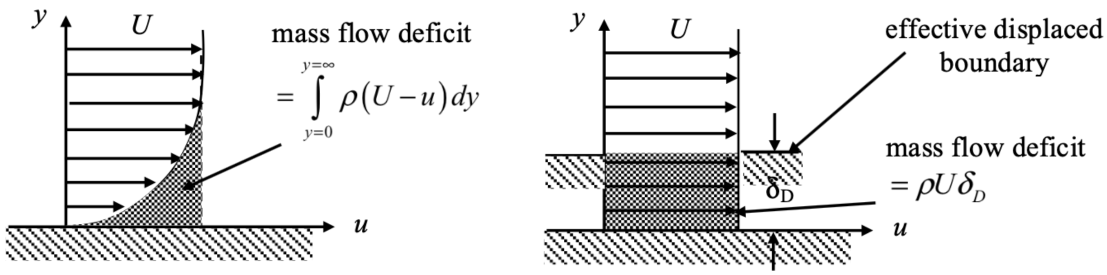


Figure 9.3. A schematic illustrating the displacement boundary layer thickness definition.

(3) *momentum thickness, δ_M or* Θ. The momentum thickness, δ_M, is the thickness of a stagnant layer that has the same momentum deficit, relative to the outer flow, as the actual boundary layer profile (Figure 9.4). This concept is similar to the one used to define the displacement thickness except instead of a mass deficit, the momentum thickness considers the momentum deficit.

$$\int_0^\infty u\rho(U-u)dy = \rho U^2 \delta_M, \tag{9.4}$$

$$\boxed{\delta_M = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \approx \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy.} \tag{9.5}$$

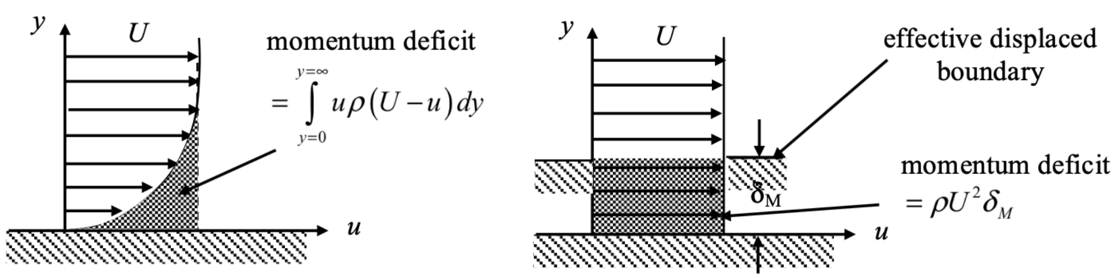
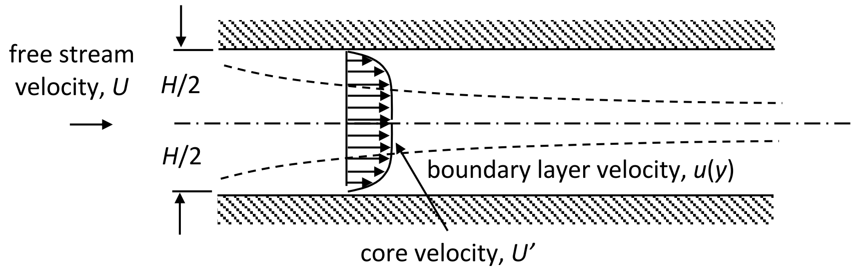


Figure 9.4. A schematic illustrating the momentum boundary layer thickness definition.

Notes:

(1) Usually, $\delta > \delta_D > \delta_M$.

Consider flow between two parallel plates in which a boundary layer has formed, as shown in the figure.



Determine the mass flow rate in terms of the displacement thickness.

SOLUTION:

Let's just consider the lower half of the flow since the upper and lower halves are symmetric. The mass flow rate in the lower half is,

$$\dot{m}_{1/2} = \underbrace{\int_0^{\delta} \rho u \, dy}_{\text{mass flow rate in BL}} + \underbrace{\int_{\delta}^{H/2} \rho U' \, dy}_{\text{mass flow rate in outer flow}} . \tag{1}$$

Note that the uniform, outer flow velocity at this cross-section is U' which is larger than U (to conserve mass). The second integral can be re-written as,

$$\int_{\delta}^{H/2} \rho U' dy = \int_0^{H/2} \rho U' dy - \int_0^{\delta} \rho U' dy , \tag{2}$$

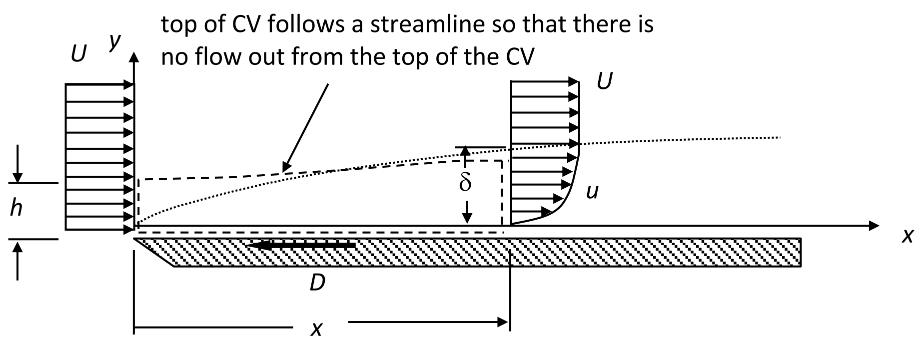
so that the mass flow rate is,

$$\begin{aligned} \dot{m}_{1/2} &= \int_0^{\delta} \rho u \, dy + \int_0^{H/2} \rho U' \, dy - \int_0^{\delta} \rho U' \, dy = \int_0^{\delta} \rho \left(u - U'\right) dy + \int_0^{H/2} \rho U' \, dy \\ &= -\rho U' \underbrace{\int_0^{\delta} \left(1 - \frac{u}{U'}\right) dy}_{=\delta_D} + \rho U' \left(H/2\right) \end{aligned} \tag{3}$$

$$\boxed{\therefore \dot{m}_{1/2} = \rho\left(H/2 - \delta_D\right) U'} \tag{4}$$

Thus, if we replace the real velocity profile by a uniform velocity profile at that cross-section (which has velocity U'), we must decrease the effective cross-sectional area (per unit depth) of each half of the pipe at that particular location by the displacement thickness, δ_D, to maintain the same mass flow rate. Note that we would need to have another relation for δ_D in order to solve for U' and visa-versa.

Using the Linear Momentum Equation, determine the viscous drag on a flat plate in terms of the momentum thickness. Assume a steady flow with the pressure everywhere equal. Make use of the given control volume.



SOLUTION:
Apply the LME in the *x*-direction to the CV shown to determine the drag acting on the fluid (or the plate) over the distance *x*,

$$-D = \frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \rho \left(\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \quad \Rightarrow \quad -D = \int_0^{\delta} \rho u^2 dy - \rho U^2 h, \tag{1}$$

where the height, *h*, is found via COM on the same CV to be,

$$\rho U h = \int_0^{\delta} \rho u dy \quad \Rightarrow \quad h = \int_0^{\delta} \frac{u}{U} dy, \tag{2}$$

so that the drag is,

$$-D = \int_0^{\delta} \rho u^2 \, dy - \rho U^2 \int_0^{\delta} \frac{u}{U} dy \quad \Rightarrow \quad D = \rho U^2 \underbrace{\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy}_{=\delta_M}, \tag{3}$$

$$\boxed{\therefore D = \rho U^2 \delta_M} \tag{4}$$

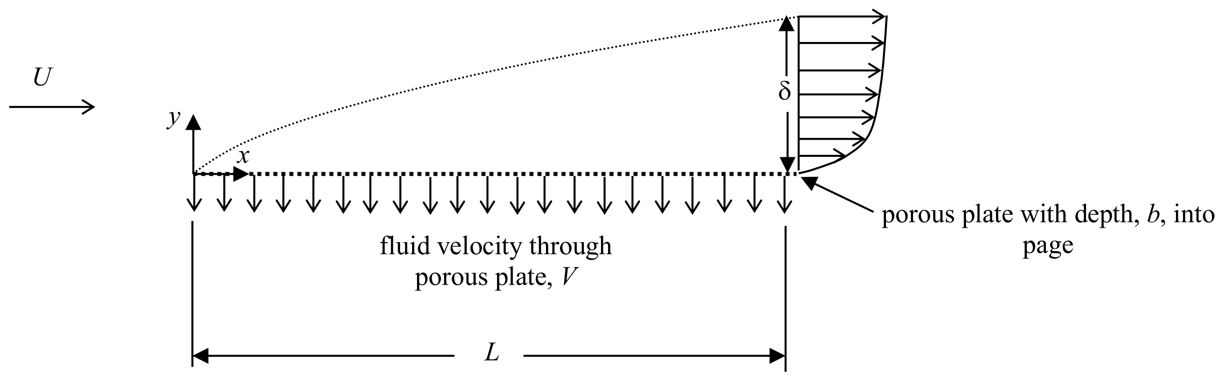
We see that the drag is related to the momentum thickness. Note that this particular example is for a situation with no pressure gradients (the pressure is a constant). The drag expression will be different if the pressure varies for the flow, but the resulting drag expression will still include the momentum thickness.

One method proposed to decrease drag and avoid boundary layer separation on aircraft is to use suction to remove the low momentum fluid near the aircraft surface. By removing the low momentum fluid near the surface, the boundary layer remains more stable and transition to a turbulent boundary layer is delayed. This method has been attempted in practice (*Aviation Week & Space Technology*, Oct. 12, 1998, pg. 42). Airbus tested a micro-perforated titanium skin on an Airbus A320 aircraft fin. The ultimate goal of Airbus' tests was to reduce wing drag by 10-16% and empenage/nacelles drag by nearly 5%. Fuel consumption was expected to decrease by as much as 13%.

To analyze this flow, consider a laminar boundary layer on a porous flat plate. Fluid is removed through the plate at a uniform velocity, V. The thickness of the boundary layer is denoted by δ and the velocity outside the boundary layer is a constant, U. Assuming that the velocity profile, u, is given by a power law expression (n is a positive constant describing the shape of the profile and y is the vertical distance from the surface of the plate):

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{n}}$$

Determine:
1. the momentum thickness of the boundary layer in terms of δ
2. the drag acting on the plate over a length L if the plate has a depth b into the page (express your answer in terms of δ_M.)



SOLUTION:

Determine the momentum thickness from its definition,

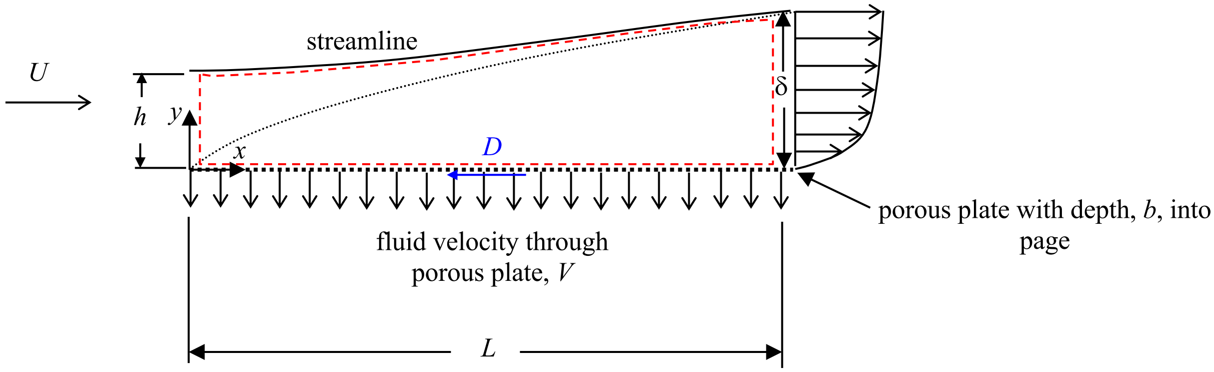
$$\delta_M = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

$$= \int_0^\delta \left(\frac{y}{\delta}\right)^{1/n}\left[1-\left(\frac{y}{\delta}\right)^{1/n}\right]dy = \delta\int_0^1 \left(\frac{y}{\delta}\right)^{1/n}\left[1-\left(\frac{y}{\delta}\right)^{1/n}\right]d\left(\frac{y}{\delta}\right) = \delta\int_0^1\left[\left(\frac{y}{\delta}\right)^{1/n}-\left(\frac{y}{\delta}\right)^{2/n}\right]d\left(\frac{y}{\delta}\right)$$

$$= \delta\left[\frac{1}{\frac{1}{n}+1}-\frac{1}{\frac{2}{n}+1}\right] = \delta\left[\frac{n}{n+1}-\frac{n}{n+2}\right] = \delta\left[\frac{n(n+2)-n(n+1)}{(n+1)(n+2)}\right]$$

$$\boxed{\therefore \delta_M = \frac{n}{(n+1)(n+2)}\delta} \tag{1}$$

Determine the drag using the linear momentum equation in the *x*-direction using the control volume shown below.



$$\frac{d}{dt}\int_{\mathrm{CV}} u_x \rho dV + \int_{\mathrm{CS}} u_x \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\mathrm{CS}} u_x \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = \rho U^2 b \int_{y=0}^{y=\delta} \left(\frac{y}{\delta}\right)^{2/n} dy - \rho U^2 bh$$

$$= \rho U^2 b\delta \int_{y/\delta=0}^{y/\delta=1} \left(\frac{y}{\delta}\right)^{2/n} d\left(\frac{y}{\delta}\right) - \rho U^2 bh$$

$$= \frac{\rho U^2 b\delta}{\frac{2}{n}+1} - \rho U^2 bh = \rho U^2 b\delta\left(\frac{n}{n+2} - \frac{h}{\delta}\right)$$

$$F_{B,x} = 0$$

$$F_{S,x} = -D \quad \text{(Since } U \text{ is a constant, the pressure will remain constant.)}$$

Substituting and simplifying gives,

$$D = \rho U^2 b\delta\left(\frac{h}{\delta} - \frac{n}{n+2}\right) \tag{2}$$

The height, h, can be found using conservation of mass on the same control volume,

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where,

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \rho U b \int_{y=0}^{y=\delta} \left(\frac{y}{\delta}\right)^{1/n} dy - \rho U b h + \rho V b L = \rho U b \delta \int_{y/\delta=0}^{y/\delta=1} \left(\frac{y}{\delta}\right)^{1/n} d\left(\frac{y}{\delta}\right) - \rho U b h + \rho V b L$$

$$= \rho U b \delta \left(\frac{1}{\frac{1}{n}+1}\right) - \rho U b h + \rho V b L = \rho U b \delta \left(\frac{n}{n+1} - \frac{h}{\delta} + \frac{V}{U}\frac{L}{\delta}\right)$$

Substituting and simplifying gives,

$$\rho U b \delta \left(\frac{n}{n+1} - \frac{h}{\delta} + \frac{V}{U}\frac{L}{\delta}\right) = 0$$

$$\frac{h}{\delta} = \frac{n}{n+1} + \frac{V}{U}\frac{L}{\delta} \tag{3}$$

Substituting Eqn. (3) into Eqn. (2) gives,

$$\boxed{D = \rho U^2 b \delta \left[\frac{n}{(n+1)(n+2)} + \frac{V}{U}\frac{L}{\delta}\right]} \tag{4}$$

or

$$\boxed{D = \rho U^2 b \left(\delta_M + \frac{V}{U} L\right)} \tag{5}$$

9.3. Boundary Layer Equations

To determine the boundary layer velocity profile for a given flow, we need to go back to the governing equations of fluid motion. Let's consider steady, incompressible, laminar flow over a sharp-edged flat plate as shown in Figure 9.5.

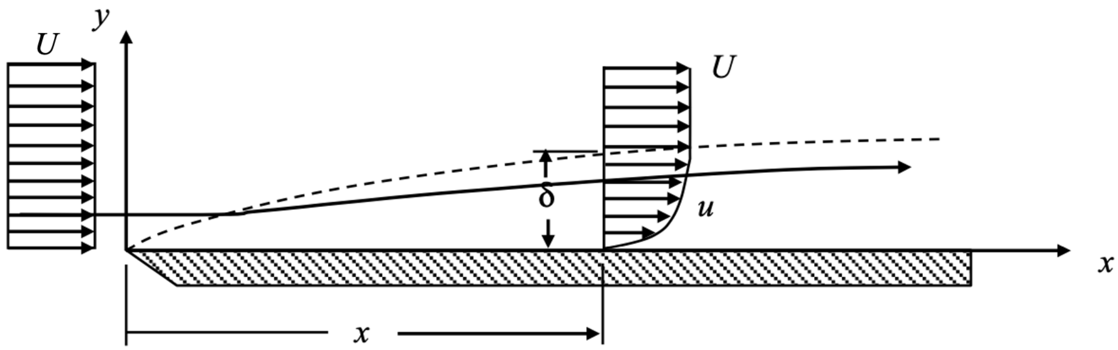


Figure 9.5. A schematic showing the formation of a boundary layer on a flat plate. The dashed line is the 99% boundary layer thickness and the solid line is a typically flow streamline.

In the following analysis, we'll assume that the boundary layer thickness, δ, is much smaller than the distance from the leading edge of the plate, x,

$$\frac{\delta}{x} \ll 1. \tag{9.6}$$

To help simplify the governing equations, let's make some estimates of the magnitudes of some of the other parameters,

$$u \sim U, \quad x \sim x, \quad y \sim \delta. \tag{9.7}$$

From the Continuity Equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies \frac{\partial v}{\partial y} \sim \frac{\partial u}{\partial x} \sim \frac{U}{x} \implies v \sim \frac{U\delta}{x}. \tag{9.8}$$

We can use these estimates in the momentum equations (Navier-Stokes equations) to determine if any terms are small in comparison to the other terms in the equations. Note that body forces are neglected in the following equations since they are typically very small in comparison to the other terms,

$$x\text{-dir:} \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{9.9}$$

$$y\text{-dir:} \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{9.10}$$

$$x\text{-dir:} \quad \frac{U^2}{x} + \frac{U^2}{x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu U}{x^2} + \frac{\nu U}{\delta^2}, \tag{9.11}$$

$$y\text{-dir:} \quad \frac{U^2}{x}\left(\frac{\delta}{x}\right) + \frac{U^2}{x}\left(\frac{\delta}{x}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\nu U}{x^2}\left(\frac{\delta}{x}\right) + \frac{\nu U}{\delta^2}\left(\frac{\delta}{x}\right). \tag{9.12}$$

Since we're assuming that $\delta \ll x$, the second viscous term in the x momentum equation is much greater than the first viscous term,

$$\frac{\nu U}{x^2} \ll \frac{\nu U}{\delta^2}, \tag{9.13}$$

so the first viscous term can be neglected. Also, since the fluid in the boundary layer is being accelerated in the x-direction despite strong viscous forces, we'll assume that the inertial terms are of the same order as the viscous term,

$$\frac{U^2}{x} \sim \frac{\nu U}{\delta^2} \implies \boxed{\delta \sim \sqrt{\frac{\nu x}{U}}}. \tag{9.14}$$

Thus, we expect the boundary layer thickness to scale with x for a laminar flow over a flat plate. Furthermore, since we assumed that $\delta \ll x$, we have,

$$\frac{\delta}{x} \ll 1 \implies \sqrt{\frac{\nu}{Ux}} \ll 1 \implies \boxed{\mathrm{Re}_x \gg 1}. \tag{9.15}$$

The assumption that $\delta \ll x$ is the same as saying that the Reynolds number based on the x-position must be much greater than one.

Now let's compare the magnitude of the pressure gradient in the x-direction to the magnitude of the pressure gradient in the y-direction. From the x-momentum equation we see that,

$$\frac{\partial p}{\partial x} \sim \frac{\rho U^2}{x} \sim \frac{\mu U}{\delta^2}, \tag{9.16}$$

From the y-momentum equation,

$$\frac{\partial p}{\partial y} \sim \frac{\rho U^2}{x}\left(\frac{\delta}{x}\right) \sim \frac{\mu U}{\delta^2}\left(\frac{\delta}{x}\right). \tag{9.17}$$

Since $\delta \ll x$,

$$\frac{\partial p}{\partial x} \gg \frac{\partial p}{\partial y}, \tag{9.18}$$

so that the pressure remains essentially constant in the y-direction in comparison to how the pressure changes in the x-direction, i.e., $p = p(x)$. Thus, we can use the pressure in the outer potential flow, determined using Bernoulli's equation, to determine how p varies with x since the pressure is essentially constant in the y-direction,

$$p + \frac{1}{2}\rho U^2 = \text{constant} \implies \frac{\partial p}{\partial x} = -\rho U \frac{\partial U}{\partial x}. \tag{9.19}$$

Substituting this relation into the x-momentum equation and simplifying gives,

$$\begin{array}{|ll|}
\hline
\text{momentum:} & u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = U\dfrac{\partial U}{\partial x} + \nu\dfrac{\partial^2 u}{\partial y^2}, \quad (9.20) \\
\text{continuity:} & \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \quad (9.21) \\
\text{boundary conditions:} & u(x, y=0) = 0 \quad \text{(no-slip at surface)} \quad (9.22) \\
 & v(x, y=0) = 0 \quad \text{(no flow through the surface)} \quad (9.23) \\
 & u(x, y \to \infty) = U \quad \text{(inner flow matches outer flow)} \quad (9.24) \\
\text{assuming:} & \mathrm{Re}_x := \dfrac{Ux}{\nu} \gg 1. \quad (9.25) \\
\hline
\end{array}$$

These are known as the Boundary Layer Equations!

Notes:

(1) The velocity U is the outer flow velocity, i.e., the velocity just outside the boundary layer. It is not necessarily the upstream velocity, U_∞.

9.4. Exact Solution for Laminar Boundary Layer Flow over a Flat Plate with No Pressure Gradient (aka the Blasius Solution)

In the previous section we developed the Boundary Layer Equations for steady, incompressible, laminar flow over a sharp-edged flat plate as shown in Figure 9.5. For flow over a flat plate, the outer potential flow velocity, U will remain constant if the displacement thickness remains small ($\mathrm{Re}_x \gg 1$ so that the outer flow is not perturbed much),

$$U = \text{constant} \implies \frac{\partial U}{\partial x} = 0 \implies \frac{\partial p}{\partial x} = 0, \tag{9.26}$$

so that the boundary layer equations simplify to,

$$\begin{aligned}
&\text{momentum:} && u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, && (9.27)\\
&\text{continuity:} && \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, && (9.28)\\
&\text{boundary conditions:} && u(x, y = 0) = 0 \quad \text{(no-slip at surface)} && (9.29)\\
& && v(x, y = 0) = 0 \quad \text{(no flow through the surface)} && (9.30)\\
& && u(x, y \to \infty) = U \quad \text{(inner flow matches outer flow)} && (9.31)\\
&\text{assuming:} && \mathrm{Re}_x := \frac{Ux}{\nu} \gg 1. && (9.32)
\end{aligned}$$

Notice that there's no characteristic length scale in the x-direction so we might expect that the flow profiles will have similar shape, but scaled in magnitude, as we move downstream. This being the case, let's look for a solution to the Boundary Layer Equations of the form,

$$\frac{u}{U} = f\left(\frac{y}{\delta}\right), \tag{9.33}$$

where the boundary layer thickness, δ, for this laminar flow will scale with (refer to the previous section),

$$\delta \sim \sqrt{\frac{\nu x}{U}}. \tag{9.34}$$

Thus, let's try looking for a similarity solution to the original PDEs using the similarity variable, η, defined as,

$$\frac{u}{U} = f(\eta) \quad \text{where} \quad \eta := y\sqrt{\frac{U}{2\nu x}}. \tag{9.35}$$

Note that the factor of "2" has been included in the similarity variable merely for convenience. It will make the resulting differential equation a little easier to work with. One other simplification we can make since we are investigating a planar flow is to write the velocity components u and v in terms of a stream function, ψ,

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = Uf(\eta), \tag{9.36}$$

$$\frac{\partial \psi}{\partial \eta} = U\frac{\partial y}{\partial \eta}f(\eta), \tag{9.37}$$

$$\psi = \int U\sqrt{\frac{2\nu x}{U}}f(\eta)d\eta + \text{constant}. \tag{9.38}$$

Since the constant is arbitrary (we really only care about the velocities so it doesn't matter what the constant is), set it equal to zero. The resulting stream function becomes,

$$\psi = \sqrt{2\nu Ux}F(\eta), \tag{9.39}$$

where $F' = f$. The velocities are found from the stream function,

$$u = \frac{\partial \psi}{\partial y} = \sqrt{2\nu Ux}F'\frac{\partial \eta}{\partial y}, \tag{9.40}$$

$$\therefore u = UF'. \tag{9.41}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{1}{2}\sqrt{\frac{2\nu U}{x}}F - \sqrt{2\nu U x}F'\frac{\partial \eta}{\partial x} = -\frac{1}{2}\sqrt{\frac{2\nu U}{x}}F + \frac{1}{2}\sqrt{2\nu U x}F' y\sqrt{\frac{U}{2\nu x^3}}, \tag{9.42}$$

$$\therefore v = \sqrt{\frac{\nu U}{2x}}(\eta F' - F). \tag{9.43}$$

Note that $f = F'$. Also,

$$\frac{\partial u}{\partial x} = \frac{\partial (UF')}{\partial \eta}\frac{\partial \eta}{\partial x} = -\frac{UF'' y}{2}\sqrt{\frac{U}{2\nu x^3}} = -\frac{U}{2x}\eta F'', \tag{9.44}$$

$$\frac{\partial u}{\partial y} = \frac{\partial (UF')}{\partial \eta}\frac{\partial \eta}{\partial y} = Uf''\sqrt{\frac{U}{2\nu x}}, \tag{9.45}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 (UF')}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 = UF''\frac{U}{2\nu x}, \tag{9.46}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial \eta}\left[\sqrt{\frac{\nu U}{2x}}(\eta F' - F)\right]\frac{\partial \eta}{\partial y} = \frac{U}{2x}\eta F''. \tag{9.47}$$

The Continuity Equation is automatically satisfied after substitution (as expected since a stream function has been used) and the original momentum PDE becomes,

$$UF'\left(-\frac{U}{2x}\eta F''\right) + \sqrt{\frac{\nu U}{2x}}(\eta F' - f)\left(UF''\sqrt{\frac{U}{2\nu x}}\right) = \nu\left(Uf'''\frac{U}{2\nu x}\right), \tag{9.48}$$

$$\therefore F''' + FF'' = 0, \tag{9.49}$$

and the boundary conditions become,

$$u(x, y = 0) = 0 \implies F'(\eta = 0) = 0, \tag{9.50}$$

$$v(x, y = 0) = 0 \implies F(\eta = 0) = 0, \tag{9.51}$$

$$u(x, y \to \infty) = 0 \implies F'(\eta \to \infty) = 1. \tag{9.52}$$

In summarizing these results, we see that the original boundary layer PDEs for laminar flow over a flat plate with no pressure gradient can be simplified to a non-linear ODE by using a similarity variable. The resulting equations and boundary conditions become,

$$\boxed{\begin{aligned} &F''' + FF'' = 0, && (9.53)\\ &F'(\eta = 0) = 0, && (9.54)\\ &F(\eta = 0) = 0, && (9.55)\\ &F'(\eta \to \infty) = 1, && (9.56)\\ &u = UF', && (9.57)\\ &v = \sqrt{\frac{\nu U}{2x}}(\eta F' - F), && (9.58)\\ &\eta = \sqrt{\frac{U}{2\nu x}}. && (9.59) \end{aligned}}$$

This is the Blasius Equation for Boundary Layer Flow Over a Flat Plate.

Notes:

(1) There is no known closed-form solution to this ODE so we resort to solving it numerically (using a Runge-Kutta method, for example). A plot of the solution is shown in Figure 9.6. The analytical results found here match experimental data very well. Note that the similarity variable in the plot does not include the square root of two term, i.e.,

$$\eta_{\text{plot}} = \sqrt{2}\eta_{\text{these notes}}. \tag{9.60}$$

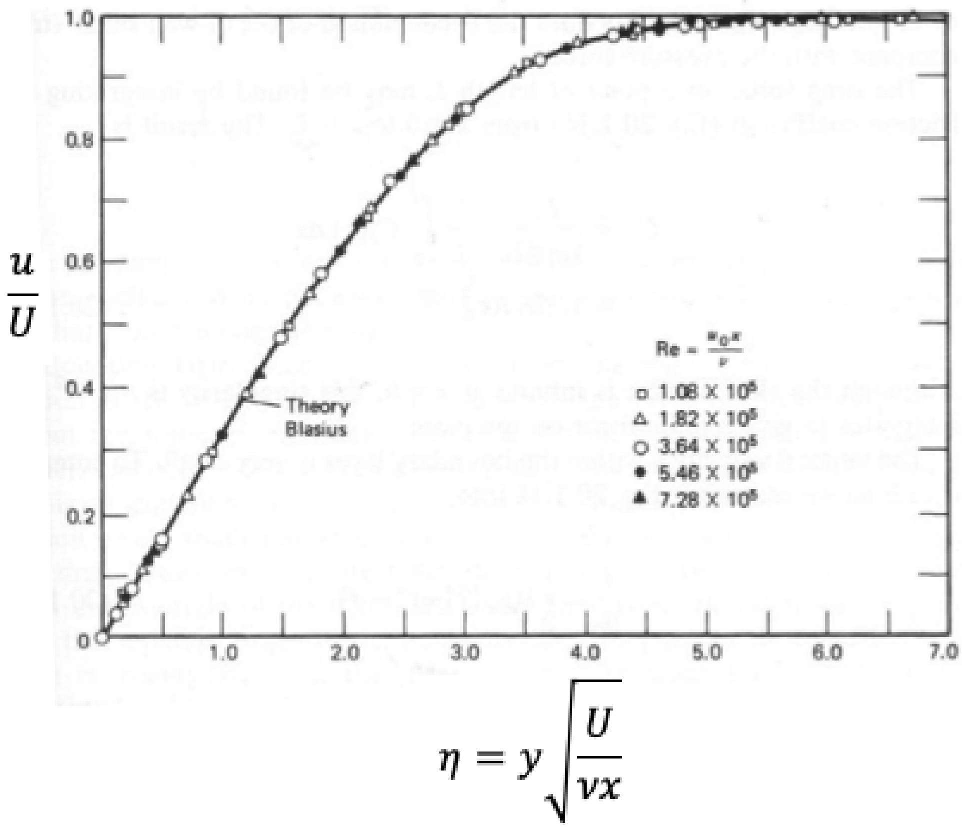


Figure 9.6. A plot showing experimental data for the dimensionless horizontal speed in a flat plate, no pressure gradient boundary layer flow plotted against the similarity variable η. The line in the plot is the numerical Blasius solution. This plot is from Panton, R.L., *Incompressible Flow*, 2nd ed., Wiley.

(2) The boundary layer thickness, δ, is found from the numerical solution to occur at $\eta = 3.5$,

$$F'(\eta = 3.5) = \frac{u}{U}(\eta = 3.5) = 0.99. \tag{9.61}$$

The boundary later thickness is then,

$$\eta = 3.5 = \delta\sqrt{\frac{U}{2\nu x}} \implies \frac{\delta}{x} = \frac{5.0}{\mathrm{Re}_x^{1/2}}. \tag{9.62}$$

(3) The displacement and momentum thicknesses can also be found numerically using their definitions,

$$\frac{\delta_D}{x} = \frac{1.72}{\mathrm{Re}_x^{1/2}}, \quad \frac{\delta_M}{x} = \frac{0.664}{\mathrm{Re}_x^{1/2}}. \tag{9.63}$$

(4) The shear stress at the plate surface in dimensionless form is known as the friction coefficient, c_f,

$$c_f := \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{U\mu\left(\frac{d(u/U)}{d\eta}\frac{\partial\eta}{\partial y}\right)\Big|_{y=0}}{\frac{1}{2}\rho U^2} = \frac{\sqrt{2}F''|_{\eta=0}}{\sqrt{\mathrm{Re}_x}} = \frac{0.664}{\mathrm{Re}_x^{1/2}}. \tag{9.64}$$

The friction coefficient is a ratio of the shear stress to the dynamic pressure in the flow.

(5) The drag coefficient, c_D, defined as the dimensionless drag acting on the plate between $x = 0$ and $x = L$, is found by integrating the shear force over the plate area,

$$c_D := \frac{\int_{x=0}^{x=L} \tau_w dx}{\frac{1}{2}\rho U^2 L} = \frac{1.328}{\mathrm{Re}_L^{1/2}} \quad \text{(where } c_D \text{ is the drag coefficient per unit depth).} \tag{9.65}$$

Note that although the boundary layer assumptions break down near the leading edge of the plate ($\mathrm{Re}_x \not\gg 1$), the distance over which this is the case is small in comparison to the typical lengths of interest. This discrepancy is generally neglected in engineering applications.

(6) This solution is only valid for laminar boundary layers. As a rule of thumb, the transition from laminar flow to turbulent flow occurs at: $\mathrm{Re}_x \approx 500,000$.

(7) Recall that the boundary layer equations on which the Blasius solution is based are valid only when $\mathrm{Re}_x \gg 1$. In practice, it has been found that the Blasius solution is accurate when $\mathrm{Re}_L > 1000$. For $1 \leq \mathrm{Re}_L \leq 1000$, the following relation developed by Imai (1957) is more appropriate,

$$c_D = \frac{1.328}{\mathrm{Re}_L^{1/2}} + \frac{2.3}{\mathrm{Re}_L}. \tag{9.66}$$

(8) In summary, for laminar flow over a flat plate with no pressure gradient, the "exact" Blasius solution is,

$$\boxed{\begin{array}{l} \dfrac{\delta}{x} = \dfrac{5.0}{\mathrm{Re}_x^{1/2}} \qquad \dfrac{\delta_D}{x} = \dfrac{1.72}{\mathrm{Re}_x^{1/2}} \qquad \dfrac{\delta_M}{x} = \dfrac{0.664}{\mathrm{Re}_x^{1/2}} \qquad (9.67) \\ c_f = \dfrac{0.664}{\mathrm{Re}_x^{1/2}} \qquad c_D = \dfrac{1.328}{\mathrm{Re}_L^{1/2}} \qquad (9.68) \\ \mathrm{Re}_x < 500,000 \qquad (9.69) \end{array}}$$

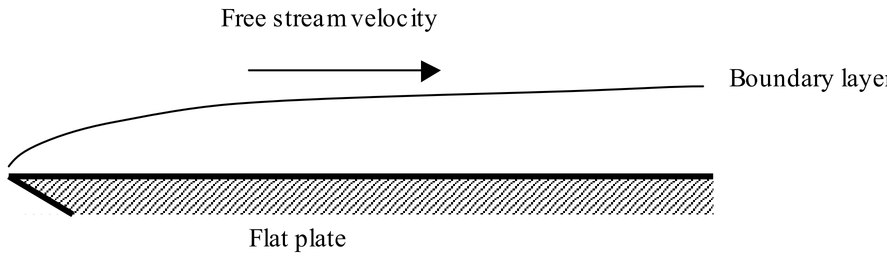
Air flows over a flat plate as shown below and forms a laminar boundary layer on the surface of the plate. Circle the letter of the statement that best represents the variation of the shear stress (force per unit area) at the wall with distance along the plate.

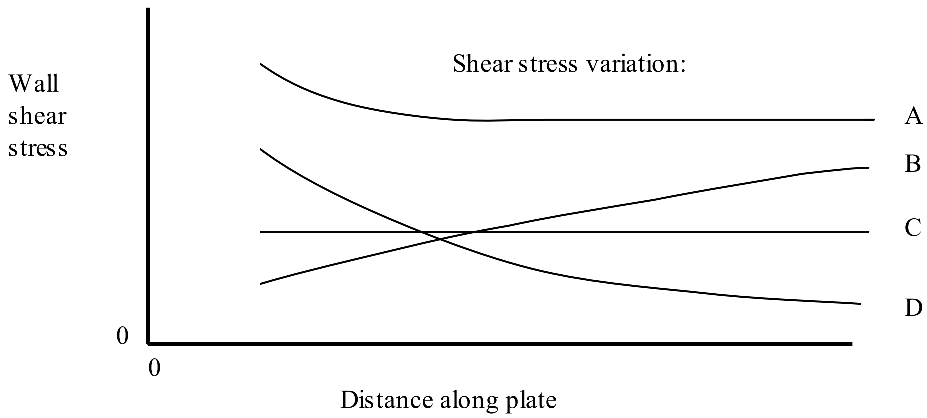
A. Shear stress variation A

B. Shear stress variation B

C. Shear stress variation C

D. Shear stress variation D





SOLUTION:

The friction coefficient for a laminar boundary layer is given by,

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\mathrm{Re}_x^{1/2}} \,. \tag{1}$$

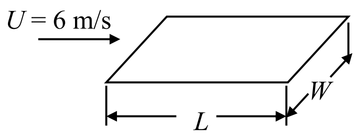
Re-arranging to solve for τ_w in terms of x gives,

$$\tau_w = \tfrac{1}{2}\rho U^2 \frac{0.664}{\left(\frac{Ux}{\nu}\right)^{1/2}} = \frac{\frac{1}{2}\cdot 0.664\,\rho \nu^{1/2} U^{3/2}}{x^{1/2}} \,. \tag{2}$$

Thus, as x increases, the wall shear stress will approach a value of zero as the inverse of the square root of x. Hence, the correct answer is $\boxed{\text{curve D}}$.

A thin flat plate 55 by 110 cm is immersed in a 6 m/s stream of SAE 10 oil at 20 °C. Compute the total skin friction drag if the stream is parallel to (a) the long side and (b) the short side.

SOLUTION:



$\nu_{\text{SAE 10 oil}} = 1.20*10^{-4}\ \text{m}^2/\text{s}$
$\rho_{\text{SAE 10 oil}} = 870\ \text{kg/m}^3$

Determine the Reynolds number at the trailing edge of the plate to see if it's laminar.

$$\text{Re}_L = \frac{UL}{\nu} \quad \text{(The flow is considered laminar if } Re < 1*10^6\text{.)} \tag{1}$$

When L = 1.10 m then Re_L = 55,200 ⇒ laminar flow. When L = 0.55 m then Re_L = 27,500 ⇒ laminar flow.

Now determine the drag on the plate using the drag coefficient, c_D, for laminar flat plate flow (the Blasius solution).

$$D = \left(\tfrac{1}{2}\rho U^2\right)\underbrace{\left(2LW\right)}_{\text{top and bottom faces}} c_D$$

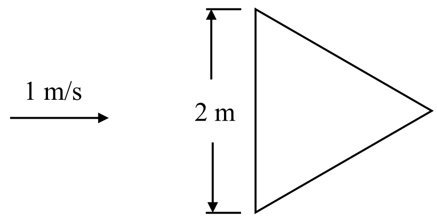
$$\therefore D = \left(\tfrac{1}{2}\rho U^2\right)\left(2LW\right)\left(\frac{1.328}{\text{Re}_L^{1/2}}\right) \tag{2}$$

When L = 1.10 m, W = 0.55 m, Re_L = 55,200, and $\boxed{D = 107\ \text{N}}$.

When L = 0.55 m, W = 1.10 m, Re_L = 27,500, and $\boxed{D = 152\ \text{N}}$.

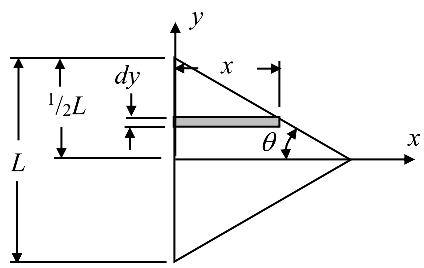
Note that the drag is greater when the short side is aligned with the flow. Why? Because from Eqn. (2) we observe that the drag varies with $\sqrt{L}$ but is proportional to W. Hence the drag will increase more rapidly with increasing width than with increasing length.

A thin equilateral triangle plate is immersed parallel to a 1 m/s stream of air at standard conditions. Estimate the skin friction drag on this plate.



SOLUTION:

Determine the drag acting on thin strips as shown in the figure below. The total drag will be the sum of each of these individual drag forces.



Since the plate is symmetric about the x-axis, consider only the $y > 0$ portion of the plate. The equation of the line defining the downstream edge of the plate is:

$$y = (-\tan\theta)x + \tfrac{1}{2}L \quad \text{where } 0 \le x \le L\cos\theta \ \ (L = 2\text{ m},\ \theta = 30^\circ) \tag{1}$$

Re-arrange to solve for x in terms of y:

$$x = \frac{\tfrac{1}{2}L - y}{\tan\theta} \tag{2}$$

Determine if the flow transitions to turbulence at any point on the plate.

$$\mathrm{Re}_{\mathrm{crit}} = \frac{Ux_{\mathrm{crit}}}{\nu} = 500{,}000 \ \Rightarrow \ x_{\mathrm{crit}} = 500{,}000\frac{\nu}{U} \tag{3}$$

Using the given data:

$\nu = 1.5*10^{-5}$ m^2/s

$U = 1$ m/s

$\Rightarrow \underline{x_{\mathrm{crit}} = 7.5\text{ m}}$

Since the longest strip length is only $L\cos\theta = (2\text{ m})\cos(30^\circ) = 1.73$ m, the flow over the entire plate will be laminar. Hence, we can use the Blasius solution relations to model the boundary layer characteristics. The drag acting on a single strip of length x and width dy is:

$$\begin{aligned} dD &= c_D \tfrac{1}{2}\rho U^2 x\,dy \\ &= \frac{1.328}{\mathrm{Re}_x^{1/2}}\tfrac{1}{2}\rho U^2 x\,dy \end{aligned} \tag{4}$$

$$\therefore dD = 0.664\sqrt{\nu}\,\rho U^{3/2} x^{1/2} dy \tag{5}$$

The total drag acting on the plate (including the top half, bottom half, and front and back surfaces) is:

$$D = 4\int_{y=0}^{y=\frac{1}{2}L} dD = 4\int_{y=0}^{y=\frac{1}{2}L} 0.664\sqrt{\nu}\rho U^{3/2} x^{1/2} dy$$

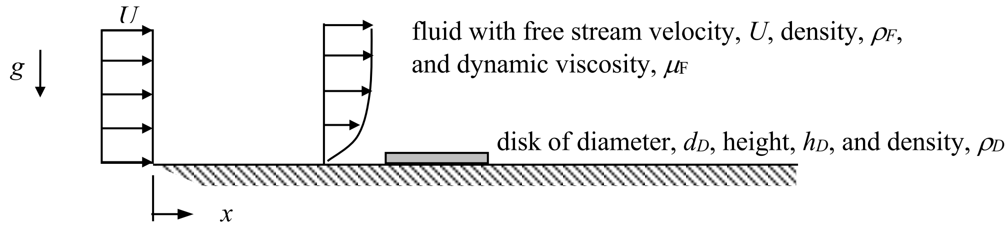
$$= 4\int_{y=0}^{y=\frac{1}{2}L} 0.664\sqrt{\nu}\rho U^{3/2}\left(\frac{\frac{1}{2}L - y}{\tan\theta}\right)^{1/2} dy \quad \text{(making use of Eqn. (2))} \qquad (6)$$

$$\boxed{\therefore D = 0.626\rho\left(UL\right)^{3/2}\sqrt{\frac{\nu}{\tan\theta}}} \qquad (7)$$

Using the given data:

ρ = 1.23 kg/m^3
U = 1 m/s
L = 2 m
ν = 1.5*10^{-5} m^2/s
θ = 30°
$\Rightarrow$ $\boxed{D = 1.1*10^{-2}\text{ N}}$

Consider a thin disk of density, ρ_D, diameter, d_D, and height, h_D, resting on a submerged flat plate as shown in the figure below. Flowing over the plate is a fluid of density, ρ_F, and dynamic viscosity, μ_F, with a free stream velocity, U. There are no pressure gradients in the flow.



Assume the flow upstream of the plate is uniform, but then results in a boundary layer when the fluid contacts the plate. The effective static friction coefficient between the disk and the plate is μ (for simplicity assume that the static and dynamic friction coefficients are equal). For the following questions, assume the following:

d_D = 2 mm
h_D = 0.5 mm
ρ_D = 2500 kg/m^3
ρ_F = 1000 kg/m^3
μ_F = 1.0e-3 kg/(m·s)
U = 1.5 m/s
μ = 0.3
g = 9.81 m/s^2

a. Determine the effective friction force acting to hold the disk in place.
b. If the disk is released at the leading edge of the plate, at what distance from the leading edge will the disk come to rest? (Neglect the inertia of the disk, *i.e.* treat the disk movement in a quasi-static manner).
c. Neglecting the flow over the disk, at what distance from the leading edge will the boundary layer separate? Justify your answer.

SOLUTION:

The disk will remain stationary when the shear force acting on it, F_S, is less than the friction force holding the disk in place, F_F, *i.e.* the disk will remain stationary when

$$F_S \le F_F \quad \text{(the disk's inertia has been neglected)} \tag{1}$$

The friction force is the friction coefficient multiplied by the effective weight.

$$\boxed{F_F = \mu\left(\rho_D - \rho_F\right)\tfrac{\pi}{4}d_D^2 h_D g} \tag{2}$$

Using the given data, $\boxed{F_F = 6.9\text{e-}6\ \text{N}}$.

The shear force acting on the disk due to the flowing fluid is (approximately*) the shear stress multiplied by the disk's projected area (when viewed from above):

$$F_S = \tau_w \tfrac{\pi}{4} d_D^2 \tag{3}$$

The shear stress may be estimated using the boundary layer relations. First assume that the boundary layer is laminar (*i.e.* $\text{Re}_x < 500{,}000$) so that the friction coefficient is:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_F U^2} = \frac{0.664}{\text{Re}_x^{1/2}} \Rightarrow \tau_w = \tfrac{1}{2}\rho_F U^2 \frac{0.664}{\text{Re}_x^{1/2}} \tag{4}$$

where

$$\text{Re}_x = \frac{\rho_F U x}{\mu_F} \tag{5}$$

Combining Eqns. (1) - (5) gives:

$$\left(\tfrac{1}{2}\rho_F U^2 \frac{0.664}{\text{Re}_x^{1/2}}\right)\tfrac{\pi}{4}d_D^2 = \mu\left(\rho_D - \rho_F\right)\tfrac{\pi}{4}d_D^2 h_D g \quad \text{(when the shear and friction forces just balance)} \tag{6}$$

$$\text{Re}_x = \left[0.332\frac{\rho_F U^2}{\mu\left(\rho_D - \rho_F\right)h_D g}\right]^2 \tag{7}$$

Using the given data:

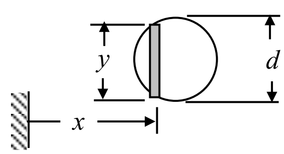
$\text{Re}_x = 1.2\text{e}5$ (laminar flow assumption ok)

$\Rightarrow \boxed{x = 0.076 \text{ m}}$

Since the flow does not have an adverse pressure gradient (in fact the flow has <u>no</u> pressure gradient), the <u>boundary layer will not separate</u>.

* A more accurate method of calculating the fluid force on the disk is to integrate the shear stress over the disk's surface.

$$F_S = \int_{x=x-\frac{1}{2}d}^{x=x+\frac{1}{2}d} \tau_w\, dA = \int_{x=x-\frac{1}{2}d}^{x=x+\frac{1}{2}d} \underbrace{\tfrac{1}{2}\rho_F U^2 \frac{0.664}{\text{Re}_x^{1/2}}}_{=\tau_w} \underbrace{y dx}_{=dA} \tag{8}$$



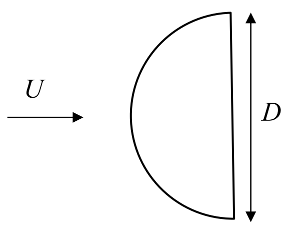
where

$$y = 2\sqrt{\tfrac{1}{4}d^2 - x^2} \tag{9}$$

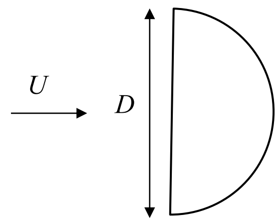
Determine the expression for the drag force acting on the thin, semi-circular plate immersed in a flow as shown below, assuming,

a. the "tip" faces the flow, and
b. the base faces the flow.

Assume the flow is laminar over the entire plate. You need not solve any integrals you encounter.

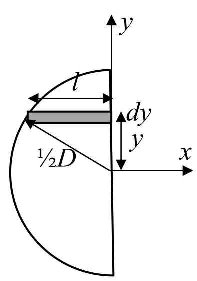


(a)



(b)

SOLUTION:



The drag force on both sides of a thin strip of the plate is,

$$dD = 2c_D \tfrac{1}{2}\rho U^2 l dy = 2\frac{1.328}{\mathrm{Re}_l^{1/2}}\tfrac{1}{2}\rho U^2 l dy = 1.328\sqrt{\frac{\nu}{Ul}}\rho U^2 l dy = 1.328\nu^{1/2}\rho U^{3/2} l^{1/2} dy \ , \tag{1}$$

where

$$l = \left(\tfrac{1}{4}D^2 - y^2\right)^{1/2} \ , \tag{2}$$

Thus, the total drag on the plate is,

$$D = \int_{y=-\frac{1}{2}D}^{y=\frac{1}{2}D} dD = \int_{y=-\frac{1}{2}D}^{y=\frac{1}{2}D} 1.328\nu^{1/2}\rho U^{3/2}\left(\tfrac{1}{4}D^2 - y^2\right)^{1/4} dy = 1.328\nu^{1/2}\rho U^{3/2}\int_{y=-\frac{1}{2}D}^{y=\frac{1}{2}D}\left(\tfrac{1}{4}D^2 - y^2\right)^{1/4} dy \ , \tag{3}$$

$$\boxed{D = 1.328\nu^{1/2}\rho U^{3/2}\int_{y=-\frac{1}{2}D}^{y=\frac{1}{2}D}\left(\tfrac{1}{4}D^2 - y^2\right)^{1/4} dy}. \tag{4}$$

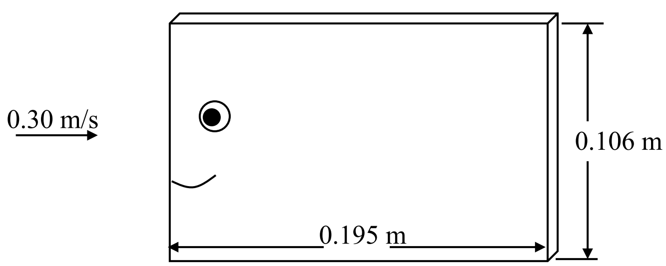
The drag force on the plate will be exactly the same if the plate is reversed since the drag on each strip is identical.

An engineer studies swimming fish in order to develop bio-inspired designs for boat and submarine hulls. These studies focus on a fish known as a "scup" (Stenotomus chrysops), shown in the following photo. The speed of the swimming fish, the body area on one side of the fish, and the length of fish from head to tail are given in the table adjacent to the photograph.

swimming speed =	0.3 m/s
body area on one side =	0.0207 m^2
body length =	0.195 m
salt water density =	1026 kg/m^3
salt water kinematic viscosity =	1.19*10^{-6} m^2/s

a. For the given conditions, determine if the flow over the entire length of the fish is laminar, turbulent, or a combination of the two. Show your work supporting your answer.

b. Estimate the total drag acting on the fish, assuming skin friction drag dominates (a good assumption based on experiments) and that the fish shape may be modeled as a thin, flat rectangular plate as shown below.



c. Calculate the power required for the fish to swim at the given conditions.

SOLUTION:

The Reynolds number based on the fish's length is:

$$\mathrm{Re}_L = \frac{UL}{\nu} = \frac{(0.30\ \mathrm{m/s})(0.195\ \mathrm{m})}{(1.19*10^{-6}\ \mathrm{m^2/s})} = 49{,}200. \qquad (1)$$

Since the Reynolds number is less than 500,000, $\boxed{\text{the boundary layer over the entire fish is laminar}}$.

The total drag acting on the (rectangular) fish may be found using the drag coefficient for a laminar boundary layer flow over a flat plate (the Blasius solution),

$$D_{2\text{ sides}} = 2D_{1\text{ side}} = 2c_D \tfrac{1}{2}\rho U^2 LW, \qquad (2)$$

where L is the length of the fish (L = 0.195 m) and H is the height of the fish (H = 0.106 m), and c_D for a laminar flat plate flow is,

$$c_D = \frac{1.328}{\mathrm{Re}_L^{1/2}}, \tag{3}$$

where the Reynolds number based on the fish's length was found in Eq. (1).

Using the given data,

$\mathrm{Re}_L = 49{,}200$

$\Rightarrow\ c_D = 5.99*10^{-3}$

Note that experimental measurements provided by Anderson *et al*. (2001) found that the drag coefficient for these conditions is $4.4*10^{-3}$. Hence, the prediction from our simplified model has a relative error of about 36%.

$\rho = 1026\ \mathrm{kg/m^3}$

$U = 0.30\ \mathrm{m/s}$

$L = 0.195\ \mathrm{m}$

$H = 0.106\ \mathrm{m}$

$\Rightarrow\ D_{1\ \mathrm{side}} = 5.72*10^{-3}\ \mathrm{N}$

$$\Rightarrow\ \boxed{D_{2\ \mathrm{sides}} = 1.14*10^{-2}\ \mathrm{N}} \tag{4}$$

The power required for the fish to swim at a speed of $U = 0.30$ m/s given the drag found in Eq. (4) is,

$$P = D_{2\ \mathrm{sides}}U \ \Rightarrow\ \boxed{P = 3.43*10^{-3}\ \mathrm{W} = 3.43\ \mathrm{mW}} \tag{5}$$

A good resource on boundary layer characteristics over swimming fish is:

Anderson, E.J., McGillis, W.R., and Grosenbaugh, M.A., 2001, "The boundary layer of swimming fish," *The Journal of Experimental Biology*, Vol. 204, pp. 81 – 102.

Air, with a density of 1.23 kg/m^3 and a kinematic viscosity of 2.5*10^{-6} m^2/s, enters a long horizontal ventilation duct of circular cross-section (radius of 0.25 m) with a velocity of 1.0 m/s. At the entrance it is assumed that this velocity is uniform over the entire cross-section. However, as the flow proceeds down the duct a thin laminar boundary develops on the inside wall of the duct.

If we first assume that this is like the boundary layer on a flat plate and that the velocity away from the boundary layer remains at 1.0 m/s, find the displacement thickness in meters at a distance *x* (in meters) from the entrance.

Having calculated this displacement thickness we recognize that the velocity outside the boundary layer cannot remain precisely constant at 1 m/s. Using the above calculated displacement thickness, find the uniform velocity outside the boundary layer at a point 200 m from the entrance. What is the pressure difference between the entrance and this point 200 m from the entrance? Describe in words how you might now proceed to a more accurate boundary layer calculation which takes this pressure gradient into account.

SOLUTION:

The displacement thickness, δ_D, for a laminar boundary layer is given by the Blasius solution to be:

$$\frac{\delta_D}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$$

$$\boxed{\delta_D = 1.72\sqrt{\frac{\nu x}{U}}} \tag{1}$$

Using the given data:

ν = 2.5*10^{-6} m^2/s
U = 1 m/s
$\boxed{\delta_D = (2.7 \times 10^{-3}\ \mathrm{m}^{1/2})\, x^{1/2}}$

From conservation of mass and the definition of the displacement thickness, the velocity in the flow outside of the boundary layer, U, is:

$$\rho U_0 \pi R^2 = \rho U \pi \left(R - \delta_D\right)^2$$

$$\boxed{U = U_0 \frac{R^2}{\left(R-\delta_D\right)^2} = U_0 \frac{1}{\left(1 - \delta_D/R\right)^2}} \tag{2}$$

where U_0 is the uniform fluid velocity at the pipe entrance. Using the given data:

U_0 = 1 m/s
R = 0.25 m
x = 200 m
$\delta_{D,x=200\ \mathrm{m}}$ = 3.9*10^{-2} m
$\boxed{U_{x=200\ \mathrm{m}} = 1.4\ \mathrm{m/s}}$

Apply Bernoulli's equation (neglecting elevation differences) in the outer, irrotational flow region.

$$p_0 + \tfrac{1}{2}\rho U_0^2 = p + \tfrac{1}{2}\rho U^2$$

$$p - p_0 = \Delta p = \tfrac{1}{2}\rho\left(U_0^2 - U^2\right) \tag{3}$$

Using the given data:

ρ = 1.23 kg/m^3
U_0 = 1 m/s
$U_{x=200\ \mathrm{m}}$ = 1.4 m/s
$\boxed{\Delta p_{x=200\ \mathrm{m}} = -0.59\ \mathrm{N/m^2}}$

For a more accurate estimate, one can iterate on the displacement thickness and core flow velocity until a converged solution is reached.

$$\delta_{D,n} = 1.72\sqrt{\frac{\nu x}{U_n}}$$

$$U_n = U_0 \frac{1}{\left(1 - \delta_{D,n}/R\right)^2}$$

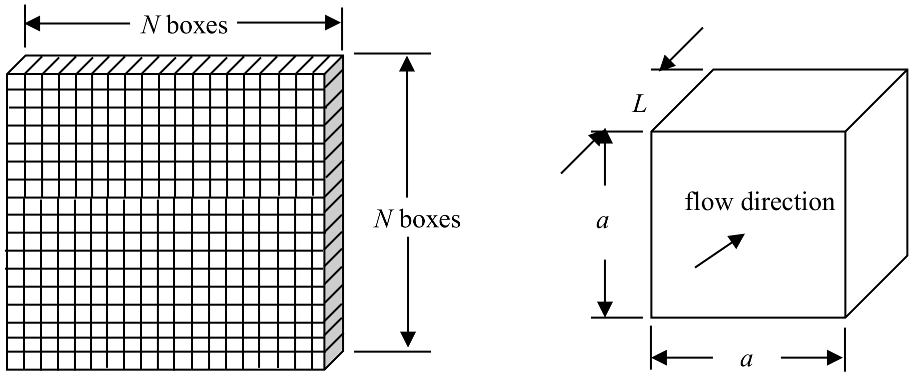
where n is the number of iterations. For example, given the previous data:

ν [m^2/s] =	2.50E-06
ρ [kg/m^3] =	1.23
x [m] =	200
U_0 [m/s] =	1.0
R[m] =	0.25

n	$\delta_{D,n}$ **[m]**	U_n **[m/s]**	Δp_n **[N/m^2]**
1	3.85E-02	1.40	-0.58
2	3.25E-02	1.32	-0.46
3	3.35E-02	1.33	-0.48
4	3.33E-02	1.33	-0.47
5	3.33E-02	1.33	-0.48
6	3.33E-02	1.33	-0.48

Flow straighteners are arrays of narrow ducts placed in wind tunnels to remove swirl and other in-plane secondary velocities. They can be idealized as square boxes constructed by vertical and horizontal plates as shown in the figure. The cross-section of the box is *a* by *a* and the box length is *L*. Assuming laminar flat plate flow and an array of *N* by *N* boxes, derive a formula for:

a. the total drag on the bundle of boxes.
b. the effective pressure drop across the bundle.



SOLUTION:

Determine the drag acting on one wall due to skin friction. From the Blasius solution, the drag coefficient for laminar, flat plate flow is:

$$c_D = \frac{1.328}{\mathrm{Re}_L^{1/2}} \tag{1}$$

where

$$c_D = \frac{D}{\frac{1}{2}\rho U^2 \left(La\right)} \tag{2}$$

$$\mathrm{Re}_L = \frac{UL}{\nu} \tag{3}$$

Note that in writing Eqn. (1) we've assumed that there is no pressure gradient in the cell's core flow. This is not exactly correct since there will, in fact, be a pressure gradient due to the growth of the boundary layer along the plate surface and hence an increase in the outer (*i.e.,* cell core) flow velocity (from conservation of mass). However, as a first estimate it is reasonable to assume a constant outer flow velocity and thus a Blasius boundary layer profile. A more precise analysis would account for the growth in the displacement thickness and the resulting increase in the outer velocity.

Using Eqns. (1) and (2), the skin friction drag acting on one wall of a cell is:

$$D_{\substack{\text{one}\\\text{wall}}} = \frac{1.328\cdot\frac{1}{2}\rho U^2\left(La\right)}{\mathrm{Re}_L^{1/2}} = \frac{0.664\rho U^2\left(La\right)}{\mathrm{Re}_L^{1/2}} \tag{4}$$

The drag acting on a single cell, which consists of four walls, is:

$$D_{\text{cell}} = 4D_{\substack{\text{one}\\\text{wall}}} = \frac{2.656\rho U^2\left(La\right)}{\mathrm{Re}_L^{1/2}} \tag{5}$$

The total drag acting on a grid of NxN cells is:

$$\boxed{D_{\substack{N\text{x}N\\ \text{cells}}} = N^2 D_{\text{cell}} = \frac{2.656\rho U^2 \left(La\right) N^2}{\text{Re}_L^{1/2}}} \tag{6}$$

where Re$_L$ is given in Eqn. (3).

As stated previously, in deriving Eqn. (1) we've assumed that there is no pressure gradient within the cell which is not entirely correct but should instead be considered a first-cut estimate. Regardless, we can still determine an effective pressure drop across the flow straightener by dividing the drag force acting on the straightener by its area.

$$\boxed{\Delta p = \frac{-D_{N\text{x}N\text{ cells}}}{N^2 a^2}} \text{ (The pressure decreases across the flow straightener.)} \tag{7}$$

A more accurate approach to solving this problem involves iteration. First, determine the velocity in the outer flow region using the displacement thickness, d_D.

$$\dot{m} = \text{constant} = \rho U a^2 = \rho U'\left(a - 2\delta_D\right)^2$$

$$U' = U\left(\frac{a}{a - 2\delta_D}\right)^2 \tag{8}$$

where, from the Blasius solution:

$$\frac{\delta_D}{L} = \frac{1.72}{\text{Re}_L^{1/2}} \tag{9}$$

and

$$\text{Re}_L = \frac{U'L}{\nu} \tag{10}$$

The pressure in the outer layer can be determined using Bernoulli's equation.

$$p + \tfrac{1}{2}\rho U^2 = p' + \tfrac{1}{2}\rho U'^2$$

$$\Delta p = p' - p = \tfrac{1}{2}\rho\left(U^2 - U'^2\right) \tag{11}$$

The iterative procedure is as follows:

1. Assume U ' = 0.
2. Evaluate the displacement thickness using Eqns. (9) and (10).
3. Evaluate the new downstream velocity, U ', using Eqn. (8).
4. Repeat steps 2 and 3 until a converged solution for U ' occurs.
5. Use Eqn. (11) to determine Δp.

9.5. Falkner-Skan Boundary Layer Solutions

Recall that a similarity approach was used to find solutions to stagnation point flow (Hiemenz Flow) and boundary layer flow over a flat plate (Blasius solution). Falkner and Skan (1931) investigated what other flows could be solved using similarity solutions. In their analysis they assumed that,

$$u(x, y) = U(x)f'(\eta), \tag{9.70}$$

where,

$$\eta \coloneqq \frac{y}{\xi(x)}. \tag{9.71}$$

Here, u is the velocity within the boundary layer, U is the outer flow (potential flow) velocity, η is the similarity variable, and ξ is a distance scaling function. Note that at this point, U and ξ are unknown functions of x.

It's convenient to work in terms of a stream function rather than velocity components so let's re-write the velocity in terms of a stream function, ψ,

$$u = \frac{\partial \psi}{\partial y} = U(x)f'(\eta) \implies \psi = \int U(x)f'(\eta)\underbrace{\frac{\partial y}{\partial \eta}}_{=\xi}\,d\eta + \text{constant}, \tag{9.72}$$

$$\therefore \psi = U(x)\xi(x)f(\eta). \tag{9.73}$$

The constant in the previous equation has been set to zero since only differences (or derivatives) in the stream function have any significance. Note also that the stream function automatically satisfies the Continuity Equation. The stream function must also satisfy the boundary layer momentum equation,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{9.74}$$

$$\implies \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^3 \psi}{\partial y^3}, \tag{9.75}$$

where,

$$\frac{\partial \psi}{\partial x} = \frac{\partial U}{\partial x}\psi f + U\frac{d\xi}{dx}f - U\xi f'\frac{y}{\xi^2}\frac{d\xi}{dx} = \frac{dU}{dx}\xi f + U\frac{d\xi}{dx}f - U\frac{d\xi}{dx}\eta f', \tag{9.76}$$

$$\frac{\partial \psi}{\partial y} = U\xi f'\frac{1}{\xi} = Uf', \tag{9.77}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{dU}{dx}f' - U\frac{y}{\xi^2}\frac{d\xi}{dx}f'' = \frac{dU}{dx}f' - \frac{U}{\xi}\frac{d\xi}{dx}\eta f'', \tag{9.78}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{Uf''}{\xi}, \tag{9.79}$$

$$\frac{\partial^3 \psi}{\partial y^3} = \frac{Uf'''}{\xi^2}. \tag{9.80}$$

Substituting and simplifying,

$$(Uf')\left(\frac{dU}{dx}f' - \frac{U}{\xi}\frac{d\xi}{dx}\eta f''\right) - \left(\frac{dU}{dx}\xi f + U\frac{d\xi}{dx}f - U\frac{d\xi}{dx}\eta f'\right)\left(\frac{Uf''}{\xi}\right) = U\frac{dU}{dx} + \nu\frac{Uf'''}{\xi^2}, \tag{9.81}$$

$$U\frac{dU}{dx}(f')^2 - \cancel{\frac{U^2}{\xi}\frac{d\xi}{dx}\eta f' f''} - \underbrace{U\frac{dU}{dx}ff'' \quad + \frac{U^2}{\xi}\frac{d\xi}{dx}ff''}_{=-\frac{U}{\xi}\frac{d}{dx}(U\xi)ff''} + \cancel{\frac{U^2}{\xi}\frac{d\xi}{dx}\eta f' f''} = U\frac{dU}{dx} + \nu\frac{Uf'''}{\xi^2}, \tag{9.82}$$

$$U\frac{dU}{dx}(f')^2 - \frac{U}{\xi}\frac{d}{dx}(U\xi)ff'' = U\frac{dU}{dx} + \nu\frac{Uf'''}{\xi^2}. \tag{9.83}$$

Multiply through by $\xi^2/(\nu U)$ and re-arrange to get,

$$f''' + \frac{\xi}{\nu}\frac{d}{dx}(U\xi)ff'' + \frac{\xi^2}{\nu}\frac{dU}{dx}\left[1-(f')^2\right] = 0. \tag{9.84}$$

If a similarity solution exists, Eq. (9.84) should be an ODE for the function f in terms of η. Thus, the coefficients in front of the second and third terms should be, at most, constants, i.e., for a similarity solution to exist, we must have,

$$\boxed{\alpha = \frac{\xi}{\nu}\frac{d}{dx}(U\xi) \quad \text{and} \quad \beta = \frac{\xi^2}{\nu}\frac{dU}{dx}}, \tag{9.85}$$

where α and β are constants. These two equations can be combined to form an expression that is more convenient to work with later in our analysis,

$$\frac{d}{dx}(U\xi^2) = U(2\xi)\frac{d\xi}{dx} + \xi^2\frac{dU}{dx}, \tag{9.86}$$

$$= 2\xi\left(U\frac{d\xi}{dx} + \xi\frac{dU}{dx}\right) - \xi^2\frac{dU}{dx}, \tag{9.87}$$

$$= \underbrace{2\xi\frac{d}{dx}(U\xi)}_{=2\nu\alpha} - \underbrace{\xi^2\frac{dU}{dx}}_{\nu\beta}, \tag{9.88}$$

$$\therefore \frac{d}{dx}(U\xi^2) = \nu(2\alpha - \beta). \tag{9.89}$$

Substituting Eq. (9.85) into Eq. (9.84) gives,

$$\boxed{f''' + \alpha ff'' + \beta\left[1-(f')^2\right] = 0}. \tag{9.90}$$

The boundary conditions for this equation are,

$$u(x, y=0) = 0 \implies f'(\eta = 0) = 0, \tag{9.91}$$

$$v(x, y=0) = 0 \implies f(\eta = 0) = 0, \tag{9.92}$$

$$u(x, y\to\infty) = U \implies f'(\eta\to\infty) = 1. \tag{9.93}$$

The procedure for determining the exact solutions to these boundary layer equations is:

(1) Select values for the constants α and β. Note that at this point we don't know what geometry we're investigating. The geometry will be determined in the next step.

(2) Determine the corresponding form for $U(x)$ and $\xi(x)$ using Eq. (9.85), or, more conveniently, part of Eq. (9.85) and Eq. (9.89) (repeated here for convenience),

$$\frac{d}{dx}(U\xi^2) = \nu(2\alpha-\beta) \quad \text{and} \quad \xi^2\frac{dU}{dx} = \nu\beta. \tag{9.94}$$

(3) Determine the function $f(\eta)$ from the ODE given in Eq. (9.90) subject to the boundary conditions in Eqs. (9.91) - (9.93). This part is usually solved numerically. Repeating the equations for convenience,

$$f''' + \alpha ff'' + \beta\left[1-(f')^2\right] = 0, \tag{9.95}$$

$$f'(\eta = 0) = 0, \tag{9.96}$$

$$f(\eta = 0) = 0, \tag{9.97}$$

$$f'(\eta\to\infty) = 1. \tag{9.98}$$

(4) Determine the stream function from Eq. (9.73) and velocity components from the stream function. The wall shear stress may also be determined,

$$\psi = U(x)\xi(x)f(\eta), \tag{9.99}$$

$$u = \frac{\partial \psi}{\partial y} = Uf', \tag{9.100}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{dU}{dx}\xi f + U\frac{d\xi}{dx}f - U\frac{d\xi}{dx}\eta f', \tag{9.101}$$

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{\partial^2 \psi}{\partial y^2}\bigg|_{y=0} = \mu \frac{Uf''|_{\eta=0}}{\xi} \quad \text{(Refer to Eq. (9.79))}. \tag{9.102}$$

Notes:

(1) Recall that $U(x)$ is the flow profile for the outer, potential flow and $\xi(x)$ is a distance scaling parameter.
(2) Consider the case when $\alpha = 1$ and β is left arbitrary. From Eq. (9.94) we have,

$$\frac{d}{dx}(U\xi^2) = \nu(2-\beta), \tag{9.103}$$

$$\implies U\xi^2 = \nu x(2-\beta) + \text{constant}. \tag{9.104}$$

The scaling factor will be zero at $x = 0$ so the constant will also be zero. Now divide the equation by the other equation given in Eq. (9.94),

$$\frac{1}{\xi^2 dU/dx}U\xi^2 = \frac{1}{\nu\beta}\nu x(2-\beta), \tag{9.105}$$

$$\frac{1}{U}\frac{dU}{dx} = \frac{\beta}{x(2-\beta)}, \tag{9.106}$$

$$\implies \ln U = \frac{\beta}{2-\beta}\ln x + c, \tag{9.107}$$

$$\therefore U(x) = cx^{\beta/(2-\beta)}, \tag{9.108}$$

where c is a constant. This outer flow velocity distribution has the same form as the potential flow over a wedge (refer back to Chapter 6). Recall that the complex potential for flow over a wedge is given by,

$$f(z) = Az^n, \tag{9.109}$$

$$\implies U - iV = \frac{df}{dz} = Anz^{n-1} = An(x+iy)^{n-1}, \tag{9.110}$$

where A is a constant and the angle between the walls of the wedge is equal to π/n as shown in Figure 9.7. Note that this f (the complex potential function) is not the same as the f in Eq. (9.90). Making use of symmetry we can produce flow over a wedge shape with a wedge angle denoted by, Δ (Figure 9.8).
Along the surface of the wedge, $y = 0$, the potential flow horizontal speed is,

$$U = Anx^{n-1}. \tag{9.111}$$

Comparing Eq. (9.111) to Eq. (9.108), we find that β and n are related,

$$\frac{\beta}{2-\beta} = n - 1 \implies \beta = \frac{2}{\pi}\left(\pi - \frac{\pi}{n}\right). \tag{9.112}$$

The full angle of the wedge, Δ, is,

$$\Delta = 2\left(\pi - \frac{\pi}{n}\right) = 2\pi\left(1 - \frac{1}{n}\right) = \pi\beta. \tag{9.113}$$

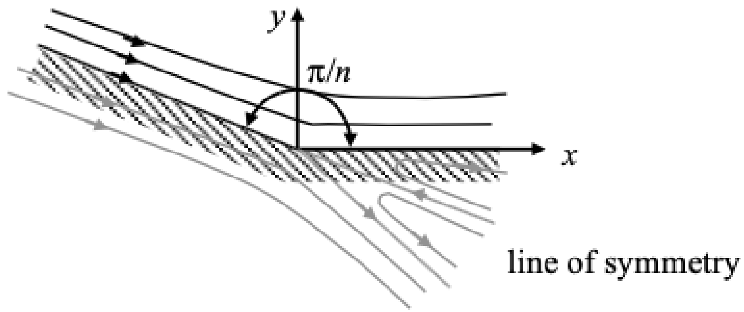


Figure 9.7. Streamlines of the Falkner-Skan boundary layer solution for flow in a corner.

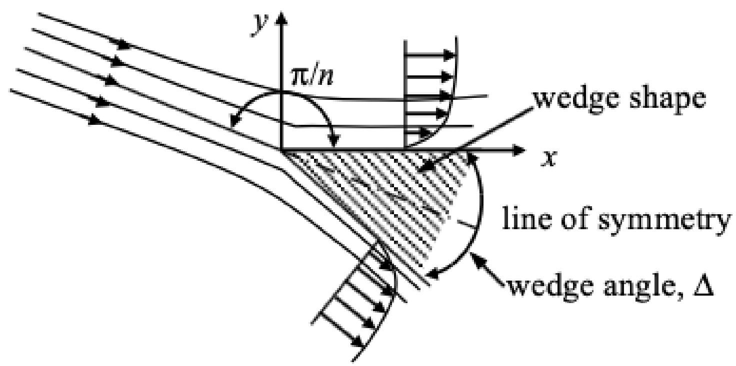


Figure 9.8. Streamlines of the Falkner-Skan boundary layer solution for flow over a wedge.

The scaling function $\xi(x)$ is found using Eqs. (9.94) and (9.111),

$$\xi^2 \frac{dU}{dx} = \nu\beta \implies \xi^2 \frac{2\beta}{2-\beta} x^{\left(\frac{\beta}{2-\beta}-1\right)} = \nu\beta, \tag{9.114}$$

$$\implies \xi(x) = \sqrt{\frac{\nu(2-\beta)}{c}} x^{\left(\frac{1-\beta}{2-\beta}\right)}, \tag{9.115}$$

$$\tau_w = \mu \frac{U f''|_{\eta=0}}{\xi} = \mu \frac{c x^{\frac{\beta}{2-\beta}}}{\sqrt{\frac{\nu(2-\beta)}{c}} x^{\frac{1-\beta}{2-\beta}}} f''|_{\eta=0} = \sqrt{\frac{\rho\mu c^3}{2-\beta}} x^{\frac{2\beta-1}{2-\beta}} f''|_{\eta=0}. \tag{9.116}$$

Now let's consider two special cases for $\alpha = 1$.

(a) *Flow over a Flat Plate (Blasius flow): $n = 1$ ($\beta = 0, \alpha = 1$).* The outer flow speed is (Eq. (9.111)),

$$U(x) = c = U_0, \tag{9.117}$$

where U_0 is a constant speed. The scaling function is found from Eq. (9.115) to be,

$$\xi(x) = \sqrt{\frac{2\nu x}{U_o}}. \tag{9.118}$$

This result is precisely the same scaling function found in our previous investigation of the Blasius solution. The governing ODE is found from Eq (9.90),

$$f''' + f f'' = 0. \tag{9.119}$$

Again, this result is identical to what was found during our investigation of the Blasius solution.

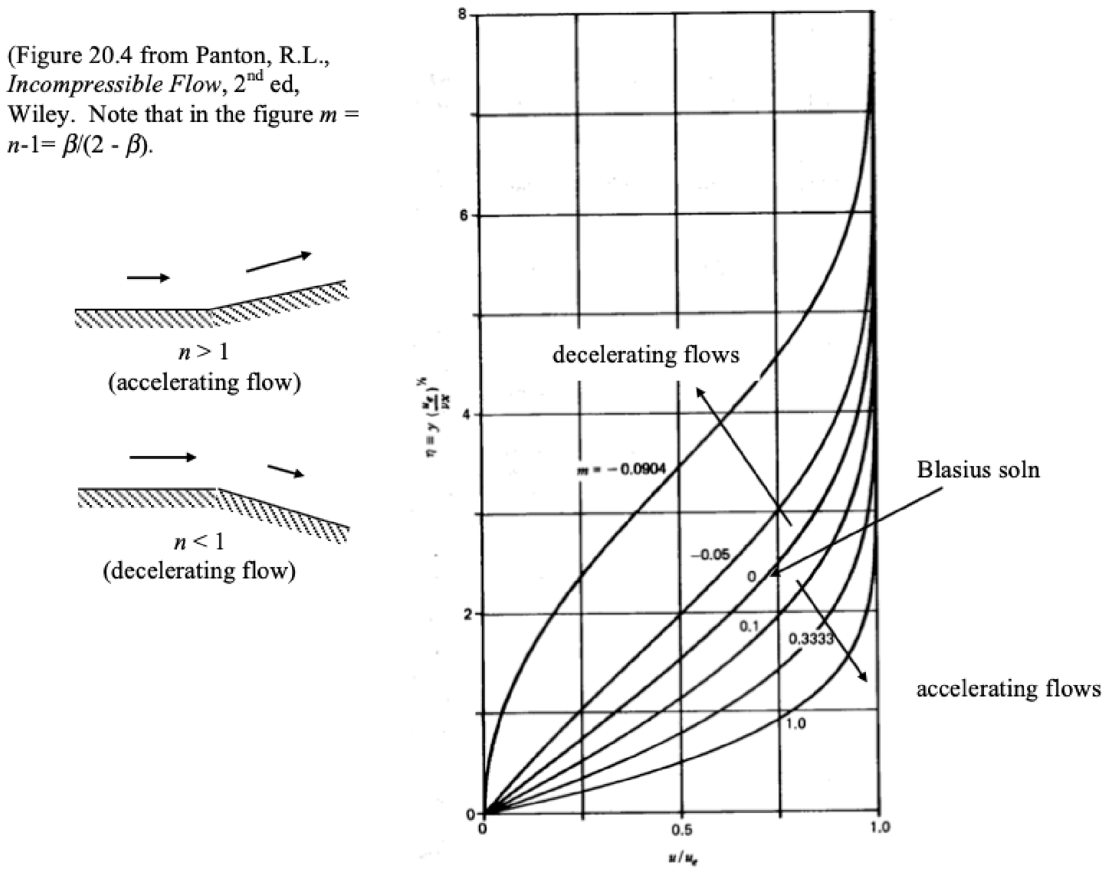


Figure 9.9. The dimensionless boundary layer profile plotted as a function of dimensionless height above the surface for different wedge shapes. Note that $m = n - 1 = \beta/(2 - \beta)$. This figure is Figure 20.4 from Panton, R.L., *Incompressible Flow*, 2nd ed., Wiley.

(b) *Stagnation Point Flow (Heimenz flow): $n = 2$ ($\beta = 1, \alpha = 1$):* The stagnation point flow solution can be recovered using $\alpha = 1$ and $\beta = 1$. Using these values, we find,

$$U(x) = cx, \tag{9.120}$$

$$\xi(x) = \sqrt{\frac{\nu}{c}}, \tag{9.121}$$

$$f''' + ff'' + \left[1 - (f')^2\right] = 0, \tag{9.122}$$

which are the same results found in our previous analysis of stagnation point flow.

Notes:

(i) Figure 9.9 shows the velocity profile for various flows over wedge shapes (produced by letting $\alpha = 1$ and varying β; refer to Figure 9.10). One item of significance observed from the figure is the fact that for accelerated flows ($n > 1 \implies dU/dx > 0$) there is no inflection point in the boundary layer profile. When the flow decelerates ($n < 1 \implies dU/dx < 0$) there is an inflection point. Of particular interest is the case when $(n - 1) = -0.0904$. For this case we observe that the inflection point occurs at the wall and, as a result, the corresponding wall shear stress is zero. It is at this point when boundary layer separation occurs. Thus, we see that a laminar boundary layer is able

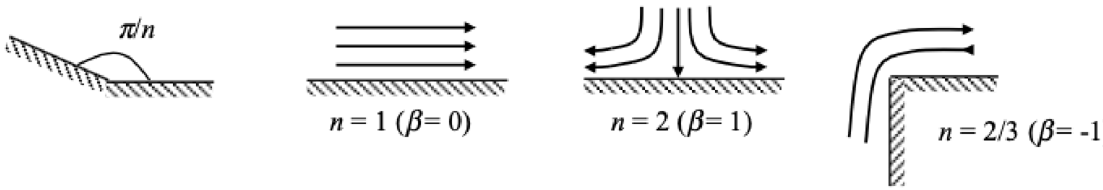
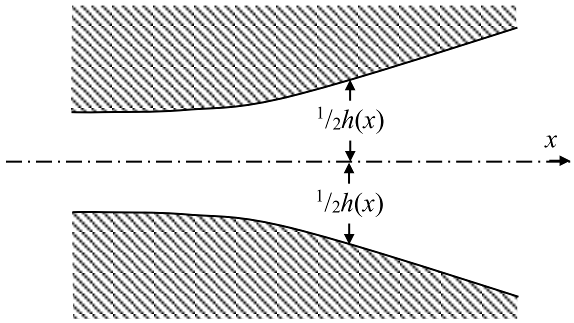


Figure 9.10. Illustrations of the various "wedge" geometries produced by $\alpha = 1$ and different β values.

to support only a very small deceleration without separation occurring. We will address the topic of boundary layer separation in Section 9.9.

The working section of a water tunnel consists of a duct with a rectangular cross-section. The width of the cross-section, b (perpendicular to the sketch), is constant but the height, $h(x)$, may vary with longitudinal distance, x, measured along the centerline of the duct:



Laminar boundary layers form on the upper and lower surfaces of the working section and would cause an acceleration of the flow outside the layers if the height h were constant (A similar effect would be caused by the front and back surfaces but we ignore this for the purposes of this problem and assume that there are no boundary layers on the front and back surfaces.) A water tunnel designer wishes to select the function $h(x)$ in order to ensure that the pressure and velocity outside the boundary layer (say, on the centerline) vary with distance, x, in a specified way. The designer decides to use functions of the form:

$$h(x) = h_0 + Hx^k$$

where h_0, H, and k are constants and the boundary layers begin at x=0. Find:

a. the value of k which produces zero longitudinal pressure gradient in the tunnel. Also find the expression for H in terms of b, the kinematic viscosity, ν, and the velocity of the flow at the centerline, U.
b. The value of k which produces a uniform acceleration, $a = dU/dx$, along the centerline of the tunnel and the relation between a and h_0.

SOLUTION:

For there to be no pressure gradient, the boundary layer displacement thickness, δ_D, must grow at the same rate as the wall moves away from the centerline. The mass flow rate in the tunnel is:

$$\dot{m} = \rho U\left(h - 2\delta_D\right)b = \text{constant} \tag{1}$$

$$\rho U\left(h - 2\delta_D\right)b = \rho U\, h\big|_{x=0}\, b \tag{2}$$

$$h_0 + Hx^k - 2\delta_D = h_0 \tag{3}$$

$$\delta_D = \tfrac{1}{2}Hx^k \tag{4}$$

For a laminar boundary layer with no pressure gradient, the Blasius solution gives:

$$\frac{\delta_D}{x} = \frac{1.72}{\mathrm{Re}_x^{1/2}} \Rightarrow \delta_D = 1.72\sqrt{\frac{\nu x}{U}} \tag{5}$$

Substitute and simplify.

$$1.72\sqrt{\frac{\nu x}{U}} = \tfrac{1}{2}Hx^k \tag{6}$$

Hence,

$$\boxed{k = 1/2} \text{ and } \boxed{H = 2 * 1.72\sqrt{\frac{\nu}{U}} = 3.44\sqrt{\frac{\nu}{U}}} \tag{7}$$

A constant acceleration implies that the outer flow velocity, U, is proportional to the position, x, *i.e.*

$$\frac{dU}{dx} = a \Rightarrow U = ax \tag{8}$$

This outer flow velocity corresponds to the case when $\alpha = 1$ and $\beta = 1$ in the Falkner-Skan boundary layer solution (which corresponds to stagnation point flow). The displacement thickness for this case is:

$$\delta_D = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \xi \int_0^\infty (1 - f')\, d\eta = \sqrt{\frac{\nu}{a}} \int_0^\infty (1 - f')\, d\eta \tag{9}$$

where for $(\alpha, \beta) = (1, 1)$, $\xi = (\nu/a)^{1/2}$ = constant. Thus, we observe that the displacement thickness for this case is a constant.

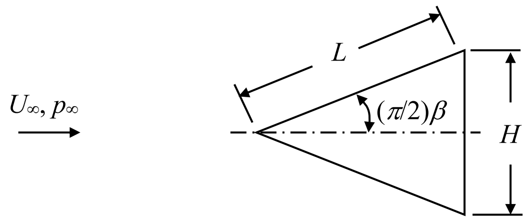
From Eqn. (2) we have:

$$\rho(ax)(h - 2\delta_D)b = \rho U_0 h_0 b \tag{10}$$

$$(ax)\left(h_0 + Hx^k - 2\delta_D\right) = U_0 h_0 \tag{11}$$

The only way the previous equation can hold true is if $\boxed{k = -1,\ h_0 = 2\delta_D, \text{ and } H = U_0 h_0 / a}$.

A uniform stream of incompressible fluid flows over a planar wedge of half-angle, $(\pi/2)\beta$, side length, L, and base length, H, as shown in the figure below. The upstream flow velocity and pressure are U_∞ and p_∞.



a. Use a Faulkner-Skan boundary layer solution to determine the skin friction drag acting on the wedge assuming laminar flow.
b. Check your solution to part (a) by calculating the drag for a flat plate ($\beta = 0$) and comparing with the Blasius solution.
c. Determine the form drag on the wedge assuming that the pressure at the back of the wedge is the same as the free stream pressure.

SOLUTION:

Recall that the Faulkner-Skan boundary layer solution for flow over a wedge with total angle, $(\pi/2)\beta$, is:

$$f''' + ff'' + \beta\left[1-(f')^2\right] = 0 \tag{1}$$

$$f'(\eta = 0) = 0 \tag{2}$$

$$f(\eta = 0) = 0$$

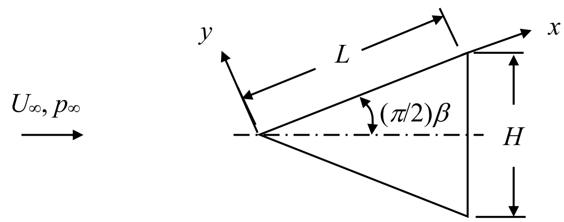
$$f'(\eta \to \infty) = 1 \tag{3}$$

$$U(x) = cx^{\frac{\beta}{2-\beta}} \tag{4}$$

$$\xi(x) = \sqrt{\frac{\nu(2-\beta)}{c}}x^{\frac{1-\beta}{2-\beta}} \tag{5}$$

$$\eta = \frac{y}{\xi(x)} \tag{6}$$

$$\tau_w = \sqrt{\frac{\rho\mu c^3}{2-\beta}}x^{\frac{2\beta-1}{2-\beta}} f''\big|_{\eta=0} \tag{7}$$



The skin friction drag over both surfaces of length, L, (assuming unit depth into the page), and also taking into account the angle of the surfaces, is:

$$D_{\substack{\text{skin}\\\text{friction}}} = 2\cos\left(\tfrac{\pi}{2}\beta\right)\int_{x=0}^{x=L} \tau_w dx \tag{8}$$

$$\begin{aligned} D_{\substack{\text{skin}\\\text{friction}}} &= 2\cos\left(\tfrac{\pi}{2}\beta\right)\int_{x=0}^{x=L} \sqrt{\frac{\rho\mu c^3}{2-\beta}}x^{\frac{2\beta-1}{2-\beta}} f''\big|_{\eta=0}\, dx = \sqrt{\frac{\rho\mu c^3}{2-\beta}} f''\big|_{\eta=0}\int_{x=0}^{x=L} x^{\frac{2\beta-1}{2-\beta}}dx \\ &= \sqrt{\frac{\rho\mu c^3}{2-\beta}} f''\big|_{\eta=0}\left(\frac{1}{\frac{2\beta-1}{2-\beta}+1}\right)x^{\frac{2\beta-1}{2-\beta}+1}\Big|_0^L \end{aligned} \tag{9}$$

$$\boxed{\therefore D_{\substack{\text{skin}\\\text{friction}}} = 2\cos\left(\tfrac{\pi}{2}\beta\right)\sqrt{\frac{\rho\mu c^3}{2-\beta}} f''\big|_{\eta=0}\left(\frac{2-\beta}{1+\beta}\right)L^{\frac{1+\beta}{2-\beta}}} \tag{10}$$

Note that when $\beta = 0$, we get Blasius flow over a flat plate.

$$D_{\substack{\text{skin}\\ \text{friction}}} = 2\sqrt{\frac{\rho\mu c^3}{2}}\, f''\big|_{\eta=0} (2) L^{1/2} \tag{11}$$

$$U(x) = c \quad \text{(from Eqn. (4))} \tag{12}$$

$$\therefore D_{\substack{\text{skin friction}\\ \text{Blasius}}} = \sqrt{8\rho\mu U^3 L}\, f''\big|_{\eta=0} = \sqrt{32}\frac{\frac{1}{2}\rho U^2 L}{\sqrt{\frac{\rho U L}{\mu}}} f''\big|_{\eta=0} \tag{13}$$

To determine $f''(\eta = 0)$, we need to solve the ODE:

$$f''' + ff'' = 0 \tag{14}$$

numerically. Performing this calculation (or using Table 4.1 from White, *Viscous Fluid Flow*, for example) gives:

$$f''(\eta = 0) = 0.46960 \tag{15}$$

and thus:

$$\boxed{\therefore D_{\substack{\text{skin friction}\\ \text{Blasius}}} = \tfrac{1}{2}\rho U^2 L\left(\frac{2.656}{\mathrm{Re}_L^{1/2}}\right)} \tag{16}$$

The Blasius drag coefficient (considering two sides of the plate) is:

$$C_{\substack{D,\text{Blasius,}\\ \text{two-sides}}} = \frac{D_{\substack{\text{skin fraction,}\\ \text{two-sides}}}}{\frac{1}{2}\rho U^2 L} = 2\left(\frac{1.328}{\mathrm{Re}_L^{1/2}}\right) \Rightarrow D_{\substack{\text{skin fraction,}\\ \text{two-sides}}} = \tfrac{1}{2}\rho U^2 L\left(\frac{2.656}{\mathrm{Re}_L^{1/2}}\right) \tag{17}$$

This is the same as Eqn. (16)!

The form drag may be found by integrating the pressure force acting on the surface and taking the component in the direction of the incoming flow. Use gage pressures to simplify the calculation. The pressure may be found using Bernoulli's equation in the outer, potential flow.

$$D_{\text{form}} = 2\sin\left(\tfrac{\pi}{2}\beta\right)\int_{x=0}^{x=L} p_{\text{gage}}\,dx \quad \text{where} \quad p_{\text{gage}} = \tfrac{1}{2}\rho\left(U_\infty^2 - U^2\right) \tag{18}$$

(Note that since a gage pressure is being used, the pressure force on the back of the wedge needn't be considered.)

$$\begin{aligned}
D_{\text{form}} &= 2\sin\left(\tfrac{\pi}{2}\beta\right)\int_{x=0}^{x=L} \tfrac{1}{2}\rho U_\infty^2\left(1 - \frac{U^2}{U_\infty^2}\right)dx \\
&= \sin\left(\tfrac{\pi}{2}\beta\right)\rho U_\infty^2 \int_{x=0}^{x=L}\left(1 - \frac{U^2}{U_\infty^2}\right)dx = \sin\left(\tfrac{\pi}{2}\beta\right)\rho U_\infty^2 \int_{x=0}^{x=L}\left(1 - \frac{c^2 x^{\frac{2\beta}{2-\beta}}}{U_\infty^2}\right)dx \\
&= \sin\left(\tfrac{\pi}{2}\beta\right)\rho U_\infty^2\left(L - \frac{c^2}{U_\infty^2}\frac{1}{\frac{2\beta}{2-\beta}+1}L^{\frac{2\beta}{2-\beta}+1}\right)
\end{aligned} \tag{19}$$

$$\boxed{\therefore D_{\text{form}} = \sin\left(\tfrac{\pi}{2}\beta\right)\rho U_\infty^2 L\left[1 - \left(\frac{c^2}{U_\infty^2}\right)\left(\frac{2-\beta}{2+\beta}\right)L^{\frac{2\beta}{2-\beta}}\right]} \tag{20}$$

9.6. The Kármán Momentum Integral Equation (KMIE)

So far we've only examined boundary layer flows that lend themselves to similarity solutions. This, of course, is very restrictive. There are many non-similar boundary layer flows that we would also like to investigate. Since the majority of fluid mechanics problems are complex, we often have to resort to empirical or semi-empirical methods for investigating the flows in greater detail. Here we'll discuss one such semi-empirical method used for investigating boundary layers called the Kármán Momentum Integral Equation (KMIE). The idea is straightforward and relies on the Linear Momentum Equation.

Consider a differential control volume as shown in Figure 9.11. The top of the control volume is defined by the line separating the boundary layer region from the outer flow region (this is *not* a streamline). Apply the

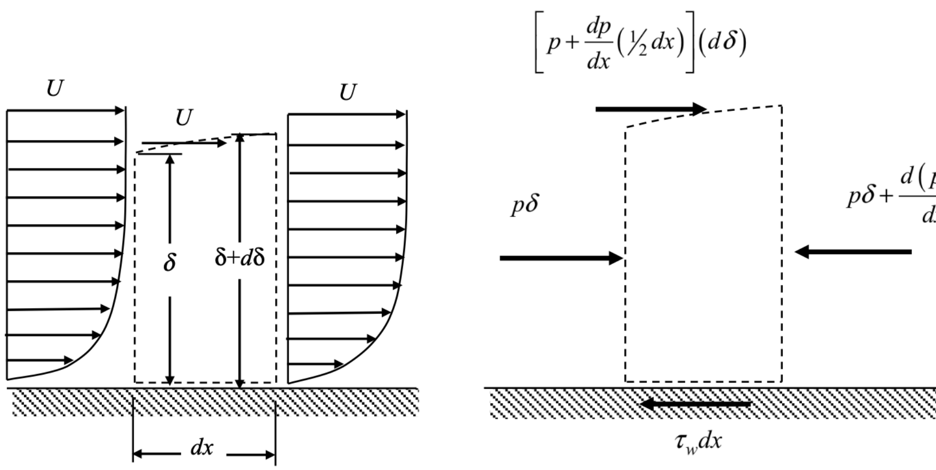


Figure 9.11. Schematics showing the control volume and free body diagram used in deriving the KMIE.

Linear Momentum Equation in the x-direction to the control volume, assuming unit depth,

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}, \tag{9.123}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady state)}, \tag{9.124}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \underbrace{-\int_{y=0}^{y=\delta} \rho u^2 dy}_{\text{left}} + \underbrace{\left[\int_0^\delta \rho u^2 dy + \frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right) dx\right]}_{\text{right}} \underbrace{-U\frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right) dx}_{\text{top}}, \tag{9.125}$$

$$F_{B,x} = 0, \quad \text{(body forces are negligibly small in boundary layers compared to other terms)}, \tag{9.126}$$

$$F_{S,x} = \underbrace{p\delta}_{\text{left}} \underbrace{-\left[p\delta + \frac{d}{dx}(p\delta)(dx)\right]}_{\text{right}} + \underbrace{\left[p + \frac{dp}{dx}(\frac{1}{2}dx)\right](d\delta)}_{\text{top}} \underbrace{-\tau_w dx}_{\text{bottom}}. \tag{9.127}$$

The mass flow rate through the top is found via Conservation of Mass on the same control volume,

$$\underbrace{-\dot{m}_{\text{top}}}_{\text{top}}\underbrace{-\int_0^\delta \rho u dy}_{\text{left}}\underbrace{+\left[\int_0^\delta \rho u dy + \frac{d}{dx}\left(\int_0^\delta \rho u dy\right)dx\right]}_{\text{right}} = 0, \tag{9.128}$$

$$\therefore \dot{m}_{\text{top}} = \frac{d}{dx}\left(\int_0^\delta \rho u dy\right)dx. \tag{9.129}$$

Substituting and simplifying, neglecting higher order terms,

$$-\frac{dp}{dx}\delta dx - \tau_w dx = \frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right)dx - U\frac{d}{dx}\left(\int_0^\delta \rho u dy\right)dx, \tag{9.130}$$

$$-\frac{dp}{dx}\delta - \tau_w = \frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right) - U\frac{d}{dx}\left(\int_0^\delta \rho u dy\right). \tag{9.131}$$

Recall that the pressure at a given x location remains constant with y position so we can find dp/dx in terms of the outer (potential) flow velocity using Bernoulli's equation outside of the boundary layer,

$$p + \frac{1}{2}\rho U^2 = \text{constant} \implies \frac{dp}{dx} + \rho U\frac{dU}{dx} = 0, \tag{9.132}$$

$$\frac{dp}{dx} = -\rho U\frac{dU}{dx}. \tag{9.133}$$

In addition, we'll re-write the boundary layer thickness in terms of an integral so that,

$$-\frac{dp}{dx}\delta = \left(\rho U\frac{dU}{dx}\right)\int_0^\delta dy = \frac{dU}{dx}\int_0^\delta \rho U dy. \tag{9.134}$$

Additional re-arranging gives,

$$U\frac{d}{dx}\left(\int_0^\delta \rho u dy\right) = \frac{d}{dx}\left(\int_0^\delta \rho u U dy\right) - \frac{dU}{dx}\int_0^\delta \rho u dy. \tag{9.135}$$

Substituting Eqs. (9.134) and (9.135) into Eq. (9.131) gives,

$$\frac{dU}{dx}\left(\int_0^\delta \rho U dy\right) - \tau_w = \frac{d}{dx}\left(\int_0^\delta \rho u^2 dy\right) - \frac{d}{dx}\int_0^\delta \rho u U dy + \frac{dU}{dx}\left(\int_0^\delta \rho u dy\right). \tag{9.136}$$

Additional re-arranging and simplifying gives,

$$\tau_w = \frac{d}{dx}\left[\int_0^\delta \rho u(U-u)dy\right] + \frac{dU}{dx}\int_0^\delta \rho(U-u)dy, \tag{9.137}$$

$$= \frac{d}{dx}\left[\rho U^2 \underbrace{\int_0^\delta \rho\frac{u}{U}\left(1-\frac{u}{U}\right)dy}_{=\delta_D}\right] + \frac{dU}{dx}\rho U\underbrace{\int_0^\delta \left(1-\frac{u}{U}\right)dy}_{=\delta_D}. \tag{9.138}$$

$$\tag{9.139}$$

Thus, if the fluid has constant density,

$$\boxed{\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\delta_M\right) + \delta_D U\frac{dU}{dx}}. \tag{9.140}$$

This equation is known as the Kármán Momentum Integral Equation (KMIE).

Notes:

(1) If the pressure remains constant, then $dU/dx = 0$ and,

$$\tau_w = \rho U^2 \frac{d\delta_m}{dx}. \tag{9.141}$$

(2) The typical methodology for using the KMIE is as follows.
   (a) Obtain an approximate expression for $U = U(x)$ from inviscid flow theory, e.g., potential flow theory. Recall that Bernoulli's equation can be used to relate the pressure and U.
   (b) Assume a velocity profile in the boundary layer subject to the appropriate boundary conditions, i.e., assume a form for,

   $$\frac{u}{U} = f\left(\frac{y}{\delta}\right), \tag{9.142}$$

   subject to the boundary conditions,

   $$\frac{u}{U}\left(\frac{y}{\delta} = 0\right) = 0 \quad \text{and} \quad \frac{u}{U}\left(\frac{y}{\delta} = 1\right) = 1. \tag{9.143}$$

   The form of the approximate velocity profile is typically found based on curve fits to experimental measurements of the boundary layer velocity profile. Higher order profiles will have additional boundary conditions. For example, a cubic curve fit will also have a boundary condition that matches the slope of the velocity profile at the free stream boundary.
   (c) The shear stress at the wall for a *laminar* flow can also be determined from the Newtonian stress-strain rate constitutive relations to be,

   $$\tau_w = \mu \left(\frac{U}{\delta}\right) \left.\frac{d(u/U)}{d(y/\delta)}\right|_{\frac{y}{\delta}=0}. \tag{9.144}$$

   For a turbulent flow, experimental data for the wall shear stress are used instead since turbulent flows use time-averaged velocity profiles. This issue is discussed in greater detail later in these notes. The laminar wall shear stress must be the same shear stress as that found using the KMIE (Eq. (9.140)). Thus, we can equate the two shear stress expressions. The resulting differential equation can then be solved for the boundary layer thickness, δ, as a function of x.

(3) This approximate technique can be used for either laminar or turbulent flows. In fact, this method is especially useful for analyzing turbulent boundary layer profiles (discussed later in these notes).

Consider laminar flow over a flat plate (U = constant). Approximate the boundary layer velocity profile using a parabolic shape,

$$\frac{u}{U} = \begin{cases} 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 & 0 \le \frac{y}{\delta} < 1 \\ 1 & \frac{y}{\delta} \ge 1 \end{cases}$$

Using the KMIE, determine the dimensionless 99% boundary layer thickness, δ/x, as a function of Reynolds number based on the distance from the leading edge, $\mathrm{Re}_x = Ux/\nu$. Compare your result to the Blasius solution.

SOLUTION:
Evaluate the momentum thickness, δ_M,

$$\delta_M = \delta \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2) d\eta \quad \text{(where } \eta = y/\delta\text{)}, \tag{1}$$

$$\delta_M = \frac{2}{15}\delta. \tag{2}$$

Now substitute this momentum thickness into the KMIE. Note that $dU/dx = 0$ since U = constant,

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2 \delta_M) + \delta_D U \underbrace{\frac{dU}{dx}}_{=0} = U^2 \frac{d\delta_M}{dx}, \tag{3}$$

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} = \frac{2}{15}\rho U^2 \frac{d\delta}{dx}. \tag{4}$$

This shear stress should be the same as the shear stress found via,

$$\tau_w = \mu \frac{du}{dy}\bigg|_{y=0} = \frac{\mu U}{\delta}\frac{d(u/U)}{d\eta}\bigg|_{\eta=0} = \frac{\mu U}{\delta}(2 - 2\eta)_{\eta=0} = 2\frac{\mu U}{\delta}. \tag{5}$$

Equating the two shear stresses and solving the resulting differential equation,

$$\frac{2}{15}\rho U^2 \frac{d\delta}{dx} = 2\frac{\mu U}{\delta}, \tag{6}$$

$$\int_0^\delta \delta d\delta = 15 \frac{\mu}{\rho U}\int_0^x dx \quad \text{(assuming } \delta = 0 \text{ when } x = 0\text{)}, \tag{7}$$

$$\frac{1}{2}\delta^2 = \frac{15\nu x}{U}, \tag{8}$$

$$\frac{\delta}{x} = \sqrt{\frac{30\nu x}{Ux^2}} = \sqrt{\frac{30}{\mathrm{Re}_x}} \quad \text{(using } \mathrm{Re}_x = Ux/\nu\text{)}, \tag{9}$$

$$\boxed{\frac{\delta}{x} \approx \frac{5.5}{\mathrm{Re}_x^{1/2}}}. \tag{10}$$

This approximate expression is only 10% different from the exact Blasius expression,

$$\frac{\delta}{x} \approx \frac{5.0}{\mathrm{Re}_x^{1/2}} \quad \text{(Blasius)}.$$

Video solution: https://www.youtube.com/watch?v=g2r4KOdA8MQ

Using the momentum integral theorem, determine the friction coefficient, c_f, dimensionless boundary layer momentum thickness, δ_M/x, and the dimensionless boundary layer displacement thickness, δ_D/x, for laminar flat plate flow with no pressure gradient assuming a sinusoidal velocity profile:

$$\frac{u}{U} \approx \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right),$$

where δ is the 99% boundary layer thickness, y is the distance from the plate surface, and U is the outer flow speed. Compare your answers with the Blasius' exact laminar boundary layer solution.

SOLUTION:

Use the Kármán Momentum Integral Equation (KMIE),

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(\delta_M U^2\right) + \delta_D U \frac{dU}{dx} \tag{1}$$

Assuming a flat plate flow with no pressure gradient,

$$U = \text{constant} \Rightarrow \frac{dU}{dx} = 0 \quad \text{(from Bernoulli's equation applied outside the boundary layer)} \tag{2}$$

Simplifying Eqn. (1) gives,

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} \tag{3}$$

The momentum thickness is given by,

$$\begin{aligned}
\delta_M &= \int_{y=0}^{y=\delta} \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \delta \int_{y/\delta=0}^{y/\delta=1} \frac{u}{U}\left(1-\frac{u}{U}\right)d\left(y/\delta\right) \\
&= \delta\int_0^1 \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\left[1-\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]d\left(y/\delta\right) \\
&= \delta\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\Bigg|_{y/\delta=0}^{y/\delta=1} - \frac{1}{2}\frac{y}{\delta}\Bigg|_{y/\delta=0}^{y/\delta=1} + \frac{1}{2\pi}\sin\left(\pi\frac{y}{\delta}\right)\Bigg|_{y/\delta=0}^{y/\delta=1}\right]
\end{aligned}$$

$$\therefore \delta_M = \delta\left(\frac{2}{\pi}-\frac{1}{2}\right) \approx 0.1367\delta \tag{4}$$

Substitute Eq. (4) into Eq. (3),

$$\tau_w = 0.1367\rho U^2 \frac{d\delta}{dx} \tag{5}$$

For a laminar flow, the shear stress can also be expressed as,

$$\tau_w = \mu \frac{du}{dy}\bigg|_{y=0}$$

$$\tau_w = \frac{\pi}{2}\frac{\mu U}{\delta} \tag{6}$$

Equate Eqs. (5) and (6) and solve for δ,

$$0.1367\rho U^2 \frac{d\delta}{dx} = \frac{\pi}{2}\frac{\mu U}{\delta}$$

$$\int_{\delta=0}^{\delta=\delta} \delta\, d\delta = 11.4908 \frac{\mu}{\rho U} \int_{x=0}^{x=x} dx$$

$$\tfrac{1}{2}\delta^2 = 11.4908 \frac{\mu}{\rho U} x$$

$$\boxed{\therefore \frac{\delta}{x} = 4.7939\sqrt{\frac{\mu}{\rho U x}} = \frac{4.7939}{\mathrm{Re}_x^{1/2}}} \tag{7}$$

Equation (7) is only 4% different from the exact Blasius solution of $\delta/x = 5.0/\mathrm{Re}_x^{1/2}$.

From Eq. (4) the momentum thickness is,

$$\boxed{\frac{\delta_M}{x} = \frac{0.6553}{\mathrm{Re}_x^{1/2}}} \tag{8}$$

This result is 1% different from the Blasius solution of $\delta_M/x = 0.664/\mathrm{Re}_x^{1/2}$.

The displacement thickness is given by,

$$\delta_D = \int_{y=0}^{y=\delta}\left(1-\frac{u}{U}\right)dy = \delta\int_{y/\delta=0}^{y/\delta=1}\left(1-\frac{u}{U}\right)d\left(y/\delta\right)$$

$$= \delta\int_0^1\left[1-\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]d\left(y/\delta\right)$$

$$= \delta\left[\left.\frac{y}{\delta}\right|_{y/\delta=0}^{y/\delta=1} + \left.\frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right|_{y/\delta=0}^{y/\delta=1}\right]$$

$$\therefore \delta_D = \delta\left(1-\frac{2}{\pi}\right) \approx 0.3634\delta \tag{9}$$

so that, when combined with Eq. (7),

$$\boxed{\frac{\delta_D}{x} = \frac{1.7420}{\mathrm{Re}_x^{1/2}}} \tag{10}$$

This result is 1% different from the Blasius solution of $\delta_D/x = 1.72/\mathrm{Re}_x^{1/2}$.

The friction coefficient can be found using Eq. (6),

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\frac{\pi}{2}\frac{\mu U}{\delta}}{\frac{1}{2}\rho U^2} = \pi\frac{\mu}{\rho U\delta} = \pi\frac{\mu}{\rho U x}\frac{x}{\delta}$$

$$\boxed{C_f = \frac{0.6553}{\mathrm{Re}_x^{1/2}}} \tag{11}$$

This result is 1% different form the Blasius solution of $C_f = 0.664/\mathrm{Re}_x^{1/2}$.

A measured dimensionless laminar boundary layer profile for flow past a flat plate is given in the table below. Use the momentum integral equation to determine the 99% boundary layer thickness. Compare your result with the exact (Blasius) result.

***y*/δ**	***u*/*U***
0.00	0.00
0.08	0.133
0.16	0.265
0.24	0.394
0.32	0.517
0.40	0.630
0.48	0.729
0.56	0.811
0.64	0.876
0.72	0.923
0.80	0.956
0.88	0.976
0.96	0.988
1.00	1.000

SOLUTION:

Apply the Kármán Momentum Integral Equation:

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(\delta_M U^2\right) + \delta_D U \frac{dU}{dx} \tag{1}$$

Assuming a flat plate flow with no pressure gradient:

$$U = \text{constant} \Rightarrow \frac{dU}{dx} = 0 \tag{2}$$

Simplifying Eqn. (1) gives:

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} \tag{3}$$

The momentum thickness is given by:

$$\delta_M = \int_{y=0}^{y=\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \delta \int_{y/\delta=0}^{y/\delta=1} \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(y/\delta\right)$$

Integrating the data numerically using the trapezoidal rule gives:

$$\delta_M \approx 0.131\delta \tag{4}$$

Substitute into Eqn. (3).

$$\tau_w = 0.131\rho U^2 \frac{d\delta}{dx} \tag{5}$$

For a laminar flow, the shear stress can also be expressed as:

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0} = \mu \frac{U}{\delta} \left.\frac{d\left(u/U\right)}{d\left(y/\delta\right)}\right|_{y/\delta=0} \tag{6}$$

Differentiating the data numerically using a 1st order finite difference scheme:

$$\tau_w \approx 1.66\mu \frac{U}{\delta} \tag{7}$$

Equating Eqns. (5) and (7) gives:

$$0.131\rho U^2 \frac{d\delta}{dx} = 1.66\mu \frac{U}{\delta}$$

$$\int_{\delta=0}^{\delta=\delta} \delta d\delta = 12.67 \frac{\mu}{\rho U} \int_{x=0}^{x=x} dx$$

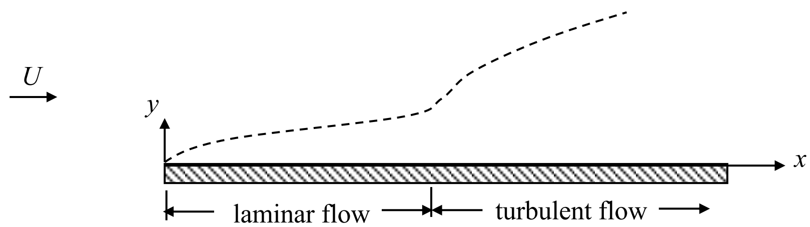
$$\tfrac{1}{2}\delta^2 = 12.67 \frac{\mu x}{\rho U}$$

$$\boxed{\therefore \frac{\delta}{x} = 5.034\sqrt{\frac{\mu}{\rho U x}} = \frac{5.034}{\mathrm{Re}_x^{1/2}}} \tag{8}$$

Equation (8) is within 1% of the exact Blasius solution of $\delta/x = 5.0/\mathrm{Re}_x^{1/2}$.

Another approach to this problem is to fit a polynomial curve to the given data rather than numerically differentiating and integrating the data.

The flat plate formulas for turbulent flow over a flat plate assume that turbulent flow begins at the leading edge ($x = 0$). In reality there is an initial region of laminar flow as shown in the figure.



1. Derive an expression for the 99% boundary layer thickness in the turbulent region by accounting for the laminar part of the flow.
2. Plot the dimensionless boundary layer thickness, δ/x, as a function of Reynolds number ($10^4 \leq \mathrm{Re}_x \leq 10^8$, use a log scale for the Re_x axis) for your derived relation and for the turbulent relation that does not consider the laminar part.

Assume a 1/7th power law velocity profile for the turbulent boundary layer and an experimental friction coefficient correlation of $C_f \approx 0.020\,\mathrm{Re}_\delta^{-1/6}$.

SOLUTION:

First determine the boundary layer thickness in the laminar flow region using the Blasius solution:

$$\frac{\delta}{x} = \frac{5.0}{\mathrm{Re}_x^{1/2}} \quad (\mathrm{Re}_x < 500{,}000) \tag{1}$$

Assume that the transition to turbulence occurs at a Reynolds number of 500,000 so that condition at the transition point is:

$$\delta_{\text{trans}} = 3.536 * 10^3 \left(\frac{\nu}{U}\right) \tag{2}$$

where

$$x_{\text{trans}} = \frac{500{,}000\nu}{U} \tag{3}$$

Now use the Karman Momentum Integral Equation to determine the boundary layer characteristics for the turbulent region. Assume that the velocity profile follows the following form:

$$\frac{u}{U_\infty} = \begin{cases} \left(\dfrac{y}{\delta}\right)^{1/7} & \dfrac{y}{\delta} < 1 \\ 1 & \dfrac{y}{\delta} \geq 1 \end{cases} \tag{4}$$

Using this velocity profile, the momentum thickness is:

$$\delta_M = \int_{y=0}^{y=\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_{y=0}^{y=\delta} \left(\frac{y}{\delta}\right)^{1/7} \left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right] dy = \frac{7}{72}\delta \tag{5}$$

To determine the shear stress, recall that from the Karman Momentum Integral Equation, with a constant outer velocity:

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} = \frac{7}{72}\rho U^2 \frac{d\delta}{dx} \tag{1.6}$$

so that the friction coefficient is:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{7}{36}\frac{d\delta}{dx} \tag{1.7}$$

Using the given experimental wall friction correlation:

$$C_f \approx 0.020\,\mathrm{Re}_\delta^{-1/6} \tag{1.8}$$

where $\mathrm{Re}_\delta = (U\delta/\nu)$, equate the two friction coefficients to give:

$$\frac{7}{36}\frac{d\delta}{dx} = 0.020\left(\frac{U\delta}{\nu}\right)^{-1/6} \tag{9}$$

$$\int_{\delta=\delta_{\text{trans}}}^{\delta=\delta} \delta^{1/6} d\delta = 0.103\left(\frac{U}{\nu}\right)^{-1/6} \int_{x=x_{\text{trans}}}^{x=x} dx \tag{10}$$

$$\delta^{7/6} - \delta_{\text{trans}}^{7/6} = 0.120\left(\frac{U}{\nu}\right)^{-1/6}\left(x - x_{\text{trans}}\right) \tag{11}$$

where Eqns. (2) and (3) are used for δ_{trans} and x_{trans}, respectively. Substituting and simplifying results in:

$$\delta^{7/6} - 1.380*10^4\left(\frac{\nu}{U}\right)^{7/6} = 0.120\left(\frac{U}{\nu}\right)^{-1/6}\left(x - 500{,}000\frac{\nu}{U}\right) \tag{12}$$

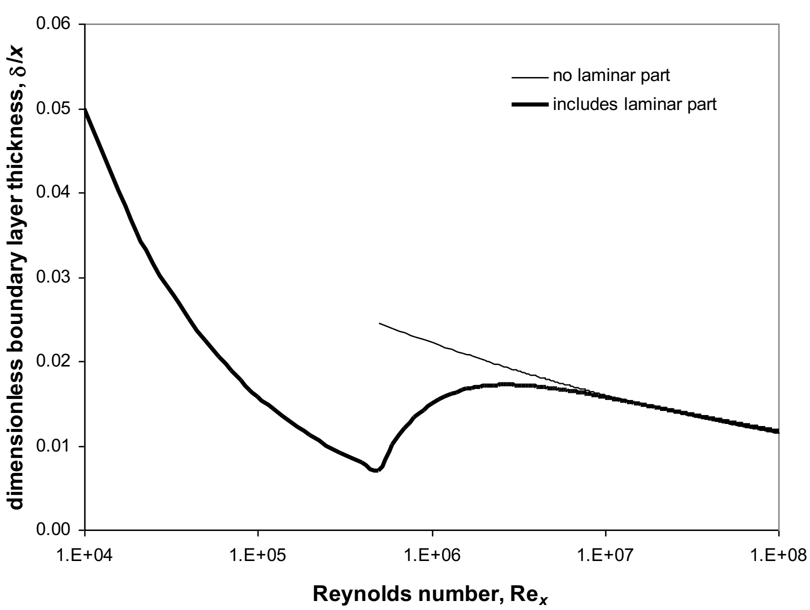
$$\delta^{7/6} = 0.120\left(\frac{\nu}{U}\right)^{1/6} x - 6.000*10^4\left(\frac{\nu}{U}\right)^{7/6} + 1.380*10^4\left(\frac{\nu}{U}\right)^{7/6} \tag{13}$$

$$\left(\frac{\delta}{x}\right)^{7/6} = 0.120\left(\frac{\nu}{Ux}\right)^{1/6} - 4.620*10^4\left(\frac{\nu}{Ux}\right)^{7/6} \tag{14}$$

$$\boxed{\frac{\delta}{x} = \left(\frac{0.120}{\mathrm{Re}_x^{1/6}} - \frac{4.620*10^4}{\mathrm{Re}_x^{7/6}}\right)^{6/7} \qquad \mathrm{Re}_x > 500{,}000} \tag{15}$$

Compare this result to one that assumes that the turbulent boundary layer starts from the leading edge:

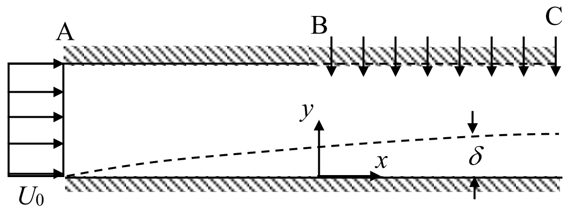
$$\frac{\delta}{x} \approx \frac{0.16}{\mathrm{Re}_x^{1/7}} \quad (\mathrm{Re}_x > 500{,}000) \tag{16}$$



Air flows between two parallel flat plates as shown in the figure below. The upper plate is porous from point B to point C and additional air is injected through this surface. As a result, the free stream speed, $U(x)$, varies as:

$$U(x) = U_0 + \alpha x$$

where U_0 is the air speed entering the channel (at point A), α is a constant, and x is the distance downstream of the point B. A boundary layer develops along the lower surface. Assuming a linear velocity distribution in the boundary layer, estimate the rate of boundary layer growth, $d\delta/dx$, in terms of δ, x, U_0, α, and the air properties.



SOLUTION:

Assuming a linear profile in the boundary layer means:

$$\frac{u}{U} = \frac{y}{\delta} \tag{1}$$

Note that with this velocity profile:

$$\frac{u}{U}\left(\frac{y}{\delta} = 0\right) = 0 \quad \text{and} \quad \frac{u}{U}\left(\frac{y}{\delta} = 1\right) = 1 \tag{2}$$

To determine the rate at which the boundary layer thickness grows with x, begin with the Karman momentum integral equation:

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2 \delta_M\right) + \delta_D U \frac{dU}{dx} \tag{3}$$

where

$$\delta_D = \int_{y=0}^{y=\delta} \left(1 - \frac{u}{U}\right) dy \tag{4}$$

$$\delta_M = \int_{y=0}^{y=\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \tag{5}$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{6}$$

$$U(x) = U_0 + \alpha x \Rightarrow \frac{dU}{dx} = \alpha \tag{7}$$

Substitute Eqn. (1) into Eqns. (4) - (6).

$$\delta_D = \int_{y=0}^{y=\delta} \left(1 - \frac{y}{\delta}\right) dy = \delta - \tfrac{1}{2}\delta = \tfrac{1}{2}\delta \tag{8}$$

$$\delta_M = \int_{y=0}^{y=\delta} \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \tfrac{1}{2}\delta - \tfrac{1}{3}\delta = \tfrac{1}{6}\delta \tag{9}$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\mu U}{\delta} = \frac{\mu\left(U_0 + \alpha x\right)}{\delta} \tag{10}$$

Substitute Eqns. (7), (8) - (10) into Eqn. (3) and simplify.

$$\frac{\nu\left(U_0 + \alpha x\right)}{\delta} = \frac{1}{6}\frac{d}{dx}\left[\left(U_0 + \alpha x\right)^2 \delta\right] + \frac{1}{2}\delta\left(U_0 + \alpha x\right)\alpha \tag{11}$$

$$\frac{\nu\left(U_0 + \alpha x\right)}{\delta} = \frac{1}{6}\left(U_0 + \alpha x\right)^2 \frac{d\delta}{dx} + \frac{1}{3}\delta\left(U_0 + \alpha x\right)\alpha + \frac{1}{2}\delta\left(U_0 + \alpha x\right)\alpha \tag{12}$$

$$\frac{\nu\left(U_0 + \alpha x\right)}{\delta} = \frac{1}{6}\left(U_0 + \alpha x\right)^2 \frac{d\delta}{dx} + \frac{5}{6}\delta\left(U_0 + \alpha x\right)\alpha \tag{13}$$

$$\boxed{\frac{d\delta}{dx} = \frac{6\nu}{\left(U_0 + \alpha x\right)}\frac{1}{\delta} - \frac{5\alpha}{\left(U_0 + \alpha x\right)}\delta} \tag{14}$$

9.7. Approximate Methods: Thwaites' Correlation

Thwaites' method (1949) is generally considered the best available one parameter method for describing laminar boundary layers. The correlation uses the Kármán Momentum Integral Equation along with dimensionless experimental laminar boundary layer data. Recall from the previous notes that the momentum thickness, δ_M, the displacement thickness, δ_D, and the shear stress at the wall, τ_w, can be related via the Kármán Momentum Integral Equation (KMIE),

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\delta_m) + \delta_D U \frac{dU}{dx}. \tag{9.145}$$

Note again that U is the outer flow speed, i.e., the flow speed just outside the boundary layer. We can re-arrange this equation into the following form,

$$\frac{\tau_w}{\rho} = 2\delta_M U \frac{dU}{dx} + \delta_D U \frac{dU}{dx}, \tag{9.146}$$

$$= (2\delta_M + \delta_D)U\frac{dU}{dx} + U^2\frac{d\delta_M}{dx}. \tag{9.147}$$

Let's write the KMIE using two dimensionless shape factors, H and T, defined in the following manner (following the approach of Holstein and Bohlen, 1940),

$$H := \frac{\delta_D}{\delta_M} \quad \text{and} \quad T := \frac{\tau_w}{\mu U/\delta_M} = \frac{\tau_W \delta_M}{\mu U}, \tag{9.148}$$

where H is the shape correlation and T is the shear correlation. These shape factors only depend on the shape of the velocity profile. Re-writing the KMIE using these shape factors gives,

$$\frac{\nu U T}{\delta_M} = U^2 \frac{d\delta_M}{dx} + (2+H)\delta_M U \frac{dU}{dx}. \tag{9.149}$$

Multiplying through by $\delta_M/(U\nu)$ and re-writing the $d\delta_M/dx$ term gives,

$$\frac{\nu U T}{\delta_M} = U^2 \frac{d\delta_M}{dx} + (2+H)\delta_M U \frac{dU}{dx}, \tag{9.150}$$

$$T = U\frac{\delta_M}{\nu}\frac{d\delta_M}{dx} + (2+H)\frac{\delta_M^2}{\nu}\frac{dU}{dx}, \tag{9.151}$$

$$0 = \frac{U}{2}\frac{d}{dx}\left(\frac{\delta_M^2}{\nu}\right) + \left(\frac{\delta_M^2}{\nu}\right)(2+H)\frac{dU}{dx} - T, \tag{9.152}$$

$$U\frac{d}{dx}\left(\frac{\delta_M^2}{\nu}\right) = 2\left[T - (2+H)\frac{dU}{dx}\left(\frac{\delta_M^2}{\nu}\right)\right]. \tag{9.153}$$

After plotting several known velocity profiles, researchers found that H and T depend almost entirely on another dimensionless quantity, λ, where,

$$\lambda := \frac{\delta_M^2}{\nu}\frac{dU}{dx}, \tag{9.154}$$

so that the shape of the boundary layer velocity profile will be determined entirely by λ. Substituting Eq. (9.153) into Eq. (9.154) gives,

$$U\frac{d}{dx}\left(\frac{\delta_M^2}{\nu}\right) = 2\left[T - (2+H)\frac{dU}{dx}\left(\frac{\delta_M^2}{\nu}\right)\right], \tag{9.155}$$

$$U\frac{d}{dx}\left(\frac{\lambda}{U'}\right) = F(\lambda) = 2[T - \lambda(2+H)]. \tag{9.156}$$

Figure 9.12 plots $F(\lambda)$ as a function of λ for a number of known profiles. Thwaites proposed a simple linear

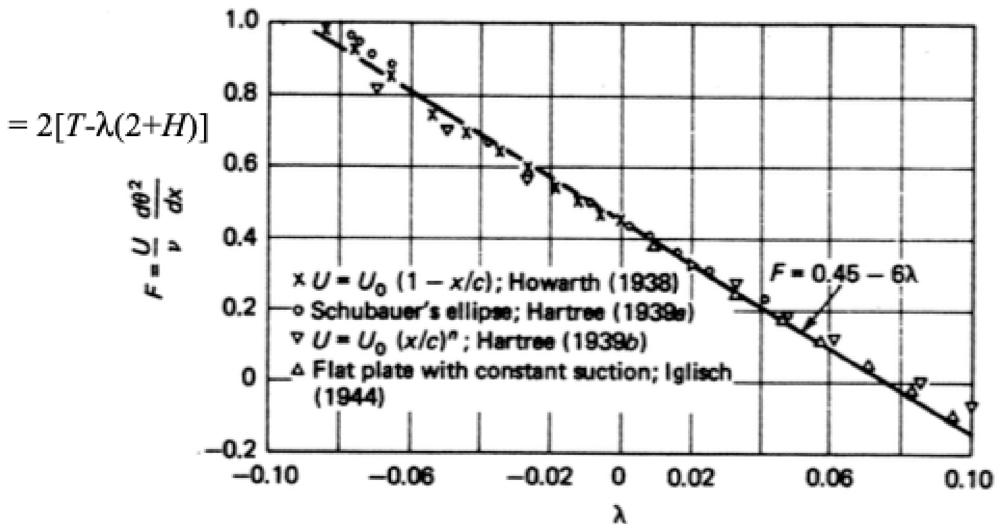


Figure 9.12. The function $F(\lambda)$ plotted as a function of λ for several known boundary layer profiles. This figure is from Figure 4-22 in White, F.M., *Viscous Fluid Flow*, 2nd ed., McGraw-Hill. Note that the θ in the plot is the momentum thickness, δ_M.

curve fit to the data,

$$U\frac{d}{dx}\left(\frac{\delta_M^2}{\nu}\right) = F(\lambda) = 0.45 - 6.0\lambda, \tag{9.157}$$

$$= 0.45 - 6.0\frac{dU}{dx}\left(\frac{\delta_M^2}{\nu}\right), \tag{9.158}$$

$$0.45\nu = U\frac{d(\delta_M^2)}{dx} + 6.0\delta_M^2\frac{dU}{dx}, \tag{9.159}$$

$$= \frac{1}{U^5}\frac{d}{dx}\left(\delta_M^2 U^6\right), \tag{9.160}$$

$$\delta_M^2 U^6 = \int_0^x 0.45\nu U^5 dx + \text{constant} \quad [\delta_M(x=0) = 0 \implies \text{constant} = 0], \tag{9.161}$$

so that the momentum thickness is related to the outer flow velocity by,

$$\boxed{\delta_M^2 = \frac{0.45\nu}{U^6}\int_0^x U^5 dx.} \tag{9.162}$$

Notes:

(1) The functions $H(\lambda)$ and $T(\lambda)$ are found from plots of several known velocity profiles. Figure 9.13 shows Thwaites' correlation data for the shape factors. The following two curve fits to the data are given by White (*Viscous Fluid Flow*, 2nd ed., McGraw-Hill),

$$T(\lambda) \approx (\lambda + 0.09)^{0.62}, \tag{9.163}$$

$$H(\lambda) \approx 2.0 + 4.14z - 83.5z^2 + 854z^3 - 3337z^4 + 4576z^5 \quad (z = 0.25 - \lambda). \tag{9.164}$$

(2) The typical procedure for using Thwaites' method is as follows:
- Determine $U = U(x)$ for the outer potential flow.
- Determine δ_M using Eq. (9.162).
- Determine λ using Eq. (9.154).

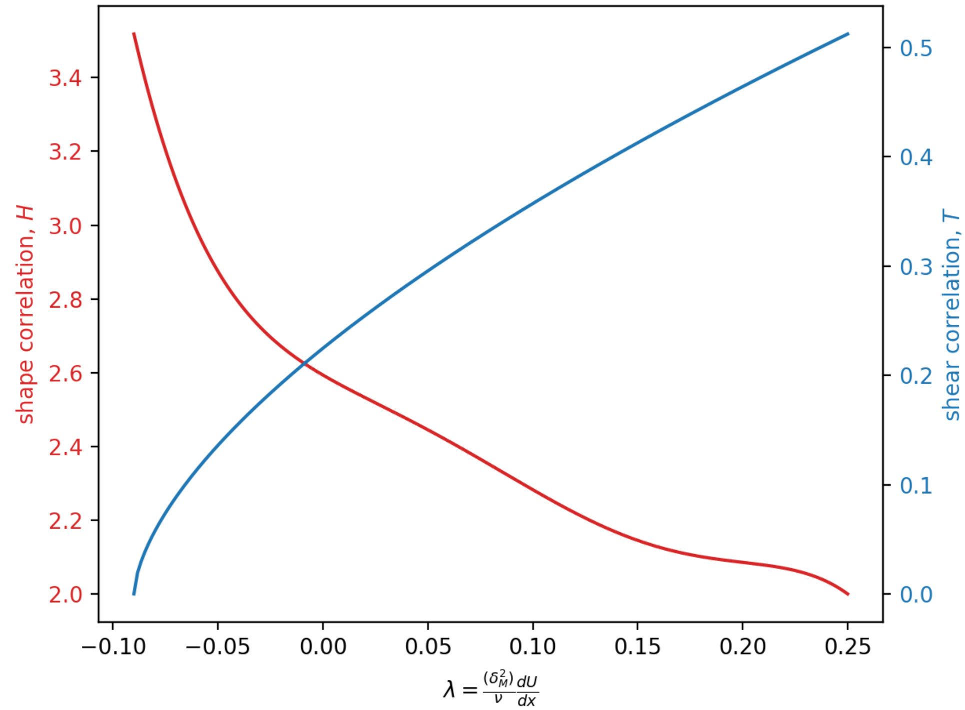


Figure 9.13. The shape factors $T(\lambda)$ and $H(\lambda)$ plotted against $\lambda = \frac{\delta_M^2}{\nu}\frac{dU}{dx}$.

- Determine τ_w and δ_D using Eq. (9.148) and the curve fits or using Figure 9.13.

(3) This method is considered one of the best methods for predicting the behavior of laminar boundary layers. It is accurate to about $\pm 5\%$ for favorable or mild adverse pressure gradients and is accurate to about $\pm 15\%$ near the separation point. When more accurate calculations are necessary, one typically turns to numerical methods for solving the boundary layer equations.

(4) Recall that the boundary layer separation point occurs when $\tau_w = 0 \implies T(\lambda) = 0$ so that,

$$\lambda_{\text{at separation pt}} = -0.090. \tag{9.165}$$

(5) Thwaites' method, as presented here, is restricted to laminar, planar flows. A similar type of method can also be derived for laminar, axi-symmetric flows.

(6) The outer flow velocity, U, may in fact be significantly different than the expected potential flow velocity profile. For example, for flow around a cylinder (or any bluff body), boundary layer separation results in a large wake region. Thus, the potential flow prediction (using a uniform stream and a doublet to model flow around a cylinder, for example) for the outer flow velocity may be greatly in error. Often we must resort to experimental data to obtain the outer flow velocity profile.

Consider the decelerating non-similar outer flow described by,
$U = U_0\left(1 - {}^{x}/_{L}\right)$.
Using Thwaites' method, determine the point at which flow separation occurs.

SOLUTION:
Using Thwaite's correlation we have,

$$\delta_M^2 = \frac{0.45\nu}{U_0^6\left(1-{}^{x}/_{L}\right)^6}\int_0^x U_0^5\left(1 - {}^{x}/_{L}\right)^5 dx, \tag{1}$$

$$\delta_M^2 = 0.075\frac{\nu L}{U_0}\left[\left(1 - {}^{x}/_{L}\right)^{-6} - 1\right], \tag{2}$$

so that the dimensionless parameter, λ, is,

$$\lambda = \frac{\delta_M^2}{\nu}\frac{dU}{dx} = -0.075\left[\left(1 - {}^{x}/_{L}\right)^{-6} - 1\right]. \tag{3}$$

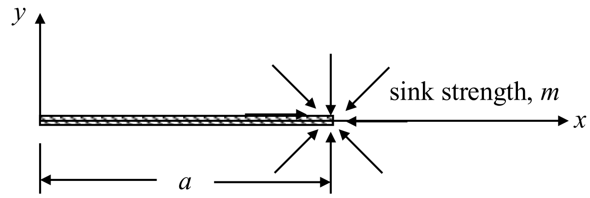
Separation occurs when λ = -0.090,

$$-0.090 = -0.075\left[\left(1 - {}^{x}/_{L}\right)^{-6} - 1\right], \tag{4}$$

$$\boxed{\frac{x_{sep}}{L} = 0.123}. \tag{5}$$

This value is within 3% of the "exact" result of x_{sep}/L, which is found numerically.

Consider the boundary layer flow resulting from a sink located at the trailing edge of a thin, flat plate as shown in the figure.



a. Using Thwaites' method, determine and plot the dimensionless momentum thickness, $\delta_M/(a^2\nu/m)^{1/2}$, over the top surface of the plate as a function of dimensionless distance along the plate, x/a.
b. Will boundary layer separation occur on the plate? If so, determine the location of the separation point. If not, explain why. [Hint: No calculations are necessary for this part.]

SOLUTION:

Determine the outer flow velocity profile by modeling the outer, potential flow as a single line sink located at $x = a$.

$$f(z) = \frac{-m}{2\pi}\log(z-a) \quad \text{(where } m > 0\text{)} \tag{1}$$

The fluid velocity is given by:

$$u_x - iu_y = \frac{df}{dz} = \frac{-m}{2\pi}\frac{1}{z-a} = \frac{-m}{2\pi}\frac{(\bar{z}-a)}{(z-a)(\bar{z}-a)} = \frac{-m}{2\pi}\frac{(\bar{z}-a)}{\left(|z|^2 - za - \bar{z}a + a^2\right)} = \frac{-m}{2\pi}\frac{(x-a)-iy}{\left(x^2+y^2-2xa+a^2\right)}$$

$$u_x - iu_y = \frac{-m}{2\pi}\frac{(x-a)}{\left[(x-a)^2+y^2\right]} - i\frac{-m}{2\pi}\frac{y}{\left[(x-a)^2+y^2\right]} \tag{2}$$

On the plate surface, $y = 0$ so that:

$$u_x = \frac{-m}{2\pi}\frac{1}{(x-a)} \quad \text{and} \quad u_y = 0 \quad \text{(Note that } 0 \le x \le a \text{ on the plate surface.)} \tag{3}$$

Use Thwaites' method to determine the momentum thickness, δ_M:

$$\delta_M^2 = \frac{0.45\nu}{U^6}\int_{x=0}^{x=x} U^5 dx = \frac{0.45\nu}{\left[\frac{-m}{2\pi}\frac{1}{(x-a)}\right]^6}\int_{x=0}^{x=x}\left[\frac{-m}{2\pi}\frac{1}{(x-a)}\right]^5 dx$$

$$= \frac{(2\pi)0.45\nu(x-a)^6}{-m}\int_{x=0}^{x=x}(x-a)^{-5}dx$$

$$= \frac{(2\pi)0.45\nu(x-a)^6}{4m}\left[(x-a)^{-4} - a^{-4}\right]$$

$$= \frac{0.45\pi\nu a^2}{2m}\left[\left(\frac{x}{a}-1\right)^2 - \left(\frac{x}{a}-1\right)^6\right]$$

$$\delta_M = \sqrt{\frac{0.45\pi\nu a^2}{2m}\left[\left(1-\frac{x}{a}\right)^2 - \left(1-\frac{x}{a}\right)^6\right]}$$

$$\therefore \frac{\delta_M}{\sqrt{\nu a^2 / m}} = \sqrt{0.225\pi\left[\left(1-\frac{x}{a}\right)^2 - \left(1-\frac{x}{a}\right)^6\right]} \tag{4}$$

Boundary layer separation will not occur on the plate since there is a favorable pressure gradient along the surface. This can be shown by examining Bernoulli's equation in the outer flow.

$$p + \tfrac{1}{2}\rho U^2 = \text{constant} \Rightarrow \frac{\partial p}{\partial x} + \rho U \frac{\partial U}{\partial x} = 0$$

$$\therefore \frac{\partial p}{\partial x} = -\rho U \frac{\partial U}{\partial x} \tag{5}$$

The velocity on the plate surface is given by Eqn. (3) so that:

$$\therefore \frac{\partial p}{\partial x} = \rho\left(\frac{m}{2\pi}\right)^2 \frac{1}{(x-a)^3} \tag{6}$$

Since $0 \le x \le a$, $\partial p/dx < 0$ (a favorable pressure gradient).

Another approach to showing that there will be no boundary layer separation is to use Thwaites' method (applicable to laminar boundary layers). Boundary layer separation occurs when the wall shear stress is zero, *i.e.* $\tau_w = 0$. In Thwaites' method this occurs when:

$$\lambda = \frac{\delta_M^2}{\nu}\frac{dU}{dx} = -0.090 \tag{7}$$

Substituting Eqn. (4) for U gives:

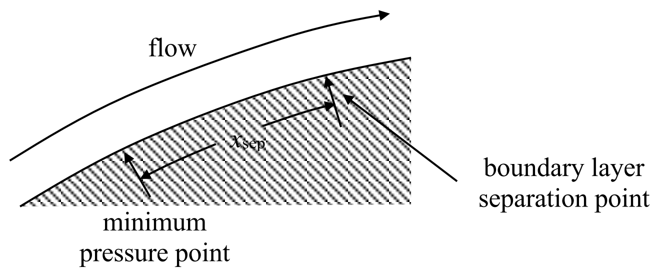
$$-0.090 = \frac{\delta_M^2}{\nu}\left[\frac{m}{2\pi}\frac{1}{(x-a)^2}\right] \tag{8}$$

Noting that the right hand side will always be positive (note that $m > 0$ as defined in Eqn. (1)), we must conclude that boundary layer separation will not occur.

Near the minimum pressure point in the steady, planar flow of an incompressible fluid past a body, the velocity outside the laminar boundary layer, U, is given by:

$$U = U_{\max}\left(1 - cx^2\right)$$

where $U_{\max}$ and c are constants and x is a coordinate measured along the body surface from the minimum pressure point:



Estimate the distance x_{sep} to the laminar boundary layer separation point assuming that $cx_{sep}^2 << 1$. Note that you should retain first and second order terms until the final result to get an answer accurate to first order. You may also assume that the momentum thickness at the minimum pressure point is negligible.

SOLUTION:

Use Thwaites' correlation to determine the boundary layer separation point. Thwaites' correlation is given by:

$$\delta_M^2 = \frac{0.45\nu}{U^6}\int_{x'=0}^{x'=x} U^5 dx' \quad \Rightarrow \quad \lambda \equiv \frac{\delta_M^2}{\nu}\frac{dU}{dx} = \frac{0.45}{U^6}\frac{dU}{dx}\int_{x'=0}^{x'=x} U^5 dx' \tag{1}$$

Substitute in for U as given in the problem statement.

$$\lambda = \frac{0.45}{U_{\max}^6\left(1-cx^2\right)^6}\left(-2cU_{\max}x\right)\int_{x'=0}^{x'=x} U_{\max}^5\left(1-cx'^2\right)^5 dx' = -0.90cx\left(1-cx^2\right)^{-6}\int_{x'=0}^{x'=x}\left(1-cx'^2\right)^5 dx' \tag{2}$$

Use a binomial expansion to evaluate the terms in parentheses:

$$\left(1-cx^2\right)^r = 1 - rcx^2 + \tfrac{1}{2}r\left(r-1\right)\left(cx^2\right)^2 - \tfrac{1}{6}r\left(r-1\right)\left(r-2\right)\left(cx^2\right)^3 + \cdots \tag{3}$$

Note that if $cx_{sep}^2 << 1$, then Eqn. (3) for $r = 5$ and $r = -6$ simplifies to:

$$\left(1-cx^2\right)^5 \approx 1 - 5cx^2 \tag{4}$$

$$\left(1-cx^2\right)^{-6} \approx 1 + 6cx^2 \tag{5}$$

Substitute into Eqn. (2) and simplify.

$$\begin{aligned}
\lambda &\approx -0.90cx\left(1+6cx^2\right)\int_{x'=0}^{x'=x}\left(1-5cx'^2\right)dx' \\
&\approx -0.90cx\left(1+6cx^2\right)\left(x - \tfrac{5}{3}cx^3\right) \\
&\approx -0.90cx^2\left(1+6cx^2\right)\left(1-\tfrac{5}{3}cx^2\right) \\
&\approx -0.90cx^2\left(1-\tfrac{5}{3}cx^2+6cx^2\right)
\end{aligned} \tag{6}$$

$$\therefore \lambda \approx -0.90cx^2\left(1+\tfrac{13}{3}cx^2\right) \approx -0.90cx^2 \tag{7}$$

Boundary layer separation occurs when $\lambda = -0.090$ so that:

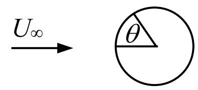
$$\lambda_{sep} = -0.090 \approx -0.90cx_{sep}^2 \quad \Rightarrow \quad \boxed{\therefore x_{sep} \approx \sqrt{\frac{1}{10c}}} \tag{8}$$

Starting with the potential flow around a cylinder, use Thwaite's method to find the location of laminar boundary layer separation from the cylinder. In the actual flow the boundary layer separation point occurs substantially upstream of the result obtained in this problem. Why?

SOLUTION:

The potential function for flow around a non-rotating cylinder of radius R is:

$$\phi = -U_\infty r\cos\theta\left(1+\frac{R^2}{r^2}\right)$$



so that the velocity on the surface of the cylinder is:

$$u_\theta\big|_{r=R} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = 2U_\infty\sin\theta \quad (1)$$

Using Thwaites' Method:

$$\delta_M^2 = \frac{0.45\nu}{U^6}\int_0^s U^5 ds \quad \text{where } s = R\theta \text{ and } ds = Rd\theta \quad (2)$$

and

$$\lambda = \frac{\delta_M^2}{\nu}\frac{dU}{ds} \quad (3)$$

so that, after combining Eqns. (2) and (3):

$$\lambda = \frac{dU}{d\theta}\frac{0.45}{U^6}\int_0^\theta U^5 d\theta \quad (4)$$

Substituting Eqn. (1) into Eqn. (4) gives:

$$\lambda = \frac{d}{d\theta}\left(2U_\infty\sin\theta\right)\frac{0.45}{\left(2U_\infty\sin\theta\right)^6}\int_0^\theta\left(2U_\infty\sin\theta\right)^5 d\theta$$

$$= \left(2U_\infty\cos\theta\right)\frac{0.45}{2U_\infty\sin^6\theta}\int_0^\theta \sin^5\theta d\theta$$

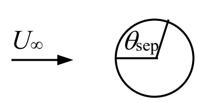
$$= \frac{0.45\cos\theta}{\sin^6\theta}\underbrace{\int_0^\theta \sin^5\theta d\theta}_{=-\frac{1}{5}\sin^4\theta\cos\theta+\frac{4}{5}\underbrace{\int_0^\theta\sin^3\theta d\theta}_{=-\frac{1}{3}\sin^2\theta\cos\theta-\frac{2}{3}\cos\theta+\frac{2}{3}}}$$

$$= \frac{0.45\cos\theta}{\sin^6\theta}\left(-\tfrac{1}{5}\sin^4\theta\cos\theta-\tfrac{4}{15}\sin^2\theta\cos\theta-\tfrac{8}{15}\cos\theta+\tfrac{8}{15}\right)$$

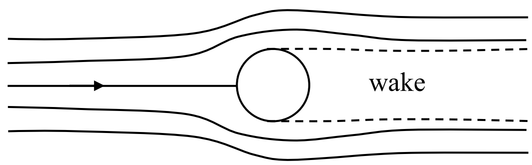
$$\therefore \lambda = \frac{0.09\cos\theta}{\sin^6\theta}\left(-\sin^4\theta\cos\theta-\tfrac{4}{3}\sin^2\theta\cos\theta-\tfrac{8}{3}\cos\theta+\tfrac{8}{3}\right) \quad (5)$$

At separation, λ = -0.090 so that, solving Eqn. (5) numerically for θ_{sep} gives:

$$\boxed{\theta_{\text{sep}} = 1.7996 \text{ rad} = 103.1^\circ} \quad (6)$$



The separation point in a real laminar flow occurs at $\theta_{sep} \approx 80.5°$ (from Hiemenz, 1911 experimental data). Our calculation is in error because we have assumed that the outer flow velocity is identical to the potential flow solution (the front and back half streamlines are symmetric.) However, in reality, boundary layer separation causes the actual outer flow streamlines to look much different over the downstream side of the cylinder as shown below.

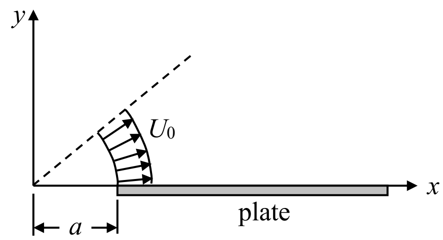


A better approximation for the separation point can be found if experimental data for the outer flow velocity is used. For example, Hiemenz (1911) found that a real outer flow laminar velocity profile can be fit well with:

$$\frac{u_\theta\big|_{r=R}}{U_\infty} = 1.814\theta - 0.271\theta^3 - 0.0471\theta^5 \quad \text{when } \mathrm{Re}_R = \frac{U_\infty R}{\nu} = 9500$$

Other correlations exist for other values of the Reynolds number.

Flow emanates from the origin of a wedge and passes over a flat plate as shown in the figure below.



The leading edge of the plate is located a distance a from the origin of the wedge. At the leading edge, the flow speed is U_0.

1. Determine the position on the plate at which the flow separates.
2. Plot the momentum thickness up to the separation point.

SOLUTION:

The outer, potential flow may be modeled as a flow from a source:

$$\phi = \frac{m}{2\pi}\ln r \tag{1}$$

so that

$$u_r = \frac{\partial \phi}{\partial r} = \frac{m}{2\pi}\frac{1}{r} \tag{2}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \tag{3}$$

The source strength m may be found by noting that the flow speed is U_0 at $r = a$:

$$u_r\left(r=a\right) = U_0 = \frac{m}{2\pi}\frac{1}{a} \Rightarrow m = 2\pi U_0 a \tag{4}$$

Thus, the outer flow velocity over the plate, U, is:

$$U = u_r = U_0\frac{a}{x} \quad \text{(Note that along the plate surface, } r = x.\text{)} \tag{5}$$

To find the separation point, use Thwaites' correlation:

$$\lambda \equiv \frac{\delta_M^2}{\nu}\frac{dU}{dx} = \frac{0.45}{U^6}\frac{dU}{dx}\int_{x'=a}^{x'=x} U^5 dx' \quad \text{(Note that the plate's leading edge begins at } x = a.\text{)} \tag{6}$$

$$\lambda = \frac{0.45}{U_0^6\left(a/x\right)^6}U_0\left(\frac{-a}{x^2}\right)\int_{x'=a}^{x'=x} U_0^5\left(a/x'\right)^5 dx' = \frac{-0.45x^4}{-4}x'^{-4}\Big|_a^x = 0.113x^4\left(\frac{1}{x^4}-\frac{1}{a^4}\right) \tag{7}$$

$$\lambda = 0.113\left[1-\left(\frac{x}{a}\right)^4\right] \tag{8}$$

At separation, λ = -0.090 so that

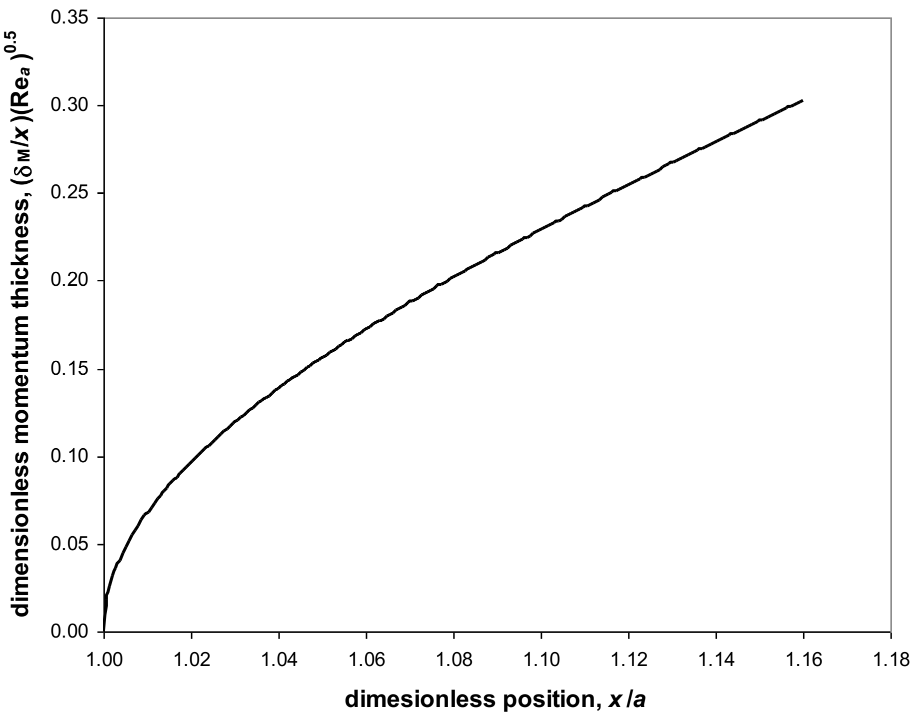
$$\lambda_{\text{sep}} = -0.090 = 0.113\left[1-\left(\frac{x_{\text{sep}}}{a}\right)^4\right] \tag{9}$$

$$\boxed{\therefore \frac{x_{\text{sep}}}{a} = \left(1+\frac{0.090}{0.113}\right)^{1/4} = 1.16} \tag{10}$$

The momentum thickness may be found using the definition of λ and Eqn. (8).

$$\frac{\delta_M^2}{\nu}\frac{dU}{dx} = 0.113\left[1-\left(\frac{x}{a}\right)^4\right] \Rightarrow \frac{\delta_M^2}{\nu}U_0\left(\frac{-a}{x^2}\right) = 0.113\left[1-\left(\frac{x}{a}\right)^4\right] \tag{11}$$

$$\boxed{\frac{\delta_M}{x}\sqrt{\mathrm{Re}_a} = \sqrt{0.113\left[\left(\frac{x}{a}\right)^4 - 1\right]}} \quad \text{where } \mathrm{Re}_a = U_0 a/\nu. \tag{12}$$



9.8. Turbulent Boundary Layer over a Flat Plate with No Pressure Gradient

To analyze a turbulent boundary layer we must use the momentum integral approach coupled with experimental data since no exact solutions are known. To approximate the velocity profile in a turbulent boundary layer, recall the Law of the Wall (refer to Chapter 10),

$$\frac{\bar{u}}{u^*} = \frac{yu^*}{\nu} \qquad \text{for } \frac{yu^*}{\nu} \leq 5, \tag{9.166}$$

$$\frac{\bar{u}}{u^*} = \frac{1}{K'} \ln\left(\frac{yu^*}{\nu}\right) + c \quad \text{for } \frac{yu^*}{\nu} > 5, \tag{9.167}$$

where $u^* = \sqrt{\tau_w/\rho}$ is the "friction velocity". We could substitute this velocity profile into the KMIE and solve. This velocity profile is cumbersome to use, however. Instead, Prandtl suggested approximating the logarithmic turbulent velocity profile using a 1/7th power-law curve fit,

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \quad \text{for } \frac{y}{\delta} \leq 1, \tag{9.168}$$

$$\frac{u}{U_\infty} = 1 \qquad \text{for } \frac{y}{\delta} > 1. \tag{9.169}$$

Using this velocity profile, the momentum thickness becomes,

$$\delta_M = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \int_0^\delta \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \left[1 - \left(\frac{y}{\delta}\right)^{\frac{1}{2}}\right] dy, \tag{9.170}$$

$$\delta_M = \frac{7}{72}\delta. \tag{9.171}$$

To determine the shear stress, recall that from the Kármán momentum integral equation,

$$\tau_w = \rho U_\infty^2 \frac{d\delta_M}{dx} = \frac{7}{72}\rho U_\infty^2 \frac{d\delta}{dx}, \tag{9.172}$$

so the friction coefficient becomes,

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{7}{36}\frac{d\delta}{dx}. \tag{9.173}$$

Experimental wall friction data for turbulent boundary layers can be fit using,

$$c_f \approx 0.020 \mathrm{Re}_\delta^{-\frac{1}{6}}, \tag{9.174}$$

where, $\mathrm{Re}_\delta = U_\infty \delta/\nu$. Note that experimental data for the wall shear stress is used instead of $\tau_w = \mu(du/dy)_{y=0}$, which was used for laminar boundary layers. The reason for the difference is that turbulent boundary layers use time-averaged data rather than instantaneous data. Equating the two friction coefficients gives,

$$\frac{7}{36}\frac{d\delta}{dx} = 0.020\left(\frac{U_\infty \delta}{\nu}\right)^{-\frac{1}{6}}, \tag{9.175}$$

$$\int_{\delta_0}^{\delta} \delta^{\frac{1}{6}} d\delta = 0.103\left(\frac{U_\infty}{\nu}\right)^{\frac{1}{6}} \int_{x_0}^{x} dx, \tag{9.176}$$

$$\delta^{\frac{7}{6}} - \delta_{)}^{\frac{7}{6}} = 0.120\left(\frac{U_\infty}{\nu}\right)^{-\frac{1}{6}} (x - x_0). \tag{9.177}$$

Assuming $\delta_0 = 0$ at $x_0 = 0$, meaning that the boundary layer starts off turbulent at the leading edge (refer to Figure 9.14), the previous equation becomes,

$$\left(\frac{\delta}{x}\right)^{\frac{7}{6}} = 0.120\left(\frac{\nu}{U_\infty x}\right)^{\frac{1}{6}}, \tag{9.178}$$

$$\boxed{\frac{\delta}{x} \approx \frac{0.163}{\mathrm{Re}_x^{\frac{1}{7}}}}. \tag{9.179}$$

From this relation we can also determine the displacement thickness, momentum thickness, friction factor,

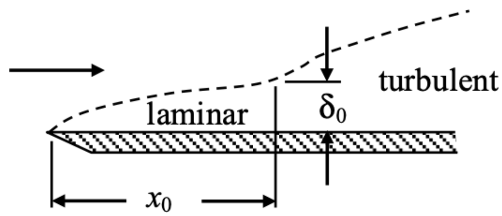


Figure 9.14. A sketch showing the boundary condition used in integrating the turbulent boundary layer thickness equation. The shear stress relationship holds strictly for the part of the boundary layer that is turbulent. If the laminar boundary layer thickness and distance downstream are small in comparison to the current thickness and location, then we may assume that the turbulent boundary layer starts approximately at the leading edge, i.e., $\delta_0 = 0$ at $x_0 = 0$.

and drag coefficient. These relations are summarized below.

$$\boxed{\frac{\delta}{x} \approx \frac{0.16}{\mathrm{Re}_x^{\frac{1}{7}}}} \quad \boxed{\frac{\delta_D}{x} \approx \frac{0.02}{\mathrm{Re}_x^{\frac{1}{7}}}} \quad \boxed{\frac{\delta_M}{x} \approx \frac{0.016}{\mathrm{Re}_x^{\frac{1}{7}}}}, \tag{9.180}$$

$$\boxed{c_f \approx \frac{0.027}{\mathrm{Re}_x^{\frac{1}{7}}}} \quad \boxed{c_D \approx \frac{0.031}{\mathrm{Re}_L^{\frac{1}{7}}}} \quad \mathrm{Re}_x = \frac{U_\infty x}{\nu}. \tag{9.181}$$

Notes:

(1) The boundary layer thickness grows as $\delta \sim x^{\frac{6}{7}}$ for a turbulent boundary layer whereas it grows as $\delta \sim x^{\frac{1}{2}}$ for a laminar boundary layer. Hence, a boundary layer grows more rapidly with downstream distance for turbulent flow than for a laminar flow. The momentum and displacement thicknesses also increase more rapidly for turbulent boundary layers.
(2) The shear stress decreases more rapidly for laminar flow than for a turbulent flow. The drag does not increase as rapidly in a laminar flow as compared to a turbulent flow.
(3) Another experimental friction curve fit that is commonly used is:

$$c_f \approx \frac{0.0466}{\mathrm{Re}_\delta^{\frac{1}{4}}}, \tag{9.182}$$

which gives,

$$\boxed{\frac{\delta}{x} \approx \frac{0.382}{\mathrm{Re}_x^{\frac{1}{5}}}} \quad \boxed{\frac{\delta_D}{x} \approx \frac{0.0478}{\mathrm{Re}_x^{\frac{1}{5}}}} \quad \boxed{\frac{\delta_M}{x} \approx \frac{0.0371}{\mathrm{Re}_x^{\frac{1}{5}}}}, \tag{9.183}$$

$$\boxed{c_f \approx \frac{0.0594}{\mathrm{Re}_x^{\frac{1}{5}}}} \quad \boxed{c_D \approx \frac{0.0742}{\mathrm{Re}_L^{\frac{1}{5}}}} \quad \mathrm{Re}_x = \frac{U_\infty x}{\nu}. \tag{9.184}$$

White (in White, F.M., *Viscous Fluid Flow*, 2nd ed., McGraw-Hill) states that the experimental curve fit given by Eq. (9.182) is based on limited data and is not as accurate as the curve fit given by Eq. (9.174). This argument is supported by the plot shown in Figure 9.15.

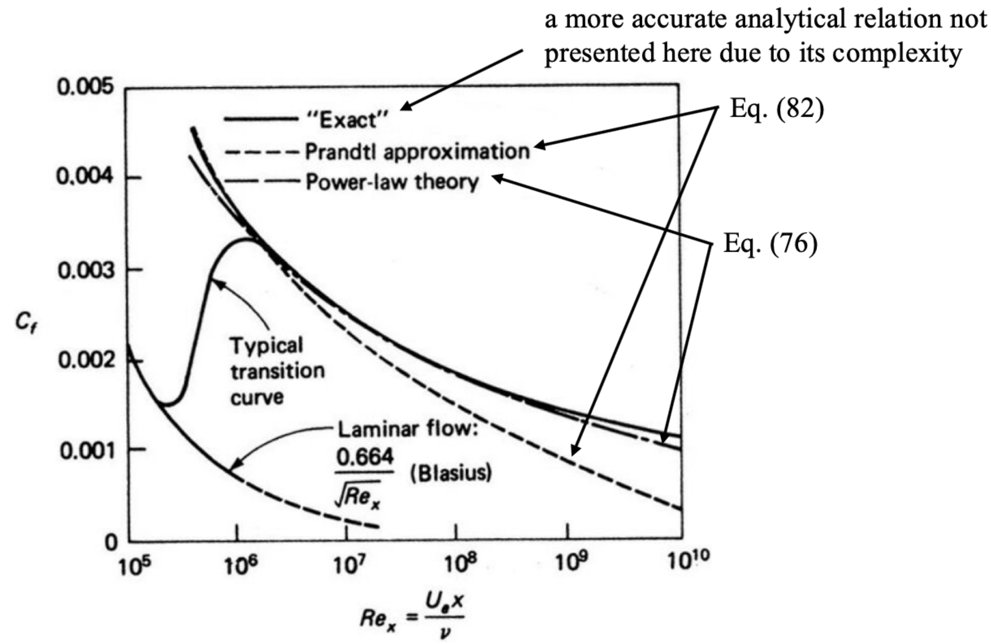
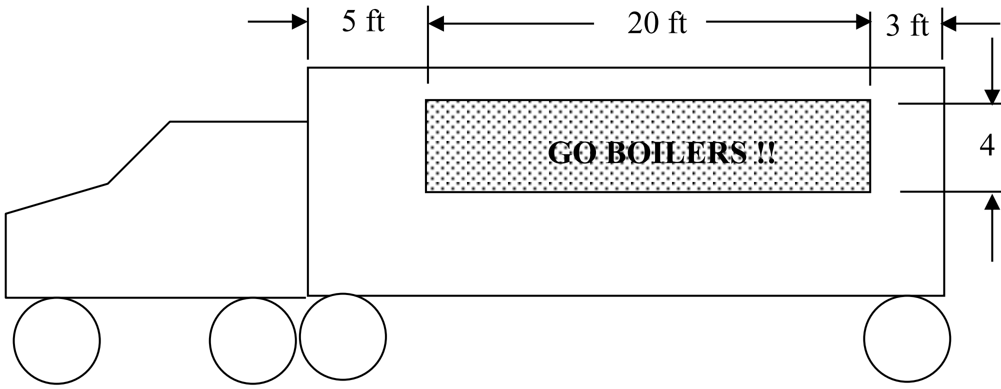


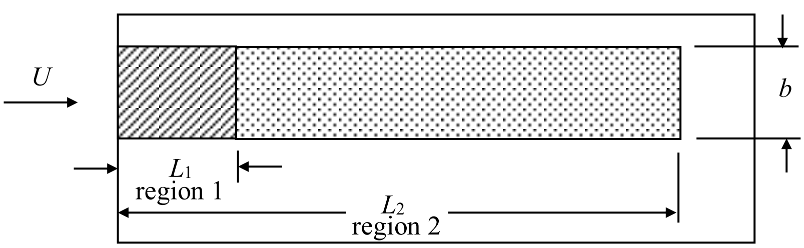
Figure 9.15. Boundary layer friction coefficients plotted against the Reynolds number based on the boundary layer thickness. Note that in the plot, Eq. (82) is actually Eq. (9.181) and Eq. (76) is actually Eq. (9.184)). Plot from White, F.M., *Viscous Fluid Flow*, 2nd ed., McGraw-Hill.

A thin smooth sign is attached to the side of a truck as shown. Estimate the skin friction drag on the sign when the truck speed is 55 mph.



SOLUTION:

Assume that the boundary layer forms at the front of the trailer.



$L_1 = 5$ ft
$L_2 = 25$ ft
$b = 4$ ft
$U = 55$ mph $= 80.7$ ft/s
$\nu_{air} = 1.57*10^{-4}$ ft^2/s
$\rho_{air} = 2.38*10^{-3}$ slugs/ft^3

To find the drag on the sign, determine the drag on region 2 and subtract the drag from region 1.

$$D_{\text{sign}} = D_2 - D_1 \tag{1}$$

where

$$D_i = c_{Di}\,\tfrac{1}{2}\rho U^2 L_i b \quad (i = 1 \text{ or } 2) \tag{2}$$

Substitute and simplify.

$$D_{\text{sign}} = \tfrac{1}{2}\rho U^2 b\left(c_{D2}L_2 - c_{D1}L_1\right) \tag{3}$$

The drag coefficients are determined from the Reynolds numbers at each region's trailing edge.

$$\text{Re}_1 = \frac{UL_1}{\nu} = \frac{(80.7\text{ ft/s})(5\text{ ft})}{\left(1.57*10^{-4}\text{ ft}^2/\text{s}\right)} = 2.6*10^6 \quad \text{(turbulent!)} \tag{4}$$

$$\text{Re}_2 = \frac{UL_2}{\nu} = \frac{(80.7\text{ ft/s})(25\text{ ft})}{\left(1.57*10^{-4}\text{ ft}^2/\text{s}\right)} = 1.3*10^7 \quad \text{(turbulent!)} \tag{5}$$

Assume that the flow is fully turbulent throughout regions 1 and 2 (neglect any laminar flow contribution) so that:

$$c_{D1} = \frac{0.0742}{\text{Re}_1^{1/5}} = \frac{0.0742}{\left(2.6*10^6\right)^{1/5}} = 3.87*10^{-3} \tag{6}$$

$$c_{D2} = \frac{0.0742}{\text{Re}_2^{1/5}} = \frac{0.0742}{\left(1.3*10^7\right)^{1/5}} = 2.80*10^{-3} \tag{7}$$

Substitute into Eqn. (3) and evaluate.

$$D_{\text{sign}} = \tfrac{1}{2}\left(2.38*10^{-3}\ \text{slugs/ft}^3\right)\left(80.7\ \text{ft/s}\right)^2\left(4\ \text{ft}\right)\left[\left(2.80*10^{-3}\right)\left(25\ \text{ft}\right)-\left(3.87*10^{-3}\right)\left(5\ \text{ft}\right)\right]$$

$$\boxed{\therefore D_{\text{sign}} = 1.57\ \text{lb}_\text{f}} \tag{8}$$

A vertical stabilizing fin on a land-speed-record car is 1.65 m long and 0.785 m tall. The automobile is to be driven at the Bonneville Salt Flats in Utah, where the elevation is 1340 m and the summer temperature reaches 50 degC. The car speed is 560 km/hr. Calculate the power required to overcome skin friction drag on the fin.

SOLUTION:

At a temperature of 313 K, the kinematic viscosity of air is $\nu = 2.0*10^{-5}$ m²/s. Thus, the Reynolds number at the trailing edge of the vertical fin is:

$$\mathrm{Re}_L = \frac{UL}{\nu} = \frac{\left(560*10^3\ \frac{\text{m}}{\text{hr}}\cdot\frac{\text{hr}}{3600\text{ s}}\right)(1.65\text{ m})}{\left(1.7*10^{-5}\ \frac{\text{m}^2}{\text{s}}\right)} \Rightarrow \mathrm{Re}_L = 1.51*10^7 \quad (1)$$

Clearly the flow is turbulent at the trailing edge of the vertical fin. At what distance from the leading edge of the fin does the flow transition from laminar to turbulent? To answer this question, calculate the distance at the transition Reynolds number,

$$\mathrm{Re}_{\text{crit}} = 500,000 = \frac{Ux}{\nu} = \frac{\left(560*10^3\ \frac{\text{m}}{\text{hr}}\cdot\frac{\text{hr}}{3600\text{ s}}\right)x}{\left(1.7*10^{-5}\ \frac{\text{m}^2}{\text{s}}\right)} \Rightarrow x = 5.5\text{ cm} \quad (2)$$

Thus, most of the flow over the fin is turbulent. Since this is the case, approximate the entire flow over the fin as being turbulent. The drag coefficient for a turbulent boundary layer over a flat plate is,

$$C_D = \frac{0.0742}{\mathrm{Re}_L^{1/5}} \Rightarrow \frac{D_{\text{1-side}}}{\frac{1}{2}\rho U^2 LH} = \frac{0.0742}{\mathrm{Re}_L^{1/5}} \Rightarrow D_{\text{2-sides}} = 2D_{\text{1-side}} = 2\cdot\frac{0.0742}{\mathrm{Re}_L^{1/5}}\cdot\frac{1}{2}\rho U^2 LH \quad (3)$$

The power is given by,

$$P = UD_{\text{2-sides}} \Rightarrow \boxed{P = \frac{0.0742}{\mathrm{Re}_L^{1/5}}\cdot\rho U^3 LH} \quad (4)$$

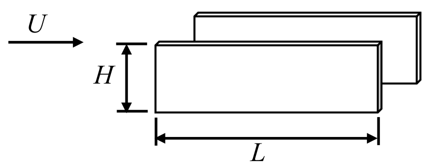
Using the given data,

ρ = 1.075 kg/m³ (standard atmosphere at an altitude of 1340 m)
Re_L = $1.51*10^7$
U = 560 km/hr = 155.6 m/s
L = 1.65 m
H = 0.785 m
$\Rightarrow$ $\boxed{P = 14.3\text{ kW}}$

Note that a speed of 540 km/hr at a temperature of 50 degC result in a Mach number of 0.43. Thus, a more accurate approach to solving this problem would assume relations for a compressible boundary layer, rather than the incompressible relations assumed in the previous solution.

The U.S. Navy has built the *Sea Shadow*, which is a *small waterplane area twin-hull* (SWATH) ship with a reduced radar profile. This catamaran is 160 ft long and its twin hulls have a draft of 14 ft. Assume that ocean turbulence triggers a fully turbulent boundary layer on the sides of each hull. Treat these as flat plate boundary layers and calculate the drag on the ship and power required to overcome this drag for speeds ranging from 5 to 13 knots.

SOLUTION:
Model the twin hulls as two flat plates with turbulent boundary layers as shown in the figure below.



Assuming turbulent boundary layer over the full length of the hull, the drag force on one side of a hull is,

$$C_D \equiv \frac{D}{\frac{1}{2}\rho U^2 LH} = \frac{0.0742}{\mathrm{Re}_L^{1/5}}, \tag{1}$$

$$D_{\substack{\text{one side}\\ \text{of hull}}} = \frac{0.0742}{\mathrm{Re}_L^{1/5}}\left(\tfrac{1}{2}\rho U^2 LH\right) \text{ where } \mathrm{Re}_L = \frac{UL}{\nu}. \tag{2}$$

The total drag acting on the ship will be four times the drag in Eq. (2) since there are two hulls, each with two sides,

$$D_{\text{total}} = 4D_{\substack{\text{one side}\\ \text{of hull}}}. \tag{3}$$

Using the given numbers,

ρ_{seawater} = 1025 kg/m^3 = 63.99 lb$_m$/ft^3,
U = 5 to 13 kn = 8.44 ft/s to 21.94 ft/s (1 kt = 1.15 mph = 1.688 ft/s),
L = 160 ft,
H = 14 ft,
μ_{seawater} = 1.08*10^{-3} Pa.s,
=> $\nu = \mu/\rho$ = 1.05*10^{-6} m^2/s = 1.13*10^{-5} ft^2/s,
=> Re$_L$ = 1.19*10^8 – 3.10*10^8 (clearly in the turbulent regime),
=> $D_{\text{one side of hull}}$ = 285 – 1590 lb$_f$ (1 lb$_f$ = 32.2 lb$_m$.ft/s^2),
=> $\boxed{D_{\text{total}} = 1140 – 6370\ \text{lb}_f}$.

The power to overcome this total drag is,
$P = D_{\text{total}}U$,
=> $\boxed{P = 17.5 – 254\ \text{hp}}$ (1 hp = 550 lb$_f$.ft/s)

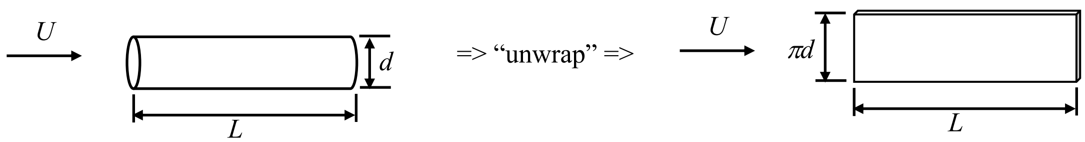
Note that the hulls for the *Sea Shadow* are more complex than the flat plates described in this simple problem. The actual hulls have cylindrical elements, which are tapered at the ends, as shown in the figure to the side.

The U.S. Navy's *Ohio*-class guided-missile submarines have a length of 170.69 m (560 ft) and a beam, i.e., width, of 12.8 m (42 ft). Assume the submarine travels at 37.0 kph (= 20 kn) when fully submerged.

a. What percentage of the submarine's surface has a laminar boundary layer for these conditions?
b. Estimate the power required for the submarine to overcome skin friction drag for these conditions.

SOLUTION:

Model the submarine as a cylinder with a diameter of d = 12.8 m and a length of L = 170.69 m. "Unwrap" the cylinder and model the flow along its length of the cylinder as flow adjacent to a flat plate as shown in the figures below.



First calculate the distance from the leading edge at which the boundary layer transitions from laminar to turbulent flow,

$$\mathrm{Re}_{x_{\mathrm{crit}}} = \frac{Ux_{\mathrm{crit}}}{\nu} = 500{,}000 \quad \Rightarrow \quad x_{\mathrm{crit}} = 500{,}000\left(\frac{\nu}{U}\right). \tag{1}$$

Using the given numbers,

U = 37.0 kph = 10.3 m/s,
ρ_{seawater} = 1025 kg/m^3,
μ_{seawater} = 1.08*10^{-3} Pa.s,
=> $\nu = \mu/\rho$ = 1.05*10^{-6} m^2/s, (2)
=> x_{crit} = 5.10*10^{-2} m = 5.1 cm!

Thus, the fraction of the length that's laminar is,

$\boxed{x_{\mathrm{crit}}/L = (5.10*10^{-2}\text{ m})/(170.69\text{ m}) = 0.030\%.}$ (3)

Clearly, the flow over the submarine can be assumed turbulent over the entire length without much error.

Assuming a turbulent boundary layer over the full length of the hull, the drag force is,

$$C_D \equiv \frac{D}{\frac{1}{2}\rho U^2 L(\pi d)} = \frac{0.0742}{\mathrm{Re}_L^{1/5}}, \tag{4}$$

$$D = \frac{0.0742}{\mathrm{Re}_L^{1/5}}\left(\tfrac{1}{2}\rho U^2 LH\right) \text{ where } \mathrm{Re}_L = \frac{UL}{\nu}. \tag{5}$$

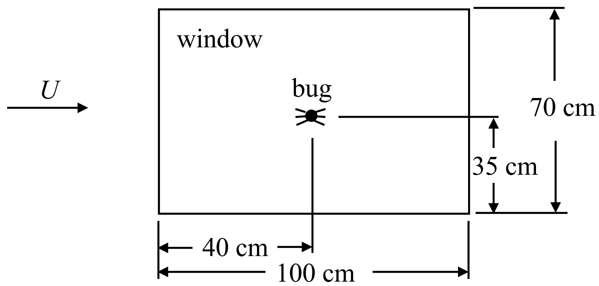
Using the given numbers,

L = 170.69 m,
$\pi d = \pi$(12.8 m) = 40.2 m,
=> Re_L = 1.67*10^9,
=> $\boxed{D = 3.96*10^5\text{ N}}$ (= 89,000 lbf!)

The power to overcome this skin friction drag is,

$P = DU$,
=> $\boxed{P = 4.08*10^6\text{ W}}$ (= 5470 hp!)

A small bug rests on the outside of a car side window as shown in the figure below. The surrounding air has a density of 1.2 kg/m^3 and kinematic viscosity of 1.5*10^{-5} m^2/s. To first order, we can approximate the flow as flat plate flow with no pressure gradient and the start of the boundary layer begins at the leading edge of the window. Also assume that the flow is turbulent over the entire length of the window (this isn't a good assumption, but for simplicity, we'll make it here).



a. Determine the minimum speed at which the bug will be sheared off of the car window if the bug can resist a shear stress of up to 1 N/m^2.
b. What is the total skin friction drag acting on the window at a speed of U = 20 m/s?
c. Ignoring the presence of the bug, at what streamwise location will the boundary layer separation point occur on the window? Justify your answer.

SOLUTION:

Assume that the flow over the window is turbulent at the bug location so that:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0594}{\mathrm{Re}_x^{1/5}}, \tag{1}$$

$$\tau_w = (0.0594)\tfrac{1}{2}\rho U^2\left(\frac{\nu}{Ux}\right)^{1/5} = (0.0594)\tfrac{1}{2}\rho U^{9/5}\left(\frac{\nu}{x}\right)^{1/5}, \tag{2}$$

$$\therefore U = \left[\frac{\tau_w}{\frac{1}{2}(0.0594)\rho}\left(\frac{x}{\nu}\right)^{1/5}\right]^{5/9}. \tag{3}$$

Using the given data:

τ_w = 1 N/m^2
ρ = 1.2 kg/m^3
x = 0.4 m
ν = 1.5*10^{-5} m^2/s
$\Rightarrow$ U = 20 m/s

Check the Reynolds number to verify the turbulent flow assumption.

$$\mathrm{Re}_x = \frac{Ux}{\nu} \approx 530{,}000 \;\Rightarrow\; \text{Turbulent flow assumption is ok!} \tag{4}$$

Hence, the minimum required speed to shear off the bug is 20 m/s.

The total skin friction drag acting on the car window (assuming turbulent flow throughout) at the given velocity is:

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 (LW)} = \frac{0.0742}{\mathrm{Re}_L^{1/5}} \quad \text{where } \mathrm{Re}_L = \frac{UL}{\nu}, \tag{5}$$

$$\therefore D = \tfrac{1}{2}\rho U^2 (LW) \frac{0.0742}{\mathrm{Re}_L^{1/5}}. \tag{6}$$

Using the given data:

U = 20 m/s
ρ = 1.2 kg/m^3
L = 1 m
W = 0.7 m
ν = 1.5*10^{-5} m^2/s
$\Rightarrow$ Re_L = 1.3*10^6
$\Rightarrow$ C_D = 4.4*10^{-3}
$\Rightarrow$ $\boxed{D = 0.7\ \mathrm{N}}$

Boundary layer separation will not occur since there is no adverse pressure gradient in the flow (zero pressure gradient was assumed).

Note that the distance from the leading edge where the flow transitions from laminar to turbulent flow at U = 20 m/s is,

$$\mathrm{Re}_{x_{\text{crit}}} = \frac{U x_{\text{crit}}}{\nu} = 500{,}000, \tag{7}$$

$$\Rightarrow \quad x_{\text{crit}} = 500{,}000\left(\frac{\nu}{U}\right), \tag{8}$$

x_{crit} = 37.5 cm,

This distance is a considerable portion of the window length. Hence, a better approach to solving this problem would be to include both the laminar and turbulent portions in the analysis rather than neglecting the laminar portion as was done in the previous analysis.

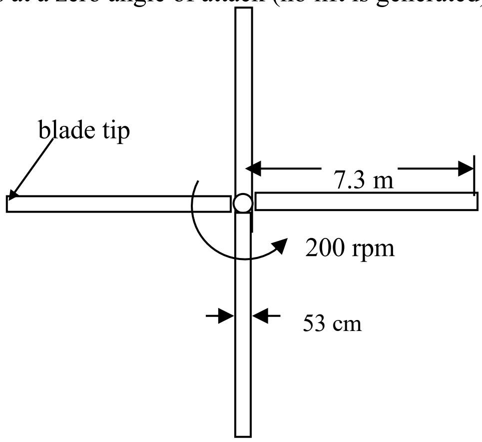
Use the drag coefficient given in Pritchard et al. (8$^{\text{th}}$ ed., Eq. 9.37a), which takes into account the skin friction drag of the laminar part and the turbulent part,

$$C_D = \frac{0.0742}{\mathrm{Re}_L^{1/5}} - \frac{1740}{\mathrm{Re}_L}. \tag{9}$$

For Re_L = 1.3*10^6 (calculated previously), C_D = 3.1*10^{-3}, which gives a drag force of D = 0.5 N. This more accurate value for the drag is approximately 42% less than drag calculated assuming turbulent flow over the full length.

Video solution: https://www.youtube.com/watch?v=UQS2bUAPF-w

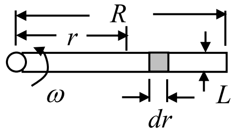
A four-bladed Apache helicopter rotor rotates at 200 rpm in air (with a density of 1.2 kg/m^3 and kinematic viscosity $1.5*10^{-5}$ m^2/s). Each blade has a chord length of 53 cm and extends a distance of 7.3 m from the center of the rotor hub. To greatly simplify the problem, assume that the blades can be modeled as very thin flat plates at a zero angle of attack (no lift is generated).



a. At what radial distance from the hub center is the flow at the blade trailing edge turbulent?
b. What is the (99%) boundary layer thickness at the blade tip trailing edge?
c. Assuming that the flow over the entire length of the four blades is turbulent, estimate the power required to drive the helicopter rotor (neglecting all other effects besides aerodynamic drag).

SOLUTION:

The transition to turbulence occurs when Re_{crit} = 500,000 where



$$\mathrm{Re}_{crit} = \frac{UL}{\nu} = \frac{(r_{crit}\omega)L}{\nu} \tag{1}$$

$$r_{crit} = \frac{\nu \mathrm{Re}_{crit}}{\omega L} = \frac{\left(1.5*10^{-5}\ \mathrm{m^2/s}\right)(500{,}000)}{\left(20.94\ \mathrm{rad/s}\right)\left(53*10^{-2}\ \mathrm{m}\right)}$$

(Note: $\omega = \left(200\ \mathrm{rot/min}\right)\left(\frac{2\pi\ \mathrm{rad}}{\mathrm{rot}}\right)\left(\frac{\mathrm{min}}{60\ \mathrm{s}}\right) = 20.94\ \mathrm{rad/s}$)

$$\boxed{\therefore r_{crit} = 0.68\ \mathrm{m}} \tag{2}$$

To determine the boundary layer thickness at the blade tip trailing edge, first calculate the Reynolds number there.

$$\mathrm{Re}_L = \frac{UL}{\nu} = \frac{(R\omega)L}{\nu} = \frac{(7.3\ \mathrm{m})\left(20.94\ \mathrm{rad/s}\right)\left(53*10^{-2}\ \mathrm{m}\right)}{\left(1.5*10^{-5}\ \mathrm{m^2/s}\right)} = 5.40*10^6 \tag{3}$$

$\therefore \mathrm{Re}_L = 5.40*10^6 \Rightarrow$ the flow is turbulent

Using the turbulent boundary layer correlations:

$$\frac{\delta}{L} = \frac{0.382}{\mathrm{Re}_L^{1/5}} \tag{4}$$

$$\delta = \frac{0.382\left(53*10^{-2}\ \mathrm{m}\right)}{\left(5.40*10^6\right)^{1/5}}$$

$$\boxed{\therefore \delta = 9.1*10^{-3}\ \mathrm{m} = 9.1\ \mathrm{mm}} \tag{5}$$

Video solution: https://www.youtube.com/watch?v=UQS2bUAPF-w

In order to determine the power required to drive the rotor, first determine the torque resulting from the skin friction drag acting on the blades.

$$dF_{\text{1-blade}} = \underbrace{2}_{\substack{\text{two}\\\text{sides}}} C_D \tfrac{1}{2}\rho \underbrace{U}_{=r\omega}{}^2 \underbrace{dA}_{=Ldr} \quad \text{where } C_D = \frac{0.0742}{\text{Re}_L^{1/5}} \text{ (turbulent BL correlation)} \tag{6}$$

$$dT_{\text{1-blade}} = r\,dF_{\text{1-blade}} \tag{7}$$

$$\begin{aligned}
T_{\text{4-blades}} &= 4\int_{r=0}^{r=R} dT_{\text{1-blade}} = 4\int_{r=0}^{r=R} r\,dF_{\text{1-blade}}\\
&= 4\int_{r=0}^{r=R} r2\frac{0.0742}{\text{Re}_L^{1/5}}\tfrac{1}{2}\rho(r\omega)^2 L\,dr\\
&= 4\int_{r=0}^{r=R} r\frac{0.0742}{\left(\frac{r\omega L}{\nu}\right)^{1/5}}\rho(r\omega)^2 L\,dr\\
&= 2.97*10^{-1}\rho\nu^{1/5}\omega^{9/5}L^{4/5}\int_{r=0}^{r=R} r^{14/5}dr
\end{aligned}$$

$$\therefore T_{\text{4-blades}} = 7.81*10^{-2}\rho\nu^{1/5}\omega^{9/5}L^{4/5}R^{19/5} \tag{8}$$

$$\boxed{P_{\text{4-blades}} = \omega T_{\text{4-blades}} = 7.81*10^{-2}\rho\nu^{1/5}\omega^{14/5}L^{4/5}R^{19/5}} \tag{9}$$

For:

ρ = 1.2 kg/m^3
ν = 1.5*10^{-5} m^2/s
ω = 20.94 rad/s
L = 53*10^{-2} m
R = 7.3 m
$\Rightarrow$ $T_{\text{4-blades}}$ = 2790 N·m $\Rightarrow$ $\boxed{P_{\text{4-blades}} = 58.3 \text{ kW}}$

Video solution: https://www.youtube.com/watch?v=TKYRmqL1xGQ

A wind tunnel has a test section 1 m square by 6 m long with air at 20°C moving at an average velocity of 30 m/s. To account for the growing boundary layer, the walls are slanted slightly outward. At what angle should the walls be slanted between x = 2 m and x = 4 m to keep the test-section velocity constant?

SOLUTION:

Determine the displacement boundary layer thickness assuming flat plate flow. First check the flow Reynolds number to determine whether or not the flow is laminar.

$$\mathrm{Re}_{x=2\text{ m}} = \frac{Ux}{\nu} = \frac{(30\text{ m/s})(2\text{ m})}{(1.5e\text{-}5\text{ m}^2/\text{s})} = 4.0e6 \tag{1}$$

$$\mathrm{Re}_{x=4\text{ m}} = \frac{Ux}{\nu} = \frac{(30\text{ m/s})(4\text{ m})}{(1.5e\text{-}5\text{ m}^2/\text{s})} = 8.0e6$$

Thus the flow in the tunnel is turbulent in the range of interest.

Use the following correlation for turbulent flat plate flow to determine the displacement boundary layer thickness.

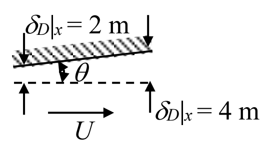
$$\frac{\delta_D}{x} = \frac{0.0478}{\mathrm{Re}_x^{1/5}} \tag{2}$$

x = 2 m: $\mathrm{Re}_x = 4.0e6 \Rightarrow \delta_D = 4.6e\text{-}3$ m

x = 4 m: $\mathrm{Re}_x = 8.0e6 \Rightarrow \delta_D = 8.0e\text{-}3$ m

As an approximation, assume that the boundary layer grows linearly between x = 2 m and x = 4 m so that the angle the walls need to be slanted outward is:

$$\tan\theta = \frac{\delta_D\big|_{x=4\text{ m}} - \delta_D\big|_{x=4\text{ m}}}{4\text{ m} - 2\text{ m}} \tag{3}$$



$$\boxed{\therefore \theta = 0.1^\circ} \tag{4}$$

9.9. Boundary Layer Separation

Consider flow around a cylinder as shown in Figure 9.16. On the front side of the cylinder the boundary layer

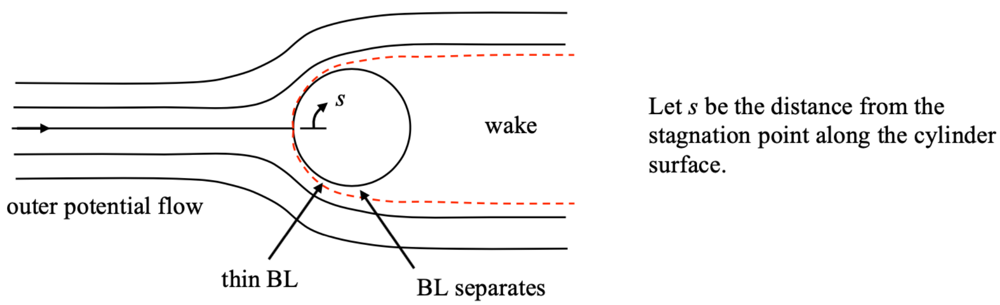


Figure 9.16. A sketch of the flow around a cylinder showing boundary layer separation.

grows with increasing distance along the surface, as we might expect. Moving toward the back half of the cylinder, however, the boundary layer no longer remains "attached" to the surface and, instead of following the cylinder contour, forms a wake behind the cylinder. The point where the boundary layer no longer follows the cylinder surface is termed the boundary layer separation point. As part of our examination of boundary layer separation, it's helpful to define the following:

$$\text{an adverse pressure gradient is one in which } dp/ds > 0 \text{, and} \tag{9.185}$$

$$\text{a favorable pressure gradient is one in which } dp/ds < 0. \tag{9.186}$$

In an adverse pressure gradient, the pressure increases moving downstream. Thus, the pressure is acting to decelerate the flow. In a favorable pressure gradient, the pressure decreases moving downstream and the pressure acts to accelerate the flow.

Now let's plot the pressure as a function of position on the cylinder surface. Figure 9.17 plots the pressure determined from potential flow theory as a function of the angle measured from the leading stagnation point. Imagine a fluid particle near the cylinder surface. It experiences the same pressure as that in the outer potential flow (recall that the pressure doesn't vary much in the spanwise direction within a boundary layer). As the fluid particle moves from point A to point B, it accelerates and gains energy as it's accelerated by the favorable pressure gradient. In going from point B to C, however, the fluid particle experiences an adverse pressure gradient that acts to decelerate the fluid particle. If the flow is inviscid, as is the case in a potential flow, then the flow on the back half of the cylinder will be symmetric to the flow on the front half and the fluid particle decelerates to zero speed at the trailing edge stagnation point located at point C. In a real flow, however, the fluid particle is subject to viscous stresses, which constantly remove energy from the flow. As a result, on the back half of the cylinder the fluid particle decelerates rapidly due to the adverse pressure gradient and viscous dissipation. At some point between B and C, the fluid particle loses its forward momentum and is no longer able to move further into the adverse pressure gradient. It is at this point the flow separates, with the fluid particle moving downstream and forming a wake rather than following the cylinder surface.

Figure 9.18 plots the dimensionless pressure on the cylinder surface for a potential (inviscid) flow along with representative curves for viscous flows with laminar and turbulent boundary layers. The inviscid analysis works well at predicting the pressure on the upstream side of the cylinder near the leading stagnation point, but becomes increasingly inaccurate moving toward the downstream side of the cylinder. The reason for the increasing inaccuracy is due to boundary layer separation, which distorts the flow streamlines so they no longer match those from a potential flow analysis. Once the boundary layer separates, a wake forms, within which the pressure remains nearly constant and equal to the pressure on the cylinder surface where the boundary layer separated.

Notes:

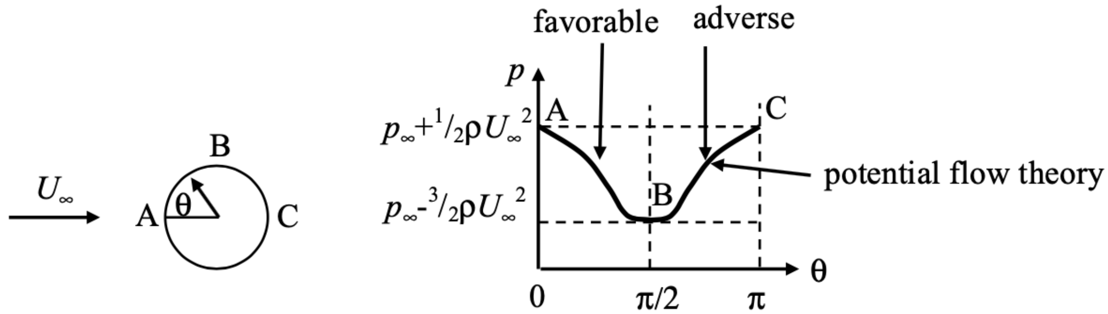
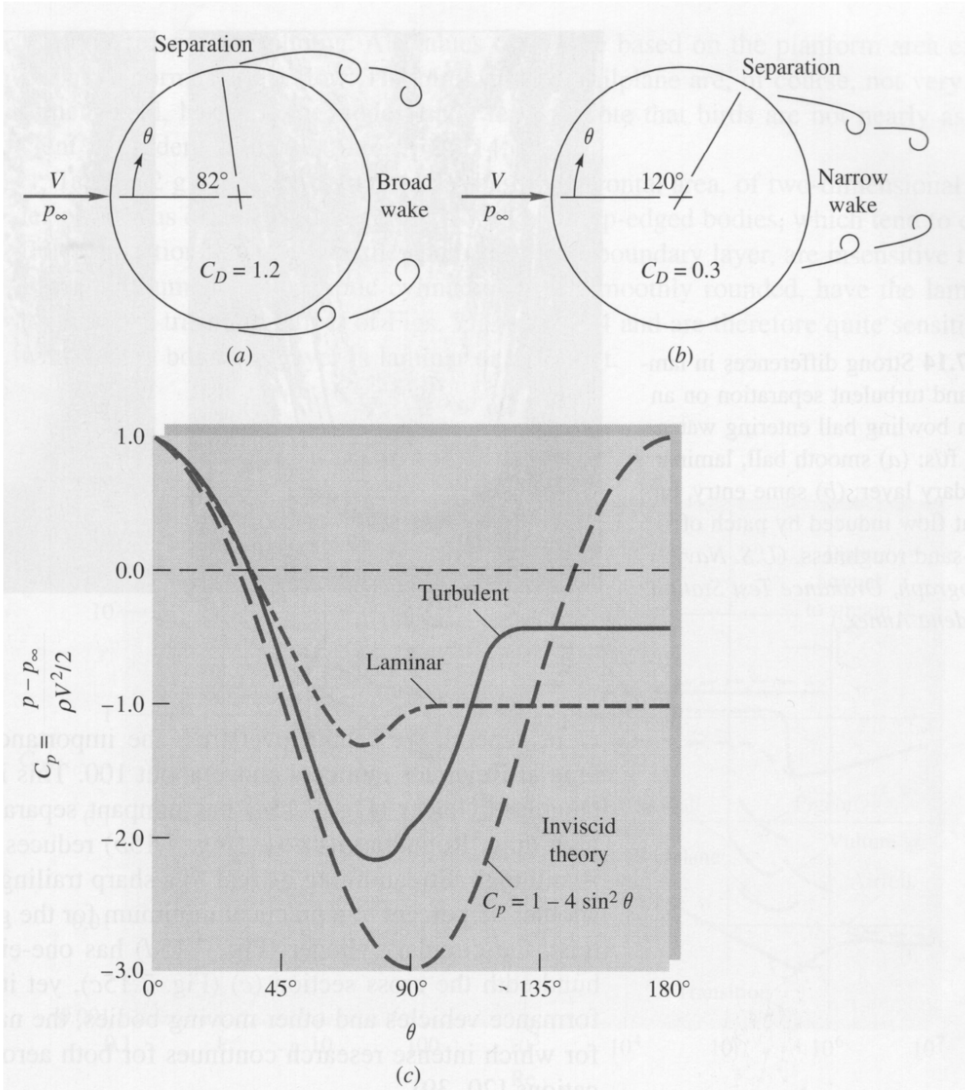


Figure 9.17. The pressure at the surface of the cylinder plotted as a function of the angle from the leading stagnation point, assuming potential flow around the cylinder. Over the front half of the cylinder the pressure gradient is favorable. Over the back half the pressure gradient is adverse.

(1) Boundary layer separation requires an adverse pressure gradient; however, an adverse pressure gradient does not necessarily cause boundary layer separation. It's possible for the boundary layer to have enough momentum to carry it through the adverse pressure gradient region.
(2) Delaying boundary layer separation results in a wake that is both smaller and with a more symmetric pressure profile (front and back halves of the cylinder). Both effects act to reduce the cylinder's form drag, i.e., the drag due to pressure forces.
(3) A turbulent boundary layer separates later than a laminar boundary layer (and, thus, has smaller form drag). The reason for the delayed separation is that a turbulent boundary layer has more momentum than a laminar boundary layer, as shown in Figure 9.19. The larger momentum in the turbulent boundary layer means that the boundary layer flow can travel further into the adverse pressure gradient region before separating. The larger momentum in the turbulent boundary layer is the result of turbulent eddies mixing free stream air, which has large momentum, into the boundary layer.
(4) Although a turbulent boundary layer results in a smaller form drag, i.e., pressure drag, due to delayed boundary layer separation, it does increase the skin friction drag, i.e., the drag due to viscous wall shear stresses on the surface. From Figure 9.19 one can see that the velocity gradient at the surface is larger for a turbulent boundary layer than for a laminar boundary layer and, thus, the shear stress will be larger. As is discussed in the following section, one must consider both form drag and skin friction drag when determining the total drag on an object.
(5) A laminar boundary layer can be "tripped" into becoming a turbulent boundary layer. Common techniques for inducing a turbulent boundary layer are to add roughness or bumps to a surface.
(6) Examples of flows with boundary layer separation are shown in Figures 9.20 - 9.23. Figure 9.20 provides a schematic of the boundary layer flow in a diverging channel. From Conservation of Mass, the velocity in the diverging channel decreases moving downstream. From Bernoulli's equation, the pressure must increase. Thus, the pressure gradient is adverse and boundary layer separation can occur. Figure 9.21 shows photographs from a corresponding flow.
Figure 9.22 shows a photograph of a laminar boundary layer separating over the top of a cylinder (top-most image). This separation point is further upstream than when the boundary layer is turbulent (image second from the top). The bottom two photographs show flow over a sharp, obtuse angle. When the boundary layer is laminar (second from the bottom), the flow separates immediately at the apex; however, when the boundary layer is turbulent, it has enough momentum to remain attached (bottom). This last photograph demonstrates the point made in the first note: Just because there's an adverse pressure gradient, it doesn't mean the boundary layer must separate. In this case the boundary layer has enough momentum to flow into the adverse pressure gradient region.



(From White, F.M., *Fluid Mechanics*, 3rd ed., McGraw-Hill.)

Figure 9.18. The pressure coefficient on the surface of a cylinder plotted as a function of angle from the leading edge stagnation point. The figure includes curves assuming potential flow (inviscid theory), viscous laminar flow, and viscous turbulent flow. This figure is from White, F.M., *Fluid Mechanics*, 3rd ed., McGraw-Hill.

The images shown in Figure 9.23 show bowling balls dropped into water. The ball in the left image is smooth and the boundary layer is laminar, resulting in separation near the ball's equator. The ball in the left image has a roughened surface at the leading edge, inducing a turbulent boundary layer and delayed separation.

Let's look at the velocity profile at different points along a flat plate for a flow with an adverse pressure gradient ($dp/dx > 0$), as shown in Figure 9.24. In an adverse pressure gradient flow the boundary layer velocity profile will always have an inflection point. This behavior can be shown by considering the boundary layer momentum equation,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{9.187}$$

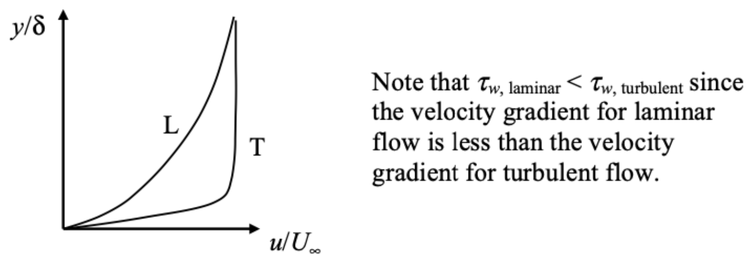
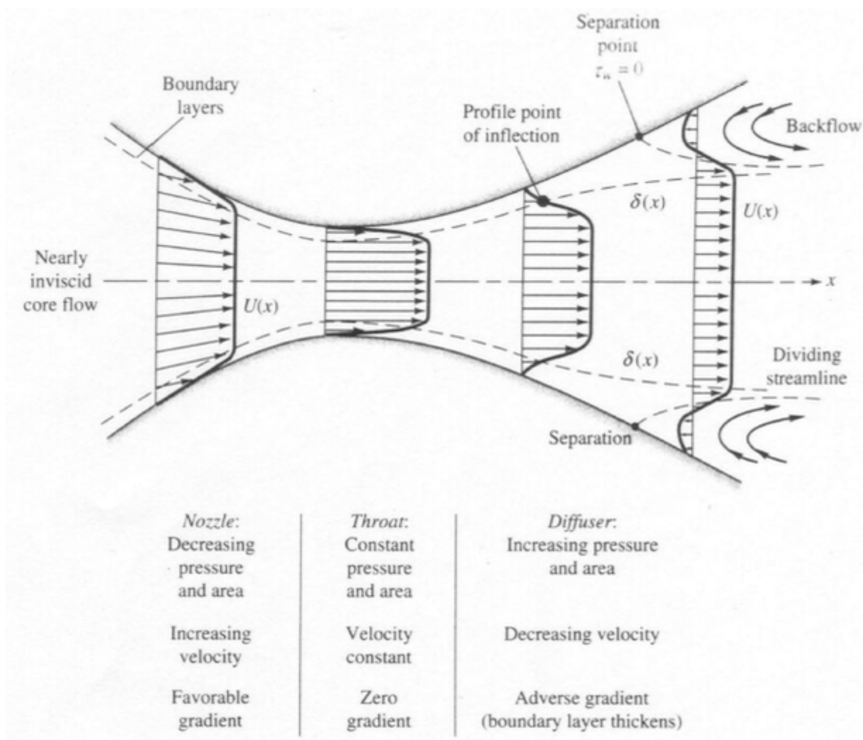


Figure 9.19. Sketches of laminar and turbulent boundary layer velocity profiles. Due to turbulent mixing, the turbulent boundary layer has more momentum than the laminar boundary layer.



(From White, F.M., *Fluid Mechanics*, 3rd ed., McGraw-Hill.)

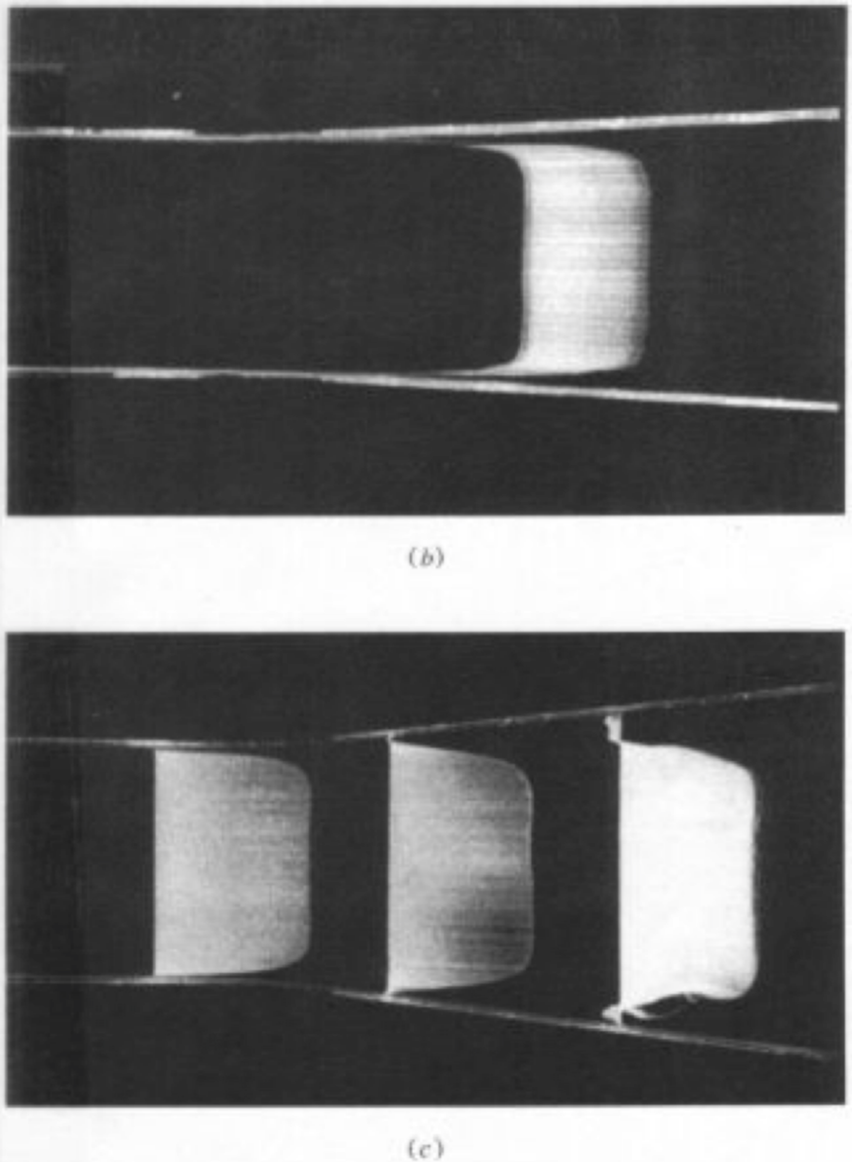
Figure 9.20. An illustration of flow in a diverging, planar channel. This figure is from White, F.M., *Fluid Mechanics*, 3rd ed., McGraw-Hill.

Note that at the wall boundary ($y = 0$), $u = v = 0$ so that,

$$\frac{dp}{dx} = \mu \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}. \tag{9.188}$$

Thus, at the wall boundary in an adverse pressure gradient ($dp/dx > 0$),

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} > 0. \tag{9.189}$$

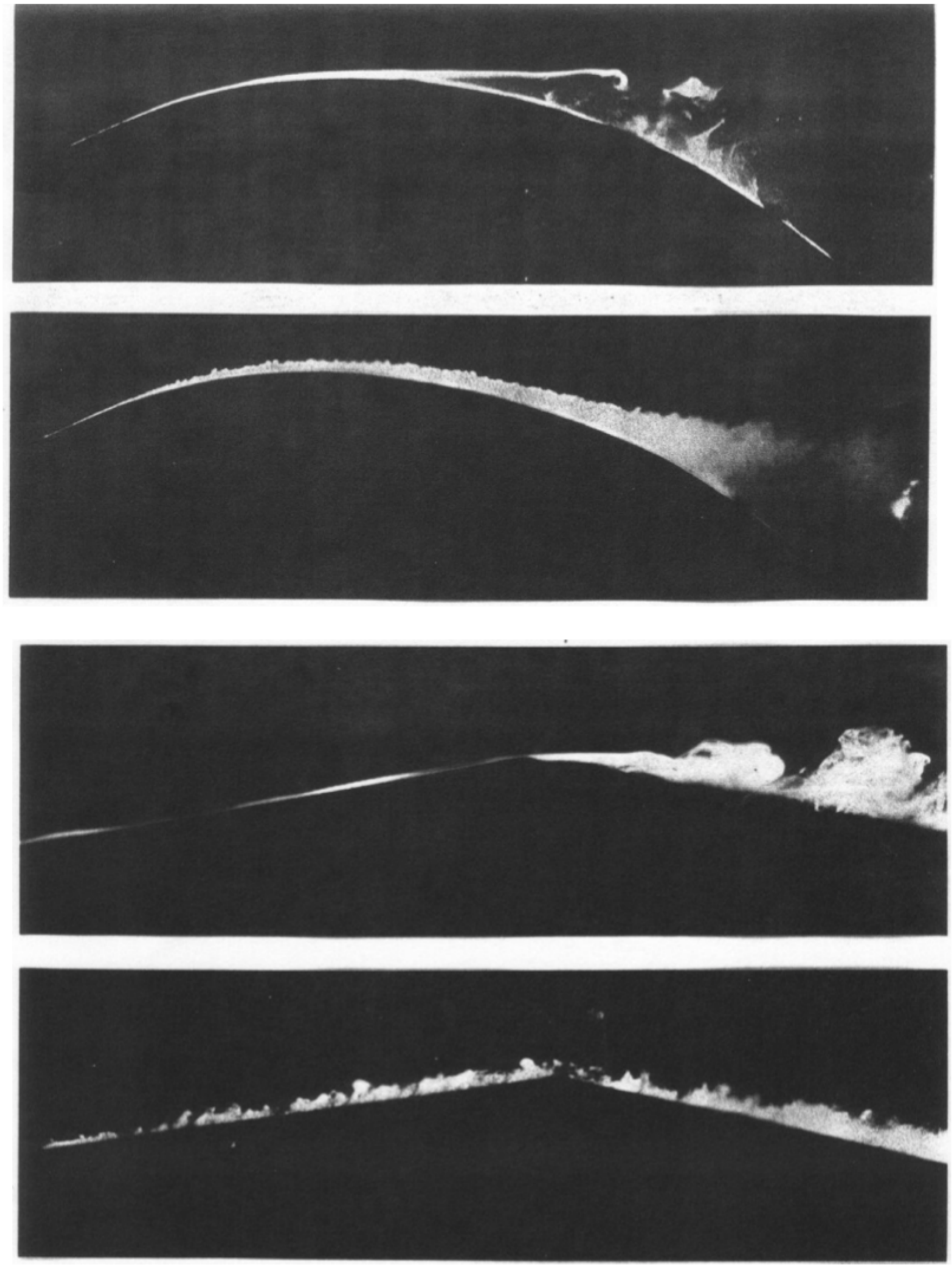


(b) Boundary-layer growth in a small-angle diffuser. (c) Boundary-layer separation in a large-angle diffuser. *[Parts (b) and (c) from the film "Fundamental of Boundary Layers," by the National Committee for Fluid Mechanics Films and the Educational Development Center.]*

Figure 9.21. Flow in a diverging, planar channel. In the top figure, the diverging angle is small and the adverse pressure gradient is sufficiently small so that boundary layer separation doesn't occur. In the bottom figure, the diverging angle is larger resulting in a larger adverse pressure gradient and separation.

At the free stream ($y = \delta$), however, we have,

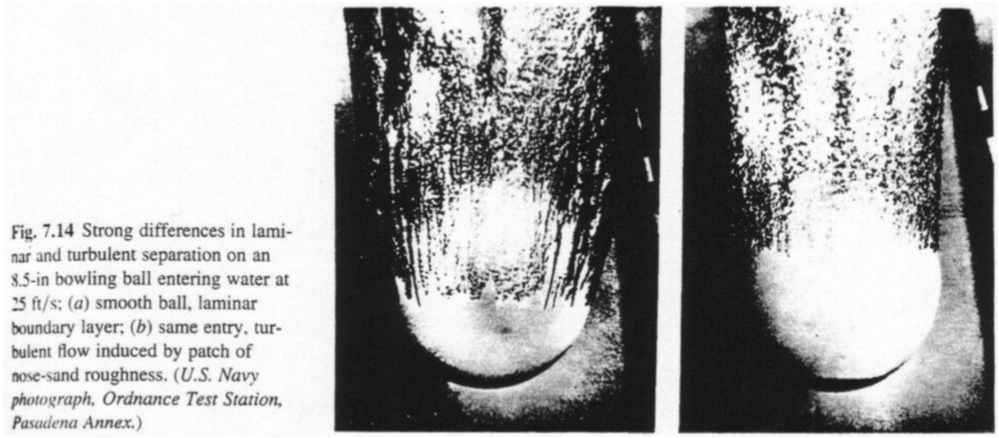
$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=\delta} < 0, \tag{9.190}$$



(From Van Dyke, M., *An Album of Fluid Motion*, Parabolic Press.)

Figure 9.22. Photographs of a laminar boundary flowing over a cylinder (top), a turbulent boundary layer flowing over a cylinder (second from top), a laminar boundary layer flowing over a sharp angle (second from bottom), and a turbulent boundary layer flowing over a sharp angle (bottom).

in order for the boundary layer profile to merge smoothly with the outer flow velocity. The change in sign of the boundary layer curvature indicates that somewhere within the boundary layer there must be a point of



(From White, F.M., *Fluid Mechanics*, 3rd ed., McGraw-Hill.)

Figure 9.23. Two bowling balls dropped into water. The left image is a smooth bowling ball and the right image is a bowling ball with the leading edge surface roughened by attaching sand paper in order to induce a turbulent boundary layer. This figure is from White, F.M., *Fluid Mechanics*, 3rd ed., McGraw-Hill.

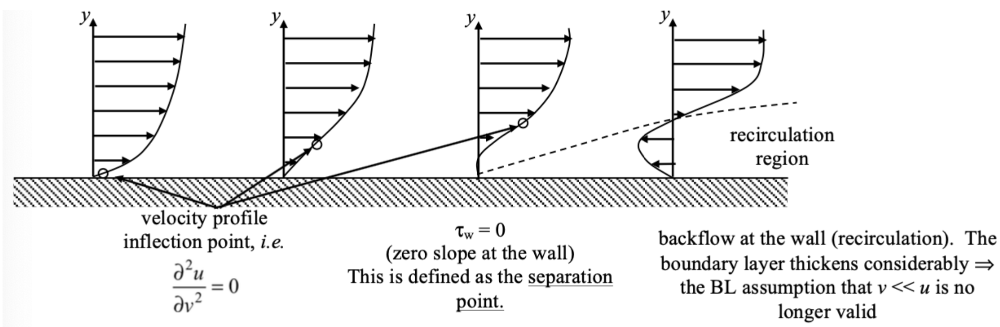


Figure 9.24. The boundary layer velocity profile at different downstream locations. Boundary layer separation is defined as occurring where the wall shear stress is zero. Downstream of this point there is flow recirculation.

inflection. The inflection point moves toward the outer flow boundary as the flow moves downstream in an adverse pressure gradient flow.

Boundary layer separation is defined as occurring where the shear stress is zero at the wall, i.e.,

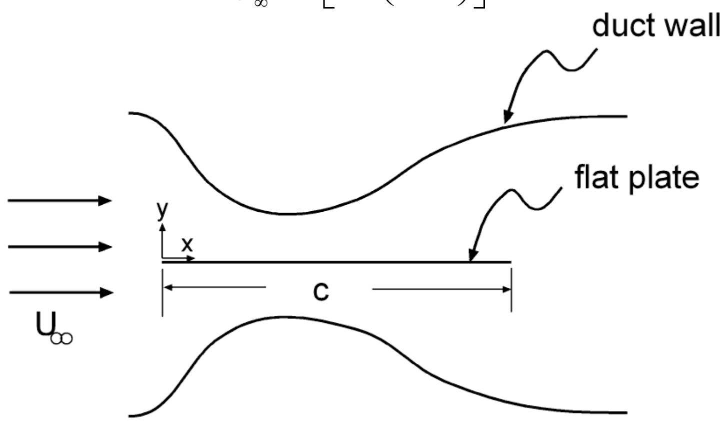
$$\tau_w = 0 \implies \mu \left.\frac{du}{dy}\right|_{y=0} = 0 \implies \underline{\text{boundary layer separation}} \tag{9.191}$$

Downstream of the separation point, a recirculation zone occurs where the flow reverses direction (being pushed by the adverse pressure gradient). When recirculation occurs, the boundary layer grows rapidly in thickness and the fundamental assumption that the boundary layer thickness is small compared to the downstream distance breaks down. Thus, the boundary layer is considered not to extend beyond the point of separation.

Video solution: https://www.youtube.com/watch?v=j1W7ylPSv-E

A flat plate of length c is placed inside a duct. By curving the walls of the duct, the pressure distribution on the flat plate can be set. Assume the walls of the duct are contoured in such a way that the outer flow over the plate gives the following velocity on the surface of the flat plate:

$$\frac{u_e(x)}{U_\infty} = \left[8\frac{x}{c}\left(1-\frac{x}{c}\right)\right]^{1/5}$$



1. Write an expression for the streamwise pressure gradient as a function of x/c.
2. Determine which portions of the plate have a favorable pressure gradient and which portions have an adverse pressure gradient.

SOLUTION:

In the outer flow region (the inviscid core), we can use Bernoulli's equation,

$$p+\tfrac{1}{2}\rho U^2 = \text{constant} \Rightarrow \frac{dp}{dx}+\rho U\frac{dU}{dx}=0 \Rightarrow \frac{dp}{dx}=-\rho U\frac{dU}{dx} \tag{1}$$

Here,

$$U = u_e(x) = U_\infty\left[8\frac{x}{c}\left(1-\frac{x}{c}\right)\right]^{1/5} \Rightarrow \frac{dU}{dx}=\frac{8}{5}U_\infty\left[8\frac{x}{c}\left(1-\frac{x}{c}\right)\right]^{-\frac{4}{5}}\left(\frac{1}{c}-\frac{2x}{c^2}\right) \Rightarrow$$

$$\frac{dU}{dx}=\frac{8}{5}\frac{U_\infty}{c}\left[8\frac{x}{c}\left(1-\frac{x}{c}\right)\right]^{-\frac{4}{5}}\left(1-2\frac{x}{c}\right) \tag{2}$$

Thus,

$$\frac{dp}{dx}=-\rho\left\{U_\infty\left[8\frac{x}{c}\left(1-\frac{x}{c}\right)\right]^{1/5}\right\}\left\{\frac{8}{5}\frac{U_\infty}{c}\left[8\frac{x}{c}\left(1-\frac{x}{c}\right)\right]^{-\frac{4}{5}}\left(1-2\frac{x}{c}\right)\right\} \tag{3}$$

$$\boxed{\frac{dp}{dx}=-\frac{8}{5}\frac{\rho U_\infty^2}{c}\left[8\frac{x}{c}\left(1-\frac{x}{c}\right)\right]^{-\frac{3}{5}}\left(1-2\frac{x}{c}\right)} \tag{4}$$

An adverse pressure gradient is one in which $dp/dx > 0$. A favorable pressure gradient is one in which $dp/dx < 0$. Note also that $0 \le x/c \le 1$.

$$-\frac{8}{5}\frac{\rho U_\infty^2}{c}\underbrace{\left[8\frac{x}{c}\left(1-\frac{x}{c}\right)\right]^{-\frac{3}{5}}}_{>0 \text{ for } 0<x/c<1}\underbrace{\left(1-2\frac{x}{c}\right)}_{>0 \text{ for } x/c<\frac{1}{2}}\begin{cases} <0 & x/c<\frac{1}{2} \\ >0 & x/c>\frac{1}{2}\end{cases} \tag{5}$$

$\boxed{\text{Thus, there is a favorable pressure gradient for } x/c < \tfrac{1}{2} \text{ and adverse pressure gradient for } x/c > \tfrac{1}{2}.}$

9.10. Lift and Drag on Immersed Objects

The force acting on an object immersed in a fluid flow is comprised of the force due to pressure over the surface and the force due to viscous wall shear stresses (Figure 9.25). If we know the pressure (p) and shear stress (τ) distribution over the object, then,

$$F_{p,i} = \int_A -p dA n_i, \tag{9.192}$$

$$F_{s,i} = \int_A \tau_{ji} dA n_i. \tag{9.193}$$

where F_p is the force due to the pressure component, F_s is the force due to the shear stress component, A is the surface area of the object, and n_i are the unit normal vector components of the differential surface area. The component of the resultant force acting in the direction *parallel to the incoming flow* is known as the drag force, F_D, and the component *perpendicular to the incoming flow* is known as the lift force, F_L.

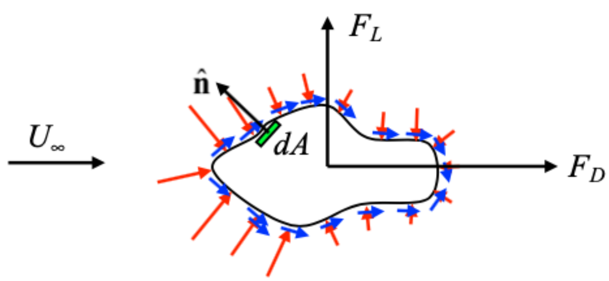


Figure 9.25. An illustration of the pressure and shear force distributions over an immersed object. The resultant lift F_L and F_D forces acting on the object are also shown.

Notes:

(1) The pressure force component of the drag is known as the form drag while the shear stress drag component is known as the skin friction drag.
(2) A streamlined body is one in which the (skin friction drag) $\gg$ (form drag) (refer to Figure 9.26).

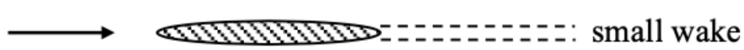


Figure 9.26. An example of a streamlined body.

A bluff body is one in which the (form drag) $\gg$ (skin friction drag) (refer to Figure 9.27).

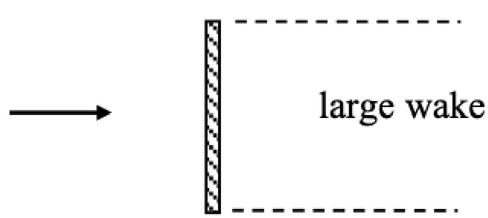


Figure 9.27. An example of a bluff body.

(3) The lift and drag are often expressed in dimensionless form as lift and drag coefficients, C_L and C_D,

$$C_L := \frac{L}{\frac{1}{2}\rho U_\infty^2 A} \quad \text{and} \quad C_D := \frac{D}{\frac{1}{2}\rho U_\infty^2 A}, \tag{9.194}$$

where A is usually the frontal projected area, i.e., the area seen from the front of the object, for a bluff body, or the planform area, i.e., the area seen from above, for a streamlined body (Figure 9.28). To avoid ambiguity, it is best to report what area is used to form the lift and drag coefficients.

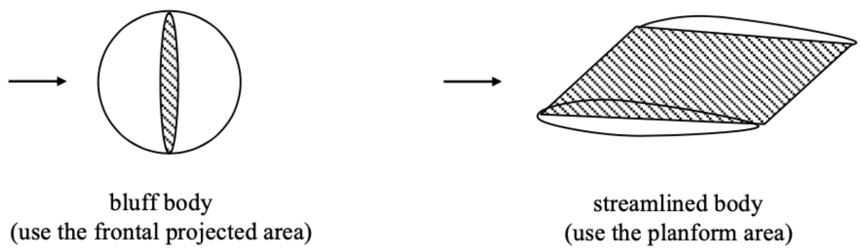


Figure 9.28. Illustrations of the frontal projected area (left) and the planform area (right).

9.10.1. Flow around a Sphere at Different Reynolds Numbers

Since flow around spheres is common in practice, it's worthwhile to examine the flow behavior around a sphere at different Reynolds numbers,

$$\mathrm{Re}_D = \frac{U_\infty D}{\nu}, \tag{9.195}$$

where U_∞ is the upstream flow speed, D is the sphere diameter, and ν is the fluid's kinematic viscosity. Although we'll specifically look at flow around a sphere, the general patterns shown here are observed for many other objects too, although the details may be different.

Figure 9.29 shows sketches of different flow regimes as a function of Reynolds number while Figure 9.30 shows corresponding photographs. At the smallest Reynolds number the flow streamlines are symmetric between the front and back halves of the sphere since fluid inertia is negligible. This regime is known as the Stokes or creeping flow regime. As the Reynolds number increases, inertia becomes more significant and a wake with fixed eddies forms downstream of the sphere. At larger Reynolds numbers, the eddies no longer remain attached behind the sphere and, instead, detach periodically from the sphere and are carried downstream. This phenomenon is known as a Kármán Vortex Street. Figure 9.31 shows striking photographs of Kármán vortex streets. At even larger Reynolds numbers the flow structure around the sphere becomes even more complex. A laminar boundary layer forms on the front half of the sphere and then separates at an angle of approximately 80° from the leading stagnation point. A laminar wake forms near the sphere downstream surface, which transitions to turbulence further downstream. At a Reynolds number of approximately 200,000 the flow undergoes what's referred to as the drag crisis. At this Reynolds number the boundary layer transitions from laminar to turbulent on the sphere surface and, thus, separates further downstream. The form drag on the sphere, which is the primary contributor to the overall drag, decreases as a result and the drag coefficient drops significantly. This decrease in drag coefficient is discussed further in the following notes. Reynolds numbers larger than 200,000 result in a nearly fully turbulent boundary layer.

Notes:

(1) The periodic shedding of vortices off an object results in periodic forces exerted on the object in the spanwise direction. The Tacoma Narrows bridge disaster from 1940 (Figure 9.32) occurred because a structural natural frequency of the bridge matched the frequency of the shedding vortices, causing the bridge to resonant and eventually collapse.

(2) Experimental measurements have shown that the dimensionless frequency of the shedding vortices, f, expressed as a Strouhal number, i.e., $\mathrm{St} = fD/V$, remains relatively constant at 0.2 for $100 < \mathrm{Re}_D < 1 \times 10^6$. The fact that the Strouhal number is insensitive to the Reynolds number over a wide range of Reynolds numbers has been used to design a type of flow velocity meter known as a vortex flow meter (Figure 9.34). By measuring the frequency of the forces acting on the obstruction (of known size) and knowing that the Strouhal number is approximately equal to 0.2, the flow velocity can be estimated.

(3) The drag coefficient acting on a sphere is shown in Figure 9.35. Commonly-used curve fits for the drag coefficient are,

$$\mathrm{Re}_D < 1: \quad C_D = \frac{24}{\mathrm{Re}_D} \quad \text{(Stokes' drag law)}, \tag{9.196}$$

$$\mathrm{Re}_D < 5: \quad C_D = \frac{24}{\mathrm{Re}_D}\left(1 + \frac{3}{16}\mathrm{Re}_D\right) \quad \text{(Oseen's approximation)}, \tag{9.197}$$

$$0 \leq \mathrm{Re}_D \leq 2 \times 10^5: \quad C_D = \frac{24}{\mathrm{Re}_D} + \frac{6}{\left(1 + \sqrt{\mathrm{Re}_D}\right)} + 0.4, \tag{9.198}$$

$$0 \leq \mathrm{Re}_D < 2 \times 10^5: \quad C_D = 0.44 \quad \text{(Newton's Law)}. \tag{9.199}$$

The only analytically-derived expression for the drag coefficient is for the Stokes flow regime and Oseen's approximation. The remainder of the drag coefficient relations are empirical curve fits. Note the abrupt decrease in the drag coefficient at the drag crisis, which occurs at a Reynolds number of 200,000. This is where the boundary layer transitions to turbulence, delaying separation and decreasing the form drag. The onset of the drag crisis is dependent on the surface roughness (Figure 9.36). Increasing roughness causes the transition to turbulence to occur sooner and moves the drag crisis to a smaller Reynolds number. The dimples on a golf ball serve this same purpose. By decreasing the drag coefficient on the ball, the ball will travel further (and make golf ball manufacturers more money!).

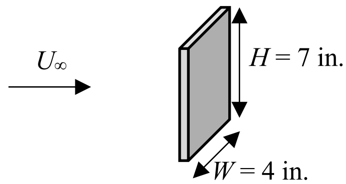
Notes:

(a) Interestingly, the Reynolds numbers for a 95 mph baseball, a 170 mph golf ball, a 100 mph cricket ball, and a 140 mph tennis ball are all near the drag crisis.

(4) Drag coefficients for irregular shapes are usually found experimentally or, in some cases, computationally. Figures 9.37 and 9.38 give drag coefficients for a variety of objects.

Estimate the wind force on your hand when you hold it out of your car window while driving 55 mph. What would the force be if you held your hand out of the window of a jet flying at 550 mph?

SOLUTION:
Model your hand as a rectangular flat plate oriented normal to the flow as shown in the following figure.



The drag force on your hand (the plate) is,

$$D = c_D \frac{1}{2}\rho U_\infty^2 A, \tag{1}$$

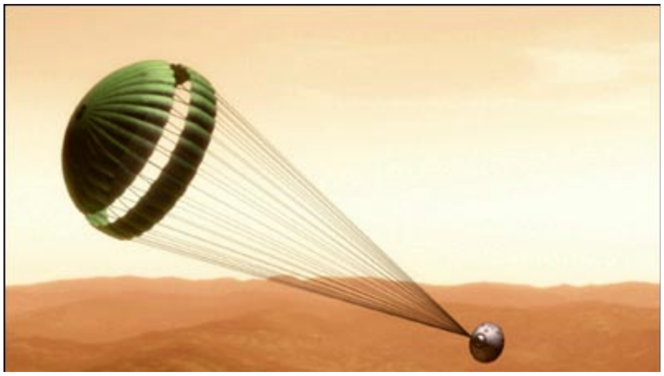
where,

$\rho_{\text{air}} = 2.38*10^{-3}$ slug/ft^3,
$\mu_{\text{air}} = 3.737*10^{-7}$ slug/(ft.s),
$U_{\infty,1}$ = 55 mph = 80.7 ft/s,
$U_{\infty,2}$ = 550 mph = 807 ft/s,
$A = WH$ = (4 in.)(7 in.) = 28 in.^{2} = 0.19 ft^2,
$c_{D,\text{flat plate}} \approx 1.2$ (obtained from a drag coefficient table for a flat plate),
for H/W = 1.75,
$\text{Re}_{Dh} = U_\infty D_h/\nu$ (Reynolds number based on a hydraulic diameter),
$D_h = 4LW/[2(W+H)]$ = 5.1 in. = 0.42 ft (hydraulic diameter),
=> $\text{Re}_{Dh,1}$ = 217,000, $\text{Re}_{Dh,2}$ = 2,170,000 (these values aren't used in the calculation other than to ensure the drag coefficient in the table is in the correct range),

=> $\boxed{D = 1.80 \text{ lb}_f \text{ at 55 mph}}$ and $\boxed{D = 180 \text{ lb}_f \text{ at 550 mph}}$.

Note that at 550 mph compressibility of the air would be significant and should be included in the drag calculations. Furthermore, the upstream air density would be smaller than the sea level value since the jet would be at an elevation higher than sea level. Thus, the drag estimate at 550 mph is questionable.

A parachute was used during part of the landing sequence to deposit the *Spirit* rover on the Martian surface. The parachute had a fully-open, projected diameter of 14.1 m and was designed to slow the landing package (lander and rover) to a terminal speed of 65 m/s (retro-rockets were used to bring the landing package to a near zero vertical velocity). If the mass of the landing package was 544 kg, what was the drag coefficient for the parachute? Assume the gravitational acceleration on Mars is 3.72 m/s^2 and that the density of the Martian atmosphere near the surface is 0.016 kg/m^3.



SOLUTION:

At terminal speed, the weight of the landing package must be balanced by the drag acting on the parachute (neglecting the drag on the landing package itself),

$$\sum F_y = 0 = D - W \, , \tag{1}$$

where,

$$D = c_D \tfrac{1}{2} \rho V_T^2 A \, , \tag{2}$$

$$W = mg \, . \tag{3}$$

Substitute and re-arrange to solve for the drag coefficient,

$$c_D \tfrac{1}{2} \rho V_T^2 A - mg = 0 \tag{4}$$

$$\boxed{c_D = \frac{mg}{\tfrac{1}{2} \rho V_T^2 A}} \tag{5}$$

Using the given data,

m = 544 kg
g = 3.72 m/s^2
ρ = 0.016 kg/m^3
V_T = 65 m/s
A = 156.1 m^2 (=π/4*(14.1 m)^2)
$\Rightarrow$ $\boxed{c_D = 0.38}$

Video solution: https://www.youtube.com/watch?v=hQ69_0ybIys

In the book/movie *The Martian*, the mission of a crew of astronauts is derailed by a massive Martian windstorm. If the Martian atmosphere has a density of 0.016 kg/m^3 and the wind speed is 26.8 m/s (= 60 mph), what is the drag force acting on astronaut Mark Watney? Based on wind tunnel testing, assume that the drag coefficient multiplied by the frontal projected area of a typical person is $C_D A$ = 0.84 m^2 (see, for example, Table 7.3 in White, F.M., *Fluid Mechanics*, 7$^{\text{th}}$ ed., McGraw-Hill).

What wind speed on Earth would produce an equivalent drag force?

SOLUTION:
The drag force is given by,

$$D = C_D \tfrac{1}{2}\rho_{\text{Mars}} V^2 A\,, \tag{1}$$

where,

$C_D A$ = 0.84 m^2 (given),
ρ_{Mars} = 0.016 kg/m^3,
V = 26.8 m/s,
=> $\boxed{D = 4.8 \text{ N } (= 1.1 \text{ lb}_f)}$

Thus, we see that the author took considerable artistic liberty in portraying the damage caused by a Martian windstorm.

To determine the wind speed on Earth that would cause the same drag force, set the drag forces for Mars and Earth equal,

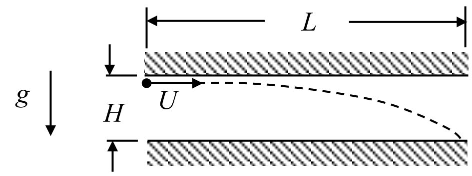
$$C_D \tfrac{1}{2}\rho_{\text{Mars}} V_{\text{Mars}}^2 A = C_D \tfrac{1}{2}\rho_{\text{Earth}} V_{\text{Earth}}^2 A\,, \tag{2}$$

$$V_{\text{Earth}} = V_{\text{Mars}}\sqrt{\frac{\rho_{\text{Mars}}}{\rho_{\text{Earth}}}}\,. \tag{3}$$

Using ρ_{Earth} = 1.23 kg/m^3, $\boxed{V_{\text{Earth}} = 3.1 \text{ m/s } (= 6.8 \text{ mph})}$, which corresponds to a light breeze.

Video solution: https://www.youtube.com/watch?v=w-uheRu1QEk

Gravity settling tanks are sometimes used to separate particles from a fluid stream. Estimate the critical length, L, for capturing a particle by gravity settling in the channel shown below. Express your answer in terms of the channel height, H, the fluid velocity, U, the fluid density, ρ_f, the fluid dynamic viscosity, μ_f, particle density, ρ_p, the particle diameter, d, and the acceleration due to gravity, g. You may assume that the particle diameter is very small and that the fluid velocity profile in the channel is uniform. How will the length L change if the particle diameter is doubled?



SOLUTION:

In order to capture the particle, we want the particle to settle on the base before passing through the device, i.e.,

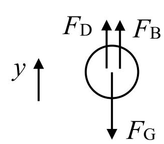
$t_{\text{settling}} < t_{\text{residence}}$

where,

$t_{\text{residence time}} = L/U$ (chamber length / fluid velocity)

$t_{\text{settling time}} = H/U_{\text{pt}}$ (settling height / particle terminal velocity)

The particle terminal velocity can be determined by considering a free body diagram acting on the particle. The forces acting on the particle include a drag force, F_D, a buoyant force, F_B, and a gravitational force, F_G.



$$\sum F_y = 0 = F_D + F_B - F_G$$

$$0 = 3\pi\mu_f U_{pt} d + \rho_f \frac{\pi}{6} d^3 g - \rho_p \frac{\pi}{6} d^3 g$$

Note that Stokes drag has been assumed for the particle drag force since the particle Reynolds number is assumed to be very small. Solving the previous equation for the terminal velocity, U_{pt}, gives,

$$U_{pt} = \frac{\left(\rho_s - \rho_f\right) g d^2}{18\mu_f}$$

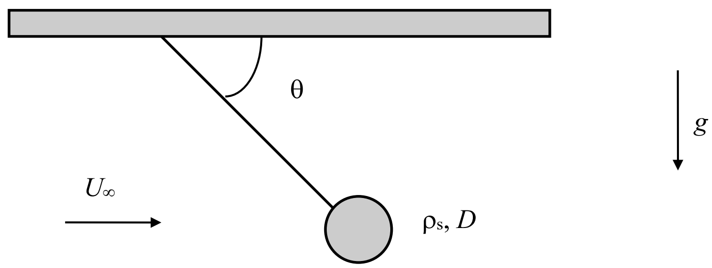
Since the settling time must be less than the residence time,

$$\frac{L}{U} > \frac{18\mu_f H}{\left(\rho_s - \rho_f\right) g d^2} \Rightarrow \boxed{\frac{L}{H} > \frac{18\mu_f U}{\left(\rho_s - \rho_f\right) g d^2}}$$

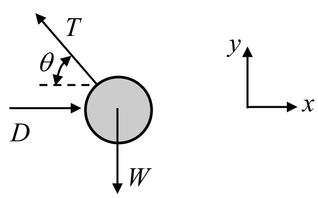
The length of the settling chamber, L, will decrease by a factor of four if the particle size doubles.

A heavy sphere attached to a string will hang at an angle, θ, when immersed in a stream of velocity U_∞ as shown in the figure.

a. Derive an expression for θ as a function of the sphere and flow properties.
b. What is θ if the sphere is steel (SG=7.86) of diameter 3 cm and the flow is sea-level standard air at U_∞ = 40 m/s? Neglect the string drag.
c. For the same parameters as in part (b), at what velocity will the angle be 45°?



SOLUTION:
Draw a free body diagram for the sphere and balance forces in the vertical and horizontal directions.



$$\sum F_y = 0 = T\sin\theta - W \;\Rightarrow\; T = \frac{W}{\sin\theta} = \frac{mg}{\sin\theta} \tag{1}$$

$$\sum F_x = 0 = -T\cos\theta + D \;\Rightarrow\; T = \frac{D}{\cos\theta} = \frac{c_D \frac{1}{2}\rho_a U_\infty^2 \frac{\pi}{4} d^2}{\cos\theta} \tag{2}$$

Set the tensions equal in Eqs. (1) and (2) and simplify.

$$\frac{mg}{\sin\theta} = \frac{c_D \frac{1}{2}\rho U_\infty^2 \frac{\pi}{4} d^2}{\cos\theta} \;\Rightarrow\; \tan\theta = \frac{mg}{c_D \frac{1}{2}\rho_a U_\infty^2 \frac{\pi}{4} d^2} = \frac{\rho_S \frac{\pi}{6} d^3 g}{c_D \frac{1}{2}\rho_a U_\infty^2 \frac{\pi}{4} d^2} \tag{3}$$

$$\boxed{\therefore \tan\theta = \frac{4}{3}\frac{1}{c_D}\left(\frac{\rho_S}{\rho_a}\right)\left(\frac{gd}{U_\infty^2}\right)} \tag{4}$$

where the drag coefficient, c_D, is a function of the Reynolds number based on the sphere diameter, *i.e.*, $\mathrm{Re}_d = U_\infty d/\nu_a$.

For the given data,

SG = 7.86 $\Rightarrow$ ρ_S = 7860 kg/m^3
U_∞ = 40 m/s
d = 0.03m
ρ_a = 1.23 kg/m^3
g = 9.81 m/s^2
ν_a = 1.1*10^{-5} m^2/s
$\Rightarrow$ Re_d = 110,000 $\Rightarrow$ c_D = 0.44
$\Rightarrow$ $\boxed{\theta = 74°}$

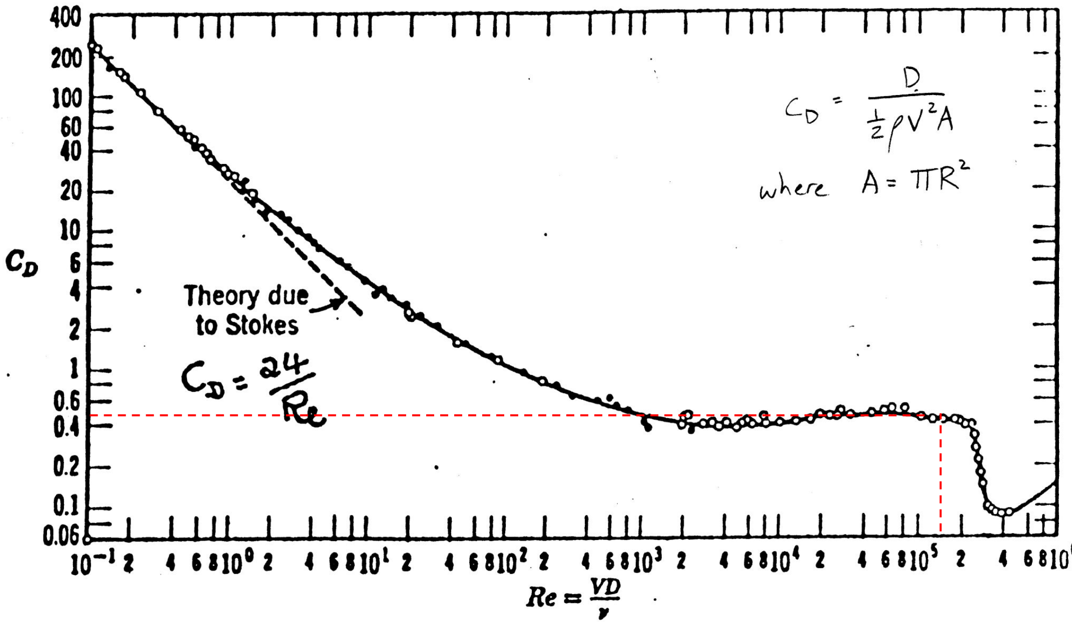


Fig. 8.32 Drag coefficient of a sphere as a function of Reynolds number (Ref. 13).

To find the wind speed corresponding to the given angle, we need to iterate to a solution since the drag coefficient is a complex function of the flow speed. The following algorithm can be used for iteration. Note that other algorithms may also be possible.

1. Guess a value for the speed $U_{\infty,\text{guess}}$.
2. Calculate the Reynolds number, $\text{Re} = U_\infty d/\nu$.
3. Use the plot shown above to determine the drag coefficient, c_D.
4. Calculate the speed $U_{\infty,\text{calc}}$ using a re-arranged Eq. (4) and the given angle,

$$U_\infty = \sqrt{\frac{4}{3}\frac{1}{c_D}\frac{gd}{\tan\theta}\left(\frac{\rho_S}{\rho_a}\right)}. \tag{5}$$

5. If $U_{\infty,\text{calc}} = U_{\infty,\text{guess}}$ (to within some acceptable tolerance), then stop the iterations because the solution has been found. If $U_{\infty,\text{calc}} \neq U_{\infty,\text{guess}}$, then let $U_{\infty,\text{guess}} = U_{\infty,\text{calc}}$ and repeat steps 2 – 5.

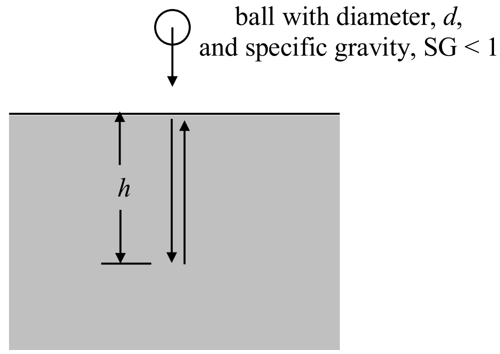
For example, starting with $U_{\infty,\text{guess}} = 1.0$ m/s.

ρ_s [kg/m^3] =	7860
d [m] =	0.03
ρ_a [kg/m$_3$] =	1.23
g [m/s^2] =	9.81
ν_a [m$_2$/s] =	0.000011
θ [deg] =	45

$U_{\text{inf,guess}}$ [m/s]	Re [-]	c_D [-]	$U_{\text{inf,calc}}$ [m/s]
1.00	2727	0.42	77.21
77.21	210578	0.40	79.51
79.51	216841	0.39	80.68
80.68	220025	0.38	81.36
81.36	221890	0.37	81.79
81.79	223065	0.37	82.07
82.07	223837	0.37	82.26
82.26	224357	0.37	82.40
82.40	224715	0.37	82.49
82.49	224964	0.37	82.55
82.55	225138	0.37	82.60
82.60	225261	0.37	82.63
82.63	225348	0.37	82.65
82.65	225410	0.37	82.67
82.67	225453	0.37	82.68
82.68	225485	0.37	82.69
82.69	225507	0.37	82.69
82.69	225523	0.37	82.70
82.70	225534	0.37	82.70

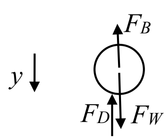
Thus, the flow speed for this case is $\boxed{U_\infty = 82.7 \text{ m/s}}$.

A buoyant ball of specific gravity, SG< 1, dropped into water at an impact speed, V_0, penetrates a distance, h, into the water and pops out again. Assuming a constant drag coefficient, derive an expression for h as a function of the system properties. How deep will a 5 cm diameter ball with SG=0.5 and C_D= 0.47 penetrate if it enters water at a speed of 10 m/s? You may neglect splashing, air entrainment, and added mass effects in your analysis.



SOLUTION:

Apply Newton's 2nd Law to the ball:



$$m\frac{dV}{dt} = F_W - F_B - F_D \tag{1}$$

where m is the ball mass, y is the depth of the ball from the free surface, F_W is the ball weight, F_B is the buoyant force acting on the ball, and F_D is the drag force acting on the ball.

$$m = \rho_S \tfrac{\pi}{6} d^3 \tag{2}$$

$$F_W = mg \tag{3}$$

$$F_B = \rho_F \tfrac{\pi}{6} d^3 g \tag{4}$$

$$F_D = C_D \tfrac{1}{2} \rho_F V^2 \tfrac{\pi}{4} d^2 \tag{5}$$

where ρ_S and ρ_F are the ball and fluid densities, respectively, d is the ball diameter, g is the acceleration due to gravity, and C_D is the drag coefficient. Substitute Eqs. (2) - (5) into Eq. (1) and simplify.

$$\rho_S \tfrac{\pi}{6} d^3 \frac{dV}{dt} = \rho_S \tfrac{\pi}{6} d^3 g - \rho_F \tfrac{\pi}{6} d^3 g - C_D \tfrac{1}{2} \rho_F V^2 \tfrac{\pi}{4} d^2 \tag{6}$$

$$\frac{dV}{dt} = \left(1 - \frac{\rho_F}{\rho_S}\right) g - \frac{3}{4} C_D \frac{\rho_F}{\rho_S} \frac{1}{d} V^2 \tag{7}$$

$$\frac{dV}{dt} = \underbrace{\left(1 - \frac{1}{SG}\right)}_{=-\alpha} g - \underbrace{\frac{3}{4} C_D \frac{1}{SG} \frac{1}{d}}_{=\beta} V^2 \tag{8}$$

$$\frac{dV}{dt} = -\left(\alpha + \beta V^2\right) \tag{9}$$

where

$$\alpha = \left(\frac{1}{SG} - 1\right) g \tag{10}$$

$$\beta = \frac{3}{4} C_D \frac{1}{SG} \frac{1}{d} \tag{11}$$

Make Eq. (9) dimensionless using a dimensionless velocity and time:

$$V' = \sqrt{\frac{\beta}{\alpha}}V \tag{12}$$

$$t' = \sqrt{\alpha\beta}t \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (9) gives:

$$\frac{d\left(\sqrt{\frac{\alpha}{\beta}}V'\right)}{d\left(t'/\sqrt{\alpha\beta}\right)} = -\left[\alpha + \beta\left(\sqrt{\frac{\alpha}{\beta}}V'\right)^2\right] \tag{14}$$

$$\frac{dV'}{dt'} = -\left(1 + V'^2\right) \tag{15}$$

The initial condition for Eq. (9) is:

$$V'\left(t' = 0\right) = V_0' \tag{16}$$

where

$$V_0' = \sqrt{\frac{\beta}{\alpha}}V_0 \tag{17}$$

Solving Eq. (9) using an integration table or a symbolic ODE solver (e.g., MAPLE) gives:

$$\int_{V'=V_0'}^{V'=V'} \frac{dV'}{1 - V'^2} = \int_{t'=0}^{t'=t'} dt' \tag{18}$$

$$-\tan^{-1}\left(V'\right) + \tan^{-1}\left(V_0'\right) = t' \tag{19}$$

$$V' = \tan\left[\tan^{-1}\left(V_0'\right) - t'\right] \tag{20}$$

Note that the maximum depth of the ball, h, occurs when $V'(t' = T') = 0$.

$$T' = \tan^{-1}\left(V_0'\right) \tag{21}$$

The maximum dimensionless depth of the ball, h' (= βh) is found by integrating Eq. (19) in time.

$$\int_{y'=0}^{y'=h'} dy' = \int_{t'=0}^{t'=T'} \tan\left[\tan^{-1}\left(V_0'\right) - t'\right]dt' \tag{22}$$

$$h' = -\frac{1}{2}\ln\left(1 + \left\{\tan\left[\tan^{-1}\left(V_0'\right) + T'\right]\right\}^2\right) + \frac{1}{2}\ln\left(1 + V_0'^2\right) \tag{23}$$

$$\boxed{\therefore h' = \frac{1}{2}\ln\left(\frac{1 + V_0'^2}{1 + \left\{\tan\left[2\tan^{-1}\left(V_0'\right)\right]\right\}^2}\right)} \tag{24}$$

A plot of the dimensionless velocity and position as functions of dimensionless time are shown in Fig. 1 using the data given in the problem statement.

SG = 0.5
d = 0.05 m
g = 9.81 m/s^2
C_D = 0.47
V_0 = 10 m/s
$\Rightarrow$ α = 9.81 m/s^2 and β = 14.1 m^{-1} and V_0' = 11.99

Using the given data, the time at which the ball achieves its maximum depth is:

$$T' = 1.49 \Rightarrow \boxed{T = 0.13\ \text{s}} \quad (25)$$

The maximum depth is:

$$h' = 2.47 \Rightarrow \boxed{h = 0.18\ \text{m}} \quad (26)$$

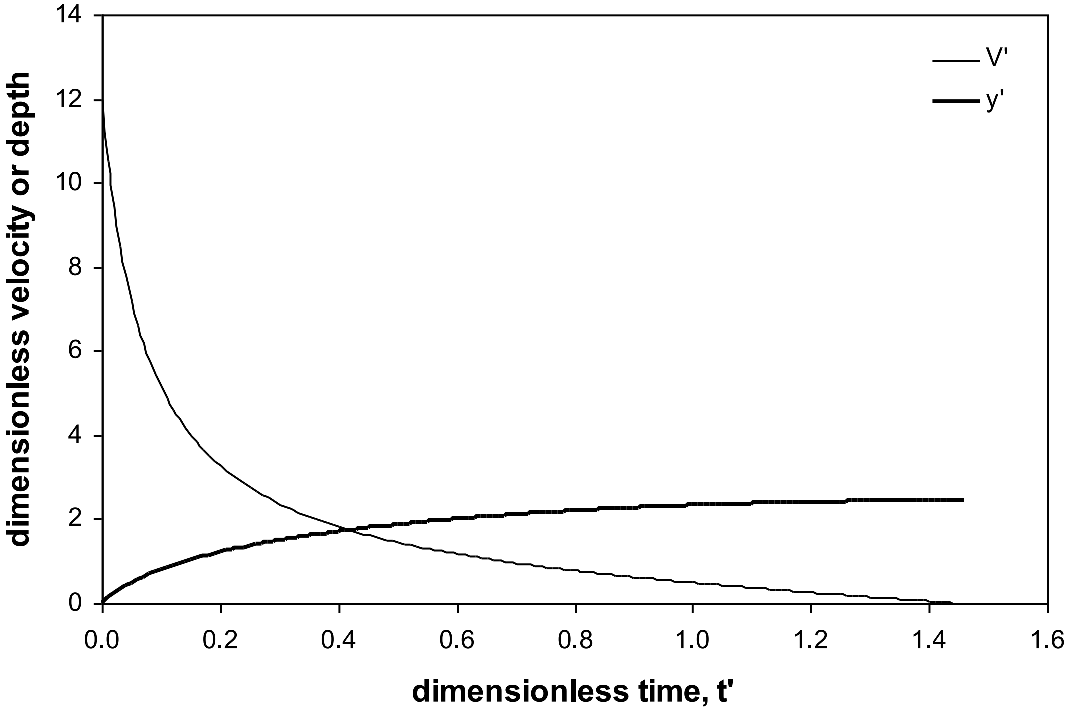
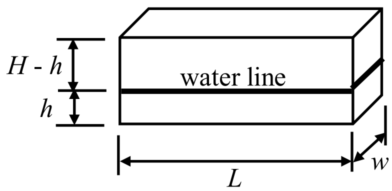


Figure 1. The dimensionless velocity, V', and dimensionless position, y', plotted as a function of dimensionless time, t', for $\alpha = 9.81$ m/s^2, $\beta = 14.1$ m^{-1}, and $V_0' = 11.99$.

In this analysis a constant drag coefficient was assumed. This is a reasonable assumption over the range $1000 < Re_d < 200{,}000$. A more accurate analysis would take into account the variation in drag coefficient with speed (and would also require a computational solution). In addition to splashing and air entrainment effects (air entrained into the wake of the ball), added mass effects should also be taken into account. When accelerating (or decelerating) an object in a fluid, we must also accelerate (or decelerate) the surrounding fluid. This extra force required to accelerate the surrounding fluid can be incorporated into the object mass and is known as an “added mass” or “virtual mass.”

A barge weighing 8820 kN that is 10 m wide, 30 m long, and 7 m tall has come free from its tug boat in the Mississippi River. It is in a section of river that has a current of 1 m/s. In addition, there is a wind blowing straight upriver at 10 m/s . Assume that the drag coefficient is 1.3 for both the part of the barge in the wind as well as the part below the water. The drag coefficients for the water-exposed and air-exposed portions of the barge are based on the water and air wetted areas, respectively. Determine the speed at which the barge will be steadily moving. Is it moving upriver or downriver?

SOLUTION:



First determine the wetted areas above and below the waterline. Balancing forces in the vertical direction on the barge,

$$\sum F_{\text{vert}} = 0 = -W + \rho_{H_2O} g L w h \, , \tag{1}$$

where W is the barge weight and the second term on the right hand side is the buoyant force, with h being the draft of the barge (the depth below the water). Solving for h gives,

$$h = \frac{W}{\rho_{H_2O} g L w} \, . \tag{2}$$

Using the given values,

$W = 8820$ kN,
$\rho_{H2O} = 1000$ kg/m^3,
$g = 9.81$ m/s^2,
$L = 30$ m,
$w = 10$ m,
=> $\underline{h = 3.00 \text{ m.}}$

Now determine the wetted areas below and above water,

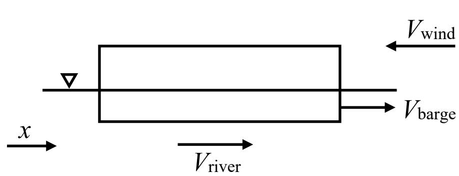
$$A_{\substack{\text{wetted,}\\ \text{below}}} = Lw + 2Lh + 2wh \, , \tag{3}$$

$$A_{\substack{\text{wetted,}\\ \text{above}}} = Lw + 2L\left(H - h\right) + 2w\left(H - h\right) . \tag{4}$$

Using the given values,

$\underline{A_{\text{wetted,below}} = 540 \text{ m}^2}$,
$\underline{A_{\text{wetted,above}} = 620 \text{ m}^2}$.

Now balance forces in the horizontal direction. These forces include the drag caused by the river and the drag caused by the wind. Assume that the barge is moving in the same direction as the river (downstream), as shown in the figure below.



$$\sum F_x = 0 = c_{D,\text{below}} \tfrac{1}{2}\rho_{\text{H}_2\text{O}}\left(V_{\text{river}} - V_{\text{barge}}\right)^2 A_{\substack{\text{wetted,}\\\text{below}}} - c_{D,\text{above}} \tfrac{1}{2}\rho_{\text{air}}\left(V_{\text{wind}} + V_{\text{barge}}\right)^2 A_{\substack{\text{wetted,}\\\text{above}}}, \tag{5}$$

Solve for V_{barge}, noting that the drag coefficients are the same above and below the waterline (given in the problem statement),

$$\rho_{\text{H}_2\text{O}}\left(V_{\text{river}} - V_{\text{barge}}\right)^2 A_{\substack{\text{wetted,}\\\text{below}}} = \rho_{\text{air}}\left(V_{\text{wind}} + V_{\text{barge}}\right)^2 A_{\substack{\text{wetted,}\\\text{above}}}, \tag{6}$$

$$V_{\text{river}}^2 - 2V_{\text{river}}V_{\text{barge}} + V_{\text{barge}}^2 = \underbrace{\left(\frac{\rho_{\text{air}}}{\rho_{\text{H}_2\text{O}}}\right)\left(\frac{A_{\substack{\text{wetted,}\\\text{above}}}}{A_{\substack{\text{wetted,}\\\text{below}}}}\right)}_{=c}\left(V_{\text{wind}}^2 + 2V_{\text{wind}}V_{\text{barge}} + V_{\text{barge}}^2\right), \tag{7}$$

$$\left(1-c\right)V_{\text{barge}}^2 - 2\left(V_{\text{river}} + cV_{\text{wind}}\right)V_{\text{barge}} + V_{\text{river}}^2 - cV_{\text{wind}}^2 = 0, \tag{8}$$

$$\underbrace{1}_{=A}V_{\text{barge}}^2 + \underbrace{\frac{-2\left(V_{\text{river}} + cV_{\text{wind}}\right)}{\left(1-c\right)}}_{=B}V_{\text{barge}} + \underbrace{\frac{\left(V_{\text{river}}^2 - cV_{\text{wind}}^2\right)}{\left(1-c\right)}}_{=C} = 0, \tag{9}$$

$$V_{\text{barge}} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{10}$$

Using the give data,

$\rho_{\text{air}} = 1.23$ kg/m^3,
$V_{\text{river}} = 1$ m/s,
$V_{\text{wind}} = 10$ m/s,
=> $c = 1.41*10^{-3}$, $A = 1$, $B = -2.03$ m/s, $C = 8.60*10^{-1}$ m^2/s^2
=> $\underline{V_{\text{barge}} = 1.43 \text{ m/s}, 0.601 \text{ m/s.}}$

Note that it's not possible for the barge to move faster than the river's speed of 1 m/s, so $V_{\text{barge}} \neq 1.43$ m/s. Thus, the correct answer is $\boxed{V_{\text{barge}} = 0.601 \text{ m/s (downstream)}}$.

If we had assumed that V_{barge} was moving upstream (same direction as V_{wind}), then Eq. (5) would be,

$$\sum F_x = 0 = c_{D,\text{below}} \tfrac{1}{2}\rho_{\text{H}_2\text{O}}\left(V_{\text{river}} + V_{\text{barge}}\right)^2 A_{\substack{\text{wetted,}\\\text{below}}} - c_{D,\text{above}} \tfrac{1}{2}\rho_{\text{air}}\left(V_{\text{wind}} - V_{\text{barge}}\right)^2 A_{\substack{\text{wetted,}\\\text{above}}}, \tag{11}$$

which would simplify to,

$$V_{\text{barge}}^2 + \frac{2\left(V_{\text{river}} + cV_{\text{wind}}\right)}{\left(1-c\right)}V_{\text{barge}} + \frac{\left(V_{\text{river}}^2 - cV_{\text{wind}}^2\right)}{\left(1-c\right)} = 0. \tag{12}$$

Solving this equation gives,

$\underline{V_{\text{barge}} = -0.601 \text{ m/s}, -1.43 \text{ m/s}}$.

Thus, we see that the original choice of direction for V_{barge} (upstream) was incorrect and the barge is actually moving downstream. As in the previous discussion, the barge cannot move faster than the river speed so the correct speed is 0.601 m/s.

Video solution: https://www.youtube.com/watch?v=OWJzjsObeNE

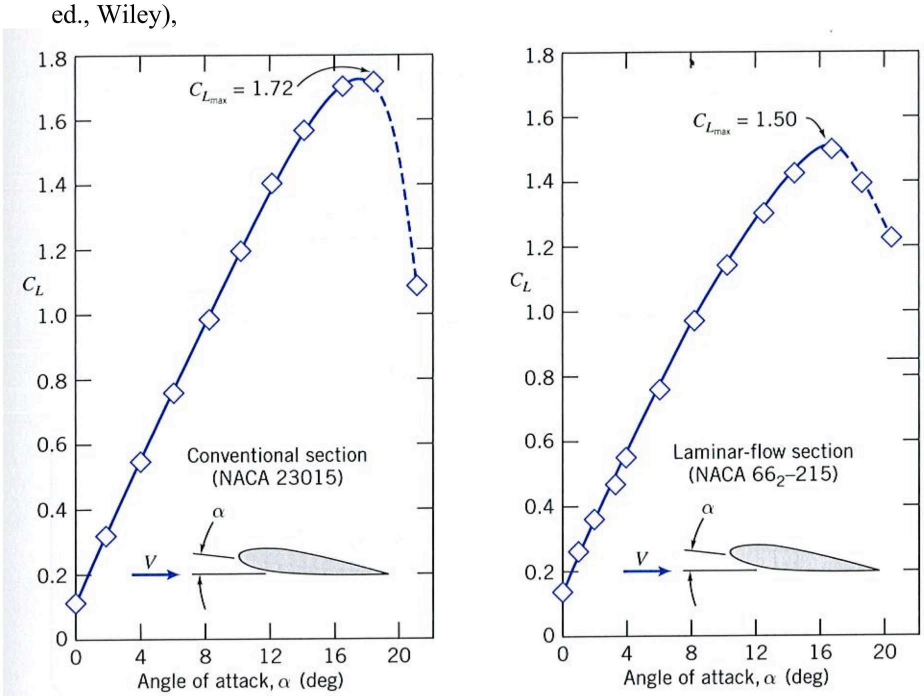
Some cars come with a rear "spoiler" (actually an upside-down airfoil) mounted on the rear of the vehicle that is supposed to increase the down force on the car and improve traction. Calculate a typical down force caused by a rear wing used on a passenger vehicle.

SOLUTION:
The lift force is given by,

$$L = C_L \tfrac{1}{2}\rho V^2 A\,, \tag{1}$$

where,

A = 2 ft^2 (= 0.186 m^2), assuming a span of 4 ft and a chord length of 0.5 ft (note that this is a planform area),
ρ = 1.23 kg/m^3,
V = 24.6 m/s (= 55 mph),
C_L = 1.4, (a typical value based on Fig. 9.17 from Pritchard et al., *Introduction to Fluid Mechanics*, 8th ed., Wiley),



(*a*) Lift coefficient vs. angle of attack

=> $\boxed{L = 96.9\text{ N } (= 21.8\text{ lb}_f)}$

Thus, we see the spoiler produces very little down force on the vehicle.

To produce a down force of 200 lb$_f$ (= 890 N), the car would need to travel at a speed of 70.7 m/s (= 158 mph).

Note that rear spoilers are sometimes used to direct airflow downward to help reduce the size of the trailing wake and thus reduce drag.

9.11. Review Questions

(1) What scaling arguments are used in deriving the boundary layer equations?
(2) What are the appropriate boundary conditions for the boundary layer equations?
(3) What restrictions are there on the Reynolds number for using the boundary layer equations?
(4) Describe how the pressure within a boundary layer is determined.
(5) Describe, in words, the approach used in deriving the Blasius solution to the boundary layer equations.
(6) What assumptions are made in the Blasius boundary layer solution? (e.g., Reynolds number limitations, pressure gradients, free stream conditions, surface curvature, etc.)
(7) At what Reynolds number (an engineering rule of thumb estimate) does a laminar boundary layer transition to a turbulent boundary layer?
(8) How does the boundary layer thickness vary with the distance from the leading edge of the boundary layer for a flat plate, no pressure gradient boundary layer flow?
(9) What is the expression for the 99% boundary layer thickness resulting from the Blasius solution?
(10) What do the Falkner-Skan boundary layer solutions represent?
(11) What are the boundary conditions used in the Falkner-Skan boundary layer solution?
(12) Give two examples of practical boundary layer solutions that are embedded within the Falkner-Skan general solution.
(13) Can the Kärmän momentum integral equation (KMIE) be used for flows with non-uniform pressure gradients? Turbulent flows? Compressible flows? Unsteady flows?
(14) How might one find the outer flow velocity, U, when using the KMIE?
(15) Describe the typical methodology used when applying the KMIE.
(16) What is the 1/7th power law profile for a turbulent boundary layer?
(17) In which type of boundary layer flow does the shear stress decrease most rapidly? Laminar or turbulent? In which type of flow does the drag increase most rapidly?
(18) Give a physical description of why boundary layer separation occurs.
(19) What defines the point at which boundary layer separation occurs?
(20) Why can't the boundary layer equations be used downstream of boundary layer separation point?
(21) Why do turbulent boundary layers separate further downstream than laminar boundary layers?
(22) What is meant by "favorable" and "adverse" pressure gradients?
(23) Can a boundary layer separate in a favorable pressure gradient flow?
(24) Must boundary layers always separate in an adverse pressure gradient flow?
(25) What are the restrictions in using Thwaites' correlation?
(26) Describe the flow behavior as a function of Reynolds number for flow over a cylinder.
(27) Sketch a plot of drag coefficient as a function of Reynolds number for flow over a sphere. Indicate points of particular interest on the plot. Identify whether the axes are linear or logarithmic.

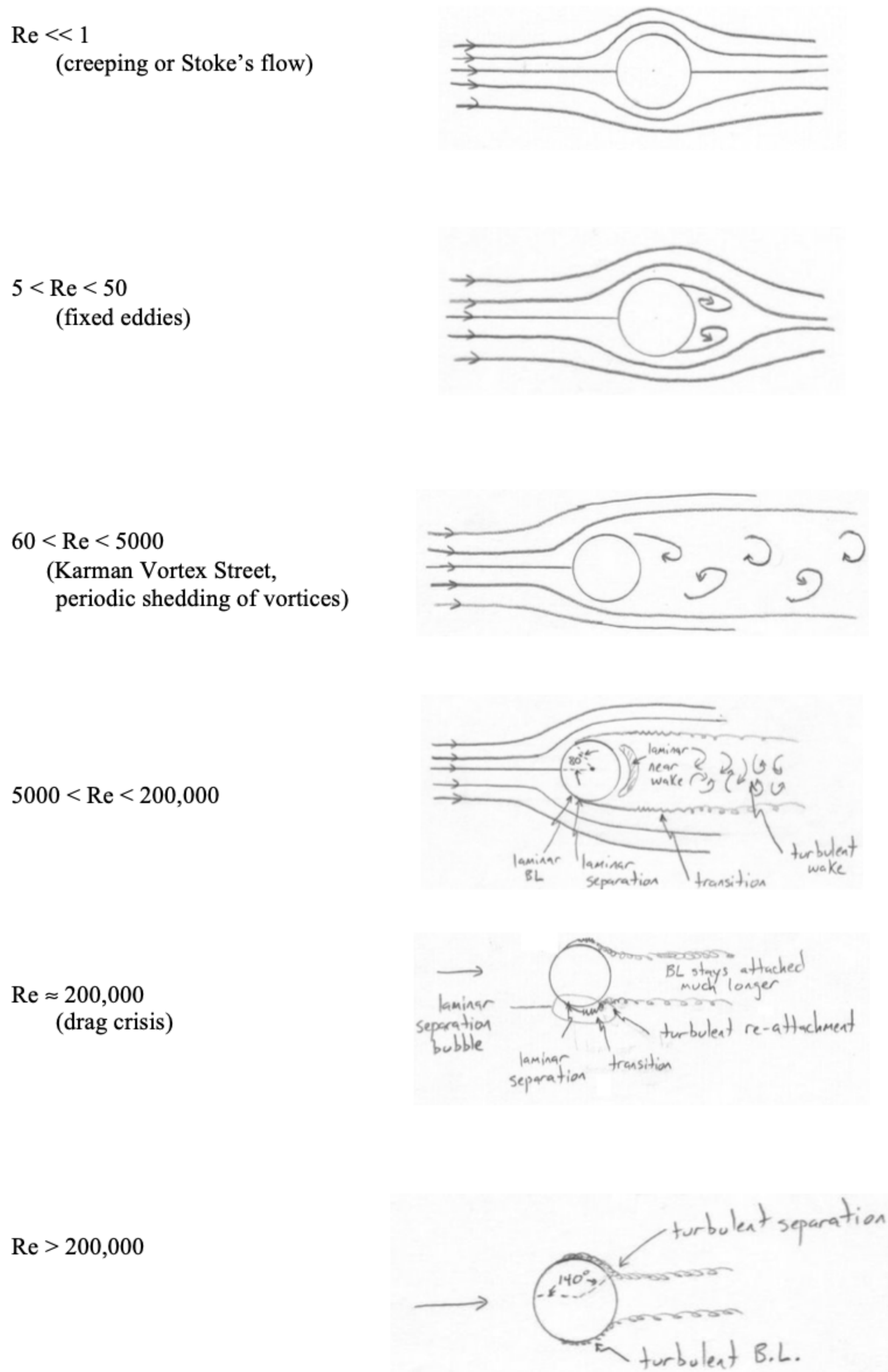


Figure 9.29. Drawings of the different regimes of flow around a sphere as a function of Reynolds number.

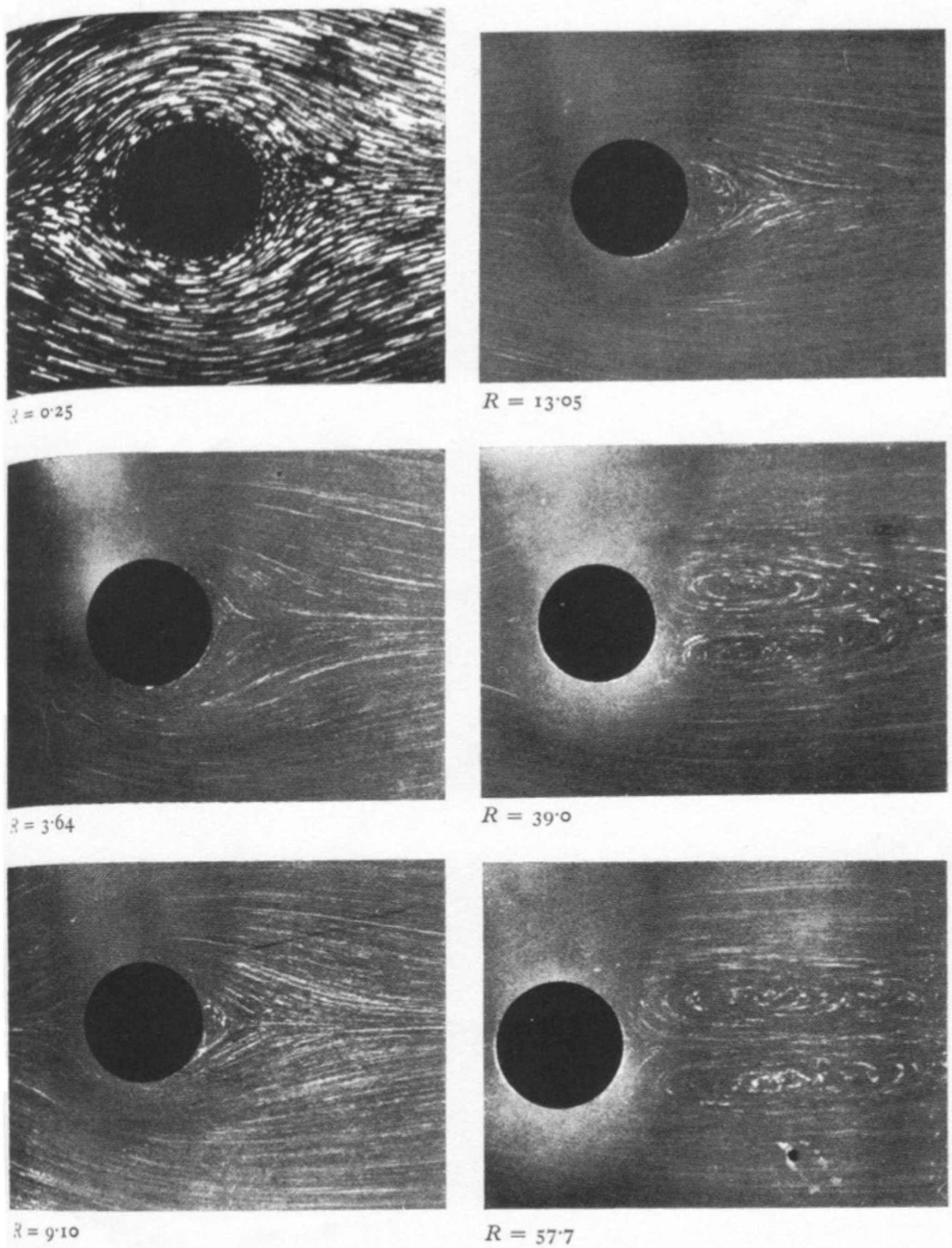
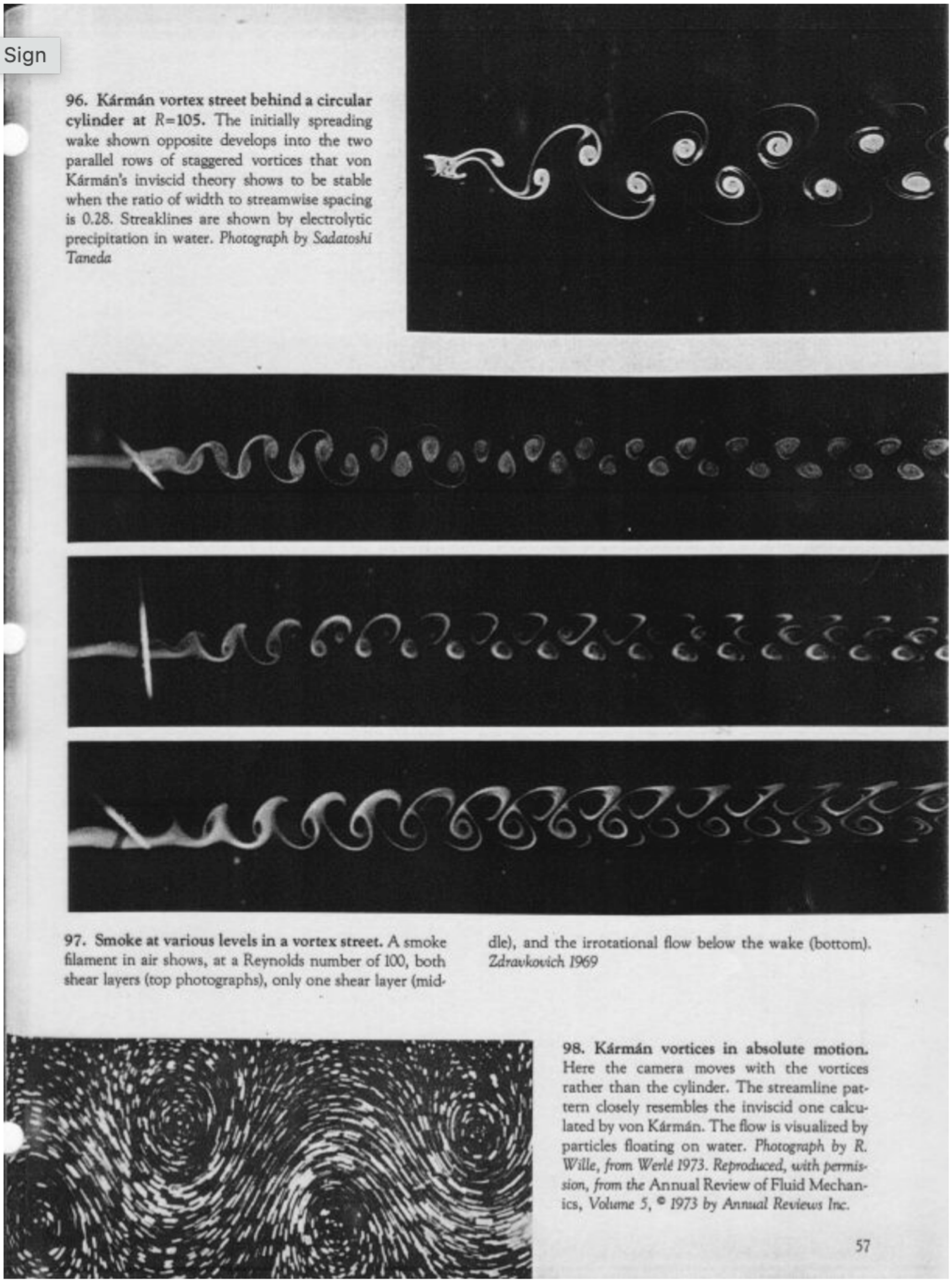


Figure 4.12.1. Streamlines of steady flow (from left to right) past a circular cylinder of radius a; $R = 2aU/\nu$. The photograph at $R = 0{\cdot}25$ (from Prandtl and Tietjens 1934) shows the movement of solid particles at a free surface, and all the others (from Taneda 1956*a*) show particles illuminated over an interior plane normal to the cylinder axis.

(From Batchelor, G.K., *An Introduction to Fluid Dynamics*, Cambridge University Press.)

Figure 9.30. Photographs showing the different regimes of flow around a sphere as a function of Reynolds number (R). These photographs are from Batchelor, G.K., *An Introduction to Fluid Dynamics*, Cambridge University Press.



96. Kármán vortex street behind a circular cylinder at R=105. The initially spreading wake shown opposite develops into the two parallel rows of staggered vortices that von Kármán's inviscid theory shows to be stable when the ratio of width to streamwise spacing is 0.28. Streaklines are shown by electrolytic precipitation in water. *Photograph by Sadatoshi Taneda*

97. Smoke at various levels in a vortex street. A smoke filament in air shows, at a Reynolds number of 100, both shear layers (top photographs), only one shear layer (middle), and the irrotational flow below the wake (bottom). *Zdravkovich 1969*

98. Kármán vortices in absolute motion. Here the camera moves with the vortices rather than the cylinder. The streamline pattern closely resembles the inviscid one calculated by von Kármán. The flow is visualized by particles floating on water. *Photograph by R. Wille, from Werlé 1973. Reproduced, with permission, from the* Annual Review of Fluid Mechanics, *Volume 5, © 1973 by Annual Reviews Inc.*

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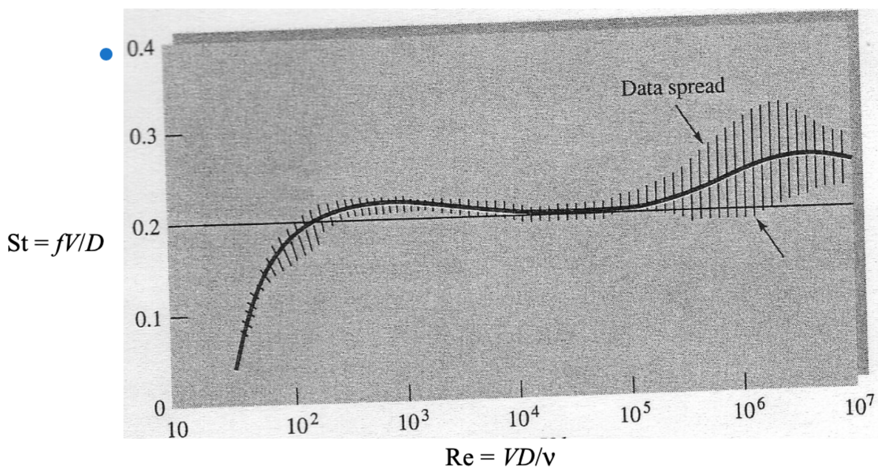
(From Van Dyke, M., *An Album of Fluid Motion*, Parabolic Press.)

FIGURE 9.31. Photographs showing Kármán vortex streets downstream of an immersed object. These photos are from Van Dyke, M., *An Album of Fluid Motion*, Parabolic Press.

Figure 9.32. Photographs of the Tacoma Narrows Bridge prior to failure.



(Figure from White, F.M., *Fluid Mechanics*, McGraw-Hill.)

Figure 9.33. The Strouhal number based on the vortex shedding frequency from a cylinder plotted as a function of the Reynolds number. This figure is from White, F.M., *Fluid Mechanics*, McGraw-Hill.

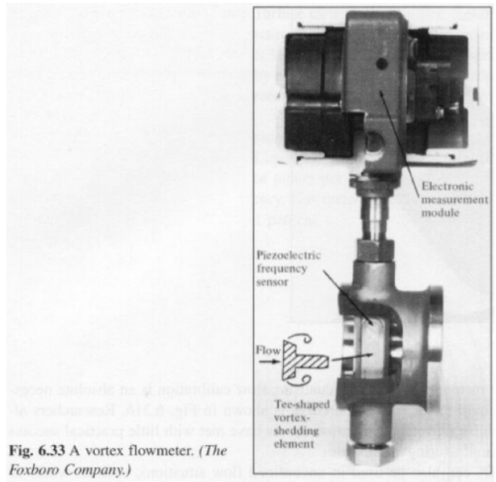


Figure 9.34. A photograph of a vortex flow meter.

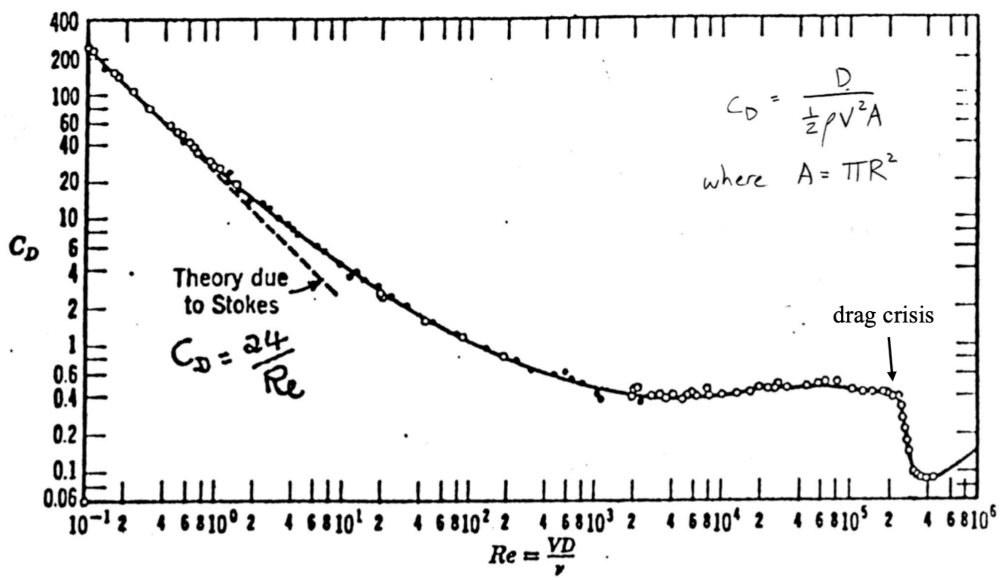


Fig. 8.32 Drag coefficient of a sphere as a function of Reynolds number (Ref. 13).

Figure 9.35. The drag coefficient for a sphere based on the sphere's frontal projected area plotted against the Reynolds number.

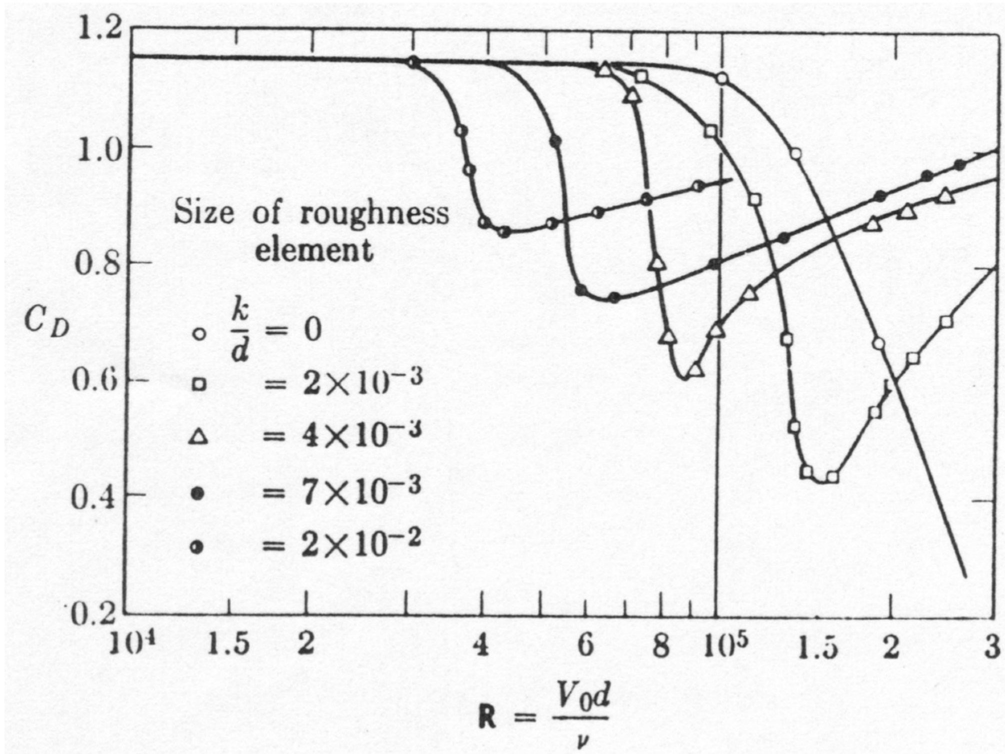


Figure 9.36. The drag coefficient on a cylinder plotted as a function of the Reynolds number for different degrees of surface roughness. Increasing roughness causes the drag crisis to occur at a smaller Reynolds number.

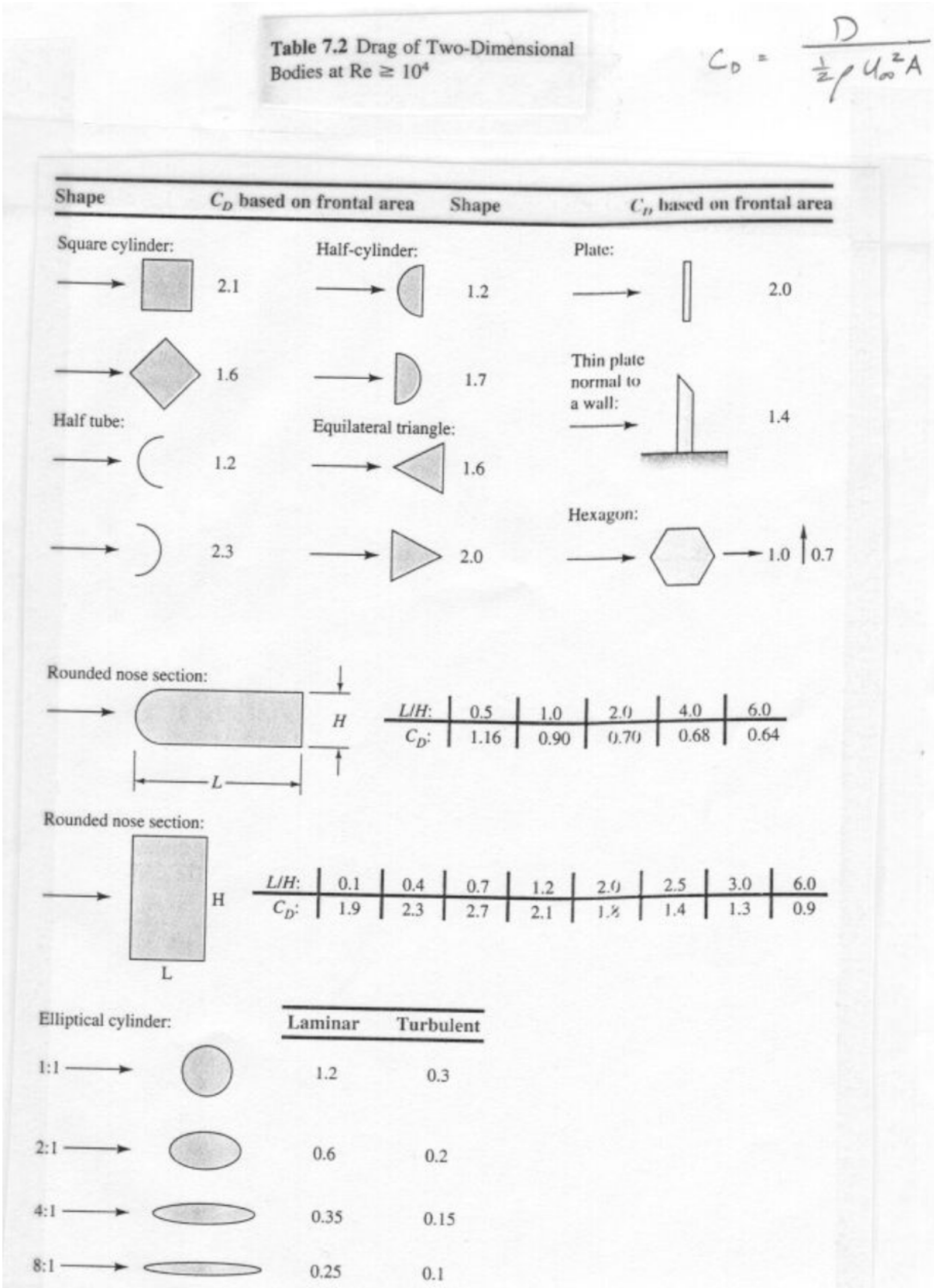


FIGURE 9.37. Drag coefficients for a variety of two-dimensional objects. This table is from White, F.M., *Fluid Mechanics*, 3rd ed., McGraw-Hill.

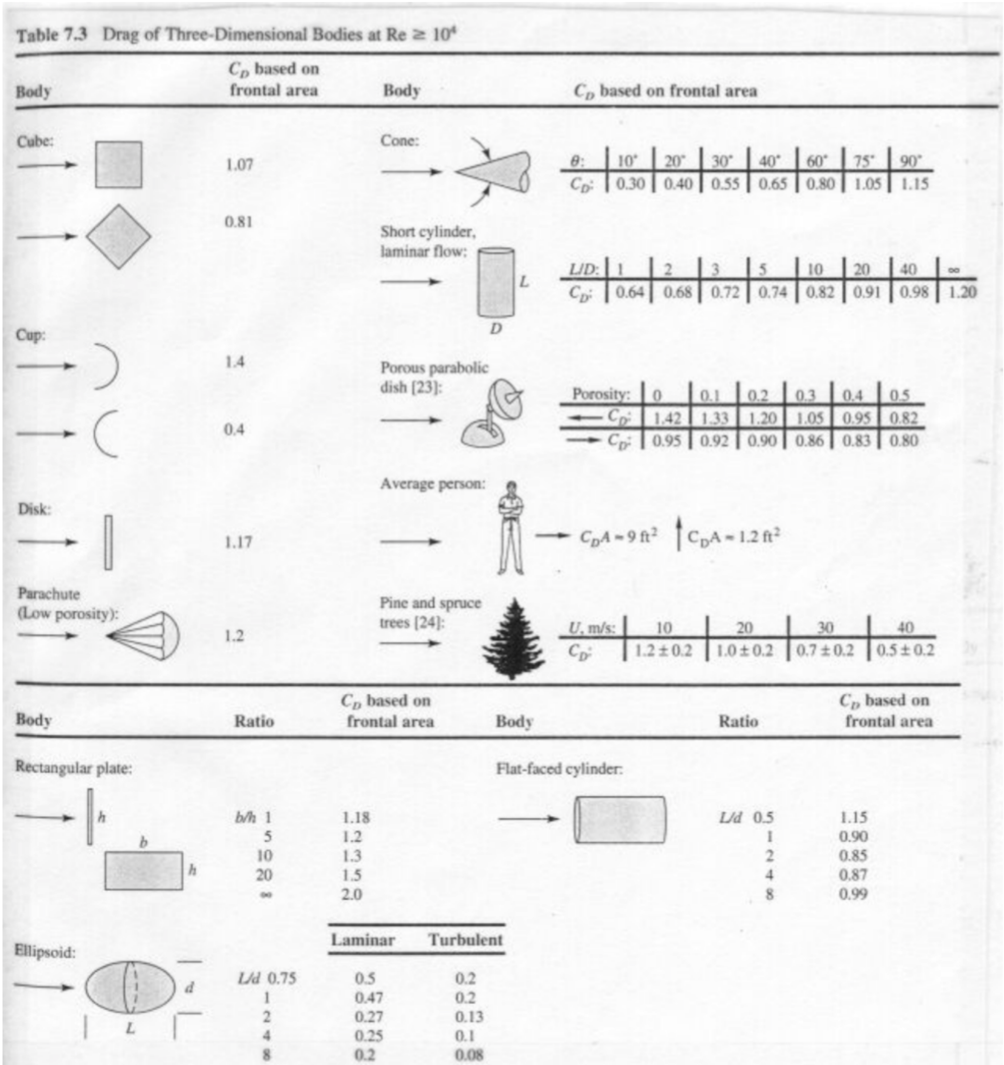


Table 7.3 Drag of Three-Dimensional Bodies at Re $\geq 10^4$

Body	C_D based on frontal area
Cube: → (face)	1.07
Cube: → (edge)	0.81
Cup: → (convex)	1.4
Cup: → (concave)	0.4
Disk:	1.17
Parachute (Low porosity):	1.2

Body	C_D based on frontal area							
Cone:	θ:	10°	20°	30°	40°	60°	75°	90°
	C_D:	0.30	0.40	0.55	0.65	0.80	1.05	1.15

Body	C_D based on frontal area								
Short cylinder, laminar flow:	L/D:	1	2	3	5	10	20	40	∞
	C_D:	0.64	0.68	0.72	0.74	0.82	0.91	0.98	1.20

Body	C_D based on frontal area						
Porous parabolic dish [23]:	Porosity:	0	0.1	0.2	0.3	0.4	0.5
	← C_D:	1.42	1.33	1.20	1.05	0.95	0.82
	→ C_D:	0.95	0.92	0.90	0.86	0.83	0.80

Average person: → $C_DA \approx 9$ ft^2 ↑ $C_DA \approx 1.2$ ft^2

Body	C_D based on frontal area				
Pine and spruce trees [24]:	U, m/s:	10	20	30	40
	C_D:	1.2 ± 0.2	1.0 ± 0.2	0.7 ± 0.2	0.5 ± 0.2

Body	Ratio		C_D based on frontal area
Rectangular plate:	b/h	1	1.18
		5	1.2
		10	1.3
		20	1.5
		∞	2.0

Body	Ratio		C_D based on frontal area
Flat-faced cylinder:	L/d	0.5	1.15
		1	0.90
		2	0.85
		4	0.87
		8	0.99

Body	Ratio		Laminar	Turbulent
Ellipsoid:	L/d	0.75	0.5	0.2
		1	0.47	0.2
		2	0.27	0.13
		4	0.25	0.1
		8	0.2	0.08

Figure 9.38. Drag coefficients for a variety of three-dimensional objects. This table is from White, F.M., *Fluid Mechanics*, 3rd ed., McGraw-Hill.

CHAPTER 10

Introduction to Turbulence

10.1. Introduction

Let's consider the following simple experiment (this thought experiment is similar to the famous dye injection experiment performed by Osbourne Reynolds). At a particular point in a circular pipe flow, let's measure a velocity component of the fluid as a function of time for varying Reynolds numbers (Figure 10.1). At

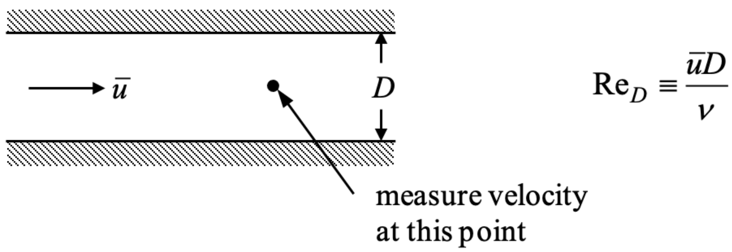


Figure 10.1. An illustration of the Reynolds laminar/turbulent pipe flow experiment.

Reynolds numbers less than approximately $\mathrm{Re}_D < 2300$, we would find that the measured speed remains nearly constant, as shown in Figure 10.2. For $2300 < \mathrm{Re}_D < 4000$ we would find a slightly different behavior,

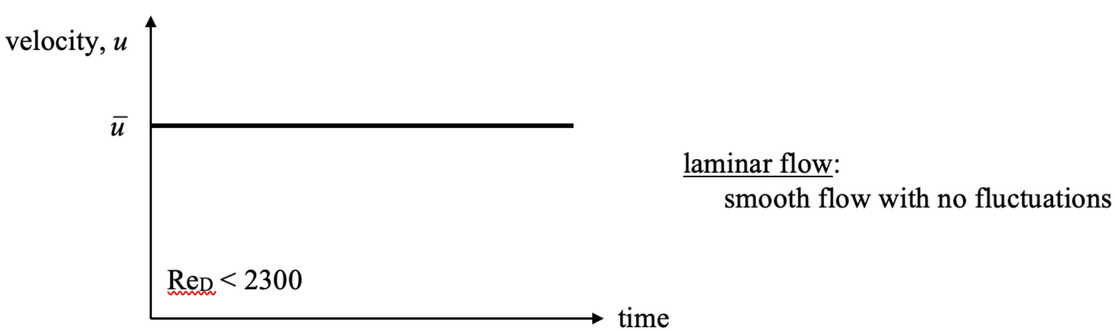


Figure 10.2. Fluid velocity plotted as a function of time for a laminar flow.

as shown in Figure 10.3. For $\mathrm{Re}_D > 4000$ we find a very different behavior, as shown in Figure 10.4.

Notes:

(1) Turbulence is a difficult phenomenon to analyze. It is typically studied using semi-empirical analyses, i.e., analyses that combine theory and experimental data. The transition region is even more difficult to analyze.
(2) The uncertainties associated with the transitional regime are also reflected in the value for the friction factor, f, shown in the Moody chart (Figure 10.5). Note that for $2300 < \mathrm{Re}_D < 4000$ the

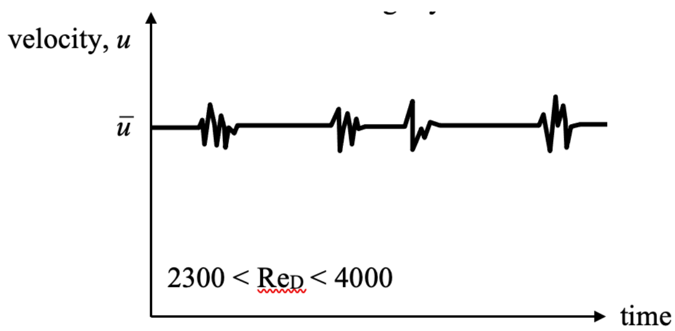


Figure 10.3. Fluid velocity plotted as a function of time for a transitional flow.

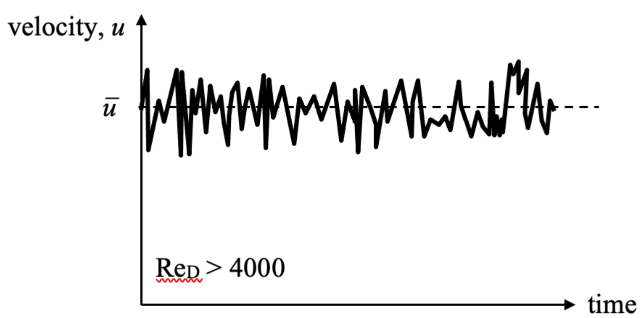


Figure 10.4. Fluid velocity plotted as a function of time for a turbulent flow.

value for the friction factor is not well defined since the friction factor varies considerably as the flow transitions between laminar and turbulent behavior.

10.2. Time-Averaged Continuity and Navier-Stokes Equations

Since the time-varying velocity data shown in the previous section example appears to consist of a fluctuating part superimposed on a mean value, let's make the following definitions. First, express the instantaneous velocity component, u_i, as the sum of a mean velocity, $\bar{u}_i$, and a fluctuating velocity, u_i', i.e.,

$$u_i = \bar{u}_i + u_i', \tag{10.1}$$

where the mean velocity over the time interval from t_0 to $t_0 + T$ is given by,

$$\bar{u}_i = \frac{1}{T}\int_{t_0}^{t_0+T} u_i dt. \tag{10.2}$$

Note that the time average of the fluctuating velocity will be zero,

$$\overline{u_i'} = \frac{1}{T}\int_{t_0}^{t_0+T}(u_i - \bar{u}_i)dt = \underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} u_i dt}_{=\bar{u}_i} - \underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} \bar{u}_i dt}_{=\bar{u}_i} = 0. \tag{10.3}$$

However, the mean of the square of the fluctuating velocities will, in general, be greater than zero,

$$\overline{(u_i')^2} = \frac{1}{T}\int_{t_0}^{t_0+T}(u_i - \bar{u}_i)^2 dt \geq 0. \tag{10.4}$$

Similarly,

$$\overline{(u_i'u_j')} = \frac{1}{T}\int_{t_0}^{t_0+T}(u_i - \bar{u}_i)(u_j - \bar{u}_j)dt \neq 0 \quad \text{(in general)} \quad (i \neq j). \tag{10.5}$$

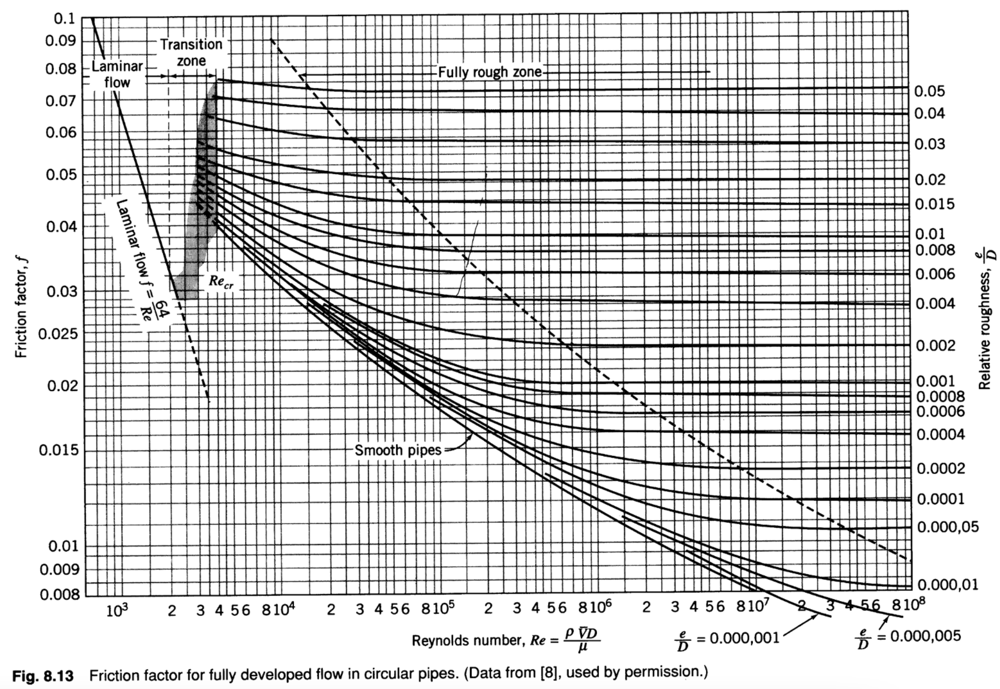


Figure 10.5. The Moody diagram, which plots the friction factor as a function of Reynolds number. Note the uncertainty in the transitional regime between laminar flow ($\mathrm{Re}_D < 2300$) and turbulent flow ($\mathrm{Re}_D > 4000$). This figure is from Fox, R.W. and McDonald, A.T., *Introduction to Fluid Mechanics*, 5th ed., Wiley.

Now let's look at the governing equations for an incompressible, Newtonian fluid, i.e., the Continuity and Navier-Stokes Equations, where we'll use the mean and fluctuating parts for our unknown variables,

$$u_i = \bar{u}_i + u_i', \tag{10.6}$$

$$p = \bar{p} + p'. \tag{10.7}$$

Substituting into the Continuity Equation gives,

$$\frac{\partial u_i}{\partial x_i} = 0 = \frac{\partial(\bar{u}_i + u_i')}{\partial x_i} = \frac{\partial \bar{u}_i}{\partial x_i} + \frac{\partial u_i'}{\partial x_i}. \tag{10.8}$$

Now take the time-average of the Continuity Equation,

$$\frac{1}{T}\int_{t_0}^{t_0+\delta T}\left(\frac{\partial \bar{u}_i}{\partial x_i} + \frac{\partial u_i'}{\partial x_i}\right)dt = 0. \tag{10.9}$$

where,

$$\frac{1}{T}\int_{t_0}^{t_0+\delta T}\frac{\partial \bar{u}_i}{\partial x_i}dt = \frac{\partial}{\partial x_i}\left(\frac{1}{T}\int_{t_0}^{t_0+T}\bar{u}_i dt\right) = \frac{\partial \bar{u}_i}{\partial x_i}, \tag{10.10}$$

$$\frac{1}{T}\int_{t_0}^{t_0+\delta T}\frac{\partial u_i'}{\partial x_i}dt = \frac{\partial}{\partial x_i}\left(\frac{1}{T}\int_{t_0}^{t_0+T}u_i' dt\right) = 0 \tag{10.11}$$

$$\tag{10.12}$$

Thus, the time-averaged Continuity Equation becomes,

$$\boxed{\frac{\partial \bar{u}_i}{\partial x_i} = 0}. \tag{10.13}$$

The time-averaged Continuity Equation looks the same as the instantaneous continuity equation except it is in terms of time-averaged velocities.

Now let's take the same approach with the Navier-Stokes equations,

$$\rho\left(\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}\right)=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_k^2}+\rho f_i. \tag{10.14}$$

To help with the upcoming analysis, let's re-write the left-hand-side using the Continuity Equation,

$$\rho\frac{\partial u_i}{\partial t}+\underbrace{\rho u_k\frac{\partial u_i}{\partial x_k}+\rho u_i\underbrace{\frac{\partial u_k}{\partial x_k}}_{=0}}_{=\rho\frac{\partial(u_i u_k)}{\partial x_k}}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_k^2}+\rho f_i. \tag{10.15}$$

Write the velocities and pressure in terms of mean and fluctuating parts,

$$\rho\frac{\partial(\bar{u}_i+u_i')}{\partial t}+\rho\frac{\partial[(\bar{u}_i+u_i')(\bar{u}_k+u_k')]}{\partial x_k}=-\frac{\partial(\bar{p}+p')}{\partial x_i}+\mu\frac{\partial^2(\bar{u}_i+u_i')}{\partial x_k^2}+\rho f_i, \tag{10.16}$$

$$\rho\frac{\partial\bar{u}_i}{\partial t}+\rho\frac{\partial u_i'}{\partial t}+\rho\frac{\partial}{\partial x_k}(\bar{u}_i\bar{u}_k+\bar{u}_iu_k'+\bar{u}_ku_i'+u_i'u_k')=-\frac{\partial\bar{p}}{\partial x_i}-\frac{\partial p'}{\partial x_i}+\mu\frac{\partial^2\bar{u}_i}{\partial x_k^2}+\mu\frac{\partial^2 u_i'}{\partial x_k^2}+\rho f_i. \tag{10.17}$$

Time-average the entire previous equation noting that for any term F:

$$\overline{\frac{\partial F}{\partial x}}=\frac{\partial\bar{F}}{\partial x}, \tag{10.18}$$

and,

$$\overline{\frac{\partial u_i'}{\partial t}}=\overline{\bar{u}_iu_k'}=\overline{\bar{u}_ku_i'}=\overline{\frac{\partial p'}{\partial x_i}}=\overline{\frac{\partial^2 u_i'}{\partial x_k^2}}, \tag{10.19}$$

so that we have,

$$\rho\frac{\partial\bar{u}_i}{\partial t}+\rho\frac{\partial}{\partial x_k}(\bar{u}_i\bar{u}_k+\overline{u_i'u_k'})=-\frac{\partial\bar{p}}{\partial x_i}+\mu\frac{\partial^2\bar{u}_i}{\partial x_k^2}+\rho f_i. \tag{10.20}$$

We can further simplify this equation by using the time-averaged Continuity Equation,

$$\frac{\partial}{\partial x_k}(\bar{u}_i\bar{u}_k)=\bar{u}_i\underbrace{\frac{\partial\bar{u}_k}{\partial x_k}}_{=0}+\bar{u}_k\frac{\partial\bar{u}_i}{\partial x_k}. \tag{10.21}$$

Thus, the time-averaged Navier-Stokes equations become,

$$\boxed{\rho\left(\frac{\partial\bar{u}_i}{\partial t}+\bar{u}_k\frac{\partial\bar{u}_i}{\partial x_k}\right)=-\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_k}\left[\mu\frac{\partial^2\bar{u}_i}{\partial x_k^2}-\rho\left(\overline{u_i'u_k'}\right)\right]+\rho f_i}. \tag{10.22}$$

If we compare Eq. (10.22) to the instantaneous Navier-Stokes equation (Eq. (10.14)), we see that an extra term appears on the right-hand side with the same dimensions as the laminar shear stress term,

$$\boxed{-\rho\frac{\partial}{\partial x_k}\left(\overline{u_i'u_k'}\right)}. \tag{10.23}$$

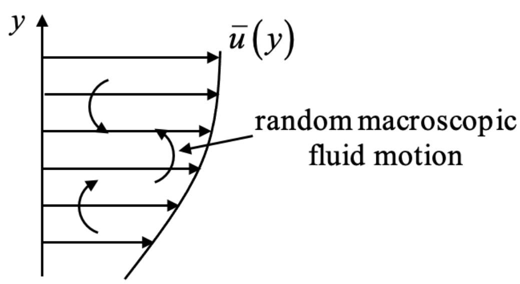
These terms are referred to as Reynolds "stresses", although they are, in fact, momentum fluxes.

Notes:

(1) There are both Reynolds normal and shear stresses. For example, the x-component of the time-averaged Navier-Stokes equations is given by:

$$\rho \frac{D\bar{u}}{Dt} = -\frac{\partial \bar{p}}{\partial x} + \rho f_x + \frac{\partial}{\partial x}\left[\mu \frac{\partial \bar{u}}{\partial x} \underbrace{-\rho\overline{(u')^2}}_{\substack{\text{Reynolds}\\\text{“normal stress”}}}\right] + \frac{\partial}{\partial y}\left[\mu \frac{\partial \bar{u}}{\partial y} \underbrace{-\rho\overline{(u'v')}}_{\substack{\text{Reynolds}\\\text{“shear stress”}}}\right] + \frac{\partial}{\partial z}\left[\mu \frac{\partial \bar{u}}{\partial z} \underbrace{-\rho\overline{(u'w')}}_{\substack{\text{Reynolds}\\\text{“shear stress”}}}\right] \tag{10.24}$$

(2) What, physically speaking, is a Reynolds “stress”? First let's examine what causes viscosity in a laminar flow. Recall that a fluid is comprised of a collection of molecules. In a flow with a velocity gradient, the molecules in a particular layer will have an average velocity in addition to some random, thermal motion. Since there is a velocity gradient in the fluid ($u \neq$ constant), molecules in adjacent layers will not have the same average velocity. Due to their random motion, molecules starting in a particular layer with a particular average velocity will move into layers with a different average velocity. When a molecule moves into a region of larger velocity, the molecules in that layer must accelerate the incoming molecule up to the new speed. In order to accelerate the new molecule, the molecules already in the layer must exert a force on the new molecule (via molecular collisions). From Newton's Third law, the new molecule exerts an equal, but opposite force, on the layer. An identical process happens when a particle moves into a region of slower average velocity. Thus, we see that the laminar shear stress is a result of the random flux of molecules into neighboring layers with different average velocities.
In a turbulent flow, this microscopic effect occurs, but there is also a macroscopic flux due to the random motion of the fluid (Figure 10.6). This macroscopic random fluid motion into regions of differing average speed gives rise to Reynolds “stresses”.



$$\tau_{\text{total}} = \underbrace{\mu \frac{\partial \bar{u}}{\partial y}}_{\text{laminar}} \underbrace{-\rho\overline{u'v'}}_{\text{turbulent}}$$

Figure 10.6. An illustration showing how turbulent mixing results in Reynolds “stresses”.

(3) Near a wall, the Reynolds shear stresses are small due to the wall restricting the random motion of the fluid. This region is termed the viscous sub-layer and,

$$\tau_{\text{viscous sub-layer}} \approx \tau_{\text{laminar}}. \tag{10.25}$$

Far from the wall, turbulent motion dominates. This region is termed the turbulent core and,

$$\tau_{\text{turbulent core}} \approx \tau_{\text{turbulent}}. \tag{10.26}$$

The region between the laminar sub-layer and the turbulent core is referred to as the transition region where the laminar and turbulent shear stresses are of the same order of magnitude. These regions are shown in Figure 10.7.

(4) The relative turbulence intensity is defined as the magnitude of the Reynolds stresses relative to some characteristic flow speed, U, e.g., an outer flow velocity,

$$TI := \frac{\sqrt{\overline{u_i' u_j'}}}{U}. \tag{10.27}$$

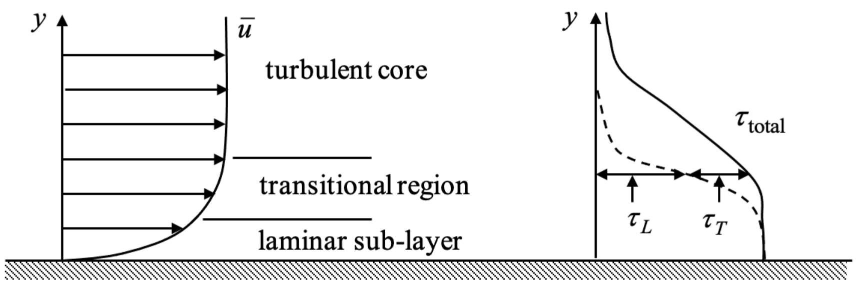


Figure 10.7. An illustration showing a time-averaged turbulent boundary layer and the corresponding shear stress profile.

(5) If the turbulence intensity is the same in all directions, then the turbulence is considered isotropic. Otherwise, the turbulence is anisotropic. As shown in Figure 10.8 for turbulent flow over a flat plate with no pressure gradient and $\mathrm{Re}_x = 10^7$, the turbulence near the wall is found to be anisotropic with the relative turbulence intensity normal (v') to the wall being smaller than the streamwise (u') and lateral (w') turbulence intensities. This effect is the result of the geometric constraint posed by the wall. Farther from the wall at approximately $y/\delta = 0.8$, the turbulence becomes nearly isotropic.

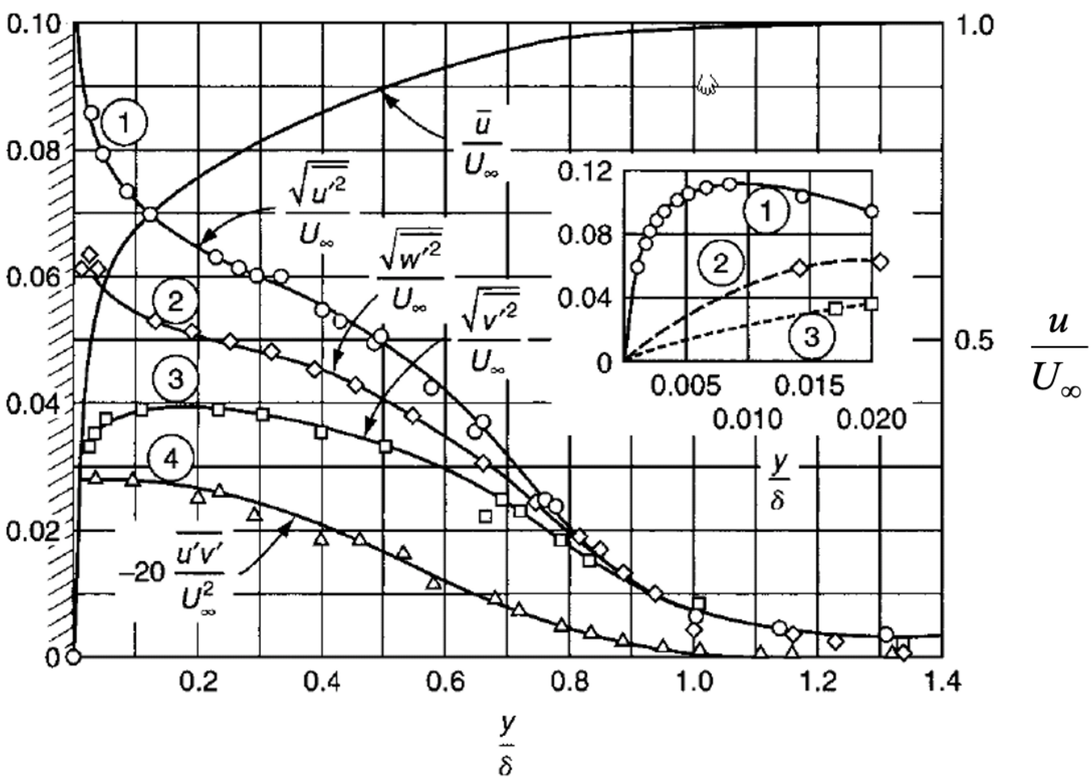


Figure 10.8. A plot of the turbulence intensities as a function of dimensionless distance from the wall. This is Figure 6.4 from White, F.M., *Viscous Fluid Flow*, McGraw-Hill, 3rd ed.

(6) The turbulent kinetic energy, K, is defined as the kinetic energy of the normal turbulent fluctuations,

$$K := \frac{1}{2}\overline{u_i' u_i'}. \tag{10.28}$$

The turbulent kinetic energy is used in models that examine the balance of energy associated with turbulent motion (beyond the scope of these notes).

10.3. Law of the Wall

Let's write the quantities that affect the mean velocity profile, $\bar{u}$, near the wall,

$$\bar{u} = f(y, \rho, \nu, \tau_w) \tag{10.29}$$

where y is the distance from the wall, ρ and ν are the fluid density and kinematic viscosity, respectively, and τ_w is the shear stress that the wall exerts on the fluid. By performing a dimensional analysis on these variables we find a relation termed the Law of the Wall,

$$\boxed{\frac{\bar{u}}{u^*} = g\left(\frac{yu^*}{\nu}\right)}, \tag{10.30}$$

where g is an unknown function and,

$$u^* := \sqrt{\frac{\tau_w}{\rho}}, \tag{10.31}$$

where u^* is referred to as the "friction velocity".

Nearest the wall, in the region known as the laminar sub-layer ($0 \leq yu^*/\delta < 5$), the wall stress is given by:

$$\tau_w = \mu \frac{d\bar{u}}{dy}. \tag{10.32}$$

Dividing through by the density and integrating, the mean velocity profile in the vicinity of the wall,

$$\int_0^y \underbrace{\frac{\tau_w}{\rho}}_{=(u^*)^2} dy = \int_0^{\bar{u}} \nu d\bar{u} \implies (u^*)^2 y = \nu \bar{u}, \tag{10.33}$$

$$\boxed{\frac{\bar{u}}{u^*} = \frac{u^* y}{\nu}}. \tag{10.34}$$

This is the time-averaged velocity profile in the laminar sub-layer ($0 \leq yu^*/\nu < 5$).

Now let's examine the turbulent core region. One simple model, known as Prandtl's Mixing Length Hypothesis, assumes that the fluctuating velocities, e.g., u' and v', are approximately equal to some typical eddy length, l, multiplied by the velocity gradient, $d\bar{u}/dy$ (Figure 10.9),

$$u', v' \approx l \frac{d\bar{u}}{dy}. \tag{10.35}$$

Thus, the Reynolds shear stress in the turbulent core is,

$$\tau = \rho u'v' \approx \rho l^2 \left(\frac{d\bar{u}}{dy}\right)^2. \tag{10.36}$$

Note that l cannot be constant since $u', v' \to 0$ as $y \to 0$. This stipulation implies that $l \to 0$ as $y \to 0$. Let's assume a simple relation that satisfies this condition,

$$l = Ky \tag{10.37}$$

where the constant K is known as Karman's Universal Mixing Length Constant.

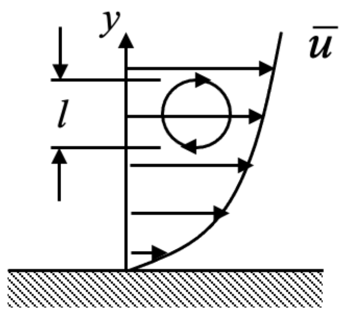


Figure 10.9. An illustration of a turbulent eddy in Prandtl's mixing length hypothesis.

Substituting Eq. (10.37) into Eq. (10.36) and simplifying gives,

$$\tau = \rho l^2 \left(\frac{d\bar{u}}{dy}\right)^2 = \rho K^2 y^2 \left(\frac{d\bar{u}}{dy}\right)^2, \tag{10.38}$$

$$\rho u^{*2} = \rho K^2 y^2 \left(\frac{d\bar{u}}{dy}\right)^2 \quad \text{since } u^* = \sqrt{\frac{\tau_w}{\rho}}, \tag{10.39}$$

$$u^* = Ky\left(\frac{d\bar{u}}{dy}\right), \tag{10.40}$$

$$\int \frac{dy}{y} = \int \frac{K}{u^*} d\bar{u} \implies \ln y = \frac{K}{u^*}\bar{u} + c \implies \frac{\bar{u}}{u^*} = \frac{1}{K}\ln y + c, \tag{10.41}$$

$$\tag{10.42}$$

$$\boxed{\frac{\bar{u}}{u^*} = \frac{1}{K'}\ln\left(\frac{yu^*}{\nu}\right) + c} \quad \left(\text{for } \frac{yu^*}{\nu} > 70, \text{ i.e., in the turbulent core}\right) \tag{10.43}$$

In summary, the Law of the Wall states:

$$\frac{\bar{u}}{u^*} = g\left(\frac{yu^*}{\nu}\right) \tag{10.44}$$

$$0 \leq \frac{yu*}{\nu} < 5: \quad \frac{\bar{u}}{u^*} = \frac{u^* y}{\nu} \quad \text{(laminar sub-layer)} \tag{10.45}$$

$$5 \leq \frac{yu^*}{\nu} < 70: \quad \text{(often use Eq. (10.43) here)} \quad \text{(transition region)} \tag{10.46}$$

$$\frac{yu^*}{\nu} > 70: \quad \frac{\bar{u}}{u^*} = \frac{1}{K'}\frac{u^* y}{\nu} + c \quad \text{(turbulent core).} \tag{10.47}$$

Notes:

(1) Experimental curve fits indicate that $K' \approx 0.41$ and $c = 5.0$.
(2) Comparing laminar and turbulent pipe flows (Figure 10.10), we observe that the turbulent velocity profile (a logarithmic curve) is blunter than the laminar profile (a parabolic curve). The blunter profile of the turbulent flow is due to mixing of the different velocity regions, which results in most of the flow having approximately the same speed.
(3) A plot of the Law of the Wall velocity relations and corresponding experimental data is shown in Figure 10.11.

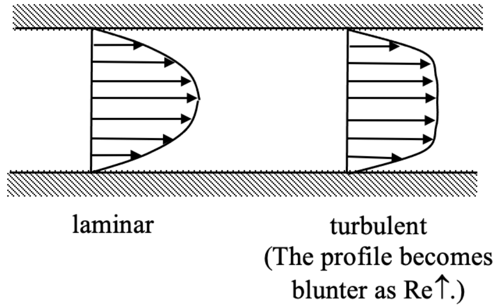


Figure 10.10. Representative velocity profiles for laminar and turbulent flows in a pipe. The laminar velocity profile is a paraboloid for flow in a circular pipe. The turbulent profile is blunter and approaches a nearly uniform profile, except near the pipe walls, as the Reynolds number increases.

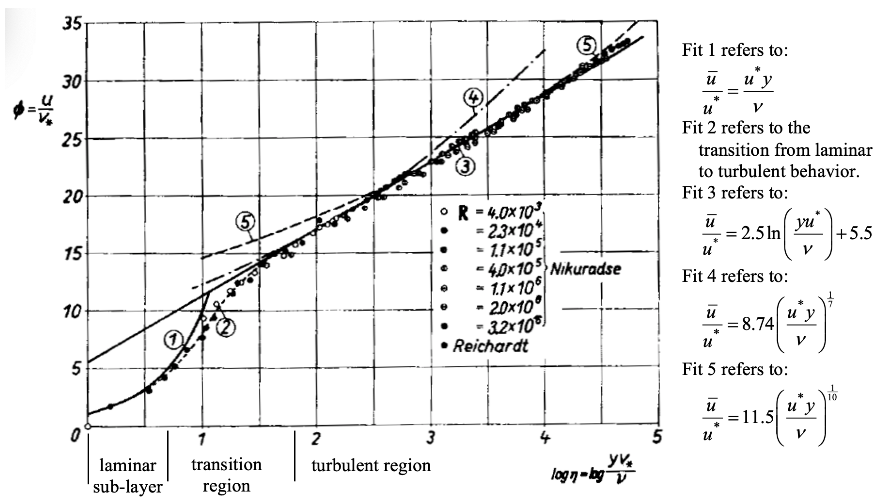


Figure 10.11. A plot comparing experimental data and Law of the Wall models. Figure 20.4 from Schlichting, H., *Boundary Layer Theory*, McGraw-Hill.)

CHAPTER 11

Pipe Flows

11.1. Entrance Region

The flow in the entrance region is complex (Figure 11.1) and will not be investigated here. Experiments have shown that the dimensionless length of the entrance region depends on whether the entering flow is laminar or turbulent, with,

$$\text{laminar flow:} \quad \frac{L}{D} \approx 0.06\text{Re}_D, \tag{11.1}$$

$$\text{turbulent flow:} \quad \frac{L}{D} \approx 4.4\text{Re}_D^{1/6}, \tag{11.2}$$

where L is the length of the entrance region and D is the pipe diameter. For many engineering flows, $1 \times 10^4 < \text{Re}_D < 1 \times 10^5 \implies 20 < L/D < 30$. The shorter entrance region length for turbulent flows is due to the fact that turbulent mixing rapidly averages the flow speeds across the pipe cross-section.

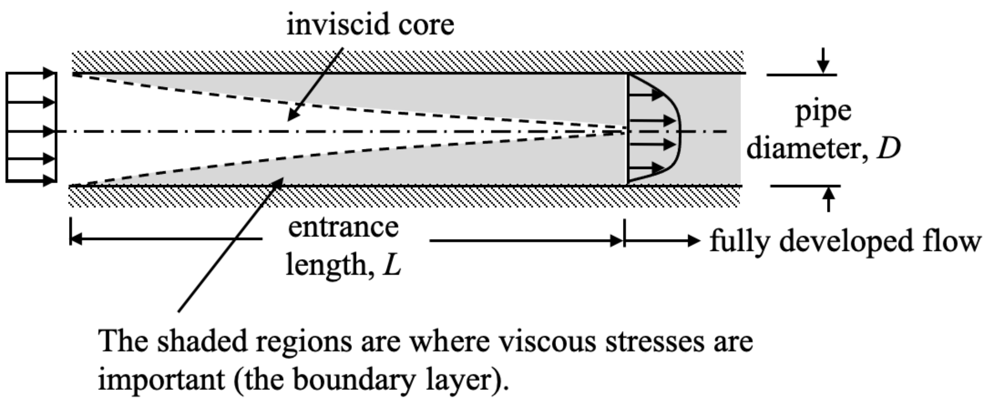


Figure 11.1. The structure of a pipe flow entrance region.

11.2. Fully Developed Laminar Circular Pipe Flow (Poiseuille Flow)

The derivation in this section was previously covered in Chapter 8 and is repeated here, in a slightly condensed form, for convenience. Consider the steady flow of an incompressible, constant viscosity, Newtonian fluid within an infinitely long, circular pipe of radius, R (Figure 11.2).

Make the following assumptions,

(1) The flow is axi-symmetric and there is no "swirl" velocity. $\implies \frac{\partial}{\partial\theta}(\dots) = 0$ and $u_\theta = 0$
(2) The flow is at steady state. $\implies \frac{\partial}{\partial t}(\dots) = 0$
(3) The flow is fully-developed in the z-direction. $\implies \frac{\partial u_r}{\partial z} = \frac{\partial u_z}{\partial z} = 0$
(4) There are no body forces. $\implies f_r = f_\theta = f_z = 0$

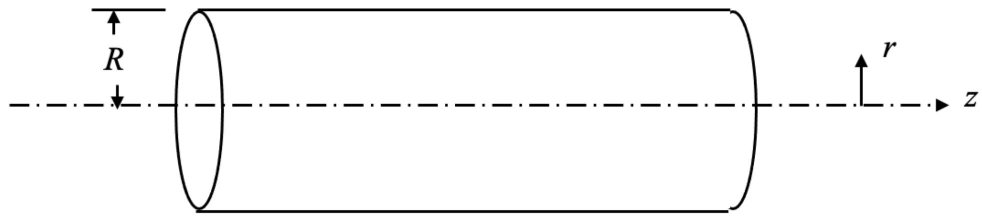


Figure 11.2. A schematic of flow through a circular pipe.

Let's first examine the Continuity Equation,

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0. \tag{11.3}$$

From assumptions #1 and #3,

$$ru_r = \text{constant}. \tag{11.4}$$

Since there is no flow through the walls, the constant must be zero and, thus,

$$u_r = 0 \quad \text{(Call this condition \# 5.)} \tag{11.5}$$

Now let's examine the Navier-Stokes equation in the z-direction,

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho f_z. \tag{11.6}$$

We can simplify this equation using our assumptions,

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(\#2)}+\underbrace{u_r}_{=0(\#5)}\frac{\partial u_z}{\partial r}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}_{=0(\#1)}+u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0(\#3)}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_z}{\partial \theta^2}}_{=0(\#1)}+\underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(\#3)}\right]+\rho\underbrace{f_z}_{=0(\#4)}, \tag{11.7}$$

$$\implies \frac{d}{dr}\left(r\frac{du_z}{dr}\right)=\frac{r}{\mu}\frac{dp}{dz}, \tag{11.8}$$

$$\implies r\frac{du_z}{dr}=\frac{r^2}{2\mu}\frac{dp}{dz}+c_1, \tag{11.9}$$

$$\implies u_z=\frac{r^2}{4\mu}\frac{dp}{dz}=c_1\ln r+c_2. \tag{11.10}$$

Note that in the previous derivation the fact that u_z is a function only of r has been used to change the partial derivatives to ordinary derivatives. Furthermore, examining the Navier-Stokes equations in the r and θ directions demonstrates that the pressure, p, is a function only of z and, thus, ordinary derivatives can be used when differentiating the pressure with respect to z.

Now let's apply boundary conditions to determine the unknown constants c_1 and c_2. First, note that the fluid velocity in a pipe must remain finite as $r \to 0$ so the constant c_1 must be zero (this is a type of kinematic boundary condition). Also, the pipe wall is fixed so we have $u_z(r=R)=0$ (no-slip condition). After applying boundary conditions we have,

$$\boxed{u_z=\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1-\left(\frac{r}{R}\right)^2\right]}. \tag{11.11}$$

This is known as <u>Poiseuille Flow in a Circular Pipe</u>.

Notes:

(1) The velocity profile is a paraboloid with the maximum velocity occurring along the centerline. The average velocity in the pipe is found from,

$$\bar{u} = \frac{1}{\pi R^2}\int_{r=0}^{r=R} u_z(2\pi r dr) = \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{D^2}{32\mu}\left(-\frac{dp}{dz}\right) = \frac{1}{2}u_{\max}, \tag{11.12}$$

where $u_{\max}$ is the maximum fluid speed and D is the pipe diameter.

(2) The volumetric flow rate through the pipe is,

$$Q = \bar{u}\left(\frac{\pi}{4}D^2\right) = \frac{\pi D^4}{128\mu}\left(-\frac{dp}{dz}\right). \tag{11.13}$$

(3) We can determine stresses using the constitutive relations for a Newtonian fluid. The shear stress that the pipe walls apply to the fluid, τ_w, is,

$$\tau_w = \frac{R}{2}\left(\frac{dp}{dz}\right) = \frac{-4\mu\bar{u}}{R}, \tag{11.14}$$

where $\bar{u}$ is the average speed in the pipe. Note that an alternate method for determining the average wall shear stress, which in this case is equal to the exact wall shear stress, is to balance shear forces and pressure forces on a small slice of the flow as shown in Figure 11.3.

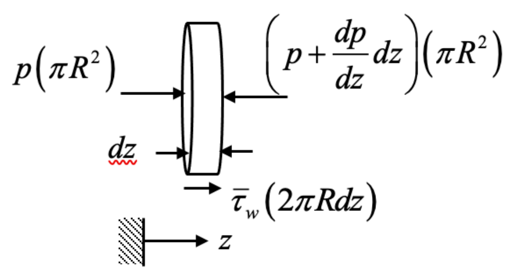


Figure 11.3. A free body diagram showing the forces on a thin disk of fluid in the pipe.

In engineering applications, it is common to express the average shear stress in terms of a dimensionless (Darcy) friction factor, f_D, which is defined as,

$$f_D \coloneqq \left|\frac{4\overline{\tau_w}}{\frac{1}{2}\rho\bar{u}^2}\right| = 64\left(\frac{\mu}{\rho\bar{u}D}\right) = \frac{64}{\mathrm{Re}_D}, \tag{11.15}$$

where $D = 2R$ is the pipe diameter and Re_D is the Reynolds number based on the pipe diameter. The Darcy friction factor appears in the Moody chart for incompressible, viscous pipe flow. Note again that this solution is only valid only for a laminar flow. The condition for the flow to remain laminar is found experimentally to be,

$$\mathrm{Re}_D = \frac{\rho\bar{u}D}{\mu} < 2300. \tag{11.16}$$

(4) Re-write Eq. (11.12),

$$\bar{u} = \frac{D^2}{32\mu}\left(-\frac{dp}{dz}\right) \implies |\bar{u}| = \frac{D^2}{32\mu}\left(\frac{\Delta p}{L}\right), \tag{11.17}$$

where, in the fully developed region, the pressure gradient remains constant and we may write dp/dz as $\Delta p/L$ where Δp is the pressure drop over a length L of the pipe (Figure 11.4). Re-arranging Eq. (11.17) and dropping the absolute value symbol for convenience,

$$\Delta p = \frac{32\mu\bar{u}L}{D^2}. \tag{11.18}$$

Make the previous equation dimensionless by dividing through by the dynamic pressure based on the average flow speed,

$$\frac{\Delta p}{\frac{1}{2}\rho\bar{u}^2} = \frac{64\mu}{\rho\bar{u}D}\left(\frac{L}{D}\right) = \left(\frac{64}{\mathrm{Re}_D}\right)\left(\frac{L}{D}\right). \tag{11.19}$$

The dimensionless pressure drop is also referred to as a loss coefficient, k. Hence, for a laminar flow, the loss coefficient corresponding to the viscous stresses at the pipe walls is,

$$k_{\substack{\text{laminar,}\\ \text{wall stresses}}} = \left(\frac{64}{\mathrm{Re}_D}\right)\left(\frac{L}{D}\right) = f_D\left(\frac{L}{D}\right). \tag{11.20}$$

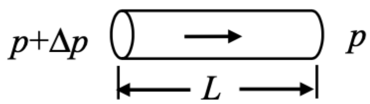


Figure 11.4. A schematic showing the change in pressure over the pipe length.

Video solution: https://www.youtube.com/watch?v=5IRg1q_BIQI

A liquid with a specific gravity of 0.95 flows steadily at an average velocity of 10 m/s through a horizontal, smooth tube of diameter 5 cm. The fluid pressure is measured at 1 m intervals along the pipe as follows:

x [m]	0	1	2	3	4	5	6
p [kPa]	304	273	255	240	226	213	200

a. Estimate the average wall shear stress, in Pa, in the fully developed region of the pipe.
b. What is the approximate wall shear stress between stations 1 and 2? State any significant assumptions you make.

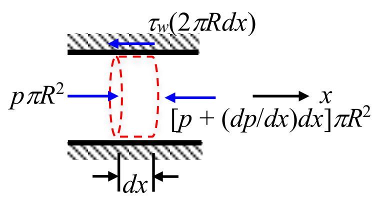
SOLUTION:

First determine the fully developed region by examining the pressure gradient in the pipe. The pressure gradient is constant in the fully developed region.

x [m]	0	1	2	3	4	5	6
p [kPa]	304	273	255	240	226	213	200
dp/dx [kPa/m]		-31	-18	-15	-14	-13	-13

Hence, the fully developed region starts at $x = 4$ m where the pressure drop remains constant at $dp/dx = -13$ kPa/m.

To determine the average wall shear stress in the pipe, apply the linear momentum equation in the x direction to the control volume shown in the figure below.



$$\frac{d}{dt}\int_{CV} u_x \rho \, dV + \int_{CS} u_x (\rho \mathbf{u}_{rel} \cdot d\mathbf{A}) = F_{S,x} + F_{B,x}, \quad (1)$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho \, dV = 0 \quad \text{(steady flow)}, \quad (2)$$

$$F_{B,x} = 0 \quad \text{(no body forces in } x\text{-direction)}, \quad (3)$$

$$F_{S,x} = p\pi R^2 - \left(p + \frac{dp}{dx}dx\right)\pi R^2 - \bar{\tau}_w (2\pi R dx) = -\frac{dp}{dx} dx \pi R^2 - \bar{\tau}_w (2\pi R dx), \quad (4)$$

$$\int_{CS} u_x (\rho \mathbf{u}_{rel} \cdot d\mathbf{A}) = 0 \quad (5)$$

(since for a fully-developed flow, the inlet and outlet velocity profiles are identical)

Substitute and simplify,

$$0 = -\frac{dp}{dx} dx \pi R^2 - \bar{\tau}_w (2\pi R dx), \quad (6)$$

$$\boxed{\bar{\tau}_w = -\frac{R}{2}\frac{dp}{dx}}. \quad (7)$$

Video solution: https://www.youtube.com/watch?v=5IRg1q_BIQI

Using the given data,

$R = (0.05/2)\text{ m} = 0.025\text{ m}$

$dp/dx = -13\text{ kPa/m}$

$\boxed{\Rightarrow \overline{\tau}_w = 163\text{ Pa}}$

For part (b), apply the same linear momentum equation, except that between stations 1 and 2, the velocity profile is not fully developed, hence the momentum flux term in the linear momentum equation (Eq. (5)) won't be zero. However, if the flow is turbulent, as would be expected for such a large velocity and assuming a liquid viscosity similar to that of water, the velocity profile will not change considerably as the flow continues downstream in the entrance region. The reason for this is that a turbulent velocity profile already looks like an average velocity profile due to the radial mixing associated with turbulence. Hence, although the momentum flux term isn't exactly zero, it is expected to be small in comparison to the pressure gradient term. As a result, even in the entrance region the average wall shear stress may be found using,

$$\boxed{\overline{\tau}_w \approx -\frac{R}{2}\frac{dp}{dx}}. \tag{8}$$

Using the given data between stations 1 and 2,

$R = (0.05/2)\text{ m} = 0.025\text{ m}$

$dp/dx = -18\text{ kPa/m}$

$\boxed{\Rightarrow \overline{\tau}_w = 225\text{ Pa}}$

11.3. Fully Developed Turbulent Circular Pipe Flow

11.3.1. Turbulent Flow in a Smooth, but Frictional, Pipe

The volumetric flow rate in a smooth pipe for turbulent flow may be estimated by integrating the time averaged velocity profile, modeled using the Law of the Wall, over the cross-sectional area of the pipe. As an engineering approximation, we'll neglect the influence of the pipe curvature and the presence of the opposite side of the pipe in this estimation,

$$Q = \int_{r=0}^{r=R} \bar{u}(2\pi r dr) = 2\pi \int_0^R u^* \left[\frac{1}{K}\ln\left(\frac{yu^*}{\nu}\right) + c\right](R-y)dy, \tag{11.21}$$

where $K \approx 0.41$ and $c = 5.0$. Note that we've switched coordinates so we can integrate out from the wall to the centerline (Figure 11.5). Also, we're neglecting the laminar sub-layer velocity profile since it is typically very thin.

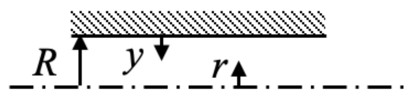


Figure 11.5. The coordinate system used in turbulent pipe flow analysis.

The average velocity in the pipe is,

$$\bar{u} = \frac{Q}{\pi R^2} \implies \frac{\bar{u}}{u^*} \approx 2.44\ln\left(\frac{Ru^*}{\nu}\right) + 1.34. \tag{11.22}$$

Recall that,

$$f_D = \frac{8\tau_w}{\rho \bar{u}^2} = \frac{8(u^*)^2}{\bar{u}^2} \quad \text{since} \quad u^* = \sqrt{\frac{\tau_w}{\rho}}. \tag{11.23}$$

Thus,

$$\frac{\bar{u}}{u^*} = \sqrt{\frac{8}{f_D}}. \tag{11.24}$$

Also note that,

$$\frac{Ru^*}{\nu} = \frac{R}{\nu}\sqrt{\frac{\tau_w}{\rho}} = \frac{R}{\nu}\sqrt{\frac{f_D \bar{u}^2}{8}} = \frac{1}{2}\frac{2R\bar{u}}{\nu}\sqrt{\frac{f_d}{8}}. \tag{11.25}$$

Thus,

$$\frac{Ru^*}{\nu} = \frac{1}{2}\mathrm{Re}_D\sqrt{\frac{f_d}{8}}. \tag{11.26}$$

Substituting Eqs. (11.24) and (11.26) into Eq. (11.22) gives,

$$\sqrt{\frac{8}{f_D}} \approx 2.44\ln\left(\frac{1}{2}\mathrm{Re}_D\sqrt{\frac{f_D}{8}}\right) + 1.34, \tag{11.27}$$

$$\sqrt{\frac{1}{f_D}} \approx 1.99\log_{10}\left(\mathrm{Re}_D\sqrt{f_D}\right) - 1.02. \tag{11.28}$$

where a "$\log_{10}$" is used in place of the "ln" in the previous equation. Prandtl derived Eq. (11.28), but altered the constants slightly to better fit the data,

$$\boxed{\frac{1}{f_D} \approx 2.0\log_{10}\left(\mathrm{Re}_D\sqrt{f_D}\right) - 0.8}. \tag{11.29}$$

This equation relates the friction factor and Reynolds number for turbulent flow in a smooth, circular pipe.

Equation (11.29) is implicit in f_D, which means that an iterative approach must be used to solve for f_D as a function of Re_D. A number of approximations to this relation have been proposed that are easier to solve. For example, Blasius, a student of Prandtl's, suggested the following approximation,

$$\boxed{f_D \approx \frac{0.316}{\mathrm{Re}_D^{1/4}}}, \tag{11.30}$$

which is valid for $4000 < \mathrm{Re}_D < 1 \times 10^5$.

11.3.2. Turbulent Flow in a Very Rough Pipe

The roughness of the pipe walls can significantly affect the friction factor for turbulent flows (roughness has a negligible effect on the friction factor for laminar flows). Recall from the Law of the Wall that the time averaged velocity in the laminar sub-layer is,

$$\frac{\bar{u}}{u^*} = \frac{yu^*}{\nu} \quad \text{for} \quad 0 \leq \frac{yu^*}{\nu} \leq 5. \tag{11.31}$$

Thus, the thickness of the laminar sub-layer, δ_{LSL}, is,

$$\frac{\delta_{\mathrm{LSL}} u^*}{\nu} = 5 \implies \delta_{\mathrm{LSL}} = \frac{5\nu}{u^*}. \tag{11.32}$$

Since,

$$u^* = \sqrt{\frac{\tau_w}{\rho}} = \bar{u}\sqrt{\frac{f_D}{8}} \quad \text{(refer to Eq. (11.24))}, \tag{11.33}$$

we have,

$$\delta_{\mathrm{LSL}} = \frac{5\nu}{\bar{u}}\sqrt{\frac{8}{f_D}} \implies \frac{\delta_{\mathrm{LSL}}}{D} = \frac{5\nu}{\bar{u}D}\sqrt{\frac{8}{f_D}} = \frac{5}{\mathrm{Re}_D}\sqrt{\frac{8}{f_D}}, \tag{11.34}$$

$$\therefore \frac{\delta_{\mathrm{LSL}}}{D} = \frac{14.1}{\mathrm{Re}_D\sqrt{f_D}}. \tag{11.35}$$

Thus, if the <u>wall roughness</u>, ϵ, is much smaller than the laminar sub-layer thickness, then we'll still have a laminar sub-layer and the flow won't be significantly affected by the wall roughness, i.e., we may treat the wall as being smooth (but still frictional). However, if $\epsilon \gg \delta_{\mathrm{LSL}}$, then the laminar sub-layer will be destroyed and the wall roughness becomes the new length scale for use in the Law of the Wall, i.e.,

$$\frac{\bar{u}}{u^*} = f\left(\frac{y}{\epsilon}\right). \tag{11.36}$$

Following the same analysis as that for turbulent flow in a smooth pipe, but using y/ϵ in place of yu^*/ν, we obtain,

$$\boxed{\sqrt{\frac{1}{f_D}} \approx -2.0\log_{10}\left(\frac{\epsilon/D}{3.7}\right)}. \tag{11.37}$$

This is the friction factor for turbulent flow in a very rough pipe. The term ϵ/D is known as the <u>relative roughness</u>. Note that this equation is independent of the Reynolds number.

11.3.3. Turbulent Flow in a Rough Pipe

For the transitional regime where ϵ/D is between "smooth" and "very rough", empirical formulas in which the friction factor is a function of both ϵ/D and Re_D have been developed,

$$\boxed{\sqrt{\frac{1}{f_D}} \approx -2.0\log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}_D\sqrt{f_D}}\right) \quad \mathrm{Re}_D > 4000} \quad \underline{\text{Colebrook Formula}}, \tag{11.38}$$

which is implicit in f_D, or the explicit empirical formula,

$$\boxed{\sqrt{\frac{1}{f_D}} \approx -1.8\log_{10}\left[\frac{6.9}{\mathrm{Re}_D} + \left(\frac{\epsilon/D}{3.7}\right)^{1.11}\right] \quad \mathrm{Re}_D > 4000} \quad \underline{\text{Haaland Formula}}. \tag{11.39}$$

The Haaland formula isn't as accurate as the Colebrook formula, but it's easier to calculate since it's explicit in f_D. To solve the Colebrook formula for f_D, an iterative algorithm must be used. An initial first guess for f_D using the Haaland formula usually results in convergence in the Colebrook formula within one or two iterations.

11.4. The Moody Plot

The previous friction factor relations have been summarized into a single plot known as the Moody Plot, which is shown in Figure 11.6.

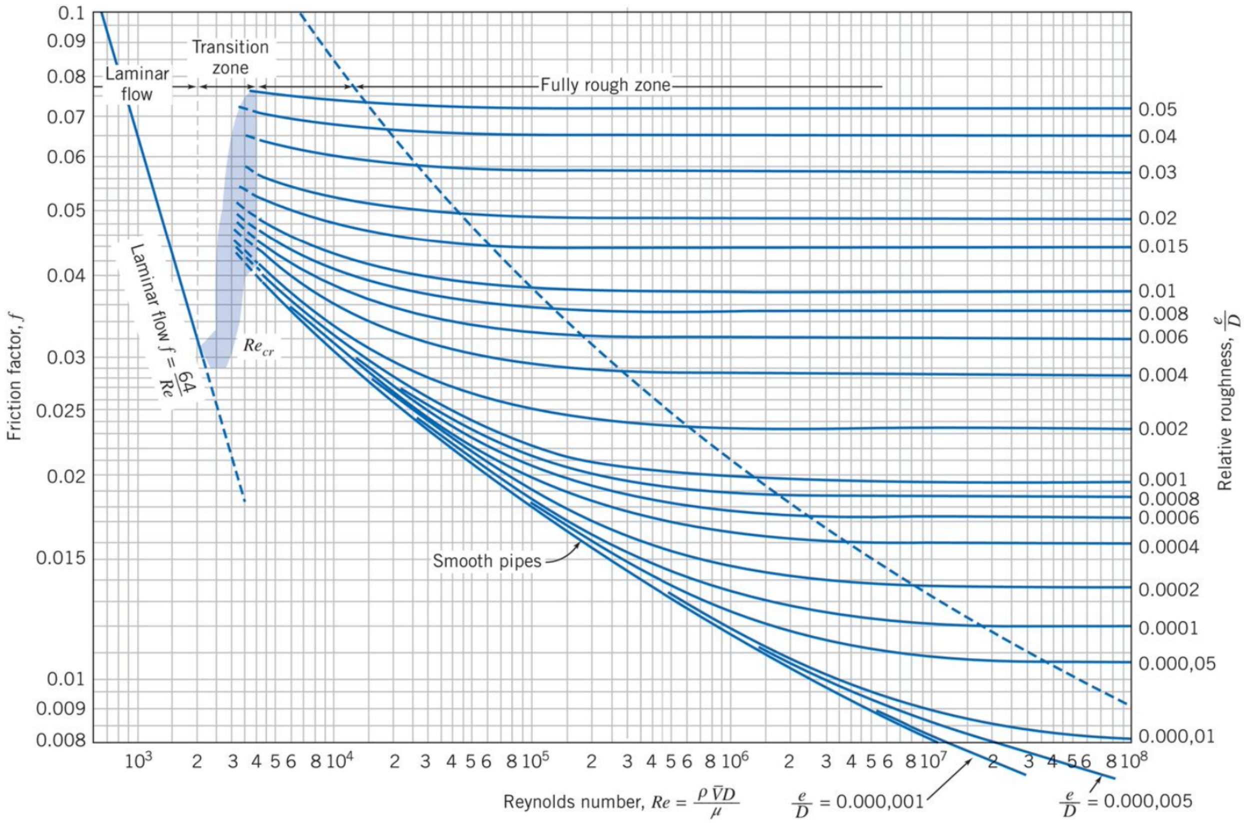


Figure 11.6. The Moody plot, which plots the (Darcy) friction factor as a function of Reynolds number for different relative roughnesses. This figure is from Pritchard, P.J. and Mitchell, J.W., *Fox and McDonald's Introduction to Fluid Mechanics*, 9th ed., Wiley.

Notes:

(1) For Reynolds numbers less than 2,300, one may use either the analytical expression for the friction factor,

$$f_D = \frac{64}{\mathrm{Re}_D}, \tag{11.40}$$

or the Moody plot.

(2) Reynolds numbers between approximately 2,300 and 4,000 correspond to the transitional regime between laminar and turbulent flow. The gray region in the Moody plot reflects the fact that the friction factor can vary significantly in this region. At best, bounds can be determined for the friction factor in this region rather than a specific value.
(3) The fully rough zone (aka wholly turbulent zone, fully turbulent zone) in the Moody plot is a region where the friction factor is a weak function of the Reynolds number, but a strong function of the relative roughness. If the Reynolds number of a flow is unknown, but is expected to be large, it is often helpful to assume that the flow is in the fully rough zone as an initial first guess.
(4) The roughnesses of various types of pipe materials have been compiled into tables such as Table 11.1. Note that "smooth" in the table does not mean frictionless.

TABLE 11.1. A table of pipe material wall roughnesses.

Material (new)	$\boldsymbol{\epsilon}$ (ft)	$\boldsymbol{\epsilon}$ (mm)
riveted steel	0.003 - 0.03	0.9 - 9.0
concrete	0.001 - 0.01	0.3 - 3.0
wood stave	0.0006 - 0.003	0.18 - 0.9
cast iron	0.0085	0.26
galvanized iron	0.0005	0.15
asphalted cast iron	0.0004	0.12
commercial steel or wrought iron	0.00015	0.046
drawn tubing	0.000005	0.0015
glass	smooth	smooth

1. Using the Moody chart, determine the friction factor for a Reynolds number of 10^5 and a relative roughness of 0.001.
2. What is the friction factor for a Reynolds number of 1000?
3. What is the friction factor for a Reynolds number of 10^6 in a smooth pipe?

SOLUTION:

1. The friction factor is $\boxed{f_D \approx 0.0225}$ (Follow the red lines in the following figure.)
2. Since the Reynolds number is less than 2300, we can use the exact laminar flow relation:

$$f_D = \frac{64}{\mathrm{Re}_D} \Rightarrow \boxed{f_D = 0.064}.$$

Alternately, we could use the Moody chart by following the blue lines in the following figure.

3. The friction factor is $\boxed{f_D \approx 0.012}$. (Follow the green lines in the following figure.)

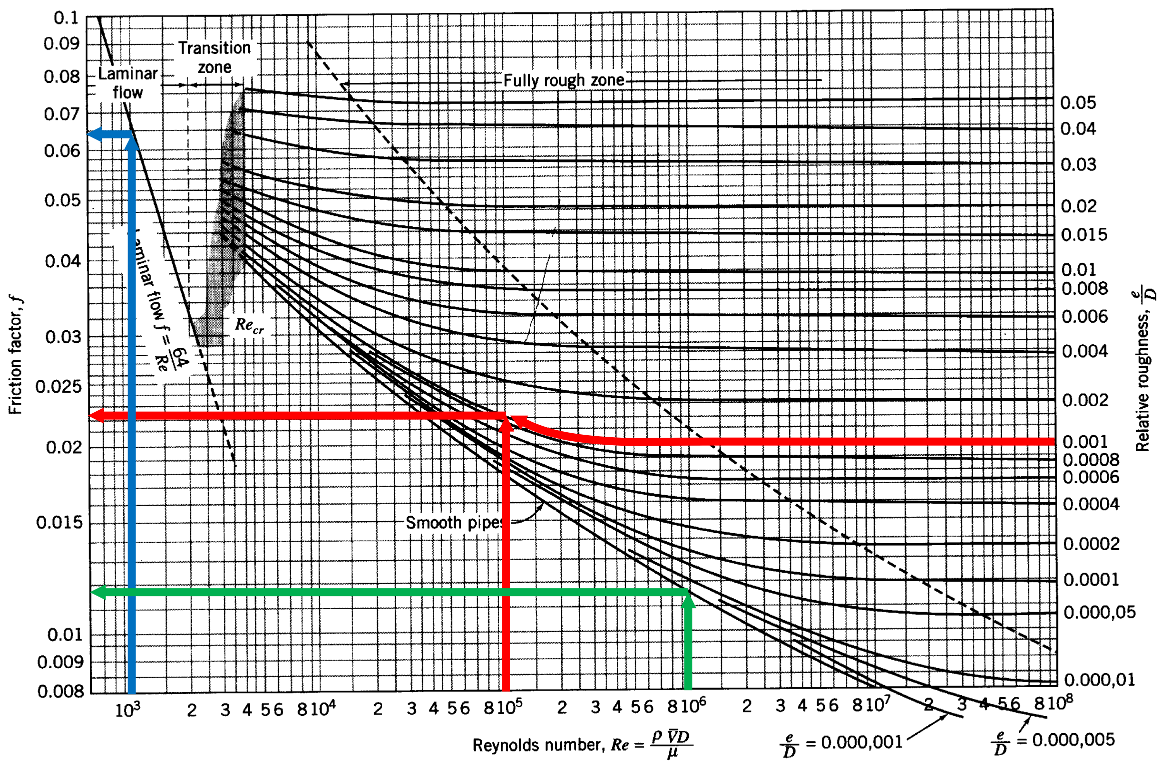


Fig. 8.13 Friction factor for fully developed flow in circular pipes. (Data from [8], used by permission.)

Create a computer program that uses the Colebrook formula to calculate a friction factor. The program inputs should be the Reynolds number and the relative roughness. The output should be the friction factor.

Use your program to create a copy of the Moody plot. Plot the friction factor for Reynolds numbers between 4000 and $1*10^8$ for 10 relative roughness values between $1*10^{-6}$ and $1*10^{-2}$. Use logarithmic horizontal and vertical axes.

SOLUTION:
The Colebrook formula is,

$$\sqrt{\frac{1}{f}} \approx -2.0\log_{10}\left(\frac{\epsilon/D}{3.7}+\frac{2.51}{\mathrm{Re}_D\sqrt{f}}\right). \tag{1}$$

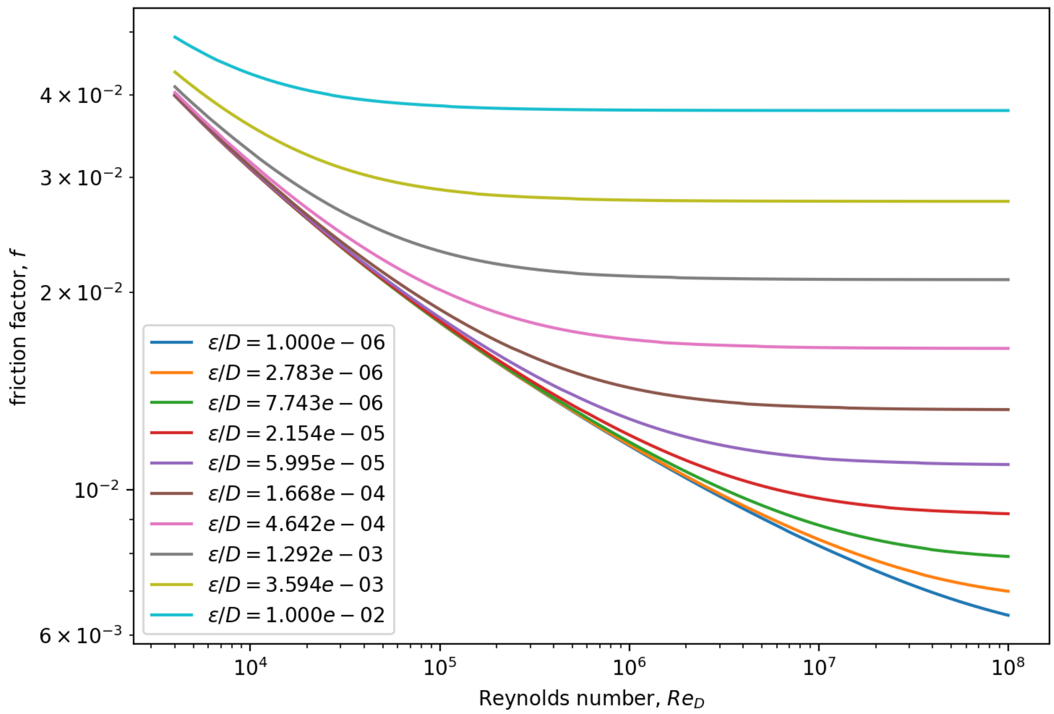
This function is implicit in *f* so an iterative scheme must be used to solve it. There are various algorithms that can be used to solve for *f*. In this solution, the following algorithm is used:

1. Guess a value for the friction factor *f*. Use the Haaland formula for this first guess,
$$\sqrt{\frac{1}{f}} \approx -1.8\log_{10}\left[\frac{6.9}{\mathrm{Re}_D}+\left(\frac{\epsilon/D}{3.7}\right)^{1.11}\right]. \tag{2}$$
2. Solve for the friction factor on the left-hand side of the Colebrook formula (Eq. (1)), call this *f'*, using the guessed value for *f* on the right-hand side.
3. Is the value of *f'* equal to *f* within an acceptable tolerance (*tol*), i.e.,
$$\text{Is } \frac{|f'-f|}{f} < tol? \tag{3}$$
If not, then let *f* = *f'* and repeat step 2. If so, then we now have our value for *f*. A counter is also included in the program to ensure that we don't iterate an unacceptably large number of times. Usually this algorithm solve for *f* within a few iterations, but the counter is a fail-safe measure since the iterative scheme isn't guaranteed to converge.

This algorithm is implemented here using the Python programming language. The plot is shown at the end of this document.

```
# pipe_50.py

# Import some helpful Python libraries.
import numpy as np  # used for numerical routines
import matplotlib.pyplot as plt  # used for making plots

# Create a function for the Haaland formula.
def f_Haaland(Re, e_D):
    return (-1.8*np.log10(6.9/Re+(e_D/3.7)**1.11))**-2

# Create a function for solving the Colebrook formula.
def f_Colebrook(Re, e_D):
    # The first guessed value for the friction factor uses the Haaland formula.
    fprime = f_Haaland(Re, e_D)

    # Set the relative difference between f and fprime large initially
    # to start the loop.
    freldiff = 1

    # Set the acceptable tolerance to be <= 0.1 percent.
    tol = 0.001

    # Only go up to 1000 iterations.
    max_counter = 1000

    # Initialize the counter.
    counter = 0

    # Loop until the relative difference is less than the tolerance or
    # until the maximum number of iterations is reached.
    while ((freldiff > tol) and (counter < max_counter)):
        counter = counter + 1  # Update the counter.
```

```
        f = fprime  # Set f equal to fprime for solving the Colebrook formula.

        # Colebrook formula.
        fprime = (-2.0*np.log10(e_D/3.7 + 2.51/Re/np.sqrt(f)))**-2

        # Calculate the relative difference
        freldiff = np.absolute((fprime-f)/f)


    # The maximum number of iterations was reached.  Print a warning
    # and exit the program.
    if (counter == max_counter):
        print("The maximum number of iterations was reached.  Did not converge on a value for 
f.")
        exit(1)

    # Return the converged value for the friction factor.
    return f


# Make arrays of Reynolds numbers and relative roughnesses values.
Re = np.geomspace(4e3, 1e8, num=100)
e_D = np.geomspace(1e-6, 1e-2, num=10)

# Create an array of friction factor values the same size as the
# Reynolds number array.  Initialize the array with zero values.
f = np.full_like(Re,0)

# Calculate the friction factor for all combinations of the Reynolds
# numbers and relative roughnesses.
for j in range(len(e_D)):
    for i in range(len(Re)):
        f[i] = f_Colebrook(Re[i], e_D[j])
    # Create some text for the plot legend.
    legendtext = '$\epsilon/D = %.3e$' % e_D[j]
    # Plot these friction factors as a function of Reynolds number for
    # this particular relative roughness.
    plt.plot(Re, f, "-", label=legendtext)

# Label the plot, change the axes to be logarithmic, and show the legend.
plt.xlabel(r"Reynolds number, $Re_D$")
plt.ylabel(r"friction factor, $f$")
plt.xscale("log")
plt.yscale("log")
plt.legend()
plt.show()
```


friction factor, f
4×10^{-2}
3×10^{-2}
2×10^{-2}
10^{-2}
6×10^{-3}
10^{4}
10^{5}
10^{6}
10^{7}
10^{8}
Reynolds number, Re_D
$\varepsilon/D = 1.000e-06$
$\varepsilon/D = 2.783e-06$
$\varepsilon/D = 7.743e-06$
$\varepsilon/D = 2.154e-05$
$\varepsilon/D = 5.995e-05$
$\varepsilon/D = 1.668e-04$
$\varepsilon/D = 4.642e-04$
$\varepsilon/D = 1.292e-03$
$\varepsilon/D = 3.594e-03$
$\varepsilon/D = 1.000e-02$

11.5. Other Losses

The loss due to the viscous resistance caused by the pipe walls is referred to as a major loss (aka the straight-run headloss). Pressure losses may occur due to viscous dissipation resulting from fluid interactions with other parts of a pipe system such as valves, bends, contractions/expansions, inlets, and connectors. These losses are known as minor losses (aka the fittings headloss). The names can be misleading since it's not uncommon in pipe systems to have most of the pressure loss resulting from the minor losses, e.g., a pipe system with a large number of bends and valves, but short sections of straight pipe. What causes these minor losses? The pressure loss results primarily from viscous dissipation in regions with large velocity gradients, such as in a recirculation zone as shown in Figure 11.7.

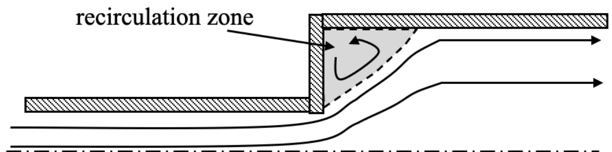


Figure 11.7. A sketch showing where viscous losses occur in a sudden pipe expansion.

A closely related phenomenon known as the vena contracta acts to effectively reduce the diameter at entrances and bends (Figure 11.8). The recirculation zone also results in a pressure loss.

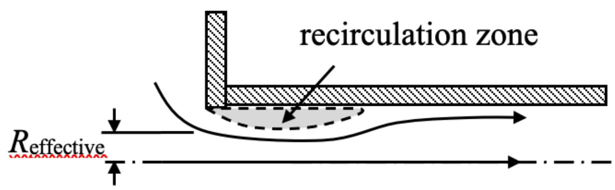


Figure 11.8. A sketch illustrating a vena contracta.

Although minor loss coefficients can be determined analytically for certain situations, most frequently the loss coefficient for a particular device is found experimentally. Essentially, one measures the pressure drop across the device, Δp, and forms the loss coefficient, k, using,

$$k = \frac{\Delta p}{\frac{1}{2}\rho \bar{u}^2}, \tag{11.41}$$

where ρ is the fluid density and $\bar{u}$ is the average speed through the device. Many tables with experimentally determined loss coefficients are available, for example in *Perry's Chemical Engineers' Handbook*.

Notes:

(1) When using a loss coefficient, it is important to know what velocity has been used to form the coefficient. For example, the loss coefficient for a contraction is typically based on the speed downstream of the contraction, while the loss coefficient for an expansion is based on the speed upstream of the expansion.
(2) Minor losses are sometimes given in terms of equivalent lengths of pipe. An equivalent minor loss of 10 pipe diameters worth of a particular type of pipe means that the major loss caused by a pipe of that type, 10 diameters in length will give the same pressure loss as the minor loss. Thus, a loss coefficient and equivalent pipe length, L_e, can be related by,

$$k = f_D \left(\frac{L_e}{D}\right). \tag{11.42}$$

(3) For non-circular pipes or pipes that are not completely filled, the same methods of determining the friction factor and loss coefficients are used, except that a hydraulic diameter, D_h, is used in place of the diameter. The hydraulic diameter is defined as,

$$D_h := \frac{4A}{P_w}, \tag{11.43}$$

where A and P_w are the cross-sectional *flow area* (not necessarily the pipe cross-sectional area) and wetted perimeter of the pipe, i.e., the part of the pipe that is in contact with the fluid. For example, consider a completely filled square pipe of side length L (Figure 11.9). The hydraulic diameter for such a pipe is,

$$D_h = \frac{4(L^2)}{4L} = L. \tag{11.44}$$

Now consider a completely filled annular pipe with outer diameter D_o and inner diameter D_i

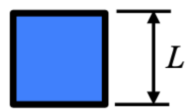


Figure 11.9. A square cross-sectioned pipe filled with fluid.

(Figure 11.10). The hydraulic diameter for this case is,

$$D_h = \frac{4\left(\frac{\pi}{4}D_o^2 - \frac{\pi}{4}D_i^2\right)}{\pi D_o - \pi D_i} = \frac{D_o^2 - D_i^2}{D_o + D_i} = D_o - D_i. \tag{11.45}$$

Now consider a half-filled circular pipe of diameter D (Figure 11.11). The hydraulic diameter for

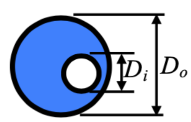


Figure 11.10. An annular cross-sectioned pipe filled with fluid.

this case is,

$$D_h = \frac{4\left(\frac{1}{2}\frac{\pi}{4}D^2\right)}{\left(\frac{1}{2}\pi D\right)} = D. \tag{11.46}$$

Notes:

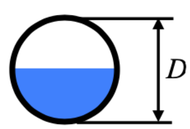


Figure 11.11. A circular cross-sectioned pipe half filled with fluid.

(a) Often a hydraulic radius, R_h, is used instead of a hydraulic diameter for flows in conduits with a free surface. The hydraulic radius is defined as,

$$R_h := \frac{A}{P_w}. \tag{11.47}$$

Using this definition, $D_h \neq 2R_h$, but is instead, $D_h = 4R_h$, which can be confusing. The Manning Formula (not covered in these notes) is frequently used in the analysis of free surface conduit flows.

11.6. The Extended Bernoulli Equation

Recall from Section 4.6 that the First Law of Thermodynamics may be written as,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12}, \tag{11.48}$$

where each of the terms in the equation has dimensions of length. The "1" and "2" subscripts refer to the inlet and outlet conditions, respectively. Note that in writing this form of the Extended Bernoulli Equation (EBE), it has been assumed that z points in the direction opposite to gravitational acceleration. If z pointed in the same direction as gravity, then there would be a $-$ sign in front of the z in the EBE.

The various terms in the equation are referred to as "head" quantities:

$$\frac{p}{\rho g} := \underline{\text{pressure head}}, \tag{11.49}$$

$$\alpha\frac{\bar{V}}{2g} := \underline{\text{velocity or dynamic head}}, \tag{11.50}$$

$$z := \underline{\text{elevation head}}, \tag{11.51}$$

$$H_L := \underline{\text{head loss}}, \tag{11.52}$$

$$H_S := \underline{\text{shaft head}}. \tag{11.53}$$

$$\tag{11.54}$$

Recall that the α in the velocity head term is the kinetic energy correction factor, which accounts for the fact that an average speed is used in the EBE rather than the real velocity profile (again, refer to Section 4.6). A value of $\alpha=2$ is used for laminar flows while $\alpha=1$ is typically assumed for turbulent flows (actually, $\alpha\to 1$ as $\mathrm{Re}_D\to\infty$).

The head loss term (H_L) accounts for both major and minor losses and may be written as,

$$H_{L,12}=\sum_{\forall i}k_i\frac{\bar{V}_i^2}{2g}, \tag{11.55}$$

where the subscript "i" accounts for every loss in the pipe system. Recall that the major loss coefficient may be written as,

$$k_{\text{major}}=f_D\left(\frac{L}{D}\right). \tag{11.56}$$

The shaft head term (H_S) accounts for the pressure addition (or reduction) resulting from the inclusion of devices such as pumps, compressors, fans, turbines, and windmills. Those devices that add head to the flow are positive (e.g., pumps), while those that extract head are negative (e.g., turbines). The shaft head term may be written in terms of shaft power, $\dot{W}_S$, as,

$$H_{S,12}=\frac{\dot{W}_{S,12}}{\dot{m}g}, \tag{11.57}$$

where $\dot{m}$ is the mass flow rate through the device.

Notes:

(1) One can visualize head quantities using manometers. For example, consider Figure 11.12, which shows a straight section of pipe with manometers at upstream (1) and downstream (2) locations. The pressure heads at the inlet and outlet are given by, respectively, $H_1=p_1/(\rho g)$ and $H_2=p_2/(\rho g)$. The major head loss between these two points is simply $H_L=H_1-H_2=(p_1-p_2)/(\rho g)$. A minor head loss through a device such as a valve can be shown in a similar way, as in Figure 11.13. The shaft head rise across a pump is shown in Figure 11.14. Reporting an equivalent length for a minor head loss can be thought of as replacing the device, e.g., the valve with the head loss shown in Figure 11.13, with a straight section of pipe (Figure 11.12) that produces the same head loss as the valve.

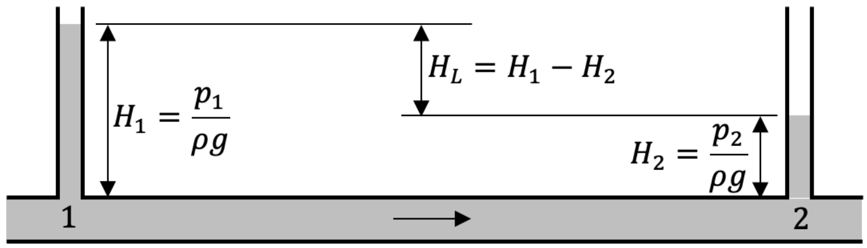


Figure 11.12. A schematic illustrating the major head loss in a pipe.

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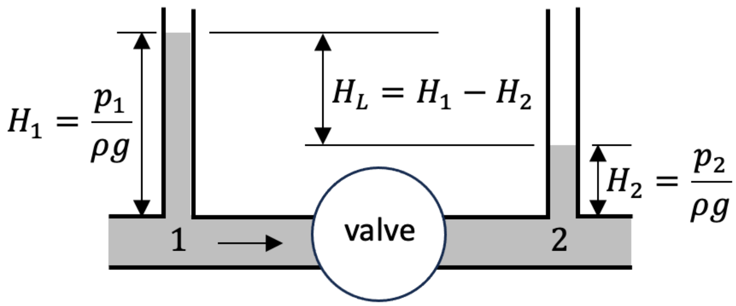

Figure 11.13. A schematic illustrating the minor head loss through a valve.

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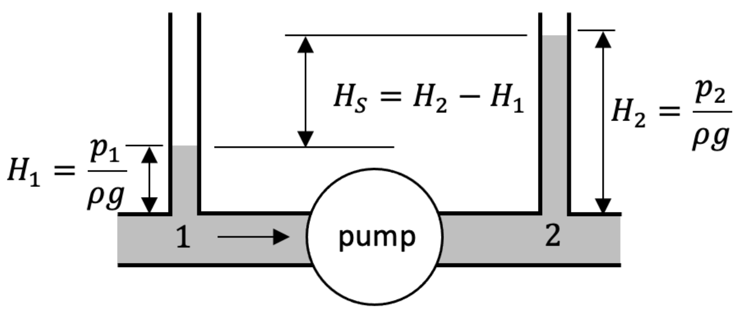

Figure 11.14. A schematic illustrating the shaft head rise across a pump.

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(2) One often must make a number of assumptions at the beginning of a pipe flow solution, e.g., the flow is laminar, the flow is turbulent, or the flow is in the fully rough zone. For example, for flow through a hypodermic needle, it's reasonable to assume that the flow will be laminar since the needle diameter is so small. Having experience with pipe flow systems helps one to make good assumptions. Regardless of what assumptions are made, it is important that one verifies that the calculated flow conditions are consistent with the assumptions that were made. For example, if one assumes laminar flow in the hypodermic needle then solves for the flow velocity, then the Reynolds number should be checked to verify that a laminar flow assumption was correct. If so, then the solution procedure is consistent. If not, then the laminar flow assumption was incorrect and a turbulent flow assumption should be made and the problem re-solved.

(3) Pipe manufacturers will often provide nomographs to quickly determine the volumetric flow rate, average speed, and, often, pressure drop due to major head losses in their pipes. A nomograph is a type of visual calculator, an example of one is shown in Figure 11.15.

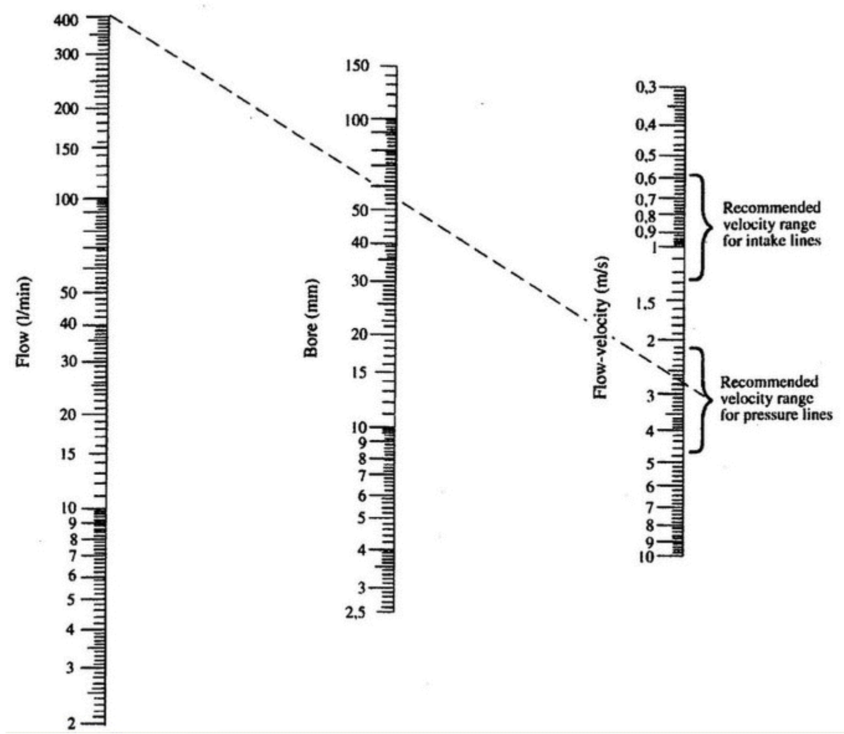

Figure 11.15. An example nomograph for relating the volumetric flow rate, average flow speed, and pipe inner diameter (figure from). To use the nomograph, locate two known quantities on the plots, then draw a straight line between them and read off the third quantity from the plot. In this figure, the flow rate is known to be 400 l/min with an inner pipe diameter (i.e., bore) of 55 mm. The average flow speed is then 2.8 m/s.

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11.7. Pipe Systems

Pipe flow systems can be classified as being of one of three types:

- Type I: The desired flow rate is specified and the required pressure drop must be determined.
- Type II: The desired pressure drop is specified and the required flow rate must be determined.
- Type III: The desired flow rate and pressure drop are specified and the required pipe diameter must be determined.

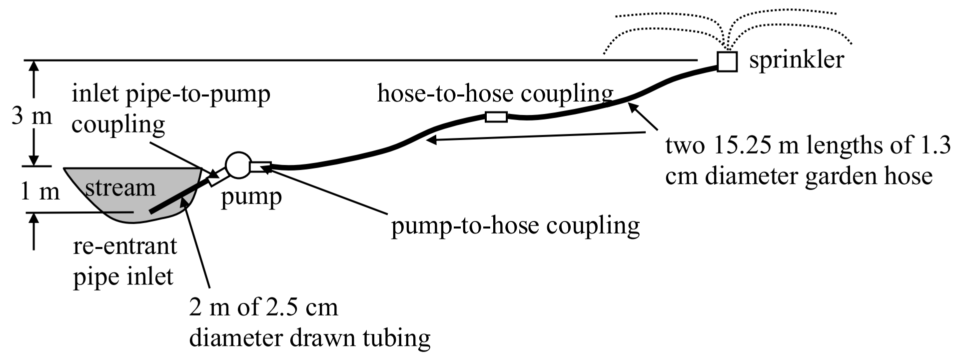
Type I pipe systems are the easiest to solve. Since the flow velocity and diameter are known, calculation of the major loss coefficient, and the friction factor in particular, is straightforward. Type II and Type III problems are more challenging to solve since the friction factor is unknown. These types of pipe systems usually require iteration to solve.

Notes:

(1) There is no unique iterative scheme that must be used to solve Type II and Type III pipe flow problems. Different people may propose different algorithms. In addition, there is no guarantee that a particular iterative scheme will converge to a solution.

(2) When using an iterative scheme, choose an initial flow rate or diameter that is reasonable. Don't start with an exceedingly small or large value. For example, for a Type II pipe system, choose a starting flow rate that corresponds to the fully turbulent zone region.
(3) It's often worthwhile to first assume that a Type II and Type III flow system is operating in the fully rough zone of the Moody plot. Using this assumption will often avoid the need for iteration. However, one must verify at the end of the solution that the assumption of fully rough flow was correct. If not, then an iterative solution should be considered with the fully rough zone conditions used as a starting point for the iterations.
(4) Following are typical flow speeds for different flow scenarios, but keep in mind that the actual speeds will depend on the specific geometry under consideration. These speeds can be used for initial speed guesses in an iterative calculation.
   - pumped liquids: 1.5 m/s,
   - gravity-fed liquids: 1 m/s,
   - gases: 20 m/s.

A homeowner plans to pump water from a stream in their backyard to water their lawn. A schematic of the pipe system is shown in the figure.



Details of the system are given in the following table. The design flow rate for the system is 2.5×10^{-4} m^3/s.

Item	**Value**
water density	1000 kg/m^3
water dynamic viscosity	1×10^{-3} kg/(m.s)
inlet pipe length	2 m
inlet pipe diameter	2.5 cm
inlet pipe material	drawn tubing
hose length	two 15.25 m lengths
hose diameter	1.3 cm
hose roughness	smooth
pipe inlet loss coefficient	0.8
inlet pipe-to-pump coupling loss coefficient	0.1
pump-to-hose coupling loss coefficient	0.2
hose-to-hose coupling loss coefficient	0.5
pressure drop across the sprinkler	210 kPa at 2.5×10^{-4} m^3/s flow rate
sprinkler nozzle exit diameter	4 mm

For the following questions, circle the answer that is most correct.

1. What is the loss coefficient for the sprinkler at design conditions? Base the sprinkler loss coefficient on the velocity just upstream of the sprinkler.
2. What is the friction factor for the hose?
3. What is the velocity head, including the kinetic energy correction factor, at the sprinkler exit?

For the next two questions, assume that the loss coefficient for the sprinkler is 100 and the friction factor for the hose is 0.01.

4. What is the total *minor* head loss in the system?
5. What is the total *major* head loss in the system?

For the next question, assume that the velocity head at the sprinkler exit, including the kinetic energy correction factor, is 10 m and the total head loss is 10 m.

6. What is the shaft head required to operate the system at the design flow rate?

For the next question, assume that the shaft head required to operate the system at the design flow rate is 100 m.

7. What power must be supplied to the pump if the pump is 65% efficient?

SOLUTION:

1. The loss coefficient for the sprinkler may be found using the definition of a loss coefficient.

$$K_{\text{sprinkler}} = \frac{\Delta p_{\text{sprinkler}}}{\frac{1}{2}\rho \bar{V}_{\text{hose}}^2} \tag{1}$$

where

$$\bar{V}_{\text{hose}} = \frac{Q}{\frac{\pi}{4}D_{\text{hose}}^2} \tag{2}$$

Using the given parameters,

$$\bar{V}_{\text{hose}} = 1.88\ \text{m/s} \quad \text{and} \quad \boxed{K_{\text{sprinkler}} = 118} \tag{3}$$

2. The friction factor for the hose may be found using the Moody plot. The Reynolds number for the flow in the hose is:

$$\text{Re}_D = \frac{\rho \bar{V}_{\text{hose}} D_{\text{hose}}}{\mu} \tag{4}$$

Using the given parameters, the hose Reynolds number is $\text{Re}_D = 24500$. From the Moody chart and using the smooth curve (we're told to consider the hose to be "smooth"), the friction factor for the hose is $\boxed{f_{\text{hose}} = 0.0244}$.

3. The velocity head, including the kinetic energy correction factor, at the sprinkler's exit is:

$$\left.\alpha \frac{\bar{V}^2}{2g}\right|_{\substack{\text{sprinkler}\\ \text{exit}}} \tag{5}$$

where

$$\bar{V}_{\substack{\text{sprinkler}\\ \text{exit}}} = \frac{Q}{\frac{\pi}{4}D_{\substack{\text{sprinkler}\\ \text{exit}}}^2} \Rightarrow \bar{V}_{\substack{\text{sprinkler}\\ \text{exit}}} = 19.9\ \text{m/s} \tag{6}$$

and

$$\alpha = 1 \tag{7}$$

since

$$\text{Re}_D = \frac{\rho \bar{V}_{\substack{\text{sprinkler}\\ \text{exit}}} D_{\substack{\text{sprinkler}\\ \text{exit}}}}{\mu} \Rightarrow \text{Re}_D = 79600 > 2300 \Rightarrow \text{Turbulent flow at the exit!} \tag{8}$$

Thus,

$$\boxed{\left.\alpha \frac{\bar{V}^2}{2g}\right|_{\substack{\text{sprinkler}\\ \text{exit}}} = 20.2\ \text{m}} \tag{9}$$

4. The total minor head loss for the system includes minor losses at the pipe inlet, the pump couplings, the hose coupling, and the sprinkler.

$$H_{L,\text{minor}} = \left[K_{\text{inlet}} + K_{\substack{\text{inlet pipe-pump}\\ \text{coupling}}}\right]\frac{\bar{V}_{\text{pipe}}^2}{2g} + \left[K_{\substack{\text{pump-hose}\\ \text{coupling}}} + K_{\substack{\text{hose-hose}\\ \text{coupling}}} + K_{\text{sprinkler}}\right]\frac{\bar{V}_{\text{hose}}^2}{2g} \tag{10}$$

Using the given parameters,

$$\boxed{H_{L,\text{minor}} = 21.5\ \text{m}}$$

5. The total major head loss for the system includes the major losses in the inlet pipe and the hoses.

$$H_{L,\text{major}} = \left.f\left(\frac{L}{D}\right)\frac{\bar{V}^2}{2g}\right|_{\text{pipe}} + \left.f\left(\frac{L}{D}\right)\frac{\bar{V}^2}{2g}\right|_{\text{hoses}} \tag{11}$$

$$\boxed{H_{L,\text{major}} = 10.4\ \text{m}}$$

6. The required shaft head may be found by applying the Extended Bernoulli Equation from the free surface of the stream to the outlet of the sprinkler.

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \tag{12}$$

$$H_S=\left(\frac{p_2-p_1}{\rho g}\right)+\left(\alpha\frac{\bar{V}_2^1}{2g}-\alpha\frac{\bar{V}_1^1}{2g}\right)+\left(z_2-z_1\right)+H_L \tag{13}$$

where

$$p_1=p_2=p_{\text{atm}} \tag{14}$$

$$\bar{V}_1\approx 0 \tag{15}$$

$$\bar{V}_2=\bar{V}_{\substack{\text{sprinkler}\\ \text{exit}}} \tag{16}$$

$$H_L=H_{L,\text{minor}}+H_{L,\text{major}} \tag{17}$$

Using the given parameters, $\boxed{H_S = 55.1\text{ m}}$.

7. The power that must be supplied to the pump with the given efficiency is:

$$\dot{W}_{\substack{\text{into}\\ \text{pump}}}=\frac{\dot{W}_{\substack{\text{into}\\ \text{water}}}}{\eta}=\frac{\dot{m}gH_S}{\eta}=\frac{\rho QgH_S}{\eta} \tag{18}$$

where, using the given parameters,

$$\boxed{\dot{W}_{\substack{\text{into}\\ \text{pump}}}=210\text{ W}}$$

It rains during the construction of a building and water fills a recently excavated pit to a depth 0.5 m. In order to continue construction, the water must first be pumped out of the pit. A hose with a length of 50 m, a diameter of $2.5*10^{-2}$m, and a surface roughness of $5.0*10^{-5}$m is attached to a pump. Note that the kinematic viscosity of the water is $1.005*10^{-6}$ m^2/s and the density is 1000 kg/m^3.

a. If the pump is placed at the pit's surface (figure A), what is the maximum depth of the pit, H, for which water can be pumped out at a velocity of 1 m/s without causing cavitation in the pipe? The vapor pressure of water for the current temperature is 2.337 kPa (abs) and atmospheric pressure is 101 kPa (abs).
b. If the pump is placed at the bottom of the pit (figure B), what is the maximum depth of the pit, H, for which water can be pumped out at a velocity of 1 m/s? Assume that the pump supplies a power of 200 W to the fluid.

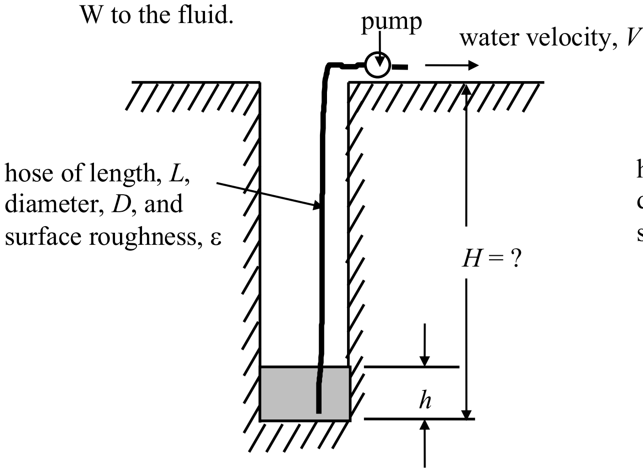


Figure A

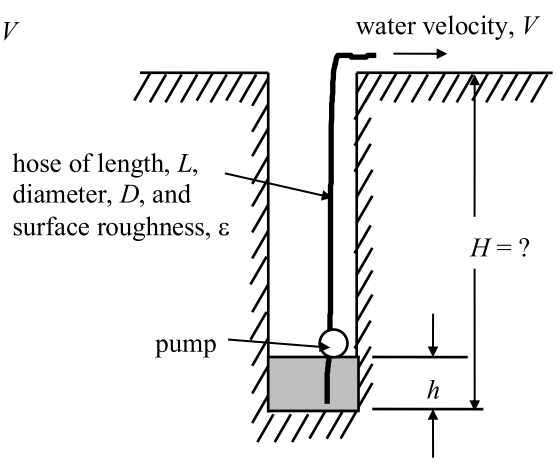
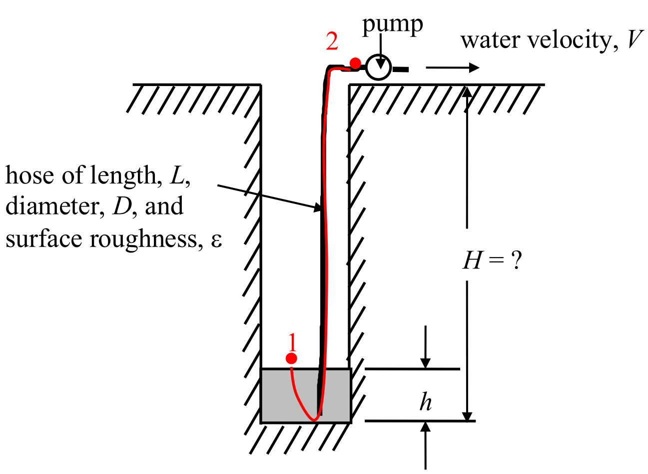


Figure B

SOLUTION:



Apply the Extended Bernoulli Equation from point 1 to point 2,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,1\to 2}+H_{S,1\to 2} \quad (1)$$

where,

$p_1 = p_{atm}$,

$p_2 = p_v$ (point 2 is located just before the pump; the lowest pressure that can be reached is vapor pressure),

$z_1 = h$,

$z_2 = H = ?$,

$\bar{V}_1 \approx 0$,

$\bar{V}_2 = V$,

$\alpha_2 \approx 1$ (assuming turbulent flow),

$H_{S,1\to 2} = 0$ (there's no pump between points 1 and 2),

$$H_{L,12} = \sum_i K_i \frac{\bar{V}_i^2}{2g} = \left(K_{\text{minor}} + K_{\text{major}}\right)\frac{\bar{V}^2}{2g}. \quad (2)$$

Now determine the flow Reynolds number,

$$\text{Re}_D = \frac{\bar{V}_2 D}{\nu} = \frac{(1\ \text{m/s})(2.5\text{e-2 m})}{(1.005\text{e-6 m}^2/\text{s})} = 25{,}000 \Rightarrow \text{turbulent flow assumption is justified!} \quad (2)$$

Also determine the relative roughness of the pipe,

$$\frac{\varepsilon}{D} = \frac{5.0\text{e-5 m}}{2.5\text{e-2 m}} = 0.002 \quad (3)$$

Use the Moody chart to determine the pipe's friction factor,

$\Rightarrow \underline{f = 0.0289}$ (4)

The major loss coefficient, $K_{\text{major}} = f(L/D)$ is,

$K_{\text{major}} = 57.7$.

Since the major loss coefficient is so large, it's reasonable to neglect the minor loss coefficients.

Substituting into the EBE and simplifying,

$$\frac{p_v}{\rho g} + \frac{V^2}{2g} + H = \frac{p_{atm}}{\rho g} + h - K_{\text{major}}\frac{V^2}{2g}, \quad (6)$$

$$H = \frac{p_{atm} - p_v}{\rho g} + h - \left[1 + K_{\text{major}}\right]\frac{V^2}{2g}. \quad (7)$$

Using the given parameters,

$p_{atm} = 101*10^3$ Pa (abs),

$p_v = 2.337*10^3$ Pa (abs),

$\rho = 1000$ kg/m^3,

$g = 9.81$ m/s^2,

$h = 0.5$ m,

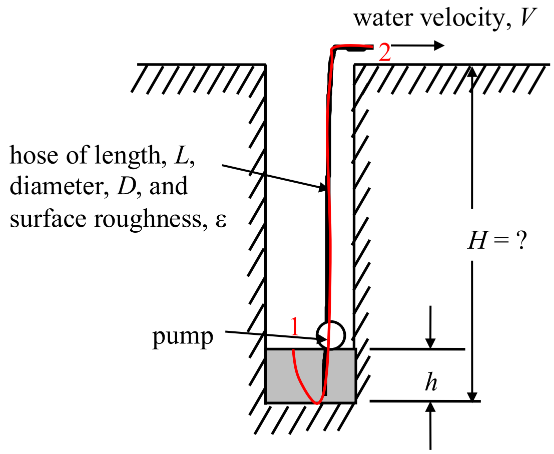
$K_{\text{major}} = 57.7$,

$V = 1$ m/s,

=> $\boxed{H = 7.57\text{ m}}$.

The height is short because we're relying on atmospheric pressure to push the water up through the hose.

Now consider the case when the pump is in the pit.



Apply the Extended Bernoulli Equation from point 1 to point 2,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,1\to 2}+H_{S,1\to 2} \tag{8}$$

where,

$p_1 = p_{atm}$,
$p_2 = p_{atm}$ (flow exits to the atmosphere),
$z_1 = h$,
$z_2 = H = ?$,
$\bar{V}_1 \approx 0$,
$\bar{V}_2 = V$,
$\alpha_2 \approx 1$ (turbulent flow, as before),
$H_{S,12} = \frac{P}{\dot{m}g}$ (there's a pump between points 1 and 2),
$H_{L,12} = \sum_i K_i \frac{\bar{V}_i^2}{2g} = K_{\text{major}}\frac{\bar{V}^2}{2g}$ (neglecting minor losses, as before) (9)

Substitute and solve for the height,

$$\frac{p_{atm}}{\rho g}+\frac{V^2}{2g}+H=\frac{p_{atm}}{\rho g}+h-K_{\text{major}}\frac{V^2}{2g}+\frac{P}{\dot{m}g}, \tag{10}$$

$$H = h-\left(1+K_{\text{major}}\right)\frac{V^2}{2g}+\frac{P}{\dot{m}g}. \tag{11}$$

Using the given values,

$h = 0.5$ m,
$K_{\text{major}} = 57.7$,
$V = 1$ m/s,
$g = 9.81$ m/s^2,
$P = 200$ W,
=> $\dot{m} = \rho V\frac{\pi D^2}{4} = 0.491$ kg/s,
=> $\boxed{H = 39.0\text{ m}}$.

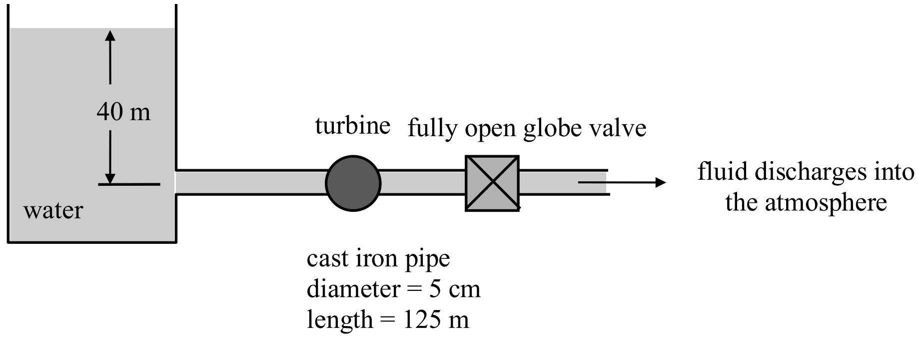
The height in part (b) is much higher than the height in part (a) because in part (b) the pump is used to increase the pressure beyond atmospheric pressure, which pushes the fluid up the hose.

Video solution: https://www.youtube.com/watch?v=6jWgPpJq_3s

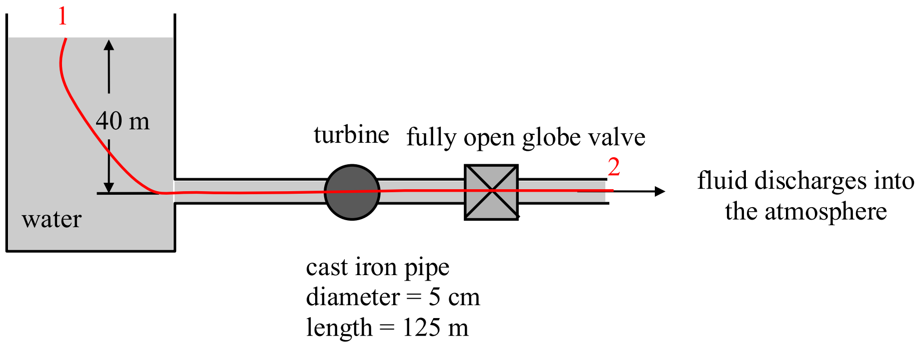
Following is python code used to perform the calculations.

```
# pipe_03.py

import numpy as np

def f_Haaland(Re, e_D):  # Calculate the friction factor using the Haaland formula
    return (-1.8*np.log10(6.9/Re+(e_D/3.7)**1.11))**-2

def f_Colebrook(Re, e_D):  # Calculate the friction factor using the Colebrook formula
    fprime = f_Haaland(Re, e_D)
    freldiff = 1
    tol = 0.001
    while (freldiff > tol):
        f = fprime
        fprime = (-2.0*np.log10(e_D/3.7 + 2.51/Re/np.sqrt(f)))**-2
        freldiff = np.absolute((fprime-f)/f)
    return f

# Initialize variable values.
g = 9.81 # m/s^2; gravitational acceleration
rho = 1000 # kg/m^3; water density
nu = 1.005e-6  # m^2/s; water kinematic viscosity
h = 0.5  # m; water free surface height
V = 1 # m/s; average flow speed
L = 50 # m; hose length
D = 2.5e-2  # m; hose diameter
e = 5.0e-5  # m; hose roughness
patm = 101e3  # Pa; atmospheric pressure
pv = 2.337e3  # Pa; vapor pressure
P = 200 # W; pump power

# Calculate the Reynolds number.
Re = V*D/nu
print("Re = %.3e" % Re)

# Calculate the relative roughness.
e_D = e/D
print("e/D = %.3e" % e_D)

# Determine the friction factor.
f = f_Colebrook(Re, e_D)
print("f = %.3e" % f)

# Calculate the viscous loss coefficient.
K_major = f*(L/D)
print("K_major = %.3e" % K_major)

# Calculate the height in part (A).
H = (patm-pv)/rho/g + h - (1 + K_major)*(V**2)/2/g
print("H = %.3e m" % H)

# Calculate the mass flow rate.
mdot = rho*V*np.pi*D*D/4
print("mdot = %.3e kg/s" % mdot)

# Now calculate the height in part (B).
H = h - (1 + K_major)*(V**2)/2/g + P/mdot/g
print("H = %.3e m" % H)
```

Determine the power, in kW, extracted by the turbine in the system shown below. The pipe entrance is sharp-edged and the volumetric flow rate is 0.004 m^3/s. The density of water is 998 kg/m^3 and the kinematic viscosity is $1.005*10^{-6}$ m^2/s.



SOLUTION:



Apply the Extended Bernoulli Equation from point 1 to point 2,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12}, \tag{1}$$

where,

$p_1 = p_2 = p_{atm}$,

$\bar{V}_1 \approx 0$,

$$\bar{V}_2=\frac{Q}{\pi D^2/4}, \tag{2}$$

$$z_2-z_1=-H, \tag{3}$$

$$H_{L,12}=\sum_i K_i\frac{\bar{V}_i^2}{2g}=\left(K_{\text{square inlet}}+K_{\text{valve}}+K_{\text{major}}\right)\frac{\bar{V}_2^2}{2g}, \tag{4}$$

$$H_{S,12}=\frac{P}{\dot{m}g}. \tag{5}$$

Substitute and solve for the power,

$$\alpha_2\frac{\bar{V}_2^{\,2}}{2g}+H=-\left(K_{\text{square inlet}}+K_{\text{valve}}+K_{\text{major}}\right)\frac{\bar{V}_2^2}{2g}+\frac{P}{\dot{m}g}, \tag{6}$$

$$P=\dot{m}g\left[H+\left(\alpha_2+K_{\text{square inlet}}+K_{\text{valve}}+K_{\text{major}}\right)\frac{\bar{V}_2^2}{2g}\right]. \tag{7}$$

Using the given data,

$\rho = 998$ kg/m^3 and $\nu = 1.005*10^{-6}$ m^2/s,

$D = 0.05$ m,

$L = 125$ m,

$Q = 0.004$ m^3/s => $\dot{m}=\rho Q$=3.992 kg/s and $\bar{V}_2=2.037$ m/s, (8)

$g = 9.81$ m/s^2,

$H = 40$ m,

$K_{\text{square inlet}} = 0.5$ (from minor loss table),

$K_{\text{valve}} = 10$ (from minor loss table),

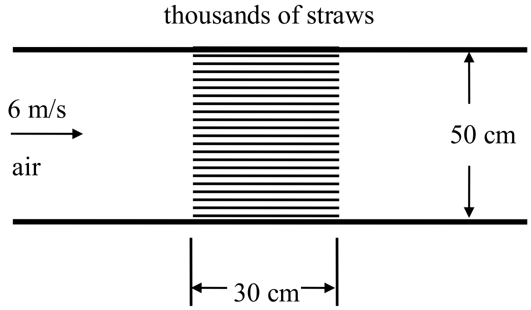
$\epsilon = 0.26*10^{-3}$ m (cast iron pipe) => $\epsilon/D = 0.0052$,

$\mathrm{Re}_D = \frac{\bar{V}_2 D}{\nu} = 101{,}400$ (turbulent flow => $\alpha_2 \approx 1$), (9)
$f = 0.0316$ (from Moody plot or Colebrook Formula),
$K_{\text{major}} = f(L/D) = 79.0$ (The inlet loss coefficient is much smaller than the valve and major loss coefficients and, thus, could be reasonably neglected.)
=> $\boxed{P = 2.32\text{ kW}}$. This is the power extracted by the turbine.

Following is a python code used to perform the calculations.

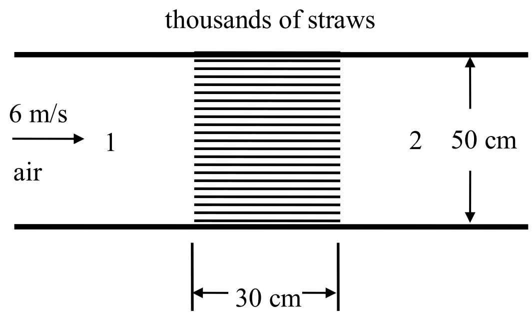
```
# pipe_04.py

import numpy as np

def f_Haaland(Re, e_D):  # Calculate the friction factor using the Haaland formula
    return (-1.8*np.log10(6.9/Re+(e_D/3.7)**1.11))**-2

def f_Colebrook(Re, e_D):  # Calculate the friction factor using the Colebrook formula
    fprime = f_Haaland(Re, e_D)
    freldiff = 1
    tol = 0.001
    while (freldiff > tol):
        f = fprime
        fprime = (-2.0*np.log10(e_D/3.7 + 2.51/Re/np.sqrt(f)))**-2
        freldiff = np.absolute((fprime-f)/f)
    return f

# Initialize variable values.
g = 9.81 # m/s^2; gravitational acceleration
rho = 998 # kg/m^3; water density
nu = 1.005e-6  # m^2/s; water kinematic viscosity
H = 40  # m; water free surface height
Q = 0.004 # m^3/s; volumetric flow rate
L = 125 # m; pipe length
D = 0.05  # m; pipe diameter
e = 0.26e-3  # m; pipe roughness (cast iron)
K_inlet = 0.5  # square-edged pipe inlet
K_valve = 10  # fully open globe valve
alpha2 = 1  # kinetic energy correction factor

# Calculate the mass flow rate.
mdot = rho*Q
print("mdot = %.3e kg/s" % mdot)

# Calculate the average flow speed.
V2 = Q/(np.pi/4*D*D)
print("V2 = %.3e m/s" % V2)

# Calculate the Reynolds number.
Re = V2*D/nu
print("Re = %.3e" % Re)

# Calculate the relative roughness.
e_D = e/D
print("e/D = %.3e" % e_D)

# Determine the friction factor.
f = f_Colebrook(Re, e_D)
print("f = %.3e" % f)

# Calculate the viscous loss coefficient.
K_major = f*(L/D)
print("K_major = %.3e" % K_major)

# Calculate the power extracted from the turbine.
P = mdot*g*(H + (alpha2 + K_inlet + K_valve + K_major)*(V2**2)/2/g)
print("P = %.3e kW" % (P/1000))
```

For straightening and smoothing an air flow in a 50 cm diameter duct, the duct is packed with a "honeycomb" of 30 cm long, 4 mm diameter thin straws. The inlet flow is air moving at an average velocity of 6 m/s. Estimate the pressure drop across the honeycomb. The density of the air is 1.2 kg/m^3 and the kinematic viscosity is 1.5e-5 m^2/s. You may neglect inlet and outlet minor losses.



SOLUTION:

Apply the Extended Bernoulli's Equation from point 1 to point 2.



$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,1\to 2}+H_{S,1\to 2} \tag{1}$$

where

$$\frac{p_2-p_1}{\rho g}=\frac{\Delta p}{\rho g}=?\quad \text{(This is what we're trying to find.)} \tag{2}$$

$$\left(\alpha\frac{\bar{V}^2}{2g}\right)_2=\left(\alpha\frac{\bar{V}^2}{2g}\right)_1 \tag{3}$$

$$z_2=z_1 \tag{4}$$

$$H_{S,1\to 2}=0\quad \text{(There is no shaft work between points 1 and 2.)} \tag{5}$$

$$H_{L,1\to 2}=\left(f\frac{L_S}{D_S}\right)\frac{\bar{V}_S^2}{2g}\quad \text{(where the subscript "}S\text{" refers to the conditions in the straw)} \tag{6}$$

Note that minor losses have been neglected.

Substitute and re-arrange.

$$\frac{\Delta p}{\rho g}=-\left(f\frac{L_S}{D_S}\right)\frac{\bar{V}_S^2}{2g} \tag{7}$$

Now determine the average flow velocity in the straw, $\bar{V}_S$. Note that the flow rate through the pipe must be the same as the flow rate through all of the straws.

$$\bar{V}_P \frac{\pi}{4} D_P^2 = N_S \bar{V}_S \frac{\pi}{4} D_S^2 \tag{8}$$

where the number of straws, N_S, is:

$$\frac{\pi}{4} D_P^2 = N_S \frac{\pi}{4} D_S^2 \;\Rightarrow\; N_S = \left(\frac{D_P}{D_S}\right)^2 \tag{9}$$

and thus:

$$\bar{V}_S = \bar{V}_P \tag{10}$$

Now determine the friction factor for the flow through the straw. First calculate the straw's Reynolds number.

$$\mathrm{Re}_S = \frac{\bar{V}_S D_S}{\nu} = \frac{\left(6 \ \mathrm{m/s}\right)\left(4e-3 \ \mathrm{m}\right)}{\left(1.5e-5 \ \mathrm{m^2/s}\right)} = 1600 \;\Rightarrow\; \text{laminar flow in the straws} \tag{11}$$

$$\Rightarrow\; f = \frac{64}{\mathrm{Re}_S} \tag{12}$$

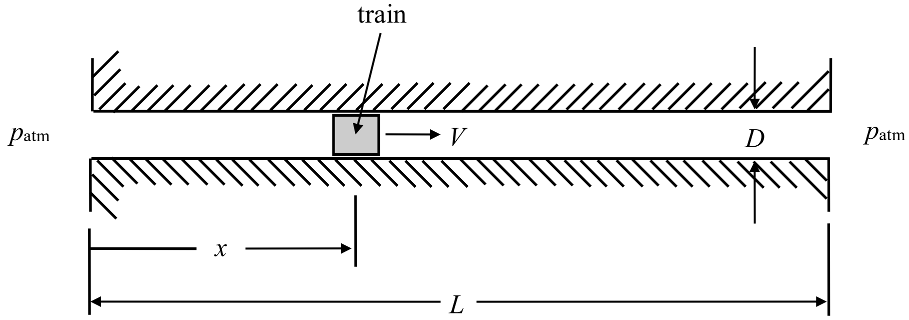
Substitute Eqn. (12) into Eqn. (7) and solve for the pressure drop. using the given data.

$$\boxed{\frac{\Delta p}{\rho g} = -\left(\frac{64}{\mathrm{Re}_S}\frac{L_S}{D_S}\right)\frac{\bar{V}_S^2}{2g}} \quad (L_S = 0.30 \text{ m}; \rho = 1.2 \text{ kg/m}^3) \tag{13}$$

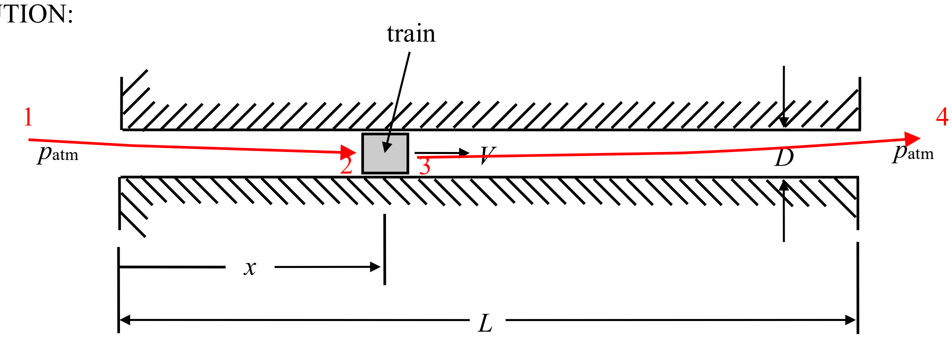
$$\boxed{\therefore \Delta p = \text{-}64.8 \text{ Pa}}$$

A train travels through a tunnel as shown in the figure. The train and tunnel are both circular in cross section. The tunnel has a diameter of $D = 3$ m, a total length of $L = 2000$ m, and walls comprised of concrete. The clearance between the train and the tunnel wall is small so that it may be assumed that the air in front of the train is pushed through the tunnel with the same speed as the train, $V = 20$ m/s.

1. Determine the pressure difference between the front and rear of the train when the train is a distance, x, from the tunnel entrance.
2. Determine the power, P, required to produce the air flow in the tunnel when the train is a distance, x, from the tunnel entrance.



SOLUTION:



Apply the Extended Bernoulli Equation from 1 to 2 and from 3 to 4 to obtain the pressures on the front and back faces of the train,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12}, \quad (1)$$

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_4=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_3-H_{L,34}+H_{S,34}, \quad (2)$$

where,

$p_1 = p_3 = p_{atm}$, (3)

$\bar{V}_1 = \bar{V}_4 = 0$, (4)

$\bar{V}_2 = \bar{V}_3 = V$ (same speed as the train), (5)

$\alpha_2 \approx \alpha_3 \approx 1$ (turbulent flow, $Re_D = VD/\nu = (20 \text{ m/s})(3 \text{ m})/(1.5*10^{-5} \text{ m}^2/\text{s}) = 4.00*10^6$), (6)

$z_1 \approx z_2 \approx z_3 \approx z_4$ (the train is moving through air so the elevation differences are negligible), (7)

$H_{L,12} = f_{D,12}\left(\frac{x}{D}\right)\frac{\bar{V}_2^2}{2g}$ (neglecting minor losses since the tunnel is long), (8)

$H_{L,34} = f_{D,34}\left(\frac{L-x}{D}\right)\frac{\bar{V}_3^2}{2g}$ (neglecting minor losses since the tunnel is long), (9)

$H_{S,12} = H_{S,34} = 0$ (no fluid machinery), (10)

Substitute and solve for the pressures on the front and back faces of the train.

$$\frac{p_2}{\rho g}+\frac{V^2}{2g}=\frac{p_{atm}}{\rho g}-f_{D,12}\left(\frac{x}{D}\right)\frac{V^2}{2g} \Rightarrow \frac{p_2-p_{atm}}{\rho g}=-\left[1+f_D\left(\frac{x}{D}\right)\right]\frac{V^2}{2g}, \quad (11)$$

$$\frac{p_{atm}}{\rho g}=\frac{p_3}{\rho g}+\frac{V^2}{2g}-f_{D,34}\left(\frac{L-x}{D}\right)\frac{V^2}{2g} \Rightarrow \frac{p_3-p_{atm}}{\rho g}=-\left[1-f_D\left(\frac{L-x}{D}\right)\right]\frac{V^2}{2g}, \quad (12)$$

Note that the friction factor will be the same along both paths. Subtract Eq. (11) from Eq. (12),

$$\frac{p_3 - p_2}{\rho g} = f_D\left(\frac{L}{D}\right)\frac{V^2}{2g} \Rightarrow p_3 - p_2 = f_D\left(\frac{L}{D}\right)\frac{1}{2}\rho V^2. \quad (13)$$

Find the friction factor from the Moody plot or the Colebrook Formula. The Reynolds number was calculated previously in Eq. (6). The relative roughness is,

$\epsilon/D = (1.7*10^{-3}\text{ m})/(3\text{ m}) = 5.67*10^{-4}$ (the wall material is concrete), (14)

=> $f_D = 0.0173$.

Using the given and previously calculated values,

$\rho = 1.23\text{ kg/m}^3$,
$L = 2000$ m,
$D = 3$ m,
$V = 20$ m/s,
=> $\boxed{p_3 - p_2 = 2.84\text{ kPa}}$.

Thus, there is a net pressure force acting on the train to resist its motion. The force is,

$$F = (p_3 - p_2)\frac{\pi D^2}{4}. \quad (15)$$

The power required to overcome this force at the given speed is,

$$P = FV = (p_3 - p_2)\frac{\pi D^2}{4}V. \quad (16)$$

Using the given and previously calculated values,

$F = 20.1$ kN,
$\boxed{P = 402\text{ kW}}$.

The following python code was used to perform the calculations.

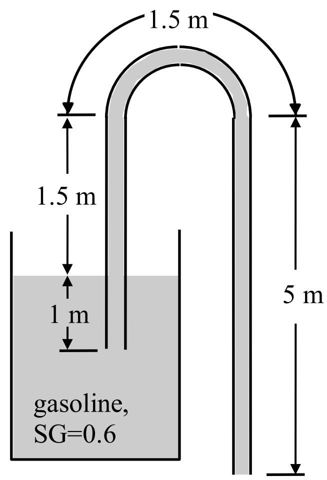
```
# pipe_08.py

import numpy as np

def f_Haaland(Re, e_D):  # Calculate the friction factor using the Haaland formula
    return (-1.8*np.log10(6.9/Re+(e_D/3.7)**1.11))**-2

def f_Colebrook(Re, e_D):  # Calculate the friction factor using the Colebrook formula
    fprime = f_Haaland(Re, e_D)
    freldiff = 1
    tol = 0.001
    while (freldiff > tol):
        f = fprime
        fprime = (-2.0*np.log10(e_D/3.7 + 2.51/Re/np.sqrt(f)))**-2
        freldiff = np.absolute((fprime-f)/f)
    return f

# Initialize variable values.
g = 9.81 # m/s^2; gravitational acceleration
rho = 1.23 # kg/m^3; air density
nu = 1.5e-5  # m^2/s; air kinematic viscosity
V = 20 # m/s; train speed
L = 2000 # m; tunnel length
D = 3  # m; tunnel diameter
e = 1.7e-3  # m; roughness of concrete

# Calculate the Reynolds number.
Re = V*D/nu
print("Re = %.3e" % Re)

# Calculate the relative roughness.
e_D = e/D
print("e/D = %.3e" % e_D)

# Determine the friction factor.
f = f_Colebrook(Re, e_D)
print("f = %.3e" % f)

# Determine the pressure difference.
Delta_p = f*(L/D)*rho*(V**2)/2
```

```
print("Delta p = %.3e kPa" % (Delta_p/1000))

# Determine the pressure force on the train.
F = Delta_p*np.pi/4*D**2
print("F = %.3e kN" % (F/1000))

# Determine the power required to overcome the pressure force at this speed.
P = F*V
print("P = %.3e kW" % (P/1000))
```

Gasoline at 20 °C is being siphoned from a tank through a rubber hose having an inside diameter of 25 mm. The roughness for the hose is 0.01 mm.

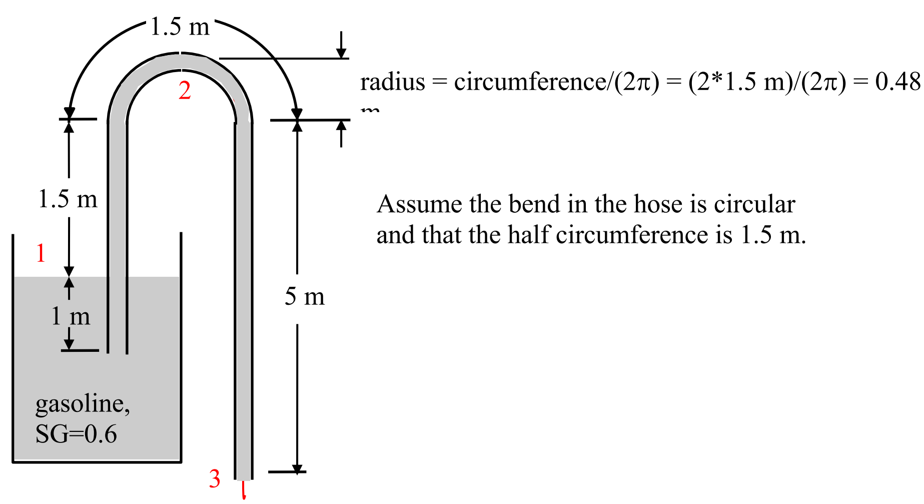
1. What is the volumetric flow rate of the gasoline through the hose?
2. What is the minimum pressure in the hose and where does it occur?

You may neglect minor losses. The kinematic viscosity of gasoline is 4.294e-7 m^2/s.



SOLUTION:

Apply the Extended Bernoulli Equation from points 1 to 3.



$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_3=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \tag{1}$$

where

$$p_1 = p_3 = p_{atm} \tag{2}$$

$$\bar{V}_1 \approx 0 \quad \text{(surface of a large tank)} \tag{3}$$

$$\bar{V}_3 = ? \quad (\alpha_3 \approx 1\text{, assuming turbulent flow in the hose}) \tag{4}$$

$$z_1 - z_3 = 3.5\text{ m} \tag{5}$$

$H_S = 0$ (6)

$$H_L = f\left(\frac{L}{D}\right)\frac{\bar{V}_3^2}{2g} \quad \text{(Neglecting minor losses.)} \tag{7}$$

where f may be found from the Moody diagram. The relative roughness is

$$\frac{\varepsilon}{D} = \frac{0.01\text{ mm}}{25\text{ mm}} = 4*10^{-4} \tag{8}$$

Assuming fully turbulent flow so that f is not a function of the Reynolds number, the Moody diagram gives, $f = 0.016$.

Substitute into Eq. (1) and solve for $\bar{V}_3$.

$$\frac{\bar{V}_3^2}{2g}\left[1+f\left(\frac{L}{D}\right)\right] = (z_1 - z_3) \tag{9}$$

$$\bar{V}_3 = \sqrt{\frac{2g(z_1-z_3)}{1+f\left(\frac{L}{D}\right)}} \tag{10}$$

Using the given data,

$g = 9.81$ m/s^2
$z_1 - z_3 = 3.5$ m
$f = 0.016$
$L = 1$ m + 1.5 m + 1.5 m + 5 m = 9 m
$D = 0.025$ m
$\Rightarrow \bar{V}_3 = 3.2$ m/s

Check the Reynolds number to verify the fully turbulent assumption.

$$\text{Re}_D = \frac{\bar{V}_3 D}{\nu} = \frac{(3.2\text{ m/s})(0.025\text{ m})}{(4.294*10^{-7}\text{ m}^2/\text{s})} = 1.9*10^5 \tag{11}$$

The given relative roughness and this Reynolds number puts the flow in the fully turbulent range so the assumptions made in the problem are consistent.

The flow rate is given by,

$$Q = \bar{V}_3\frac{\pi D^2}{4} = (3.2\text{ m/s})\frac{\pi(0.025\text{ m})^2}{4} \Rightarrow \boxed{Q = 1.57*10^{-3}\text{ m}^3/\text{s}} \tag{12}$$

The minimum pressure occurs near point 2 in the figure shown previously. Apply the Extended Bernoulli Equation from points 2 to 3.

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_3 = \left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2 - H_L + H_S \tag{13}$$

where

$p_2 = ?$
$p_3 = p_{atm}$ (14)
$\bar{V}_2 = \bar{V}_3$ ($\alpha_2 \approx \alpha_3 \approx 1$ (15)
$z_2 - z_3 = 0.48$ m + 5 m = 5.48 m (16)
$H_S = 0$ (17)

$$H_L = f\left(\frac{L}{D}\right)\frac{\bar{V}_3^2}{2g} \tag{18}$$

where $f = 0.016$ was found previously.

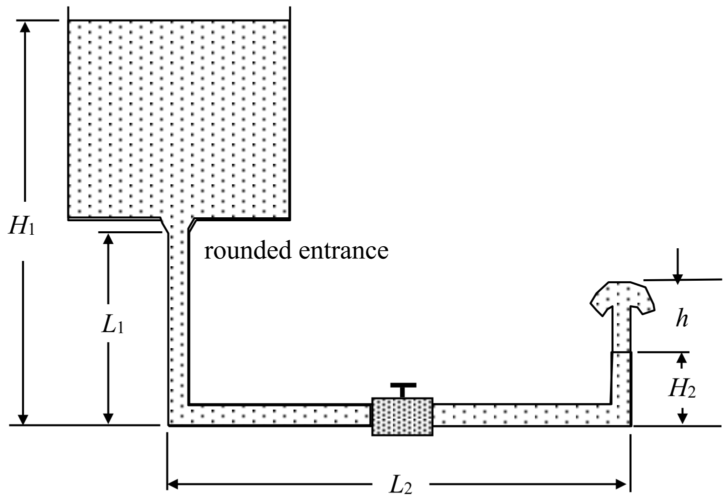
Substitute into Eq. (13) and solve for p_2.

$$p_2 = p_3 + \rho g\left(z_3 - z_2\right) + f\left(\frac{L}{D}\right)\frac{1}{2}\rho\bar{V}_3^2 \tag{19}$$

Using the given data,

p_3 = 101 kPa (abs)
ρ = 0.6(1000 kg/m^3) = 600 kg/m^3
g = 9.81 m/s^2
$z_3 - z_2$ = -5.48 m
f = 0.016
L = (1.5 m)/2 + 5 m = 5.75 m
D = 0.025 m
$\bar{V}_3 = 3.2\ \text{m/s}$
$\Rightarrow$ $\boxed{p_2 = 80.0 \text{ kPa (abs)}}$

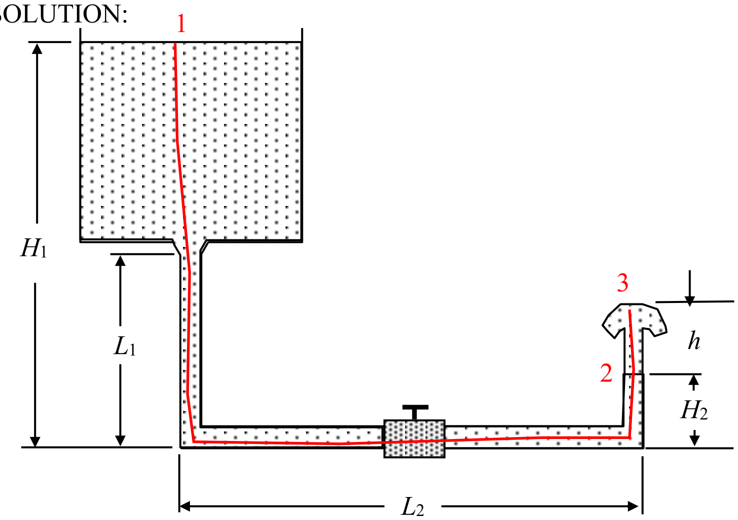
Water flows from a container as shown in the figure. Determine the loss coefficient needed in the valve if the water is to "bubble up" a distance h above the outlet pipe.



$H_1 = 45$ in
$L_1 = 18$ in
$L_2 = 32$ in
$H_2 = 2$ in
$h = 3$ in

The pipe is ½ in diameter galvanized iron pipe with threaded fittings.

SOLUTION:



Apply the Extended Bernoulli Equation from point 1 to point 2,

$$\left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_2 = \left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_1 - H_{L,12} + H_{S,12}, \tag{1}$$

where,

$$p_1 = p_2 = p_{\text{atm}}, \tag{2}$$
$$\bar{V}_1 \approx 0, \tag{3}$$
$$\bar{V}_2 = ?, \tag{4}$$
$$\alpha_2 \approx 1 \text{ (assume turbulent flow)}, \tag{5}$$
$$z_1 = H_1 \text{ and } z_2 = H_2 \tag{6}$$
$$H_{L,12} = \sum_i K_i \frac{\bar{V}_i^2}{2g} = \left(K_{\text{inlet}} + 2K_{\text{elbow}} + K_{\text{valve}} + K_{\text{major}}\right)\frac{\bar{V}_2^2}{2g}, \tag{7}$$
$$H_{S,12} = 0. \tag{8}$$

Substitute and solve for the valve loss coefficient,

$$\frac{\bar{V}_2^2}{2g} + H_2 = H_1 - \left(K_{\text{inlet}} + 2K_{\text{elbow}} + K_{\text{valve}} + K_{\text{major}}\right)\frac{\bar{V}_2^2}{2g}, \tag{9}$$

$$K_{\text{valve}} = \frac{2g(H_1-H_2)}{\bar{V}_2^2} - 1 - K_{\text{inlet}} - 2K_{\text{elbow}} - K_{\text{major}}. \tag{10}$$

The major loss coefficient is,

$$K_{\text{major}} = f\left(\frac{L_1+L_2+H_2}{D}\right), \tag{11}$$

where the friction factor is found using the Moody plot or the Colebrook formula using,

$$\text{Re}_D = \frac{\bar{V}_2 D}{\nu}, \tag{12}$$

and the relative roughness ϵ/D.

The velocity in the pipe (and at location 2) can be found by applying Bernoulli's equation from point 2 to point 3,

$$\left(\frac{p}{\rho g} + \frac{\bar{V}^2}{2g} + z\right)_3 = \left(\frac{p}{\rho g} + \frac{\bar{V}^2}{2g} + z\right)_2, \tag{13}$$

where,

$p_3 = p_{\text{atm}}$,
$\bar{V}_3 \approx 0$,
$z_3 = H_2 + h$.

Substitute and solve for the average velocity at 2,

$$\bar{V}_2 = \sqrt{2gh}. \tag{14}$$

Note that the "ordinary" Bernoulli's equation was used from 2 to 3 since these points are outside the pipe.

Using the given parameters,

$g = 32.2$ ft/s^2,
$\nu = 1.21*10^{-5}$ ft^2/s,
$H_1 = 45$ in. = 3.75 ft,
$H_2 = 2$ in. = 0.167 ft,
$h = 3$ in. = 0.25 ft,
$L_1 = 18$ in. = 1.5 ft,
$L_2 = 32$ in. = 2.67 ft,
$D = 0.5$ in. = 0.0417 ft,
$\epsilon = 0.0005$ ft (galvanized iron pipe) => $\epsilon/D = 0.0120$,
$K_{\text{inlet}} = 0.05$ (rounded inlet; from minor loss table),
$K_{\text{elbow}} = 1.5$ (90 deg threaded elbow; from minor loss table).
=> $\bar{V}_2 = 4.012$ ft/s,
=> $\text{Re}_D = 13{,}820$ (turbulent flow assumption is ok!)
=> $f = 0.04387$,
=> $K_{\text{major}} = 4.562$,
=> $\boxed{K_{\text{valve}} = 5.72}$.

The following python code was used to perform the computations.

```
# pipe_12.py

import numpy as np

def f_Haaland(Re, e_D):  # Calculate the friction factor using the Haaland formula
    return (-1.8*np.log10(6.9/Re+(e_D/3.7)**1.11))**-2

def f_Colebrook(Re, e_D):  # Calculate the friction factor using the Colebrook formula
    fprime = f_Haaland(Re, e_D)
    freldiff = 1
    tol = 0.001
    while (freldiff > tol):
        f = fprime
        fprime = (-2.0*np.log10(e_D/3.7 + 2.51/Re/np.sqrt(f)))**-2
        freldiff = np.absolute((fprime-f)/f)
    return f

# Initialize variable values.
g = 32.2 # ft/s^2; gravitational acceleration
```

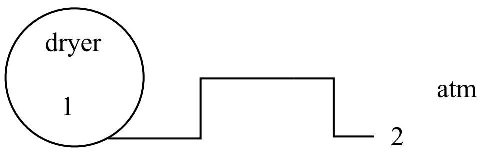
```
nu = 1.21e-5  # ft^2/s; water kinematic viscosity
H1 = 45/12  # ft; tank free surface height
H2 = 2/12  # ft; exit pipe height
h = 3/12  # ft; fountain height
L1 = 18/12 # ft; vertical pipe length
L2 = 32/12 # ft; horizontal pipe length
D = 0.5/12  # ft; pipe diameter
e = 0.0005  # ft; pipe roughness (galvanized iron)
K_inlet = 0.05  # rounded pipe inlet
K_elbow = 1.5  # 90 deg threaded elbow

# Print the lengths in feet.
print("H1 = %.3e ft, H2 = %.3e ft, h = %.3e ft, L1 = %.3e ft, L2 = %.3e ft, D = %.3e
ft" % (H1, H2, h, L1, L2, D))

# Calculate the average velocity in the pipe.
V2 = np.sqrt(2*g*h)
print("V2 = %.3e ft/s" % V2)

# Calculate the Reynolds number.
Re = V2*D/nu
print("Re = %.3e" % Re)

# Calculate the relative roughness.
e_D = e/D
print("e/D = %.3e" % e_D)

# Determine the friction factor.
f = f_Colebrook(Re, e_D)
print("f = %.3e" % f)

# Calculate the viscous loss coefficient.
K_major = f*((L1+L2+H2)/D)
print("K_major = %.3e" % K_major)

# Calculate the valve loss coefficient.
K_valve = 2*g*(H1-H2)/V2/V2 - 1 - K_inlet - 2*K_elbow - K_major
print("K_valve = %.3e" % K_valve)
```

According to an appliance manufacturer, the 4 in diameter galvanized iron vent on a clothes dryer is not to contain more than 20 ft of pipe and four 90° elbows. Under these conditions, determine the air flow rate if the gage pressure within the dryer is 1.04 psf. You may assume the following:

kinematic viscosity of air: 1.79e-4 ft^2/s
density of air: 2.20e-3 $slugs/ft^3$
$K_{90° \text{ bend}} = 1.5$
$K_{\text{entrance}} = 0.5$

SOLUTION:

Apply the Extended Bernoulli Equation from point 1 to point 2 as shown in the figure below.



$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \quad (1)$$

where

$p_2 = 0$ (gage) and $p_1 = 1.04$ psfg
$\bar{V}_2 = ?$ (Assume turbulent flow so that $\alpha_2 \approx 1$.)
$\bar{V}_1 << \bar{V}_2$ (The air in the dryer is relatively stagnant compared to the outflowing air.)
$z_2 - z_1 \approx 0$
$H_S = 0$

$$H_L = f\left(\frac{L}{D}\right)\frac{\bar{V}_P^2}{2g}+K_{\text{entrance}}\frac{\bar{V}_P^2}{2g}+4K_{\text{bends}}\frac{\bar{V}_P^2}{2g} \quad (2)$$

Note that $\bar{V}_P = \bar{V}_2$ since the pipe and exit diameters are the same. Also, there is no exit loss since point 2 is located just at the exit of the pipe. The air has not undergone any exit losses at this point.

Substitute and simplify.

$$\frac{\bar{V}_2^2}{2}\left[f\left(\frac{L}{D}\right)+K_{\text{entrance}}+4K_{\text{bends}}+1\right]=\frac{p_{1,g}}{\rho} \quad (3)$$

It's given that

L = 20 ft
D = 4 in. = 0.33 ft
K_{elbow} = 1.5
K_{entrance} = 0.5
$p_{1,g}$ = 1.04 lb_f/ft^2
ρ = $2.20*10^{-3}$ $slug/ft^3$

Using these parameters, Eq. (3) becomes,

$$\bar{V}_2^2\left[60f+7.5\right]=945.5 \text{ ft}^2/\text{s}^2 \quad (4)$$

Note that f is dependent on the Reynolds number and relative roughness,

$$\mathrm{Re}_D = \frac{\bar{V}_P D}{\nu} = (1862\ \mathrm{s/ft})\bar{V}_2 \tag{5}$$

where $\nu_{\text{air}} = 1.79{*}10^{-4}$ ft^2/s and $\bar{V}_P = \bar{V}_2$. The roughness of galvanized iron pipe is $e = 0.0005$ ft so that the relative roughness is,

$$\frac{e}{D} = 0.0015 \tag{6}$$

To solve for $\bar{V}_2$, we must iterate to a solution since f is also a (complex) function of $\bar{V}_2$ because of the Reynolds number dependence. One iterative procedure that can be used is given below.

1. Choose a value for f.
2. Calculate $\bar{V}_2$ using Eq. (4).
3. Calculate Re_D using Eq. (5).
4. Use the Moody diagram with the Re_D calculated from Step 3 and the relative roughness given in Eq. (6) to find f'.
5. Is $f' = f$? If so, then the iterations are complete and $\bar{V}_2$ is the value found in Step 2. Otherwise, use f' as the new value for f and go to Step 2.

Using this iterative algorithm and an initial guess of $f = 0.025$,

1. $f = 0.025$
   a. $\bar{V}_2 = 10.25$ ft/s
   b. $\mathrm{Re}_D = 19{,}000$
   c. $f' = 0.029$ (This value is different than our original guess, must continue iterations.)
2. $f = 0.029$
   a. $\bar{V}_2 = 10.11$ ft/s
   b. $\mathrm{Re}_D = 18{,}800$
   c. $f' = 0.029$ (This value matches our initial guess! Iterations complete!)

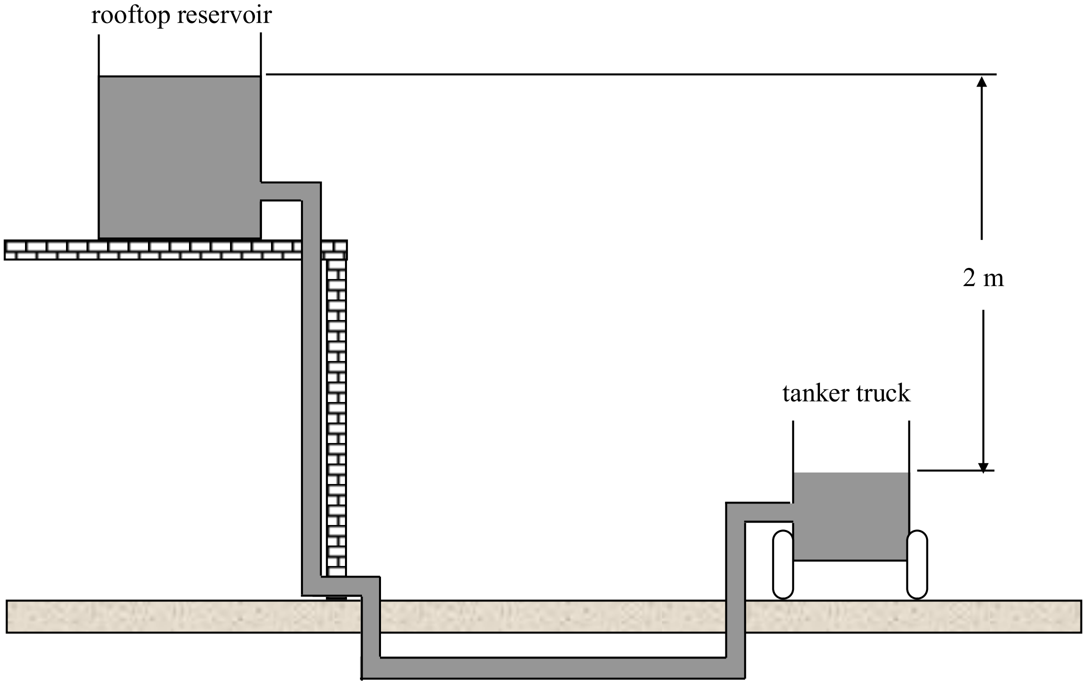
Note that the flow is turbulent, which is consistent with the assumption that $\alpha_2 \approx 1$.

The volumetric flow rate may be found using,

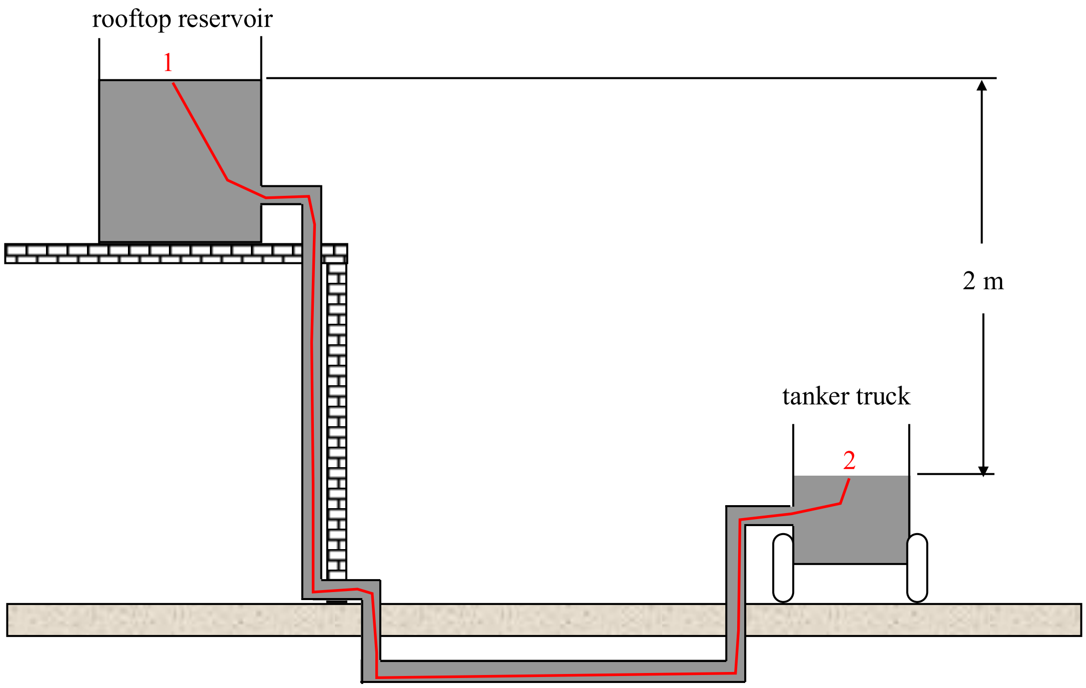
$$Q = \bar{V}_2 \frac{\pi D^2}{4} \quad \Rightarrow \quad \boxed{Q = 0.882\ \mathrm{ft^3/s}} \tag{7}$$

where $\bar{V}_2 = 10.11$ ft/s and $D = 0.33$ ft.

Water at 10 °C (kinematic viscosity of $1.307*10^{-6}$ m^2/s) is to flow from a roof-top reservoir to a tanker truck through a cast iron pipe (roughness of 0.26 mm) of length 20 m at a flow rate of 0.0020 m^3/s. The roof-top tank water level is located 2 m above the tanker truck fluid level. The system contains a sharp-edged entrance, six threaded 90° elbows, and a sharp-edged exit. Determine the required pipe diameter for the given flow conditions.



SOLUTION:



Apply the Extended Bernoulli Equation from point 1 to point 2,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12}, \tag{1}$$

where,

$p_1 = p_2 = p_{atm}$,

$\bar{V}_1 \approx \bar{V}_2 \approx 0$,

$$z_2 - z_1 = -H, \tag{2}$$

$$H_{L,12}=\sum_i K_i\frac{\bar{V}_i^2}{2g}=\left(K_{\text{square inlet}}+4K_{\text{elbow}}+K_{\text{exit}}+K_{\text{major}}\right)\frac{\bar{V}^2}{2g}, \tag{3}$$

$$H_{S,12}=0, \tag{4}$$

$$\bar{V}=\frac{Q}{\pi D^2/4} \text{ (relating the average flow speed in the pipe to the volumetric flow rate).} \tag{5}$$

Substitute and simplify,

$$H=\left(K_{\text{square inlet}}+4K_{\text{elbow}}+K_{\text{exit}}+K_{\text{major}}\right)\frac{\bar{V}^2}{2g}, \tag{6}$$

$$H=\left(K_{\text{square inlet}}+4K_{\text{elbow}}+K_{\text{exit}}+K_{\text{major}}\right)\frac{8Q^2}{\pi^2 D^4 g}, \tag{7}$$

$$D^4=\left(K_{\text{square inlet}}+4K_{\text{elbow}}+K_{\text{exit}}+K_{\text{major}}\right)\frac{8Q^2}{\pi^2 gH}. \tag{8}$$

The major loss coefficient is,

$$K_{\text{major}}=f\left(\frac{L}{D}\right), \tag{9}$$

where the friction factor is a function of the relative roughness and the Reynolds number,

$$f=f\left(\frac{\epsilon}{D},\text{Re}_D\right), \tag{10}$$

where,

$$\text{Re}_D=\frac{\bar{V}D}{\nu}. \tag{11}$$

Since the velocity, Reynolds number, and relative roughness are all functions of the pipe diameter, Eq. (5) cannot be solved explicitly for the pipe diameter. Instead, an iterative solution must be used. The following algorithm is implemented in the python code at the end of this solution.

1. Choose a value for the diameter, D.
2. Calculate the average pipe speed using Eq. (5).
3. Calculate the Reynolds number using Eq. (11).
4. Calculate the relative roughness.
5. Determine the friction factor using the Colebrook Formula (or use the Moody plot).
6. Calculate the major loss coefficient using Eq. (9).
7. Use the EBE (Eq. (8)) to solve for the pipe diameter, D'.
8. Is D' equal to D? If so, then the iterations are complete. If not, let $D = D'$ and return to step 2.

The given parameters are,

$\nu = 1.307*10^{-6}$ m^2/s,
$g = 9.81$ m/s^2,
$H = 2.0$ m,
$L = 20$ m,
$Q = 0.0020$ m^3/s,
$\epsilon = 0.26$ mm (cast iron pipe),
$K_{\text{inlet}} = 0.5$ (sharp-edge entrance; from a minor loss table),
$K_{\text{elbow}} = 1.5$ (90 deg, threaded elbow; from a minor loss table),
$K_{\text{exit}} = 1$.

Running the python code generates the following output,

```
Iterating:  D= 4.000e-02 m, Re= 4.871e+04, e/D= 6.500e-03, f= 3.451e-02, K_major= 1.725e+01, D' = 4.497e-02 m
Iterating:  D= 4.497e-02 m, Re= 4.332e+04, e/D= 5.781e-03, f= 3.364e-02, K_major= 1.496e+01, D' = 4.389e-02 m
Iterating:  D= 4.389e-02 m, Re= 4.439e+04, e/D= 5.924e-03, f= 3.381e-02, K_major= 1.541e+01, D' = 4.411e-02 m
Iterating:  D= 4.411e-02 m, Re= 4.417e+04, e/D= 5.895e-03, f= 3.378e-02, K_major= 1.531e+01, D' = 4.406e-02 m
Final:  D = 4.411e-02 m, Re = 4.417e+04, e/D = 5.895e-03, f = 3.378e-02, D' = 4.406e-02 m
```

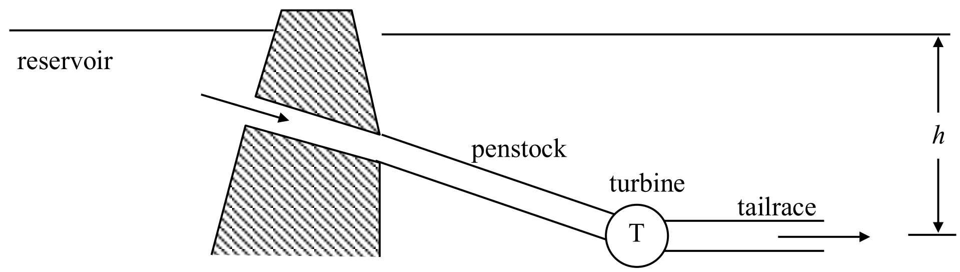
Thus, the pipe diameter should be $\boxed{D = 4.41 \text{ cm}}$.

Following is the python code used for the computations.

```
# pipe_11.py

import numpy as np

# Initialize variable values.
g = 9.81  # m/s^2, gravitational acceleration
H = 2.0  # m, height difference
L = 20  # m, pipe length
Q = 0.0020  # m^3/s, volumetric flow rate
K_inlet = 0.5  # sharp-edge entrance loss coefficient
K_elbow = 1.5  # 90 deg threaded elbow loss coefficient
K_exit = 1.0  # exit loss coefficient
nu = 1.307e-6  # m^2/s, kinematic viscosity
e = 0.26e-3  # m, roughness (cast iron pipe)

def Re_fcn(D):  # Calculate the Reynolds number given the diameter
    return 4*Q/np.pi/nu/D

def f_Haaland(Re, e_D):  # Calculate the friction factor using the Haaland formula
    return (-1.8*np.log10(6.9/Re+(e_D/3.7)**1.11))**-2

def f_Colebrook(Re, e_D):  # Calculate the friction factor using the Colebrook formula
    fprime = f_Haaland(Re, e_D)
    freldiff = 1
    tol = 0.001
    while (freldiff > tol):
        f = fprime
        fprime = (-2.0*np.log10(e_D/3.7 + 2.51/Re/np.sqrt(f)))**-2
        freldiff = np.absolute((fprime-f)/f)
    return f

def D_fcn(f, K_major):  # Calculate the diameter from the EBE
    D = ((K_inlet + 4*K_elbow + K_exit + K_major)*8*Q*Q/np.pi/np.pi/g/H)**(1/4)
    return D
```

```
# Initial guess for D and set the iteration tolerance.
Dprime = 0.04  # m, first guess at the diameter
Dreldiff = 1  # a large number to start with
tol = 0.001  # tolerance
countmax = 1000  # maximum number of iterations before giving up

# Iterate until D and Dprime are nearly equal or we reach the maximum
# number of iterations.
count = 0
while ((Dreldiff > tol) and (count < countmax)):
    count = count + 1
    D = Dprime
    Re = Re_fcn(D)
    e_D = e/D
    f = f_Colebrook(Re, e_D)
    K_major = f*(L/D)
    Dprime = D_fcn(f, K_major)
    Dreldiff = np.absolute((D-Dprime)/D)  # find the relative difference
    print("Iterating:  D= %.3e m," % D, "Re= %.3e," % Re, "e/D= %.3e," %
e_D,"f= %.3e," % f, "K_major= %.3e," % K_major, "D\' = %.3e m" % Dprime)


if (count == countmax):
    print("Didn't converge to a solution after %d iterations." % countmax)
else:
    print("Final:  D = %.3e m," % D, "Re = %.3e," % Re, "e/D = %.3e," % e_D,"f
= %.3e," % f, "D\' = %.3e m" % Dprime)
```

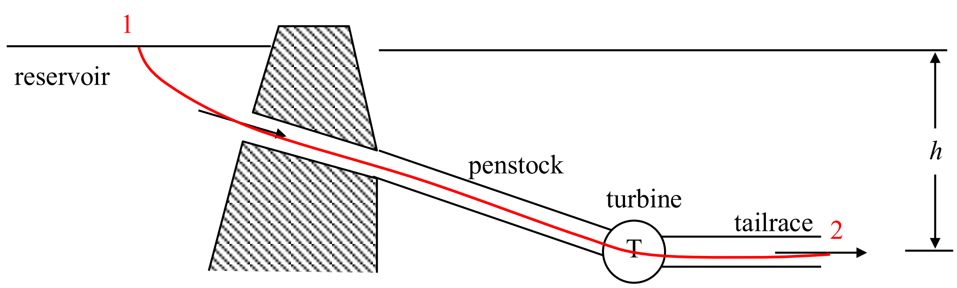
The tailrace (discharge pipe) of a hydro-electric turbine installation is at an elevation, h, below the water level in the reservoir:



The losses in the penstock (the pipe leading to the turbine) and the tailrace are represented by the loss coefficient, K, based on the mean velocity, U, in those pipes, which have the same cross-sectional area, A. The flow discharges to atmospheric pressure at the exit from the tailrace. The water density is denoted by ρ and the acceleration due to gravity by g. Assume turbulent flow conditions.

a. What is the drop in total head across the turbine?
b. What is the power developed by the turbine assuming that it has an efficiency η?
c. What is the optimum velocity, U_{opt}, that will produce the maximum power output from the turbine assuming that h, K, A, ρ, and g are constant?

SOLUTION:



Apply the Extended Bernoulli Equation from 1 to 2,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12}, \tag{1}$$

where,

$$p_1=p_2=p_{atm}, \tag{2}$$

$$\bar{V}_1\approx 0, \tag{3}$$

$$\bar{V}_2=U \quad (\text{assume } \alpha_2\approx 1), \tag{4}$$

$$z_2-z_1=-h, \tag{5}$$

$$H_{L,12}=\sum_i K_i\frac{\bar{V}_i^2}{2g}=K\frac{U^2}{2g}, \tag{6}$$

$$H_{S,12}=\frac{P}{\rho Qg}=\frac{P}{\rho UAg}. \tag{7}$$

Substitute and simplify,

$$\frac{U^2}{2g}-h=-K\frac{U^2}{2g}+\frac{P}{\rho gUA}, \tag{8}$$

$$\boxed{\frac{P}{\rho gUA}=(1+K)\frac{U^2}{2g}-h}. \tag{9}$$

Note that since it's a turbine, we expect the right-hand side of Eq. (9) to be negative. Thus, the total drop in head across the turbine is equal to the absolute value of the right-hand side of Eq. (9).

The power developed by the turbine, assuming an efficiency of η is,

$$\boxed{P = \eta\rho g U A\left[(1+K)\frac{U^2}{2g} - h\right]}. \tag{10}$$

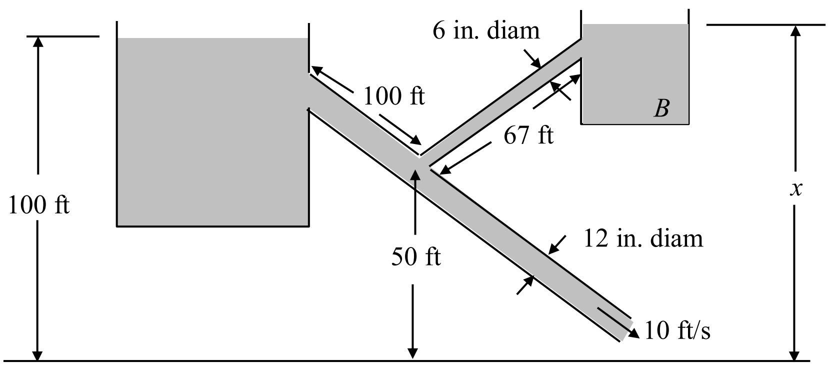
To optimize the power output, take the derivative of Eq. (10) and set it equal to zero,

$$\frac{dP}{dU} = 0 = \eta\rho g A\frac{d}{dU}\left[(1+K)\frac{U^3}{2g} - hU\right], \tag{11}$$

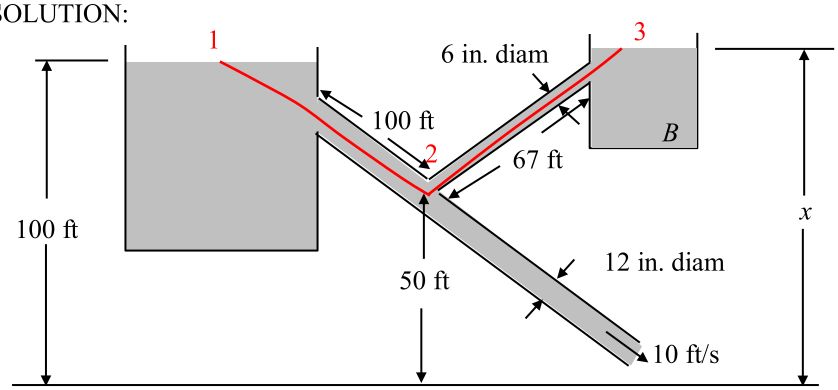
$$0 = 3(1+K)\frac{U^2}{2g} - h, \tag{12}$$

$$\boxed{U_{opt} = \sqrt{\frac{2gh}{3(1+K)}}}. \tag{13}$$

In the water flow system shown, reservoir *B* has variable elevation, *x*. Determine the water level in reservoir *B* so that no water flows into or out of that reservoir. The speed in the 12 in. diameter pipe is 10 ft/s. Assume the pipes are constructed of cast iron and that the entrances are sharp-edged.



SOLUTION:



Apply the Extended Bernoulli Equation from point 1 to point 2,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12}, \tag{1}$$

where,

$$p_1 = p_{atm}, \tag{2}$$

$$\bar{V}_1 \approx 0, \tag{3}$$

$$\bar{V}_2 \text{ is given,} \tag{4}$$

$$z_2 - z_1 \text{ is given,} \tag{5}$$

$$H_{L,12} = \sum_i K_i \frac{\bar{V}_i^2}{2g} = \left(K_{\text{inlet}} + K_{\text{major}}\right)\frac{\bar{V}_2^2}{2g}, \tag{6}$$

$$H_{S,12} = 0. \tag{7}$$

The pressure at point 2 can be found using hydrostatics since there is no flow from point 3 to point 2,

$$p_2 = p_{atm} + \rho g(x - z_2). \tag{8}$$

Substitute and solve for *x*,

$$\frac{p_{atm}}{\rho g} + (x - z_2) + \alpha_2\frac{\bar{V}_2^2}{2g} + z_2 = \frac{p_{atm}}{\rho g} + z_1 - \left(K_{\text{inlet}} + K_{\text{major}}\right)\frac{\bar{V}_2^2}{2g}, \tag{9}$$

$$x = z_1 - \left(K_{\text{inlet}} + K_{\text{major}} + \alpha_2\right)\frac{\bar{V}_2^2}{2g}, \tag{10}$$

Using the given parameters,

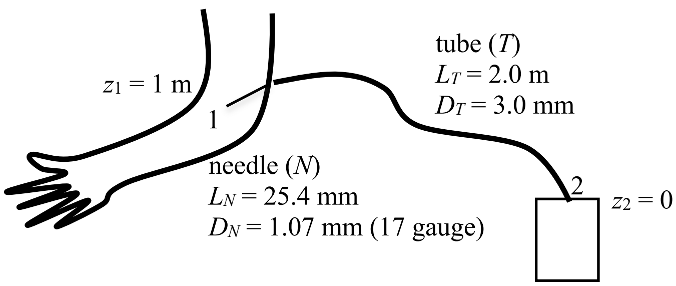
$z_1 = 100$ ft,
$K_{\text{inlet}} = 0.5$ (sharp-edge inlet; from a minor loss table),
$\bar{V}_2 = 10$ ft/s,
$\nu = 1.08 \times 10^{-5}$ ft^2/s,
$g = 32.2$ ft/s^2,
$D = 12$ in. = 1 ft,
$L_{12} = 100$ ft,
$\epsilon = 0.00085$ ft (cast iron) => $\epsilon/D = 0.00085$,
$\text{Re}_D = \bar{V}_2 D/\nu = 925{,}900$ (turbulent flow => $\alpha_2 \approx 1$),
=> $f = 0.01924$ (from the Moody plot or the Colebrook formula),
=> $K_{\text{major}} = f(L_{12}/D) = 1.924$,
=> $\boxed{x = 94.7 \text{ ft}}$.

The following python code was used for the computations.

```
# pipe_15.py

import numpy as np

def f_Haaland(Re, e_D):  # Calculate the friction factor using the Haaland formula
    return (-1.8*np.log10(6.9/Re+(e_D/3.7)**1.11))**-2

def f_Colebrook(Re, e_D):  # Calculate the friction factor using the Colebrook formula
    fprime = f_Haaland(Re, e_D)
    freldiff = 1
    tol = 0.001
    while (freldiff > tol):
        f = fprime
        fprime = (-2.0*np.log10(e_D/3.7 + 2.51/Re/np.sqrt(f)))**-2
        freldiff = np.absolute((fprime-f)/f)
    return f

# Initialize variable values.
g = 32.2 # ft/s^2; gravitational acceleration
nu = 1.08e-5  # ft^2/s; kinematic viscosity
z1 = 100  # ft; free surface height
V2 = 10 # ft/s; flow rate
L12 = 100 # ft; pipe length
D = 12/12  # ft; pipe diameter
e = 0.00085  # ft; pipe roughness (cast iron)
K_inlet = 0.5  # square-edged pipe inlet

# Calculate the Reynolds number.
Re = V2*D/nu
print("Re = %.3e" % Re)

# Calculate the relative roughness.
e_D = e/D
print("e/D = %.3e" % e_D)

# Determine the friction factor.
f = f_Colebrook(Re, e_D)
print("f = %.3e" % f)

# Calculate the viscous loss coefficient.
K_major = f*(L12/D)
print("K_major = %.3e" % K_major)

# Calculate the elevation x.
x = z1 - (K_inlet + K_major + 1)*(V2**2)/2/g
print("x = %.3e ft" % x)
```

Consider the process of donating blood. Blood flows from a vein in which the pressure is greater than atmospheric, through a long small-diameter tube, and into a plastic bag that is essentially at atmospheric pressure. Based on fluid mechanics principles, estimate the amount of time it takes to donate a pint of blood. List all assumptions and show calculations.

SOLUTION:



First, a few assumptions:

1. Treat blood as a Newtonian fluid. Blood is actually slightly non-Newtonian with shear-thinning behavior, but we'll model it as Newtonian here for simplicity. Assume the density of blood is ρ = 1060 kg/m^3 and its dynamic viscosity is μ = 3.5 cP = 3.5*10^{-3} kg/(m.s). Hence, the kinematic viscosity is $\nu = \mu/\rho$ = 3.30*10^{-6} m^2/s.
2. Steady flow through the needle and tube. In fact, the real flow will be pulsatile due to the fluctuating pressure in the vein.
3. Constant pressure in the arm and collection bag. Again, the real flow will have periodic pressure variations in the arm.
4. The mean arterial pressure in the arm is at 93.3 mm Hg (gage). In practice, there are two values given for blood pressure: a systolic pressure and a diastolic pressure. The systolic pressure is the pressure when the heart is contracted while the diastolic pressure is when the heart is relaxed. A value of (systolic/diastolic) 120/80 mm Hg (gage) is within the normal range of blood pressures. The mean arterial pressure (MAP) is the average pressure over a cardiac cycle and can be approximated as: MAP = $p_{\text{diastolic}}$ + (1/3)*($p_{\text{systolic}} - p_{\text{diastolic}}$).
5. The pipe system consists of a needle, plastic tubing, and a plastic collection bag. A 17 gauge (1.07 mm inner diameter) needle diameter is often used for collecting blood. These needles are approximately 2.54 cm (1 in.) in length. The plastic tubing is assumed to be approximately 2.0 m in length with an inner diameter of 3.0 mm.
6. The flow in the needle and tube is laminar since the diameters are small.
7. The collection bag is located below the person's arm. We'll assume an elevation difference of 0.5 m.

Apply the Extended Bernoulli Equation from point 1 (in the vein) to point 2 (just upstream of the tube exit leading into the bag),

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,1\to 2}+H_{S,1\to 2}\,, \qquad (1)$$

where

p_1 = p_{arm} = 93.3 mm Hg (gage) (use the mean arterial pressure in the arm)
p_2 = 0 (gage) (discharging into a bag that is at atmospheric pressure)

$\bar{V}_1 \approx 0$ (blood speed in the vein is small compared to the speed in the needle and tube)
$\bar{V}_2 = \bar{V}_T$ (blood speed just before the exit of the tube)
$\alpha_2 = \alpha_T = 2$ (because the tube diameter is small, assume the flow is laminar at the tube exit)
$z_1 = 0.5$ m (assume the person's arm is 0.5 m above the bag)
$z_2 = 0$
$H_{S,1\to 2} = 0$ (no fluid machinery in the process)

$$H_{L,1\to 2} = \frac{\bar{V}_N^2}{2g}\left[f_N\left(\frac{L_N}{D_N}\right) + K_{\text{entrance}} + K_{\text{expansion}}\right] + \frac{\bar{V}_T^2}{2g}\left[f_T\left(\frac{L_T}{D_T}\right)\right] \tag{2}$$

(major losses in the needle and tube and a minor loss at the inlet and at the transition from the need to the tube)

Note that,

$$\bar{V}_N = \frac{Q}{\frac{\pi}{4}D_N^2} = \frac{4Q}{\pi D_N^2} \quad \text{and} \quad \bar{V}_T = \frac{Q}{\frac{\pi}{4}D_T^2} = \frac{4Q}{\pi D_T^2} \tag{3}$$

$$f_N = \frac{64}{\text{Re}_{D_N}} = \frac{64\nu}{\bar{V}_N D_N} = \frac{16\pi\nu D_N}{Q} \quad \text{and} \quad f_T = \frac{64}{\text{Re}_{D_T}} = \frac{64\nu}{\bar{V}_T D_T} = \frac{16\pi\nu D_T}{Q} \quad \text{(since the flow is laminar)} \tag{4}$$

Substitute and simplify,

$$\alpha_T\frac{\bar{V}_T^2}{2g} = \left(\frac{p_{g,\text{arm}}}{\rho g} + z_1\right) - \frac{\bar{V}_N^2}{2g}\left[f_N\left(\frac{L_N}{D_N}\right) + K_{\text{entrance}} + K_{\text{expansion}}\right] - \frac{\bar{V}_T^2}{2g}\left[f_T\left(\frac{L_T}{D_T}\right)\right], \tag{5}$$

$$\left(\frac{p_{g,\text{arm}}}{\rho g} + z_1\right) = \frac{8Q^2}{\pi^2 g D_N^4}\left(\frac{16\pi\nu L_N}{Q} + K_{\text{entrance}} + K_{\text{expansion}}\right) + \frac{8Q^2}{\pi^2 g D_T^4}\left(\frac{16\pi\nu L_T}{Q} + \alpha_T\right), \tag{6}$$

$$\left(\frac{p_{g,\text{arm}}}{\rho g} + z_1\right) = \frac{8Q^2}{\pi^2 g D_N^4}\frac{16\pi\nu L_N}{Q} + \frac{8Q^2}{\pi^2 g D_T^4}\frac{16\pi\nu L_T}{Q} + \frac{8Q^2}{\pi^2 g D_N^4}\left(K_{\text{entrance}} + K_{\text{expansion}}\right) + \frac{8Q^2}{\pi^2 g D_T^4}\alpha_T, \tag{7}$$

$$\left(\frac{p_{g,\text{arm}}}{\rho g} + z_1\right) = \frac{128\nu Q}{\pi g}\left(\frac{L_N}{D_N^4} + \frac{L_T}{D_T^4}\right) + \frac{8Q^2}{\pi^2 g}\left(\frac{K_{\text{entrance}} + K_{\text{expansion}}}{D_N^4} + \frac{\alpha_T}{D_T^4}\right), \tag{8}$$

$$\underbrace{\frac{8}{\pi^2 g}\left(\frac{K_{\text{entrance}} + K_{\text{expansion}}}{D_N^4} + \frac{\alpha_T}{D_T^4}\right)}_{=A}Q^2 + \underbrace{\frac{128\nu}{\pi g}\left(\frac{L_N}{D_N^4} + \frac{L_T}{D_T^4}\right)}_{=B}Q + \underbrace{-\left(\frac{p_{g,\text{arm}}}{\rho g} + z_1\right)}_{=C} = 0\,. \tag{9}$$

Using the given data,

g	= 9.81 m/s^2
K_{entrance}	= 0.78 (re-entrant inlet)
$K_{\text{expansion}}$	= 0.8 (area ratio = $(D_N/D_T)^2$ = 0.126)
D_N	= 1.07*10^{-3} m
α_T	= 2 (laminar flow at tube exit)
D_T	= 3.0*10^{-3} m
ν	= 3.30*10^{-6} m^2/s
L_N	= 2.54*10^{-2} m
L_T	= 2.0 m
$p_{g,\text{arm}}$	= $\rho g H$ = (13.6*1000 kg/m^3)(9.81 m/s^2)(93.3*10^{-3} m Hg) = 12.5*10^3 Pa
ρ	= 1060 kg/m^3
z_1	= 0.5 m

$\Rightarrow$ A = 1.03*10^{11} s^2/m^5, B = 6.07*10^5 s/m^2, C = -1.70 m

$\Rightarrow$ $\underline{Q = 2.07*10^{-6}\ \text{m}^3/\text{s}}$ $\Rightarrow$ $\bar{V}_N$ = 2.31 m/s, $\bar{V}_T$ = 0.29 m/s $\Rightarrow$ Re_{DN} = 748, Re_{DT} = 266 $\Rightarrow$ The laminar flow assumptions are good ones!

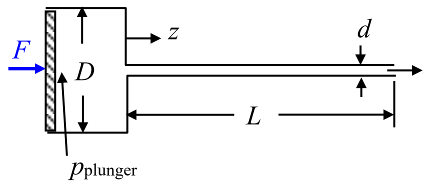
One pint is equivalent to V_{collect} = 4.73*10^{-4} m^3, thus the expected time required to collect one pint of blood is,

$$T = \frac{V_{\text{collect}}}{Q} \Rightarrow \boxed{T = 229\text{ s} = 3.8\text{ min.}}$$

In practice, the time required to donate a pint of blood is approximately 8 – 10 minutes, so this prediction, although in the right ballpark, is too small when compared to reality. Two assumptions likely factor into this error. First, we've assumed fully developed flow in the needle, which is most likely not the case. The pressure drop in the needle will be larger than what we've predicted from our fully developed flow model and thus the flow rate will decrease and the predicted donation time will increase. Secondly, there will be additional losses due to the bends in the tubing and, especially, due to the clamp located on the tubing to make it easier to stop the flow, if needed. These additional minor losses aren't negligible and will contribute to make the flow rate smaller.

A hypodermic needle, with an inside diameter of 0.1 mm and a length of 25 mm is used to inject saline solution with a dynamic viscosity five times that of water. The plunger diameter is 10 mm and the maximum force that can be exerted by a thumb on the plunger is 45 N. Estimate the volume flow rate of saline that can be produced.

SOLUTION:



For a viscous, laminar, fully developed flow in a circular pipe (Poiseuille flow), the average velocity is

$$\bar{u} = \frac{d^2}{32\mu}\left(-\frac{dp}{dz}\right) \tag{1}$$

and the volumetric flow rate is:

$$Q = \bar{u}\frac{\pi d^2}{4} = \frac{\pi d^4}{128\mu}\left(-\frac{dp}{dz}\right) \tag{2}$$

The pressure gradient, assuming fully developed flow in the needle, is:

$$\frac{dp}{dz} = \frac{\Delta p}{L} = \frac{p_{\text{atm}} - p_{\text{plunger}}}{L} = \frac{-p_{\text{plunger,gage}}}{L} \tag{3}$$

where $p_{\text{plunger,gage}}$ is:

$$p_{\text{plunger,gage}} = \frac{F}{\left(\pi D^2/4\right)} \tag{4}$$

Using the given data:

d = 0.1e-3 m
D = 10e-3 m
L = 25e-3 m
F = 45 N
μ = 5e-3 N·m/s
$\Rightarrow$ $p_{\text{plunger,gage}}$ = 5.73e5 Pa
$\Rightarrow$ dp/dz = -2.29e7 Pa/m
$\Rightarrow$ $\bar{u}$ = 1.43 m/s
$\Rightarrow$ $\boxed{Q = 1.13\text{e-}8 \text{ m}^3/\text{s} = 11.3 \text{ mm}^3/\text{s}}$

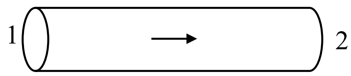
Check the Reynolds number to verify that the laminar flow assumption is ok.

$$\text{Re} = \frac{\rho \bar{u} d}{\mu} \quad (\text{Use } \rho \approx 1000 \text{ kg/m}^3.) \tag{5}$$

$\Rightarrow$ Re = 28.8 < 2300 $\Rightarrow$ The laminar flow assumption is justified!

The average flow speed in a constant-diameter section of the Alaskan pipeline is 8.27 ft/s. At the inlet, the pressure is 1200 psig and the elevation is 150 ft; at the outlet, the pressure is 50 psig and the elevation is 375 ft. Calculate the head loss in this section of pipeline.

SOLUTION:



Apply the Extended Bernoulli's Equation from 1 to 2.

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,1\to 2}+H_{S,1\to 2} \quad (1)$$

where

$$\frac{p_2-p_1}{\rho g}=\frac{(50-1200)\ \frac{\text{lb}_\text{f}}{\text{in}^2}\cdot 144\ \frac{\text{in}^2}{\text{ft}^2}}{\underbrace{(0.9)}_{=\text{SG}_{\substack{\text{crude}\\ \text{oil}}}}\left(1.94\ \frac{\text{slug}}{\text{ft}^3}\right)\left(32.2\ \frac{\text{ft}}{\text{s}^2}\right)}=-2940\text{ ft} \quad (2)$$

$$\left(\alpha\frac{\bar{V}^2}{2g}\right)_2=\left(\alpha\frac{\bar{V}^2}{2g}\right)_1 \quad \text{(since the flow is fully developed and mass is conserved)} \quad (3)$$

$$z_2-z_1=(375-150)\text{ ft}=225\text{ ft} \quad (4)$$

$$H_{L,1\to 2}=?\quad \text{(This is what we're trying to find.)} \quad (5)$$

$$H_{S,1\to 2}=0\quad \text{(There is no shaft work between points 1 and 2.)} \quad (6)$$

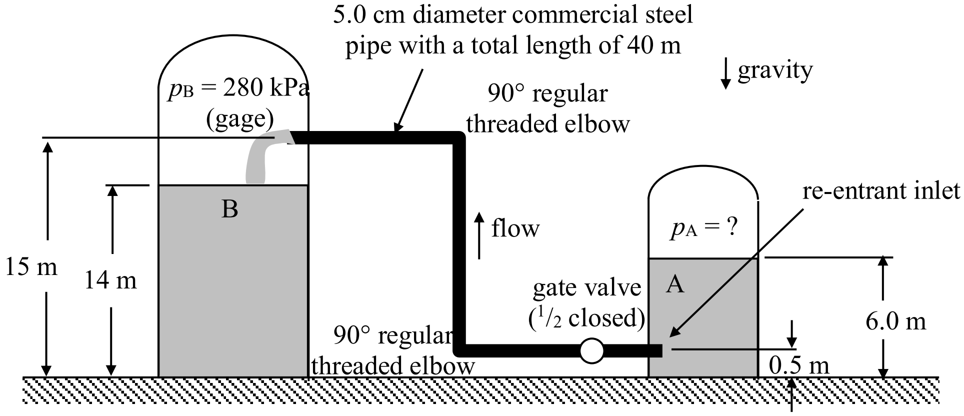
Substitute and solve for H_L.

$$\boxed{H_{L,1\to 2}=-\left(\frac{p_2-p_1}{\rho g}\right)-(z_2-z_1)} \quad (7)$$

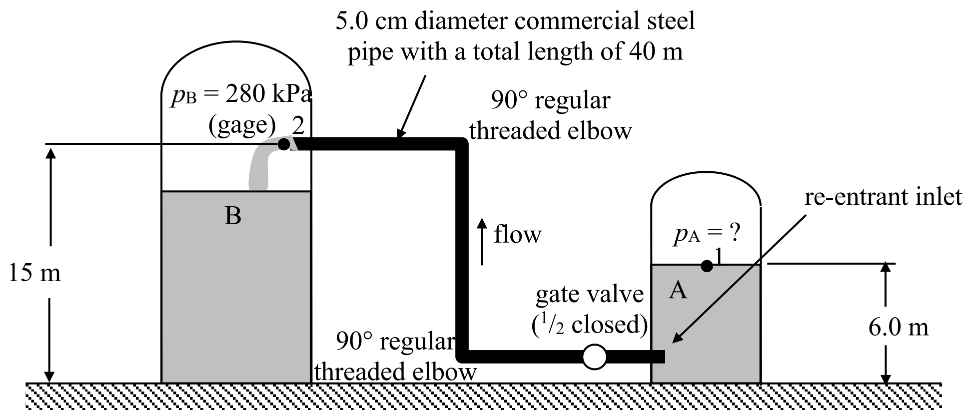
$$\boxed{\therefore H_{L,1\to 2}=2940-225\text{ ft}=2720\text{ ft}} \quad (8)$$

Video solution: https://www.youtube.com/watch?v=dIk7W8oWFSA

Consider the pipe system shown below in which water (with a density of 1.0E3 kg/m^3 and a dynamic viscosity of 1.3E-3 Pa·s) flows from the tank A to tank B. If the required flow rate is 1.0E-2 m^3/s, what is the required gage pressure in tank A?



SOLUTION:



Apply the Extended Bernoulli Equation from the free surface of tank A (point 1) to the end of the pipe leading into tank B (point 2).

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12} \tag{1}$$

where

$p_1 = p_A = ?$ and $p_2 = p_B = 2.8\text{E}5$ Pa (gage) (given)

$\bar{V}_1 \approx 0$ (large tank) $\quad \bar{V}_2 = \bar{V} \quad \alpha_2 \approx 1$ (Assume turbulent flow.)

$z_1 = 6.0\text{E}0$ m and $z_2 = 1.5\text{E}1$ m (given)

$H_{S,12} = 0$ (no fluid machinery between points 1 and 2)

$$H_{L,12} = f\left(\frac{L}{D}\right)\frac{\bar{V}^2}{2g}+K_{\substack{\text{re-entrant}\\ \text{inlet}}}\frac{\bar{V}^2}{2g}+K_{\substack{\text{1/2 open}\\ \text{gate valve}}}\frac{\bar{V}^2}{2g}+2K_{\substack{90^\circ\text{ threaded}\\ \text{elbow}}}\frac{\bar{V}^2}{2g}$$

$$H_{L,12} = \left[f\left(\frac{L}{D}\right)+K_{\substack{\text{re-entrant}\\ \text{inlet}}}+K_{\substack{\text{1/2 open}\\ \text{gate valve}}}+2K_{\substack{90^\circ\text{ threaded}\\ \text{elbow}}}\right]\frac{\bar{V}^2}{2g} \tag{2}$$

(Note that there are no exit losses at point 2 since no mixing occurs there.)

The mean velocity in the pipe is determined from the volumetric flow rate and the pipe area.

$$\bar{V} = \frac{Q}{\frac{\pi}{4}D^2}$$

Using the given data:

$$\bar{V} = \frac{1.0\text{E-2 m}^3/\text{s}}{\frac{\pi}{4}\left(5.0\text{E-2 m}\right)^2} = 5.1\text{E0 m/s}$$

The friction factor, f, is determined from the Moody chart using the Reynolds number in the pipe, Re, and the relative roughness, ε/D.

$$\text{Re} = \frac{\rho\bar{V}D}{\mu} = \frac{\left(1.0\text{E3 kg/m}^3\right)\left(5.1\text{E0 m/s}\right)\left(5.0\text{E-2 m}\right)}{\left(1.3\text{E-3 Pa}\cdot\text{s}\right)} = 2.0\text{E5} \quad \text{(Turbulent flow assumption ok.)}$$

$$\frac{\varepsilon}{D} = \frac{\left(4.5\text{E-5 m}\right)}{\left(5.0\text{E-2 m}\right)} = 9.0\text{E-4}$$

$$f = 2.1\text{E-2}$$

Hence, the major loss coefficient for the system is:

$$K_{\text{major}} = f\left(\frac{L}{D}\right) = \left(2.1E-2\right)\left(\frac{4.0\text{E1 m}}{5.0\text{E-2 m}}\right) = 1.7\text{E1}$$

The minor loss coefficients are found from minor loss tables to be:

$$K_{\substack{\text{re-entrant}\\ \text{inlet}}} = 8.0\text{E-1}$$

$$K_{\substack{\text{half open}\\ \text{gate valve}}} = 2.1\text{E0}$$

$$K_{\substack{90^\circ\ \text{threaded}\\ \text{elbow}}} = 1.5\text{E0}$$

Using the given data, the total head loss (from Eqn. (2)) is:

$$H_{L,12} = \left[\left(2.1E-2\right)\left(\frac{4.0\text{E1 m}}{5.0\text{E-2 m}}\right) + 8.0\text{E-1} + 2.1\text{E0} + 2\left(1.5\text{E0}\right)\right]\frac{\left(5.1\text{E0 m/s}\right)^2}{2\left(9.8\text{E0 m/s}^2\right)}$$

$$\therefore H_{L,12} = 3.0\text{E1 m}$$

Re-arranging Eqn. (1) to solve for p_1 gives:

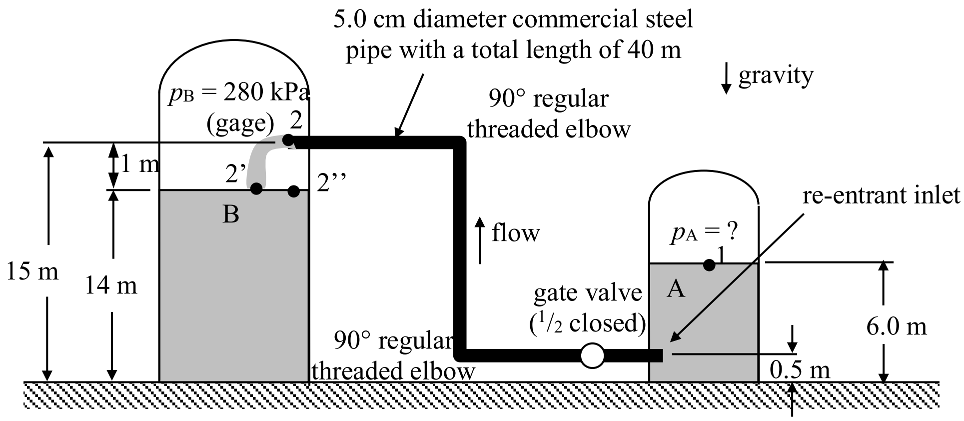
$$p_A = p_B + \tfrac{1}{2}\rho\bar{V}^2 + \rho g\left(z_2 - z_1 + H_{L,12}\right)$$

$$= \left(2.8\text{E5 Pa}\right) + \tfrac{1}{2}\left(1.0\text{E3 kg/m}^3\right)\left(9.5\text{E0 m/s}\right)^2 + \left(1.0\text{E3 kg/m}^3\right)\left(9.8\text{E0 m/s}^2\right)\left(1.5\text{E1 m} - 6.0\text{E0 m} + 3.0\text{E1 m}\right)$$

$$\boxed{p_A = 6.8\text{E5 Pa}}$$

Video solution: https://www.youtube.com/watch?v=dIk7W8oWFSA

Now let's solve the problem using points 2' and 2'' as shown in the figure below.



Apply the Extended Bernoulli Equation from the free surface of tank A (point 1) to the end of the stream at the surface of the free jet (point 2').

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_{2'}=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12'}+H_{S,12'} \tag{3}$$

where

$p_1 = p_A = ?$ and $p_{2'} = p_B = 2.8\text{E}5$ Pa (gage) (given)

$\bar{V}_1 \approx 0$ (large tank)

$\bar{V}_{2'}^2 = \bar{V}_2^2 + 2g(1\text{ m})$ (using Bernoulli's Eqn applied from the end of the pipe to the surface of tank B)

$\bar{V}_{2'} = 6.7\text{E}0$ m/s

$\alpha_{2'} \approx 1$ (Assume turbulent flow.)

$z_1 = 6.0\text{E}0$ m and $z_{2'} = 1.4\text{E}1$ m (given)

$H_{S,12'} = 0$ (no fluid machinery between points 1 and 2)

$$H_{L,12'} = f\left(\frac{L}{D}\right)\frac{\bar{V}_2^2}{2g}+K_{\substack{\text{re-entrant}\\ \text{inlet}}}\frac{\bar{V}_2^2}{2g}+K_{\substack{\text{1/2 open}\\ \text{gate valve}}}\frac{\bar{V}_2^2}{2g}+2K_{\substack{90^\circ\text{ threaded}\\ \text{elbow}}}\frac{\bar{V}_2^2}{2g}$$

$$H_{L,12'}=\left[f\left(\frac{L}{D}\right)+K_{\substack{\text{re-entrant}\\ \text{inlet}}}+K_{\substack{\text{1/2 open}\\ \text{gate valve}}}+2K_{\substack{90^\circ\text{ threaded}\\ \text{elbow}}}\right]\frac{\bar{V}_2^2}{2g} \tag{4}$$

(Note that there are no exit losses from point 2 to point 2' since the kinetic energy in the stream hasn't been dissipated.)

Using the same data as in the previous solution, except for the velocity at 2' and the elevation at 2',

$$p_A = p_B + \tfrac{1}{2}\rho\bar{V}_{2'}^2 + \rho g\left(z_{2'} - z_1 + H_{L,12'}\right)$$

$$= (2.8\text{E}5\text{ Pa}) + \tfrac{1}{2}(1.0\text{E}3\text{ kg/m}^3)(6.7\text{E}0\text{ m/s})^2 + (1.0\text{E}3\text{ kg/m}^3)(9.8\text{E}0\text{ m/s}^2)(1.4\text{E}1\text{ m} - 6.0\text{E}0\text{ m} + 3.0\text{E}1\text{ m})$$

$\boxed{p_A = 6.8\text{E}5\text{ Pa}}$ (Same answer as found previously!)

Now apply the Extended Bernoulli Equation from the free surface of tank A (point 1) to the surface of the tank (point 2'').

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_{2''}=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12''}+H_{S,12''} \tag{5}$$

where

$p_1 = p_A = ?$ and $p_{2'} = p_B = 2.8\text{E}5$ Pa (gage) (given)

$\bar{V}_1 \approx 0$ (large tank)

$\bar{V}_{2''} \approx 0$ (surface of large tank)

$z_1 = 6.0\text{E}0$ m and $z_{2''} = 1.4\text{E}1$ m (given)

$H_{S,12''} = 0$ (no fluid machinery between points 1 and 2)

$$H_{L,12''}=f\left(\frac{L}{D}\right)\frac{\bar{V}_2^2}{2g}+K_{\substack{\text{re-entrant}\\\text{inlet}}}\frac{\bar{V}_2^2}{2g}+K_{\substack{\text{1/2 open}\\\text{gate valve}}}\frac{\bar{V}_2^2}{2g}+2K_{\substack{90^\circ\text{ threaded}\\\text{elbow}}}\frac{\bar{V}_2^2}{2g}+K_{\text{exit}}\frac{\bar{V}_{2'}^2}{2g}$$

$$H_{L,12''}=\left[f\left(\frac{L}{D}\right)+K_{\substack{\text{re-entrant}\\\text{inlet}}}+K_{\substack{\text{1/2 open}\\\text{gate valve}}}+2K_{\substack{90^\circ\text{ threaded}\\\text{elbow}}}\right]\frac{\bar{V}_2^2}{2g}+K_{\text{exit}}\frac{\bar{V}_{2'}^2}{2g} \tag{6}$$

(Note that the kinetic energy in the stream is dissipated when going from point 2' to point 2''. Thus, the correct velocity to use in the velocity head term is the velocity at 2'.)

Using the same data as in the previous solution, except for the velocity at 2' and the elevation at 2',

$$p_A = p_B + \rho g\left(z_{2''} - z_1 + H_{L,12''}\right)$$

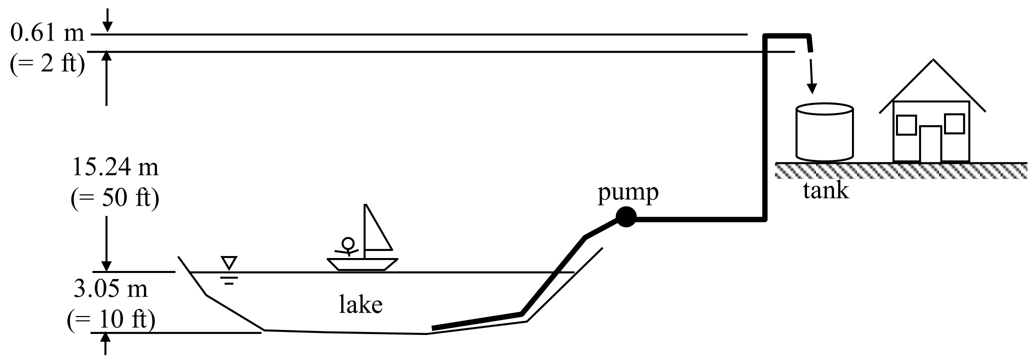
$$= \left(2.8\text{E}5\text{ Pa}\right)+\left(1.0\text{E}3\text{ kg/m}^3\right)\left(9.8\text{E}0\text{ m/s}^2\right)\left(1.4\text{E}1\text{ m}-6.0\text{E}0\text{ m}+3.2\text{E}1\text{ m}\right)$$

$\boxed{p_A = 6.8\text{E}5\text{ Pa}}$ (Same answer as found previously!)

You purchase a cottage at a lake and need to install a pump to feed water to the house. You plan to pump water at night to fill a storage tank you've installed next to the cottage. The pipes and fittings you have chosen to use for the installation are listed in the table below.

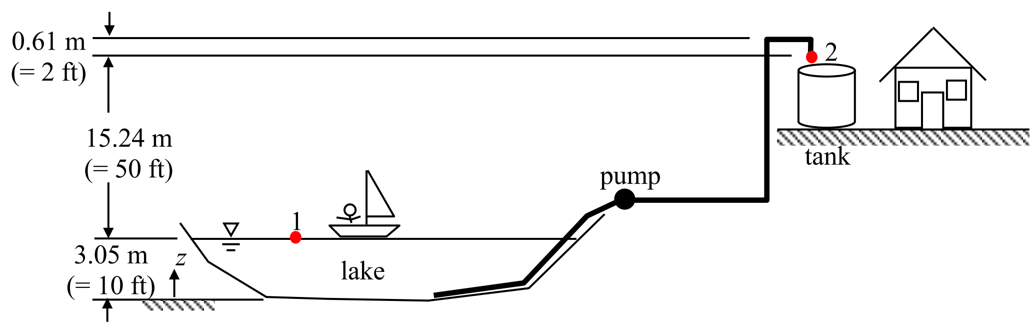
a. What is the minimum head rise across a pump that is capable of providing a flow rate of 18.93 liters per minute (= 5 gpm) of water to the tank?

b. What power should be supplied to the pump assuming the pump efficiency is 65%.



System Component	Amount
straight, smooth plastic pipe, 5.08 cm (= 2 in.) diameter	28.96 m (=95 ft)
re-entrant inlet	1
regular 90º flanged elbow	10
union	8
fully open globe valve	1
fully open gate valve	4

SOLUTION:



Apply the Extended Bernoulli Equation from point 1 to point 2.

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,1\to 2}+H_{S,1\to 2} \tag{1}$$

where

$$p_2=p_1=p_{\text{atm}} \tag{2}$$

$$\bar{V}_1\approx 0 \tag{3}$$

$$\bar{V}_2=\bar{V}_P=\frac{Q}{\frac{\pi D^2}{4}} \quad \text{and} \quad \alpha_2\approx 1 \text{ (assuming turbulent flow)} \tag{4}$$

$$z_2-z_1=15.24\text{ m} \tag{5}$$

$$H_{L,1\to2} = \sum_i K_i \frac{\bar{V}_i^2}{2g} = K_{\text{major}}\frac{\bar{V}_P^2}{2g} + K_{\text{inlet}}\frac{\bar{V}_P^2}{2g} + 10K_{\text{elbow}}\frac{\bar{V}_P^2}{2g} + 8K_{\text{union}}\frac{\bar{V}_P^2}{2g} + K_{\substack{\text{globe}\\\text{valve}}}\frac{\bar{V}_P^2}{2g} + 4K_{\substack{\text{gate}\\\text{valve}}}\frac{\bar{V}_P^2}{2g}$$
$$= \left(f\frac{L}{D} + K_{\text{inlet}} + 10K_{\text{elbow}} + 8K_{\text{union}} + K_{\substack{\text{globe}\\\text{valve}}} + 4K_{\substack{\text{gate}\\\text{valve}}}\right)\frac{\bar{V}_P^2}{2g} \tag{6}$$

(where $\bar{V}_P = \bar{V}_2 = \frac{Q}{\frac{\pi D^2}{4}}$)

$$H_{S,1\to2} = ? \tag{7}$$

Substitute into the Extended Bernoulli Equation and re-arrange.

$$\frac{\bar{V}_P^2}{2g} + z_2 = z_1 - \left(f\frac{L}{D} + K_{\text{inlet}} + 10K_{\text{elbow}} + 8K_{\text{union}} + K_{\substack{\text{globe}\\\text{valve}}} + 4K_{\substack{\text{gate}\\\text{valve}}}\right)\frac{\bar{V}_P^2}{2g} + H_{S,1\to2} \tag{8}$$

$$H_{S,1\to2} = (z_2 - z_1) + \left(1 + f\frac{L}{D} + K_{\text{inlet}} + 10K_{\text{elbow}} + 8K_{\text{union}} + K_{\substack{\text{globe}\\\text{valve}}} + 4K_{\substack{\text{gate}\\\text{valve}}}\right)\frac{\bar{V}_P^2}{2g} \tag{9}$$

$$\boxed{H_{S,1\to2} = (z_2 - z_1) + \left(1 + f\frac{L}{D} + K_{\text{inlet}} + 10K_{\text{elbow}} + 8K_{\text{union}} + K_{\substack{\text{globe}\\\text{valve}}} + 4K_{\substack{\text{gate}\\\text{valve}}}\right)\frac{8Q^2}{\pi^2 D^4 g}} \tag{10}$$

Use the given data to determine $H_{S,1\to2}$.

$z_2 - z_1 = 15.24$ m
$L = 28.96$ m
$D = 5.08$ cm $= 5.08\text{e-}2$ m
$K_{\text{inlet}} = 0.8$
$K_{\text{elbow}} = 0.3$
$K_{\text{threaded union}} = 0.06$
$K_{\text{globe valve}} = 10$
$K_{\text{gate valve}} = 0.15$
$Q = 18.93$ L/min $= 3.154\text{e-}4$ m^3/s (= 5 gpm)
$g = 9.81$ m/s^2 (= 32.2 ft/s^2)

The friction factor is found using the Moody chart for a smooth pipe and a Reynolds number of:

$$\text{Re} = \frac{\bar{V}_P D}{\nu} = \frac{QD}{\frac{\pi D^2}{4}\nu} = \frac{4Q}{\pi D \nu} = \frac{4(3.154\text{e-}4\ \text{m}^3/\text{s})}{\pi(5.08\text{e-}2\ \text{m})(1\text{e-}6\ \text{m}^2/\text{s})} \approx 7900$$

(The turbulent flow assumption is valid!) (11)

$\Rightarrow f \approx 0.033$ (from the Moody diagram) (12)

$\boxed{\therefore H_{S,1\to2} = 15.28\ \text{m}}$ (= 50.14 ft) (13)

Note that the head loss is much smaller than the elevation head.

The power is related to the shaft head by:

$$H_{S,1\to2} = \frac{\dot{W}_{S,1\to2}}{\dot{m}g} = \frac{\dot{W}_{S,1\to2}}{\rho Q g} \tag{14}$$

$$\therefore \dot{W}_{S,1\to2} = \rho Q g H_{S,1\to2} \tag{15}$$

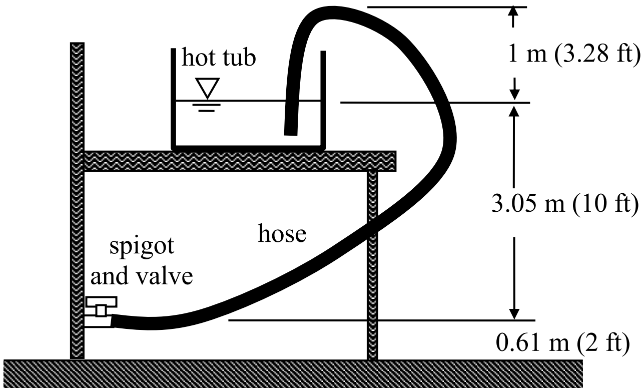
Using the given data ($\rho = 1000$ kg/m^3):

$$\therefore \dot{W}_{S,1\to2} = 47.3\ \text{W}\ (= 0.06\ \text{hp}) \tag{16}$$

Since the efficiency is $\eta = 65\%$, the power that must be supplied to the pump is:

$$\therefore \dot{W}_{\text{supply}} = \frac{\dot{W}_{S,1\to2}}{\eta} \Rightarrow \boxed{\therefore \dot{W}_{\text{supply}} = 72.7\ \text{W}}\ (= 0.1\ \text{hp}) \tag{17}$$

A hot tub sits on a deck as shown in the figure below. A homeowner plans to fill the hot tub with water from a 1.91 cm (0.75 in.) diameter, 7.62 m (25 ft) length of old garden hose attached to an outdoor spigot (aka faucet) located underneath the deck. The hose has an internal roughness of 0.5 mm ($1.97*10^{-2}$ in.) and the gage water pressure just upstream of the spigot valve is 379 kPa (55 psig).

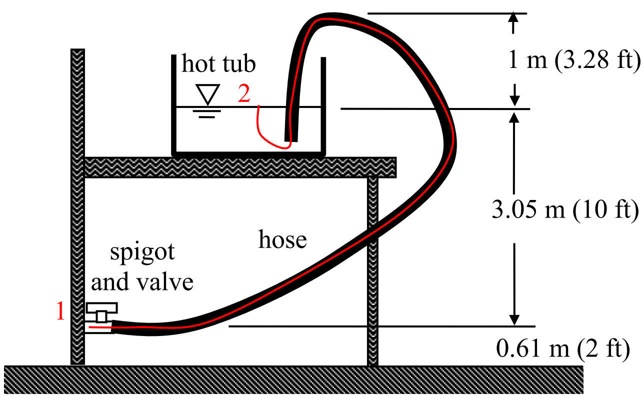


water density = 1000 kg/m^3 (62.4 lb$_m$/ft^3)
water kinematic viscosity = $1.00*10^{-6}$ m^2/s
($1.05*10^{-5}$ ft^2/s)

The minor losses due to the bends in the hose are much smaller than the minor loss due to the valve, which has a loss coefficient of 2.

Determine the volumetric flow rate of water into the hot tub. Clearly state and justify all significant assumptions.

SOLUTION:



Apply the Extended Bernoulli Equation from 1 to 2.

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \tag{1}$$

where

$$p_{1,g}=3.79*10^3\text{ Pa}\ \text{ and }\ p_{2,g}=0 \tag{2}$$

$$\bar{V}_1=\bar{V}_h\ \text{ and }\ \bar{V}_2\approx 0 \tag{3}$$

$$z_2-z_1=\Delta z=3.05\text{ m} \tag{4}$$

$$H_L=\left[f\left(\frac{L}{D}\right)+K_{\text{valve}}+K_{\text{exit}}+K_{\text{bends}}\right]\frac{\bar{V}_h^2}{2g} \tag{5}$$

$$H_S=0 \tag{6}$$

Assume the flow in the hose is fully turbulent so that,

$$\alpha_1 \approx 1, \tag{7}$$

and the friction factor is solely a function of the relative roughness,

$$\frac{e}{D} = \frac{0.5*10^{-3}\ \text{m}}{1.91*10^{-2}\ \text{m}} = 2.62*10^{-2} \tag{8}$$

Using the Moody diagram, the friction factor is,

$$f \approx 0.054 \tag{9}$$

Substitute into Eq. (1) and solve for the average water speed in the hose.

$$\Delta z = \frac{p_{1,g}}{\rho g} + \frac{\bar{V}_h^2}{2g} - \left[f\left(\frac{L}{D}\right) + K_{\text{valve}} + K_{\text{exit}} + K_{\text{bends}}\right]\frac{\bar{V}_h^2}{2g} \tag{10}$$

Note that the major loss coefficient of,

$$K_{\text{major}} = f\left(\frac{L}{D}\right) = 21.65, \tag{11}$$

is larger than the valve and exit losses (recall that from the problem statement, $K_{\text{bends}} << K_{\text{valve}}$, and thus the bend losses may be neglected), with $K_{\text{valve}}/K_{\text{major}} = 0.09$ and $K_{\text{exit}}/K_{\text{major}} = 0.05$. We should retain these minor losses since they are not small enough to be neglected.

Solving Eq. (10) for the hose average velocity gives,

$$\bar{V}_h = \sqrt{\frac{2g\left(\Delta z - \frac{p_{1,g}}{\rho g}\right)}{1 - f\left(\frac{L}{D}\right) - K_{\text{valve}} - K_{\text{exit}}}} \Rightarrow \bar{V}_h = 5.4\ \text{m/s} \tag{12}$$

The volumetric flow rate is,

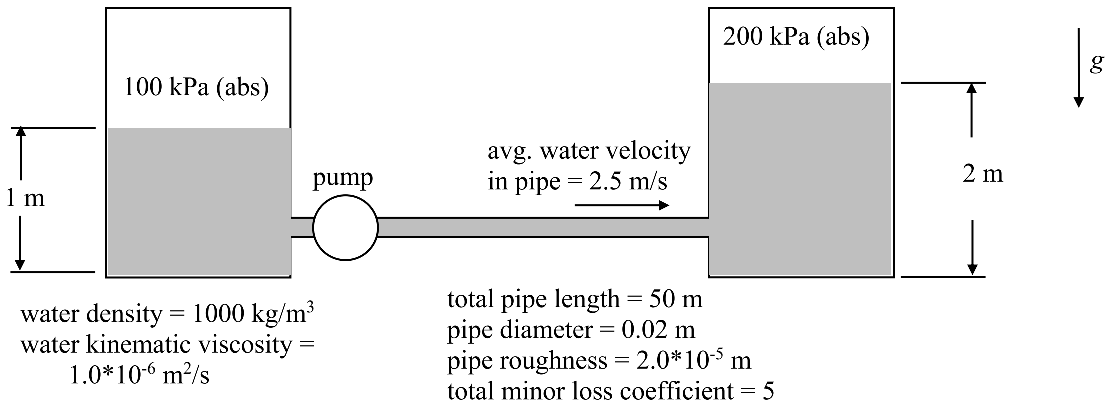
$$Q = \bar{V}_h \tfrac{\pi}{4} D^2 \Rightarrow \boxed{Q = 1.6*10^{-3}\ \text{m}^3/\text{s}} \tag{13}$$

Verify that the flow is indeed in the fully turbulent zone.

$$\text{Re}_D = \frac{\bar{V}_h D}{\nu} = \frac{(5.4\ \text{m/s})(1.91*10^{-2}\ \text{m})}{1.0*10^{-6}\ \text{m}^2/\text{s}} = 1.0*10^5 \tag{14}$$

This value of the Reynolds number is in the fully turbulent zone, so the assumption of fully turbulent flow was a good one.

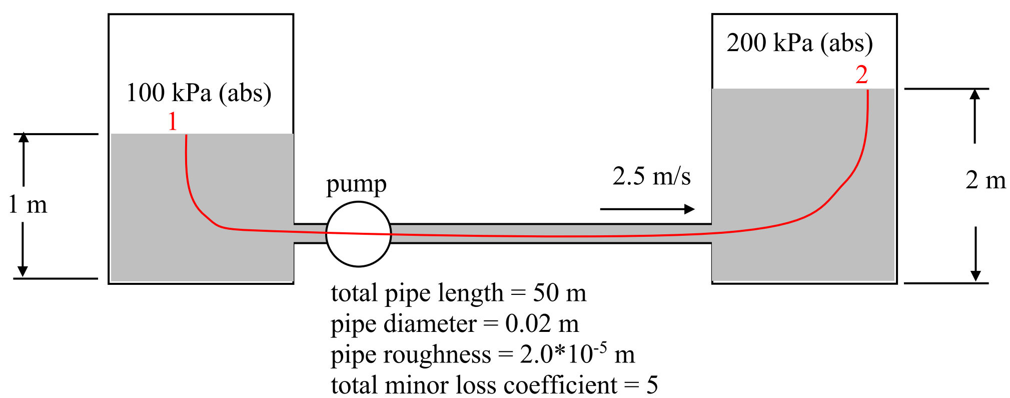
Consider the pipe system shown in the figure below.



Determine the power the pump must provide to the water to maintain the given conditions.

SOLUTION:

Apply the Extended Bernoulli Equation from point 1 to point 2.



$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \tag{1}$$

Re-arrange to solve for the shaft head term,

$$H_S=\left(\frac{p_2-p_1}{\rho g}\right)+\left(\alpha_2\frac{\bar{V}_2^2}{2g}-\alpha_1\frac{\bar{V}_1^2}{2g}\right)+\left(z_2-z_1\right)+H_L \tag{2}$$

where

$$g=9.81\text{ m/s}^2 \text{ and } \rho=1000\text{ kg/m}^3 \tag{3}$$

$$p_1=100\text{ kPa (abs) and } p_2=200\text{ kPa (abs)} \tag{4}$$

$$\bar{V}_1\approx 0 \text{ and } \bar{V}_2\approx 0 \tag{5}$$

$$z_1=1\text{ m and } z_2=2\text{ m} \tag{6}$$

$$H_L=\left[f\left(\frac{L}{D}\right)+K_{\text{minor}}\right]\frac{\bar{V}_{\text{pipe}}^2}{2g} \tag{7}$$

and

$L = 50\text{ m}$ and $D = 0.02\text{ m}$ (8)

$K_{\text{minor}} = 5$ (9)

$$e/D = \left(2.0*10^{-5}\text{ m}\right)/\left(0.02\text{ m}\right) = 0.001 \quad (10)$$

$$\text{Re} = \frac{\bar{V}_{\text{pipe}}D}{\nu} = \frac{\left(2.5\text{ m/s}\right)\left(0.02\text{ m}\right)}{\left(1.0*10^{-6}\text{ m}^2/\text{s}\right)} = 50,000 \quad (11)$$

Use the Moody diagram to find the friction factor for this Reynolds number and relative roughness,

$f = 0.024$ (12)

Using the given data,

$H_S = 31.7\text{ m}$ (13)

The power may be found from the shaft head term using,

$$\dot{W}_S = \rho Q g H_S = \rho V_{\text{pipe}} \tfrac{\pi}{4} D^2 g H_S \quad (14)$$

Thus,

$$\boxed{\dot{W}_S = 244\text{ W}} \quad (15)$$

11.7.1. Serial Pipe Systems

Serial pipe systems have multiple pipes that have the same inlet conditions and the same outlet conditions (Figure 11.16). For these systems one simply applies the EBE separately for each pipe.

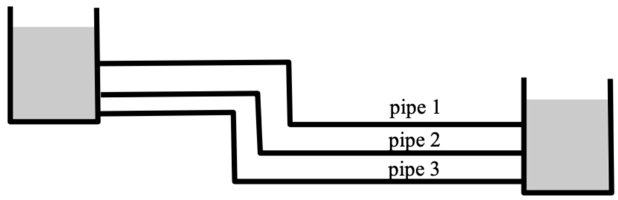


Figure 11.16. An example of a serial pipe system, which has multiple pipes, but the same inlet and outlet conditions.

11.7.2. Parallel Pipe Systems

Parallel pipe systems involve pipes that have intersections, i.e., nodes (Figure 11.17). These pipe systems are more challenging to solve. The EBE can be used between nodes and between nodes and inlets and outlets. Conservation of Mass should be applied at each node. The result will be a system of non-linear equations (due to velocity squared terms that appear in the EBE) that must be solved simultaneously. Often these systems of equations are solved computationally using iterative techniques.

Interestingly, pipe networks have many similarities with electrical networks, with pipe resistances corresponding to electrical resistances, flow rates corresponding to current, and head differences (due to elevation differences or pumps) corresponding to voltage differences. There are other electrical analogies too. For example surge tanks have properties similar to capacitors, heavy paddle wheels have properties similar to inductors, and ball and check valves act as diodes.

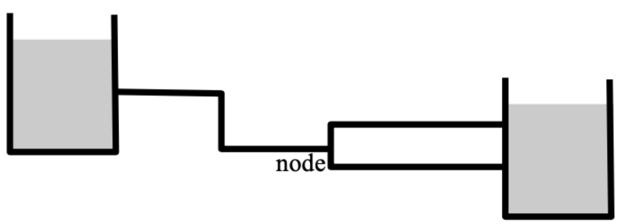
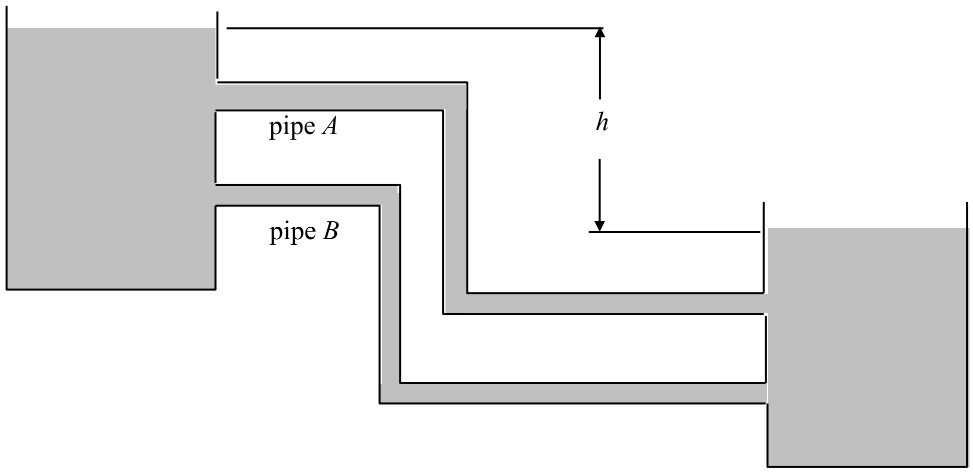


Figure 11.17. An example of a parallel pipe system, which has multiple, interconnecting pipes. The location at which pipes intersect is known as a “node”.

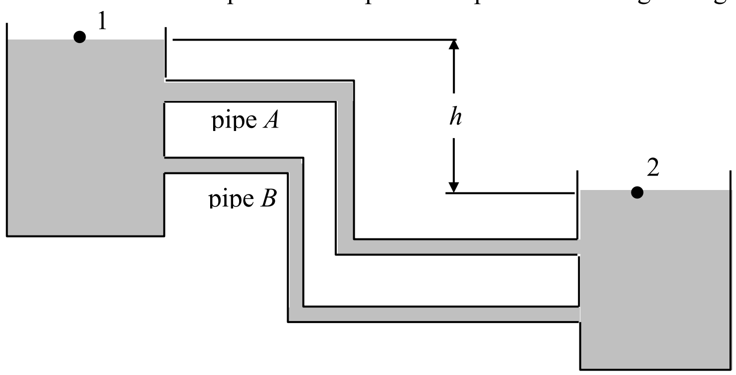
Video solution: https://www.youtube.com/watch?v=hEtlZQde5x4

Two water reservoirs are connected by galvanized iron pipes. Assume D_A = 75 mm, D_B = 50 mm, and h = 10.5 m. The length of both pipes is 100 m. Compare the head losses in pipes *A* and *B*. Compute the volume flow rate in each pipe.



SOLUTION:

Apply the Extended Bernoulli Equation from point 1 to point 2 traveling through each pipe.



$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12}$$

where

$p_1 = p_2 = p_{atm}$ (free surfaces)

$\bar{V}_1 \approx \bar{V}_2 \approx 0$ (surface of large tanks)

$z_1 - z_2 = h$ (given)

$H_{S,12} = 0$ (no fluid machines between points 1 and 2)

$$H_{L,12,A} = K_{\text{major},A}\frac{\bar{V}_A^2}{2g}+K_{\text{entrance}}\frac{\bar{V}_A^2}{2g}+2K_{\text{elbow}}\frac{\bar{V}_A^2}{2g}+K_{\text{exit}}\frac{\bar{V}_A^2}{2g}$$

$$H_{L,12,B} = K_{\text{major},B}\frac{\bar{V}_B^2}{2g}+K_{\text{entrance}}\frac{\bar{V}_B^2}{2g}+2K_{\text{elbow}}\frac{\bar{V}_B^2}{2g}+K_{\text{exit}}\frac{\bar{V}_B^2}{2g}$$

and

$$K_{\text{major},A} = f_A\left(\frac{L_A}{D_A}\right) \quad \text{and} \quad K_{\text{major},B} = f_B\left(\frac{L_B}{D_B}\right)$$

Substituting into the Extended Bernoulli Equation gives:

$$h = \left[f_A \left(\frac{L_A}{D_A} \right) + K_{\text{entrance}} + 2K_{\text{elbow}} + K_{\text{exit}} \right] \frac{\bar{V}_A^2}{2g} \tag{1}$$

$$h = \left[f_B \left(\frac{L_B}{D_B} \right) + K_{\text{entrance}} + 2K_{\text{elbow}} + K_{\text{exit}} \right] \frac{\bar{V}_B^2}{2g} \tag{2}$$

From Eqns. (1) and (2) we observe that $\boxed{\text{the head loss in each pipe is the same and equal to 10.5 m}}$.

The pipes are made of galvanized iron so the roughness of the pipes is ε = 0.15 mm (found from an average roughness table). Hence, the relative roughness in each pipe is:

$$\left.\frac{\varepsilon}{D}\right|_A = \frac{0.15\text{ mm}}{75\text{ mm}} = 0.0020 \quad \text{and} \quad \left.\frac{\varepsilon}{D}\right|_B = \frac{0.15\text{ mm}}{50\text{ mm}} = 0.0030$$

Since we don't yet know the velocity in each pipe, assume that the flows are in the wholly turbulent flow region so that the friction factor is independent of the Reynolds number. For this case, the Moody chart (or the Colebrook relation) indicates that the friction factors corresponding to the relative roughnesses determined above are:

$$f_A = 0.0234 \quad \text{and} \quad f_B = 0.0262$$

Substitute the given data into Eqns. (1) and (2).

h	=	10.5 m
K_{entrance}	=	0.5 (assuming a sharp-edged entrance)
K_{elbow}	=	1.5 (assuming 90° threaded elbows)
K_{exit}	=	1.0
g	=	9.81 m/s^2
L_A	=	100 m
L_B	=	100 m
D_A	=	75 mm = 7.5*10^{-2} m
D_B	=	50 mm = 5.0*10^{-2} m
f_A	=	0.0234 (from above)
f_B	=	0.0262 (from above)
ν_{H20}	=	1.01*10^{-6} m^2/s (water at 20 °C)

Note that the sum of the minor loss coefficients (= 4.5 = 0.5 + 2*1.5 + 1) are not insignificant compared to the major loss coefficients so the minor losses cannot be neglected without significant error.

$$K_{\text{major},A} = 31.2 \quad \text{and} \quad K_{\text{major,B}} = 52.4$$

Solving for the average velocities gives:

$$\bar{V}_A = 2.40\text{ m/s} \quad \text{and} \quad \bar{V}_B = 1.90\text{ m/s}$$

The corresponding Reynolds numbers are:

$$\text{Re}_A = \frac{\bar{V}_A D_A}{\nu_{\text{H20}}} = 1.78*10^5 \quad \text{and} \quad \text{Re}_B = \frac{\bar{V}_B D_B}{\nu_{\text{H20}}} = 9.42*10^4 \tag{3}$$

Unfortunately, these Reynolds numbers do not put us in the wholly turbulent zone on the Moody chart (although it's very close) so we must try iterating to a solution instead. For a new choice of friction factors, use the Reynolds number given in Eqn. (3) and consult the Moody chart (or the Colebrook formula).

$$f_A = 0.0244 \quad \text{and} \quad f_B = 0.0275$$

Using these friction factors we find:

$\bar{V}_A = 2.36$ m/s and $\bar{V}_B = 1.86$ m/s

and

$$\mathrm{Re}_A = \frac{\bar{V}_A D_A}{\nu_{\mathrm{H20}}} = 1.75*10^5 \quad \text{and} \quad \mathrm{Re}_B = \frac{\bar{V}_B D_B}{\nu_{\mathrm{H20}}} = 9.21*10^4$$

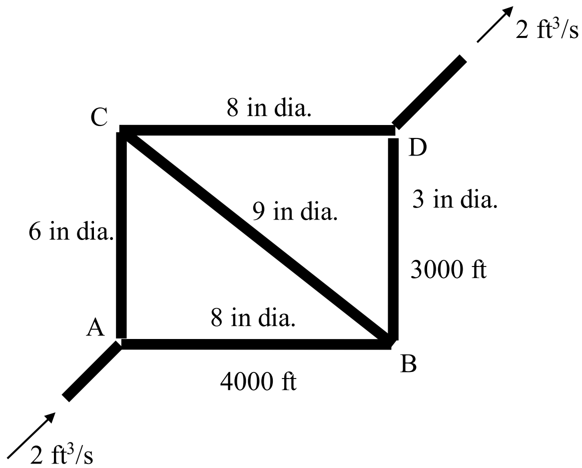
Fortunately these Reynolds numbers give the same friction factors that we started with.

Thus, the volumetric flow rate through each pipe is:

$$\boxed{Q_A = \bar{V}_A \frac{\pi D_A^2}{4} = 1.04*10^{-2} \text{ m}^3/\text{s}}$$

$$\boxed{Q_B = \bar{V}_B \frac{\pi D_B^2}{4} = 3.65*10^{-3} \text{ m}^3/\text{s}}$$

In the five-pipe horizontal network shown in the figure, assume that all pipes have a friction factor f = 0.025. For the given inlet and exit flow rate of 2 ft^3/s of water at 20 °C, determine the flow rate and direction in all pipes. If p_A=120 lb$_f$/in^2 (gage), determine the pressures at points B, C, and D.



SOLUTION:

Apply the Extended Bernoulli Equation around the loops ABCA and DBCD.

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_A=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_A-H_{L,ABCA}+H_{S,ABCA} \tag{1}$$

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_D=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_D-H_{L,DBCD}+H_{S,DBCD} \tag{2}$$

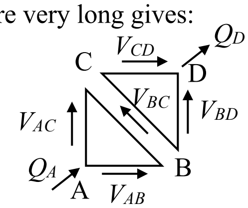
Note that the shaft head terms ($H_{S,ABCA}$ and $H_{S,DBCD}$) are zero since there are no fluid machines in the loops. Simplifying Eqns. (1) and (2) gives:

$$H_{L,ABCA}=0 \tag{3}$$

$$H_{L,DBCD}=0 \tag{4}$$

Expanding the head loss term and neglecting minor losses since the pipes are very long gives:

$$H_{L,ABCA}=f_{AB}\left(\frac{L_{AB}}{D_{AB}}\right)\frac{\bar{V}_{AB}^2}{2g}+f_{BC}\left(\frac{L_{BC}}{D_{BC}}\right)\frac{\bar{V}_{BC}^2}{2g}-f_{AC}\left(\frac{L_{AC}}{D_{AC}}\right)\frac{\bar{V}_{AC}^2}{2g}=0 \tag{5}$$



$$H_{L,DBCD}=-f_{BD}\left(\frac{L_{BD}}{D_{BD}}\right)\frac{\bar{V}_{BD}^2}{2g}+f_{BC}\left(\frac{L_{BC}}{D_{BC}}\right)\frac{\bar{V}_{BC}^2}{2g}+f_{CD}\left(\frac{L_{CD}}{D_{CD}}\right)\frac{\bar{V}_{CD}^2}{2g}=0 \tag{6}$$

Note that particular velocity directions have been assumed in the head loss expressions.

At each pipe node the volumetric flow rate must be conserved (conservation of mass). Hence:

at node A: $Q_A=Q_{AB}+Q_{AC} \Rightarrow Q_A=\bar{V}_{AB}\frac{\pi D_{AB}^2}{4}+\bar{V}_{AC}\frac{\pi D_{AC}^2}{4}$ (7)

at node B: $Q_{AB}=Q_{BC}+Q_{BD} \Rightarrow \bar{V}_{AB}\frac{\pi D_{AB}^2}{4}=\bar{V}_{BC}\frac{\pi D_{BC}^2}{4}+\bar{V}_{BD}\frac{\pi D_{BD}^2}{4}$ (8)

at node C: $Q_{CD}=Q_{AC}+Q_{BC} \Rightarrow \bar{V}_{CD}\frac{\pi D_{CD}^2}{4}=\bar{V}_{AC}\frac{\pi D_{AC}^2}{4}+\bar{V}_{BC}\frac{\pi D_{BC}^2}{4}$ (9)

at node D: $Q_D=Q_{BD}+Q_{CD} \Rightarrow Q_D=\bar{V}_{BD}\frac{\pi D_{BD}^2}{4}+\bar{V}_{CD}\frac{\pi D_{CD}^2}{4}$ (10)

Simplify and summarize Eqns. (5) - (10).

$$f_{AB}\left(\frac{L_{AB}}{D_{AB}}\right)\bar{V}_{AB}^2 + f_{BC}\left(\frac{L_{BC}}{D_{BC}}\right)\bar{V}_{BC}^2 - f_{AC}\left(\frac{L_{AC}}{D_{AC}}\right)\bar{V}_{AC}^2 = 0 \tag{11}$$

$$-f_{BD}\left(\frac{L_{BD}}{D_{BD}}\right)\bar{V}_{BD}^2 + f_{BC}\left(\frac{L_{BC}}{D_{BC}}\right)\bar{V}_{BC}^2 + f_{CD}\left(\frac{L_{CD}}{D_{CD}}\right)\bar{V}_{CD}^2 = 0 \tag{12}$$

$$\left(\frac{\pi D_{AB}^2}{4}\right)\bar{V}_{AB} + \left(\frac{\pi D_{AC}^2}{4}\right)\bar{V}_{AC} = Q_A \tag{13}$$

$$\left(\frac{\pi D_{BC}^2}{4}\right)\bar{V}_{BC} + \left(\frac{\pi D_{BD}^2}{4}\right)\bar{V}_{BD} - \left(\frac{\pi D_{AB}^2}{4}\right)\bar{V}_{AB} = 0 \tag{14}$$

$$\left(\frac{\pi D_{AC}^2}{4}\right)\bar{V}_{AC} + \left(\frac{\pi D_{BC}^2}{4}\right)\bar{V}_{BC} - \left(\frac{\pi D_{CD}^2}{4}\right)\bar{V}_{CD} = 0 \tag{15}$$

$$\left(\frac{\pi D_{BD}^2}{4}\right)\bar{V}_{BD} + \left(\frac{\pi D_{CD}^2}{4}\right)\bar{V}_{CD} = Q_D \tag{16}$$

Note that Eqn. (16) is not independent since it can be formed by adding Eqns. (13) and (14), subtracting Eqn. (15) and noting that $Q_D = Q_A$. Hence, Eqns. (11) - (15) represent five equations with five unknowns ($\bar{V}_{AB}, \bar{V}_{BC}, \bar{V}_{AC}, \bar{V}_{BD}$, and $\bar{V}_{CD}$). Note that f_{AB}, f_{BC}, f_{AC}, f_{BD}, and f_{CD} are given in the problem statement along with each pipe's length and diameter and the volumetric flowrate Q_A.

Using the given data:

$f_{\text{all pipes}}$	=	0.025
$L_{AB} = L_{CD}$	=	4000 ft
$L_{AC} = L_{BD}$	=	3000 ft
L_{BC}	=	5000 ft (from the Pythagorean theorem)
$D_{AB} = D_{CD}$	=	8/12 ft
D_{AC}	=	6/12 ft
D_{BD}	=	3/12 ft
D_{BC}	=	9/12 ft
Q_A	=	2 ft^3/s

The system of non-linear algebraic equations (Eqns. (11) - (15)) can be solved iteratively. One approach is given below.

1. Assume a value of $\bar{V}_{AB}$.
2. Solve for $\bar{V}_{AC}$ using Eqn. (13).
3. Solve for $\bar{V}_{BC}$ using Eqn. (11).
4. Solve for $\bar{V}_{CD}$ using Eqn. (15).
5. Solve for $\bar{V}_{BD}$ using Eqn. (12).
6. Solve for $\bar{V}_{AB}$ using Eqn. (14).
7. Are the $\bar{V}_{AB}$ s from step 6 and step 1 equal? If so, then the iterations are finished. If not, then choose a new value for $\bar{V}_{AB}$ and go to step 2.

After some iteration.

$\bar{V}_{AB}$	=	$3.40*10^{0}$ ft/s	$\Rightarrow$ Q_{AB}	=	$1.19*10^{0}$ ft^3/s
$\bar{V}_{AC}$	=	$4.14*10^{0}$ ft/s	$\Rightarrow$ Q_{AC}	=	$8.13*10^{-1}$ ft^3/s
$\bar{V}_{BC}$	=	$2.24*10^{0}$ ft/s	$\Rightarrow$ Q_{BC}	=	$9.90*10^{-1}$ ft^3/s
$\bar{V}_{CD}$	=	$5.17*10^{0}$ ft/s	$\Rightarrow$ Q_{CD}	=	$1.80*10^{0}$ ft^3/s
$\bar{V}_{BD}$	=	$4.02*10^{0}$ ft/s	$\Rightarrow$ Q_{BD}	=	$1.97*10^{-1}$ ft^3/s

To find the pressure at the various nodes, apply the Extended Bernoulli Equation between the nodes.

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_B=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_A-H_{L,AB}+H_{S,AB}$$

where

$p_A = 120$ psig

$p_B = ?$

$V_A = V_B$ (the velocity just upstream of point B is equal to the velocity just downstream of point A)

$z_A = z_B$

$H_{S,AB} = 0$

$\rho_{H20\ @\ 20\ °C} = 1.94$ slug/ft^3 (Note: 1 lb$_f$ = 1 slug·ft/s^2)

$$H_{L,AB}=f_{AB}\left(\frac{L_{AB}}{D_{AB}}\right)\frac{\bar{V}_{AB}^2}{2g}$$

$$\Rightarrow\ p_B=p_A-\tfrac{1}{2}\rho\bar{V}_{AB}^2 f_{AB}\left(\frac{L_{AB}}{D_{AB}}\right)$$

$\boxed{p_B = 1.08*10^{2}\text{ psig}}$

Using a similar approach from A to C (or from B to C):

$\boxed{p_C = 1.03*10^{2}\text{ psig}}$

Using a similar approach from B to D (or from C to D):

$\boxed{p_D = 7.57*10^{1}\text{ psig}}$

CHAPTER 12

Fluid Machinery

12.1. Introduction

There are two categories of fluid machines.

(1) Those that do work on the fluid:
    (a) pumps (used for liquids),
    (b) fans (used for gases/vapor; $\Delta p <$ a few inches of H2O),
    (c) blowers (used for gases/vapor; a few inches of H2O $< \Delta p < 1\,\text{atm}$),
    (d) compressors (used for gases/vapor; $\Delta p > 1\,\text{atm}$)
(2) Those that extract work from the fluid:
    (a) turbines

Fluid machines that do work on the fluid will be the focus of this chapter. These fluid machines may be further categorized into two types:

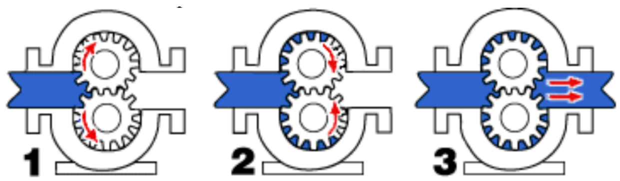
(1) Positive Displacement Pumps (PDPs)
    (a) Fluid movement is generated by using changes in volume.
    (b) Examples include reciprocating piston engines, the heart, gear pumps, rotating screw pumps, and bellows.
    (c) PDPs typically produce a periodic flow rate since the volume changes occur periodically.
    (d) PDPs can produce large Δp (pressure rise), but usually have a small Q (flow rate).
(2) Dynamic Pumps
    (a) Dynamic pumps do not have closed volumes like PDPs. Dynamic pumps move the fluid by changing the fluid's momentum.
    (b) Examples include axial flow and radial flow pumps (aka turbomachines), jet pumps, and electromagnetic pumps.
    (c) The pressure change across dynamic pumps is usually smaller than the pressure change across a PDP, i.e., $\Delta p_{\text{dynamic pumps}}$ typically $< \Delta p_{\text{PDP}}$. However, the flow rate through a dynamic pump is usually larger than the flow rate through a PDP, i.e., $Q_{\text{dynamic pumps}}$ typically $> Q_{\text{PDP}}$.

These notes will only serve as an introduction to pumps and focus mainly on centrifugal pumps, which are one of the most common pump types found in engineering applications. There are many different pump types and numerous books and online resources are available describing their operation.

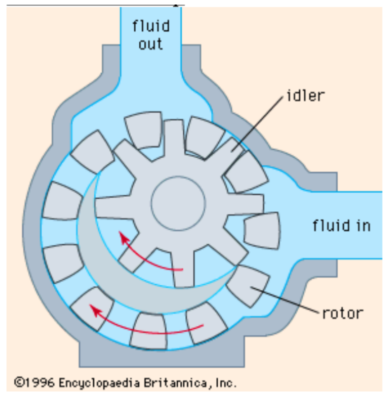
12.2. Some Examples of Positive Displacement Pumps

Positive displacement pumps (PDPs) operate by using changes in a cavity's volume to move fluid downstream. Large pressure changes can be achieved across a PDP, but the flow rates are typically small compared to dynamic pumps and the flow rates are often periodic since the volume changes occur periodically.

A large number of PDP designs have been proposed. The following figures (Figures 12.1 - 12.15) provide just a few examples.



(a) An external gear pump. Image from http://www.pumpschool.com/principles/external.htm.



(b) An internal gear pump. Image from http://media-2.web.britannica.com/eb-media/56/3656-004-5CA4041D.gif.

Figure 12.1. Examples of gear pumps. Gear pumps are often used in automatic transmissions.

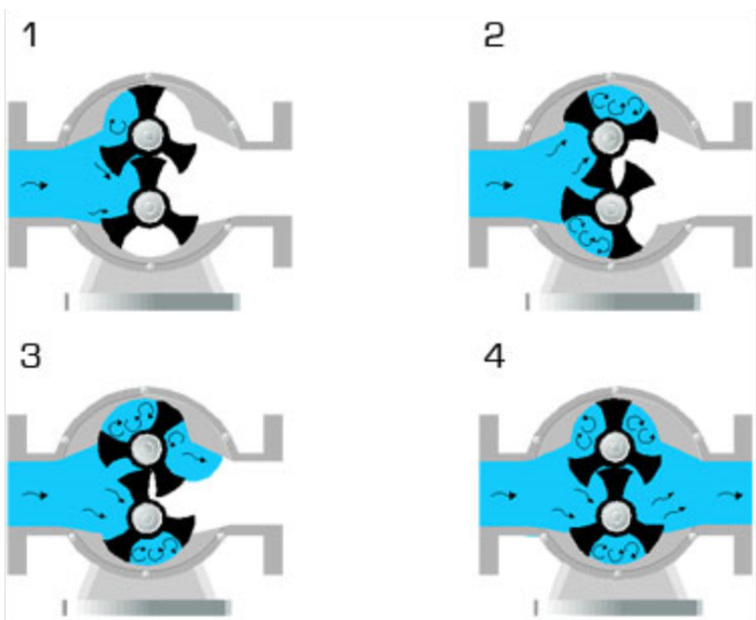


Figure 12.2. An example of a lobe pump. Lobe pumps often have two or three lobes and are often used in in diesel superchargers. Image from http://www.megator.co.uk/lobe_pump.htm.

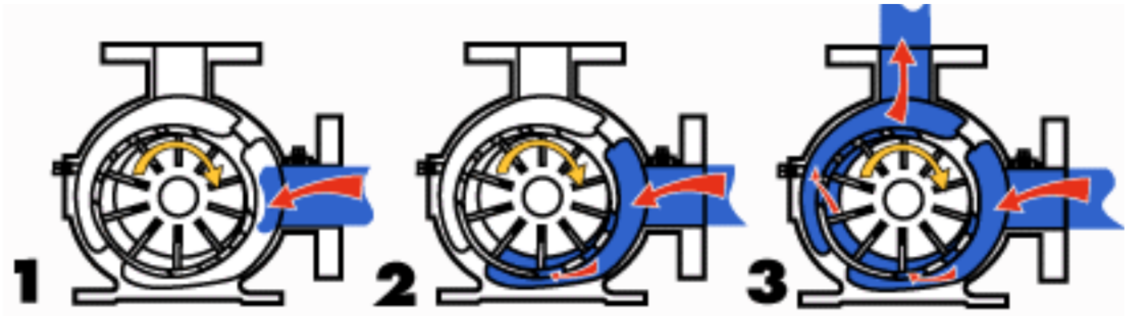


Figure 12.3. An example of a vane pump. Centrifugal force or springs are used to push out the vanes. These are often used as power steering pumps and in automatic transmissions. Image from http://www.pumpschool.com/principles/vane.htm.

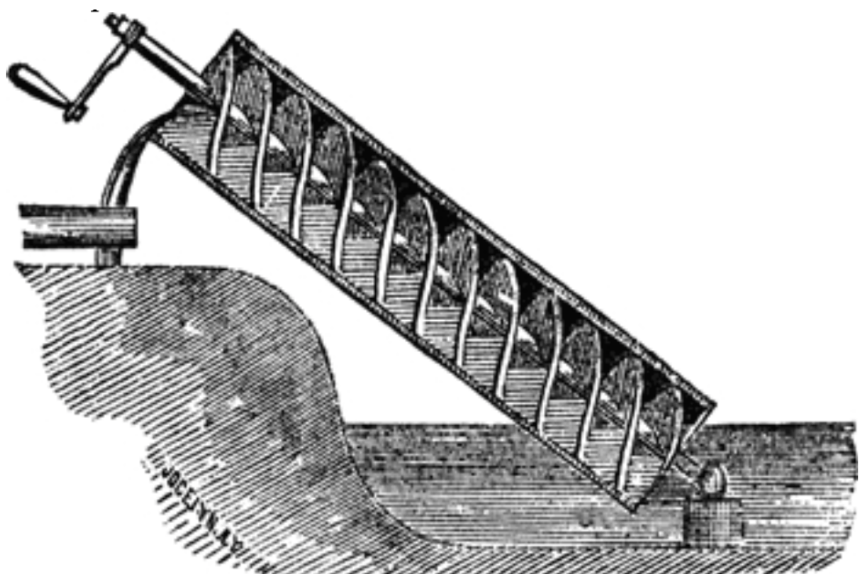
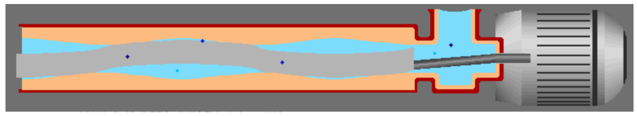
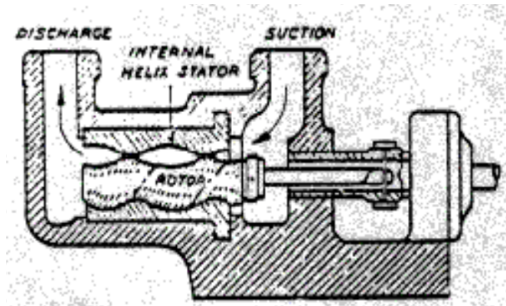


Figure 12.4. An example of a screw pump. Archemiedes screw pumps were first used more than 2000 years ago. They're still in use for irrigation purposes. Image from http://en.wikipedia.org/wiki/File:Archimedes_screw.JPG.



(A) Image from http://www.animatedsoftware.com/pumpglos/progrssv.htm.



(B) Image from http://www.roymech.co.uk/Related/Pumps/Rotary%20Positive%20Displacement.html.

Figure 12.5. Examples of cavity pumps.

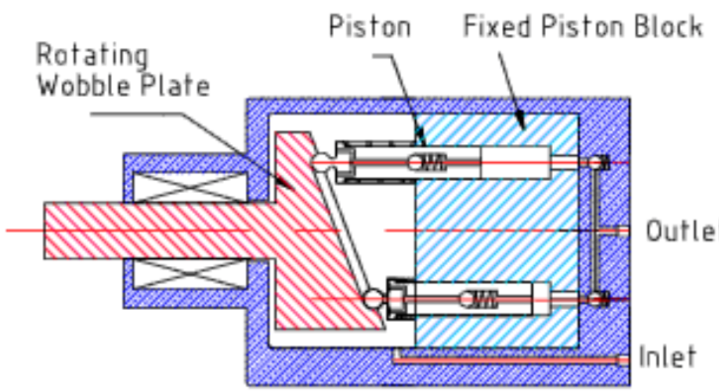


Figure 12.6. An example of a wobble plate piston pump. Image from http://www.roymech.co.uk/Related/Pumps/Rotary%20Positive%20Displacement.html.

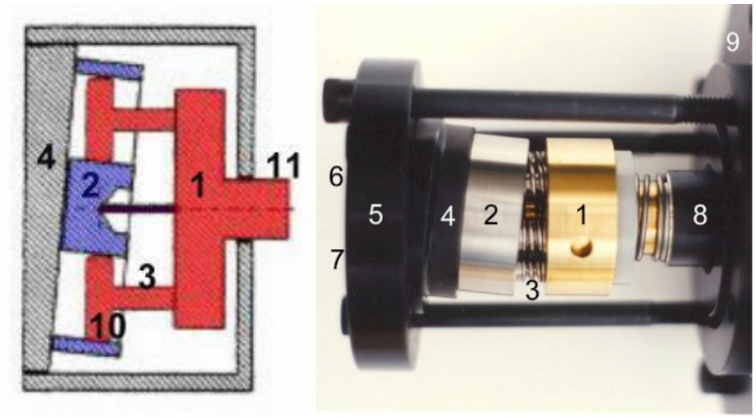


Figure 12.7. An example of a Wolfhart Principle pump. Image from http://www.allstar.fiu.edu/aero/wolfhart_pump_principle.htm.

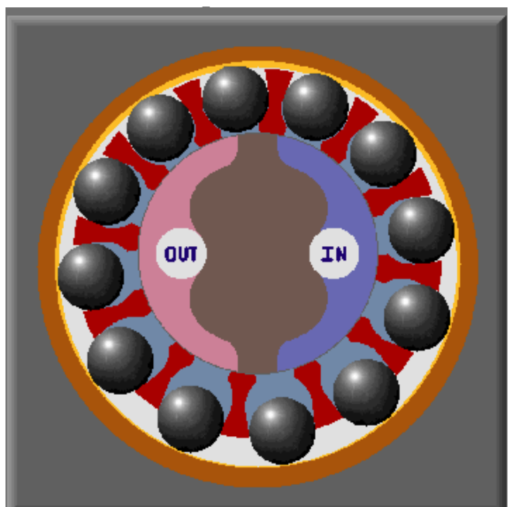


Figure 12.8. An example of a ball piston pump. Image from http://www.animatedsoftware.com/pumpglos/ballpist.htm.

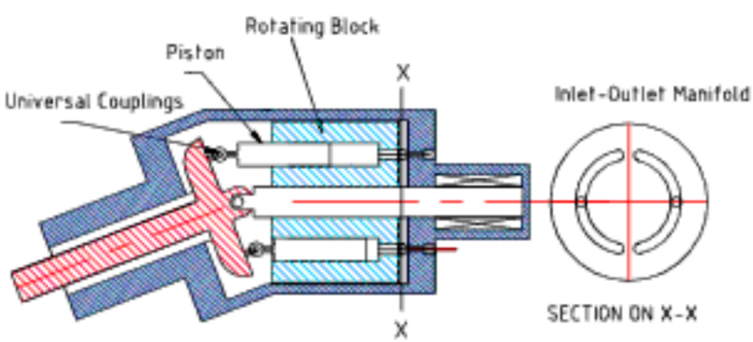


Figure 12.9. An example of a bent axis piston pump. Image from http://www.roymech.co.uk/Related/Pumps/Rotary%20Positive%20Displacement.html.

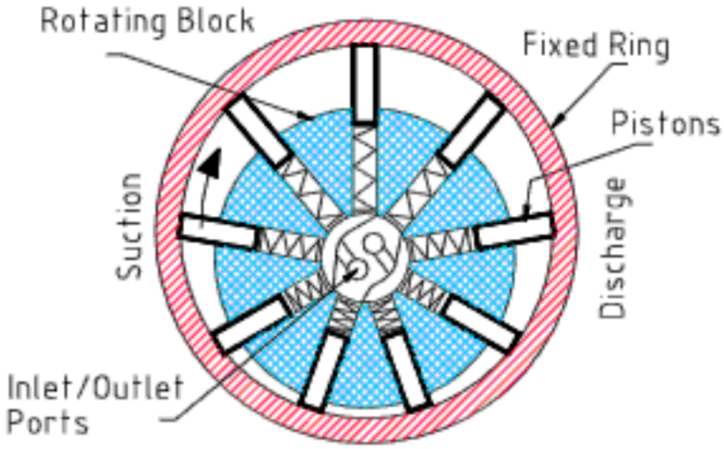


Figure 12.10. An example of a radial piston pump. Image from http://www.roymech.co.uk/Related/Pumps/Rotary%20Positive%20Displacement.html.

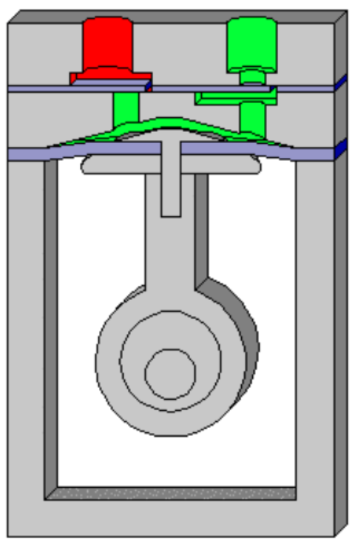


Figure 12.11. An example of a rotary cam pump. Image from http://www.labpump.co.kr/data/aboutpump.htm.

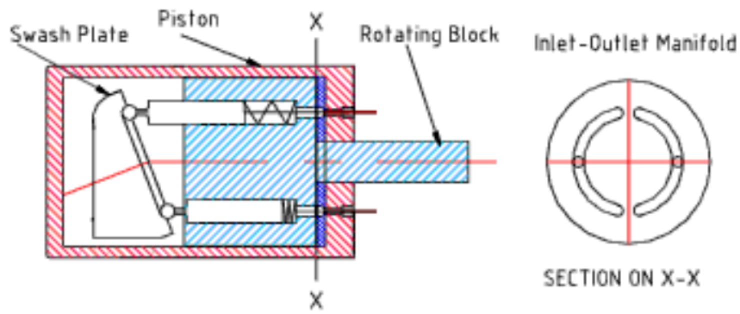


Figure 12.12. An example of a swash plate piston pump. Image from http://www.roymech.co.uk/Related/Pumps/Rotary%20Positive%20Displacement.html.

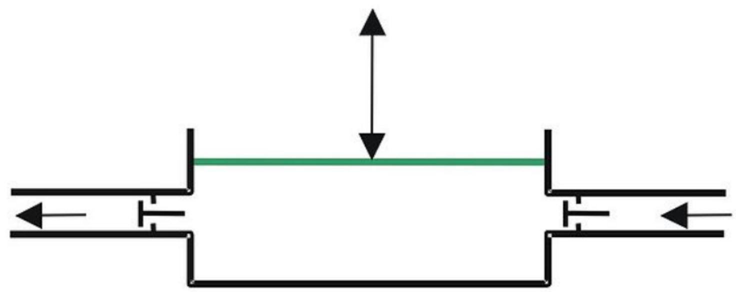


Figure 12.13. An example of a diaphragm pump. Image from http://en.wikipedia.org/wiki/File:Bomba_diafragma.jpg.

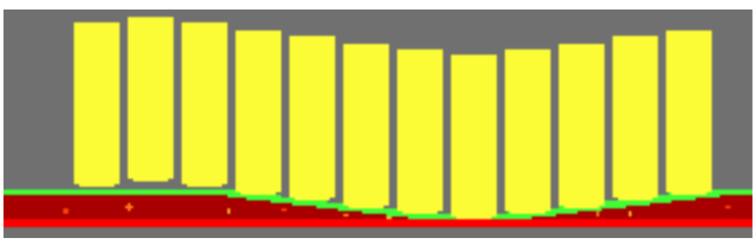
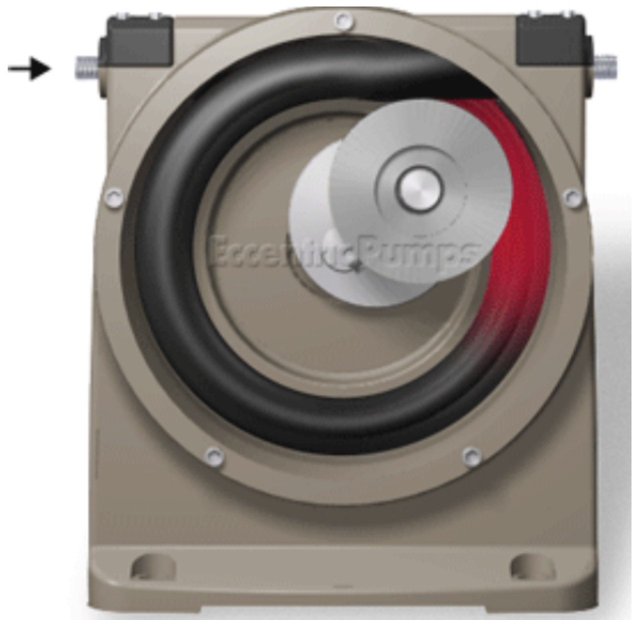
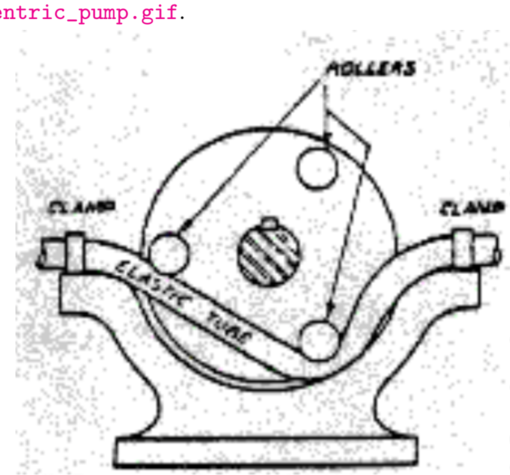


Figure 12.14. An example of a finger pump. Image from http://www.animatedsoftware.com/pumpglos/fingerpu.htm.



(a) Image from http://en.wikipedia.org/wiki/File:Eccentric_pump.gif.



(b) Image from http://www.roymech.co.uk/Related/Pumps/Rotary%20Positive%20Displacement.html.

Figure 12.15. Examples of peristaltic pumps. Peristaltic pumps are used in a wide variety of applications, including fuel pumps.

12.3. Some Examples of Dynamic Pumps

Dynamic pumps operate by using momentum changes to increase the fluid pressure. The pressure changes across dynamic pumps are generally smaller than what can be achieved by PDPs, but the flow rates are typically larger and not periodic.

Like PDPs, there are a large number of dynamic pump designs. The following figures (Figures 12.16 - 12.21) provide just a few examples.

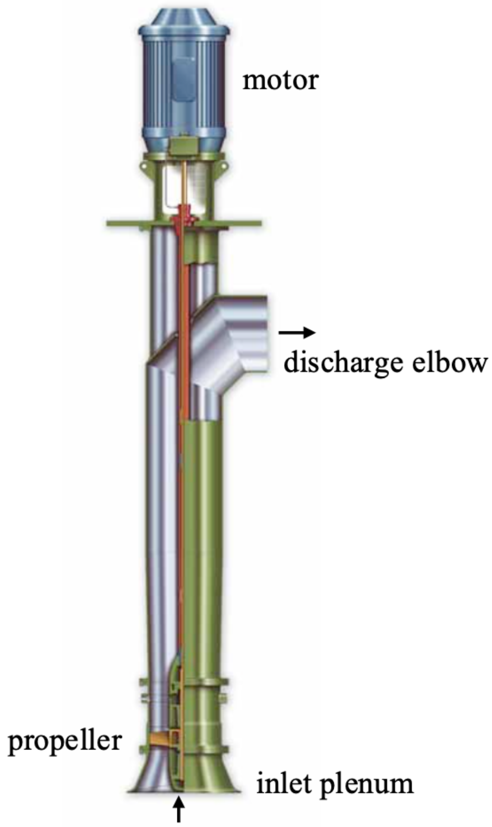
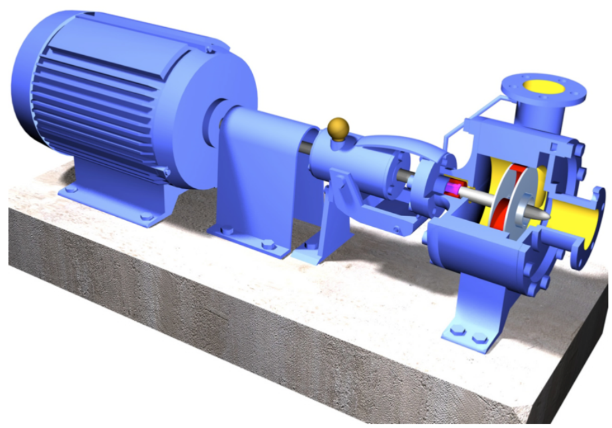
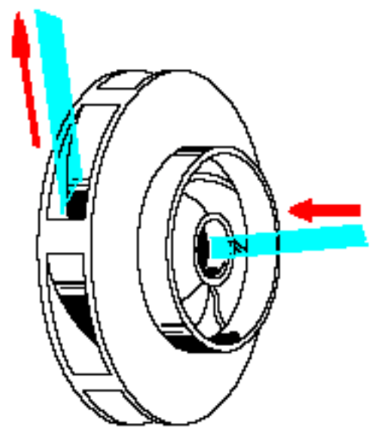


Figure 12.16. An example of a propeller pump. Image from http://www.sulzerpumps.com/Portaldata/9/Resources/brochures/power/vertical/JP_Vertical_E00635.pdf.



(a) Image from http://commons.wikimedia.org/wiki/File:CetriFugal_Pump.jpg.



(b) Image from http://www.motorera.com/dictionary/pics/r/Radial-flow_pump.gif.

Figure 12.17. Examples of radial dynamic pumps.

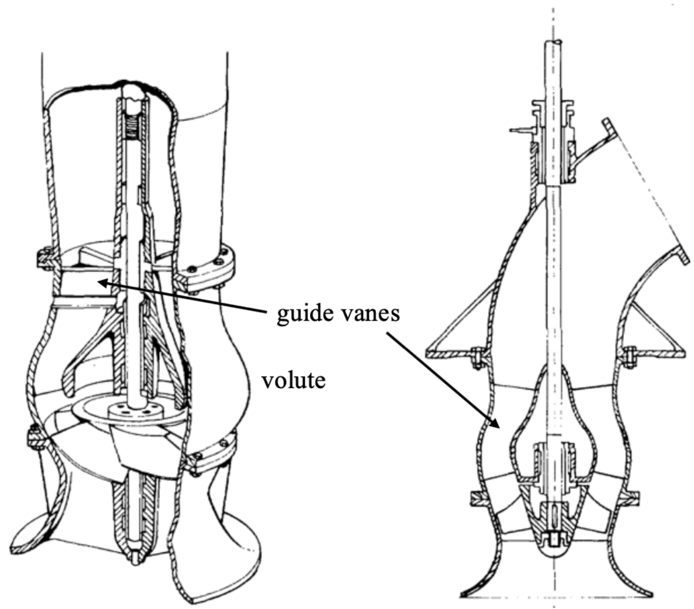


Figure 12.18. An example of a mixed pump. Image from http://www.fao.org/docrep/010/ah810e/AH810E07.htm.

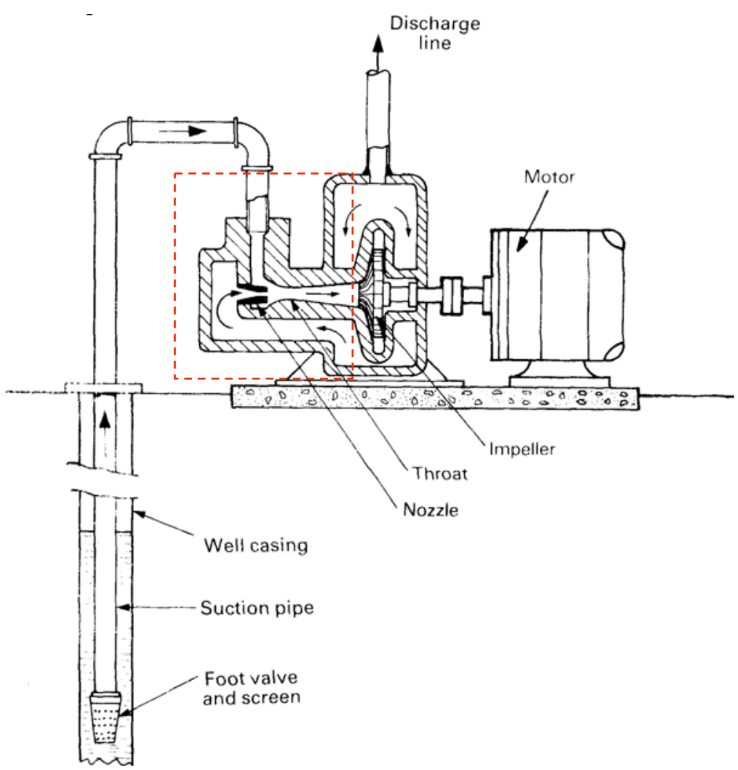


Figure 12.19. An example of a jet pump. Image from http://www.fao.org/docrep/010/ah810e/AH810E07.htm.

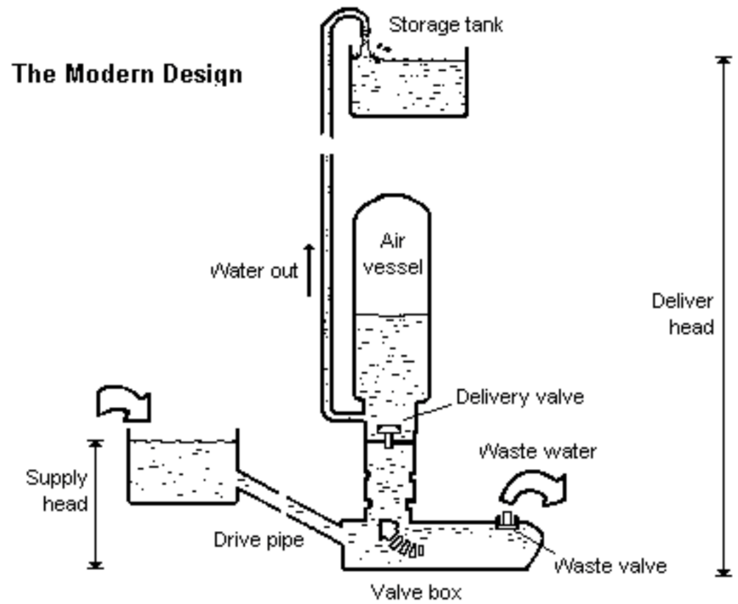


Figure 12.20. An example of a ram pump. Image from http://www.lifewater.ca/ram_pump.htm.

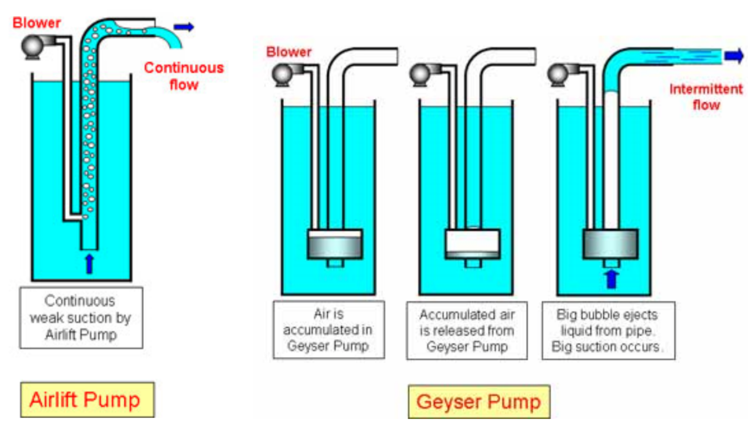


Figure 12.21. Examples of air lift pumps. Image from http://www.airliftpump.com/index.htm.

12.4. Elementary Dynamic Pump Theory for Rotating Pumps

To determine the work that a dynamic, rotating pump does on the fluid passing through it, we'll use the Moment of Momentum Equation, which relates torque to momentum fluxes.

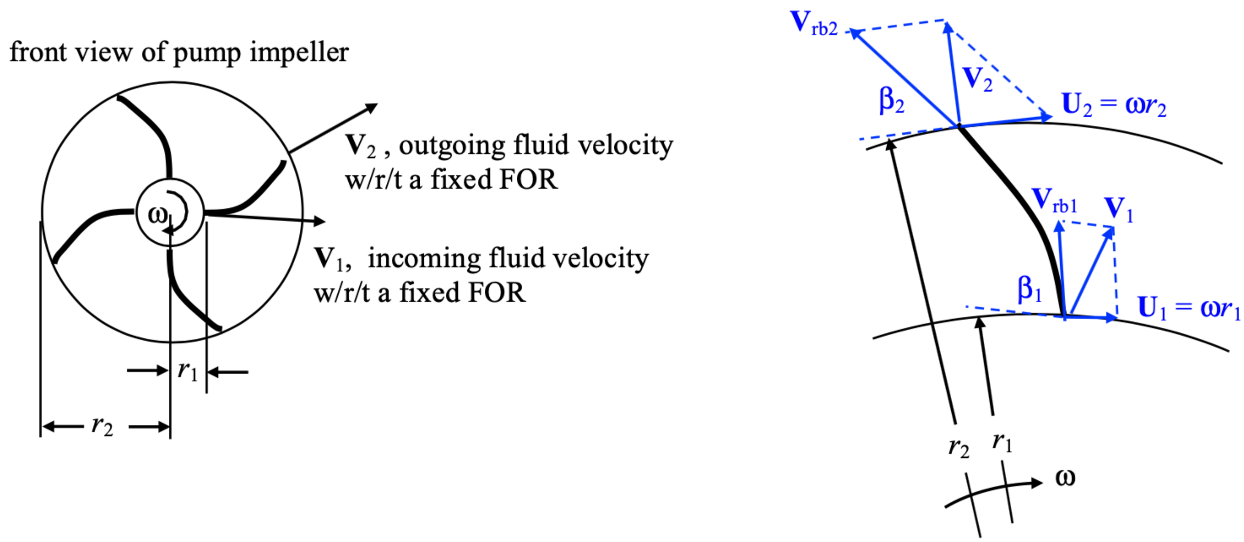


Figure 12.22. A sketch of a pump impeller. The right-hand figure presents velocity vectors and angles for an individual impeller blade.

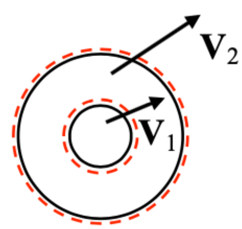


Figure 12.23. A sketch of the flow into and out of the pump impeller. The velocities $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are measured relative to a fixed coordinate system.

Consider flow through the rotating pump impeller shown in Figures 12.22 and 12.23. In these figures, define the following variables,

$$\begin{aligned} \beta_1,\, \beta_2 &:= \text{entrance/exit blade angles w/r/t the hub} \\ \boldsymbol{V}_1, \boldsymbol{V}_2 &:= \text{fluid velocities w/r/t a coordinate system fixed to the ground} \\ \boldsymbol{U}_1, \boldsymbol{U}_2 &:= \text{blade velocities w/r/t a coordinate system fixed to the ground} \\ \boldsymbol{V}_{rb1}, \boldsymbol{V}_{rb2} &:= \text{fluid velocities w/r/t the blade} \end{aligned}$$

From geometry we have,

$$\boldsymbol{V}_1 = \boldsymbol{U}_1 + \boldsymbol{V}_{rb1}, \tag{12.1}$$

$$\boldsymbol{V}_2 = \boldsymbol{U}_2 + \boldsymbol{V}_{rb2}. \tag{12.2}$$

To determine the torque, T, that must be applied to the shaft in order to rotate the impeller at angular velocity, ω, we use the Moment of Momentum Equation,

$$\boldsymbol{M}_{\text{on CV}} = \frac{d}{dt}\int_{CV} (\boldsymbol{r} \times \boldsymbol{u})\rho dV + \int_{CS} (\boldsymbol{r} \times \boldsymbol{u})\,(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A})\,. \tag{12.3}$$

When simplifying this equation, assume the following,

(1) steady-state conditions,
(2) uniform flow between blades,
(3) incompressible flow,
(4) the only moment applied to the CV is the shaft torque, $T_{\text{on CV}}$, which points in the same direction as the shaft rotation, ω,
(5) the fluid velocities are measured with respect to an inertial coordinate system.

Using these assumptions, the Moment-of-Momentum Equation simplifies to,

$$\boxed{T_{\text{on CV}} = \dot{m}\left(r_2 V_{t2} - r_1 V_{t1}\right)} \quad \underline{\text{Euler's Turbomachinery Equation}} \tag{12.4}$$

where V_{t1} and V_{t2} are the fluid velocities tangent to the impeller hub measured with respect to a fixed coordinate system.

Notes:

(1) The power required to drive the impeller is,

$$\dot{W}_{\text{on CV}} = \boldsymbol{T}_{\text{on CV}} \cdot \boldsymbol{\omega}, \tag{12.5}$$

$$\dot{W}_{\text{on CV}} = \dot{m}\omega\left(r_2 V_{t2} - r_1 V_{t1}\right), \tag{12.6}$$

but $U_1 = r_1\omega$ and $U_2 = r_2\omega$ so that,

$$\dot{W}_{\text{on CV}} = \dot{m}\left(U_2 V_{t2} - U_1 V_{t1}\right). \tag{12.7}$$

In terms of the shaft head added to the fluid,

$$\boxed{H_{\text{added to CV}} = \frac{\dot{W}}{\dot{m}g} = \frac{\left(U_2 V_{t2} - U_1 V_{t1}\right)}{g}}. \tag{12.8}$$

(2) Only the absolute fluid velocity tangential to the impeller contributes to the increase in fluid head.
(3) For an <u>ideal centrifugal pump</u>, the incoming flow has no tangential component $\implies V_{t1} = 0$,

$$H_{\text{added}} = \frac{U_2 V_{t2}}{g}. \tag{12.9}$$

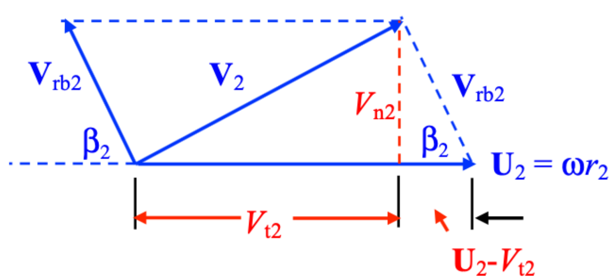


Figure 12.24. The fluid vector geometry at the exit of the blade. This figure is a zoomed-in version of the blade outlet tip shown in Figure 12.22.

(4) Figure 12.24 shows the fluid velocity vector geometry at the exit of the blade. From the geometry,

$$\tan\beta_2 = \frac{v_{n1}}{U_2 - V_{t2}}, \tag{12.10}$$

$$V_{t2} = U_2 - V_{n2}\cot\beta_2. \tag{12.11}$$

Substituting into the shaft head relation for an ideal centrifugal pump (Eq. (12.9)),

$$H_{\text{added}} = \frac{U_2\left(U_2 - V_{n2}\cot\beta_2\right)}{g} = \frac{U_2^2}{g} - \frac{U_2 V_{n2}\cot\beta_2}{g}. \tag{12.12}$$

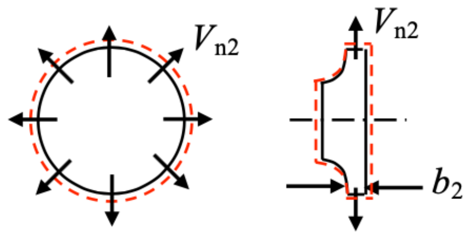


Figure 12.25. The control volume surrounding the pump impeller used when applying Conservation of Mass.

Note that the volumetric flow rate through the pump is related to the radial fluid velocity from Conservation of Mass (Figure 12.25),

$$Q = V_{n2}\left(\underbrace{2\pi r_2}_{\substack{\text{impeller}\\\text{circumference}}}\ \underbrace{b_2}_{\substack{\text{impeller}\\\text{thickness}}}\right), \tag{12.13}$$

Substituting and noting that $U_2 = r_2\omega$ gives,

$$H_{\text{added}} = \frac{(r_2\omega)^2}{g} - \frac{r_2\omega Q\cot\beta_2}{g2\pi r_2 b_2}, \tag{12.14}$$

$$\boxed{H_{\text{added}} = \frac{(r_2\omega)^2}{g} - \left(\frac{\omega\cot\beta_2}{g2\pi b_2}\right)Q}. \tag{12.15}$$

This is the theoretical head rise across an idealized centrifugal pump.

Notes:

(a) Equation (12.15) is an equation of a line (Figure 12.26).

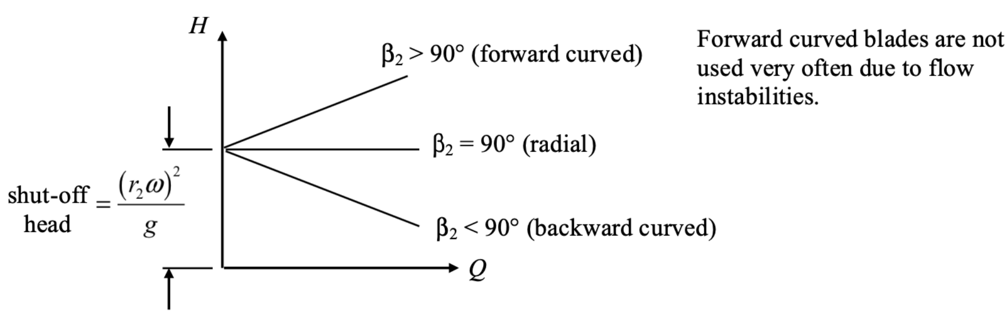


Figure 12.26. Theoretical head rise across a pump plotted as a function of volumetric flow rate for an idealized centrifugal pump.

(b) In an actual flow, losses occur within the pump due to viscous gradients created by the blades, which scale as Q^2, flow separation, impeller blade-shroud clearance flows, and other 3D flow effects (Figure 12.27). A quadratic curve is often used to fit experimental pump head curves,

$$H = H_0 + AQ^2. \tag{12.16}$$

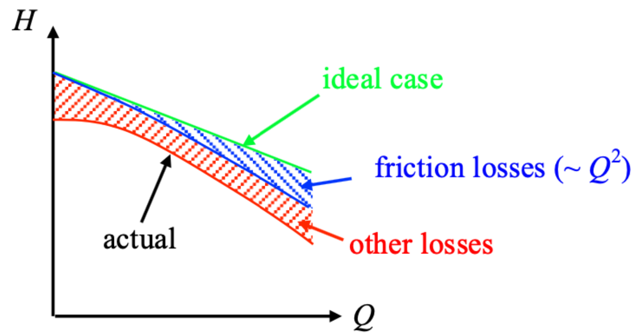


Figure 12.27. Effects of losses on the theoretical head rise across an idealized centrifugal pump.

(5) Pump efficiency is defined as,

$$\eta_P \coloneqq \frac{\dot{m}gH}{\omega T}, \tag{12.17}$$

where $\dot{m}gH$ is the known as the water or hydraulic horsepower, i.e., the power that makes it into the fluid, and ωT is known as the brake horsepower, i.e., the power put into the pump.

(a) Typical pump efficiencies are between $\eta_P = 60\% - 85\%$. If you do not know know the efficiency, a value of $\eta_P = 70\%$ is a reasonable estimate.

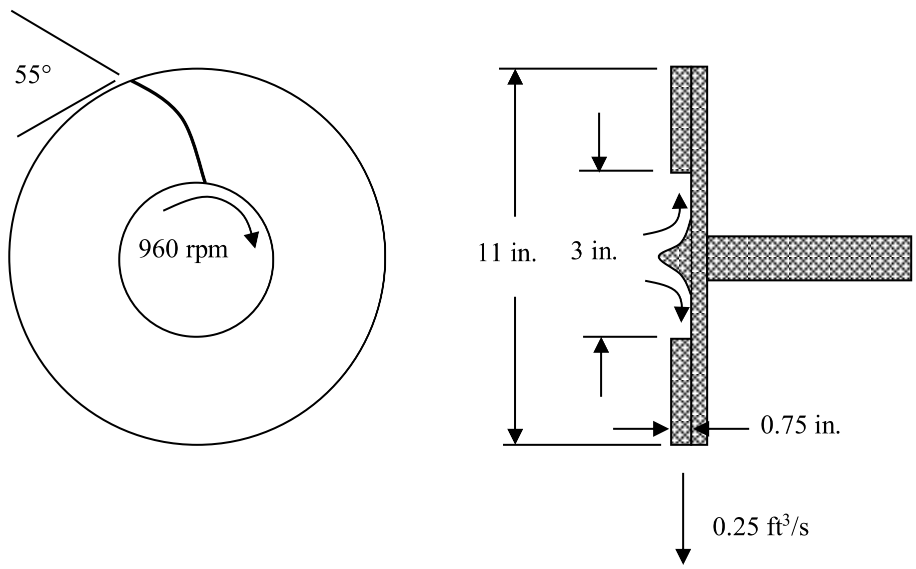
(b) As pump size decreases, the ratio of surface area to volume increases $\implies$ frictional losses increase and the pump efficiency decreases.

(6) The head rise, brake horsepower, and efficiency for a pump are provided in a pump performance plot, as shown in Figure 12.28. The top figure shows the Best Efficiency Point (BEP) for the pump, which is the flow rate at which the efficiency is a maximum. Ideally, the pump would operate at this flow rate since it has the highest efficiency of converting the shaft work into an increase in fluid pressure, but the pump can operate at other flow rates.

The top figure also shows the shut-off head (aka dead-head), which is the head rise across the pump when there is no flow through the pump. One would have this situation if a valve placed downstream of the pump was closed, but the pump was still operating. The pump would continue to do work on the fluid, but that work would go mainly into increasing the fluid pressure (and the temperature) rather than increasing kinetic energy.

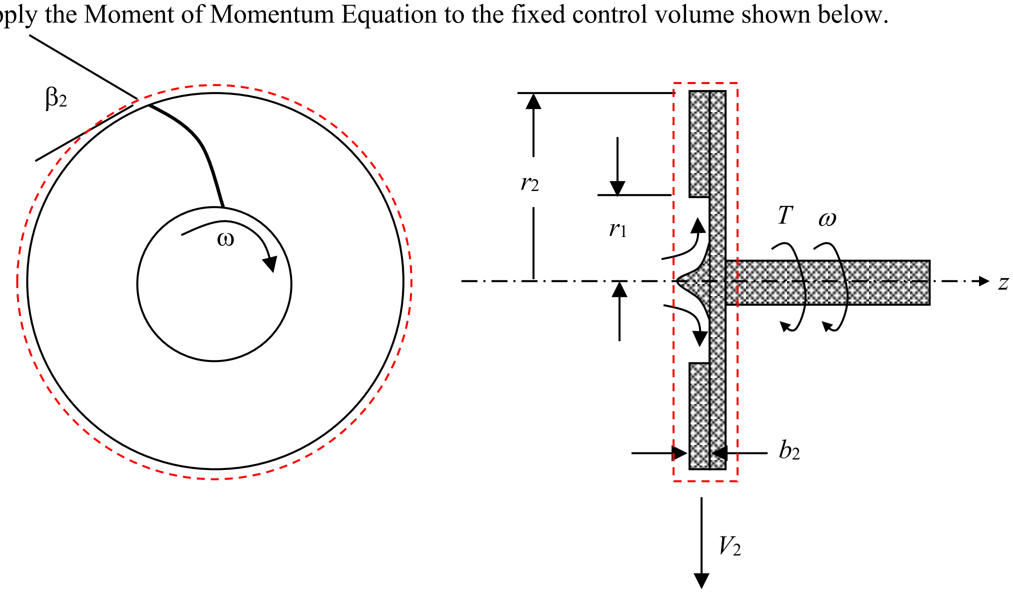
The bottom chart in Figure 12.28 shows a different, and more common, pump performance plot style. Here, there are three different head curves: one for three different impeller diameters. In practice, it's common to make available several pump impellers that can fit within the same pump housing to expand the range of operation of a pump. There are three different head curves for this particular pump housing corresponding to the three different impellers. The solid lines with numbers along the top are the efficiency curves and the dashed lines with numbers along the bottom right are the brake horsepower curves. The dashed line at the very bottom with the label "NPSHR" is discussed in the next section.

An idealized centrifugal water pump is shown below. The volumetric flow rate through the pump is 0.25 ft^3/s and the angular speed of the impeller is 960 rpm. Calculate the power required to drive the pump.



SOLUTION:

Apply the Moment of Momentum Equation to the fixed control volume shown below.



$$\frac{d}{dt}\int_{CV}(\boldsymbol{r}\times\boldsymbol{u})\rho dV + \int_{CS}(\boldsymbol{r}\times\boldsymbol{u})(\rho\boldsymbol{u}_{rel}\cdot d\boldsymbol{A}) = \boldsymbol{M}_{on\ CV}, \quad (1)$$

where,

$$\frac{d}{dt}\int_{CV}(\boldsymbol{r}\times\boldsymbol{u})\rho dV = \boldsymbol{0} \quad \text{(steady-state)}, \quad (2)$$

$$\int_{CS}(\boldsymbol{r}\times\boldsymbol{u})(\rho\boldsymbol{u}_{rel}\cdot d\boldsymbol{A}) = \left\{r_2\hat{\boldsymbol{e}}_r \times \underbrace{[V_2(\sin\beta_2\,\hat{\boldsymbol{e}}_r - \cos\beta_2\,\hat{\boldsymbol{e}}_\theta) + \omega r_2\hat{\boldsymbol{e}}_\theta]}_{\text{exiting fluid velocity relative to ground}}\right\}\rho\,\underbrace{(V_2\sin\beta_2)}_{\substack{\text{fluid velocity}\\ \text{exiting the CV}}}\,\underbrace{(2\pi r_2 b_2)}_{\text{exit area}} \quad (3)$$

$$= (\omega r_2^2 - r_2V_2\cos\beta_2)\rho V_2\sin\beta_2\,(2\pi r_2 b_2)\hat{\boldsymbol{e}}_z \quad (4)$$

(Note that at the inlet $\boldsymbol{r}_1$ and $\boldsymbol{V}_1$ are parallel so the cross-product is zero.)

$$\boldsymbol{M}_{on\ CV} = T\hat{\boldsymbol{e}}_z. \quad (5)$$

Substitute and simplify.

$$T = (\omega r_2^2 - r_2V_2\cos\beta_2)\rho V_2\sin\beta_2\,(2\pi r_2 b_2). \quad (6)$$

The velocity V_2 can be related to the volumetric flow rate, Q, by considering the flow at the impeller exit,

$$Q = V_2\sin\beta_2\,(2\pi r_2 b_2), \quad (7)$$

$$V_2 = \frac{Q}{\sin\beta_2(2\pi r_2 b_2)}. \quad (8)$$

Substitute Eq. (8) into Eq. (6) and simplify,

$$T = \left(\omega r_2^2 - r_2\frac{Q}{\sin\beta_2(2\pi r_2 b_2)}\cos\beta_2\right)\rho\frac{Q}{\sin\beta_2(2\pi r_2 b_2)}\sin\beta_2\,(2\pi r_2 b_2), \quad (9)$$

$$\boxed{T = \left(\omega r_2^2 - \frac{Q}{2\pi b_2\tan\beta_2}\right)\frac{\rho g Q}{g}}. \quad (10)$$

Using the given data:

$(\rho g)_{H20}$ = 62.4 lb_f/ft^3
g = 32.2 ft/s^2
Q = 0.25 ft^3/s
b_2 = 0.75 in = $6.25*10^{-2}$ ft
ω = 960 rpm = 100.5 rad/s
β_2 = 55°
r_2 = 5.5 in = $4.58*10^{-1}$ ft
$\Rightarrow$ $\boxed{T = 10.0\ \text{ft}\cdot\text{lb}_f}$

The power required to drive the impeller is,

$$\dot{W} = T\omega. \quad (11)$$

Using the given data:

$\boxed{\dot{W} = 1005\ \text{ft}\cdot\text{lb}_f/\text{s} = 1.83\ \text{hp}}$

The problem could have also been worked out using velocity polygons. Since the absolute inlet velocity has no tangential component,

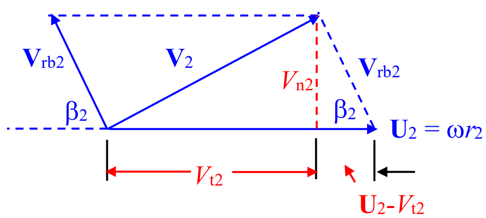
$$H = \frac{U_2 V_{t2}}{g} \quad \text{where } U_2 = \omega r_2 \tag{12}$$

and,

$$\dot{W} = \dot{m} g H \tag{13}$$

$$\Rightarrow \dot{W} = \frac{\dot{m} g \omega r_2 V_{t2}}{g}. \tag{14}$$

Use a velocity polygon at the exit to determine V_{t2}.



From the geometry:

$$V_{t2} = U_2 - V_{n2} \cot \beta_2, \tag{15}$$

where V_{n2} is found from Conservation of Mass,

$$V_{n2} = \frac{Q}{2\pi r_2 b_2}. \tag{16}$$

Using the given data:

U_2 = 46.3 ft/s
V_{n2} = 1.39 ft/s
H = 65.1 ft
$\dot{W}$ = 1005 ft·lb$_f$/s = 1.83 hp (Same answer as before!)

Data measured during tests of a centrifugal pump at 3500 rpm are given in the table below:

Parameter	Inlet Section	Outlet Section
gage pressure, p [kPa]	95.2	412
elevation above datum, z [m]	1.25	2.75
avg speed of flow, V [m/s]	2.35	3.62

The working fluid is water. The flow rate is 11.5 m^3/hr and the torque applied to the pump shaft is 3.68 N·m. Evaluate the head rise across the pump, the hydraulic power input to the fluid, and the pump efficiency. If the electric motor efficiency is 85%, calculate the electric power requirement.

SOLUTION:

First determine the total heads at the inlet and outlet to the pump. The total head is given by,

$$H = \frac{p}{\rho g} + \alpha \frac{\bar{V}^2}{2g} + z \,. \tag{1}$$

Using the given data (and noting that $D = [Q/(^{\pi}/_{4}V)]^{1/2}$ and $\mathrm{Re} = VD/\nu$ so that $\mathrm{Re}_{\mathrm{inlet}} = 9.78\mathrm{e}4$ and $\mathrm{Re}_{\mathrm{outlet}} = 1.21\mathrm{e}5 \Rightarrow \alpha_{\mathrm{inlet}} \approx \alpha_{\mathrm{outlet}} \approx 1$) and using absolute pressures when calculating the head:

H_{inlet} = 21.6 m

H_{outlet} = 55.7 m

$\boxed{\Delta H = 34.1 \text{ m}}$

The hydraulic power input to the fluid is given by,

$$\dot{W}_{\mathrm{fluid}} = \dot{m} g \left(H_{\mathrm{outlet}} - H_{\mathrm{inlet}} \right), \tag{2}$$

$$\boxed{\dot{W}_{\mathrm{fluid}} = 1.07 \text{ kW}}$$

The power required to drive the pump is,

$$\dot{W}_{\mathrm{shaft}} = T\omega \,, \tag{3}$$

$$\dot{W}_{\mathrm{shaft}} = 1.35 \text{ kW}$$

The efficiency of the pump is given by,

$$\eta_{\mathrm{pump}} = \frac{\dot{W}_{\mathrm{fluid}}}{\dot{W}_{\mathrm{shaft}}} \,, \tag{4}$$

$$\boxed{\eta_{\mathrm{pump}} = 79.4\%}$$

The electric power required is,

$$\dot{W}_{\substack{\text{required}\\ \text{for motor}}} = {}^{\dot{W}_{\mathrm{shaft}}}\!\big/_{\eta_{\mathrm{motor}}} \,, \tag{5}$$

$$\boxed{\dot{W}_{\substack{\text{required}\\ \text{for motor}}} = 1.59 \text{ kW}}$$

Video solution: https://www.youtube.com/watch?v=yh5RIZT6t1A

Data measured during tests of a centrifugal pump at 3500 rpm are given in the table below.

Parameter	Inlet Section	Outlet Section
gage pressure, p [kPa]	85.2	412
elevation above datum, z [m]	1.25	2.75
avg speed of flow, V [m/s]	2.35	3.62

The flow rate is 11.5 m^3/hr and the torque applied to the pump shaft is 3.68 N·m. Evaluate the total heads at the pump inlet and outlet, the hydraulic power input to the fluid, and the pump efficiency. Specify the electric motor size needed to drive the pump. If the electric motor efficiency is 85%, calculate the electric power requirement.

SOLUTION:
At the outlet,

$$H_{\text{outlet}} = \left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_{\text{outlet}}, \quad (1)$$

where,

$p_{\text{outlet}} = 85.2$ kPa (gage),
$\rho = 1000$ kg/m^3,
$g = 9.81$ m/s^2,
$\alpha_{\text{outlet}} \approx 1$ (assuming turbulent flow),
$\bar{V}_{\text{outlet}} = 2.35$ m/s,
$z_{\text{outlet}} = 1.25$ m,
=> $\boxed{H_{\text{outlet}} = 10.2 \text{ m}}$.

At the inlet,

$$H_{\text{inlet}} = \left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_{\text{inlet}}, \quad (2)$$

where,

$p_{\text{inlet}} = 412$ kPa (gage),
$\rho = 1000$ kg/m^3,
$g = 9.81$ m/s^2,
$\alpha_{\text{inlet}} \approx 1$ (assuming turbulent flow),
$\bar{V}_{\text{inlet}} = 3.62$ m/s,
$z_{\text{inlet}} = 2.75$ m,
=> $\boxed{H_{\text{inlet}} = 45.4 \text{ m}}$.

The rate at which energy is put into the fluid is,

$$\dot{W}_{\text{on fluid}} = \rho g Q H, \quad (3)$$

where,

$Q = 11.5$ m^3/hr $= 3.19 \times 10^{-3}$ m^3/s,
$H = H_{\text{outlet}} - H_{\text{inlet}} = 35.2$ m (head rise across the pump),
=> $\boxed{\dot{W}_{\text{on fluid}} = 1100 \text{ W}}$.

The rate at which energy is put into the pump is,

$$\dot{W}_{\text{on pump}} = \omega T, \quad (4)$$

where,

$\omega = 3500$ rpm $= 366.5$ rad/s,
$T = 3.68$ N.m,
=> $\dot{W}_{\text{on pump}} = 1349$ W.

The pump efficiency is,

$$\eta_{\text{pump}} = \frac{\dot{W}_{\text{on fluid}}}{\dot{W}_{\text{on pump}}}, \tag{5}$$

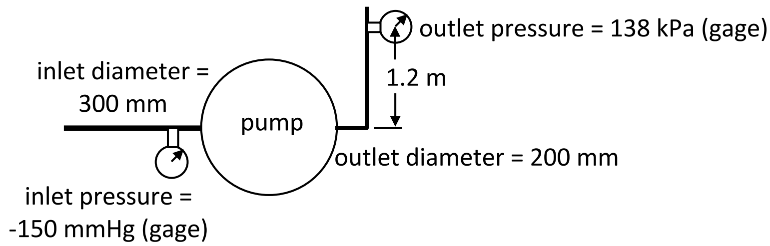
Using the previously calculated values, $\boxed{\eta_{\text{pump}} = 0.82 = 82\%.}$

The electric motor size required for operation is $\boxed{1350 \text{ W} = 1.8 \text{ hp}}$.

If the motor is 85% efficient, then we need to supply the motor with,

$$\dot{W}_{\text{on motor}} = \frac{\dot{W}_{\text{on pump}}}{\eta_{\text{motor}}} \;\Rightarrow\; \boxed{\dot{W}_{\text{on motor}} = 1590 \text{ W} = 2.1 \text{ hp}}. \tag{6}$$

Brine, with a specific gravity of 1.2, passes through an 85% efficient pump at a flow rate of 125 L/s. The centerlines of the pump's 300 mm diameter inlet and 200 mm diameter outlet are at the same elevation. The inlet suction gage pressure is 150 mm of mercury (specific gravity of 13.6) below atmospheric pressure. The discharge pressure is measured 1.2 m above the centerline of the pump's outlet and indicates 138 kPa (gage). Neglecting losses in the pipes, what is the input power to the pump?



SOLUTION:

The power into the pump may be found from the head rise across the pump, the flow rate through the pump, the brine properties, and the pump efficiency,

$$\eta = \frac{\dot{W}_{\text{into fluid}}}{\dot{W}_{\text{into pump}}} \Rightarrow \dot{W}_{\text{into pump}} = \frac{\dot{W}_{\text{into fluid}}}{\eta}, \tag{1}$$

where,

$$\dot{W}_{\text{into fluid}} = \rho Q g H, \tag{2}$$

and,

$$H = \left(\frac{p_{\text{outlet}} - p_{\text{inlet}}}{\rho g}\right) + \left(\frac{V_{\text{outlet}}^2 - V_{\text{inlet}}^2}{2g}\right) + \left(z_{\text{outlet}} - z_{\text{inlet}}\right). \tag{3}$$

Using the given data:

η = 85%

ρ = (1.2)(1000 kg/m^3) = 1200 kg/m^3 (4)

g = 9.81 m/s^2

$z_{\text{outlet}} - z_{\text{inlet}}$ = 1.2 m

p_{outlet} = 138*10^3 Pa (gage)

p_{inlet} = $\rho_{\text{Hg}} g h$ = -(13.6)(1000 kg/m^3)(9.81 m/s^2)(0.150 m) = -20000 Pa (gage) (5)

Q = 125 L/s = 0.125 m^3/s

D_{outlet} = 0.200 m

D_{inlet} = 0.300 m

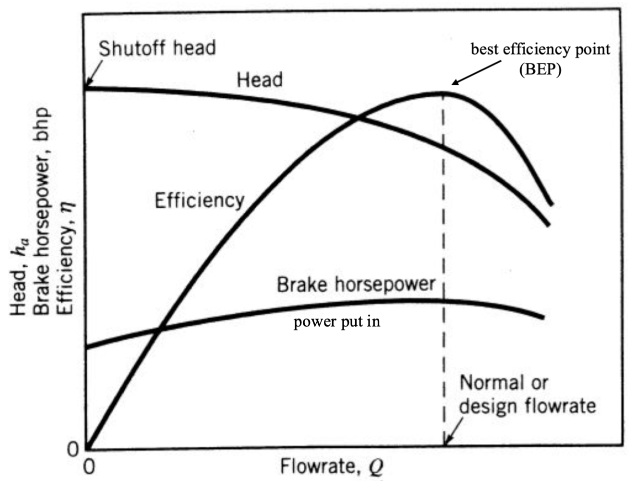
V_{outlet} = $Q\Big/\left(\frac{\pi}{4}D_{\text{outlet}}^2\right)$ = 3.98 m/s (6)

V_{outlet} = $Q\Big/\left(\frac{\pi}{4}D_{\text{inlet}}^2\right)$ = 1.77 m/s (7)

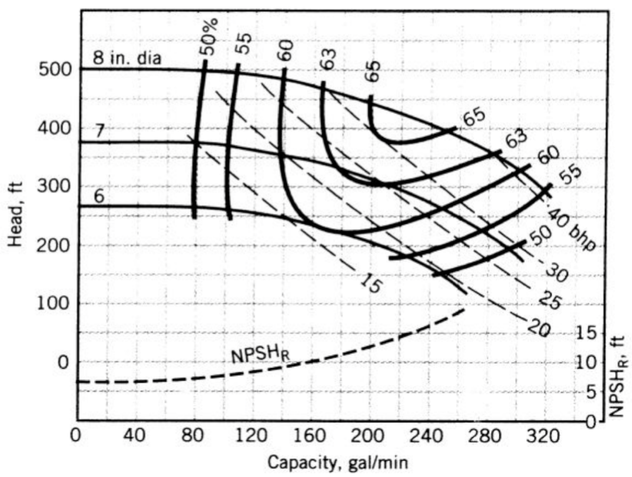
$\Rightarrow H$ = 15.3 m

$\Rightarrow \dot{W}_{\text{into fluid}} = 22.5\text{ kW}$

$\therefore \boxed{\dot{W}_{\text{into pump}} = 26.4\text{ kW}}$



(a) This figure is from Munson, B.R., Young, D.F., and Okiishi, T.H., *Fundamentals of Fluid Mechanics*, 3rd ed., Wiley.



■ **FIGURE 12.12 Performance curves for a two-stage centrifugal pump operating at 3500 rpm. Data given for three different impeller diameters.**

(b) This figure is from Munson, B.R., Young, D.F., and Okiishi, T.H., *Fundamentals of Fluid Mechanics*, 3rd ed., Wiley.

Figure 12.28. Two examples of pump performance plots.

12.5. Net Positive Suction Head (NPSH)

Along the suction side of the impeller blade near the pump inlet are regions of low pressure (Figure 12.29).

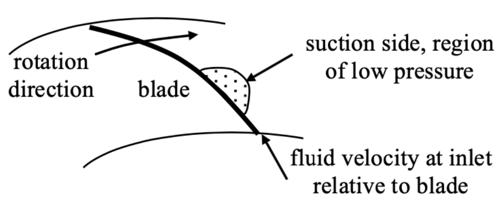


Figure 12.29. A sketch showing the region of low pressure on the suction side of an impeller blade.

If the local pressure is less than the vapor pressure of the liquid, then cavitation will occur: $p \leq p_v \implies$ cavitation. Recall that cavitation is “boiling” (liquid turning to vapor) that occurs when the pressure is less than the liquid’s vapor pressure. Cavitation can not only decrease the performance of a pump, but it can also cause pump damage, vibration, and noise. Vapor bubbles caused by cavitation move into regions of higher pressure, collapse violently, and produce localized regions of very high pressure or high speed jets of water that can chip away at surfaces. The result is that the pump material erodes away. One can often hear when cavitation occurs because of the noise generated by the collapsing vapor bubbles.

Let’s define a quantity that will aid us in determining when cavitation in a pump will occur,

$$\underline{\text{Net Positive Suction Head}}, \text{NPSH} := \left(\frac{p}{\rho g} + \frac{\bar{V}^2}{2g}\right)_S - \frac{p_v}{\rho g}. \tag{12.18}$$

Notes:

(1) The first term in NPSH is the head at the suction side of the pump near the impeller inlet. This region is where we expect to have the smallest head. Note that we define our reference plane for elevation along the centerline of the pump: $z = 0$.

(2) The second term in NPSH is the vapor pressure head. This head is when the liquid turns to vapor. The vapor pressure is typically given in terms of an absolute pressure so the suction pressure should also be an absolute pressure. Note that vapor pressure increases as temperature increases.

(3) NPSHR := Net Positive Suction Head Required to avoid cavitation. This quantity is a *pump property* and is determined experimentally.

(4) NPSHA := Net Positive Suction Head Available to the pump. This quantity is a *system property* and can be determined via analysis or experiments. NPSHA is related to the total head available to the pump at the pump inlet (minus the vapor head).

(5) We must have NPSHA > NPSHR to avoid cavitation. Regulations typically recommend at least a 10% margin for safety. For critical applications such as for power generation or flood control, a 100% margin is often used.

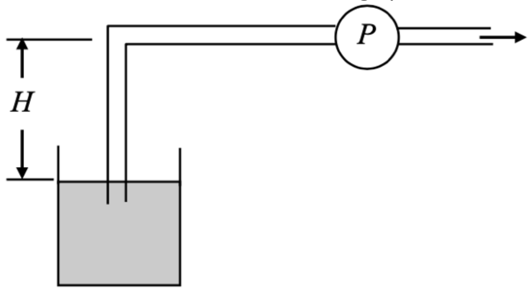
(6) NPSHR increases with increasing flow rate since the pressure at the suction side of the pump blade near the pump inlet will decrease (consider Bernoulli’s equation). Similarly, the pressure will decrease with increasing blade rotation speed resulting in an increased NPSHR.

(7) The vapor pressure may be estimated using the Antoine Equation,

$$\log_{10} p_v = A - \frac{B}{C + T}, \tag{12.19}$$

where p_v is the vapor pressure in bar (abs), A, B, and C are constants for the particular liquid of interest, and T is the fluid’s absolute temperature in Kelvin. For example, for pure water in the temperature range 379 - 573 K, $A = 3.55959$, $B = 643.748$, and $C = -198.043$. Constants for other temperature ranges and other liquids are available from the National Institute of Standards and Technology (NIST) Chemistry WebBook.

Determine the NPSHA for the following system.



SOLUTION:
Choose point 1 to be on the surface of the tank and point 2 to be just upstream of the pump. Apply the EBE from 1 to 2,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12}, \tag{1}$$

where,

$p_2 = p_s$,
$p_1 = p_{\text{atm}}$,
$\bar{V}_2=\bar{V}_s$ (assume turbulent flow => $\alpha_1 \approx 1$),
$\bar{V}_2=0$,
$z_2=z_1+H$,
$H_{L,12}=H_{L,12}$,
$H_{S,12}=0$.

Substitute and simplify,

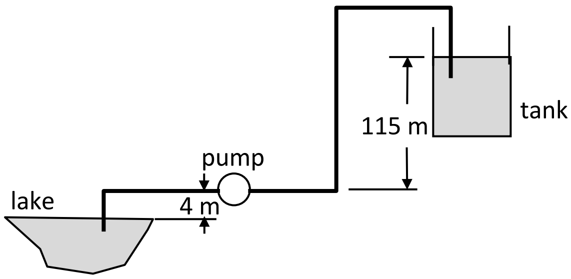
$$\left(\frac{p}{\rho g}+\frac{\bar{V}^2}{2g}\right)_s=\frac{p_{atm}}{\rho g}+\underbrace{(z_1-z_2)}_{=-H}-H_{L,12}. \tag{2}$$

From the definition of NPSH we have,

$$\boxed{NPSHA=\left(\frac{p}{\rho g}+\frac{\bar{V}^2}{2g}\right)_s-\frac{p_v}{\rho g}=\frac{p_{atm}-p_v}{\rho g}-H-H_{L,12}}. \tag{3}$$

Notes:
1. Increasing H, H_{L12}, or p_v decreases NPSHA and increases the likelihood of cavitation since the difference between NPSHA and NPSHR is reduced.
2. Increasing p_{atm} increases NPSHA and decreases the likelihood of cavitation.
3. The vapor pressure p_v varies with temperature.
4. The vapor pressure p_v is usually given in terms of absolute pressure and, thus, the atmospheric pressure should also be an absolute pressure.

A pump station is used to fill a tank on a hill using water from a lake. The flow rate is 10.5 L/s and atmospheric pressure is 101 kPa (abs). The pump is located 4 m above the lake, and the tank surface level is 115 m above the pump. The suction and discharge lines are 10.2 cm diameter commercial steel pipe. The equivalent length of the inlet line between the lake and the pump is 100 m. The total equivalent length between the lake and the tank is 2300 m, including all fittings, bends, screens, and valves. The overall efficiency of the pump and motor set is 70%.



water density = 1000 kg/m^3
water dynamic viscosity = 1*10^{-3} Pa.s
water vapor pressure = 1820 Pa (abs)

What is the net positive suction head available for this pump?

SOLUTION:

Apply the Extended Bernoulli Equation between the lake surface (1) and the pump inlet (2).

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \quad (1)$$

where

ρ = 1000 kg/m^3
μ = 1*10^{-3} Pa.s
p_V = 1820 Pa (abs)
g = 9.81 m/s^2
p_1 = p_{atm} = 101 kPa (abs)
V_1 ≈ 0
$z_2 - z_1$ = 4 m
H_S = 0 (Point 2 is located upstream of the pump.)
D = 0.102 m
Q = 10.5 L/s = 0.0105 m^3/s
V_2 = $Q/(\pi/4D^2)$ = 1.28 m/s (2)
Re$_D$ = $\rho V_2 D/\mu$ = 131,000 (3)
α_2 ≈ 1 (turbulent flow)
ε = 0.045*10^{-3} m (commercial steel)
H_L = $f\left(\frac{L_e}{D}\right)\frac{\bar{V}_2^2}{2g}$ = 1.61 m (4)

where Re$_D$ = 131,000 and ε/D = 0.0004 $\Rightarrow f$ = 0.0195 (from the Moody chart) (5)
and L_e = 100 m

Re-arrange Eqn. (1) to solve for the NPSHA:

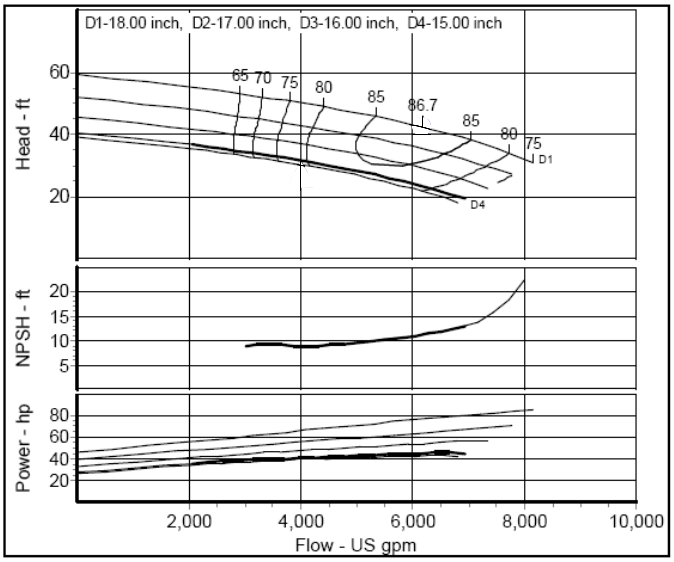
$$NPSHA=\left(\frac{p}{\rho g}+\frac{\bar{V}^2}{2g}\right)_S-\frac{p_V}{\rho g}=\frac{p_{atm}-p_V}{\rho g}+z_1-z_2-H_L \quad (6)$$

$\therefore \boxed{\text{NPSHA = 4.5 m}}$

A Peerless Model 16A 18B pump is proposed as the supply unit for the Purdue Engineering Mall fountain. The following requirements have been provided by the architectural firm:

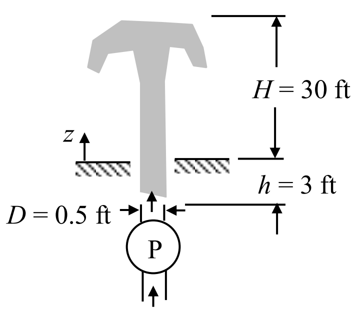
- The pump outlet is to be located 3 feet below ground level.
- The water flow is to reach a peak height of 30 feet above ground level.
- The discharge from the pump is 6 inches in diameter.

The pump characteristics are given in the following plot.



a. What head must be supplied by the pump? Report your answer in ft.
b. What flow rate must be supplied by the pump? Report your answer in gal/min (gpm).
c. What pump impeller diameter should be used? (either 15.00, 16.00, 17.00, or 18.00 inch diameter)
d. What is the pump efficiency? Report your answer in terms of a percentage.
e. What power is required to drive the pump? Report your answer in horsepower (hp).
f. What range of NPSH is acceptable at the pump inlet? Report your answer in ft.

SOLUTION:



The head that must be supplied by the pump is:

$$H_S = H + h \Rightarrow \boxed{H_S = 33 \text{ ft}} \quad (1)$$

Video solution: https://www.youtube.com/watch?v=9UnZ6V66k_4

The flow rate may be found using the pump outlet diameter and the velocity required to achieve the desired height. Apply Bernoulli's Equation from point 1 to point 2.

$$\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_1 \tag{2}$$

where

$p_1 = p_2 = p_{atm}$

$V_1 = 0$

$z_1 = -h$

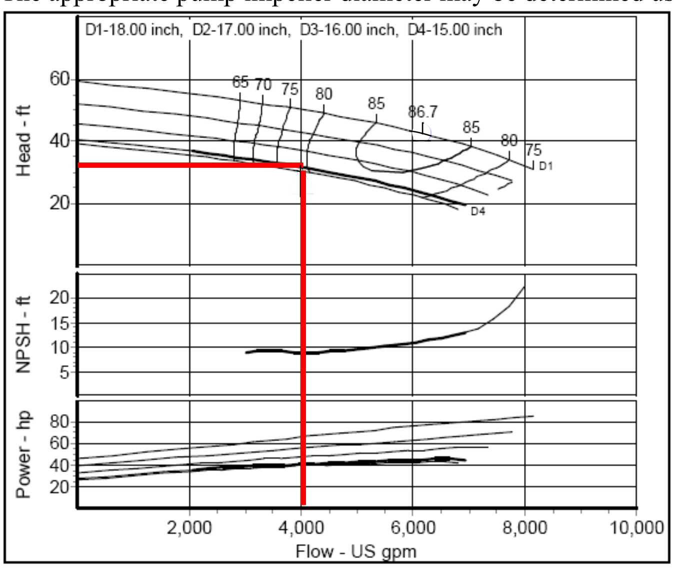
$z_2 = H$

$$\frac{V_1^2}{2g}=z_2-z_1 \Rightarrow V_1=\sqrt{2g\left(z_2-z_1\right)} \quad \text{(using the given data: } \underline{V_1 = 46.1 \text{ ft/s}}\text{)} \tag{3}$$

The flow rate is thus:

$$Q=V_1\tfrac{\pi}{4}D^2 \quad \text{(using the given data: } \boxed{Q = 9.05 \text{ ft}^3\text{/s} = 4060 \text{ gpm}}\text{)} \tag{4}$$

The appropriate pump impeller diameter may be determined using the given pump characteristics plot.



The nearest impeller diameter is the $\boxed{\text{15.00 inch}}$.

The pump efficiency may also be found from the pump characteristic plot and is $\boxed{\sim 80\%}$.

The power required to drive the pump is:

$$\dot{W}_{\substack{\text{input into}\\ \text{pump}}}=\frac{\rho QgH}{\eta}=\frac{1}{0.80}\left(62.4\ \frac{\text{lb}_\text{m}}{\text{ft}^3}\right)\left(9.05\ \frac{\text{ft}^3}{\text{s}}\right)\left(32.2\ \frac{\text{ft}}{\text{s}^2}\right)(33 \text{ ft})\left(\frac{\text{lb}_\text{f}}{32.2\frac{\text{lb}_\text{m}\cdot\text{ft}}{\text{s}^2}}\right)\left(\frac{\text{hp}}{550\ \frac{\text{ft}\cdot\text{lb}_\text{f}}{\text{s}}}\right) \tag{5}$$

$$\boxed{\therefore \dot{W}_{\substack{\text{input into}\\ \text{pump}}}=42.4 \text{ hp}}$$

The required NPSH to avoid cavitation at this flow rate is (from the pump plot) ~9 ft so the range of acceptable NPSH is $\boxed{> \sim 9 \text{ ft}}$.

12.6. Pump Similarity

Most pump performance data (H-Q curves) are given only for one value of the pump rotational speed and one pump impeller diameter. Is there some way to determine the pump performance data for other speeds and diameters without requiring additional testing? There is...using dimensional analysis! Recall that we typically are interested in knowing the head rise across a pump, H ($= h/g$ where h is the specific energy rise across the pump), power required to operate the pump, $\dot{W}$ (bhp), and pump efficiency, η, as a function of the volumetric flow rate through the pump, Q:

$$h, \dot{W}, \eta = fcns(Q, \rho, \mu, D, \omega), \tag{12.20}$$

where ρ and μ are the fluid density and dynamic viscosity, D is the pump impeller diameter, and ω is the pump rotational speed.

Performing a dimensional analysis we find the following,

$$\Psi, \Pi, \eta = fcns(\Phi, \mathrm{Re}), \tag{12.21}$$

where,

$$\Psi \coloneqq \text{dimensionless head coefficient} = \frac{gH}{\omega^2 D^2}, \tag{12.22}$$

$$\Pi \coloneqq \text{dimensionless power coefficient} = \frac{\dot{W}}{\rho\omega^3 D^5}, \tag{12.23}$$

$$\eta \coloneqq \text{efficiency} = \frac{\rho Q g H}{\dot{W}}, \tag{12.24}$$

$$\Phi \coloneqq \text{dimensionless flow coefficient} = \frac{Q}{\omega D^3}, \tag{12.25}$$

$$\mathrm{Re} \coloneqq \text{Reynolds number} = \frac{\rho\omega D^2}{\mu}. \tag{12.26}$$

Notes:

(1) In most pump flows, Re is very large $\implies$ the variations in viscous effects from one flow to another are small $\implies$ Re similarity can be neglected.
   (a) If Re is considerably different from one flow to another, e.g., pump water vs. pumping molasses, then Re effects cannot be ignored. The flow physics for large Re, where viscous forces $\ll$ inertial forces, are different than for small Re where viscous forces $\gg$ inertial forces.
   (b) Thus, for large Re:

$$\Psi, \Pi, \eta = fcns(\Phi). \tag{12.27}$$

(2)

$$\eta = \frac{\rho Q g H}{\dot{W}} = \frac{\Psi\Phi}{\Pi}. \tag{12.28}$$

(3) For similarity between geometrically similar flows (assuming large Re), we have the following <u>pump scaling laws</u>:

$$\Phi_1 = \Phi_2 \implies \left(\frac{Q}{\omega D^3}\right)_1 = \left(\frac{Q}{\omega D^3}\right)_2, \tag{12.29}$$

$$\Psi_1 = \Psi_2 \implies \left(\frac{gH}{\omega^2 D^2}\right)_1 = \left(\frac{gH}{\omega^2 D^2}\right)_2, \tag{12.30}$$

$$\Pi_1 = \Pi_2 \implies \left(\frac{\dot{W}}{\rho\omega^3 D^5}\right)_1 = \left(\frac{\dot{W}}{\rho\omega^3 D^5}\right)_2, \tag{12.31}$$

$$\eta_1 = \eta_2 \qquad (\text{since } \Phi_1 = \Phi_2,\ \Psi_1 = \Psi_2,\ \Pi_1 = \Pi_2 \text{ and } \eta = \Psi\Phi/\Pi). \tag{12.32}$$

(a) For a given pump ($D =$ constant) using the same fluid ($\rho, \mu =$constants) and the same gravity ($g =$constant),

$$\left(\frac{Q}{\omega}\right)_1 = \left(\frac{Q}{\omega}\right)_2, \tag{12.33}$$

$$\left(\frac{H}{\omega^2}\right)_1 = \left(\frac{H}{\omega^2}\right)_2, \tag{12.34}$$

$$\left(\frac{\dot{W}}{\omega^3}\right)_1 = \left(\frac{\dot{W}}{\omega^3}\right)_2. \tag{12.35}$$

The efficiency remains relatively constant when only changing the pump rotational speed (as given at the start of this note).

(b) For a given pump speed ($\omega =$constant), but varying diameters (assuming a geometrically similar family of pumps), and using the same fluid ($\rho, \mu =$constants) and the same gravity ($g =$constant),

$$\left(\frac{Q}{D^3}\right)_1 = \left(\frac{Q}{D^3}\right)_2, \tag{12.36}$$

$$\left(\frac{H}{D^2}\right)_1 = \left(\frac{H}{D^2}\right)_2, \tag{12.37}$$

$$\left(\frac{\dot{W}}{D^5}\right)_1 = \left(\frac{\dot{W}}{D^5}\right)_2. \tag{12.38}$$

Note that we are assuming that all length scales within the pump are scaled in the same way to maintain geometric similarity. This is not true in practice since pump impellers with different diameters are often put in the same pump casing. Also, surface roughness isn't scaled proportionally. The result is that the pump scaling laws are only approximations.

Since geometric scaling isn't completely satisfied, researchers have proposed the following empirical scaling rules that produce more accurate predictions than the ones given previously,

$$\left(\frac{Q}{D^2}\right)_1 = \left(\frac{Q}{D^2}\right)_2, \tag{12.39}$$

$$\left(\frac{H}{D^2}\right)_1 = \left(\frac{H}{D^2}\right)_2, \tag{12.40}$$

$$\left(\frac{\dot{W}}{D^4}\right)_1 = \left(\frac{\dot{W}}{D^4}\right)_2. \tag{12.41}$$

Since the scaling isn't perfect, the efficiency does not remain the same when scaling impeller size. As was done with the other dimensionless quantities, an empirical relationship can be used to scale the pump efficiencies, such as the one proposed by Moody,

$$\frac{1-\eta_2}{1-\eta_1} = \left(\frac{D_1}{D_2}\right)^{1/5}. \tag{12.42}$$

A centrifugal pump with a 12 in. diameter impeller requires a power input of 60 hp when the flowrate is 3200 gpm against a 60 ft head. The impeller is changed to one with a 10 in. diameter. Determine the expected flowrate, head, and input power if the pump speed remains the same.

SOLUTION:
Since the pump speed remains the same and assuming geometrically similar pumps, the pump scaling laws are,

$$\frac{Q_1}{Q_2} = \left(\frac{D_1}{D_2}\right)^3 \qquad \frac{H_1}{H_2} = \left(\frac{D_1}{D_2}\right)^2 \qquad \frac{\dot{W}_1}{\dot{W}_2} = \left(\frac{D_1}{D_2}\right)^5$$

Using the given parameters,

$$\begin{aligned} Q_1 &= 3200 \text{ gpm}, \\ D_1 &= 12 \text{ in}, \\ D_2 &= 10 \text{ in}, \\ H_1 &= 60 \text{ ft}, \\ \dot{W}_1 &= 60 \text{ hp}, \end{aligned}$$

$$\boxed{\begin{aligned} Q_2 &= 1850 \text{ gpm} \\ H_2 &= 41.7 \text{ ft} \\ \dot{W}_2 &= 24.1 \text{ hp} \end{aligned}}$$

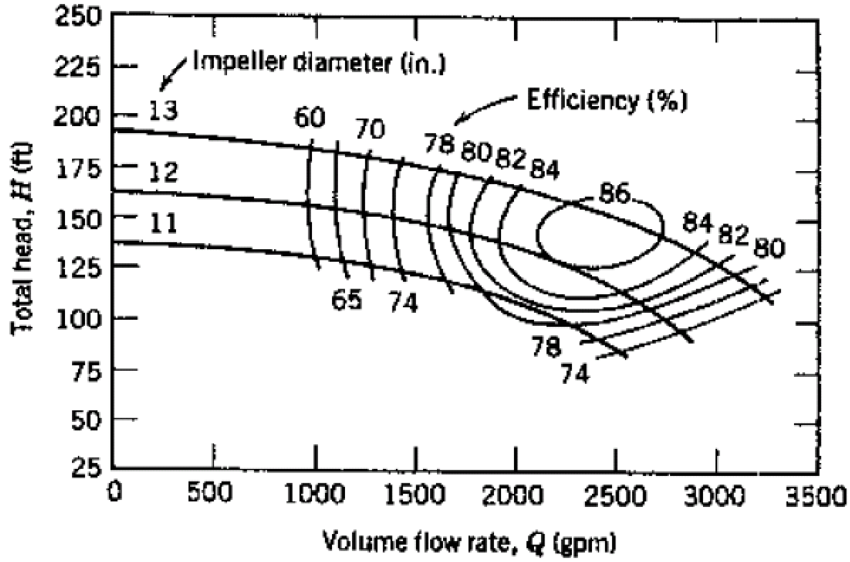
If the empirical (and more accurate) scaling laws are used,

$$\frac{Q_1}{Q_2} = \left(\frac{D_1}{D_2}\right)^2 \qquad \frac{H_1}{H_2} = \left(\frac{D_1}{D_2}\right)^2 \qquad \frac{\dot{W}_1}{\dot{W}_2} = \left(\frac{D_1}{D_2}\right)^4$$

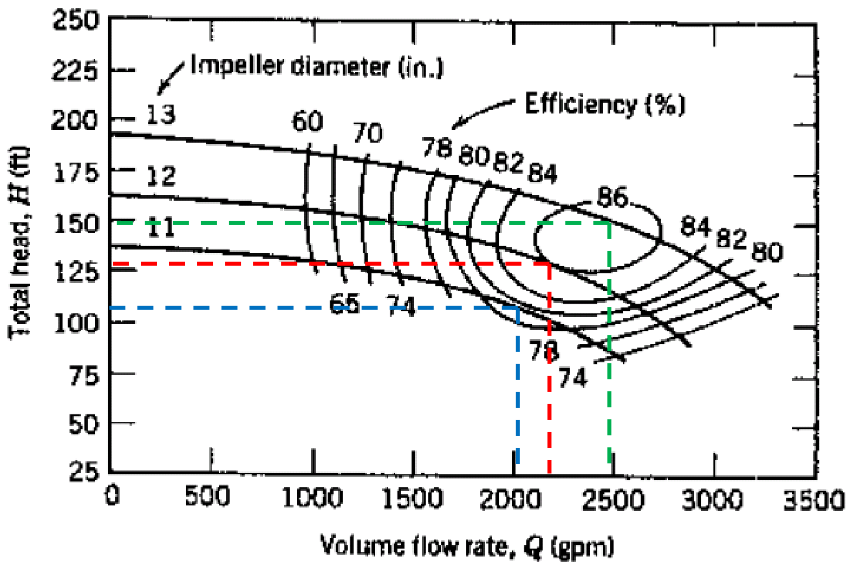
$$\boxed{\begin{aligned} Q_2 &= 2220 \text{ gpm} \\ H_2 &= 41.7 \text{ ft} \\ \dot{W}_2 &= 28.9 \text{ hp} \end{aligned}}$$

Video solution: https://www.youtube.com/watch?v=wvRz_63C4Bk

Typical performance curves for a centrifugal pump, tested with three different impeller diameters in a single casing, are shown in the figure below. Specify the flow rate and head produced by the pump at its best efficiency point with a 12 in. diameter impeller. Scale these data to predict the performance of this pump when tested with 11 in. and 13 in. impellers. Comment on the accuracy of the scaling procedure.



SOLUTION:



From the pump performance diagram,

$Q_{12\text{ in.},BEP} = 2200$ gpm

$H_{12\text{ in.},BEP} = 130$ ft

$\eta_{12\text{ in.},BEP} = 86\%$

Using the geometric scaling rule,

$$Q_2 = Q_1\left(\frac{D_2}{D_1}\right)^3 \tag{1}$$

For D_2 = 11 in. and D_1 = 12 in., $Q_{12\text{ in.}}$ = 2200 gpm, $\underline{Q_{11\text{ in.}} = 1690\text{ gpm}}$.
For D_2 = 13 in. and D_1 = 12 in., $Q_{12\text{ in.}}$ = 2200 gpm, $\underline{Q_{13\text{ in.}} = 2800\text{ gpm}}$.

Using the alternate scaling rule that takes into account imperfect geometric similarity,

$$Q_2 = Q_1\left(\frac{D_2}{D_1}\right)^2, \tag{2}$$

$\underline{Q_{11\text{ in.}} = 1850\text{ gpm}}$
$\underline{Q_{13\text{ in.}} = 2580\text{ gpm}}$

From the pump performance diagram, $Q_{11\text{ in.}} \approx 2000$ gpm.
From the pump performance diagram, $Q_{13\text{ in.}} \approx 2500$ gpm.

The alternate scaling predicts the volumetric flow rate much better than the geometric scaling rule. Indeed, the alternate scaling rule predictions are off by approximately 8% (11 in.) and 3% (13 in.) while the geometric scaling rule is off by 16% (11 in.) and 12% (13 in.).

From the pump scaling rules,

$$H_2 = H_1\left(\frac{D_2}{D_1}\right)^2 \tag{3}$$

For D_2 = 11 in. and D_1 = 12 in., H_1 = 130 ft, $\underline{H_{11\text{ in.}} = 109\text{ ft}}$.
For D_2 = 13 in. and D_1 = 12 in., H_1 = 130 ft, $\underline{H_{13\text{ in.}} = 153\text{ ft}}$.

From the pump performance diagram, $H_{11\text{ in.}} \approx 110$ ft.
From the pump performance diagram, $H_{13\text{ in.}} \approx 150$ ft.

The geometric scaling rule predictions are excellent with errors of only 1% (11 in.) and 2% (13 in.).

The best efficiencies may be found via the Moody empirical formula,

$$\frac{1-\eta_2}{1-\eta_1} = \left(\frac{D_1}{D_2}\right)^{1/5}, \tag{4}$$

For D_2 = 11 in. and D_1 = 12 in., $\eta_{1,BEP}$ = 86%, $\underline{\eta_{11\text{ in.},BEP} = 86\%}$.
For D_2 = 13 in. and D_1 = 12 in., $\eta_{1,BEP}$ = 86%, $\underline{\eta_{13\text{ in.},BEP} = 86\%}$.

From the pump performance diagram, $\eta_{11\text{ in.},BEP} \approx 82\%$ and $\eta_{11\text{ in.},BEP} \approx 87\%$. The Moody formula does a good job of predicting the best efficiency values for both impeller (5% and 1% relative errors for the 11 in. and 13 in. impellers, respectively).

12.7. Specific Speed

The first step in pump selection is to decide what class of pump (radial, mixed, or axial) will be most efficient for the given application. We usually know the flow rate Q, head rise H, and shaft rotational speed ω for the application, but not the pump size. We can form a useful dimensionless group from these quantities (including gravitational acceleration g since that's also known),

$$N_s := \text{specific speed} = \frac{\Phi^{1/2}}{\Psi^{3/4}} = \frac{\omega Q^{1/2}}{(gH)^{3/4}}. \tag{12.43}$$

This dimensionless parameter is known as the specific speed. It's customary to characterize a fluid machine by its specific speed at the design point, i.e., N_s is only given for the Best Efficiency Operating (BEP) conditions. Thus, by calculating the specific speeds for a variety of different pump types, we can create a plot that allows us to select what class of pump would be most efficient early in the design stage when we only know the desired flow rate, head rise, and shaft speed.

Notes:

(1) low Q, high $H \implies$ low $N_s \implies$ centrifugal pumps,
(2) high Q, low $H \implies$ high $N_s \implies$ axial pumps.
(3) In practice (especially in the U.S.), a combination of units are used to describe ω, Q, and H such that N_s is dimensional (signified by N_{sd}),

$$N_{sd} := \frac{\omega(\text{rpm})\sqrt{Q(\text{gpm})}}{[H(\text{ft})]^{3/4}} \tag{12.44}$$

The specific speed (N_s) and dimensional specific speed (N_{sd}) have the same physical meaning, but are different in magnitude by a constant factor,

$$N_{sd} = (2733\,\text{rpm}\cdot\text{gpm}^{1/2}/\text{ft}^{3/4})N_s. \tag{12.45}$$

(4) Given ω, Q, and H, we can calculate N_s (or N_{sd}) and, using the chart shown in Figure 12.30, determine which type of pump would be most efficient for the given conditions.

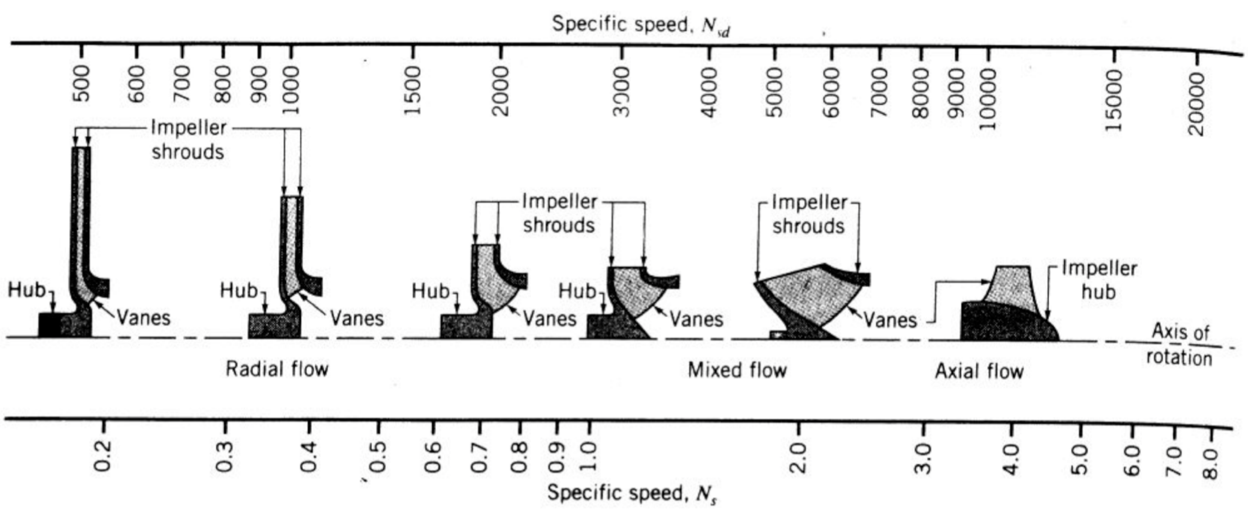


Figure 12.30. A plot of pump type as a function of specific speed. This plot is from Munson, B.R., Young, D.F, and Okiishi, T.H., *Fundamentals of Fluid Mechanics*, 3rd ed., Wiley.

Following are some rules of thumb:
(a) Positive displacement pumps are used for small flow rates (Q) and large head rises (H).

(b) Centrifugal pumps are for moderate H and large Q. Axial flow pumps are for larger Q and small H.

(c) For very large head rises, pumps are often combined in series (aka multi-stage pumps).

A small centrifugal pump, when tested at 2875 rpm with water, delivered a flowrate of 252 gpm and a head of 138 ft at its best efficiency point (efficiency is 76%). Determine the specific speed of the pump at this test condition. Sketch the impeller shape you expect. Compute the required power input to the pump.

SOLUTION:

The dimensional specific speed is given by:

$$N_{sd} = \frac{\omega\ (\text{rpm})\sqrt{Q\ (\text{gpm})}}{\left[H\ (\text{ft})\right]^{3/4}}$$

Using the given data:

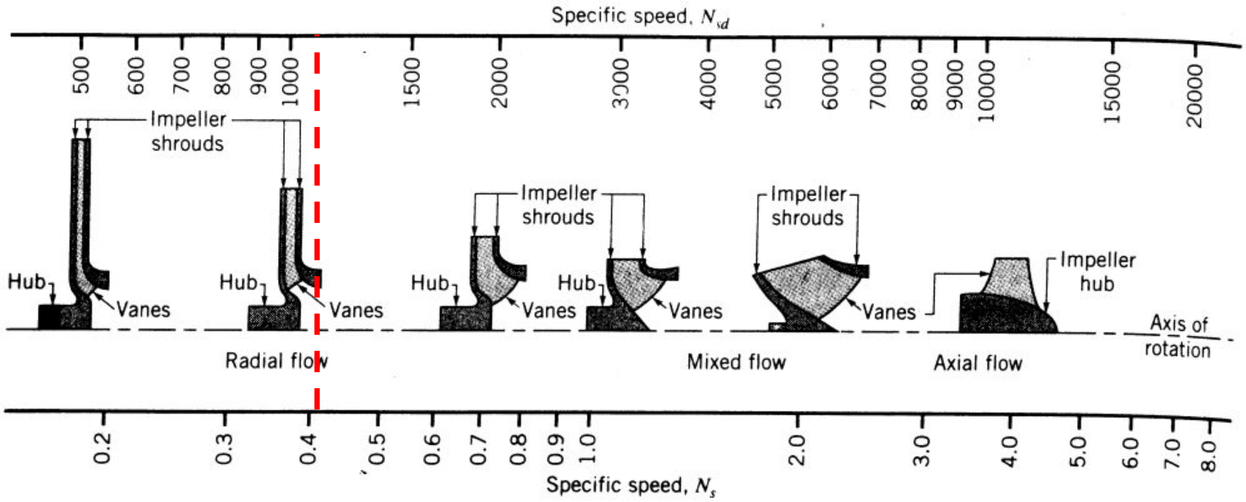
$$\boxed{N_{sd} = 1130\ \text{rpm}\cdot\text{gpm}^{1/2}/\text{ft}^{3/4}}$$

The dimensionless specific speed is:

$$N_s = \frac{N_{sd}}{2733\ \dfrac{\text{rpm}\cdot\text{gpm}^{1/2}}{\text{ft}^{3/4}}}$$

$$\boxed{N_s = 0.414}$$

The expected impeller shape is radial as shown in the figure below.



(Figure from Munson, B.R., Young, D.F., and Okiishi, T.H., *Fundamentals of Fluid Mechanics*, 3rd ed., Wiley.)

The power input to the pump is given by:

$$\dot{W}_{\text{shaft}} = \dot{W}_{\text{fluid}} / \eta_P$$

where

$$\dot{W}_{\text{fluid}} = \dot{m}gH = \rho QgH$$ (Note: 1 ft^3 = 7.48 gal, 1 hp = 550 $\text{lb}_f\cdot\text{ft/s}$, and 1 lb_f = 1 $\text{slug}\cdot\text{ft/s}^2$.)

$$\dot{W}_{\text{fluid}} = 8.80\ \text{hp}$$

$$\boxed{\dot{W}_{\text{shaft}} = 11.6\ \text{hp}}$$

12.8. System Operating Point (aka Duty Point)

How do we select a pump for a given system? Analyze the system to determine the shaft head required to give a specified volumetric flow rate. Compare this equation to a pump performance curve (H-Q curve) to determine if the pump operates efficiently at this Q. If so, then the choice of pump is appropriate.

For example, consider the system shown in Figure 12.31. Apply the Extended Bernoulli Equation from point 1 to point 2,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12}, \tag{12.46}$$

where $p_1 = p_2 = p_{atm}$ and $\bar{V}_1 \approx \bar{V}_2 \approx 0$. Thus, the head required from the pump is,

$$H_{S,12}=(z_2-z_1)+H_{L,12}. \tag{12.47}$$

Recall that,

$$H_{L,12}=\sum_i K_i\frac{\bar{V}_i^2}{2g}=\sum_i K_i\frac{Q_i^2}{2gA_i^2}, \tag{12.48}$$

so that,

$$H_{S,12}\approx(z_2-z_1)+cQ^2, \tag{12.49}$$

where c is a constant that incorporates the loss coefficients and area ratios, and an "$\approx$" is used since the loss coefficient may depend on the flow speeds.

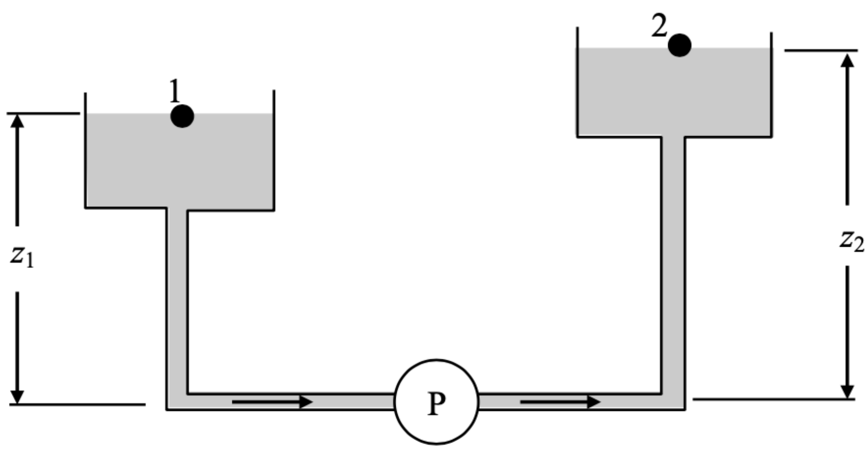


Figure 12.31. The system used in the operating point example.

The flow rate at which the system operates is at the intersection of the system head curve with the pump performance curve, as shown in Figure 12.32.

Notes:

(1) Ideally we would want the operating point to occur near the Best Efficiency Point for the pump.
(2) For laminar flow,

$$K_{\text{major}}=\frac{64}{\text{Re}}\left(\frac{L}{D}\right)=\frac{64\nu}{\bar{V}D}\frac{L}{D}=\frac{c}{Q}, \tag{12.50}$$

where c is a constant. Thus,

$$H_L\sim Q, \tag{12.51}$$

$$\implies \text{system curve is:}\quad H_S=c_1+c_2Q\quad\text{(a line instead of a parabola).} \tag{12.52}$$

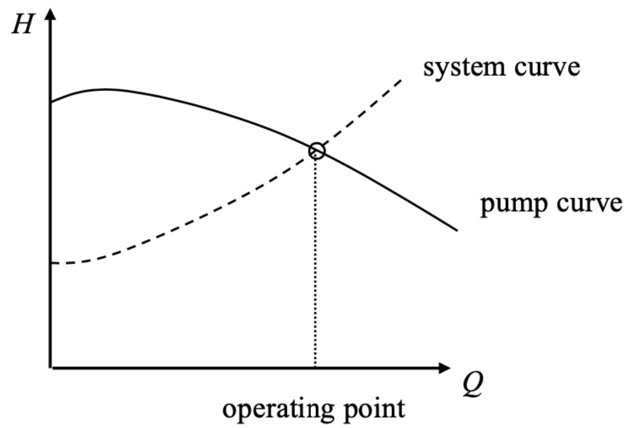


Figure 12.32. The head (H) required for the system to operate at a given flow rate (Q), i.e., the system head curve, and the head rise generated by the pump (H) at a given flow rate (Q), i.e., the pump performance curve. The flow rate at which the two curves intersect is the system operating flow rate.

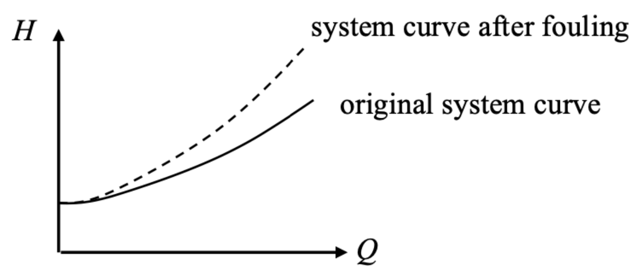


Figure 12.33. An illustration of the original system head curve and the system head curve after fouling.

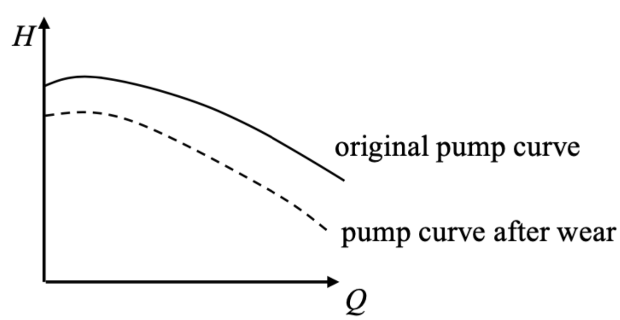


Figure 12.34. An illustration of the original pump head curve and the pump head curve after wear.

(3) The system curve may change over time due to fouling of the pipes and other factors $\Longrightarrow$ increased losses $\Longrightarrow$ the system curve becomes steeper, as shown in Figure 12.33. The pump curve may also change due to wear on the bearings, impeller, etc., as shown in Figure 12.34.

(4) Stability issues become significant when the pump has a flat or falling performance curve, which is defined as a performance curve in which H decreases as Q decreases (Figure 12.35). For example, Figure 12.36 shows the system curve intersecting the pump curve at two different flow rates. The flow rate in the falling portion of the pump curve (the left point in the figure) is unstable since a slight perturbation results in the flow rate diverging way from the point. The operating point in the rising portion of the pump curve (right point in the figure), however, is stable since conditions resulting from a small perturbation from this point will drive the flow rate back to the operating point.

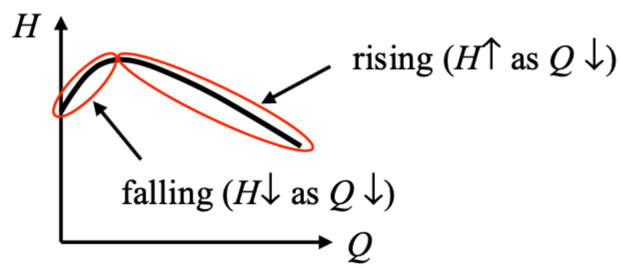


Figure 12.35. An illustration of a "falling" pump performance curve.

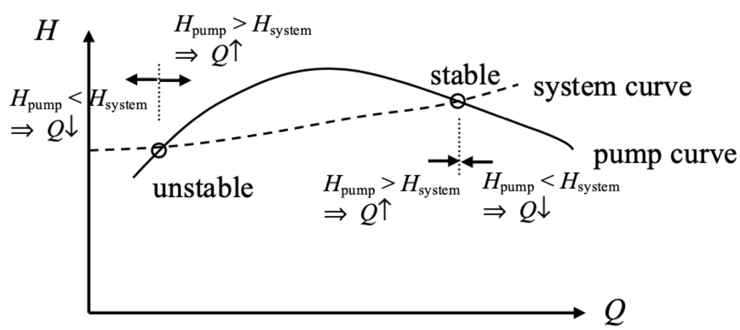


Figure 12.36. An illustration showing stable and unstable operating points.

Figure 12.37 shows a more complex pump curve with two rising sections and one falling section. The operating points in the rising sections are stable while the operating point in the falling portion is unstable. Usually this type of situation is undesirable since in engineering we typically prefer to have an unambiguous, stable operating point rather than the possibility that the operating point might suddenly change if a sufficiently large perturbation occurs.

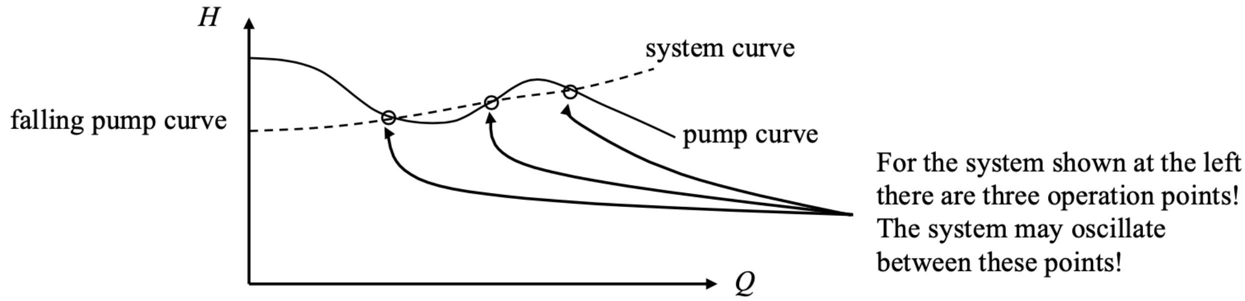


Figure 12.37. An illustration showing a pump curve resulting in two stable operating points and one unstable operating point.

Figure 12.38 shows a situation in which the system and pump curves remain close to each other for a range of flow rates. This condition is also undesirable since a perturbation from the operating point will take a long time to come back to equilibrium and, as a result, the flow rate will drift over a range of values. Instead, it is better to have a situation in which the system and pump curves intersect with a large angle between the curves so there's a large potential (i.e., head difference) driving the system back into equilibrium if there's a perturbation.

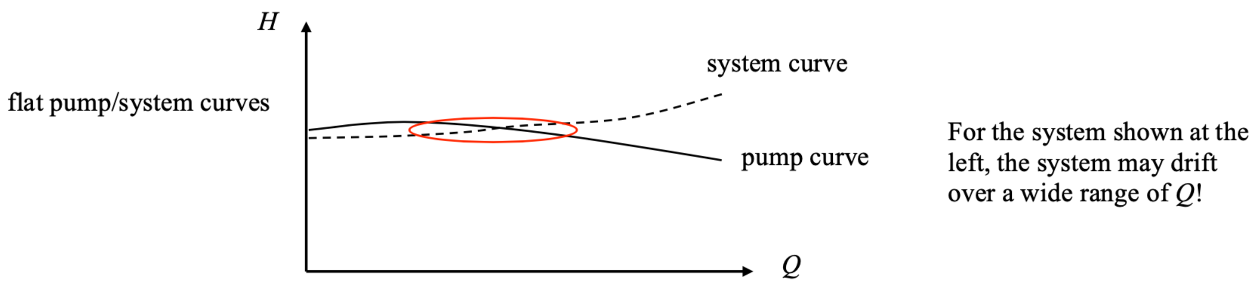
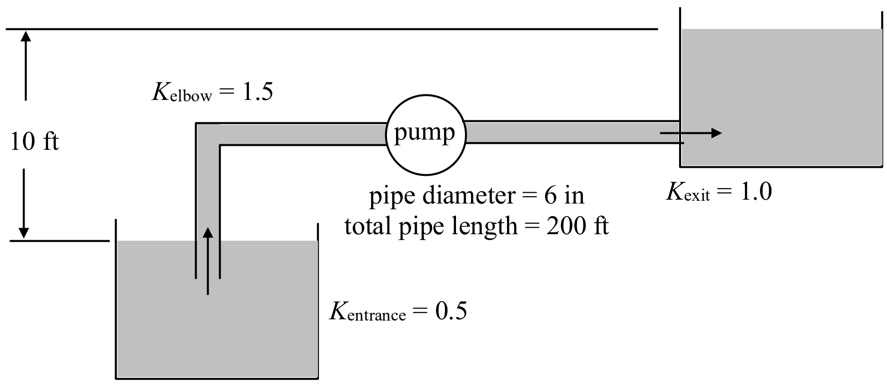


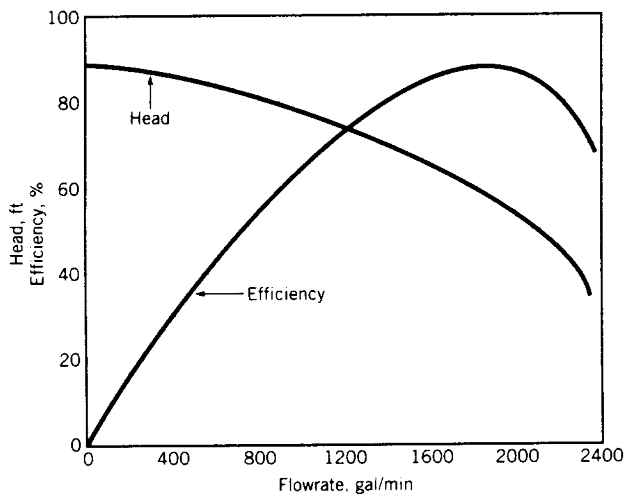
Figure 12.38. An illustration demonstrating that the operating flow rate for a flat pump performance curve can vary considerably.

Water is to be pumped from one large open tank to a second large open tank. The pipe diameter throughout is 6 in. and the total length of the pipe between the pipe entrance and exit is 200 ft. Minor loss coefficients for the entrance, exit, and the elbow are shown on the figure and the friction factor can be assumed constant and equal to 0.02. A certain centrifugal pump having the performance characteristics shown is suggested as a good pump for this flow system.

a. With this pump, what would be the flow rate between the tanks?

b. Do you think this pump would be a good choice?

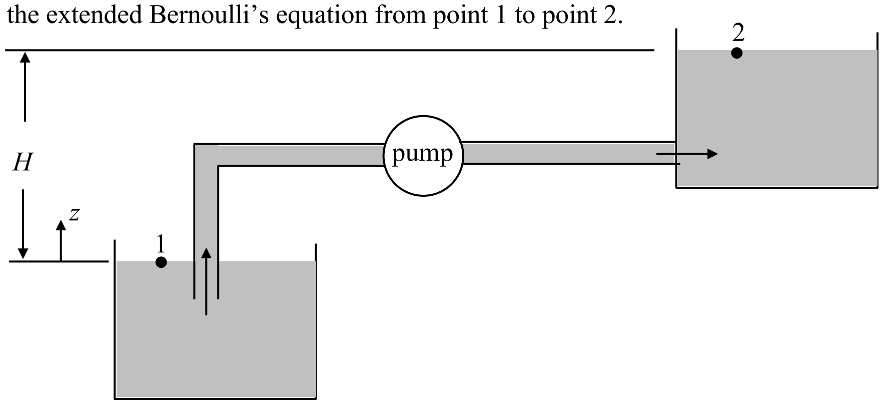




(b)

SOLUTION:

Apply the extended Bernoulli's equation from point 1 to point 2.



$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \tag{1}$$

where

$p_2 = p_1 = p_{\text{atm}}$ (free surface)

$\bar{V}_2 \approx \bar{V}_1 \approx 0$ (large tanks)

$z_2 - z_1 = H$

$$H_L=\frac{\bar{V}^2}{2g}\left[f\left(\frac{L}{D}\right)+K_{\text{entrance}}+K_{\text{exit}}+K_{\text{elbow}}\right] \text{ (where } \bar{V} \text{ is the mean velocity in the pipe)} \tag{2}$$

Note that the mean pipe velocity can be expressed in terms of the volumetric flow rate.

$$\bar{V}=\frac{Q}{\pi D^2/4}$$

Substitute and simplify.

$$H_S=H+\frac{8Q^2}{\pi^2 gD^4}\left[f\left(\frac{L}{D}\right)+K_{\text{entrance}}+K_{\text{exit}}+K_{\text{elbow}}\right] \tag{3}$$

For the given problem:

H	=	10 ft
g	=	32.2 ft/s^2
f	=	0.02
D	=	6 in = 0.5 ft
L	=	200 ft (Note: $K_{\text{major}} = f(L/D) = 8.0$)
K_{entrance}	=	0.5
K_{exit}	=	1.0
K_{elbow}	=	1.5

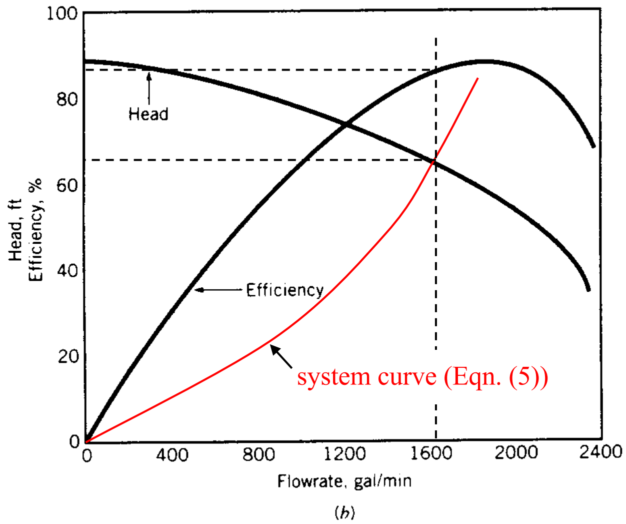
$\Rightarrow$ $H_S = (10 + 4.43Q^2)$ ft Note that $[Q]$ = ft^3/s. (4)

This is the head that must be added to the fluid by the pump in order to move the fluid at the volumetric flow rate Q.

With $[Q]$ = gpm, Eqn. (4) becomes:

$\boxed{H_S = (10 + 2.25*10^{-5}Q^2)\text{ ft}}$ Note that $[Q]$ = gpm. (5)

Plot Eqn. (5) on the pump performance curve to determine the operating point.



From the figure we observe that the operating point occurs at:

$\boxed{Q \approx 1600 \text{ gpm}}$

corresponding to a head rise and efficiency of

$H \approx 67$ ft

$\eta \approx 84\%$

The operating efficiency is close to the optimal efficiency of 86% so this is a good pump to use.

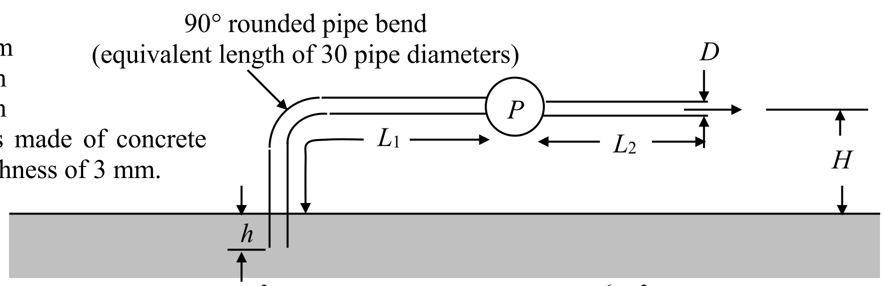
The power required to operate this pump is,

$$\dot{W} = \frac{\rho Q g H}{\eta} = \frac{\left(62.4\ \frac{\text{lb}_f}{\text{ft}^3}\right)\left(1600\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{\text{min}}{60\text{ s}}\right)\left(\frac{\text{ft}^3}{7.48\text{ gal}}\right)\left(66.5\text{ ft}\right)\left(\frac{\text{hp}}{550\text{ ft.lb}_f/\text{s}}\right)}{0.84}$$

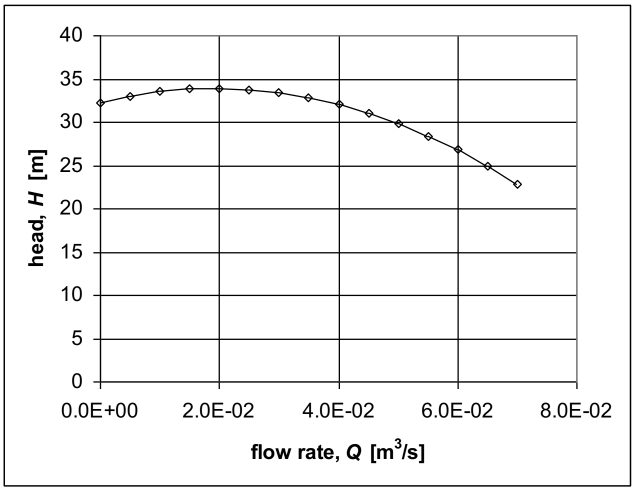
$\boxed{\dot{W} = 32.0 \text{ hp}}$

Consider the pipe/pump system shown in the figure below.

h = 0.5 m
H = 2 m
D = 0.2 m
L_1 = 10 m
L_2 = 20 m
The pipe is made of concrete with a roughness of 3 mm.



water with density of 1000 kg/m^3, kinematic viscosity of $1.0*10^{-6}$ m^2/s, and vapor pressure of 2.34 kPa



The pump performance head curve is approximated as:

$$H = (3.23*10^1 \text{ m}) + (1.65*10^2 \text{ s/m}^2)Q - (4.82*10^3 \text{ s}^2/\text{m}^5)Q^2$$

where $[H]$ = m and $[Q]$ = m^3/s.

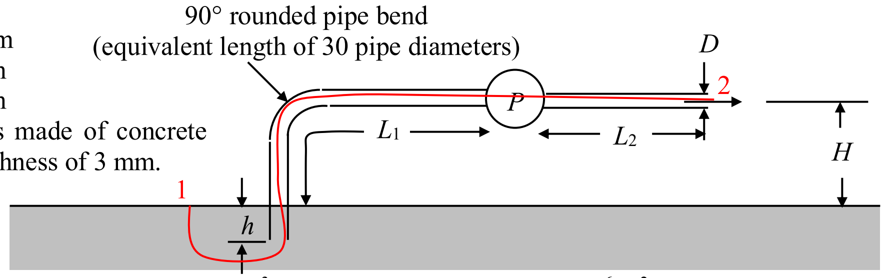
a. Determine the system head curve for the pipe system.
b. Determine the operating point for the system.
c. How will the flow rate within the pipe change over time if the pipe carries "hard" water and lime deposits form on the interior pipe walls? Explain your answer. You should assume that the deposits do not significantly affect the pipe diameter.
d. Calculate the net positive suction head available at the pump inlet.
e. If we wanted to add a valve to control the flow rate in the pipe, would it be better to put the valve upstream or downstream of the pump? Explain your answer.

Video solution: https://www.youtube.com/watch?v=ne0mde21ChA

SOLUTION:

Apply the Extended Bernoulli Equation from points 1 to 2.

h = 0.5 m
H = 2 m
D = 0.2 m
L_1 = 10 m
L_2 = 20 m
The pipe is made of concrete with a roughness of 3 mm.



water with density of 1000 kg/m^3, kinematic viscosity of 1.0*10^{-6} m^2/s, and vapor pressure of 2.34 kPa

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \tag{1}$$

where

$$p_1 = p_2 = p_{atm} \tag{2}$$

$$\bar{V}_1 \approx 0 \quad \text{and} \quad \bar{V}_2=\frac{4Q}{\pi D^2} \quad \text{(Also assume turbulent flow, } \alpha_2 \approx 1.) \tag{3}$$

$$z_2 - z_1 = H \tag{4}$$

$$H_L=\left(K_{\text{major}}+K_{\text{inlet}}+K_{\text{elbow}}\right)\frac{\bar{V}_2^2}{2g} \tag{5}$$

Solve for H_S.

$$H_S=H+\left[1+f\left(\frac{L}{D}\right)+K_{\text{inlet}}+f\left(\frac{L_e}{D}\right)\right]\frac{8Q^2}{\pi^2 gD^4} \tag{6}$$

Here,

H = 2 m
L = $L_1 + L_2$ = 30 m
D = 0.2 m
K_{inlet} = 0.78 (re-entrant inlet)
L_e/D = 30
g = 9.81 m/s^2

The relative roughness is:

$$e/D = (3*10^{-3}\text{ m})/(0.2\text{ m}) = 0.015 \tag{7}$$

Assume the flow Reynolds number is large enough so that it is in the fully rough zone and the friction factor is independent of the Reynolds number.

$$e/D = 0.015 \text{ in fully rough zone (Re > 70,000)} \Rightarrow f \approx 0.044 \tag{8}$$

Substitute and simplify.

$$H_S = 2\text{ m}+\left[1+\underbrace{6.6}_{\substack{\text{major}\\\text{losses}}}+\underbrace{0.78+1.32}_{\substack{\text{minor}\\\text{losses}}}\right]\left(5.16*10^1\ \tfrac{\text{s}}{\text{m}^5}\right)Q^2 \tag{9}$$

(Note that the minor losses are not negligible compared to the major loss.)

$$\boxed{H_S = 2\text{ m}+\left(5.01*10^2\ \tfrac{\text{s}}{\text{m}^5}\right)Q^2} \quad \text{(This is the system head curve.)} \tag{10}$$

Video solution: https://www.youtube.com/watch?v=ne0mde2lChA

The operating point occurs where the system and pump curves intersect.

$$\underbrace{2\text{ m}+\left(5.01*10^{2}\ \tfrac{\text{s}}{\text{m}^{5}}\right)Q^{2}}_{\text{system curve}}=\underbrace{\left(3.23*10^{1}\text{ m}\right)+\left(1.65*10^{2}\ \tfrac{\text{s}}{\text{m}^{2}}\right)Q-\left(4.82*10^{3}\ \tfrac{\text{s}}{\text{m}^{5}}\right)Q^{2}}_{\text{pump curve}} \tag{11}$$

$$\left(5.32*10^{3}\ \tfrac{\text{s}}{\text{m}^{5}}\right)Q^{2}-\left(1.65*10^{2}\ \tfrac{\text{s}}{\text{m}^{2}}\right)Q-\left(3.03*10^{1}\text{ m}\right)=0 \tag{12}$$

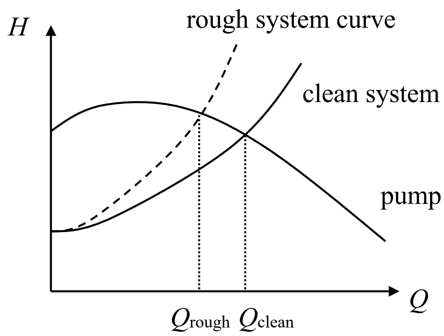
$$\boxed{Q=9.26*10^{-2}\ \tfrac{\text{m}^{3}}{\text{s}}} \tag{13}$$

Verify the Reynolds number assumption.

$$\bar{V}_2=\frac{4Q}{\pi D^2}=\frac{4\left(9.26*10^{-3}\ \frac{\text{m}^3}{\text{s}}\right)}{\pi\left(0.2\text{ m}\right)^2}=2.95\ \tfrac{\text{m}}{\text{s}} \tag{14}$$

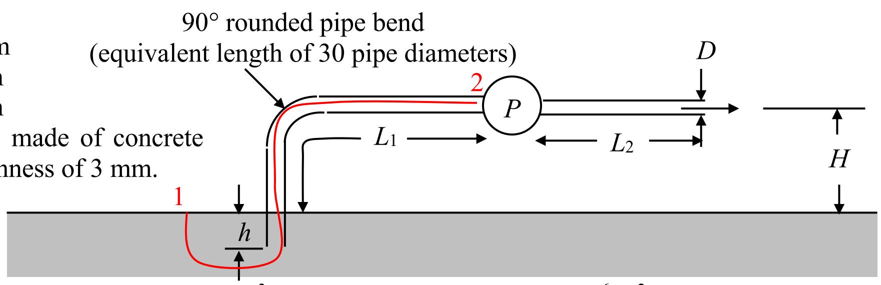
$$\text{Re}=\frac{\bar{V}_2 D}{\nu}=\frac{\left(2.95\ \frac{\text{m}}{\text{s}}\right)\left(0.2\text{ m}\right)}{\left(1*10^{-6}\ \frac{\text{m}^2}{\text{s}}\right)}=590{,}000\ \Rightarrow\ \text{The assumption of fully turbulent flow is ok!} \tag{15}$$

As lime deposits collect, the relative roughness will increase resulting in an increase in the friction factor. Thus, the system curve will steepen over time and the operating flow rate will decrease.



Apply the Extended Bernoulli Equation from points 1 to 2 in the figure below.

h = 0.5 m
H = 2 m
D = 0.2 m
L_1 = 10 m
L_2 = 20 m
The pipe is made of concrete with a roughness of 3 mm.



water with density of 1000 kg/m^3, kinematic viscosity of 1.0*10^{-6} m^2/s, and vapor pressure of 2.34 kPa

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \tag{16}$$

where

$$p_1 = p_{atm} \tag{17}$$

$$\bar{V}_1 \approx 0 \quad \text{(The flow has been shown to be turbulent} \Rightarrow \alpha_2 \approx 1.) \tag{18}$$

$$z_2 - z_1 = H \tag{19}$$

$$H_L=\left(K_{\text{major}}+K_{\text{inlet}}+K_{\text{elbow}}\right)\frac{\bar{V}_2^2}{2g} \tag{20}$$

$$H_S = 0 \quad \text{(There is no pump between points 1 and 2.)} \tag{21}$$

Re-arrange to put in terms of *NPSHA*.

$$NPSHA \equiv \left(\frac{p}{\rho g}+\frac{\bar{V}^2}{2g}\right)_2-\frac{p_v}{\rho g}=\frac{p_1-p_v}{\rho g}-H-\left[f\left(\frac{L_1}{D}\right)+K_{\text{inlet}}+f\left(\frac{L_e}{D}\right)\right]\frac{8Q^2}{\pi^2 g D^4} \tag{22}$$

(Note that the major loss is based on L_1.)

Here,

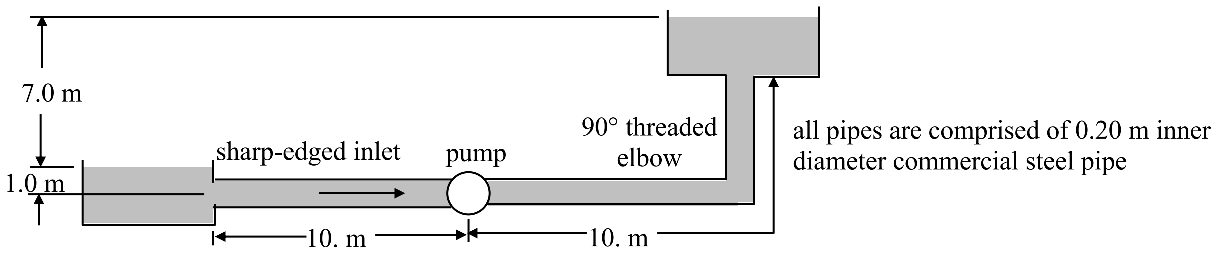
p_1 = p_{atm} = 101 kPa (abs)
p_v = 2.34 kPa (abs)
ρ = 1000 kg/m^3
g = 9.81 m/s^2
H = 2 m
f ≈ 0.044 (from previous work)
L_1 = 10 m
D = 0.2 m
K_{inlet} = 0.78 (re-entrant inlet)
L_e/D = 30
Q = 9.26*10^{-2} m^3/s

Substitute and simplify.

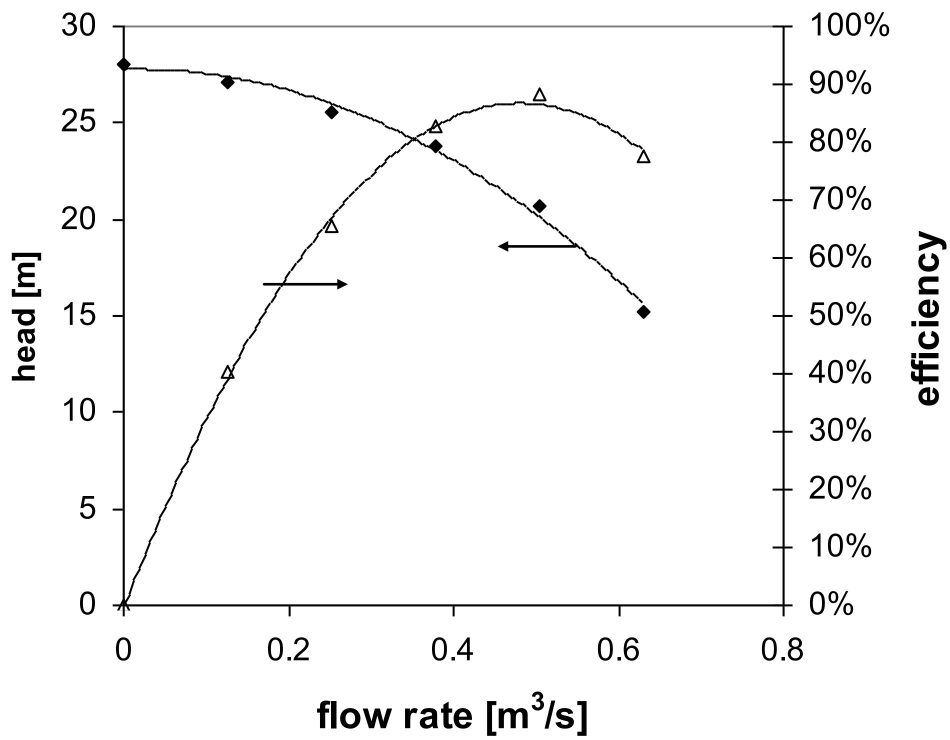
$$\boxed{NPSHA = 6.16\text{ m}} \tag{23}$$

We would be better off putting the valve downstream of the pump so that the NPSHA remains as large as possible to avoid cavitation in the pump.

Consider the pipe system shown in the figure below. The fluid to be pumped is water with a density of 1.0*E*3 kg/m^3, a kinematic viscosity of 1.0E-6 m^2/s, and a vapor pressure of 2.3E3 Pa.



The pump used in this system has the performance plot shown below.



Curve fits to the pump performance data are given below:

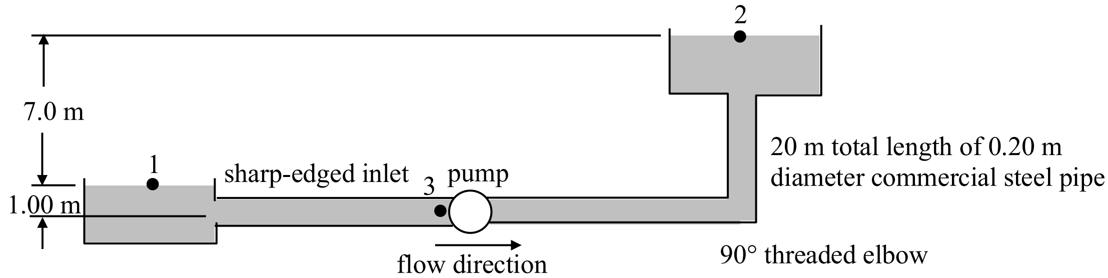
H [m] = (-3.25E1 s^2/m^5) Q^2 + (1.23E0 s/m^2) Q + (2.78E1 m)

η_P = (-3.74E0 s^2/m^6) Q^2 + (3.60E0 s/m^3) Q

a. Determine the operating volumetric flow rate of the system.
b. Is the given pump a good choice for this system? Explain your answer.
c. Determine the NPSHA to the pump for the flow rate determined in part (a).
d. Give one specific modification to the pipe system that could be employed to decrease the likelihood that cavitation will occur in the pump.

SOLUTION:

Apply the Extended Bernoulli Equation from point 1 to point 2.



$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12} \tag{1}$$

where

$p_1 = p_2 = p_{\text{atm}}$ (free surfaces exposed to the atmosphere)

$\bar{V}_1 \approx \bar{V}_2 \approx 0$ (large tanks)

$z_2 - z_1 = 7.00\text{E}0\text{ m}$ (given)

$$H_{L,12}=f\left(\frac{L}{D}\right)\frac{\bar{V}^2}{2g}+K_{\substack{\text{sharp-edged}\\ \text{entrance}}}\frac{\bar{V}^2}{2g}+K_{\substack{90^\circ\text{ threaded}\\ \text{elbow}}}\frac{\bar{V}^2}{2g}+K_{\text{exit}}\frac{\bar{V}^2}{2g}$$

$$=\left[f\left(\frac{L}{D}\right)+K_{\substack{\text{sharp-edged}\\ \text{entrance}}}+K_{\substack{90^\circ\text{ threaded}\\ \text{elbow}}}+K_{\text{exit}}\right]\frac{\bar{V}^2}{2g}$$

$$=\left[f\left(\frac{L}{D}\right)+K_{\substack{\text{sharp-edged}\\ \text{entrance}}}+K_{\substack{90^\circ\text{ threaded}\\ \text{elbow}}}+K_{\text{exit}}\right]\frac{Q^2}{2g\left(\frac{\pi}{4}D^2\right)^2}$$

Re-arrange Eqn. (1) to solve for $H_{S,12}$.

$$H_{S,12}=\left(z_2-z_1\right)+\left[f\left(\frac{L}{D}\right)+K_{\substack{\text{sharp-edged}\\ \text{entrance}}}+K_{\substack{90^\circ\text{ threaded}\\ \text{elbow}}}+K_{\text{exit}}\right]\frac{Q^2}{2g\left(\frac{\pi}{4}D^2\right)^2} \tag{2}$$

Assume that the flow is in the fully rough zone where the friction factor is independent of the Reynolds number. The pipe roughness, ε, is $\varepsilon = 0.0450E\text{-}3$ m so that the relative roughness is:

$$\frac{\varepsilon}{D}=\frac{4.50\text{E-5 m}}{2.00\text{E-2 m}}=2.25\text{E-4}$$

From the Moody chart in the fully rough zone:

$f =$ 1.41E-2

Substitute in the given data into Eqn. (2):

z_2-z_1 = 7.00E0 m
L = 2.00E1 m
D = 2.00E-1 m
K_{inlet} = 5.00E-1
K_{elbow} = 1.50E0
K_{exit} = 1.00E0
g = 9.81E0 m/s^2

$$H_{S,12} = (7.00\text{E0 m}) + \left(2.28\text{E2 } {}^{\text{s}^2}\!\!\diagup_{\text{m}^5}\right)Q^2 \tag{3}$$

Equate the system head curve (Eqn. (3)) to the given curve fit for the pump head curve to solve for the operating point flow rate.

$$\left(\text{-3.25E1 s}^2/\text{m}^5\right)Q^2 + \left(1.23\text{E0 s/m}^2\right)Q + \left(2.78\text{E1 m}\right) = \left(7.00\text{E0 m}\right) + \left(2.28\text{E2 } {}^{\text{s}^2}\!\!\diagup_{\text{m}^5}\right)Q^2$$

$$\left(\text{-2.60E2 s}^2/\text{m}^5\right)Q^2 + \left(1.23\text{E0 s/m}^2\right)Q + \left(2.08\text{E1 m}\right) = 0$$

$$\boxed{\therefore Q = 2.85\text{E-1 m}^3/\text{s}}$$

Check that the Reynolds number is in the fully rough zone as assumed.

$$\bar{V} = \frac{Q}{\frac{\pi}{4}D^2} = \frac{2.85\text{E-1 m}^3/\text{s}}{\frac{\pi}{4}\left(2.0\text{E-2 m}\right)} = 9.08 \text{ m/s}$$

$$\text{Re} = \frac{\bar{V}D}{\nu} = \frac{\left(9.08\text{E0 m/s}\right)\left(2.0\text{E-1 m}\right)}{\left(1.0\text{E-6 m}^2/\text{s}\right)} = 1.82\text{E6}$$

The Reynolds number and the relative roughness put the flow in the fully rough zone so the assumption was a good one.

The efficiency is determined using the given curve fit for the efficiency and the calculated volumetric flow rate.

$$\eta_P = 72\%$$

This efficiency is not near the Best Efficiency Point for the pump (BEP = 90%) so this is <u>not a good pump to use for this application</u>.

The NPSHA to the pump is found by applying the Extended Bernoulli Equation between points 1 and 3 and utilizing the definition of NPSH.

$$\left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_3 = \left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_1 - H_{L,13} + H_{S,13} \tag{4}$$

where

$p_1 = p_{\text{atm}}$ (free surface exposed to the atmosphere)

$\bar{V}_1 \approx 0$ (large tank)

$z_1 - z_3 = 1.00\text{E0 m}$ (given)

$\alpha_3 \approx 1$ (turbulent flow based on the Reynolds number calculated previously)

$H_{S,13} = 0$

$$H_{S,13} = \left[f\left(\frac{L_{13}}{D}\right) + K_{\substack{\text{sharp-edged}\\\text{entrance}}}\right]\frac{Q^2}{2g\left(\frac{\pi}{4}D^2\right)^2}$$

Substitute into the definition of NPSH.

$$NPSH = \left(\frac{p}{\rho g} + \frac{\bar{V}^2}{2g}\right)_s - \frac{p_v}{\rho g}$$

$$NPSHA = \left(\frac{p_{\text{atm}} - p_v}{\rho g}\right) + \left(z_1 - z_3\right) - \left[f\left(\frac{L_{13}}{D}\right) + K_{\substack{\text{sharp-edged}\\\text{entrance}}}\right]\frac{Q^2}{2g\left(\frac{\pi}{4}D^2\right)^2}$$

Using the given data:

p_{atm} = 1.01E5 Pa
p_v = 2.30E3 Pa
ρ = 1.00E3 kg/m^3
g = 9.81E0 m/s^2
z_1-z_3 = 1.00E0 m
f = 1.41E-1 (found previously)
L_{13} = 1.0E1 m
D = 2.0E-1 m
K_{inlet} = 5.0E-1
Q = 2.85E-1 m^3/s (found previously)
$\Rightarrow$ $\boxed{\text{NPSHA} = 6.01\text{E}0 \text{ m}}$

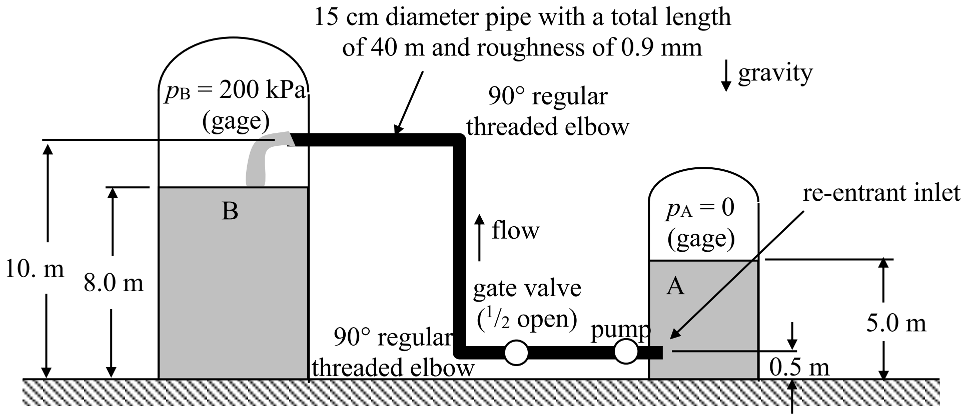
In order to avoid cavitating the pump, we would need to make sure that NPSHA > NPSHR for the pump.

If NPSHA < NPSHR, then the following could be easily implemented to increase NPSHA:

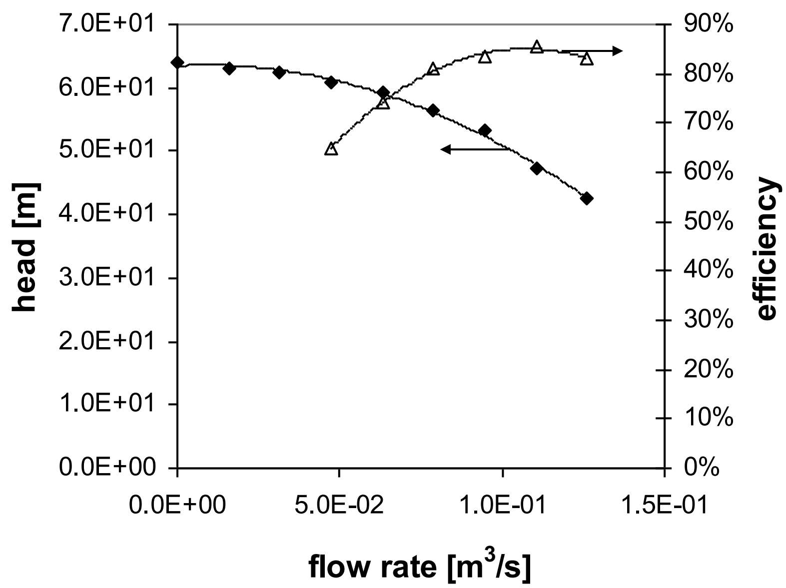
1. Decrease the elevation of the pump inlet so that $z_1 - z_3$ increases.
2. Decrease the losses from 1 to 3 by:
   a. decreasing the pipe length from 1 to 3 and
   b. using a rounded inlet into the pipe.

Note that increasing the pipe diameter from 1 to 3 or changing the pipe material from 1 to 3 might be difficult to implement and would also change the system operating point. Although they would be difficult to implement, increasing the pressure in tank 1 or decreasing the flow temperature to decrease the vapor pressure would also act to increase NPSHA.

Consider the pipe/pump system shown below in which water (with a density of 1.0E3 kg/m^3 and dynamic viscosity of 1.3E-3 Pa·s) is pumped from tank A to tank B.



The pump to be used in the system has the following pump performance curve.



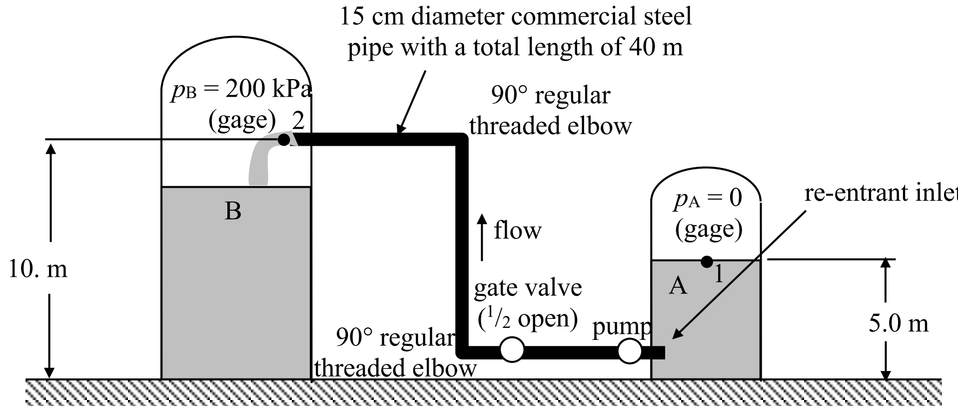
Curve fits to the pump performance data are given below:

H [m] = (-1.5E3 s^2/m^5) Q^2 + (2.8E1 s/m^2) Q + (6.3E1 m)

η_P = (-5.6E1 s^2/m^6) Q^2 + (1.2E1 s/m^3) Q + (2.1E-1)

a. Determine the operating point for the system.
b. Is the given pump efficient for this application? Explain your answer.
c. Do you anticipate that cavitation in the pump will be an issue? Explain your answer.

SOLUTION:



Apply the Extended Bernoulli Equation from the free surface of tank A (point 1) to the end of the pipe leading into tank B (point 2).

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,12}+H_{S,12} \tag{1}$$

where

$p_1 = p_A = 0$ (gage) and $p_2 = p_B = 2.0\text{E}5$ Pa (gage) (given)

$\bar{V}_1 \approx 0$ (large tank)

$\bar{V}_2=\bar{V}_P=\dfrac{Q}{\frac{\pi}{4}D^2}$ and $\alpha_2 \approx 1$ (assuming turbulent flow)

$z_1 = 5.0\text{E}0$ m and $z_2 = 1.0\text{E}1$ m (given)

$$H_{L,12}=f\left(\frac{L}{D}\right)\frac{\bar{V}_P^2}{2g}+K_{\substack{\text{re-entrant}\\ \text{inlet}}}\frac{\bar{V}_P^2}{2g}+K_{\substack{\text{1/2 open}\\ \text{gate valve}}}\frac{\bar{V}_P^2}{2g}+2K_{\substack{90^\circ\ \text{threaded}\\ \text{elbow}}}\frac{\bar{V}_P^2}{2g}$$

$$H_{L,12}=\left[f\left(\frac{L}{D}\right)+K_{\substack{\text{re-entrant}\\ \text{inlet}}}+K_{\substack{\text{1/2 open}\\ \text{gate valve}}}+2K_{\substack{90^\circ\ \text{threaded}\\ \text{elbow}}}\right]\frac{\bar{V}_P^2}{2g} \tag{2}$$

(Note that there are no exit losses at point 2.)

The friction factor, f, is determined from the Moody chart using the Reynolds number in the pipe, Re, and the relative roughness, ε/D. Since the Reynolds number is unknown at this point (since the flow rate and hence velocity are unknown), assume that the flow occurs in the fully rough zone. The pipe has a roughness of 0.9 mm. Hence:

$$\frac{\varepsilon}{D} = \frac{(9.0\text{E-4 m})}{(1.5\text{E-1 m})} = 6.0\text{E-3}$$

$$f = 3.2\text{E-2}$$

Hence, the major loss coefficient for the system is:

$$K_{\text{major}} = f\left(\frac{L}{D}\right) = (3.2\text{E-2})\left(\frac{4.0\text{E1 m}}{1.5\text{E-1 m}}\right) = 8.6\text{E0}$$

The minor loss coefficients are found from minor loss tables to be:

$$K_{\substack{\text{re-entrant}\\ \text{inlet}}} = 8.0\text{E-1}$$

$$K_{\substack{\text{half open}\\ \text{gate valve}}} = 2.1\text{E0}$$

$$K_{\substack{90^\circ\text{ threaded}\\ \text{elbow}}} = 1.5\text{E0}$$

Re-arrange Eqn. (1) to solve for $H_{S,12}$ and substitute the values given above.

$$H_{S,12} = \frac{p_2 - p_1}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} - \alpha_1 \frac{\bar{V}_1^2}{2g} + z_2 - z_1 + H_{L,12}$$

$$= \frac{p_2}{\rho g} + \frac{Q^2}{2g\left(\frac{\pi}{4}D^2\right)^2} + z_2 - z_1 + H_{L,12}$$

$$= \frac{(2.0\text{E5 Pa})}{(1.0\text{E3 kg/m}^3)(9.8\text{E0 m/s}^2)} + \frac{Q^2}{2(9.8\text{E0 m/s}^2)\left[\frac{\pi}{4}(1.5\text{E-1 m})^2\right]^2} + (1.0\text{E1 m}) - (5.0\text{E0 m})$$

$$+\left[8.6\text{E0} + 8.0\text{E-1} + 2.1\text{E0} + 2(1.5\text{E0})\right]\frac{Q^2}{2(9.8\text{E0 m/s}^2)\left[\frac{\pi}{4}(1.5\text{E-1 m})^2\right]^2}$$

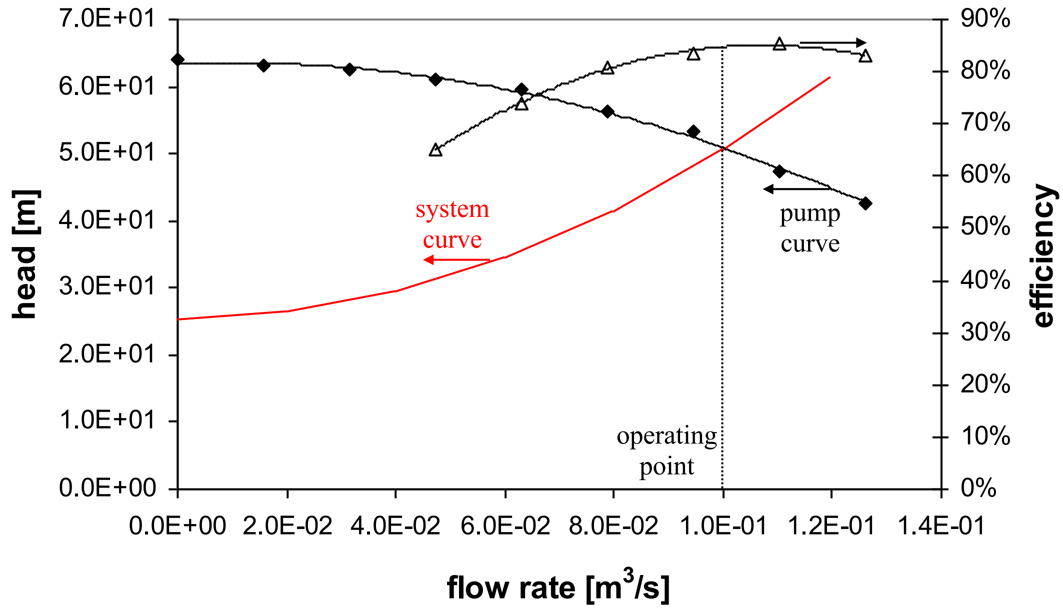
$$H_{S,12} = (2.5\text{E1 m}) + (2.5\text{E3 s}^2/\text{m}^5)Q^2$$

Equate the system head curve with the given pump head curve to determine the operating point.

$$\underbrace{(2.5\text{E1 m}) + (2.5\text{E3 s}^2/\text{m}^5)Q^2}_{\text{system curve}} = \underbrace{(6.3\text{E1 m}) + (2.8\text{E1 s/m}^2)Q + (\text{-}1.5\text{E3 s}^2/\text{m}^5)Q^2}_{\text{pump curve}}$$

$$(4.0\text{E3 s}^2/\text{m}^5)Q^2 + (\text{-}2.8\text{E1 s/m}^2)Q + (\text{-}3.8\text{E1 m}) = 0$$

$$\boxed{Q = 1.0\text{E-1 m}^3/\text{s}}$$



The velocity corresponding to this flow rate is:

$$\bar{V}_2 = \frac{Q}{\frac{\pi}{4}D^2} = 5.7\text{E0 m/s}$$

and the corresponding Reynolds number is:

$$\text{Re} = \frac{\bar{V}_2 D}{\nu} = 6.5\text{E5}$$

Hence, the assumption of fully rough turbulent flow is ok.

The pump efficiency at this flow rate is found using the given pump efficiency curve.

$$\eta_P = \left(\text{-5.6E1 s}^2/\text{m}^6\right)Q^2 + \left(1.2\text{E1 s/m}^3\right)Q + \left(2.1\text{E-1}\right)$$

$$\eta_P = 85\%$$

Since this efficiency is very close to the best efficiency point, this is an efficient pump for this application.

To determine if cavitation will occur in the pump, we would need to compare the NPSH available at the pump inlet to the NPSH required by the pump (need NPSHA > NPSHR to avoid cavitation). The NPSHA can be determined by apply the Extended Bernoulli Equation from point 1 to a point at the inlet of the pump and using the definition of NPSH.

$$\left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_{\text{inlet}} = \left(\frac{p}{\rho g} + \alpha\frac{\bar{V}^2}{2g} + z\right)_1 - H_{L,1\text{-inlet}} + H_{S,1\text{-inlet}}$$

$$NPSH \equiv \left(\frac{p}{\rho g} + \frac{\bar{V}^2}{2g}\right)_{\text{inlet}} - \frac{p_v}{\rho g}$$

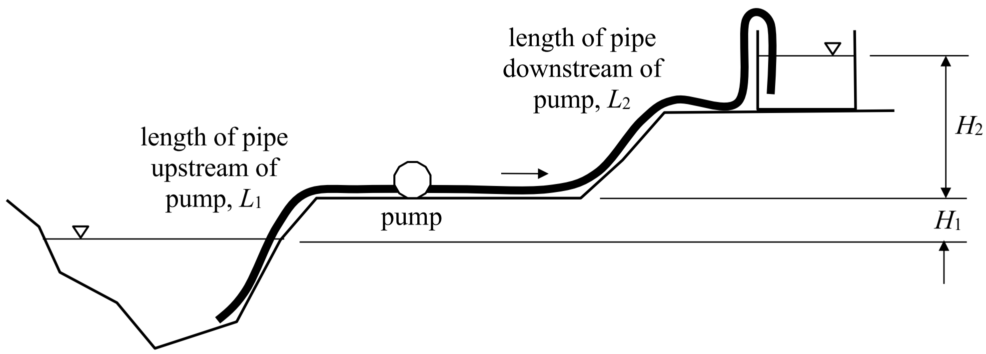
where

$$\alpha_{\text{inlet}} \approx 1 \qquad p_1 = p_{\text{atm}} \qquad H_{S,1\text{-inlet}} = 0$$

$$NPSHA = \frac{p_{\text{atm}} - p_v}{\rho g} + z_1 - z_2 - H_{L,1\text{-inlet}}$$

Since $p_{atm} > p_v$, $z_1 > z_2$, and $H_{L,1\text{-inlet}}$ will be relatively small since there are few loss mechanisms occurring upstream of the pump, cavitation in the pump will most likely not be an issue.

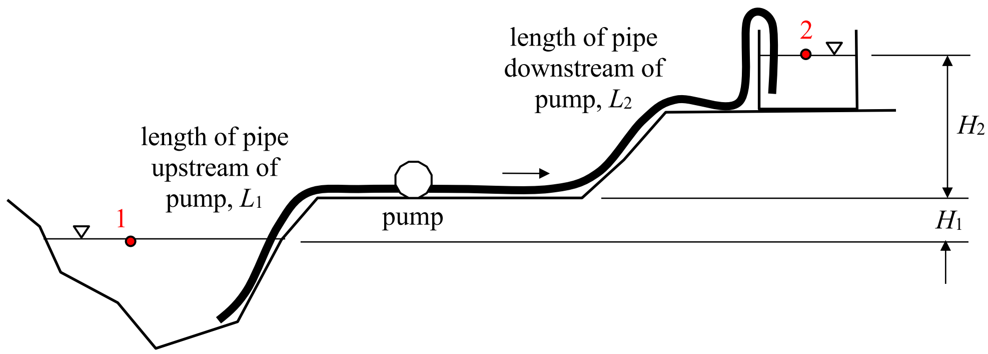
Consider the pipe system containing a pump shown in the figure below. The fluid being pumped from the lake to the tank is water (density = 1000 kg/m^3, kinematic viscosity = 1.0*10^{-6} m^2/s).



diameter of both lengths of pipe, $D_1 = D_2$ = 10 cm
length of pipe upstream of pump, L_1 = 5 m
length of pipe downstream of pump, L_2 = 15 m
roughness of both lengths of pipe, $\varepsilon_1 = \varepsilon_2$ = 1.5*10^{-4} m
total minor loss upstream of pump, $K_{\text{minor},1}$ = 1.0
total minor loss downstream of pump, $K_{\text{minor},2}$ = 2.0
H_1 = 3 m
H_2 = 10 m
pump head rise curve: H [m] = (-1.5*10^3 s^2/m^5)Q^2 + (2.8*10^1 s/m^2)Q + (6.3*10^1 m)
pump efficiency curve: η = (-5.6*10^1 s^2/m^6)Q^2 + (1.2*10^1 s/m^3)Q + (2.1*10^{-1})

a. Determine the operating flow rate for the system.
b. What power must be supplied to the pump by the motor to operate at the flow rate found in part (a)?

SOLUTION:
To determine the operating flow rate, first determine the system head curve by applying the extended Bernoulli equation from point 1 to point 2.



$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_L+H_S \ , \tag{1}$$

where

$p_1 = p_{atm}$ $\quad$ $p_2 = p_{atm}$

$\bar{V}_1 \approx 0$ $\quad$ $\bar{V}_2 \approx 0$

$z_1 = -H_1$ $\quad$ $z_2 = H_2$

$$\begin{aligned} H_L &= \sum_i K_i\frac{\bar{V}_i^2}{2g} \\ &= \left[f\left(\frac{L_1+L_2}{D}\right)+K_{\text{minor},1}+K_{\text{minor},2}\right]\frac{\bar{V}_p^2}{2g}=\left[f\left(\frac{L_1+L_2}{D}\right)+K_{\text{minor},1}+K_{\text{minor},2}\right]\frac{Q^2}{2g\left(\frac{\pi}{4}D^2\right)^2} \ . \end{aligned} \tag{2}$$

The friction factor may be found from the Moody diagram. Since the flow rate is unknown, try assuming that the flow is in the fully turbulent region of the Moody diagram (this assumption will need to be verified). In this region, the friction factor is only a function of the pipe's relative roughness,

$$\frac{\varepsilon}{D}=\frac{1.5*10^{-4}\text{ m}}{0.10\text{ m}}=1.5*10^{-3} \tag{3}$$

From the Moody diagram, $f = 0.022$.

Combining these relations gives the system head curve,

$$H_{S,\text{system}}=\left(z_2-z_1\right)+\left[f\left(\frac{L_1+L_2}{D}\right)+K_{\text{minor},1}+K_{\text{minor},2}\right]\frac{Q^2}{2g\left(\frac{\pi}{4}D^2\right)^2}=s_0+s_2Q^2 \ , \tag{4}$$

where,

$$s_0=z_2-z_1 \ , \text{ and} \tag{5}$$

$$s_2=\frac{1}{2g\left(\frac{\pi}{4}D^2\right)^2}\left[f\left(\frac{L_1+L_2}{D}\right)+K_{\text{minor},1}+K_{\text{minor},2}\right] \ . \tag{6}$$

Determine the operating point by equating the system head curve to the pump head curve,

$$s_0+s_2Q^2=p_0+p_1Q+p_2Q^2 \ , \tag{7}$$

$$\left(p_2-s_2\right)Q^2+p_1Q+\left(p_0-s_0\right)=0 \ , \tag{8}$$

$$Q=\frac{-p_1\pm\sqrt{p_1^2-4\left(p_2-s_2\right)\left(p_0-s_0\right)}}{2\left(p_2-s_2\right)} \ . \tag{9}$$

Using the given data,

s_0 = 13 m
s_2 = 6.08*10^3 s^2/m^5
p_0 = 6.30*10^1 m
p_1 = 2.80*10^1 s/m^2
p_2 = -1.50*10^3 s^2/m^5

$\boxed{Q = 8.3*10^{-2}\ \text{m}^3/\text{s}}$

Check the Reynolds number assumption of fully turbulent flow,

$$\bar{V} = \frac{Q}{\frac{\pi}{4}D^2} \quad \Rightarrow \quad \bar{V} = 10.6\ \text{m/s}\ , \tag{10}$$

$$\text{Re} = \frac{\bar{V}D}{\nu} \quad \Rightarrow \quad \text{Re} = 1.1*10^6 \tag{11}$$

Checking the Moody diagram shows that the flow is in the fully rough zone for this Reynolds number and relative roughness. Thus, our assumption of fully rough flow was a good one.

The power input to the fluid by the pump at these conditions is,

$$\dot{W}_{\substack{\text{into}\\\text{fluid}}} = \rho Q g H\ , \tag{12}$$

where H = 55.0 m at the operating flow rate of 8.3*10^{-2} m^3/s (found using either the system or pump head curves). Hence,

$$\Rightarrow \quad \dot{W}_{\substack{\text{into}\\\text{fluid}}} = 44.8\ \text{kW}\ . \tag{13}$$

Since the pump isn't 100% efficient, the power that must be supplied to the pump is,

$$\dot{W}_{\substack{\text{into}\\\text{pump}}} = \frac{\dot{W}_{\substack{\text{into}\\\text{fluid}}}}{\eta} \quad \Rightarrow \quad \boxed{\dot{W}_{\substack{\text{into}\\\text{pump}}} = 54.6\ \text{kW}}. \tag{14}$$

where the pump efficiency at the operating point is η = 82% (using the given efficiency curve for the pump, η = (-5.6*10^1 s^2/m^6)Q^2 + (1.2*10^1 s/m^3)Q + (2.1*10^{-1})).

12.9. Review Questions

(1) What class of pumps generates large Δps? What class of pumps generates large Q?
(2) What is the fundamental principle for the operation of positive displacement pumps?
(3) Describe the fundamental principle behind the operation of dynamic pumps.
(4) What information is presented on a pump performance plot?
(5) Why do some pump performance plots have different impeller diameter curves?
(6) What is meant by the "Best Efficiency Point"?
(7) Can a pump operate at a flow rate different from the BEP?
(8) Assuming the same diameter inlet and outlet pipe, how does the average flow speed change across a pump? How does the pressure change?
(9) Describe the various components in a centrifugal pump.
(10) What is meant by the "shut-off head"?
(11) How does one determine the operating flow rate for a pump in a pipe system?
(12) What is the definition of NPSH and how is it used?
(13) What is the difference between NPSHA and NPSHR?

CHAPTER 13

Compressible Flow (aka Gas Dynamics)

13.1. Introduction to Gas Dynamics

13.1.1. What is Gas Dynamics?

Gas dynamics is a branch of fluid mechanics that examines the dynamics of compressible fluid flows and of gases in particular.

13.1.2. What is the motivation for studying compressible fluids?

Although topics regarding compressible fluid mechanics have been studied since the 1800s, few scientists and engineers were interested in the topic apart from those studying ballistics and steam turbine design. It wasn't until WW II with the development of high speed planes, rockets, and energetic explosives that the study of compressible flows became widespread. Ever since, the understanding of compressible fluid mechanics has been important in the development of not only the previously mentioned topics, but also of jet engines, rocketry, re-entry spacecraft, gas pipelines, combustion, and gas turbines.

13.1.3. What is special about compressible fluids?

Compressibility of a fluid results in several important phenomena that are not observed in incompressible fluids. Two of the most significant of these phenomena are shock waves and "choked" flow conditions. Both of these phenomena are the result of the fact that in compressible fluids, pressure disturbances propagate at a finite speed. For example, if one claps their hands, the pressure disturbance caused by the colliding hands propagates into the surrounding atmosphere with a finite speed (equal to the speed of sound). Thus, a finite amount of time passes before the surrounding air recognizes the effects of the clapping hands. In contrast, in a truly incompressible fluid, pressure disturbances propagate at an infinite speed. Thus, pressure disturbances are felt instantly everywhere in the fluid domain.

The fact that disturbances travel at finite speed raises the question of what happens if the cause of the pressure disturbance travels faster than the pressure disturbance itself? As an example, let's consider an aircraft flying in the atmosphere. When the aircraft moves slower than the disturbances propagate, pressure disturbances travel ahead of the aircraft and "inform" the air in front that the aircraft is about to arrive. Thus, the air can move smoothly out of the way as the aircraft approaches. However, if the aircraft travels faster than the speed of propagation, then the air in front of the aircraft can't move out of the way and begins to "pile up" in front of the aircraft. The result is the formation of a shock wave across which there is a rapid change in the air pressure, temperature, density and velocity.

Now let's consider a different situation. Imagine a large, pressurized tank with a converging nozzle that empties into another large tank (refer to the Figure 13.1). While holding the pressure in the left tank constant, let's begin to reduce the pressure in the right-hand tank.

When we lower the pressure in the right-hand tank, a pressure disturbance propagates upstream to the constant pressure tank and "informs" the fluid upstream that the pressure in the right-hand tank has dropped. As a result, the flow rate between tanks increases. As we continue to lower the pressure in the right-hand tank, the flow rate continues to increase until we reach a speed in the converging nozzle where the fluid speed is equal to the speed at which pressure disturbances propagate. Now if we continue to lower the pressure in the right hand tank, that pressure information can no longer propagate upstream since the fluid is flowing in

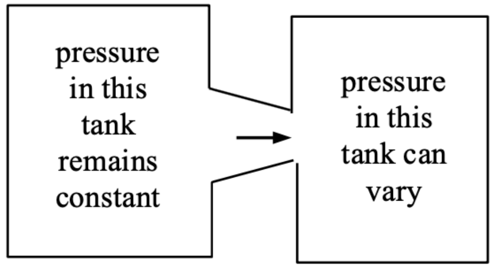


Figure 13.1. Discharge from a pressurized tank.

the opposite direction at the same speed. Thus, we have a "choked" flow condition where we can no longer increase the flow rate between the tanks.

In addition to these two phenomena, compressible flows have other counter-intuitive behaviors regarding how the fluid velocity varies with the area through which the fluid flows and how the speed is affected by frictional effects. We'll investigate all of these phenomena in this chapter.

13.1.4. What tools are required to study compressible fluid mechanics?

Several basic concepts are used in studying compressible fluid mechanics. These include:

- Conservation of Mass,
- The Linear Momentum Equations,
- The First Law of Thermodynamics,
- The Second Law of Thermodynamics,
- equations of state, e.g., the ideal gas law, and
- various concepts from thermodynamics.

In addition, we'll require knowledge of calculus, vector calculus, and differential equations (ODEs and PDEs).

13.2. Equations of State

Rather than duplicate what has been previously presented, the reader is encouraged to review Chapter 3 and, specifically, the sections on thermodynamic properties applied to ideal gases. Since compressibility effects become significant when the flow speed is larger than approximately one-third the speed of sound in the fluid, the compressibility of liquids is rarely considered. The speed of sound in water, for example, is nearly $1500\,\mathrm{m\,s^{-1}}$ and, thus, a flow speed of larger than approximately $500\,\mathrm{m\,s^{-1}}$ would be needed for compressibility to become a factor. Such high speed flows for liquids are uncommon. In contrast, the speed of sound in air at typical conditions is $340\,\mathrm{m\,s^{-1}}$ and so the compressibility of air becomes significant when the flow speed is larger than approximately $100\,\mathrm{m\,s^{-1}}$. A $100\,\mathrm{m\,s^{-1}}$ flow speed is easily achieved in normal engineering applications. Thus, compressible flow analyses typically focus on gases, which we generally model as ideal gases. Indeed, the study of compressible flow is sometimes referred to as “gas dynamics”, reflecting the emphasis on gases.

13.3. One-dimensional Flow

The reader is encouraged to review the section on flow dimensionality in Chapter 1. Much of our analysis of gas dynamics in conduits, e.g., pipes and converging and diverging ducts, will assume one-dimensional flow as an engineering approximation. Of course the flow of a real fluid through a pipe is not one-dimensional due to the no-slip condition at the pipe walls. If the Reynolds number of the flow is large enough, however, the flow may be approximated to be 1D with reasonable accuracy. As a flow’s Reynolds number increases, the velocity profile becomes more blunt-shaped and more closely approaches that of a uniform profile (Figure 13.2). Compressible flows are typically high speed so the Reynolds numbers are large and the 1D assumption is a good one.

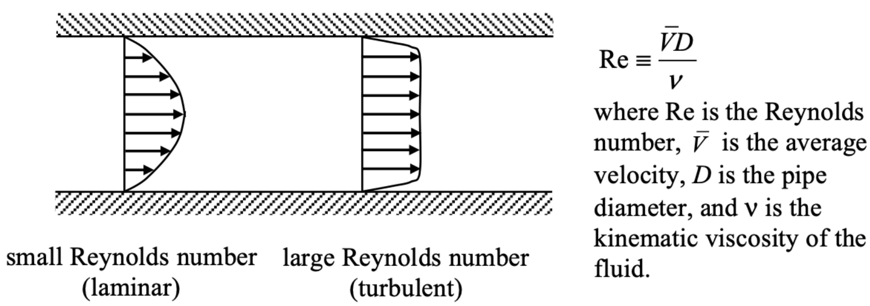


Figure 13.2. Example velocity profiles in a channel flow.

13.3.1. Governing Equations for 1D, Steady Flow

In this section we’ll write the governing equations (Conservation of Mass, Linear Momentum, and the First and Second Laws) for a 1D, steady flow. Most of what we’ll cover in this chapter will make these two assumptions, which are reasonable ones to make in many practical engineering situations.

Conservation of Mass: Apply Conservation of Mass to the control volume shown in Figure 13.3,

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}(\rho\mathbf{u}_{\mathrm{rel}}\cdot d\mathbf{A}) = 0, \tag{13.1}$$

where,

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow)}, \tag{13.2}$$

$$\int_{CS}(\rho\mathbf{u}_{\mathrm{rel}}\cdot d\mathbf{A}) = (-\rho VA) + [\rho VA + d(\rho VA)] = d(\rho VA). \tag{13.3}$$

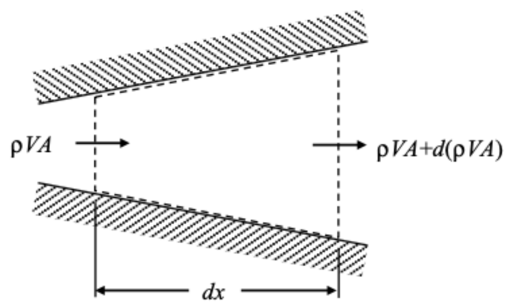


Figure 13.3. The control volume for applying Conservation of Mass.

Substituting and simplifying,

$$\boxed{d\left(\rho V A\right) = 0 \quad \text{or} \quad \dot{m} = \text{constant}}. \tag{13.4}$$

Linear Momentum Equations: Apply the Linear Momentum Equation in the x direction to the control volume

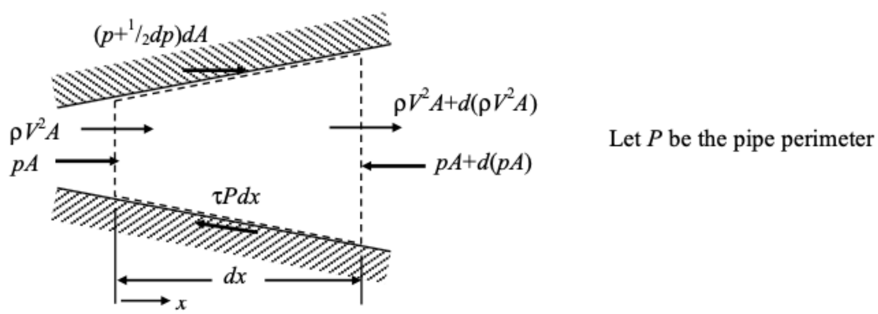


Figure 13.4. The control volume for applying the Linear Momentum Equation in the x direction.

shown in Figure 13.4,

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}, \tag{13.5}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}, \tag{13.6}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \left(-\rho V^2 A\right) + \left[\rho V^2 A + d\left(\rho V^2 A\right)\right] = d\left(\rho V^2 A\right), \tag{13.7}$$

$$F_{B,x} = 0 \quad \text{(most compressible flows involve gases and, thus, the body forces are negligible)}, \tag{13.8}$$

$$F_{S,x} = (pA) - \left[pA + d\left(pA\right)\right] + \left[\left(p + \frac{1}{2}dp\right) dA\right] - \tau P dx = -d\left(pA\right) + pdA - \tau P dx = -Adp - \tau P dx. \tag{13.9}$$

Note that higher-order terms have been neglected in the previous expression and that the friction force acts only in the x-direction since the boundaries vary smoothly (the slope is small, no discontinuities).

Re-write the friction force term using a hydraulic diameter, D_H, defined as,

$$D_H := \frac{4A}{P}, \tag{13.10}$$

and friction factor,

$$\tau = f_F\left(\frac{1}{2}\rho V^2\right) \quad f_F \text{ is known as a Fanning friction factor,} \tag{13.11}$$

$$\tau = \frac{1}{4}f_D\left(\frac{1}{2}\rho V^2\right) \quad f_D \text{ is known as a Darcy friction factor } (f_F = 4f_D), \tag{13.12}$$

so that the Linear Momentum Equation becomes,

$$dp + \rho V dV + \left(\frac{1}{2}\rho V^2\right)\left(\frac{4f_F}{D_H}\right)dx = 0, \tag{13.13}$$

or,

$$\boxed{\frac{dp}{\rho} + V dV + \left(\frac{1}{2}V^2\right)\left(\frac{4f_F}{D_H}\right)dx = 0}. \tag{13.14}$$

Notes:

(1) Gravitational effects have been neglected in the previous analysis since, when dealing with gases, gravitational effects are typically very small compared to other terms in the Momentum Equation.
(2) The Darcy friction factor, f_D, is the friction factor used in the Moody diagram for pipe flows.
(3) For a frictionless flow, Eq. (13.14) integrates to,

$$\int \frac{dp}{\rho} + \frac{1}{2}V^2 = \text{constant}. \tag{13.15}$$

The integral appears because the pressure can, in general, be a function of the density.
(4) For an incompressible fluid, ρ = constant so that after integration along a streamline (recall this is 1D flow),

$$\int \frac{dp}{\rho} = \frac{p}{\rho} + \text{constant}. \tag{13.16}$$

(5) For an ideal gas ($p = \rho RT$) undergoing an isothermal process (T = constant),

$$\int \frac{dp}{\rho} = \int \frac{d(\rho RT)}{\rho} = RT_0 \int \frac{d\rho}{\rho}, \tag{13.17}$$

$$\therefore \int \frac{dp}{\rho} = RT_0 \ln\left(\frac{\rho}{\rho_0}\right), \tag{13.18}$$

where T_0 and ρ_0 are a reference temperature and density, respectively.
(6) For an ideal gas undergoing an isentropic process (s = constant),

$$ds = 0 = c_p(T)\frac{dT}{T} - R\frac{dp}{p}, \tag{13.19}$$

$$dp = \frac{c_p(T)}{R}(\rho RT)\frac{dT}{T}, \tag{13.20}$$

$$dp = \rho c_p(T) dT. \tag{13.21}$$

Thus,

$$\int \frac{dp}{\rho} = \int c_p(T)dT = \int dh = h + \text{constant}, \tag{13.22}$$

where h is the specific enthalpy. Note that if the ideal gas has constant specific heats, i.e., is a "perfect" gas, then $\Delta h = c_p \Delta T$.

First Law of Thermodynamics Apply the First Law of Thermodynamics to the control volume shown in Figure 13.5,

$$\frac{d}{dt}\int_{CV} e\rho dV + \int_{CS}\left(h + \frac{1}{2}V^2 + gz\right)(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \delta\dot{Q}_{\text{into CV}} + \dot{W}_{\text{other,on CV}}, \tag{13.23}$$

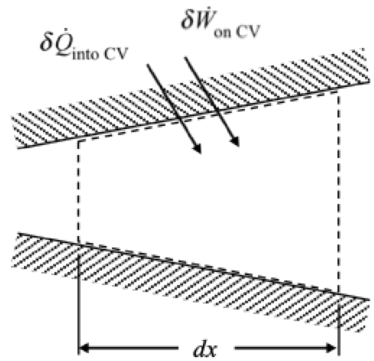


Figure 13.5. The control volume for applying the First Law of Thermodynamics.

where,

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady flow)}, \tag{13.24}$$

$$\begin{aligned}\int_{CS}\left(h+\frac{1}{2}V^2+gz\right)(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}) &= \left\{-\left(h+\frac{1}{2}V^2\right)+\left[\left(h+\frac{1}{2}V^2\right)+d\left(h+\frac{1}{2}V^2\right)\right]\right\}\dot{m}\\ &= d\left(h+\frac{1}{2}V^2\right)\dot{m} \quad (gz \text{ is negligible for gases}),\end{aligned} \tag{13.25}$$

$$\int_{CS}\delta\dot{Q}_{\text{into CV}} = \delta\dot{q}_{\text{into CV}} \quad (\delta\dot{q}_{\text{into CV}} \text{ is the rate of energy addition via heat transfer per unit volume}), \tag{13.26}$$

$$\dot{W}_{\text{other,on CV}} = 0 \quad \text{(assuming no work other than pressure work)}. \tag{13.27}$$

Substitute and simplify,

$$d\left(h+\frac{1}{2}V^2\right)\dot{m} = \delta\dot{q}_{\text{into CV}} \quad \text{or} \quad d\left(h+\frac{1}{2}V^2\right) = \delta q_{\text{into CV}}, \tag{13.28}$$

where $\delta q_{\text{into CV}}$ is the rate of energy transfer via heat transfer into the control volume per unit mass of the fluid.

Notes:

(1) For an adiabatic flow ($\delta q_{\text{into CV}} = 0$), the First Law becomes,

$$h+\frac{1}{2}V^2 = \text{constant}, \tag{13.29}$$

which is the same expression as what is obtained from the Linear Momentum Equation for an ideal gas undergoing an isentropic process (refer to Eqs. (13.15) and (13.22)).

Second Law of Thermodynamics Apply the Second Law of Thermodynamics to the control volume shown in Figure 13.6,

$$\frac{d}{dt}\int_{CV} s\rho dV + \int_{CS} s(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}) = \int_{CS}\frac{\delta\dot{Q}_{\text{into CV}}}{T} + \dot{\sigma}, \tag{13.30}$$

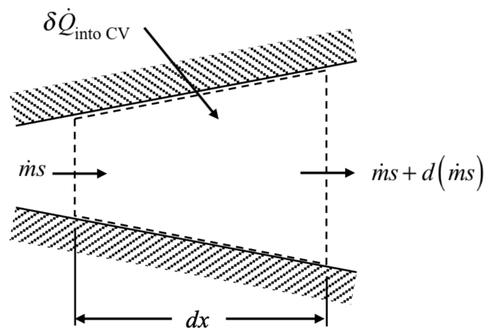


Figure 13.6. The control volume for applying the Second Law of Thermodynamics.

where,

$$\frac{d}{dt}\int_{CV} s\rho dV = 0 \quad \text{(steady flow)}, \tag{13.31}$$

$$\int_{CS} s\left(\rho\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \left[-s + (s + ds)\right]\dot{m} = (ds)\dot{m}, \tag{13.32}$$

$$\int_{CS} \frac{\delta\dot{Q}_{\text{into CV}}}{T} = \frac{\delta\dot{q}_{\text{into CV}}}{T}, \tag{13.33}$$

$$\dot{\sigma} = \dot{\sigma}. \tag{13.34}$$

Substitute and simplify,

$$(ds)\dot{m} = \frac{\delta\dot{q}_{\text{into CV}}}{T} + \dot{\sigma} \quad \text{or} \quad ds = \frac{\delta q_{\text{into CV}}}{T} + \sigma, \tag{13.35}$$

where $\delta q_{\text{into CV}}$ is the heat added to the control volume per unit mass of the flowing gas and σ is the rate of entropy generation per unit mass of the flowing gas.

Notes:

(1) For an adiabatic flow, $\delta\dot{q}_{\text{into CV}} = 0$, and so,

$$ds = \sigma \geq 0. \tag{13.36}$$

The equality in this equation only holds if the flow is internally reversible.

13.4. Speed of Sound

The speed of sound, c, in a substance is the speed at which infinitesimal pressure disturbances propagate through the surrounding substance. To understand how the speed of sound depends on the substance properties, let's examine the following model.

Consider a wave moving at velocity, c, through a stagnant fluid. Across the wave, the fluid properties such as pressure, p, density, ρ, temperature, T, and the velocity, V, can all change as shown in Figure 13.7. Now let's

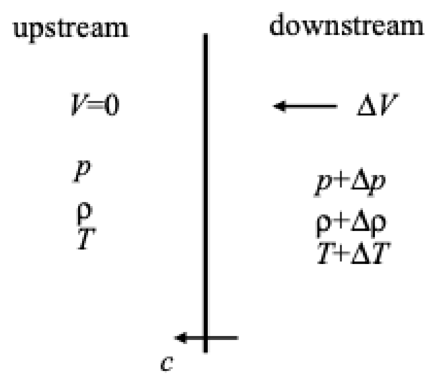


Figure 13.7. Flow across a pressure wave viewed from a frame of reference fixed to the ground.

change our frame of reference such that it moves with the wave, as shown in Figure 13.8. Apply Conservation

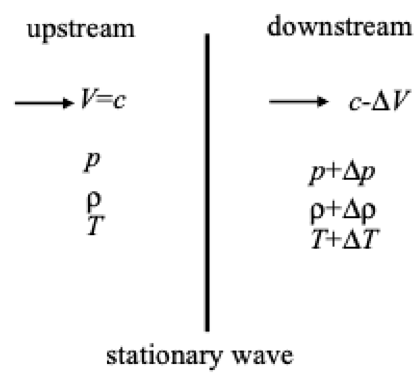


Figure 13.8. Flow across a pressure wave viewed from a frame of reference fixed to the wave.

of Mass and the Linear Momentum Equation to a thin control volume of cross-sectional area A surrounding the wave (Figure 13.9). From Conservation of Mass,

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0, \tag{13.37}$$

where,

$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(the flow is steady in the frame of reference fixed to the wave)}, \tag{13.38}$$

$$\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = -\rho c A + (\rho + \Delta\rho)(c - \Delta V)A. \tag{13.39}$$

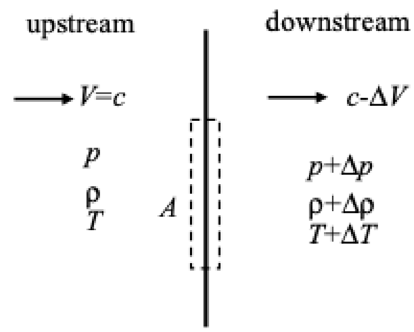


Figure 13.9. A thin control volume applied across the pressure wave.

Substitute and simplify,

$$\rho c A = (\rho + \Delta\rho)(c - \Delta V)A, \tag{13.40}$$

$$\rho c = \rho c - \rho\Delta V + \Delta\rho c - \Delta\rho\Delta V, \tag{13.41}$$

$$\Delta V = \frac{c\Delta\rho}{\rho + \Delta\rho}. \tag{13.42}$$

From the Linear Momentum Equation applied in the streamwise direction,

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}, \tag{13.43}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(the flow is steady in the frame of reference fixed to the wave)}, \tag{13.44}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}c + \dot{m}(c - \Delta V) = -\dot{m}\Delta V = -\rho c A \Delta V, \tag{13.45}$$

$$F_{B,x} = 0, \tag{13.46}$$

$$F_{S,x} = pA - (p + \Delta p)A = -\Delta p A. \tag{13.47}$$

Substituting and simplifying,

$$-\rho c A \Delta V = -\Delta p A, \tag{13.48}$$

$$c = \frac{\Delta p}{\rho \Delta V}. \tag{13.49}$$

Making use of the relation derived from Conservation of Mass (Eq. (13.42)),

$$c = \frac{\Delta p(\rho + \Delta\rho)}{\rho c \Delta\rho}, \tag{13.50}$$

$$c^2 = \frac{\Delta p}{\Delta\rho}\left(1 + \frac{\Delta\rho}{\rho}\right). \tag{13.51}$$

For a sound wave, the changes across the wave are infinitesimally small (sound waves are defined as being infinitesimally weak pressure waves) so the previous equation becomes,

$$c^2 = \lim_{\Delta \to d} \frac{\Delta p}{\Delta\rho}\left(1 + \frac{\Delta\rho}{\rho}\right) = \frac{\partial p}{\partial\rho}. \tag{13.52}$$

We also need to specify the process by which these changes occur since pressure, in general, is a function of not only the density, but other properties as well, such as temperature. Since the changes across the wave are infinitesimally small and, thus, the velocity and temperature gradients are infinitesimally small, we can regard the wave as an internally reversible process. Additionally, the temperature gradient on either side of the wave is small so there is negligible heat transfer into the control volume. Hence, the process is adiabatic. As a result, the changes across a sound wave occur isentropically (an adiabatic, internally reversible process is isentropic),

$$\boxed{c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s} \quad \underline{\text{speed of sound in a continuous substance}} \tag{13.53}$$

Notes:

(1) Note that Eq. (13.53) is the speed of sound in *any* substance. It's not limited to just fluids.
(2) If the wave is not "weak", i.e., the changes in the flow properties across the wave are not infinitesimal, then viscous effects and temperature gradients within the wave will be significant and the process can no longer be considered isentropic. We will discuss this situation later when examining shock waves (Section 13.17).
(3) Note that according to Eq. (13.51) the stronger the wave, i.e., the greater $\Delta\rho$, the faster the wave will propagate. This effect will also be examined when discussing shock waves.
(4) For an ideal gas undergoing an isentropic process ($ds = 0$),

$$ds = 0 = c_p \frac{dT}{T} - R\frac{dp}{p} = c_v \frac{dT}{T} - R\frac{d\rho}{\rho}, \tag{13.54}$$

$$\frac{R}{c_v}\frac{d\rho}{\rho} = \frac{R}{c_p}\frac{dp}{p} \implies \left.\frac{\partial p}{\partial \rho}\right|_s = \frac{c_p}{c_v}\frac{p}{\rho} = k\frac{p}{\rho} = kRT. \tag{13.55}$$

Substituting into Eq. (13.53), the speed of sound for an ideal gas is,

$$\boxed{c = \sqrt{kRT}} \quad \underline{\text{speed of sound in an ideal gas}}. \tag{13.56}$$

(a) The absolute temperature must be used when calculating the speed of sound since the Ideal Gas Law was used in its derivation.
(b) The speed of sound in air ($k = 1.4, R = 287\,\mathrm{J\,kg^{-1}\,K^{-1}}$) at standard conditions ($T = 288\,\mathrm{K}$) is $340\,\mathrm{m\,s^{-1}}$ ($= 1115\,\mathrm{ft\,s^{-1}} \approx 1/5\,\mathrm{mi\,s^{-1}}$). This value helps explain the rule of thumb whereby the distance to a thunderstorm in miles is roughly equal to the number of seconds between a lightening flash and the corresponding thunder clap divided by five.
(c) It is not unexpected that the speed of sound is proportional to the square root of the temperature. Since disturbances travel through the gas as a result of molecular impacts, we should expect the speed of the disturbance to be proportional to the speed of the molecules. The temperature is equal to the random kinetic energy of the molecules and so the molecular speed is proportional to the square root of the temperature. Thus, the speed of sound is proportional to the square root of the temperature.

(5) Equation (13.53) can also be written in terms of the bulk modulus. The bulk modulus, E_v, of a substance is a measure of the compressibility of the substance. It is defined as the ratio of a differential applied pressure to the resulting differential change in volume of a substance at a given volume (refer to Figure 13.10),

$$E_v \coloneqq \frac{\partial p}{(-\partial V/V)} = \rho\frac{\partial p}{\partial \rho}. \tag{13.57}$$

Notes:

(a) $dp > 0 \implies dV < 0 \implies E_v > 0$.
(b) From Conservation of Mass, $dV/V = -d\rho/\rho$.
(c) $E_v \uparrow \implies$ compressibility $\downarrow$.

The isentropic bulk modulus, $E_v|_s$, is defined as,

$$E_v|_s \coloneqq \left.\frac{\partial p}{-(dV/V)}\right|_s = \rho\left.\frac{\partial p}{\partial \rho}\right|_s. \tag{13.58}$$

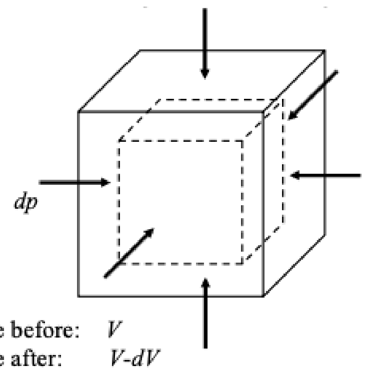


Figure 13.10. A schematic illustrating the concept of the bulk modulus.

Thus, the speed of sound can also be written as,

$$\boxed{c^2 = \frac{E_v|_s}{\rho}} \quad \underline{\text{alternate speed of sound expression}}. \tag{13.59}$$

Notes:

(a) The isentropic bulk modulus for air is $E_v|_s = k\rho RT$.
(b) The isentropic bulk modulus for water is 2.19 GPa. Thus, the speed of sound in water ($\rho = 1000\,\mathrm{kg/m^3}$) is $1480\,\mathrm{m\,s^{-1}}$ ($= 4900\,\mathrm{ft\,s^{-1}} \approx 1\,\mathrm{mi\,s^{-1}} \approx$ 5X faster than the speed of sound in air at standard conditions).
(c) For solids, the bulk modulus, E_v, is related to the modulus of elasticity, E, and Poisson's ratio, ν, by,

$$\frac{E_v}{E} = 3(1 - 2\nu). \tag{13.60}$$

For many metals, e.g., steel and aluminium, the Poisson's ratio is approximately $\nu \approx 1/3$ so that $E_v/E \approx 1$. The speed of sound in stainless steel ($E = 163\,\mathrm{GPa}; \rho = 7800\,\mathrm{kg/m^3}$) is $4570\,\mathrm{m\,s^{-1}}$ ($= 15\,000\,\mathrm{ft\,s^{-1}} \approx 3\,\mathrm{mi\,s^{-1}} \approx$ 3X faster than the speed of sound in water).

(6) The Mach number, Ma is a dimensionless parameter that is commonly used in the discussion of compressible flows. The Mach number is defined as,

$$\boxed{\mathrm{Ma} := \frac{V}{c}}, \tag{13.61}$$

where V is the flow speed and c is the speed of sound in the flow.

Notes:

(a) Compressible flows are often classified by their Mach number:
$\mathrm{Ma} < 1$ subsonic
$\mathrm{Ma} = 1$ sonic
$\mathrm{Ma} > 1$ supersonic
Additional sub-classifications include:
$\mathrm{Ma} < 0.3$ incompressible
$\mathrm{Ma} \approx 1$ transonic
$\mathrm{Ma} > 5$ hypersonic
(b) The square of the Mach number, Ma^2, is a measure of a flow's macroscopic kinetic energy to its microscopic kinetic energy.

(7) The change in the properties across a sound wave can be found from the following analysis. From Conservation of Mass applied to the control volume shown in Figure 13.9 and Eq. (13.42), making

use of the fact that the property changes across the sound wave are infinitesimally small,

$$\frac{dV}{c} = \frac{d\rho}{\rho}. \tag{13.62}$$

For an ideal gas,

$$p = \rho RT \implies \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}. \tag{13.63}$$

Combining Eqs. (13.53), (13.56), and (13.62) and simplifying,

$$\frac{\rho}{p}c^2 = \frac{dp}{p}\frac{\rho}{d\rho} \implies \frac{kRT}{RT} = \frac{dp}{p}\frac{c}{dV}, \tag{13.64}$$

$$\boxed{\frac{dV}{c} = \frac{d\rho}{\rho} = \frac{1}{k}\frac{dp}{p}}. \tag{13.65}$$

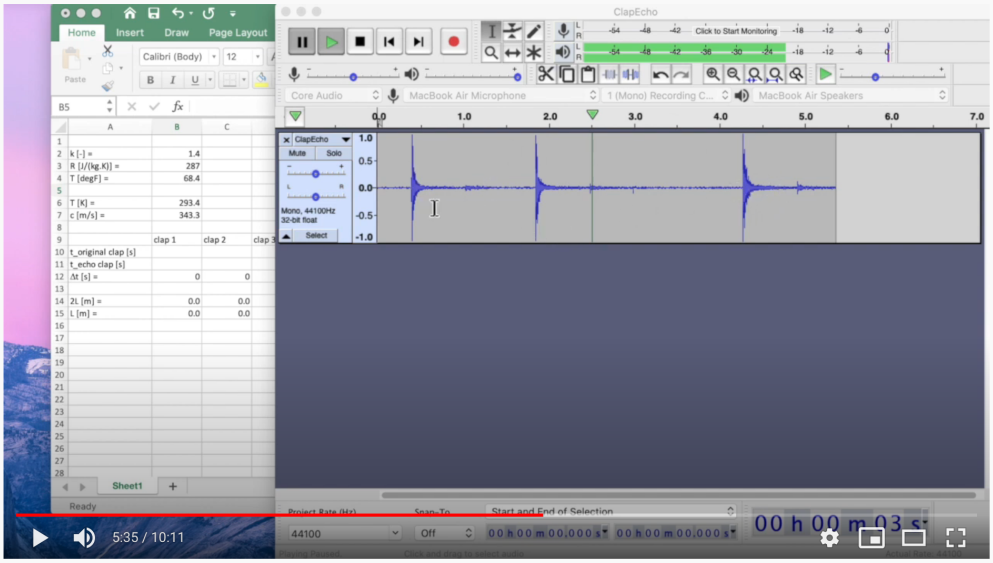
Now combine Eqs. (13.63) and (13.65),

$$\frac{dp}{p} = \frac{1}{k}\frac{dp}{p} + \frac{dT}{T}, \tag{13.66}$$

$$\boxed{\frac{dT}{T} = \left(\frac{k-1}{k}\right)\frac{dp}{p}}. \tag{13.67}$$

Thus, across a compression sound wave ($dp > 0$): $dV > 0, d\rho > 0$, and $dT > 0$. Across a rarefaction sound wave, also known as an expansion wave ($dp < 0$): $dV < 0, d\rho < 0$, and $dT < 0$.

Example:
Refer to https://www.youtube.com/watch?v=9R4xhCoBz9Y for a video demonstration that shows how a distance is calculated using a speed of sound analysis and temperature and transit time measurements.

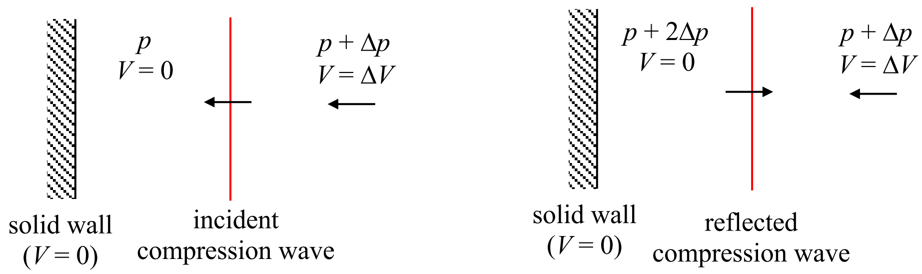


Speed of Sound Demonstration

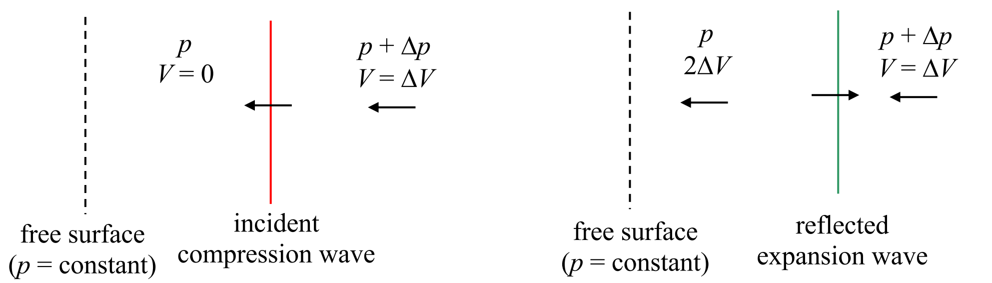
A weak compression pressure wave of magnitude Δp propagates through still air. Discuss the type of reflected wave that occurs (compression or expansion) and the boundary conditions that must be satisfied when the wave strikes normal to, and is reflected from:

a. a solid wall and
b. a free surface boundary (i.e., a surface where the pressure remains constant).

SOLUTION:



At a solid boundary the incident compression wave reflects as a compression wave in order to maintain zero air velocity at the wall. The pressure behind the reflected wave will be $p + 2\Delta p$ (two compressions).



At a free surface boundary the incident compression wave reflects as an expansion wave in order to maintain the free surface pressure. The velocity behind the reflected wave will be $2\Delta V$ in the direction opposite the wave.

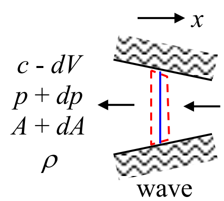
Consider a straight pipe filled with an incompressible liquid. The walls of the pipe are elastic so that the cross-sectional area, A, changes with the internal pressure, p, according to the relation:

$$A = A_0 + A_1 p$$

Thus, the pipe may have different cross-sectional areas at different axial positions depending on the internal pressure at each position. Find the speed of propagation, c, of a small pressure wave traveling along the pipe assuming A_0 and A_1 are known constants and that $A_1 p$ is always small compared with A_0. Give your answer in terms of A_0, A_1, and the density, ρ, of the liquid.

SOLUTION:

Apply conservation of mass and the linear momentum equation to the thin control volume shown below. Use a frame of reference that is fixed to the wave so that the flow appears steady.



Conservation of mass:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho c A + \rho\left(c - dV\right)\left(A + dA\right) \tag{1}$$

Note that the area is a function of the pressure.

$$A = A_0 + A_1 p \quad \text{and} \quad A + dA = A_0 + A_1\left(p + dp\right) \tag{2}$$

Substitute and simplify.

$$-\rho c A + \rho\left(c - dV\right)\left(A + dA\right) = 0$$

$$-\rho c\left(A_0 + A_1 p\right) + \rho\left(c - dV\right)\left[A_0 + A_1\left(p + dp\right)\right] = 0$$

$$-cA_0 - cA_1 p + cA_0 + cA_1\left(p + dp\right) - A_0 dV - dVA_1\left(p + dp\right) = 0$$

$$cA_1 dp - A_0 dV - dVA_1\left(p + dp\right) = 0$$

$$dV = \frac{cA_1 dp}{A_0 + A_1\left(p + dp\right)}$$

$$dV = \frac{cA_1 dp}{A_0 + A_1 p} \quad \text{(Note that } dp \ll p.\text{)} \tag{3}$$

Now apply the linear momentum equation in the x-direction to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m}c - \dot{m}\left(c - dV\right) = \dot{m}dV = \rho c A dV = \rho c\left(A_0 + A_1 p\right)dV \tag{4}$$

$$F_{B,x} = 0 \quad \text{(no body forces since the control volume is infinitesimally thin)}$$

$$\begin{aligned} F_{S,x} &= -pA + \left(p + dp\right)\left(A + dA\right) - \left(p + \tfrac{1}{2}dp\right)dA = -p\left(A_0 + A_1 p\right) + \left(p + dp\right)\left[A_0 + A_1\left(p + dp\right)\right] - pA_1 dp \\ &= pA_1 dp + dpA_0 + pA_1 dp - pA_1 dp \\ &= pA_1 dp + dpA_0 \end{aligned} \tag{5}$$

Substitute and simplify.

$$\rho c\left(A_0 + A_1 p\right)dV = pA_1 dp + dpA_0 = dp\left(A_0 + A_1 p\right)$$

$$\rho c dV = dp \tag{6}$$

Substitute in for dV using Eqn. (3).

$$\rho c\left(\frac{cA_1 dp}{A_0 + A_1 p}\right) = dp$$

$$c^2 = \frac{A_0 + A_1 p}{\rho A_1} \tag{7}$$

(8)

Since $A_1 p \ll A_0$ (given in the problem statement), Eqn. (8) becomes:

$$\boxed{c^2 = \frac{A_0}{\rho A_1}} \tag{9}$$

13.5. The Mach Cone

Consider the propagation of infinitesimal pressure waves, i.e., sound waves, emanating from an object at rest (Figure 13.11). The waves will travel at the speed of sound, c, and form circles (spheres in 3D) with radii depending on the time when the sound wave was emitted.

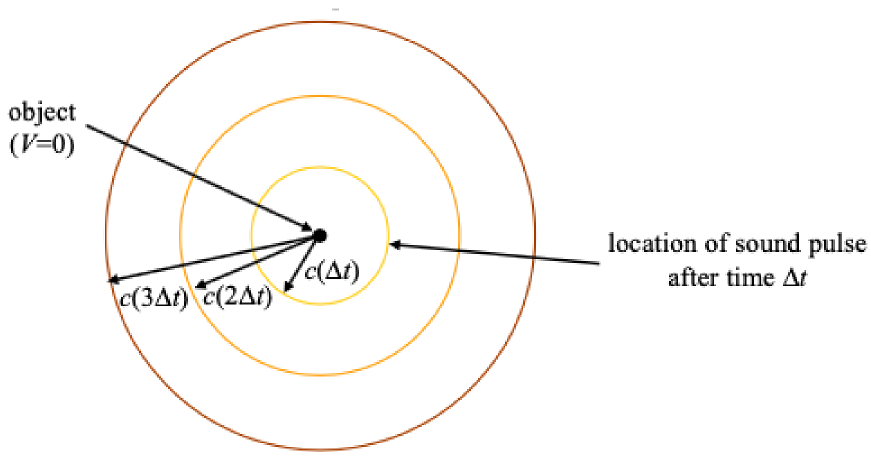


Figure 13.11. Sound waves emanating from an object at rest.

Now consider an object moving at a subsonic speed, $V < c \implies \text{Ma} < 1$ (Figure 13.12). For this case the pressure pulses are more closely spaced in the direction of the object's motion and more widely spaced behind the object. Thus, the frequency of the sound in front of the object increases, while the frequency behind the object decreases. This phenomenon is known as the Doppler Shift.

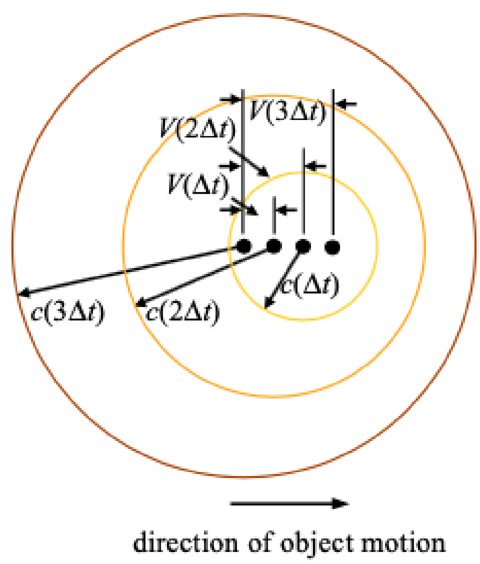


Figure 13.12. Sound waves emanating from an object moving to the right at a subsonic speed ($V < c \implies \text{Ma} < 1$).

Now consider an object traveling at a sonic speed, $V = c \implies \text{Ma} = 1$. Since no wave fronts propagate ahead of the object, an observer standing in front of the object won't hear it approaching until the object reaches the observer. Note that the infinitesimal pressure changes in front of the object begin to "pile up" on one another, producing a sudden, finite pressure change, also known as a shock wave.

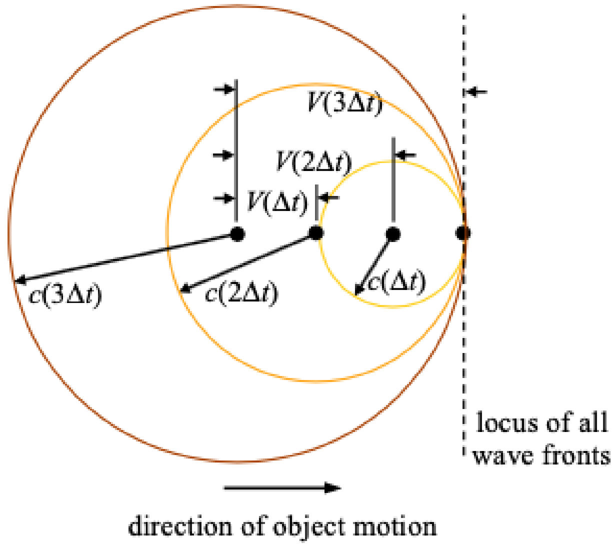


Figure 13.13. Sound waves emanating from an object moving to the right at the sonic speed ($V = c \implies \text{Ma} = 1$).

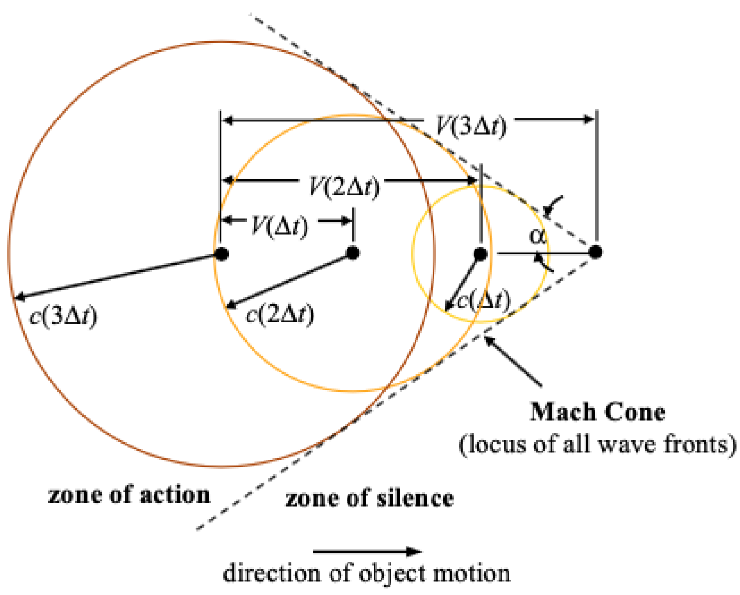


Figure 13.14. Sound waves emanating from an object moving to the right at a supersonic speed ($V > c \implies \text{Ma} > 1$).

Lastly, consider an object travelling at supersonic speeds, $V > c \implies \text{Ma} > 1$ (Figure 13.14). For this case, the object outruns the pressure pulses it generates. The locus of wave fronts forms a cone, which is known as the Mach Cone. The object cannot be heard outside of the Mach Cone and, thus, this region is termed the zone of silence. Inside the cone, which is known as the zone of action, the object can be heard. The angle of the cone, known as the Mach angle, α, is given by,

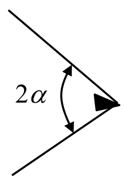
$$\sin\alpha = \frac{c\Delta t}{V\Delta t} = \frac{c}{V} = \frac{1}{\mathrm{Ma}}. \tag{13.68}$$

A projectile in flight carries with it a more or less conical-shaped shock front. From physical reasoning it appears that at great distances from the projectile this shock wave becomes truly conical and changes in velocity and density across the shock become vanishingly small.

Photographs of a bullet in flight show that at a great distance from the bullet the total included angle of the cone is 50.3°. The pressure and temperature of the undisturbed air are 14.62 psia and 73 °F, respectively. Calculate:

a. the velocity of the bullet, in ft/sec, and
b. the Mach number of the bullet relative to the undisturbed air.

SOLUTION:



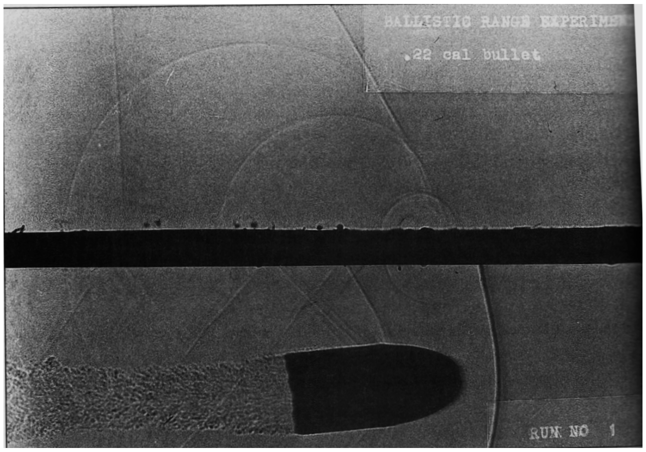
2α = 50.3°
p_{atm} = 14.62 psia
T_{atm} = 73 °F = 533 °R
γ_{air} = 1.4
R_{air} = 53.3 (ft·lb$_f$)/(lb$_m$·°R)

$$\boxed{\sin\alpha = \frac{1}{\text{Ma}}} \Rightarrow \boxed{\text{Ma} = 2.35}$$

$$\text{Ma} = \frac{V}{c} = \frac{V}{\sqrt{\gamma R T}} \Rightarrow \boxed{V = \text{Ma}\sqrt{\gamma R T_{\text{atm}}}} \Rightarrow \boxed{V = 2660\ \text{ft/s}}$$

Video solution: https://www.youtube.com/watch?v=0wf2oeEk9go

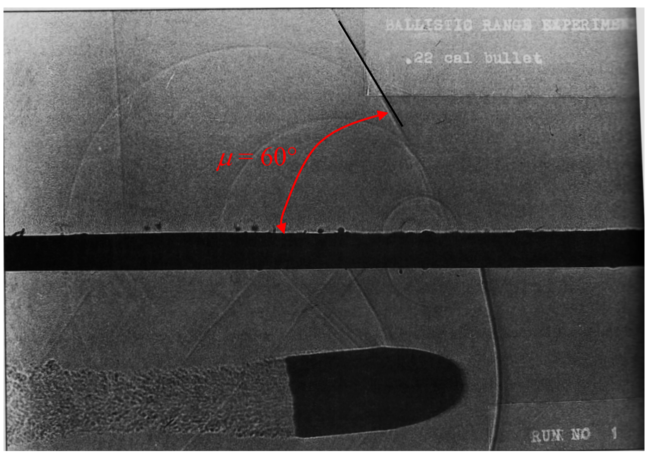
Determine the Mach number of the .22 caliber bullet shown below. Note that the plate in the figure has holes through which weak pressure disturbances can propagate.



If the temperature of the air at which the test is conducted is 70 °F, determine the speed of the bullet.

SOLUTION:

Determine the Mach angle from the photograph. Note that since the waves above the plate are very weak, they will be Mach waves.



The angle of the Mach waves is related to the Mach number via:

$$\boxed{\sin\mu = \frac{1}{\text{Ma}}} \text{ and } \mu = 60° \Rightarrow \boxed{\text{Ma} = 1.2} \quad (1)$$

Now determine the speed of the bullet from the definition of the Mach number.

$$\text{Ma} = \frac{V}{c} \Rightarrow V = c\text{Ma} \quad (2)$$

$$\boxed{\therefore V = \text{Ma}\sqrt{\gamma RT}} \quad (3)$$

For air at 70 °F (530 °R) and Ma = 1.2, $\gamma = 1.4$, $R = 53.3$ (lb$_f$·ft)/(lb$_m$·°R) $\Rightarrow$ $\boxed{V = 1350 \text{ ft/s}}$

13.6. Adiabatic, Steady, 1D Compressible Flow of a Perfect Gas

Now let's consider the 1D, steady, adiabatic flow of a compressible gas. Recall that from the Energy Equation we have,

$$dh + VdV = 0. \tag{13.69}$$

For an ideal gas we can re-write the specific enthalpy in terms of the specific heat at constant pressure, c_p, which is a function of temperature, in general, and the absolute temperature, T,

$$dh = c_p(T)dT. \tag{13.70}$$

Substituting,

$$c_p(T)dT + VdV = 0. \tag{13.71}$$

Integrating along a stream tube,

$$\int_{T_{\text{in}}}^{T_{\text{out}}} c_p(T)dT + \frac{1}{2}V^2 = \text{constant}. \tag{13.72}$$

If the gas can be assumed perfect, i.e., $c_p = \text{constant}$, then the previous equation becomes,

$$c_pT + \frac{1}{2}V^2 = \text{constant}, \tag{13.73}$$

$$\boxed{T + \frac{V^2}{2c_p} = \text{constant}}. \tag{13.74}$$

We can re-write this equation in terms of the Mach number,

$$\text{Ma} = \frac{V}{c} = \frac{V}{\sqrt{kRT}} \implies V^2 = (kRT)\text{Ma}^2 \implies T + \frac{kRT\text{Ma}^2}{2c_p} = \text{constant}. \tag{13.75}$$

Substituting the following ideal gas relation,

$$\frac{R}{c_p} = \frac{k-1}{k}, \tag{13.76}$$

results in,

$$\boxed{T\left(1 + \frac{k-1}{2}\text{Ma}^2\right) = \text{constant}} \quad \underline{\text{adiabatic, 1D, steady flow of a perfect gas}}. \tag{13.77}$$

If the flow can also be considered internally reversible, making the flow isentropic, then we can use the isentropic relations for a perfect gas,

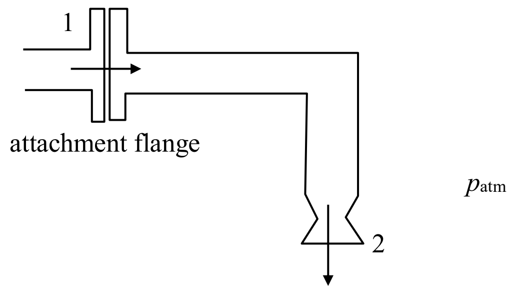
$$p = (\text{constant})T^{\frac{k}{k-1}} \quad \text{and} \quad \rho = (\text{constant})T^{\frac{1}{k-1}}, \tag{13.78}$$

to give,

$$\boxed{p\left(1 + \frac{k-1}{2}\text{Ma}^2\right)^{\frac{k}{k-1}} = \text{constant}} \quad \text{and} \quad \boxed{\rho\left(1 + \frac{k-1}{2}\text{Ma}^2\right)^{\frac{1}{k-1}} = \text{constant}}. \tag{13.79}$$

These relations are for <u>isentropic, 1D, steady flow of a perfect gas</u>.

A steady flow of air passes through the elbow-nozzle assembly shown. At the inlet (1), the pipe diameter is $D_1 = 0.1524$ m and the air properties are $p_1 = 871.7$ kPa (abs), $T_1 = 529.0$ K, and $V_1 = 230.4$ m/s. The air is expanded through a converging-diverging nozzle discharging into the atmosphere where $p_{atm} = 101.3$ kPa (abs). At the nozzle exit (2), the nozzle diameter is $D_2 = 0.3221$ m and the air properties are $T_2 = 475.7$ K and $V_2 = 400.0$ m/s.



a. Is the flow through the elbow-nozzle assembly adiabatic?
b. Determine the components of the force in the attachment flange required to hold the elbow-nozzle assembly in place. You may neglect the effects of gravity.

SOLUTION:

If the flow is adiabatic in going from 1 to 2, then the energy equation will give:

$$T_1 + \frac{V_1^2}{2c_P} = T_2 + \frac{V_2^2}{2c_P} \tag{1}$$

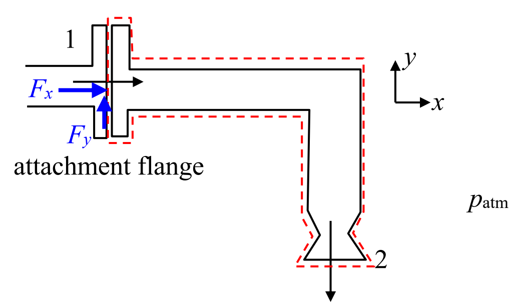
Using the given data:

$T_1 = 529.0$ K
$V_1 = 230.4$ m/s
$c_P = 1004$ J/(kg·K)
$T_2 = 475.7$ K
$V_2 = 400.0$ m/s

$$\Rightarrow \quad T_1 + \frac{V_1^2}{2c_P} = 555.4\text{ K and } \quad T_2 + \frac{V_2^2}{2c_P} = 555.1\text{ K}$$

Since the stagnation temperatures are approximately the same, $\boxed{\text{the flow can be considered adiabatic in going from 1 to 2.}}$

To determine the force components, apply the linear momentum equation to the control volume shown below using the indicated fixed frame of reference.



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = \left(V_1\right)\left(-\rho_1 V_1 \frac{\pi D_1^2}{4}\right) = -\rho_1 V_1^2 \frac{\pi D_1^2}{4}$$

$$F_{B,x} = 0$$

$$F_{S,x} = \left(p_1 - p_{atm}\right)\frac{\pi D_1^2}{4} + F_x$$

Substitute and simplify.

$$-\rho_1 V_1^2 \frac{\pi D_1^2}{4} = \left(p_1 - p_{atm}\right)\frac{\pi D_1^2}{4} + F_x$$

$$\boxed{F_x = -\rho_1 V_1^2 \frac{\pi D_1^2}{4} - \left(p_1 - p_{atm}\right)\frac{\pi D_1^2}{4}} \tag{2}$$

Using the given numerical data:

ρ_1 = $p_1/(RT_1)$ = (871.7 kPa)/[287 J/(kg·K) ·529 K] = 5.738 kg/m^3
V_1 = 230.4 m/s
D_1 = 0.1524 m
p_1 = 871.7 kPa (abs)
p_{atm} = 101.3 kPa (abs)
$\Rightarrow \boxed{F_x = \text{-19.60 kN}}$

Also note that:

$$\dot{m}_1 = \rho_1 V_1 \frac{\pi D_1^2}{4} = 24.11 \text{ kg/s} \tag{3}$$

Now consider the y-direction.

$$\frac{d}{dt}\int_{CV} u_y \rho dV + \int_{CS} u_y \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,y} + F_{S,y}$$

where

$$\frac{d}{dt}\int_{CV} u_y \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u_y \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = \left(-V_2\right)\left(\rho_2 V_2 \frac{\pi D_2^2}{4}\right) = -\rho_2 V_2^2 \frac{\pi D_2^2}{4}$$

$$F_{B,y} = 0$$

$$F_{S,y} = \left(p_2 - p_{atm}\right)\frac{\pi D_2^2}{4} + F_y$$

Substitute and simplify.

$$-\rho_2 V_2^2 \frac{\pi D_2^2}{4} = \left(p_2 - p_{atm}\right)\frac{\pi D_2^2}{4} + F_y$$

$$\boxed{F_y = -\rho_2 V_2^2 \frac{\pi D_2^2}{4} - \left(p_2 - p_{atm}\right)\frac{\pi D_2^2}{4}} \tag{4}$$

Note that from conservation of mass on the same control volume:

$$\dot{m}_1 = \dot{m}_2 = \rho_2 V_2 \frac{\pi D_2^2}{4} \tag{5}$$

The pressure at point 2 will depend on whether the flow at that point is subsonic or supersonic.

$$\mathrm{Ma}_2 = \frac{V_2}{c_2} = \frac{V_2}{\sqrt{\gamma R T_2}} \tag{6}$$

Using the given data:

V_2 = 400.0 m/s

γ = 1.4

R = 287 J/(kg·K)

T_2 = 475.7 K

$\Rightarrow$ c_2 = 437.2 m/s $\Rightarrow$ Ma_2 = 0.9148 $\Rightarrow$ The flow at point 2 is subsonic. $\Rightarrow$ The pressure at point 2 is equal to atmospheric pressure, i.e., p_2 = 101.3 kPa (abs).

Substituting the given numerical data in Eq. (4) gives $\boxed{F_y = -9.644 \text{ kN}}$.

13.7. Stagnation and Sonic Conditions

It's convenient to choose some useful reference points in the flow where we can evaluate the constants in Eqs. (13.77) and (13.79). Two such reference points are commonly used in compressible flows: stagnation conditions and sonic conditions.

Stagnation Conditions: Stagnation conditions are those conditions that would occur if the fluid is brought to rest ($V = 0 \implies \mathrm{Ma} = 0$). These conditions are typically indicated by the subscript "$_0$". Equations (13.77) and (13.79) can be written in terms of stagnation conditions as,

$$\boxed{\frac{T}{T_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{-1}} \quad \underline{\text{adiabatic, steady, 1D flow of a perfect gas}}, \tag{13.80}$$

$$\boxed{\frac{p}{p_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{k}{1-k}}} \quad \underline{\text{isentropic, steady, 1D flow of a perfect gas}}, \tag{13.81}$$

$$\boxed{\frac{\rho}{\rho_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{1}{1-k}}} \quad \underline{\text{isentropic, steady, 1D flow of a perfect gas}}. \tag{13.82}$$

We can also determine the speed of sound at stagnation conditions using the fact that $c = \sqrt{kRT}$,

$$\boxed{\frac{c}{c_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{-\frac{1}{2}}} \quad \underline{\text{adiabatic, steady, 1D flow of a perfect gas}}. \tag{13.83}$$

These various stagnation ratios are plotted in Figure 13.15 as a function of Mach number for air.

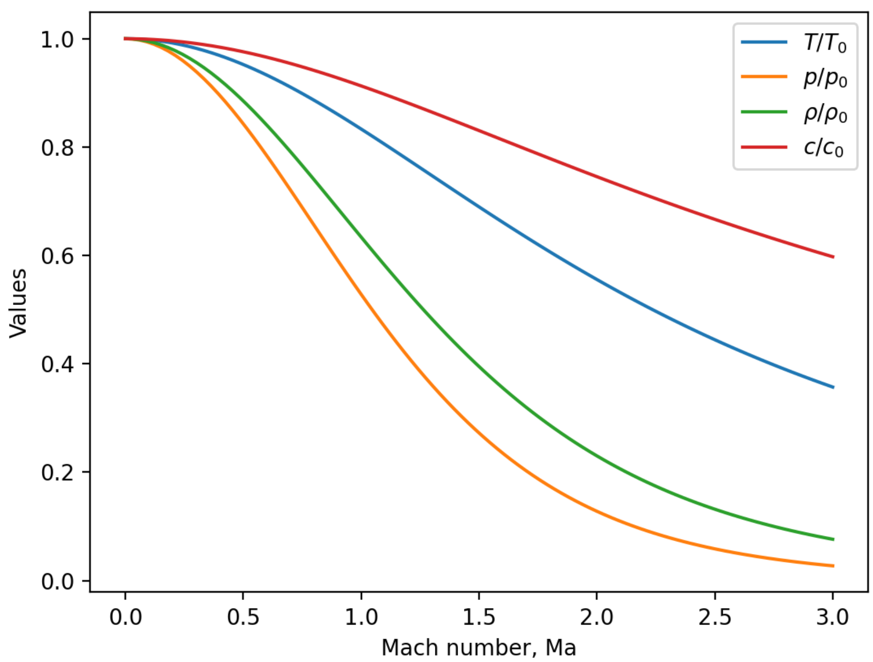


Figure 13.15. Various stagnation ratios, referring to Eqs. (13.80) - (13.83), plotted as a function of the Mach number. These ratios are for a specific heat ratio of $k = 1.4$, which corresponds to air.

Notes:

(1) Stagnation conditions are also commonly referred to as total conditions, given by the subscript "T".
(2) Stagnation conditions can be determined even for a moving fluid. The fluid doesn't necessarily have to be at rest to state its stagnation conditions. To determine stagnation conditions we only need to imagine the conditions *if* the flow is brought to rest.
(3) Equations (13.81) and (13.82) are for a flow brought to rest isentropically. Equations (13.80) and (13.83) are for a flow brought to rest adiabatically.
(4) Tables listing the values of Eqs. (13.80) - (13.83) for various Mach numbers are typically given in the back of most textbooks concerning compressible flows.
(5) Note that the stagnation temperature is greater than the flow temperature since when the flow is decelerated to zero velocity, the macroscopic kinetic energy is converted into internal energy (microscopic kinetic energy) and, thus, the temperature increases.
(6) The stagnation pressure is a significant property for a flow because it's directly related to the amount of work that can be extracted from the flow. For example, imagine bringing a flow to rest so we have stagnation conditions within the tank shown in Figure 13.16. The larger the stagnation pressure in the tank, the greater the force we can exert on the piston that can be used to perform useful work.

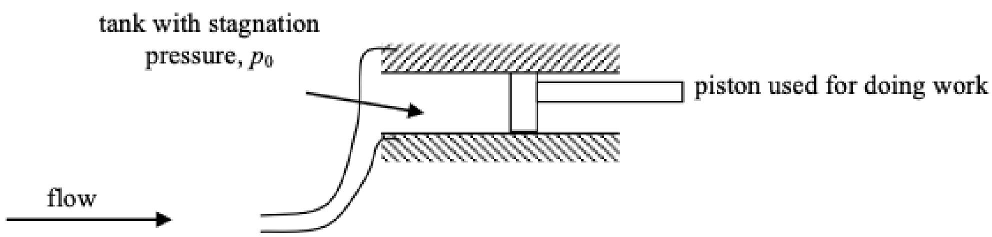


Figure 13.16. An illustration showing how stagnation pressure is related to the ability to do work.

Sonic Conditions: Another convenient reference point is where the flow has a Mach number of one (Ma = 1). Conditions where the Mach number is one are known as sonic conditions and are typically specified using the superscript "*".

Equations (13.80) - (13.83) evaluated at sonic conditions are:

$$\boxed{\frac{T^*}{T_0} = \left(1 + \frac{k-1}{2}\right)^{-1}} \quad \text{adiabatic, steady, 1D flow of a perfect gas,} \tag{13.84}$$

$$\boxed{\frac{p^*}{p_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{1-k}}} \quad \text{isentropic, steady, 1D flow of a perfect gas,} \tag{13.85}$$

$$\boxed{\frac{\rho^*}{\rho_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{1}{1-k}}} \quad \text{isentropic, steady, 1D flow of a perfect gas,} \tag{13.86}$$

$$\boxed{\frac{c^*}{c_0} = \left(1 + \frac{k-1}{2}\right)^{-\frac{1}{2}}} \quad \text{adiabatic, steady, 1D flow of a perfect gas.} \tag{13.87}$$

Notes:

(1) For air ($k = 1.4$), Eqs. (13.84) - (13.87) become,

$$\boxed{\frac{T^*}{T_0} = 0.8333} \quad \underline{\text{adiabatic, steady, 1D flow of air as a perfect gas}}, \tag{13.88}$$

$$\boxed{\frac{p^*}{p_0} = 0.5283} \quad \underline{\text{isentropic, steady, 1D flow of air as a perfect gas}}, \tag{13.89}$$

$$\boxed{\frac{\rho^*}{\rho_0} = 0.6339} \quad \underline{\text{isentropic, steady, 1D flow of air as a perfect gas}}, \tag{13.90}$$

$$\boxed{\frac{c^*}{c_0} = 0.9129} \quad \underline{\text{adiabatic, steady, 1D flow of air as a perfect gas}}. \tag{13.91}$$

The p^*/p_0 ratio value is a useful one to memorize since differences in pressure are what drive most compressible flows.

On the assumption of isentropic flow: In many engineering gas dynamics flows, the assumption that the entropy of the fluid remains constant (an isentropic process) is a good one. If viscous and heat transfer effects can be neglected, then we can reasonably assume that the flow is isentropic (isentropic = internally reversible + adiabatic). This situation is often the case for flows through short, insulated ducts or through stream tubes not passing through a boundary layer or a shock wave (strong viscous effects occur in both cases). Experiments have verified that the isentropic assumption under these conditions is reasonable.

13.8. Mollier (aka h-s) Diagrams

Mollier diagrams are diagrams that plot the enthalpy (h) as a function of entropy (s) for a process. They are often useful in visualizing trends.

Notes:

(1) Sketches of constant pressure and constant volume (or density) curves are shown in Figure 13.17.

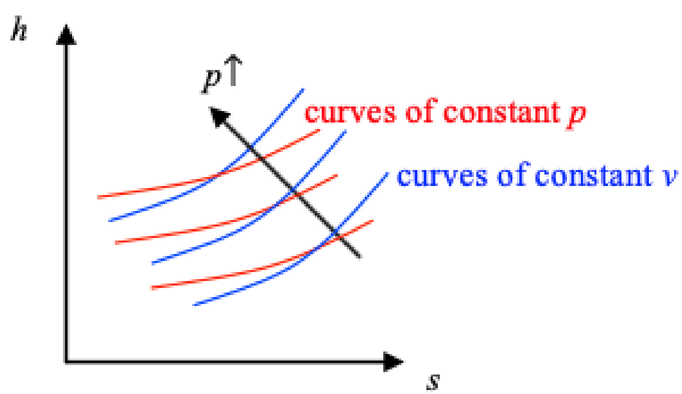


Figure 13.17. An example Mollier plot showing isobars (curves of constant pressure) and isochores (curves of constant specific volume). The curves become steeper as the specific enthalpy increases. The pressure and specific volume increase as one moves to curves approaching the upper left of the plot.

(2) For a perfect gas, curves of constant volume (or density) and constant pressure have slopes given, respectively, by,

$$Tds = du + pdv, \tag{13.92}$$

$$Tds = \frac{c_v}{c_p}\underbrace{(c_p dT)}_{=dh} + pdv, \tag{13.93}$$

$$\therefore \left.\frac{dh}{ds}\right|_v = kT, \tag{13.94}$$

and,

$$Tds = dh - vdp, \tag{13.95}$$

$$\therefore \left.\frac{dh}{ds}\right|_p = T. \tag{13.96}$$

Thus, larger temperatures (and, thus, larger specific enthalpies) will result in steeper slopes for curves of constant pressure and constant volume.

(3) From the Energy Equation (refer to Eq. (13.69)), the difference between the flow specific enthalpy and the stagnation specific enthalpy for an isentropic process is equal to the specific kinetic energy,

$$h_0 = h + \frac{1}{2}V^2. \tag{13.97}$$

Figure 13.18 shows this relationship specifically for an isentropic process, but the relationship holds for non-isentropic processes too.

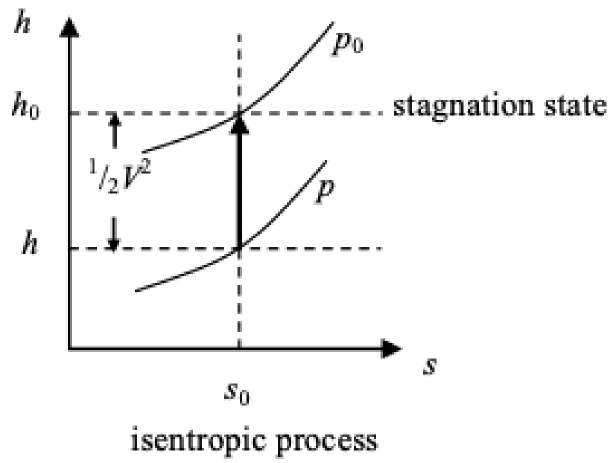


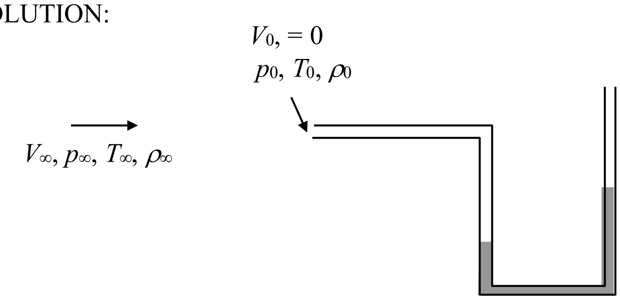
Figure 13.18. A Mollier plot showing the difference between the flow specific enthalpy and specific stagnation enthalpy, in this case for an isentropic process. The difference between the different specific enthalpies is equal to the specific kinetic energy in the flow. This relationship is true even for non-isentropic processes.

(4) For perfect gases, the h-s plots are usually shown as T-s plots since $\Delta h = c_p \Delta T$.

Video solution: https://www.youtube.com/watch?v=Pqf9mptLe08

A pitot tube is used to measure the velocity of air. At low speeds, we can reasonably treat the air as an incompressible fluid; however, at high speeds this assumption is not very good due to compressibility effects. At what Mach number does the incompressibility assumption become inaccurate for engineering calculations? Justify your answer with appropriate calculations.

SOLUTION:



First use the incompressible form of Bernoulli's equation to determine the incoming velocity.

$$p_\infty + \tfrac{1}{2}\rho_\infty V_\infty^2 = p_0 \tag{1}$$

$$\therefore \left(V_\infty\right)_{\text{incompressible}} = \sqrt{\frac{2\left(p_0 - p_\infty\right)}{\rho_\infty}} \tag{2}$$

Now consider the pressure difference for a perfect gas brought to rest isentropically (a reasonable model as long as a shock wave does not form in front of the tube).

$$p_0 - p_\infty = p_\infty\left(\frac{p_0}{p_\infty} - 1\right) \tag{3}$$

where

$$\frac{p_0}{p_\infty} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} \tag{4}$$

Substitute and simplify.

$$p_0 - p_\infty = p_\infty\left[\left(1 + \frac{\gamma-1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

$$p_0 - p_\infty = \left(\tfrac{1}{2}\rho_\infty V_\infty^2\right)\frac{p_\infty}{\left(\tfrac{1}{2}\rho_\infty V_\infty^2\right)}\left[\left(1 + \frac{\gamma-1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \tag{5}$$

Note that for an ideal gas:

$$\frac{p_\infty}{\tfrac{1}{2}\rho_\infty V_\infty^2} = \frac{RT_\infty}{\tfrac{1}{2}V_\infty^2} = \frac{2\gamma RT_\infty}{\gamma V_\infty^2} = \frac{2}{\gamma \text{Ma}_\infty^2} \tag{6}$$

Substitute and simplify.

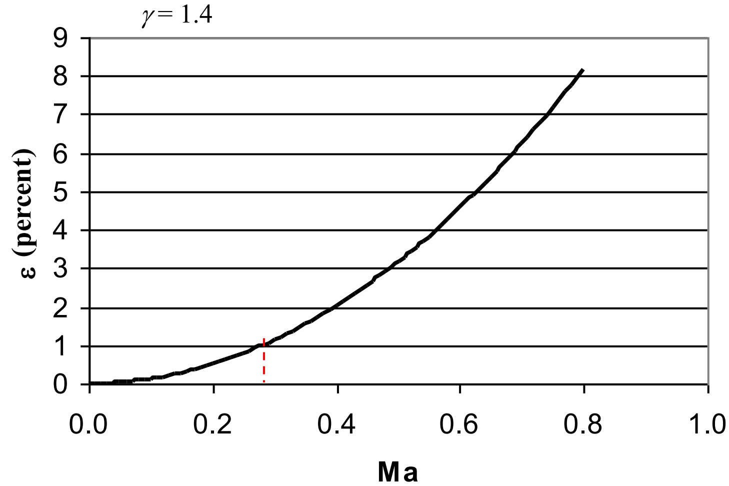
$$p_0 - p_\infty = \left(\tfrac{1}{2}\rho_\infty V_\infty^2\right)\frac{2}{\gamma\text{Ma}_\infty^2}\left[\left(1 + \frac{\gamma-1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

$$\therefore \left(V_\infty\right)_{\substack{\text{isentropic,}\\ \text{ideal gas}}} = \sqrt{\frac{2\left(p_0 - p_\infty\right)}{\rho_\infty}\left\{\frac{2}{\gamma\text{Ma}_\infty^2}\left[\left(1 + \frac{\gamma-1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]\right\}^{-1}} \tag{7}$$

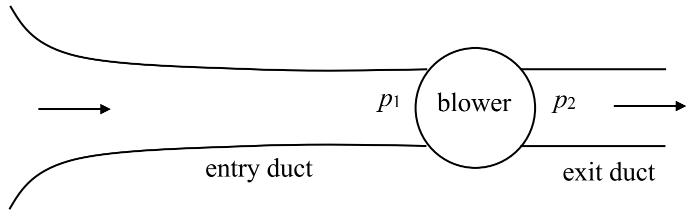
Define the relative error as:

$$\boxed{\varepsilon \equiv \frac{\left(V_\infty\right)_{\substack{\text{isentropic}\\ \text{ideal gas}}} - \left(V_\infty\right)_{\text{incompressible}}}{\left(V_\infty\right)_{\substack{\text{isentropic}\\ \text{ideal gas}}}} = 1 - \sqrt{\frac{2}{\gamma \mathrm{Ma}_\infty^2}\left[\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]}} \tag{8}$$

Thus, the error is a function only of the upstream Mach number and the specific heat ratio. Plotting the error as a function of upstream Mach number (for air, $\gamma = 1.4$) shows that, if we consider <1% error acceptable, the incompressibility assumption is valid for $\mathrm{Ma}_\infty <\approx 0.3$.



An air blower takes air from the atmosphere (100 kPa (abs) and 293 K) and ingests it through a smooth entry duct so that the losses are negligible. The cross-sectional area of the entry duct just upstream of the blower and that of the exit duct are both 0.01 m^2.

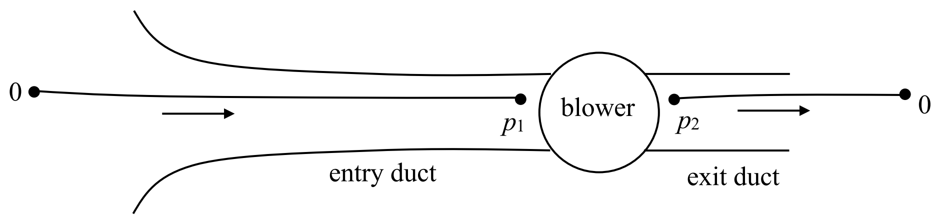


The pressure ratio, p_2/p_1, across the blower is 1.05 and the exit pressure is equal to atmospheric pressure. The air is assumed to behave isentropically upstream of the blower. Find:

a. the velocity of the air entering the blower, and
b. the mass flow rate of air through the system.

SOLUTION:

Apply the First Law between points 0 and 1 (refer to the figure below). Assume 1D, steady, isentropic ($\Rightarrow$ adiabatic) flow. Also neglect potential energy changes since a gas is the working fluid.



$$\dot{m}_1\left(h+\tfrac{1}{2}V^2\right)_1-\dot{m}_0\left(h+\tfrac{1}{2}V^2\right)_0=0 \tag{1}$$

The velocity far upstream is negligible ($V_0 \approx 0$) and $\dot{m}_1=\dot{m}_0$. Substitute and simplify.

$$h_0=h_1+\tfrac{1}{2}V_1^2 \tag{2}$$

Assume that the air behaves as a perfect gas ($\Delta h = c_P\Delta T$) and solve for V_1.

$$V_1=\sqrt{2c_P\left(T_0-T_1\right)} \tag{3}$$

Express the temperature at point 1 in terms of a pressure ratio making use of the fact that the flow is isentropic.

$$\frac{T_1}{T_0}=\left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}} \tag{4}$$

The pressure rise across the blower is specified in the problem statement. Furthermore, the pressure at point 2 is equal to atmospheric pressure, i.e., $p_2 = p_0$. Re-write Eq. (4) using this information.

$$\frac{T_1}{T_0}=\left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}=\left(\frac{p_2}{p_1}\right)^{\frac{1-\gamma}{\gamma}} \tag{5}$$

Combine Eqs. (3) and (5).

$$V_1 = \sqrt{2c_P\left(T_0 - T_1\right)} = \sqrt{2c_P T_0\left(1 - \frac{T_1}{T_0}\right)}$$

$$\boxed{\therefore V_1 = \sqrt{2c_P T_0\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{1-\gamma}{\gamma}}\right]}} \tag{6}$$

Using the given numerical data:

c_P = 1004.5 J/(kg·K) (for air)
γ = 1.4 (for air)
T_0 = 293 K
p_2/p_1 = 1.05
$\Rightarrow$ $\boxed{V_1 = 90 \text{ m/s}}$

The mass flow rate can be found from the conditions at point 1:

$$\boxed{\dot{m} = \rho_1 V_1 A_1 = \left(\frac{p_1}{RT_1}\right) V_1 A_1} \tag{7}$$

where the ideal gas law has been used. The pressure at point 1 can be found from the blower pressure ratio and the fact that p_2 is equal to atmospheric pressure.

$$p_1 = \left(\frac{p_1}{p_2}\right) p_0 \tag{8}$$

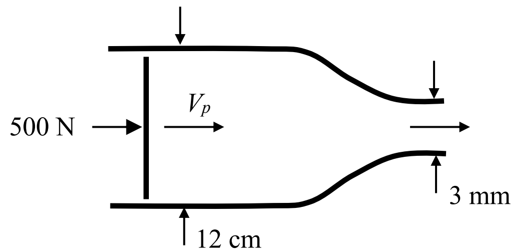
The temperature at point 1 can be found using Eq. (5). Using the given numerical data:

p_0 = 100 kPa (abs)
p_2/p_1 = 1.05
p_1 = 95.2 kPa (abs)
T_0 = 293 K
T_1 = 289 K
R = 287 J/(kg·K) (for air)
V_1 = 90 m/s
A_1 = 0.01 m^2
$\Rightarrow$ $\boxed{\dot{m} = 1.03 \text{ kg/s}}$

Video solution: https://www.youtube.com/watch?v=qnrTPKmKQz8

A force of 500 N pushes a piston of diameter 12 cm through an insulated cylinder containing air at 20 °C. The exit diameter is 3 mm and the atmospheric pressure is 1 atm (abs). Estimate:

1. the exit velocity,
2. the velocity near the piston (V_p), and
3. mass flow rate out of the device.



SOLUTION:

The pressure at the piston face may be found from the piston force and piston diameter.

$$p_1 = \frac{F}{\frac{\pi}{4}d_1^2} + p_{\text{atm}} = \frac{(500\text{ N})}{\frac{\pi}{4}(0.12\text{ m})^2} + 101\text{ kPa} = 145\text{ kPa} \tag{1}$$

Assume the flow through the piston is isentropic. The velocity at the exit may be found by applying the First Law to the air inside the piston with 1 signifying the location adjacent to the piston face and 2 signifying the device's exit.

$$\left(h + \tfrac{1}{2}V^2\right)_2 - \left(h + \tfrac{1}{2}V^2\right)_1 = \dot{Q}_{\substack{\text{into}\\\text{air}}} + \dot{W}_{\substack{\text{on}\\\text{air}}} \tag{2}$$

Assuming perfect gas behavior, adiabatic conditions, and that $V_2 >> V_1$ (since the areas are so different):

$$c_P(T_2 - T_1) + \tfrac{1}{2}V_2^2 = 0 \tag{3}$$

$$V_2 = \sqrt{2c_P(T_1 - T_2)} \tag{4}$$

Also assume that the flow is isentropic so that:

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \Rightarrow T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \tag{5}$$

Using the given data:

c_P = 1004 J/(kg·K)
T_1 = (20 + 273) K = 293 K
p_1 = 145×10^3 Pa (from Eq. (1))
p_2 = 101×10^3 Pa (discharging into the atmosphere, assuming the exit Mach number is subsonic)
$\Rightarrow T_2$ = 264 K
$\boxed{\therefore V_2 = 241\text{ m/s}}$

Check that the exit Mach number is subsonic.

$$c_2 = \sqrt{\gamma R T_2} \Rightarrow c_2 = 326\text{ m/s} \tag{6}$$

Since $V_2 < c_2$, the exit flow is subsonic and the assumption that $p_2 = p_{\text{atm}}$ is a good one.

From conservation of mass applied to the same control volume:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \Rightarrow V_1 = V_2\left(\frac{\rho_2}{\rho_1}\right)\left(\frac{D_2}{D_1}\right)^2 = V_2\left(\frac{p_2}{p_1}\right)\left(\frac{T_1}{T_2}\right)\left(\frac{D_2}{D_1}\right)^2 \tag{7}$$

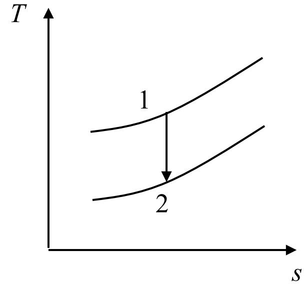
$\boxed{\therefore V_1 = 0.116\text{ m/s}}$ Clearly the assumption that $V_2 >> V_1$ was a good one.

Video solution: https://www.youtube.com/watch?v=qnrTPKmKQz8

The mass flow rate is:

$$\dot{m} = \rho_2 V_2 A_2 = \frac{p_2}{RT_2} V_2 \tfrac{\pi}{4} D_2^2 \tag{8}$$

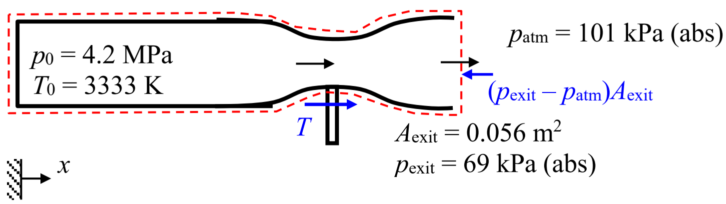
$$\boxed{\therefore \dot{m} = 2.27 * 10^{-3} \text{ kg/s}}$$



A small, solid fuel rocket motor is tested on a horizontal thrust stand at atmospheric conditions. The chamber (essentially a large tank) absolute pressure and temperature are maintained at 4.2 MPa (abs) and 3333 K, respectively. The rocket's converging-diverging nozzle is designed to expand the exhaust gas isentropically to an absolute pressure of 69 kPa. The nozzle exit area is 0.056 m^2. The gas may be treated as a perfect gas with a specific heat ratio of 1.2 and an ideal gas constant of 300 J/(kg·K). Determine, for design conditions:

a. the mass flow rate of propellant gas, and
b. the thrust force exerted on the test stand.

SOLUTION:



Determine the mass flow rate using the conditions at the exit. The Mach number at the exit may be found from the isentropic stagnation pressure ratio:

$$\frac{p_{exit}}{p_0} = \left(1 + \frac{k-1}{2}\text{Ma}_{exit}^2\right)^{\frac{k}{1-k}} \tag{1}$$

Using p_{exit} = 69e3 Pa (abs), p_0 = 4.2e6 Pa (abs), and k = 1.2:

$$\therefore \text{Ma}_{exit} = 3.136 \tag{2}$$

The exit temperature may be found using the stagnation temperature ratio:

$$\frac{T_{exit}}{T_0} = \left(1 + \frac{k-1}{2}\text{Ma}_{exit}^2\right)^{-1} \tag{3}$$

Using T_0 = 3333 K, Ma$_{exit}$ = 3.136, and k = 1.2:

$$\therefore T_{exit} = 1680 \text{ K} \tag{4}$$

The exit density may be found using the ideal gas law:

$$p_{exit} = \rho_{exit} R T_{exit} \tag{5}$$

Using p_{exit} = 69e3 Pa, T_{exit} = 1680 K, and R = 300 J/(kg·K):

$$\therefore \rho_{exit} = 0.1369 \text{ kg/m}^3 \tag{6}$$

The exit velocity may be found using the speed of sound at the exit and the Mach number definition:

$$c_{exit} = \sqrt{kRT_{exit}} \tag{7}$$

$$V_{exit} = c_{exit}\text{Ma}_{exit} \tag{8}$$

Using k = 1.2, R = 300 J/(kg·K), T_{exit} = 1680 K, and Ma$_{exit}$ = 3.136:

$$\therefore c_{exit} = 777.8 \text{ m/s} \tag{9}$$

$$\therefore V_{exit} = 2439 \text{ m/s} \tag{10}$$

The mass flow rate through the nozzle is:

$$\dot{m} = \rho_{exit} V_{exit} A_{exit} \tag{11}$$

Using ρ_{exit} = 0.1369 kg/m^3, V_{exit} = 2439 m/s, and A_{exit} = 0.056 m^2:

$$\boxed{\therefore \dot{m} = 18.69 \text{ kg/s}} \tag{12}$$

The thrust force, T, acting on the stand may be determined using the Linear Momentum Equation in the x-direction for the control volume shown in the figure.

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{13}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)} \tag{14}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m} V_{\text{exit}} \tag{15}$$

$$F_{B,x} = 0 \tag{16}$$

$$F_{S,x} = T - \left(p_{\text{exit}} - p_{\text{atm}}\right) A_{\text{exit}} \tag{17}$$

Substitute and simplify.

$$\dot{m} V_{\text{exit}} = T - \left(p_{\text{exit}} - p_{\text{atm}}\right) A_{\text{exit}} \tag{18}$$

$$\therefore T = \dot{m} V_{\text{exit}} + \left(p_{\text{exit}} - p_{\text{atm}}\right) A_{\text{exit}} \tag{19}$$

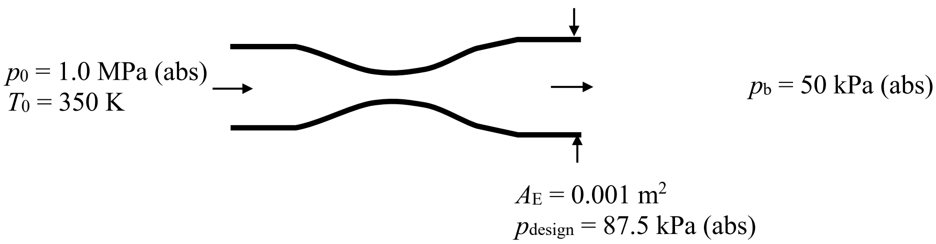
Using $\dot{m} = 18.69$ kg/s, $V_{\text{exit}} = 2439$ m/s, $p_{\text{exit}} = 69\text{e}3$ Pa (abs), $p_{\text{atm}} = 101\text{e}3$ Pa (abs), and $A_{\text{exit}} = 0.056$ m^2:

$$\boxed{\therefore T = 4.380\text{e}4 \text{ N}} \tag{20}$$

Air flows isentropically in a converging-diverging nozzle, with exit area of 0.001 m^2. The nozzle is fed from a large plenum where the stagnation conditions are 350 K and 1.0 MPa (abs). The nozzle has a design back pressure of 87.5 kPa (abs) but is operating at a back pressure of 50.0 kPa (abs). Assuming the flow within the nozzle is isentropic, determine:

a. the exit Mach number, and
b. the mass flow rate through the nozzle.

SOLUTION:



The exit Mach number may be found using the isentropic pressure ratio at the exit. Since the back pressure is less than the design pressure (underexpanded conditions), the exit pressure will be equal to the design pressure.

$$\frac{p_E}{p_0} = \left(1 + \frac{k-1}{2}\text{Ma}_E^2\right)^{\frac{k}{1-k}} \tag{1}$$

Using p_E = 87.5 kPa, p_0 = 1.0 MPa, and k = 1.4, the exit Mach number is: $\boxed{\text{Ma}_E = 2.24}$.

The mass flow rate through the nozzle may be found using the exit conditions. First, determine the temperature at the exit using the adiabatic stagnation temperature ratio:

$$\frac{T_E}{T_0} = \left(1 + \frac{k-1}{2}\text{Ma}_E^2\right)^{-1} \tag{2}$$

Using T_0 = 350 K, k = 1.4, and Ma$_E$ = 2.24, the exit temperature is: $\underline{T_E = 174.5\text{ K}}$.

The speed of sound at the exit is:

$$c_E = \sqrt{kRT_E} \tag{3}$$

Using k = 1.4, R = 287 J/(kg·K), and T_E = 174.5 K, the speed of sound at the exit is: $\underline{c_E = 264.8\text{ m/s}}$.

The velocity of the air at the exit is:

$$V_E = c_E \text{Ma}_E \tag{4}$$

Using c_E = 264.8 m/s and Ma$_E$ = 2.24: $\underline{V_E = 593.8\text{ m/s}}$.

The density at the exit may be found using the ideal gas law:

$$\rho_E = \frac{p_E}{RT_E} \tag{5}$$

With p_E = 87.5 kPa, R = 287 J/(kg·K), and T_E = 174.5 K: $\underline{\rho_E = 1.747\text{ kg/m}^3}$.

The mass flow rate through the nozzle is:

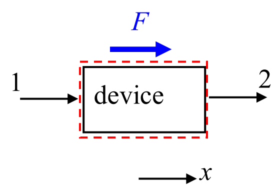
$$\dot{m} = \rho_E V_E A_E \tag{6}$$

Using the previous data, $\boxed{\dot{m} = 1.04\text{ kg/s}}$.

Oxygen (not air) enters a device with a cross-sectional area of 1 ft^2 (refer to this location as section 1) with a stagnation temperature of 1000 °R, stagnation pressure of 100 psia, and Mach number of 0.2. There is no heat transfer, work transfer, or losses as the gas passes through the device and expands to a pressure of 14.7 psia (section 2).

a. Determine the density, velocity, and mass flow rate at section 1.
b. Determine the Mach number, temperature, velocity, density, and area at section 2.
c. What force does the fluid exert on the device?

SOLUTION:



First determine the properties at section 1.

$$\rho_0 = \frac{p_0}{RT_0} \tag{1}$$

where

$$\rho_1 = \rho_0 \left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right)^{\frac{1}{1-\gamma}} \tag{2}$$

Using p_0 = 100 psia = 14400 lb_f/ft^2, T_0 = 1000 °R, R = 48.291 $(\text{ft}\cdot\text{lb}_f)/(\text{lb}_m\cdot°\text{R})$ = 1553.7 $\text{ft}^2/(\text{s}^2\cdot°\text{R})$, ρ_0 = 0.298 lb_m/ft^3. In addition, with γ = 1.395, and Ma_1 = 0.2, $\boxed{\rho_1 = 0.292\ \text{lb}_m/\text{ft}^3}$.

$$V_1 = c_1\text{Ma}_1 = \sqrt{\gamma RT_1}\,\text{Ma}_1 \tag{3}$$

where,

$$T_1 = T_0\left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right)^{-1} \tag{4}$$

Using the given values, T_1 = 992.2 °R and $\boxed{V_1 = 293.3\ \text{ft/s}}$.

$$\dot{m} = \rho_1 V_1 A_1 \tag{5}$$

Using the given values, $\boxed{\dot{m} = 85.6\ \text{lb}_m/\text{s}}$.

Now use the isentropic relations to determine the properties at section 2.

$$p_2 = p_0\left(1 + \frac{\gamma - 1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \text{Ma}_2 = \left\{\frac{2}{\gamma - 1}\left[\left(\frac{p_2}{p_0}\right)^{\frac{1-\gamma}{\gamma}} - 1\right]\right\}^{\frac{1}{2}} \tag{6}$$

Using p_2 = 14.7 psia and p_0 = 100 psia, $\boxed{\text{Ma}_2 = 1.91}$.

$$T_2 = T_0\left(1 + \frac{\gamma - 1}{2}\text{Ma}_2^2\right)^{-1} \tag{7}$$

Using the given data, $\boxed{T_2 = 581.2\ °\text{R}}$.

$$V_2 = c_2\text{Ma}_2 = \sqrt{\gamma RT_2}\,\text{Ma}_2 \tag{8}$$

Using the given data, $\boxed{V_2 = 2144\ \text{ft/s}}$.

$$\rho_2 = \frac{p_2}{RT_2} \tag{9}$$

Using the given data, $\boxed{\rho_2 = 0.0754\ \text{lb}_\text{m}/\text{ft}^3}$.

$$\dot{m} = \rho_2 V_2 A_2 \Rightarrow A_2 = \frac{\dot{m}}{\rho_2 V_2} \tag{10}$$

Using the given data, $\boxed{A_2 = 0.53\ \text{ft}^2}$.

To determine the force the fluid exerts on the device, apply the Linear Momentum Equation in the x-direction to the control volume shown in the figure,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{11}$$

where,

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)} \tag{12}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}V_1 + \dot{m}V_2 = \dot{m}\left(V_2 - V_1\right) \tag{13}$$

$$F_{B,x} = 0 \tag{14}$$

$$F_{S,x} = F + p_1 A_1 - p_2 A_2 \tag{15}$$

Substitute and simplify.

$$\dot{m}\left(V_2 - V_1\right) = F + p_1 A_1 - p_2 A_2$$

$$\boxed{F = \dot{m}\left(V_2 - V_1\right) - p_1 A_1 + p_2 A} \tag{16}$$

Substitute the given values to find F = -7950 lb_f. Note that this is the force that the device exerts on the fluid. Hence, the $\boxed{\text{force the fluid exerts on the device is 7950 lb}_\text{f}\text{ acting in the }+x\text{-direction}}$.

The Concorde aircraft flies at Ma≈2.3 at 11 km standard altitude. Estimate the temperature in °C at the front stagnation point. At what Mach number would it have a front stagnation point temperature of 450 °C?

SOLUTION:

The temperature at the stagnation point is determined using:

$$\frac{T}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_\infty^2\right)^{-1} \tag{1}$$

where

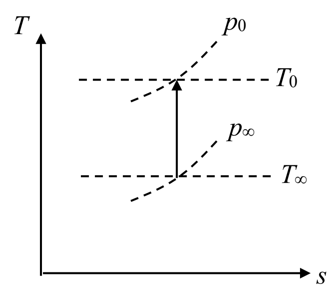
γ = 1.4
Ma_∞ = 2.3
T_∞ = 217 K (from standard atmosphere tables at an altitude of 11 km)
$\Rightarrow$ $\boxed{T_0 = 447\text{ K} = 174\text{ °C}}$

For the next part of the problem, re-arrange Eqn. (1):

$$\mathrm{Ma}_\infty = \sqrt{\frac{2}{\gamma - 1}\left(\frac{T_0}{T} - 1\right)} \tag{2}$$

Using the given data:

γ = 1.4
T_∞ = 217 K
T_0 = 450 °C = 723 K
$\Rightarrow$ $\boxed{\mathrm{Ma}_\infty = 3.4}$



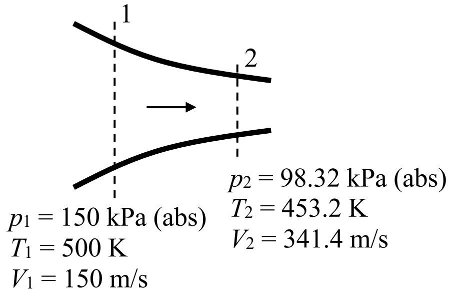
Video solution: https://www.youtube.com/watch?v=H7G2f5zl0LQ

A steady flow of air passes through a converging nozzle. At the nozzle inlet, the static pressure and temperature are $p_1 = 150$ kPa (abs), $T_1 = 500$ K, and $V_1 = 150$ m/s. At the nozzle exit, $p_2 = 98.32$ kPa (abs), $T_2 = 453.2$ K, and $V_2 = 341.4$ m/s. Assume steady, uniform flow, and that the air behaves as a perfect gas with $\gamma = 1.4$, $R = 287$ J/(kg·K), and $c_p = 1005$ J/(kg·K).

a. Is the flow through the nozzle adiabatic?
b. Is the flow through the nozzle isentropic?
c. Is the flow through the nozzle frictionless?

Support all of your answers.

SOLUTION:



If the flow is adiabatic then the stagnation temperature will remain constant, i.e., $T_{02} = T_{01}$, where:

$$T_0 = T + \frac{V^2}{2c_p} \tag{1}$$

Using the given data:

$T_1 = 500$ K
$V_1 = 150$ m/s
$T_2 = 453.2$ K
$V_2 = 341.4$ m/s
$c_p = 1005$ J/(kg·K)
$\Rightarrow$ $T_{01} = 511.2$ K and $T_{02} = 511.2$ K

Since the stagnation temperatures are equal, $\boxed{\text{the flow must be adiabatic}}$.

If the flow is isentropic, then the stagnation pressure will remain constant, i.e., $p_{02} = p_{01}$ (the stagnation density will also remain constant, i.e., $\rho_{02} = \rho_{01}$).

$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}^2\right)^{\gamma/(1-\gamma)} \tag{2}$$

Using the given data:

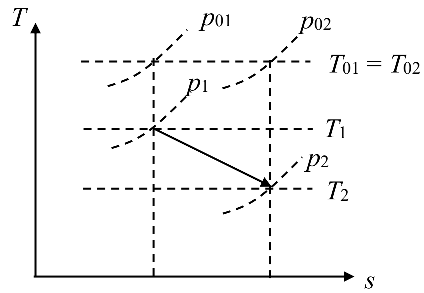
$p_1 = 150$ kPa (abs)
$V_1 = 150$ m/s
$T_1 = 500$ K

$\Rightarrow$ $\text{Ma}_1 = 0.34$ where $\text{Ma} = \frac{V}{c} = \frac{V}{\sqrt{\gamma R T}}$

$p_2 = 98.32$ kPa (abs)
$V_2 = 341.4$ m/s
$T_2 = 453.2$ K
$\Rightarrow$ $\text{Ma}_2 = 0.80$
$\Rightarrow$ $p_{01} = 162.1$ kPa (abs) and $p_{02} = 149.9$ kPa (abs)

Since the stagnation pressures are not equal, $\boxed{\text{the flow is not isentropic}}$.

Since the flow is adiabatic but non-isentropic, then some other irreversible process must take place. Two common irreversible processes that occur in gas dynamics are frictional effects and shock waves. Shock waves cannot be the source of the entropy since shock waves only occur in supersonic flows and the flow in this converging nozzle remains subsonic throughout. Hence, we can conclude that
$\boxed{\text{the flow in this nozzle is not frictionless}}$.



A supersonic wind tunnel test section is designed to have a Mach number of 2.5 at a temperature of 60 °F and 5 psia. The fluid is air.

a. Determine the required inlet stagnation temperature and pressure.
b. Calculate the required mass flow rate for a test section area of 2.0 ft^2.

SOLUTION:

The stagnation properties may be found using the isentropic relations:

$$\frac{p_{TS}}{p_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_{\mathrm{TS}}^2\right)^{\frac{\gamma}{1-\gamma}} \tag{1}$$

$$\frac{T_{TS}}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_{\mathrm{TS}}^2\right)^{-1} \tag{2}$$

where

p_{TS} = 5 psia = 720 lb$_f$/ft^2
T_{TS} = (60 + 459) °R = 519 °R
Ma_{TS} = 2.5
γ_{air} = 1.4
$\therefore \boxed{p_0 = 85.4 \text{ psia}}$ and $\boxed{T_0 = 1170\ °\text{R}}$

The mass flow rate may be found using:

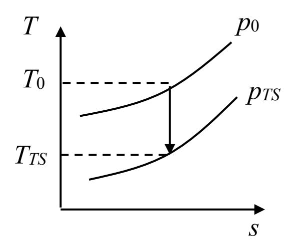
$$\dot{m}_{TS} = \rho_{TS} V_{TS} A_{TS} = \left(\frac{p_{TS}}{RT_{TS}}\right)\left(c_{TS}\mathrm{Ma}_{TS}\right)A_{TS} \tag{3}$$

where the speed of sound in the test section, c_{TS}, is:

$$c_{TS} = \sqrt{\gamma R T_{TS}} \tag{4}$$

Using the given data:

R_{air} = 53.3 (ft·lb$_f$)/(lb$_m$·°R)
A_{TS} = 2 ft^2
$\Rightarrow c_{TS}$ = 1120 ft/s
ρ_{TS} = 0.0260 lb$_m$/ft^3
$\boxed{\dot{m}_{TS} = 145 \text{ lb}_\text{m}/\text{s}}$



13.9. Effects of Area Change on Steady, 1D, Isentropic Flow

Mass conservation states that for a steady, 1D, incompressible flow, a decrease in the area will result in an increase in speed (and visa-versa). This behavior is not necessarily true, however, for a compressible flow as will be shown in this section.

Consider Conservation of Mass for a steady, 1D flow,

$$\dot{m} = \rho V A = \text{constant}, \tag{13.98}$$

$$d(\rho V A) = 0, \tag{13.99}$$

$$V A d\rho + \rho V dA + \rho A dV = 0, \tag{13.100}$$

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0. \tag{13.101}$$

Notes:

(1) If the flow is incompressible, then $d\rho = 0$ and we see that: $dV/V = -dA/A$. Thus, if the area decreases ($dA < 0$), then the speed must increase ($dV > 0$).
(2) For a compressible fluid, the density may change so we need an additional relationship between density and either area or speed to draw any conclusions about how changes in area affect changes in speed.

Recall that the speed of sound is,

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s. \tag{13.102}$$

Let's concern ourselves with an isentropic flow (assume the flow is internally reversible and adiabatic so that $s = \text{constant}$) so we can re-write this expression as,

$$d\rho = \frac{dp}{c^2}. \tag{13.103}$$

We'll also make use of Bernoulli's equation (which comes from the Linear Momentum Equation; refer to Eq. (13.14)),

$$\frac{dp}{\rho} + V dV = 0. \tag{13.104}$$

Substituting Eqs. (13.103) and (13.104) into (13.101) and simplifying,

$$\frac{1}{c^2}\frac{dp}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0, \tag{13.105}$$

$$-\frac{V^2}{c^2}\frac{dV}{V} + \frac{dA}{A} + \frac{dV}{V} = 0, \tag{13.106}$$

$$\left(\frac{V^2}{c^2} - 1\right)\frac{dV}{V} = \frac{dA}{A}, \tag{13.107}$$

$$\boxed{\left(\mathrm{Ma}^2 - 1\right)\frac{dV}{V} = \frac{dA}{A}}. \tag{13.108}$$

Note that the trends for pressure and density are opposite to the trends for speed. From Bernoulli's equation (Eq. (13.104)),

$$\frac{dp}{\rho V^2} = -\frac{dV}{V}, \tag{13.109}$$

and from Eqs. (13.108) and (13.101),

$$\frac{d\rho}{\rho} = -\mathrm{Ma}^2\frac{dV}{V}. \tag{13.110}$$

Thus,

$$dV < 0 \implies dp > 0 \quad \text{and} \quad d\rho > 0, \tag{13.111}$$
$$dV > 0 \implies dp < 0 \quad \text{and} \quad d\rho < 0. \tag{13.112}$$

The changes in temperature and Mach number can also be related to changes in the speed. Recall that for an ideal gas undergoing an isentropic process,

$$ds = 0 = c_v \frac{dT}{T} - R\frac{d\rho}{\rho}, \tag{13.113}$$
$$\frac{dT}{T} = \frac{R}{c_v}\frac{d\rho}{\rho}, \tag{13.114}$$
$$\frac{dT}{T} = (1-k)\mathrm{Ma}^2\frac{dV}{V} \quad (k > 1). \tag{13.115}$$

Thus,

$$dV < 0 \implies dT > 0 \implies d(\mathrm{Ma}) < 0, \tag{13.116}$$
$$dV > 0 \implies dT < 0 \implies d(\mathrm{Ma}) > 0. \tag{13.117}$$

where the change in Mach number is found by considering $\mathrm{Ma} = V/c = V/\sqrt{kRT}$.

Let's interpret Eq. (13.108) more closely. Consider the following cases:

<u>$\mathrm{Ma} < 1$ (subsonic flow):</u>
$dA < 0 \implies dV > 0 \implies d(\mathrm{Ma}) > 0$ (as $A \downarrow \implies V \uparrow$ and $\mathrm{Ma} \uparrow$)
$dA > 0 \implies dV < 0 \implies d(\mathrm{Ma}) < 0$ (as $A \uparrow \implies V \downarrow$ and $\mathrm{Ma} \downarrow$)

Notes:

(1) A subsonic nozzle should have a decreasing area.
(2) A subsonic diffuser should have an increasing area.
(3) The area-speed relationships for subsonic flow are identical to those for incompressible flow.

<u>$\mathrm{Ma} > 1$ (supersonic flow):</u>
$dA < 0 \implies dV < 0 \implies d(\mathrm{Ma}) < 0$ ($A \downarrow \implies V \downarrow$ and $\mathrm{Ma} \downarrow$)
$dA > 0 \implies dV > 0 \implies d(\mathrm{Ma}) > 0$ ($A \uparrow \implies V \uparrow$ and $\mathrm{Ma} \uparrow$)

Notes:

(1) A supersonic nozzle should have an increasing area.
(2) A supersonic diffuser should have a decreasing area.
(3) The area-speed relationships for supersonic flow are the opposite to those for subsonic flow.

<u>$\mathrm{Ma} = 1$ (sonic flow):</u>
$dA = 0$ (sonic conditions must occur at an inflection point in the area)

Based on the previous relationships for subsonic and supersonic flow, the <u>area at which $\mathrm{Ma} = 1$ must be a minimum</u>. Referring to Figure 13.19, if the flow starts off subsonic and the area is decreasing, then the flow will accelerate and approach $\mathrm{Ma} = 1$. Similarly, if the flow is initially supersonic, a decreasing area will decelerate the flow and it will again approach sonic conditions. The Mach number conditions downstream of the minimum area are ambiguous. In both cases the downstream flow could either be subsonic or supersonic, depending on the downstream boundary conditions. This topic is addressed in this chapter when discussing converging-diverging nozzles (Section 13.18).

Notes:

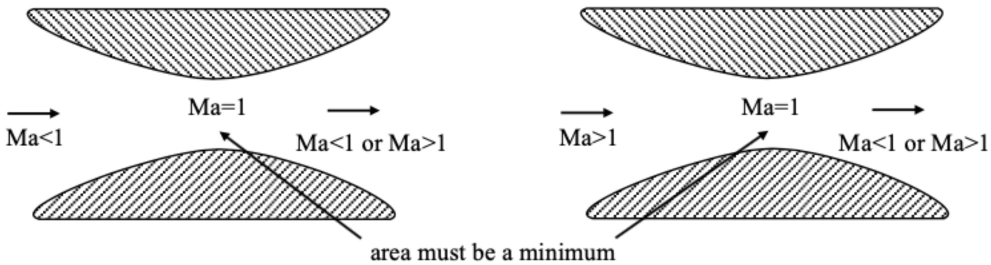


Figure 13.19. An illustration demonstrating that sonic flow occurs at a minimum area.

(1) Nothing can be said about how the speed changes when the Ma = 1 using Eq. (13.108). The speed can either decrease, remain constant, or increase. As mentioned in the previous paragraph, the flow downstream of Ma = 1 depends on the downstream boundary condition.
(2) From Eq. (13.108), a minimum area ($dA = 0$) does not necessarily imply that the Mach number is one. It could be that Ma = 1 or simply that the speed doesn't change ($dV = 0$). Thus,

$$\boxed{\mathrm{Ma} = 1 \implies \text{minimum area},} \tag{13.118}$$

$$\boxed{\text{minimum area} \centernot\implies \mathrm{Ma} = 1} \tag{13.119}$$

Now let's examine some other consequences resulting from mass conservation. Since the mass flow rate must remain constant in 1D, steady flow, we can write,

$$\dot{m} = \rho V A = \rho^* V^* A^*, \tag{13.120}$$

where the "*" quantities are the sonic conditions. Let's re-arrange this equation and substitute the isentropic relations derived in the previous section,

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V}, \tag{13.121}$$

$$= \frac{\rho^*}{\rho}\frac{\rho}{\rho_0}\frac{c^*}{c\mathrm{Ma}}, \tag{13.122}$$

$$= \frac{\rho^*}{\rho}\frac{\rho}{\rho_0}\frac{c^*}{c_0}\frac{c_0}{c}\frac{1}{\mathrm{Ma}}, \tag{13.123}$$

where, from the previous section,

$$\frac{\rho^*}{\rho_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{1}{1-k}}, \tag{13.124}$$

$$\frac{\rho}{\rho_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{1}{1-k}}, \tag{13.125}$$

$$\frac{c^*}{c_0} = \left(1 + \frac{k-1}{2}\right)^{-\frac{1}{2}}, \tag{13.126}$$

$$\frac{c}{c_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{-\frac{1}{2}}. \tag{13.127}$$

Substituting and simplifying (the algebra isn't shown here),

$$\boxed{\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left(\frac{1 + \frac{k-1}{2}\mathrm{Ma}^2}{1 + \frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}}}. \tag{13.128}$$

Notes:

(1) Equation (13.128) tells us what area we would need to contract to to get sonic conditions (Ma $=1, A=A^*$) given the current Mach number, Ma, and area, A.
(2) We could also interpret Eq. (13.128) as saying, given the area for sonic conditions, A^*, the Mach number, Ma, and area, A, are directly related for an isentropic flow. Recall that this relationship results from Conservation of Mass and the assumption of an isentropic flow.
(3) Values for A/A^* as a function of Mach number are typically included in compressible flow tables found in the appendices of most fluid mechanics textbooks.
(4) What happens if we constrict the area to a value less than A^*? For a subsonic flow, the new area information can propagate upstream and downstream and, as a result, the conditions everywhere change (i.e., the Mach numbers change according to Eq. (13.128) where the new area would be A^*). If the upstream flow is supersonic, then some non-isentropic process must occur upstream (a shock wave) so that the constricted area is no longer less than A^*.
(5) A plot of Eq. (13.128) is shown in Figure 13.20. Two important features can be observed in the plot. First, the minimum value of A/A^* is equal to one and this minimum occurs at Ma $=1$, as expected. Second, there are two values of Mach number for a given value of A/A^* – a subsonic value and a supersonic value.

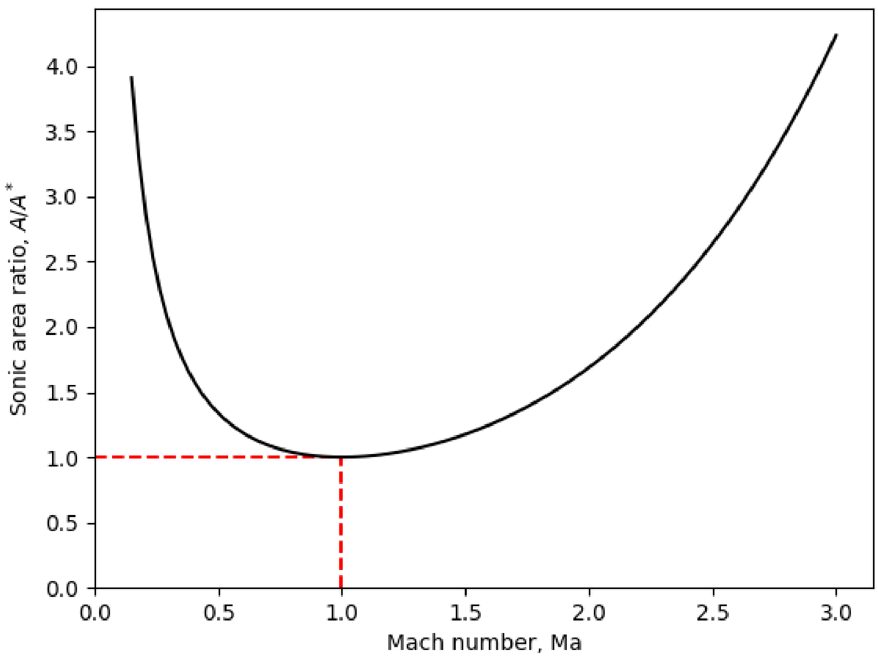


Figure 13.20. A plot of the sonic area ratio A/A^* as a function of Mach number for $k=1.4$.

13.10. Choked Flow

Consider the flow of a compressible fluid from a large reservoir into the surroundings, as shown in Figure 13.21. Let the pressure of the surroundings, called the back pressure, p_B, be controllable.

When $p_B=p_0$ there will be no flow from the reservoir since there is no driving pressure gradient. When the back pressure, p_B, is decreased, a pressure wave, i.e., a sound wave, propagates through the fluid in the nozzle and into the tank (Figure 13.22). Thus, the fluid in the tank "is informed" that the pressure outside has been lowered and a pressure gradient is established resulting in fluid being pushed out of the tank.

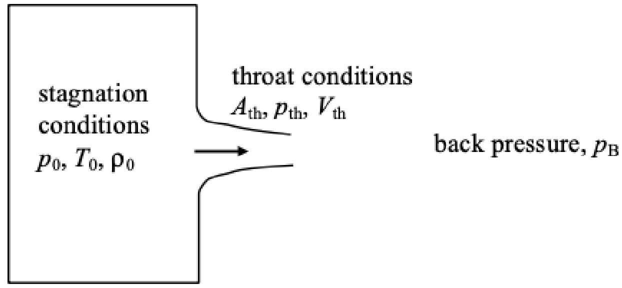


Figure 13.21. An illustration showing flow from a large tank through a converging nozzle into the surroundings.

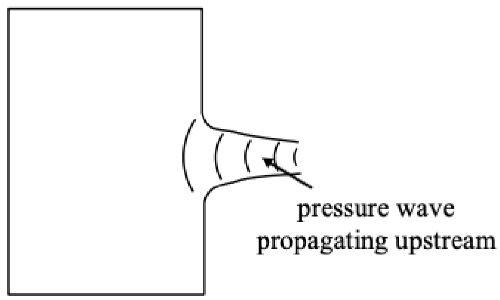


Figure 13.22. An illustration showing sound waves propagating upstream from the surroundings into the tank.

Thus, when $p_B < p_0$, the fluid will begin to flow out of the reservoir. Furthermore, as $p_B/p_0 \downarrow, V_{th} \uparrow$, and the mass flow rate increases. Note that the flow through the nozzle will be subsonic (Ma < 1) since the fluid starts from stagnation conditions and doesn't pass through a minimum area until reaching the throat. Additionally, since the flow is subsonic, the pressure at the throat will be the same as the back pressure, i.e., $p_{th} = p_B$. That this is so can be seen by noting that if $p_{th} > p_B$, then the flow would expand upon leaving the nozzle and as a result, the jet velocity would decrease and the pressure would increase. Thus, the jet pressure would diverge from the surrounding pressure. But the jet must eventually reach the surrounding pressure so the assumption that $p_{th} > p_B$ must be incorrect. A similar argument can be made for $p_{th} < p_B$.
As we continue to decrease p_B/p_0, we'll eventually reach a state where the velocity at the throat will reach $\mathrm{Ma} = 1$ ($V_{th} = V^* = c^*$). The pressure ratio at the throat will then be,

$$\frac{p_{th}}{p_0} = \frac{p_B}{p_0} = \frac{p^*}{p_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{1-k}}, \tag{13.129}$$

and the fluid speed will be,

$$V_{th} = V^* = c^* = \sqrt{kRT^*}. \tag{13.130}$$

Any further decrease in p_B has no effect on the speed at the throat since the pressure information can no longer propagate upstream into the reservoir. The fluid speed out of the tank is the same as the speed of the sound wave into the tank so the pressure information can't propagate upstream of the throat. Thus, all flow conditions upstream of the throat will remain unchanged. As a result, we can no longer increase the mass flow rate from the tank by changing the back pressure. This condition is referred to as choked flow conditions. The maximum, or choked, mass flow rate will be the same as the mass flow rate at the throat where sonic conditions occur,

$$\dot{m}_{\text{choked}} = \rho^* V^* A^* = \frac{p^*}{RT^*} V^* A^*, \tag{13.131}$$

$$= p^* \sqrt{\frac{k}{RT^*}} A^*, \tag{13.132}$$

where Eq. (13.130) has been used. Substituting the following relations,

$$p^* = \frac{p^*}{p_0} p_0 = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{1-k}} p_0, \tag{13.133}$$

$$T^* = \frac{T^*}{T_0} T_0 = \left(1 + \frac{k-1}{2}\right)^{-1} T_0, \tag{13.134}$$

$$\tag{13.135}$$

and simplifying results in,

$$\boxed{\dot{m}_{\text{choked}} = \left(1 + \frac{k-1}{2}\right)^{\frac{1+k}{2(1-k)}} p_0 \sqrt{\frac{k}{RT_0}} A^*}. \tag{13.136}$$

Notes:

(1) The choked mass flow rate (Eq. (13.136)) is the maximum mass flow rate that can be achieved from the reservoir.

(2) A quick check to see if the flow will be choked or not for the converging nozzle case is to check if the back pressure is less than or equal to the sonic pressure, i.e.,

$$\text{If } \frac{p_B}{p_0} \leq \frac{p^*}{p_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{1-k}}, \text{ then the flow will be choked and } \frac{p_{th}}{p_0} = \frac{p^*}{p_0}. \tag{13.137}$$

Note that the criterion for checking for choked flow in a converging-diverging nozzle is different, as is discussed in Section 13.18. The key concept to keep in mind is, if the flow anywhere in the channel is equal to or greater than the speed of sound, then sound waves cannot propagate upstream into the reservoir.

(3) What happens outside of the nozzle if the back pressure is less than the sonic pressure? In that case the flow must eventually adjust to the surrounding pressure. It does so by expanding in a two-dimensional process known as an expansion fan, a topic addressed in Section 13.24 (Figure 13.23).

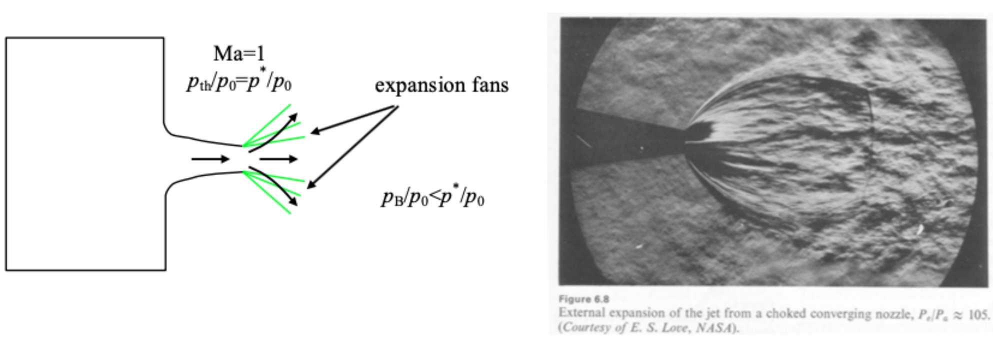


Figure 13.23. Illustration and photograph showing an expansion fan downstream of the exit of a converging nozzle. For this case, the back pressure is less than the sonic pressure and, thus, the flow rapidly expands from sonic conditions at the throat to the surrounding (lower) back pressure after leaving the nozzle.

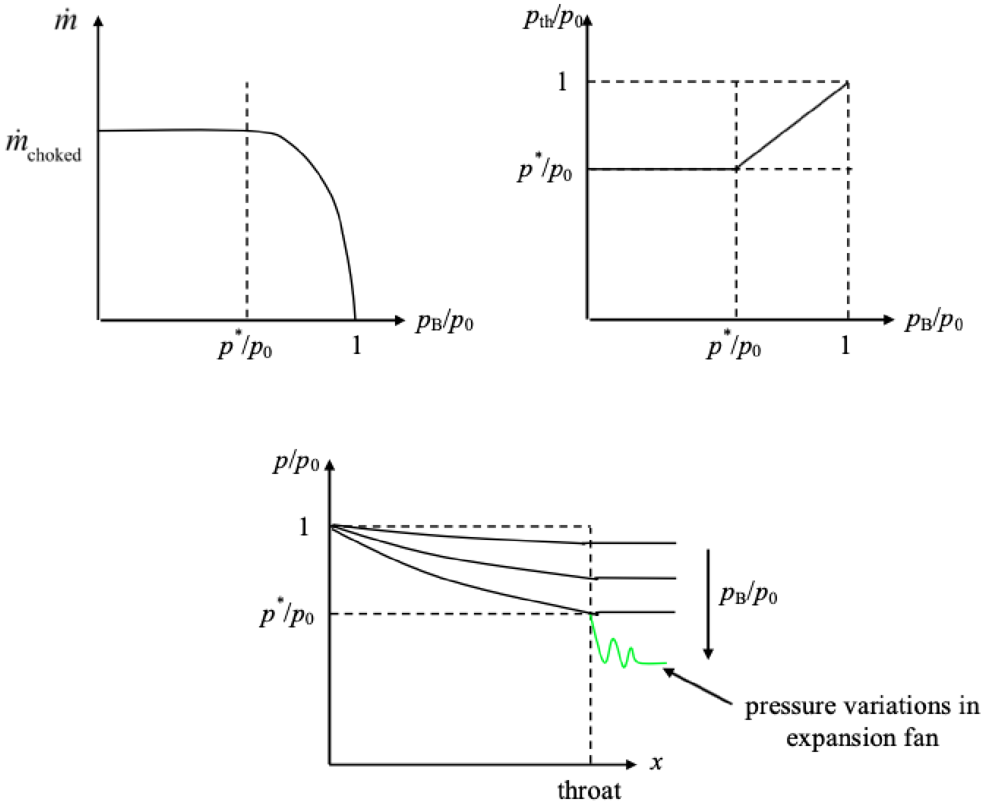


Figure 13.24. Plots showing how mass flow rate in a converging nozzle varies with the back pressure to stagnation pressure ratio (upper left), throat pressure ratio varies with the back pressure ratio (upper right), and pressure ratio within the nozzle to the position (bottom).

(4) The previously described processes are sketched in the plots shown in Figure 13.24. The upper left plot shows that, as the back pressure ratio decreases from one, the mass flow rate decreases until the back pressure reaches sonic conditions. At this point the mass flow rate equals, and remains, at the choked mass flow rate since further decreases in back pressure can't propagate upstream of the throat, where the Mach number equals one.

The upper right plot shows that the pressure at the throat equals the back pressure as the back pressure decreases (since the flow at the nozzle exit is subsonic) until the back pressure reaches the sonic pressure. At this point the Mach number at the nozzle exit equals one. Further decreases in the back pressure no longer change the conditions at or upstream of the throat since the pressure information can't propagate upstream of where the Mach number equals one (at the throat).

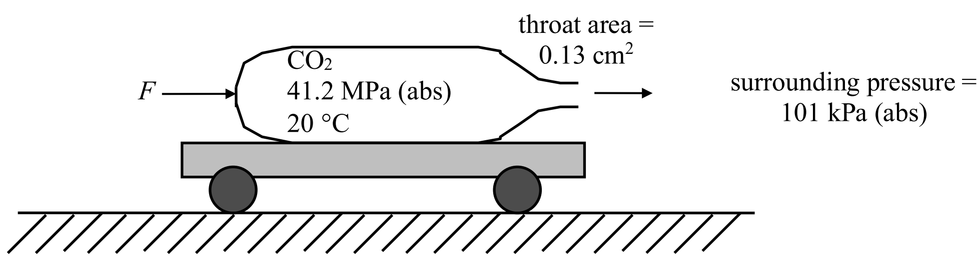
The bottom plot shows the pressure profile within the (converging) nozzle. As the back pressure decreases, the pressure decreases moving toward the throat since for a subsonic flow, a decreasing area results in an increasing speed and, from Bernoulli's equation, a decreasing pressure. At the nozzle exit the exit pressure equals the back pressure when the exit flow is subsonic. When the back pressure is equal to the sonic pressure, the pressure at the nozzle exit is also equal to the sonic pressure and the flow becomes choked. Further decreases in the back pressure aren't propagated upstream of the throat (where the Mach number is one) and, thus, the flow in the converging section remains unchanged. However, once the flow leaves the nozzle exit, it must expand in order to come into equilibrium with the smaller back pressure. It does so through a phenomenon known as an expansion fan, which is a topic covered in Section 13.24.

Video solution: https://www.youtube.com/watch?v=yCge25EldjY

A CO_2 cartridge is used to propel a small rocket cart. Compressed CO_2, stored at a pressure of 41.2 MPa (abs) and a temperature of 20 °C, is expanded through a smoothly contoured converging nozzle with a throat area of 0.13 cm^2. Assume that the cartridge is well insulated and that the pressure surrounding the cartridge is 101 kPa (abs). For the given conditions,

a. Calculate the pressure at the nozzle throat.
b. Evaluate the mass flow rate of carbon dioxide through the nozzle.
c. Determine the force, *F*, required to hold the cart stationary.
d. Sketch the process on a *T*-*s* diagram.
e. For what range of cartridge pressures will the flow through the nozzle be choked?
f. Will the mass flow rate from the cartridge remain constant for the range of cartridge pressures you found in part (e)? Explain your answer.
g. Write down (but do not solve) the differential equations describing how the pressure within the tank varies with time while the flow is choked.

Note: For CO_2, the ideal gas constant is 189 J/(kg-K) and the specific heat ratio is 1.30.



SOLUTION:

First check to see if the flow is choked upon leaving the cartridge.

$$\frac{p_b}{p_0} \overset{?}{\leq} \frac{p^*}{p_0} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{\gamma}{1-\gamma}} = 0.5457 \ \left(\text{using } \gamma\text{=1.3}\right) \quad (1)$$

Since

$$\frac{p_b}{p_0} = \frac{101*10^3 \text{ Pa}}{41.2*10^6 \text{ Pa}} = 2.45*10^{-3} < \frac{p^*}{p_0} = 0.5457 \ \Rightarrow \underline{\text{The flow is choked!}} \quad (2)$$

Because the flow is choked, the throat (exit) pressure will be the sonic pressure:

$$p_E = p^* = 0.5457 p_0 = (0.5457)(41.2 \text{ MPa}) \quad (3)$$

$$\boxed{\therefore p_E = 22.5 \text{ MPa}} \quad (4)$$

The mass flow rate will be the choked flow mass flow rate:

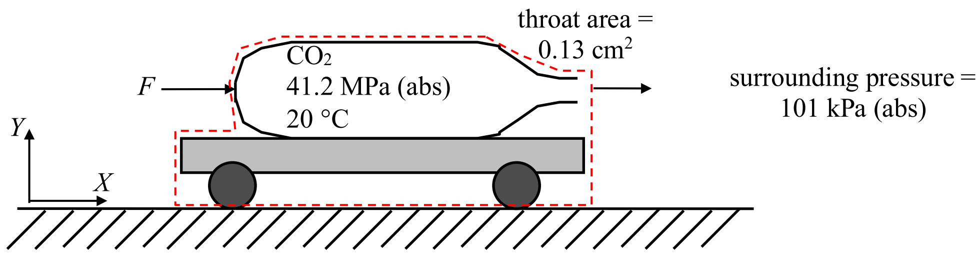
$$\dot{m} = \dot{m}_{\text{choked}} = \left(1+\tfrac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \quad (5)$$

$$\boxed{\therefore \dot{m} = 1.52 \text{ kg/s}} \quad (6)$$

where

γ = 1.3
p_0 = 41.2 MPa
R = 189 J/(kg·K)
T_0 = 20 °C = 293 K
A^* = 0.13 cm^2 = 1.3*10^{-5} m^2 (The throat area is the sonic area since the flow is choked there.)

The force required to hold the cart stationary may be found using the linear momentum equation in the x-direction applied to the control volume shown below using a fixed frame of reference.



$$\frac{d}{dt}\int_{CV} u_X \rho dV + \int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X} \tag{7}$$

where

$$\frac{d}{dt}\int_{CV} u_X \rho dV \approx 0 \quad \text{(The CV is stationary so the fluid essentially has zero velocity in the CV.)} \tag{8}$$

$$\int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = V_E \dot{m} \tag{9}$$

$$F_{B,X} = 0 \quad \text{(No gravity in the } X\text{-direction.)} \tag{10}$$

$$F_{S,X} = F + \left(p_{\text{atm}} - p_E\right) A_E \quad \text{(Need to include pressure forces in the surface force balance.)} \tag{11}$$

Substitute and simplify.

$$V_E \dot{m} = F + \left(p_{\text{atm}} - p_E\right) A_E \tag{12}$$

$$\boxed{F = V_E \dot{m} + \left(p_E - p_{\text{atm}}\right) A_E} \tag{13}$$

$$\boxed{\therefore F = 671 \text{ N}} \tag{14}$$

where

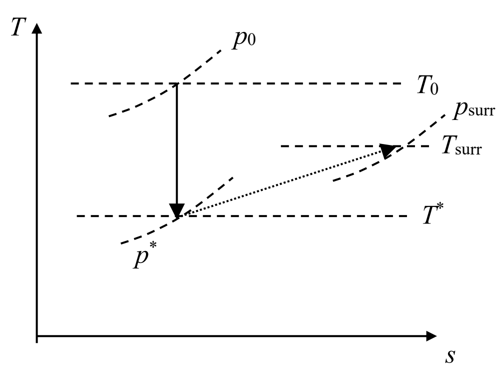
$\dot{m}$ = 1.52 kg/s (from part b)
p_E = 22.5*10^6 Pa (from part a)
p_{atm} = 101*10^3 Pa
A_E = 0.13 cm^2 = 1.3*10^{-5} m^2

and

$$V_E = c_E \underbrace{\text{Ma}_E}_{=1} = \sqrt{\gamma R T_E} = 250 \text{ m/s} \quad \text{(using } \gamma = 1.3,\ R = 189 \text{ J/(kg·K), and} \tag{15}$$

$$T_E = T^* = T_0 \left(1 + \tfrac{\gamma - 1}{2}\right)^{-1} = \left(293 \text{ K}\right)\left(0.8696\right) = 255 \text{ K} \tag{16}$$

The T-s diagram for the process is:



The flow will be choked when the back pressure is less than or equal to the sonic pressure:

$$\frac{p_{\text{surr}}}{p_0} \leq \frac{p^*}{p_0} = 0.5457 \quad (\text{using } \gamma = 1.3) \tag{17}$$

$$\boxed{\therefore p_0 \geq 185 \text{ kPa}} \quad (\text{using } p_{\text{back}} = 101 \text{ kPa}) \tag{18}$$

The <u>mass flow rate from the cartridge will not, in general, be constant</u> since the choked flow mass flow rate depends both on the stagnation pressure and stagnation temperature, *i.e.*

$$\dot{m}_{\text{choked}} = \left(1+\tfrac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \tag{19}$$

The stagnation pressure and temperature in the cartridge will vary in time (as shown below).

From conservation of mass on the previously shown control volume:

$$\frac{dM_0}{dt} = -\dot{m} = -\left(1+\tfrac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* = -\left(1+\tfrac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} \underbrace{\rho_0}_{=M_0/V_0} \sqrt{\gamma R T_0} A^* \tag{20}$$

From conservation of energy on the same control volume:

$$\frac{d}{dt}\left(M_0 c_v T_0\right) + \dot{m}\left(c_p T_E + \tfrac{1}{2}V_E^2\right) = 0 \quad (\text{the cartridge is insulated so there is no heat transfer}) \tag{21}$$

where perfect gas behavior has been assumed and

$$T_E = T^* = T_0\left(1+\tfrac{\gamma-1}{2}\right)^{-1} \tag{22}$$

$$V_E = c^* = \sqrt{\gamma R T^*} \tag{23}$$

Equations (19) - (23) present a coupled set of ordinary differential equations which would be solved numerically subject to the initial conditions:

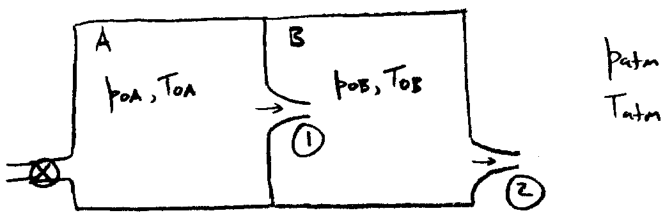
$$T_0(t=0) = 293 \text{ K} \tag{24}$$

$$M_0(t=0) = \rho_0 V_0 = p_0 V_0/(RT_0) \tag{25}$$

The stagnation pressure, p_0, and temperature, T_0, in a large tank (tank A) are maintained through a regulator valve. Tank A exhausts into tank B through a converging nozzle. The exit of the nozzle is station 1. Tank B exhausts through another converging nozzle to the atmosphere. The exit of that nozzle is station 2. The atmospheric temperature and pressure are p_{atm} (1 atm) and T_{atm} (400 K). Determine Ma_1, Ma_2, p_1, p_2, and $\dot{m}$ for the conditions stated below. Assume that the fluid is a perfect gas (air) and that the flow through the converging nozzles is isentropic.

a. p_{0A}=10 atm, T_{0A}=1000 K, A_1=0.01 m^2, A_2=0.03 m^2
b. p_{0A}=4 atm, T_{0A}=1000 K, A_1=0.01 m^2, A_2=0.03 m^2
c. p_{0A}=10 atm, T_{0A}=1000 K, A_1=0.03 m^2, A_2=0.01 m^2
d. p_{0A}=1.5 atm, T_{0A}=1000 K, A_1=0.03 m^2, A_2=0.01 m^2

SOLUTION:



- Assume steady flow conditions: $\dot{m}_1 = \dot{m}_2$

where
$$\dot{m} = \rho V A = \frac{\gamma p}{\gamma R T}(VA) = \frac{\gamma p}{c}\left(\frac{V}{c}\right)A = \frac{\gamma p}{c}(Ma)A$$
$$= \gamma\, Ma\, A\, \frac{p_0\left(1+\frac{\gamma-1}{2}Ma^2\right)^{\frac{\gamma}{1-\gamma}}}{c_0\left(1+\frac{\gamma-1}{2}Ma^2\right)^{-\frac{1}{2}}}$$
$$= \frac{\gamma p_0 A\, Ma}{\sqrt{\gamma R T_0}}\left(1+\frac{\gamma-1}{2}Ma^2\right)^{\frac{\gamma}{1-\gamma}+\frac{1}{2}}$$
$$\therefore\ \dot{m} = p_0 A\, Ma\sqrt{\frac{\gamma}{RT_0}}\left(1+\frac{\gamma-1}{2}Ma^2\right)^{\frac{\gamma+1}{2(1-\gamma)}}$$

$$\dot{m}_1 = \dot{m}_2 \Rightarrow p_{01}A_1 Ma_1\sqrt{\frac{\gamma}{RT_{01}}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma+1}{2(1-\gamma)}} = p_{02}A_2 Ma_2\sqrt{\frac{\gamma}{RT_{02}}}\left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{\frac{\gamma+1}{2(1-\gamma)}}$$

$$\Rightarrow \frac{p_{01}A_1}{p_{02}A_2} = \sqrt{\frac{T_{01}}{T_{02}}}\,\frac{Ma_2}{Ma_1}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}Ma_1^2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

Assuming the flow in the tanks is adiabatic, $T_{01} = T_{02}$

$$\therefore\ \frac{p_{01}A_1}{p_{02}A_2} = \frac{Ma_2}{Ma_1}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}Ma_1^2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

SOLUTION...

a) Assume the flow is choked at both ① and ②:

$\Rightarrow Ma_1 = Ma_2 = 1$

$$\frac{p_{01} A_1}{p_{02} A_2} = 1 \Rightarrow \underline{p_{02}} = p_{01}\left(\frac{A_1}{A_2}\right) = (10\,\text{atm})\left(\frac{0.01\,\text{m}^2}{0.03\,\text{m}^2}\right) = \underline{3.33\,\text{atm}}$$

$$\frac{p_1}{p_{01}} = \frac{p^*}{p_{01}} = 0.5283 \Rightarrow \underline{p_1} = 0.5283\,(10\,\text{atm}) = \underline{5.283\,\text{atm}}$$

- Since $p_1 = 5.283\,\text{atm} > p_{02} = 3.33\,\text{atm}$, then choked flow assumption at ① is good.

$$\frac{p_2}{p_{02}} = \frac{p^*}{p_{02}} = 0.5283 \Rightarrow \underline{p_2} = 0.5283\,(3.33\,\text{atm}) = \underline{1.759\,\text{atm}}$$

- Since $p_2 = 1.759\,\text{atm} > p_{atm} = 1\,\text{atm}$, then choked flow assumption at ② is good.

- Calculate mass flow rate: $\dot{m} = p_{01} A_1 Ma_1 \sqrt{\frac{\gamma}{RT_{01}}}\left(1 + \frac{\gamma - 1}{2} Ma_1^2\right)^{\frac{\gamma+1}{2(1-\gamma)}}$

$\Rightarrow \dot{m} = 12.91\,\text{kg/s}$

$$\boxed{\begin{array}{ll} \therefore\ Ma_1 = 1 & p_1 = 5.283\,\text{atm} \\ Ma_2 = 1 & p_2 = 1.759\,\text{atm} \\ \multicolumn{2}{c}{\dot{m} = 12.91\,\text{kg/s}} \end{array}}$$

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SOLUTION...

b) • Assume flow is choked at ① & ②:

$$p_{02} = p_{01}\left(\frac{A_1}{A_2}\right) = (4\,\text{atm})\left(\frac{0.01\,\text{m}^2}{0.03\,\text{m}^2}\right) = 1.333\,\text{atm} > p_{atm} = 1\,\text{atm}$$

$\therefore$ flow not choked at ②

• Assume flow is choked at ① and subsonic at ②:

$$\frac{p_1}{p_{01}} = \frac{p^*}{p_{01}} = 0.5283 \Rightarrow \underline{p_1} = 0.5283\,(4\,\text{atm}) = \underline{2.113\,\text{atm}}$$

$$p_2 = p_{atm} = 1\,\text{atm} \quad \text{(since subsonic)}$$

$$\dot{m}_1 = p_{01} A_1 Ma_1 \sqrt{\frac{\gamma}{RT_{01}}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$= (4.04\times10^5\,\text{Pa})(0.01\,\text{m}^2)(1)\sqrt{\frac{1.4}{(287\,\text{J/kg}\cdot\text{K})(1000\,\text{K})}}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$\therefore \underline{\dot{m}_1 = 5.164\,\text{kg/s}}$$

$$\underline{\dot{m}_2} = \rho_2 V_2 A_2 = \left(\frac{p_2}{RT_2}\right)(c_2 Ma_2)A_2 = \frac{p_2}{\sqrt{RT_2}}\sqrt{\gamma}\,Ma_2 A_2$$

$$= p_2 A_2 Ma_2\sqrt{\frac{\gamma}{RT_2}} = \underline{p_2 A_2 Ma_2\sqrt{\frac{\gamma}{RT_{02}}}\left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{1/2}}$$

• Since $\dot{m}_1 = \dot{m}_2$

$$\Rightarrow \quad p_2 A_2\sqrt{\frac{\gamma}{RT_{02}}}\,Ma_2\left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{1/2} = \dot{m}_1$$

$$\Rightarrow \quad Ma_2^2\left(1+\frac{\gamma-1}{2}Ma_2^2\right) = \left(\frac{\dot{m}_1}{p_2 A_2}\right)^2\frac{RT_{02}}{\gamma}$$

$$\Rightarrow \quad \underline{Ma_2^4 + \frac{2}{\gamma-1}Ma_2^2 - \left(\frac{2}{\gamma-1}\right)\left(\frac{RT_{02}}{\gamma}\right)\left(\frac{\dot{m}_1}{p_2 A_2}\right)^2 = 0}$$

• Solve for Ma_2 given $\gamma = 1.4$, $R = 287\,\text{J/kg}\cdot\text{K}$, $T_{02} = T_{01} = 1000\,\text{K}$
$\dot{m}_1 = 5.164\,\text{kg/s}$, $p_2 = 101\times10^3\,\text{Pa}$, $A_2 = 0.03\,\text{m}^2$

$$Ma_2^2 = 0.538$$

$$\therefore \underline{Ma_2 = 0.733} \quad \text{(subsonic as assumed)}$$

$\therefore$ $Ma_1 = 1$ $\quad p_1 = 2.113\,\text{atm}$
$Ma_2 = 0.733$ $\quad p_2 = 1\,\text{atm}$
$\dot{m} = 5.164\,\text{kg/s}$

SOLUTION...

c) • Assume the flow is choked at 1 and 2:

$$p_{02} = p_{01}\left(\frac{A_1}{A_2}\right) = (10\,\text{atm})\left(\frac{0.03\,\text{m}^2}{0.01\,\text{m}^2}\right) = 30\,\text{atm} > p_{01} = 10\,\text{atm}$$

but p_{02} cannot be larger than p_{01} (2nd Law)

• Assume that flow is subsonic at 1 and choked at 2:

$$\frac{p_1}{p_{01}} = \frac{p_{02}}{p_{01}} = \left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \underline{p_{02} = p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}}}$$

• $$\dot m_1 = p_{01}A_1 Ma_1\sqrt{\frac{\gamma}{RT_{01}}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$\dot m_2 = p_{02}A_2\sqrt{\frac{\gamma}{RT_{02}}}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} \qquad (Ma_2 = 1 \text{ since assuming choked})$$

$\dot m_1 = \dot m_2$

$$\Rightarrow p_{01}A_1 Ma_1\sqrt{\frac{\gamma}{RT_{01}}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1+\gamma}{2(1-\gamma)}} = p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}}A_2\sqrt{\frac{\gamma}{RT_{02}}}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$\Rightarrow Ma_1\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1+\gamma}{2(1-\gamma)}-\frac{\gamma}{1-\gamma}} = \frac{A_2}{A_1}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$Ma_1\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1}{2}} = \frac{A_2}{A_1}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$\underline{Ma_1^4 + \frac{2}{\gamma-1}Ma_1^2 - \frac{2}{\gamma-1}\left(\frac{A_2}{A_1}\right)^2\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{1-\gamma}} = 0}$$

• Solve for Ma_1 using $\gamma = 1.4$, $A_1 = 0.03\,\text{m}^2$, $A_2 = 0.01\,\text{m}^2$

$\Rightarrow Ma_1^2 = 0.0369$

$\therefore$ $\underline{Ma_1 = 0.192}$ (subsonic as assumed)

$$\Rightarrow \underline{p_1} = (10\,\text{atm})\left(1+\frac{\gamma-1}{2}(0.192)^2\right)^{\frac{\gamma}{1-\gamma}} = \underline{9.746\,\text{atm}} = p_{02}$$

$$\underline{p_2} = \frac{p^*}{p_{02}}p_{02} = (0.5283)(9.746\,\text{atm}) = \underline{5.149\,\text{atm}}$$

$\underline{\dot m = 12.58\,\text{kg/s}}$

$$\boxed{\therefore \begin{array}{ll} Ma_1 = 0.192 & p_1 = 9.746\,\text{atm} \\ Ma_2 = 1 & p_2 = 5.149\,\text{atm} \\ \multicolumn{2}{c}{\dot m = 12.58\,\text{kg/s}} \end{array}}$$

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SOLUTION...

d) • Assume the flow is subsonic at both 1 and 2:

$$p_1 = p_{02}$$
$$p_2 = p_{atm}$$

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1 = p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} = p_{02}$$

$$\frac{p_2}{p_{02}} = \left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \frac{p_{atm}}{p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}}} = \left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\Rightarrow \underline{Ma_2 = \left\{\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{p_{atm}}{p_{01}}\right)^{\frac{1-\gamma}{\gamma}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{-1} - 1\right]\right\}^{1/2}} \quad \text{A}$$

$$\dot{m}_1 = \dot{m}_2$$

$$\Rightarrow \frac{p_{01}A_1}{p_{02}A_2} = \frac{Ma_1}{Ma_2}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}Ma_1^2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$\text{but } p_{02} = p_1 = p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \frac{p_{01}}{p_{02}} = \left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\Rightarrow \underline{\frac{Ma_2}{Ma_1}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}Ma_1^2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} = \frac{A_1}{A_2}} \quad \text{B}$$

• Solve eqns A and B numerically for Ma_1 and Ma_2 with:
$\gamma = 1.4$, $p_{atm} = 101\times10^3$ Pa, $p_{01} = 151.5\times10^3$ Pa, $A_1 = 0.03\text{ m}^2$, $A_2 = 0.01\text{ m}^2$

$$\left.\begin{aligned} Ma_1 &= 0.182 \\ Ma_2 &= 0.760 \end{aligned}\right\} \text{ subsonic as assumed}$$

$$p_1 = 1.47\text{ atm}$$
$$p_2 = 1\text{ atm}$$

$$\dot{m} = p_{01}A_1Ma_1\sqrt{\frac{\gamma}{RT_{01}}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma+1}{2(1-\gamma)}}$$
$$= 1.79\text{ kg/s}$$

$$\boxed{\begin{aligned} \therefore\ Ma_1 &= 0.182 \quad & p_1 &= 1.47\text{ atm} \\ Ma_2 &= 0.760 \quad & p_2 &= 1\text{ atm} \\ \dot{m} &= 1.79\text{ kg/s} \end{aligned}}$$

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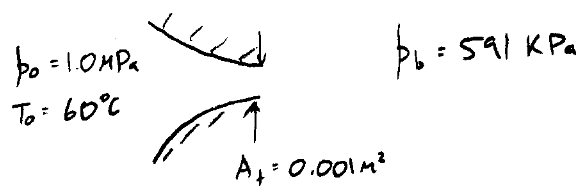
A converging nozzle, with a throat area of 0.001 m^2, is operated with air at a back pressure of 591 kPa (abs). The nozzle is fed from a large plenum chamber where the absolute stagnation pressure and temperature are 1.0 MPa and 60 °C. The exit Mach number and mass flow rate are to be determined.

SOLUTION:



- First check to see if the flow is choked.

$$\text{Is} \quad \frac{p_b}{p_o} = \frac{591\times10^3\,Pa}{1.0\times10^6\,Pa} = 0.591 \overset{?}{<} \frac{p^*}{p_o} = 0.5283$$

No. Thus, the flow is <u>not</u> choked.

Since the flow at the exit is subsonic, the exit pressure will equal the back pressure.

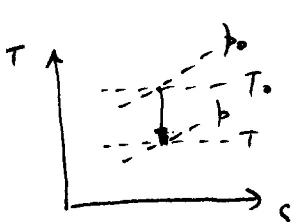
$$p_e = p_b = 591\,KPa$$

- To determine the exit Mach #, use the isentropic flow relations:

$$\frac{p_e}{p_o} = \left(1+\frac{\gamma-1}{2}Ma_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\Rightarrow \boxed{Ma_e = 0.90} \quad \text{using} \begin{cases} \gamma = 1.4 \\ p_e = 591\,KPa \\ p_o = 1.0\,MPa \end{cases}$$

- $\dot{m} = \rho_e V_e A_e$



$$= \left(\frac{p_e}{RT_e}\right)\left(Ma_e\sqrt{\gamma R T_e}\right)A_e \quad \text{where} \quad \frac{T_e}{T_o} = \left(1+\frac{\gamma-1}{2}Ma_e^2\right)^{-1}$$

$$\Rightarrow T_e = 287\,K \quad \text{using} \begin{cases} \gamma = 1.4 \\ T_o = 60°C = 333\,K \\ Ma_e = 0.90 \end{cases}$$

$$\therefore \boxed{\dot{m} = 2.19\,\tfrac{kg}{s}}$$

$$\text{using} \begin{cases} p_e = 591\,KPa & \gamma = 1.4 \\ R = 287\,\frac{J}{kg\cdot K} & A_e = 0.001\,m^2 \\ T_e = 287\,K & \\ Ma_e = 0.90 & \end{cases}$$

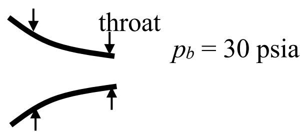
1 of 1

Video solution: https://www.youtube.com/watch?v=6uPUWzn0jE4

Air flows isentropically through a converging nozzle. At a section where the nozzle area is 0.013 ft^2, the local pressure, temperature, and Mach number are 60 psia, 40 °F, and 0.52, respectively. The back pressure is 30 psia. Determine:

a. the Mach number at the throat,
b. the mass flow rate, and
c. the throat area.

SOLUTION:



First determine if the flow is choked by checking the pressure ratio at the exit.

$$\frac{p}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_0 = 72.15 \text{ psia} \quad (1)$$

using p = 60 psia, γ = 1.4, and Ma = 0.52.

$$\frac{p_b}{p_0} = \frac{30 \text{ psia}}{72.15 \text{ psia}} = 0.4158 < \frac{p^*}{p_0} = 0.5283 \Rightarrow \text{The flow is choked!} \quad (2)$$

Since the flow is choked, $\boxed{\text{Ma}_T = 1}$ and the throat area will equal the sonic area:

$$\frac{A}{A_T} = \frac{A}{A^*} = \frac{1}{\text{Ma}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{A_T = A^* = 9.97 \times 10^{-3} \text{ ft}^2} \quad (3)$$

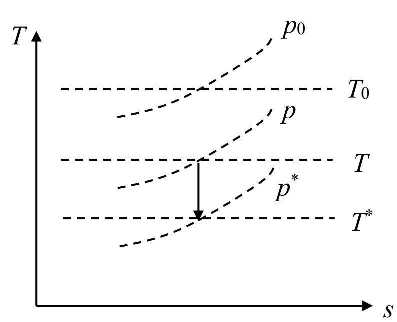
where A = 0.013 ft^2, γ = 1.4, and Ma = 0.52.

The mass flow rate will be the choked mass flow rate:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \Rightarrow \boxed{\dot{m}_{\text{choked}} = 2.40 \text{ lb}_\text{m}/\text{s}} \quad (4)$$

where γ = 1.4, R = 53.3 (lb$_f$·ft)/(lb$_m$·°R), p_0 = 72.15 psia = 1.04*10^4 lb$_f$/ft^2, A^* = 9.97*10^{-3} ft^2 and

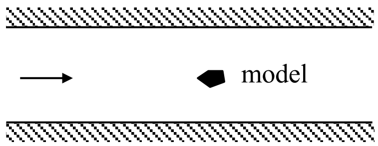
$$\frac{T}{T_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{-1} \Rightarrow \underline{T_0 = 527 \text{ °R}} \text{ (Ma = 0.52 and } T = 500 \text{ °R)} \quad (5)$$



In wind-tunnel testing near Ma = 1, a small area decrease caused by model blockage can be important. Suppose the test section area is 1 m^2, with unblocked test conditions Ma = 1.10 and T = 20 °C.

a. What model area will first cause the test section to choke?
b. If the model cross section is 0.004 m^2 (0.4 % blockage), what percentage change in test section velocity results?

SOLUTION:



First determine the area when the test section will choke. This area will be the sonic area.

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{1}$$

Using A_{TS} = 1 m^2, Ma = 1.10, and γ_{air} = 1.4, $\underline{A^* = 0.992\ \text{m}^2}$. Thus, the model area that will cause the test section to choke is $A_{\text{model}} = A_{\text{TS}} - A^* = (1 - 0.992)\ \text{m}^2 = \boxed{0.008\ \text{m}^2}$.

Using Eqn. (1) with A = (1 – 0.004) m^2 = 0.996 m^2 and A^* = 0.992 m^2, the Mach number in the test section with the blockage is $\underline{\text{Ma} = 1.07}$.

The velocity corresponding to a given Mach number is given by:

$$V = c\text{Ma} = \sqrt{\gamma RT}\,\text{Ma} \tag{2}$$

where the local temperature is found using:

$$T = T_0\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{-1} \tag{3}$$

The percent change in the test section velocity is:

$$\%\ \text{change} = \frac{V_{\text{w/ blockage}} - V_{\text{w/o blockage}}}{V_{\text{w/o blockage}}} = \frac{V_{\text{w/ blockage}}}{V_{\text{w/o blockage}}} - 1$$

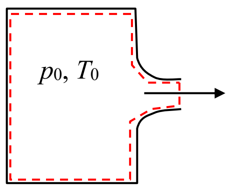
$$= \frac{\text{Ma}_{\text{w/ blockage}}}{\text{Ma}_{\text{w/o blockage}}}\sqrt{\frac{T_{\text{w/ blockage}}}{T_{\text{w/ blockage}}}} - 1$$

$$\boxed{\therefore \%\ \text{change} = \frac{\text{Ma}_{\text{w/ blockage}}}{\text{Ma}_{\text{w/o blockage}}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}^2_{\text{w/ blockage}}}{1+\frac{\gamma-1}{2}\text{Ma}^2_{\text{w/o blockage}}}\right)^{-\frac{1}{2}} - 1} \tag{4}$$

Using $\text{Ma}_{\text{w/ blockage}}$ = 1.07, $\text{Ma}_{\text{w/o blockage}}$ = 1.10, and γ_{air} = 1.4, $\boxed{\%\ \text{change} = -2.2\%.}$

A tank having a volume of 100 ft^3 is initially filled with air at 100 psia and 140 °F. Suddenly the air is allowed to escape to the atmosphere (14.7 psia) through a frictionless converging nozzle of 1 in. diameter. The tank is to be considered as insulated perfectly against heat conduction and as having no heat capacity. Plot the pressure in the tank as a function of time.

SOLUTION:



Assume perfect gas behavior.

First determine the range of tank stagnation pressures that will result in choked flow from the tank.

For $\dfrac{p_B}{p_0} \le \dfrac{p^*}{p_0} = \left(1+\dfrac{\gamma-1}{2}\right)^{\frac{\gamma}{1-\gamma}} = 0.5283$, the flow will be choked. (1)

With p_B = 14.7 psia, the flow will be choked when $p_0 \ge 27.8$ psia. Thus, the flow from the tank is initially choked.

The rate of change of mass within the tank can be found from conservation of mass applied to the control volume shown in the figure.

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

$$\Rightarrow \frac{dM_{\text{tank}}}{dt} = -\dot{m} \tag{2}$$

where $\dot{m}$ is the mass flow rate leaving the tank.

The mass flow rate for a choked flow is given by:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \tag{3}$$

where the pressure p_0 and T_0 are the pressure and temperature inside the tank and A^* is the area of the nozzle exit area (for choked flow conditions, the nozzle exit area is the sonic area). Note that the derivation for Eq. (3) assumes 1D, steady flow. In this problem we'll assume that the steady form of the isentropic flow relations can be used; however, unsteady effects will still be retained for determining the time rate of change of properties within the tank. The pressure within the tank can be related to the temperature and mass within the tank using the ideal gas law.

$$p_0 = \rho_0 RT_0 = \frac{M_{\text{tank}}}{V_{\text{tank}}} RT_0 \tag{4}$$

Hence, Eqn. (3) becomes:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} \left(\frac{M_{\text{tank}}}{V_{\text{tank}}}\right) \sqrt{\gamma RT_0} A^* \tag{5}$$

When the flow is unchoked, the mass flow rate can be found from the conditions at the nozzle exit.

$$\dot{m} = \rho_E V_E A_E = \left[\left(\frac{\rho_E}{\rho_0}\right)\rho_0\right]\left[\mathrm{Ma}_E\sqrt{\gamma R T_0\left(\frac{T_E}{T_0}\right)}\right]A_E$$

$$= \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{1}{1-\gamma}}\rho_0 \mathrm{Ma}_E\sqrt{\gamma R T_0}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{-\frac{1}{2}}A_E$$

$$\dot{m} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{1+\gamma}{2(1-\gamma)}}\left(\frac{M_{\text{tank}}}{V_{\text{tank}}}\right)\mathrm{Ma}_E\sqrt{\gamma R T_0}A_E \tag{6}$$

The Mach number at the exit can be found by combining Eq. (4) with:

$$\frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \quad \text{where } p_E = p_B \text{ since the exit flow is subsonic} \tag{7}$$

to get:

$$\mathrm{Ma}_E = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{p_B}{\frac{M_{\text{tank}}}{V_{\text{tank}}}RT_0}\right)^{\frac{1-\gamma}{\gamma}}-1\right]} \tag{8}$$

To determine the temperature in the tank, apply conservation of energy to the control volume shown in the figure.

$$\frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}} h_0\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{on}\\ \text{CV}}}$$

$$\Rightarrow \frac{d}{dt}\left(uM\right)_{\text{tank}} + \dot{m}h_{0,E} = 0 \quad \text{(where } u = c_V T \text{ is the specific internal energy)}$$

$$M_{\text{tank}}c_V\frac{dT_0}{dt} + c_V T_0\frac{dM_{\text{tank}}}{dt} + \dot{m}\left(c_P T_E + \tfrac{1}{2}V_E^2\right) = 0 \quad \text{(where } h_{0,E} = c_P T_E + \tfrac{1}{2}V_E^2\text{)}$$

$$\frac{dT_0}{dt} - \frac{\dot{m}}{M_{\text{tank}}}T_0 + \frac{\dot{m}}{M_{\text{tank}}}\left(\gamma T_E + \tfrac{1}{2c_V}\gamma R T_E \mathrm{Ma}_E^2\right) = 0 \quad \text{(using Eqn. (2))}$$

$$\frac{dT_0}{dt} - \frac{\dot{m}}{M_{\text{tank}}}T_0 + \frac{\dot{m}}{M_{\text{tank}}}\left[\gamma T_E + \tfrac{(\gamma-1)c_P}{2c_V}T_E \mathrm{Ma}_E^2\right] = 0 \quad \text{(using } \gamma R = (\gamma\text{-}1)c_P\text{)}$$

$$\frac{dT_0}{dt} - \frac{\dot{m}}{M_{\text{tank}}}T_0 + \frac{\dot{m}}{M_{\text{tank}}}\gamma T_E\left[1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right] = 0$$

$$\frac{dT_0}{dt} - \frac{\dot{m}}{M_{\text{tank}}}T_0 + \frac{\dot{m}}{M_{\text{tank}}}\gamma T_0 = 0$$

$$\frac{dT_0}{dt} + \frac{\dot{m}}{M_{\text{tank}}}\left(\gamma-1\right)T_0 = 0 \tag{9}$$

To solve for the tank pressure (p_0) as a function of time, use the following algorithm.

1. Determine the mass flow rate at time step n.
   a. If $p_0 \geq 27.8$ psia:
   $$\dot{m}\big|_n = \dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\left(\frac{M_{\text{tank}}\big|_n}{V_{\text{tank}}}\right)\sqrt{\gamma R T_0\big|_n}\,A^* \tag{10}$$
   b. If $p_0 < 27.8$ psia:
   $$\dot{m}\big|_n = \left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\big|_n\right)^{\frac{1+\gamma}{2(1-\gamma)}}\left(\frac{M_{\text{tank}}\big|_n}{V_{\text{tank}}}\right)\text{Ma}_E\big|_n\sqrt{\gamma R T_0\big|_n}\,A_E \tag{11}$$
   where
   $$\text{Ma}_E\big|_n = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{p_B}{\frac{M_{\text{tank}}\big|_n}{V_{\text{tank}}}R T_0\big|_n}\right)^{\frac{1-\gamma}{\gamma}}-1\right]} \tag{12}$$

2. Determine the change in the tank temperature.
   $$T_0\big|_{n+1} \approx T_0\big|_n + \left.\frac{dT_0}{dt}\right|_n \Delta t \quad \text{(this is a simple Euler integration scheme)} \tag{13}$$
   where Δt is the time step (assumed sufficiently small for stability and accuracy) and
   $$\left.\frac{dT_0}{dt}\right|_n = \frac{\dot{m}\big|_n}{M_{\text{tank}}\big|_n}(1-\gamma)T_0\big|_n \tag{14}$$

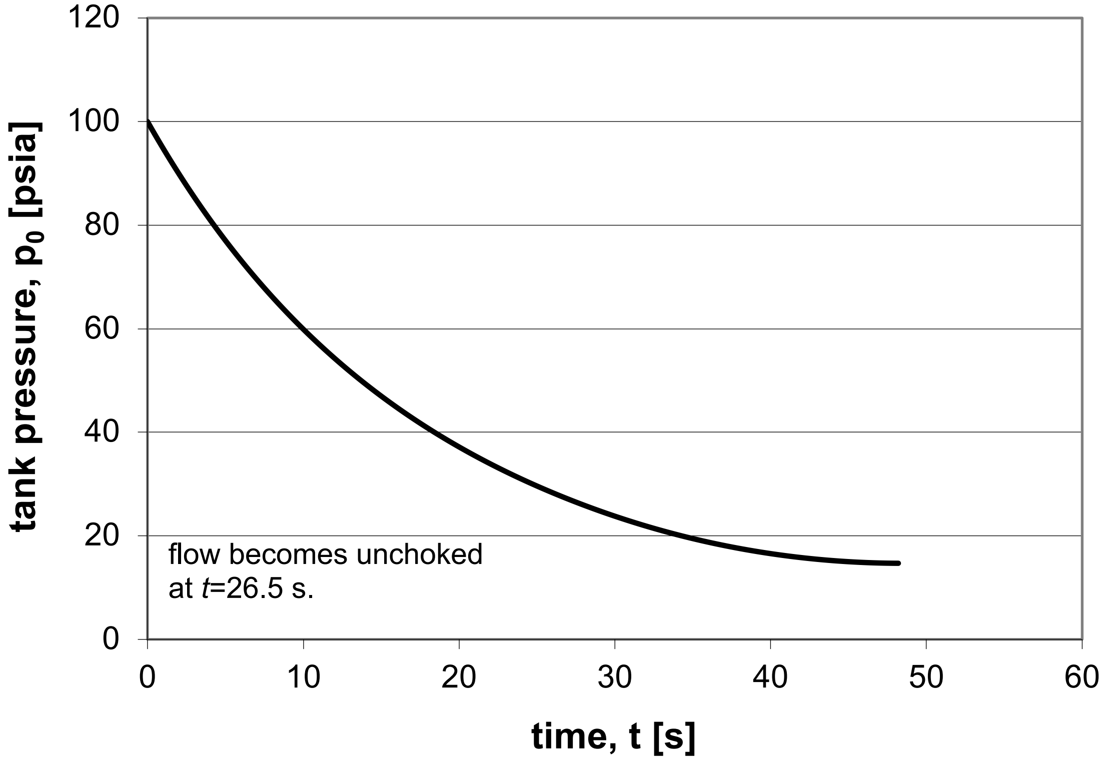
3. Determine the new mass within the tank.
   $$M_{\text{tank}}\big|_{n+1} \approx M_{\text{tank}}\big|_n + \left.\frac{dM_{\text{tank}}}{dt}\right|_n \Delta t \tag{15}$$
   where
   $$\left.\frac{dM_{\text{tank}}}{dt}\right|_n = -\dot{m}_n \tag{16}$$
   and $\dot{m}\big|_n$ is the mass flow rate found in step 1.

4. Determine the new pressure in the tank.
   $$p_0\big|_{n+1} = \frac{M_{\text{tank}}\big|_{n+1}}{V_{\text{tank}}}R T_0\big|_{n+1} \tag{17}$$

5. Repeat the steps 1-5 until the tank pressure equals the back pressure, i.e., $p_0 = p_B$.

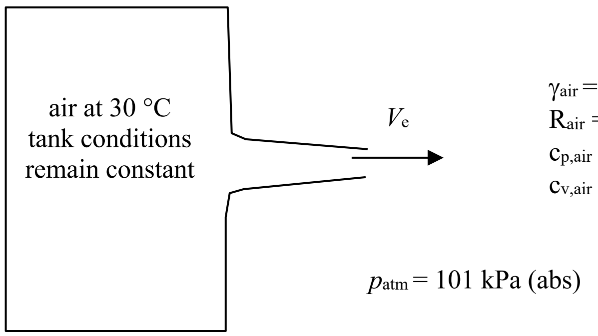
Use the following given data:

$p_0(t = 0)$	=	100 psia
$T_0(t = 0)$	=	140 °F = 500 °R
V	=	100 ft^3
A_E	=	5.45e-3 ft^2
p_B	=	14.7 psia
γ_{air}	=	1.4
R_{air}	=	53.3 (lb$_f$·ft)/(lb$_m$·°R)



The large compressed-air tank shown in the figure exhausts from a nozzle at an exit velocity of $V_e = 235$ m/s. Assuming isentropic flow, compute:

a. the pressure in the tank
b. the exit Mach number
c. Now consider a case where the exit velocity is not given and the tank pressure is 300 kPa (abs). For these conditions, determine the exit flow speed, V_E.



SOLUTION:

First determine the exit Mach number using:

$$\mathrm{Ma}_e = \frac{V_e}{c_e} \tag{1}$$

The exit speed of sound, assuming ideal gas behavior, is given by:

$$c_e = \sqrt{\gamma R T_e} \tag{2}$$

where, for an adiabatic flow:

$$T_0 = T_e + \frac{V_e^2}{2c_p} \tag{3}$$

Using the given data:

γ = 1.4
R = 287 J/(kg·K)
T_0 = 30 °C = 303 K
V_e = 235 m/s
c_p = 1005 J/(kg·K)
$\Rightarrow$ T_e = 276 K
$\Rightarrow$ c_e = 333 m/s
$\Rightarrow$ $\boxed{\mathrm{Ma}_e = 0.71}$

Since the exit Mach number is subsonic, the exit pressure will be equal to the back pressure, *i.e.*

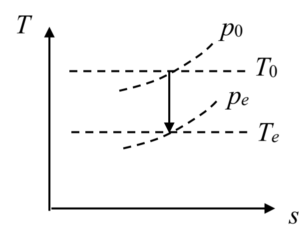
$p_e = p_{atm} = 101$ kPa (abs)

Assuming isentropic flow:

$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_e^2\right)^{\gamma/(1-\gamma)} \tag{4}$$

Using the given data:

$\Rightarrow$ $\boxed{p_0 = 141 \text{ kPa (abs)}}$



Now consider the case where the exit velocity is not given, but the tank pressure is given as p_0 = 300 kPa (abs). First determine whether or not the flow is choked. For a converging nozzle, the flow is choked if,

$$\frac{p_B}{p_0} \le \frac{p^*}{p_0} = \left(1+\frac{k-1}{2}\right)^{\frac{k}{1-k}} \underset{k=1.4}{=} 0.5283 \tag{5}$$

Using the given data (p_0 = 300 kPa (abs) and p_B = 101 kPa (abs)), p_B/p_0 = 0.3367. Thus, the flow is choked for the given conditions and $\text{Ma}_E = 1$.

Since the exit is at sonic conditions, the speed of the flow there is,

$$V_E = V^* = c^* \underbrace{\text{Ma}^*}_{=1} = \sqrt{kRT^*} \tag{6}$$

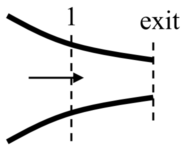
where

$$\frac{T^*}{T_0} = \left(1+\frac{k-1}{2}\right)^{-1} \underbrace{=}_{k=1.4} 0.8333 \tag{7}$$

Using the given data (T_0 = 303 K, k = 1.4, R = 287 J/(kg.K)), T^* = 253 K, and $\boxed{V_E = 319 \text{ m/s}}$.

Air flows isentropically through a converging nozzle. At a section where the nozzle area is 0.013 ft^2, the local pressure, temperature, and Mach number are 60 psia, 40 °F, and 0.52, respectively. The back pressure is 30 psia. The Mach number at the exit, the mass flow rate, and the exit area are to be determined.

SOLUTION:



$p_b = 30$ psia

$A_1 = 0.013$ ft^2
$p_1 = 60$ psia
$T_1 = 40$ °F $= 500$ °R
$\text{Ma}_1 = 0.52$

First determine whether or not the flow is choked by checking the pressure ratio at the exit. In order to do this, we must first determine the flow stagnation pressure (we'll also calculate the stagnation pressure while we're at it). Note that the flow remains subsonic in the nozzle (subsonic Mach number and no minimum area) so that there will be no shock waves in the flow to modify the flow's stagnation pressure.

$$\frac{p_1}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\gamma/(1-\gamma)} \tag{1}$$

$$\frac{T_1}{T_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{-1} \tag{2}$$

Using the given data:

$p_1 = 60$ psia
$T_1 = 500$ °R
$\gamma = 1.4$
$\text{Ma}_1 = 0.52$
$\Rightarrow p_0 = 72.2$ psia
$\Rightarrow T_0 = 527$ °R

From the ideal gas law:

$$\rho_0 = \frac{p_0}{RT_0} \quad \text{(where } R = 53.3\ (\text{ft}\cdot\text{lb}_f)/(\text{lb}_m\cdot°\text{R}) = 1716\ \text{ft}^2/(\text{s}^2\cdot°\text{R})\text{)} \tag{3}$$

$\Rightarrow \rho_0 = 1.21*10^{-3}$ slug/ft^3

Now check to see if $p_b/p_0 < p^*/p_0$.

$$\frac{p_b}{p_0} = \frac{30.0\ \text{psia}}{72.2\ \text{psia}} = 0.4155 < \frac{p^*}{p_0} = 0.5283 \tag{4}$$

$\Rightarrow$ The flow must be sonic at the exit, i.e., $\boxed{\text{Ma}_e = 1}$!

Since the flow is sonic at the exit, we know that the exit area must be the sonic area.

$$\frac{A_1}{A_e} = \frac{A_1}{A^*} = \frac{1}{\text{Ma}_1}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{5}$$

Using the given data:

γ = 1.4
A_1 = 0.013 ft^2
Ma_1 = 0.52

$$\frac{A_1}{A_e} = 1.3034 \quad \Rightarrow \boxed{A_e = 9.97*10^{-3}\ \text{ft}^2}$$

Since the flow is choked, the mass flow rate is:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \tag{6}$$

γ = 1.4
R = 53.3 (ft·lb$_f$)/(lb$_m$·°R) = 1716 ft^2/(s^2·°R)
A_e = 9.97*10^{-3} ft^2 (= A^*)
p_0 = 72.2 psia
T_0 = 527 °R

$$\Rightarrow \boxed{\dot{m} = 7.46*10^{-2}\ \text{slug/s}}$$

We could have also found the mass flow rate using:

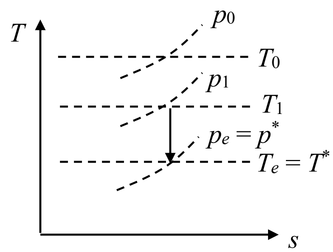
$$\dot{m} = \rho_e V_e A_e$$

where

$$V_e = c_e = \sqrt{\gamma R T_e}$$

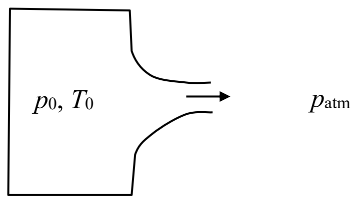
$$\frac{T_e}{T_0} = \frac{T^*}{T_0} = 0.8333$$

$$\frac{\rho_e}{\rho_0} = \frac{\rho^*}{\rho_0} = 0.6339$$



A fixed amount of gaseous fuel is to be fed steadily from a heated tank to the atmosphere through a converging nozzle. The temperature of fuel in the tank remains constant. A young engineer comes to you with the following scheme: "Pressurize the tank to a pressure considerably higher than atmospheric pressure. At the fuel nozzle outlet, the Mach number will then be equal to one. As long as the Mach number is one at the nozzle outlet, we will have the same mass flow rate." Do you agree with the young engineer? Explain your answer.

SOLUTION:



If $p_0 >> p_{atm}$, then the flow will be choked at the nozzle exit. Although the Mach number at the exit plane will remain sonic (i.e., $Ma_E = 1$) while the flow is choked, the mass flow rate will not remain constant since the stagnation pressure within the tank will decrease as mass leaves the tank. Over time, the mass flow rate from the tank will decrease.

$$\dot{m}_{choked} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \tag{1}$$

In this expression, A^* is the throat area (while the flow is choked) and T_0 remains constant, as given in the problem statement. However, the stagnation pressure decreases since,

$$p_0 = \rho_0 RT_0 = \left(\frac{M_{tank}}{\forall_{tank}}\right)RT_0. \tag{2}$$

From Conservation of Mass applied to the tank,

$$\frac{dM_{tank}}{dt} = -\dot{m} \tag{3}$$

Thus, as mass escapes from the tank, the tank mass decreases (Eq. (3)) and, from Eq. (2), the stagnation pressure decreases. Thus, from Eq. (1), the mass flow rate decreases.

A large tank contains 0.7 MPa (abs), 27 °C air. The tank feeds a converging-diverging nozzle with a throat area of $6.45*10^{-4}$ m^2. At a particular point in the nozzle, the Mach number is 2.

a. What is the area at that point?
b. What is the mass flow rate at that point?

SOLUTION:

Use the isentropic relations to determine the downstream Mach number.

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left(\frac{1+\frac{k-1}{2}\text{Ma}^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \boxed{A = 1.09*10^{-3}\ \text{m}^2} \tag{1}$$

where k = 1.4, Ma = 2, and A^* = $6.45*10^{-4}$ m^2 (the throat must be at sonic conditions since the flow goes from stagnation conditions to supersonic conditions).

Since the flow is sonic at the throat, the mass flow rate is choked:

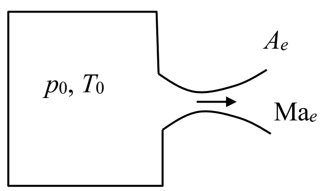
$$\dot{m}_{\text{choked}} = \left(1+\frac{k-1}{2}\right)^{\frac{k+1}{2(1-k)}} p_0\sqrt{\frac{k}{RT_0}}A^* \Rightarrow \boxed{\dot{m} = 1.05\ \text{kg/s}} \tag{2}$$

where R = 287 J/(kg.K).

A large tank supplies helium through a converging-diverging nozzle to the atmosphere. Pressure in the tank remains constant at 8.00 MPa (abs) and temperature remains constant at 1000 K. There are no shock waves in the nozzle. The nozzle is designed to discharge at an exit Mach number of 3.5. The exit area of the nozzle is 100 mm^2. Note that for helium the specific heat ratio is 1.66 and the ideal gas constant is 2077 J/(kg·K).

a. Determine the pressure at the exit of the converging/diverging nozzle.
b. Determine the mass flow rate through the device.
c. Sketch the flow process from the tank through the converging/diverging nozzle to the exit on a *T*-*s* diagram.

SOLUTION:



p_0 = 8.00e6 Pa (abs)
T_0 = 1000 K
Ma_e = 3.5
A_e = 1.0e-4 m^2
γ = 1.66
R = 2077 J/(kg.K)
Assume isentropic flow.

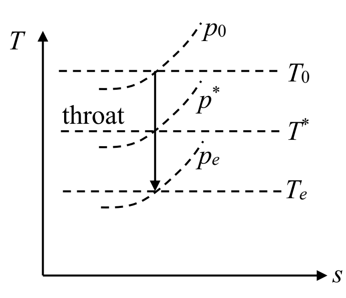
$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_e = 137 \text{ kPa}} \tag{1}$$

$$\frac{T_e}{T_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_e^2\right)^{-1} \Rightarrow T_e = 198 \text{ K} \tag{2}$$

$$\rho_e = \frac{p_e}{RT_e} \Rightarrow \rho_e = 0.332 \text{ kg/m}^3 \tag{3}$$

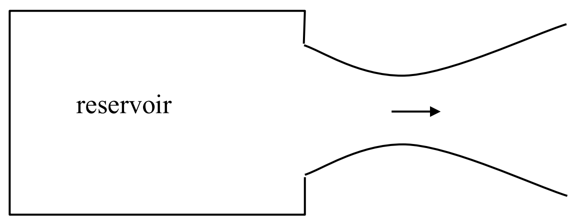
$$V_e = \text{Ma}_e\sqrt{\gamma R T_e} \Rightarrow V_e = 2890 \text{ m/s} \tag{4}$$

$$\dot{m} = \rho_e V_e A_e \Rightarrow \boxed{\dot{m} = 9.59\text{e-6 kg/s}} \tag{5}$$



Video solution: https://www.youtube.com/watch?v=emglT-vKVmA

A rocket engine can be modeled as a reservoir of gas at high temperature feeding gas to a convergent/divergent nozzle as shown in the figure below.



For the questions below, assume the following:

1. The temperature in the reservoir is 3000 K.
2. The exhaust gases have the same properties as air: γ=1.4, R=287 J/(kg·K).
3. The exit Mach number is 2.5.
4. The rocket operates at design conditions (no shock waves or expansion waves present) where the surrounding pressure is $1*10^5$ Pa (abs).
5. The area of the exit is $1*10^{-4}$ m^2.

Determine:

a. the temperature of the flow at the exit,
b. the pressure in the reservoir,
c. the throat area,
d. the mass flow rate out of the rocket,
e. the thrust produced by the rocket, and
f. sketch the process on a T-s diagram.

SOLUTION:

First determine the exit temperature using the adiabatic flow relation for stagnation temperature:

$$\frac{T_E}{T_0} = \left(1 + \frac{\gamma-1}{2}\text{Ma}_E^2\right)^{-1} \Rightarrow \boxed{T_E = 1333\text{ K}} \quad (1)$$

using T_0 = 3000 K, γ = 1.4, and Ma_E = 2.5.

Now determine the pressure in the reservoir using the isentropic stagnation pressure relation:

$$\frac{p_E}{p_0} = \left(1 + \frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_0 = 1.709*10^6\text{ Pa (abs)}} \quad (2)$$

where $p_E = p_B = 1*10^5$ Pa (since the nozzle operates at design conditions, the exit pressure is equal to the back pressure), γ = 1.4, and Ma_E = 2.5.

The throat area may be found using the isentropic sonic area ratio:

$$\frac{A_E}{A^*} = \frac{1}{\text{Ma}_E}\left(\frac{1 + \frac{\gamma-1}{2}\text{Ma}_E^2}{1 + \frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow A^* = \boxed{A_T = 3.79*10^{-5}\text{ m}^2} \quad (3)$$

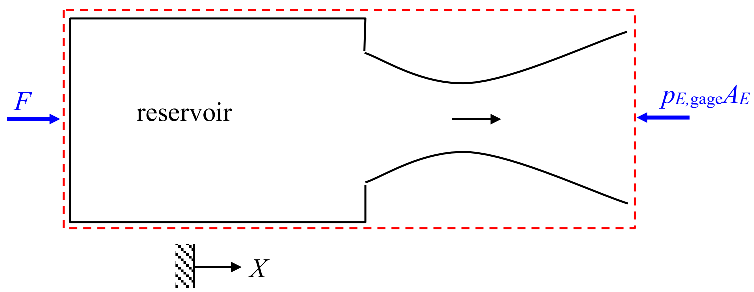
where $A_E = 1*10^{-4}$ m^2, γ = 1.4, and Ma_E = 2.5. Note that since the flow starts from stagnation conditions and is supersonic at the exit, the throat area must also be the sonic area.

The mass flow rate may be found by considering the conditions at the exit:

$$\dot{m} = \rho_E V_E A_E = \left(\frac{p_E}{RT_E}\right)\left(\text{Ma}_E\sqrt{\gamma RT_E}\right)A_E \Rightarrow \boxed{\dot{m} = 4.776*10^{-2}\ \text{kg/s}} \tag{4}$$

where $p_E = p_B = 1*10^5$ Pa (abs), $R = 287$ J/(kg·K), $T_E = 1333$ K, $\text{Ma}_E = 2.5$, $\gamma = 1.4$, and $A_E = 1.0*10^{-4}$ m^2.

The thrust on the rocket may be found by applying the linear momentum equation in the x-direction on the control volume shown below.



$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{CS}} u_X\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X} \tag{5}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV \approx 0 \quad \text{(most of the rocket mass inside the CV remains stationary)} \tag{6}$$

$$\int_{\text{CS}} u_X\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m}V_E = \dot{m}\left(\text{Ma}_E\sqrt{\gamma RT_E}\right) \tag{7}$$

$$F_{B,X} = 0 \tag{8}$$

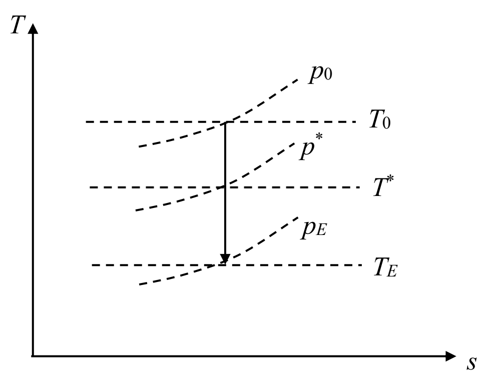
$$F_{S,X} = F - p_{E,\text{gage}}A_E = F - \left(p_E - p_{\text{atm}}\right)A_E \tag{9}$$

However, since the rocket is operating at design conditions, $p_E = p_B = p_{\text{atm}}$.

Substitute and simplify.

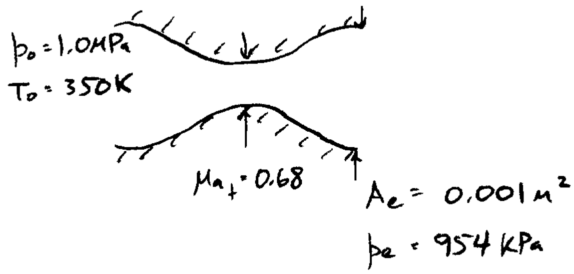
$$F = \dot{m}V_E = \dot{m}\left(\text{Ma}_E\sqrt{\gamma RT_E}\right) \Rightarrow \boxed{F = 87.4\ \text{N}} \tag{10}$$

where $\dot{m} = 4.776*10^{-2}$ kg/s, $\text{Ma}_E = 2.5$, $\gamma = 1.4$, $R = 287$ J/(kg·K), and $T_E = 1333$ K.



Air flows isentropically in a converging-diverging nozzle, with exit area of 0.001 m^2. The nozzle is fed from a large plenum where the stagnation conditions are 350 K and 1.0 MPa (abs). The exit pressure is 954 kPa (abs) and the Mach number at the throat is 0.68. Fluid properties and area at the nozzle throat and the exit Mach number are to be determined.

SOLUTION:



- The flow in the nozzle remains subsonic since the throat does not reach sonic conditions.

$$\Rightarrow \frac{p_e}{p_0} = \left(1 + \frac{\gamma-1}{2} Ma_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\boxed{\therefore Ma_e = 0.26} \quad \text{using} \quad p_e = 954 \text{ kPa},\ p_0 = 1.0 \text{ MPa},\ \gamma = 1.4$$

$$\frac{T_t}{T_0} = \left(1 + \frac{\gamma-1}{2} Ma_t^2\right)^{-1} \quad \text{using} \quad Ma_t = 0.68,\ T_0 = 350\text{K}$$

$$\boxed{\therefore T_t = 320\text{K}}$$

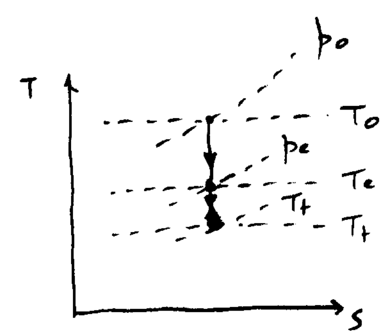
$$\frac{\rho_t}{\rho_0} = \left(1 + \frac{\gamma-1}{2} Ma_t^2\right)^{\frac{1}{1-\gamma}} \quad \text{using} \quad Ma_t = 0.68,\ \rho_0 = p_0/RT_0 = 9.96 \text{ kg/m}^3$$

$$\boxed{\therefore \rho_t = 7.98 \text{ kg/m}^3}$$

$$\frac{A_e}{A^*} = \frac{1}{Ma_e}\left(\frac{1 + \frac{\gamma-1}{2} Ma_e^2}{1 + \frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\Rightarrow \underline{A^* = 4.32 \times 10^{-4} \text{ m}^2}$$

$$\frac{A_t}{A^*} = \frac{1}{Ma_t}\left(\frac{1 + \frac{\gamma-1}{2} Ma_t^2}{1 + \frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$



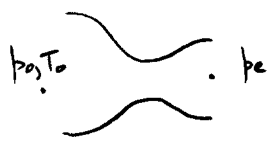
$$\boxed{\therefore A_t = 4.79 \times 10^{-4} \text{ m}^2}$$

$$\frac{p_t}{p_0} = \left(1 + \frac{\gamma-1}{2} Ma_t^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\boxed{\therefore p_t = 734 \text{ kPa}}$$

The control system for some smaller space vehicles uses nitrogen from a high-pressure bottle. When the vehicle has to be maneuvered, a valve is opened allowing nitrogen to flow out through a nozzle thus generating a thrust in the direction required to maneuver the vehicle. In a typical system, the pressure and temperature in the system ahead of the nozzle are about 1.6 MPa (abs) and 30 °C, respectively, while the pressure in the jet on the nozzle exit plane is about 6 kPa (abs). Assuming that the flow through the nozzle is isentropic and the gas velocity ahead of the nozzle is negligible, find the temperature and the velocity of the nitrogen ~~ion~~ in the nozzle exit plane. If the thrust required to maneuver the vehicle is 1 kN, find the area of the nozzle exit plane and the required mass flow rate of nitrogen. It can be assumed that the vehicle is effectively operating in a vacuum.

SOLUTION:



$p_0 = 1.6 \times 10^6$ Pa

$T_0 = 30\,°C = 303$ K

$p_e = 6 \times 10^3$ Pa

$\gamma_{N_2} = 1.4$

$R_{N_2} = 296.8 \frac{J}{kg \cdot K}$

$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma - 1}{2} Ma_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\Rightarrow \quad Ma_e = 4.43$$

$$\frac{T_e}{T_0} = \left(1 + \frac{\gamma - 1}{2} Ma_e^2\right)^{-1}$$

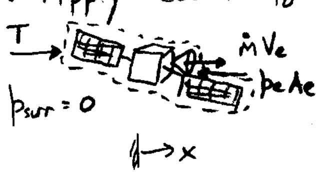
$$\Rightarrow \quad \boxed{T_e = 61.5\text{ K}}$$

$$\rho_e = \frac{p_e}{RT_e} \rightarrow \rho_e = 0.329\ \text{kg/m}^3$$

$$Ma_e = \frac{V_e}{c_e} \quad \Rightarrow \quad V_e = Ma_e c_e = Ma_e \sqrt{\gamma R T_e}$$

$$\Rightarrow \boxed{V_e = 708\ \text{m/s}}$$

- Apply COLM to the spacecraft:



$$\frac{d}{dt}\int_{CV} u_x \rho \, dV + \int_{CS} u_x \rho (\underline{u}_{rel} \cdot \hat{n})\, dS = F_{B,x} + F_{S,x}$$

where $\frac{d}{dt}\int_{CV} u_x \rho \, dV = 0$ (spacecraft initially has zero velocity)

$$\int_{CS} u_x \rho (\underline{u}_{rel} \cdot \hat{n})\, dS = \dot{m} V_e = \rho_e V_e^2 A_e$$

$$F_{B,x} = 0$$

$$F_{S,x} = T - p_e A_e$$

$$\Rightarrow \rho_e V_e^2 A_e = T - p_e A_e$$

SOLUTION...

$$\therefore \quad A_e = \frac{T}{p_e + \rho_e V_e^2}$$

$T = 1\times10^3\,\text{N}$
$p_e = 6\times10^3\,\text{Pa}$
$\rho_e = 0.329\,\text{kg/m}^3$
$V_e = 708\,\text{m/s}$

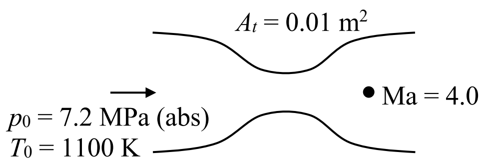
$$\therefore \boxed{A_e = 5.85\times10^{-3}\,\text{m}^2}$$

$$\dot{m} = \rho_e V_e A_e$$

$$\therefore \boxed{\dot{m} = 1.36\,\text{kg/s}}$$

Air, at a stagnation pressure of 7.20 MPa (abs) and a stagnation temperature of 1100 K, flows isentropically through a converging-diverging nozzle having a throat area of 0.01 m^2. Determine the speed and the mass flow rate at the downstream section where the Mach number is 4.0.

SOLUTION:



At the section where Ma = 4.0:

$$\frac{T}{T_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}^2\right)^{-1} \Rightarrow \underline{T = 261.9\text{ K}} \quad (1)$$

where $\gamma = 1.4$, $T_0 = 1100$ K, and Ma = 4.0.

The velocity at the section may be found from the Mach number and speed of sound.

$$V = c\text{Ma} = \sqrt{\gamma RT}\text{Ma} \Rightarrow \boxed{V = 1298\text{ m/s}} \quad (2)$$

where $R = 287$ J/(kg·K).

That mass flow rate is given by:

$$\dot{m} = \rho VA = \left(\frac{p}{RT}\right)VA \Rightarrow \boxed{\dot{m} = 87.6\text{ kg/s}} \quad (3)$$

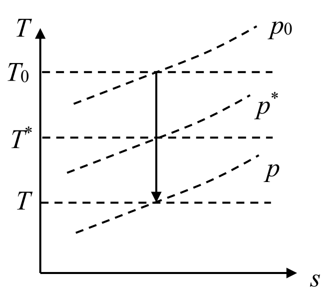
where

$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \underline{p = 4.742*10^4\text{ Pa}} \text{ (using } p_0 = 7.20\text{ MPa)} \quad (4)$$

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left(\frac{1 + \frac{\gamma - 1}{2}\text{Ma}^2}{1 + \frac{\gamma - 1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \underline{A = 0.107\text{ m}^2} \text{ (using } A^* = A_t = 0.01\text{ m}^2) \quad (5)$$

An alternate method for determine the mass flow rate is to use the choked flow mass flow rate expression.

$$\dot{m} = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \Rightarrow \boxed{\dot{m} = 87.7\text{ kg/s}} \text{ (Same result as before, within numerical error!)} \quad (6)$$



Steady, 1D, Isentropic Flow of an Imperfect Gas

Imperfect gas effects become significant when dealing with high temperature and high Mach number flows. Examples of imperfect gas effects include:

1. variations in specific heats due to the activation of intra-molecular energy modes (i.e., rotational, vibration, electronic modes),
2. dissociation of molecules, e.g.,
   for T > 2000 K, $O_2 \to 2O$ and
   for T > 4000 K, $N_2 \to 2N$
3. ionization of atoms, *e.g.*
   for T > 9000 K, $O \to O^+ + e^-$ and $N \to N^+ + e^-$
4. chemical reactions.

Non-equilibrium effects can also be significant at high Mach numbers.

In these notes, we'll focus only on the variations in specific heats due to temperature.

Recall that the governing equations are:

COM: $d(\rho VA) = 0$ (56)

LME: $dp + \rho V dV = 0$ (57)

COE: $dh + VdV = 0$ (58)

2^{nd} Law: $ds = 0$ (59)

Notes:

1. These equations are independent of the type of fluid being considered.

2. The COE and LME statements are equivalent for an isentropic flow. Recall that combining the first and second laws for a simple, compressible system with reversible *pdv* work gives:
$$du = Tds - pdv$$
$$du + d(pv) = Tds - pdv + d(pv) \tag{60}$$
$$dh = Tds + vdp$$
but since ds = 0 and v = 1/ρ:
$$dh = \frac{dp}{\rho} \tag{61}$$
Substituting into COE gives:
$$dp + \rho V dV = 0 \tag{62}$$
This is the same relation that we had from the LME! Hence, the COE and LME expressions are equivalent.

3. Equations (56)-(59) (three equations, recall from note 2 that Eqs. (57) and (58) are equivalent) have a total of five unknowns (ρ, p, V, h, s). Note that area, A, is typically a known quantity. We need two additional extra relations to close the system of equations. These relations are the equations of state for the substance of interest.
$$\rho = \rho(p, s)$$
$$h = h(p, s)$$
The equations of state may be given in equation, tabular, or graphical form. To solve the system of equations, one must generally employ an appropriate numerical scheme.
   a. Note that only two independent properties are required to fix the state of a simple compressible system at equilibrium.

4. This section remains incomplete.
Empirical specific heat data is often expressed in terms of curve fits. Exponential and polynomial curve fits are common. For example, Zucrow and Hoffman use a 4th-order polynomial curve fit:

$$\frac{c_p}{R} = a + bT + cT^2 + dT^3 + eT^4 \quad (63)$$

where $[T]$ = K and the constants $a - e$ are given in the table below for various substances and for various temperature ranges.

gas	***a***	***b**10^3**	***c**10^6**	***d**10^9**	***e**10^{12}**	***h$_0$**10^{-3}**	***s$^0_0$***	***T* range [K]**
O_2	3.62560	-1.87822	7.05545	-6.76351	2.15560	-1.04752	4.30528	300-1000
	3.62195	0.736183	-0.196522	0.0362016	-0.00289456	-1.120198	3.61510	1000-5000
N_2	3.67483	-1.20815	2.32401	-0.632176	-0.225773	-1.06116	2.35804	300-1000
	2.89632	1.51549	-0.572353	0.0998074	-0.00652236	-0.905862	6.16151	1000-5000
CO	3.71009	-1.61910	3.69236	-2.03197	0.239533	-14.3563	2.95554	300-1000
	2.98407	1.48914	-0.578997	0.103646	-0.00693536	-14.2452	6.34792	1000-5000
CO_2	2.40078	8.73510	-6.60709	2.00219	0.000632740	-48.3775	9.69515	300-1000
	4.46080	3.09817	-1.23926	0.227413	-0.0155260	-48.9614	-0.986360	1000-5000
A	2.50000	0	0	0	0	-0.745375	4.36600	300-1000
	2.50000	0	0	0	0	-0.745375	4.36600	1000-5000
H_2	3.05745	2.67652	-5.80992	5.52104	-1.81227	-0.988905	-2.29971	300-1000
	3.10019	0.511195	0.0526442	-0.0349100	0.00369453	-0.877380	-1.96294	1000-5000
H_2O	4.07013	-1.10845	4.15212	-2.96374	0.807021	-30.2797	-0.322700	300-1000
	2.71676	2.94514	-0.802243	0.102267	-0.00484721	-29.9058	6.63057	1000-5000
CH_4	3.82619	-3.97946	24.5583	-22.7329	6.96270	-10.1450	0.866901	300-1000
	1.50271	10.4168	-3.91815	0.677779	-0.0442837	-9.97871	10.7071	1000-5000
C_2H_4	1.42568	11.3831	7.98900	-16.2537	6.74913	5.33708	14.6218	300-1000
	3.45522	11.4918	-4.36518	0.761551	-0.0501232	4.47731	2.69879	1000-5000

a. Note that table also includes data for calculating the specific enthalpy, h, as a function of temperature:

$$h = \int_{T_0}^{T} c_p dT$$

$$\frac{h}{R} = h_0 + aT + \tfrac{1}{2}bT^2 + \tfrac{1}{3}cT^3 + \tfrac{1}{4}dT^4 + \tfrac{1}{5}eT^5 \quad (64)$$

where $h = h_0 R$ when $T = T_0$.

b. The table also includes data for calculating s^0, a quantity that will be discussed in the following note (note #5).

$$s^0 = \int_{T_0}^{T} c_p \frac{dT}{T}$$

$$\frac{s^0}{R} = s_0^0 + a\ln T + bT + \tfrac{1}{2}cT^2 + \tfrac{1}{3}dT^3 + \tfrac{1}{4}eT^4 \quad (65)$$

where $s^0 = s^0_0 R$ when $T = T_0$.

c. The specific internal energy can be determined using the definition for enthalpy and the ideal gas law:

$$u = h - pv = h - RT \quad (66)$$

d. The specific heat at constant volume, c_v, and the specific heat ratio, γ, are determined from:

$$c_v = c_p - R \tag{67}$$

$$\gamma = \frac{c_p}{c_v} \tag{68}$$

e. Include calculations for dry air – incomplete.

The composition of clean dry atmospheric air near sea level.

Component	**% by Volume**
N_2	78.084
O_2	20.9476
A	0.934
CO_2	0.0314
H_2	0.00005
Ne	0.001818
Kr	0.000114
Xe	0.0000087
He	0.000524
CH_4	0.0002
NO	0.00005

f. Alternate c_p expressions for monatomic and diatomic molecules – incomplete.

5. In order to simplify matters when dealing with isentropic flow of an ideal, but imperfect gas (one where the specific heats vary with temperature), we'll define some additional useful properties.

Note that from Eq. (60):

$$ds = \frac{dh}{T} - \frac{dp}{\rho T} \tag{69}$$

For an ideal gas ($dh = c_p dT$ and $p = \rho RT$):

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p} \tag{70}$$

Integrating and using some reference state as the lower limit of integration gives:

$$s_2 - s_1 = \int_{T_1}^{T_2} c_p \frac{dT}{T} - R\ln\left(\frac{p_2}{p_1}\right) \tag{71}$$

Define s^0:

$$s^0 = \int_{T_{\text{ref}}}^{T} c_p \frac{dT}{T} \tag{72}$$

where T_{ref} is some reference state so that equation can be written as:

$$\boxed{s_2 - s_1 = s_2^0 - s_1^0 - R\ln\left(\frac{p_2}{p_1}\right)} \tag{73}$$

Note that Eq. (72) is a function only of temperature. Tabulated values of s^0 are commonly given in the tables of most thermodynamics or gas dynamics texts.

For an isentropic process, Eq. (73) can be written as:

$$\boxed{\frac{p_2}{p_1} = \exp\left(\frac{s_2^0 - s_1^0}{R}\right)} \tag{74}$$

If we define state 1 as the reference state, then Eq. (74) simplifies to:

$$\boxed{p_r = \frac{p}{p_{\text{ref}}} = \exp\left(\frac{s^0}{R}\right)} \tag{75}$$

where p_r is known as the **relative pressure.** Tabulated values of p_r are commonly given in the tables of most thermodynamics or gas dynamics texts. Note that for an isentropic process:

$$\boxed{\frac{p_2}{p_1} = \frac{p_{r2}}{p_{r1}}} \tag{76}$$

Consider the steady, 1D, isentropic flow of air from a tank. Within the tank the pressure and temperature are 10 MPa and 3000 K, respectively. Determine the temperature and velocity at a location downstream of the tank where the pressure is 100 kPa if:

a. the air is assumed to behave as a perfect gas, and
b. the air is ideal but has variable specific heats.

SOLUTION:

$p_0 = 10\times10^6$ Pa $p = 100\times10^3$ Pa
$T_0 = 3000$ K $T = ?$
$V = ?$

- Assume air behaves as a perfect gas:

$$p/p_0 = \left(1+\frac{\gamma-1}{2}Ma^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \underline{Ma = 3.693}$$

using $p = 100\times10^3$ Pa, $p_0 = 10\times10^6$ Pa, $\gamma = 1.4$

$$T/T_0 = \left(1+\frac{\gamma-1}{2}Ma^2\right)^{-1} \Rightarrow \boxed{T = 804.8\text{ K}}$$

using $T_0 = 3000$ K

$$V = Ma\sqrt{\gamma RT} \Rightarrow \boxed{V = 2100.\text{ m/s}}$$

using $R = 287$ J/kg·K

- Now assume air behaves as an imperfect gas:

$$p/p_0 = p_r/p_{0r} \Rightarrow \underline{p_r = (p/p_0)\,p_{0r}} \Rightarrow \underline{p_r = 92.135}$$

using $p/p_0 = \frac{100\times10^3\text{ Pa}}{10\times10^6\text{ Pa}} = 0.0100$

$p_{0r} = 0.92135\times10^4$
from air properties table with $T_0 = 3000$K

- Also from air property tables at $T_0 = 3000$K:

$$\underline{h_0 = 3529.098\ \frac{\text{kJ}}{\text{kg}}}$$

1 of 2

SOLUTION...

- Knowing that $p_r = 92.135$ and using an air properties table:

$$p_r = 92.135 \Rightarrow \boxed{T = 1023.1\text{ K}}$$
$$h = 1073.7\ \text{kJ/kg}$$

- From COE:

$$h_0 = h + \tfrac{1}{2}V^2$$
$$\Rightarrow V = \sqrt{2(h_0 - h)} \Rightarrow \boxed{V = 2216.0\ \text{m/s}}$$

using $h_0 = 3529.098\times10^3\ \frac{\text{J}}{\text{kg}}$
$h = 1073.7\times10^3\ \text{J/kg}$

$\therefore$

	perfect gas	imperfect gas	|% error|
T [K]	804.8	1023.1	21.3%
V [m/s]	2100	2216.0	5.2%

2 of 2

Air is flowing through a duct at a Mach number of 3, a static temperature of 800 K, and a static pressure of 70 kPa (abs). Determine the temperature and pressure of the flow if it is brought to rest isentropically:

a. assuming perfect gas behavior, and
b. assuming imperfect gas behavior.

SOLUTION:

Assuming perfect gas behavior with $\gamma_{\text{air}} = 1.4$:

$$\frac{T}{T_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}^2\right)^{-1} \tag{1}$$

$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}^2\right)^{\frac{\gamma}{1-\gamma}} \tag{2}$$

Using Ma = 3, T = 800 K, and p = 70 kPa:

$\boxed{T_0 = 2240\text{ K}}$

$\boxed{p_0 = 2.57\text{ MPa}}$

Assuming imperfect gas behavior:

$T = 800$ K $\Rightarrow$ $p_r = 0.35216\text{e}2$ (from thermodynamic property tables for air) (3)

$h = 823.112$ kJ/kg

$c = 557.54$ m/s

$V = \text{Ma}c$ $\Rightarrow$ $V = 1672.62$ m/s (4)

$h_0 = h + \frac{1}{2}V^2$ $\Rightarrow$ $h_0 = 2221.941$ kJ/kg (5)

$\Rightarrow$ $\boxed{T_0 = 1977\text{ K}}$ (from thermodynamic property tables for air)

$\Rightarrow$ $p_{0r} = 0.14461\text{e}4$

$$\frac{p}{p_0} = \frac{p_r}{p_{0r}} \Rightarrow p_0 = p\left(\frac{p_{0r}}{p_r}\right) \tag{6}$$

Using p = 70 kPa, p_{0r} = 0.14461e4, and p_r = 0.35216e2, $\boxed{p_0 = 2.87\text{ MPa}}$.

A wind tunnel using air operates with a stagnation pressure and temperature of 10 atm (abs) and 1000 K, respectively. The test section of the tunnel is designed to operate at a Mach number of 5. Determine the pressure and temperature in the wind tunnel at this design Mach number:

a. assuming the perfect gas behavior, and
b. assuming imperfect gas behavior.

SOLUTION:

Assuming perfect gas behavior with $\gamma = 1.4$:

$$p = p_0\left(\frac{p}{p_0}\right) = p_0\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{\frac{\gamma}{1-\gamma}} \tag{1}$$

$$T = T_0\left(\frac{T}{T_0}\right) = T_0\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{-1} \tag{2}$$

With p_0 = 10 atm (abs) (1 atm = 101.3 kPa), T_0 = 1000 K, and Ma = 5, $\boxed{p = 1.915 \text{ kPa (abs)}}$ and $\boxed{T = 166.7 \text{ K}}$.

Assuming imperfect gas behavior:

T_0 = 1000 K $\Rightarrow$ p_{0r} = 8.4110e1
h_0 = 1047.248 kJ/kg
γ_0 = 1.3361

(Using thermodynamics tables for air, e.g., Table C.4 from Zucrow and Hoffman.)

Iterate to determine the pressure and temperature.

1. As an initial first guess for the pressure, assume perfect gas behavior.

$$p = p_0\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{\frac{\gamma}{1-\gamma}} \tag{3}$$

Using $\gamma = 1.3361$, Ma = 5.0, and p_0 = 10 atm = 1.013e6 Pa
p = 1441.7 Pa (abs)

2. Determine the relative pressure, p_r.

$$p_r = p_{0r}\left(\frac{p}{p_0}\right) \quad \text{(since for an isentropic flow, } p_2/p_1 = p_{r2}/p_{r1}\text{)} \tag{4}$$

Using p/p_0 = 1.4232e-3 and p_{0r} = 8.411e1:
p_r = 1.1970e-1 $\Rightarrow$ T = 162.57 K
γ = 1.3976
h = 163.231 kJ/kg
c = 255.37 m/s

(Using thermodynamics tables for air, e.g., Table C.4 from Zucrow and Hoffman.)

3. Determine the velocity, V, from conservation of energy.

$$h_0 = h + \tfrac{1}{2}V^2 \Rightarrow V = \sqrt{2(h_0 - h)} \tag{5}$$

Using h_0 = 1047.248 kJ/kg and h = 163.231 kJ/kg $\Rightarrow$ V = 1329.675 m/s

4. Determine the Mach number from the velocity and speed of sound.

$$\text{Ma} = \frac{V}{c} \tag{6}$$

Using V = 1329.675 m/s and c = 255.37 m/s $\Rightarrow$ Ma = 5.207

5. The calculated Mach number isn't equal to the given Mach number (=5) so we must iterate again. Choose the next guess for the pressure in step 1 using the first guess and the perfect gas ratio required to attain Ma = 5 from Ma = 5.207.

$$\frac{p_{\text{new}}}{p_{\text{prev}}} = \frac{\left(p/p_0\right)_{\text{Ma}=5.0}}{\left(p/p_0\right)_{\text{Ma}=5.207}} \quad \text{where} \quad \frac{p}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{\frac{\gamma}{1-\gamma}} \tag{7}$$

Using γ = 1.3976 (found in step 2) and p_0 = 10 atm (= 1.013e6 Pa):
p_{new} = 1830.7 Pa (abs)

Now repeat steps 2 – 5 until the calculated Mach number is sufficiently close to the given Mach number of 5.0.

Following is a spreadsheet showing the iterations.

T_0 [K] =	1000
p_0 [Pa] =	1.01E+06
Ma =	5.0
p_{0r} =	8.4110E+01
h_0 [kJ/kg] =	1047.248
γ_0 =	1.3361

Ma	**p**	**p_r**	**T [K]**	**γ**	**c [m/s]**	**h [kJ/kg]**	**V [m/s]**	**Ma**	**% error Ma**
5.000	1.4417E+03	1.1970E-01	162.57	1.3976	255.37	163.231	1329.675	5.207	4.13820%
5.207	1.8307E+03	1.5201E-01	174.04	1.3982	264.28	174.791	1320.952	4.998	-0.03436%
4.998	1.8270E+03	1.5170E-01	173.94	1.3982	264.20	174.689	1321.029	5.000	0.00109%
5.000	1.8272E+03	1.5171E-01	173.94	1.3982	264.21	174.692	1321.027	5.000	-0.00003%

perfect gas assumption:

γ =	1.4	% error
T [K] =	166.67	-4.2%
p [Pa] =	1914.61	4.8%

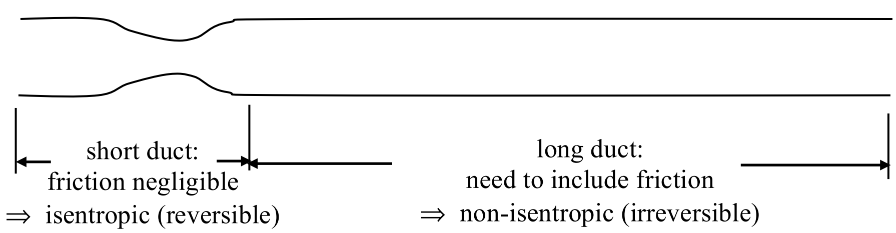
Thus, assuming imperfect gas behavior:

$\boxed{p = 1.827 \text{ kPa (abs)}}$

$\boxed{T = 173.94 \text{ K}}$

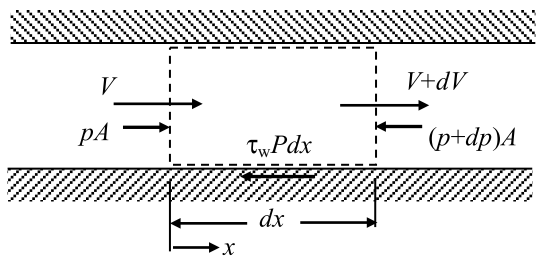
7. Adiabatic Flow with Friction (Fanno Flow)

So far we've examined adiabatic, reversible flows (isentropic flows). For flows in long ducts, the frictional, or viscous, effects can be significant and the flow can no longer be modeled as isentropic.



To examine the effects of friction, let's consider the case of a steady, 1D, adiabatic flow in a constant area duct that has irreversible effects due to friction (viscous) effects.

First let's derive the governing equations for the following differential control volume:



P is the perimeter of the duct

From COM we have:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \qquad \text{(steady flow)}$$

$$\int_{\text{CS}} \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\rho VA + \left[\rho VA + d\left(\rho VA\right)\right]$$

$$= d\left(\rho VA\right)$$

Thus,

$$d\left(\rho VA\right) = \rho AdV + Vd\rho A = 0 \quad \text{(recall that } A\text{=constant)}$$

$$\boxed{\frac{d\rho}{\rho} + \frac{dV}{V} = 0} \tag{77}$$

From the LME in the x-direction we have:

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{\substack{x,\text{body} \\ \text{on CV}}} + F_{\substack{x,\text{surface} \\ \text{on CV}}}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \qquad \text{(steady flow)}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}V + \dot{m}\left(V + dV\right)$$

$$= \dot{m}dV = \rho AVdV$$

$$F_{\substack{x,\text{body} \\ \text{on CV}}} = 0$$

$$F_{\substack{x,\text{surface} \\ \text{on CV}}} = pA - \left(p + dp\right)A - \tau_{\text{w}} P dx$$

$$= -dpA - \frac{4\tau_{\text{w}} A}{D_{\text{H}}} dx$$

where $D_{\text{H}} \equiv 4A/P$ is a hydraulic diameter

Thus,

$$\rho VAdV = -dpA - \frac{4\tau_{\text{w}} A}{D_{\text{H}}} dx$$

$$\boxed{dp + \rho VdV + \frac{4\tau_{\text{w}}}{D_{\text{H}}} dx = 0} \tag{78}$$

From COE we have:

$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS} \left(h + \tfrac{1}{2}V^2\right)\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{on CV}}$$

where

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \qquad \text{(steady flow)}$$

$$\int_{CS} \left(h + \tfrac{1}{2}V^2\right)\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}\left(h + \tfrac{1}{2}V^2\right) + \dot{m}\left[\left(h + \tfrac{1}{2}V^2\right) + d\left(h + \tfrac{1}{2}V^2\right)\right]$$

$$= \dot{m}d\left(h + \tfrac{1}{2}V^2\right) = \dot{m}\left(dh + VdV\right)$$

$$\dot{Q}_{\text{into CV}} = 0 \qquad \text{(adiabatic conditions)}$$

$$\dot{W}_{\text{on CV}} = 0$$

(no work is done on the control volume, recall that although the walls exert a force on the control volume, they do no work since there is no displacement at the walls due to the no-slip condition)

Thus,

$$\boxed{dh + VdV = 0} \tag{79}$$

The Second Law gives:

$$\frac{d}{dt}\int_{\text{CV}} s\rho\, dV + \int_{\text{CS}} s\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \geq \int_{\text{CV}} \frac{\delta \dot{q}_{\text{into CV}}}{T}$$

where

$$\frac{d}{dt}\int_{\text{CV}} s\rho dV = 0 \qquad \text{(steady flow)}$$

$$\int_{\text{CS}} s\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}s + \dot{m}\left(s + ds\right) = \dot{m}ds$$

$$\int_{\text{CV}} \frac{\delta \dot{q}_{\text{into CV}}}{T} = 0 \qquad \text{(adiabatic conditions)}$$

Thus,

$$\boxed{ds > 0} \tag{80}$$

Note that only the ">" has been retained since the friction results in irreversible conditions.

Now let's also include the equations of state. We'll concern ourselves only with an ideal gas.

The Thermal Equation of State (the ideal gas law) gives:

$$p = \rho RT \quad \Rightarrow \quad dp = RTd\rho + \rho RdT$$

so that

$$\boxed{\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}} \tag{81}$$

The Caloric Equation of State for an ideal gas can be written as:

$$\boxed{dh = c_p dT} \tag{82}$$

We can also combine the 1st and 2nd Laws with the ideal gas law to write:

$$\boxed{ds = c_p \frac{dT}{T} - R\frac{dp}{p}} \tag{83}$$

Finally, from the Mach number relation for an ideal gas we have:

$$\text{Ma}^2 = \frac{V^2}{c^2} = \frac{V^2}{\gamma RT}$$

so that

$$2\text{Ma}d\left(\text{Ma}\right) = \frac{2VdV}{\gamma RT} - \frac{V^2 dT}{\gamma RT^2}$$

$$\Rightarrow \frac{d\left(\text{Ma}\right)}{\text{Ma}} = \frac{VdV}{\gamma RT\text{Ma}^2} - \frac{V^2 dT}{2\gamma RT^2\text{Ma}^2}$$

$$\boxed{\frac{d\left(\text{Ma}\right)}{\text{Ma}} = \frac{dV}{V} - \frac{dT}{2T}} \tag{84}$$

Now let's combine Eqs. (77) through (79) and (81) through (84) (we won't go through all of the algebra here) to get the following relations:

$$\boxed{\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \frac{\gamma \mathrm{Ma}^2 \left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)}{1-\mathrm{Ma}^2}\left(\frac{4\tau_\mathrm{w}}{\rho V^2 D_\mathrm{H}}dx\right)} \tag{85}$$

$$\boxed{\frac{dV}{V} = \frac{\gamma \mathrm{Ma}^2}{2\left(1-\mathrm{Ma}^2\right)}\left(\frac{4\tau_\mathrm{w}}{\rho V^2 D_\mathrm{H}}dx\right)} \tag{86}$$

$$\boxed{\frac{dT}{T} = \frac{-\gamma(\gamma-1)\mathrm{Ma}^4}{1-\mathrm{Ma}^2}\left(\frac{4\tau_\mathrm{w}}{\rho V^2 D_\mathrm{H}}dx\right)} \tag{87}$$

$$\boxed{\frac{dp}{p} = \frac{-\gamma \mathrm{Ma}^2\left[1+(\gamma-1)\mathrm{Ma}^2\right]}{1-\mathrm{Ma}^2}\left(\frac{4\tau_\mathrm{w}}{\rho V^2 D_\mathrm{H}}dx\right)} \tag{88}$$

$$\boxed{\frac{d\rho}{\rho} = \frac{-\gamma \mathrm{Ma}^2}{1-\mathrm{Ma}^2}\left(\frac{4\tau_\mathrm{w}}{\rho V^2 D_\mathrm{H}}dx\right)} \tag{89}$$

$$\boxed{\frac{ds}{c_p} = (\gamma-1)\mathrm{Ma}^2\left(\frac{4\tau_\mathrm{w}}{\rho V^2 D_\mathrm{H}}dx\right)} \tag{90}$$

It is useful to consider how the isentropic stagnation pressure, i.e., the pressure we would have if we brought the flow to rest isentropically, varies in a Fanno flow. Recall that the isentropic stagnation pressure is given by:

$$\frac{p}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{\gamma/1-\gamma}$$

so that, after differentiating and doing some algebra,

$$\boxed{\frac{dp_0}{p_0} = -\frac{\gamma \mathrm{Ma}^2}{2}\left(\frac{4\tau_\mathrm{w}}{\rho V^2 D_\mathrm{H}}dx\right)} \tag{91}$$

Note that we could also derive Eq. (91) by noting that Eq. (83) may be written in terms of stagnation quantities:

$$ds = c_p \frac{dT_0}{T_0} - R\frac{dp_0}{p_0}$$

and incorporating COE (Eq. (79)) written for an ideal gas:

$$dh + VdV = dh_0 = dT_0 = 0$$

$$\boxed{dT_0 = 0} \tag{92}$$

gives:

$$\boxed{\frac{dp_0}{p_0} = -\frac{ds}{R}} \tag{93}$$

Now let's examine the trends indicated by these equations. Note that:

$$\frac{4\tau_w}{\rho V^2 D_H} dx > 0$$

<u>Ma < 1 (subsonic flow)</u>		<u>Ma > 1 (supersonic flow)</u>
$d(\text{Ma}) > 0$	Eq. (85)	$d(\text{Ma}) < 0$
$dV > 0$	Eq. (86)	$dV < 0$
$dT < 0$	Eq. (87)	$dT > 0$
$dp < 0$	Eq. (88)	$dp > 0$
$d\rho < 0$	Eq. (89)	$d\rho > 0$
$ds > 0$	Eq. (90)	$ds > 0$
$dT_0 = 0$	Eq. (92)	$dT_0 = 0$
$dp_0 < 0$	Eq. (91)	$dp_0 < 0$

Notes:

1. Usually we write the shear stress in terms of a friction factor, f:

   $\tau_w = f_F\left(\tfrac{1}{2}\rho V^2\right)$ Fanning friction factor

   $\tau_w = \tfrac{1}{4} f_D\left(\tfrac{1}{2}\rho V^2\right)$ D'Arcy friction factor

   $\Rightarrow f_F = \frac{1}{4} f_D$

2. The D'Arcy friction factor is used most often in the analysis of incompressible pipe flows while the Fanning friction factor is often used for compressible flows.

3. In general, the friction factor, f, is a function of the flow Reynolds number, Re, the relative roughness of the pipe walls, ε/D_H, and the Mach number, Ma (recall the Moody chart from undergraduate incompressible pipe flow problems). Typically, Mach number effects are small in comparison to the effects of Re and ε/D_H so they are often neglected. Furthermore, the Reynolds number for compressible flows is often quite large. As a result, the friction factor remains essentially constant for a compressible flow.

4. The stagnation temperature remains constant and the stagnation pressure always decreases with friction.

Substituting the (Fanning) friction factor into Eq. (85) and integrating over a particular length of pipe:

$$\int_{\mathrm{Ma}_1}^{\mathrm{Ma}_2} \frac{2\left(1-\mathrm{Ma}^2\right)d\left(\mathrm{Ma}\right)}{\gamma \mathrm{Ma}^3\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)} = \int_{x_1}^{x_2} \frac{4f_{\mathrm{F}}}{D_{\mathrm{H}}}dx$$

$$\boxed{\frac{4\bar{f}_F}{D_{\mathrm{H}}}L_{1\to 2} = \frac{1}{\gamma}\left(\frac{1}{\mathrm{Ma}_1^2}-\frac{1}{\mathrm{Ma}_2^2}\right)+\frac{\gamma+1}{2\gamma}\ln\frac{\mathrm{Ma}_1^2}{\mathrm{Ma}_2^2}\frac{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)}} \tag{94}$$

where $L_{1\to 2} = x_2\text{-}x_1$. Note that the overbar (implying a mean value) on the friction factor will be dropped in subsequent equations for convenience.

Equation (94) relates the conditions between two arbitrary points in a frictional duct flow. It's often more convenient to make one of the points a well-defined reference point. The most logical reference would be where sonic conditions occur, i.e., $\mathrm{Ma}_2=1$, since in Fanno flow the Mach number approaches unity. Using this reference point Eq. (94) becomes:

$$\boxed{\frac{4f_F}{D_{\mathrm{H}}}L^* = \frac{1}{\gamma}\left(\frac{1-\mathrm{Ma}^2}{\mathrm{Ma}^2}\right)+\frac{\gamma+1}{2\gamma}\ln\frac{\left(1+\gamma\right)\mathrm{Ma}^2}{2\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)}} \tag{95}$$

where L^* is the length of duct required, real or imaginary, for the flow to reach Ma = 1.

Equations (86) through (90) can be integrated in a similar manner:

$$\boxed{\frac{V}{V^*} = \mathrm{Ma}\left(\frac{\gamma+1}{2}\right)^{1/2}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{-1/2}} \tag{96}$$

$$\boxed{\frac{T}{T^*} = \left(\frac{c}{c^*}\right)^2 = \left(\frac{\gamma+1}{2}\right)\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{-1}} \tag{97}$$

$$\boxed{\frac{p}{p^*} = \frac{1}{\mathrm{Ma}}\left(\frac{\gamma+1}{2}\right)^{1/2}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{-1/2}} \tag{98}$$

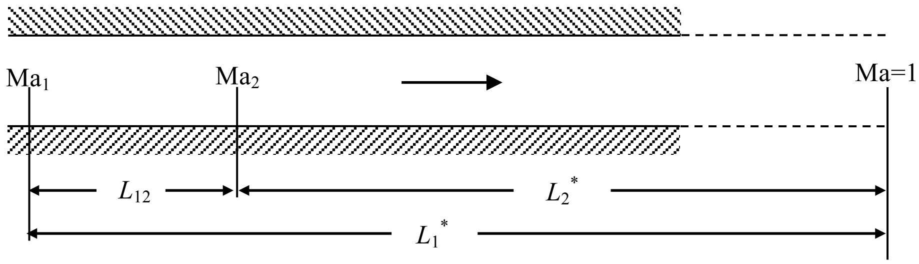
$$\boxed{\frac{\rho}{\rho^*} = \frac{1}{\mathrm{Ma}}\left(\frac{\gamma+1}{2}\right)^{-1/2}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{1/2}} \tag{99}$$

$$\boxed{\frac{s-s^*}{R} = -\ln\left\{\frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)\right]^{(\gamma+1)/2(\gamma-1)}\right\}} \tag{100}$$

$$\boxed{\frac{p_0}{p_0^*} = \frac{1}{\mathrm{Ma}}\left(\frac{\gamma+1}{2}\right)^{-(\gamma+1)/2(\gamma-1)}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{(\gamma+1)/2(\gamma-1)}} \tag{101}$$

Notes:

1. The length of duct, L_{12}, required for the flow to go from a given initial Mach number, Ma_1, to a given final Mach number, Ma_2, is found by:
$$\frac{4f_F}{D_\mathrm{H}}L_{12}=\frac{4f_F}{D_\mathrm{H}}L_1^*-\frac{4f_F}{D_\mathrm{H}}L_2^*$$
where $4f_F L^*_1/D_H$ and $4f_F L^*_2/D_H$ are found from Eq. (95) for the given Mach numbers.



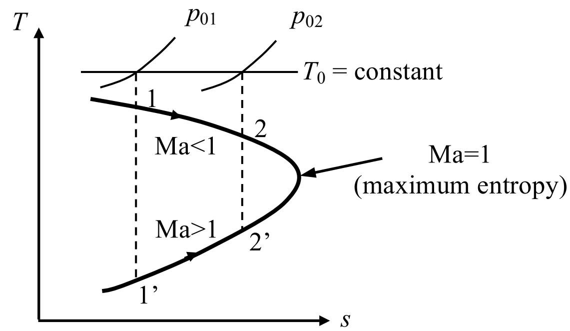
2. In order to find the change in some flow property, e.g., the pressure, between two locations where the Mach numbers are known, simply take the ratio:
$$\frac{p_2}{p_1}=\frac{\left(p_2/p^*\right)}{\left(p_1/p^*\right)}$$
3. Equations (95) through (101) are tabulated as a function of Mach number for air in the appendices of most compressible flow texts.
4. What happens if the duct is longer than L^*. Assuming the back pressure is low enough, for a subsonic flow the flow adjusts upstream so that the length of duct becomes L^*. If the flow is supersonic, then a shock wave forms upstream at a location where the length from the shock to the end of the duct is L^*. We'll discuss this in more detail in the next set of notes.
5. L^* is typically on the order of 10-100 duct diameters for a supersonic flow, depending on the friction factor. For example, for a friction factor of f_F = 0.0025, the length of duct required to reduce a flow with Ma = ∞ to Ma = 1 is 82 pipe diameters. Thus, supersonic flows reach sonic conditions in a short distance. Furthermore, large losses in stagnation pressure also occur.
6. In a Fanno flow with no shock waves, there can be no transition between supersonic and subsonic flows since both types of flow tend toward sonic conditions.

The Fanno Line

The Fanno Line is the locus of all possible Fanno flow states shown in a T-s plot. To determine how T and s are related in a Fanno flow, we can combine Eqs. (79), (81), (82), and (83) then integrate to give:

$$\frac{s-s_1}{c_p}=\ln\left[\left(\frac{T}{T_1}\right)^{1/\gamma}\left(\frac{T_0-T}{T_0-T_1}\right)^{\gamma-1/2\gamma}\right] \tag{102}$$

Equation (102) allows us to relate the entropy, s, to an arbitrarily chosen temperature, T, for a given stagnation temperature, T_0. A plot of this relation is shown below:



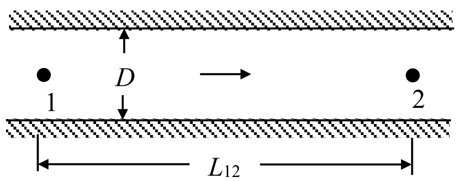
The Fanno line shows all possible combinations of entropy and temperature that can exist in an adiabatic flow with friction in a constant area duct at a given stagnation temperature (recall that the stagnation temperature remains constant in an adiabatic flow).

Notes:
1. The processes always move in a direction that increases the entropy.
2. Again for a Fanno flow:

Ma < 1 (subsonic flow)		Ma > 1 (supersonic flow)
$d(\mathrm{Ma}) > 0$	Eq. (85)	$d(\mathrm{Ma}) < 0$
$dV > 0$	Eq. (86)	$dV < 0$
$dT < 0$	Eq. (87)	$dT > 0$
$dp < 0$	Eq. (88)	$dp > 0$
$d\rho < 0$	Eq. (89)	$d\rho > 0$
$ds > 0$	Eq. (90)	$ds > 0$
$dT_0 = 0$	Eq. (92)	$dT_0 = 0$
$dp_0 < 0$	Eq. (91)	$dp_0 < 0$

Air flows in a 0.100 m ID duct under adiabatic conditions. Calculate the length of duct required to raise the Mach number of the air from Ma_1=0.5 to Ma_2=0.9 if the average value of the (Fanning) friction coefficient is 0.005.

SOLUTION:

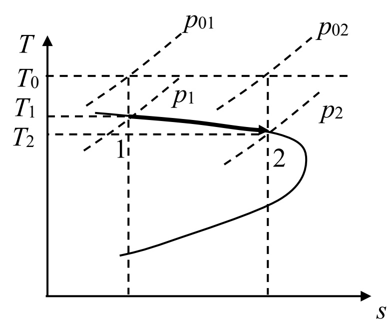


$$\frac{4f_F L_1^*}{D} = 1.06906 \text{ using the Fanno flow relations with } Ma_1 = 0.5 \text{ and } \gamma = 1.4 \tag{1}$$

$$\frac{4f_F L_2^*}{D} = 0.01451 \text{ using the Fanno flow relations with } Ma_2 = 0.9 \text{ and } \gamma = 1.4 \tag{2}$$

$$\frac{4f_F L_{12}}{D} = \frac{4f_F L_1^*}{D} - \frac{4f_F L_2^*}{D} = 1.05455 \tag{3}$$

$$\boxed{\therefore L_{12} = 5.27 \text{ m}} \text{ using } f_F = 0.005 \text{ and } D = 0.100 \text{ m} \tag{4}$$



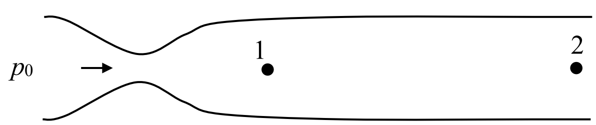
An experiment is designed to measure friction coefficients for the supersonic flow of air. Measurements from an experimental apparatus, consisting of a well-insulated, converging-diverging nozzle attached to a smooth, round tube, give the following data:

pressure upstream of the nozzle = 516 cm Hg abs
temperature upstream of the nozzle = 107.3 °F
throat diameter = 0.2416 in
diameter of nozzle exit and of tube = 0.5009 in
pressure at a point 29.60 diameters from the tube inlet = 37.1 cm Hg abs

What is the average friction factor for the tube with these conditions?

SOLUTION:

Assume the flow in the converging-diverging nozzle is isentropic since the distance is short.



Using the isentropic flow relations.

$$\frac{A_1}{A_1^*} = \left(\frac{0.5009\text{ in.}}{0.2416\text{ in.}}\right)^2 = 4.2984 \;\Rightarrow\; \text{Ma}_1 = 3.016 \;\Rightarrow\; p_1/p_{01} = 0.0266 \;\Rightarrow\; p_1 = 13.73\text{ cm Hg abs} \tag{1}$$

where p_{01} = 516 cm Hg abs.

Using the Fanno flow relations:

$$\text{Ma}_1 = 3.016 \;\Rightarrow\; p_1/p^* = 0.2163 \text{ and } \frac{4 f_F L_1^*}{D} = 0.5246 \tag{2}$$

At station 2:

$$\frac{p_2}{p^*} = \left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p^*}\right) = \left(\frac{37.1\text{ cm Hg abs}}{13.73\text{ cm Hg abs}}\right)(0.2163) = 0.5845 \tag{3}$$

Using the Fanno flow relations:

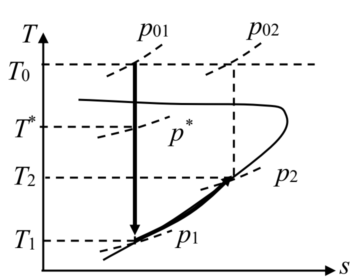
$$p_2/p^* = 0.5845 \;\Rightarrow\; \text{Ma}_2 = 1.54 \;\Rightarrow\; \frac{4 f_F L_2^*}{D} = 0.1506 \tag{4}$$

The friction factor may be found using the dimensionless sonic lengths and the distance between the stations.

$$\frac{4 f_F L_{12}}{D} = \frac{4 f_F L_1^*}{D} - \frac{4 f_F L_2^*}{D} = 0.5246 - 0.1506 = 0.3740 \tag{5}$$

$$f_F = 0.3740 \frac{D}{4L_{12}} = 0.3740 \frac{1}{4(29.60)} \tag{6}$$

$$\boxed{\therefore f_F = 0.0032} \tag{7}$$

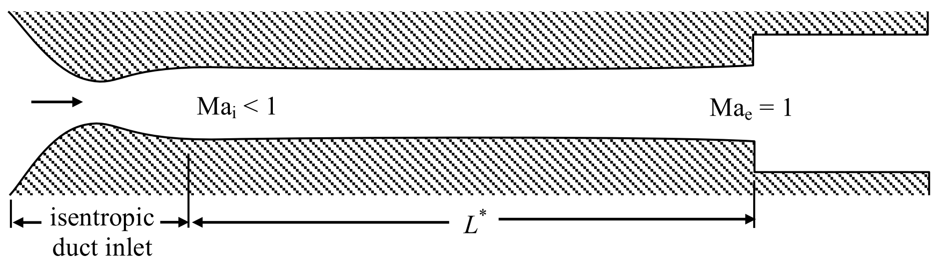


Choking in Fanno Flow

Consider what happens if the duct length has its maximum value for a given duct inlet Mach number (so that the exit Mach number is unity) and then we increase the duct length beyond this value. We'll assume that the back pressure remains low enough to produce sonic exit conditions, i.e., $p_B \leq p^*$.

Subsonic Flow:

The duct inlet Mach number will decrease until a steady-state solution again becomes possible with sonic exit conditions. This decrease results in a reduction in the flow rate, i.e., the flow is "choked" by friction. The mass flow rate through the duct can be found by the conditions at the duct inlet, which is assumed to have isentropic flow.



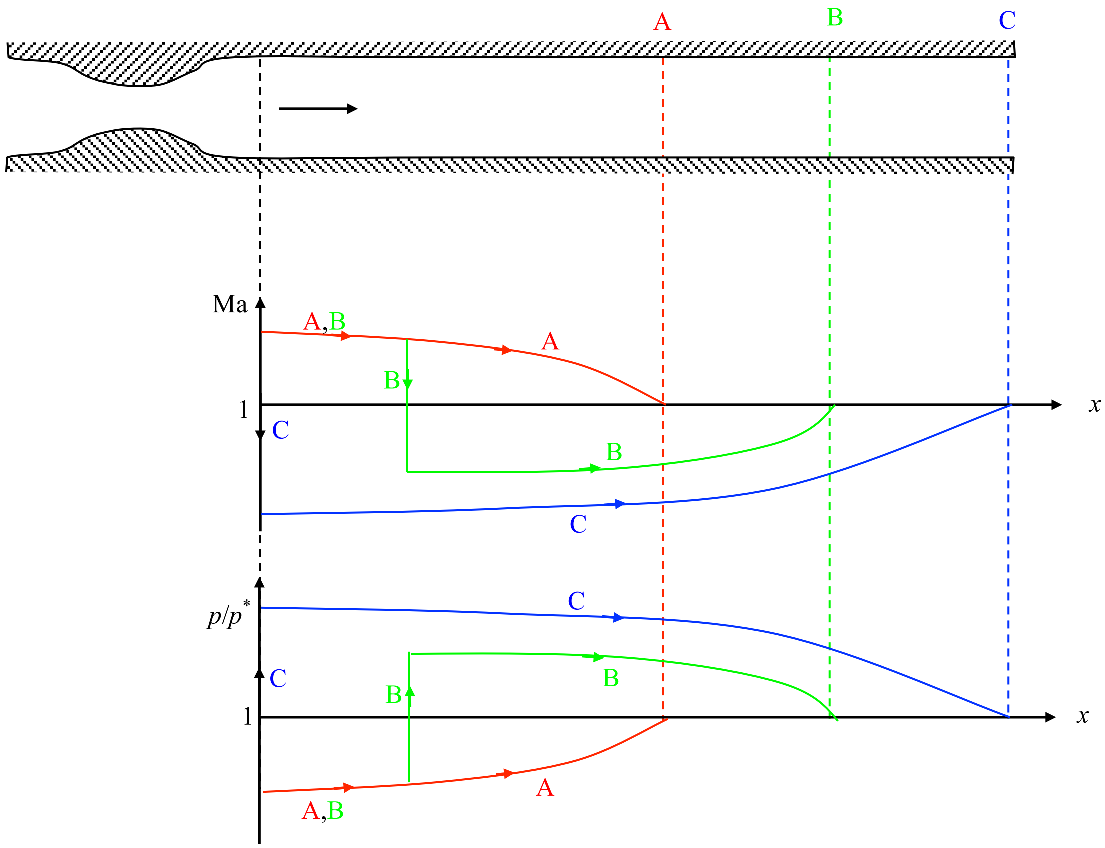
$$\dot{m} = \rho V A = \left[\left(\frac{\rho}{\rho_0}\right)\rho_0\right]\left[\mathrm{Ma}\left(\frac{c}{c_0}\right)c_0\right]A = \rho_0 c_0 A \mathrm{Ma}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

(As the Mach number decreases, the mass flow rate also decreases, i.e., $\partial\dot{m}/\partial(\mathrm{Ma}) > 0$.)

Further increases in the duct length result in decreases in the duct inlet Mach number and mass flow rate.

Supersonic Flow:

- First consider the case where the duct length has its maximum value for the inlet Mach number. The flow within the duct will be supersonic and the exit conditions will be sonic (refer to curve A shown in the plots below).
- Now increase the duct length to position B as shown in the figure. A shock wave forms in the duct at a location such that the exit conditions are sonic. The mass flow rate through the device remains unaffected since the flow at the nozzle throat is sonic.
- As the duct length is increased further, the shock wave moves upstream until finally at the duct length indicated by position C, the shock wave is located at the duct inlet. Now the flow throughout the duct is subsonic. The mass flow rate, however, remains unaffected since the throat conditions are sonic.
- Further increases in the duct length cause the shock wave to move into the diverging section of the nozzle toward the nozzle throat. The duct inlet Mach number will continue to decrease but the mass flow rate is unaffected.
- If the duct length is increased further, the shock vanishes at the nozzle throat and the flow throughout the entire device becomes subsonic and further increases in the duct length will result in decreasing duct inlet Mach number and decreasing mass flow rate through the device (refer to the previous discussion concerning subsonic flow).

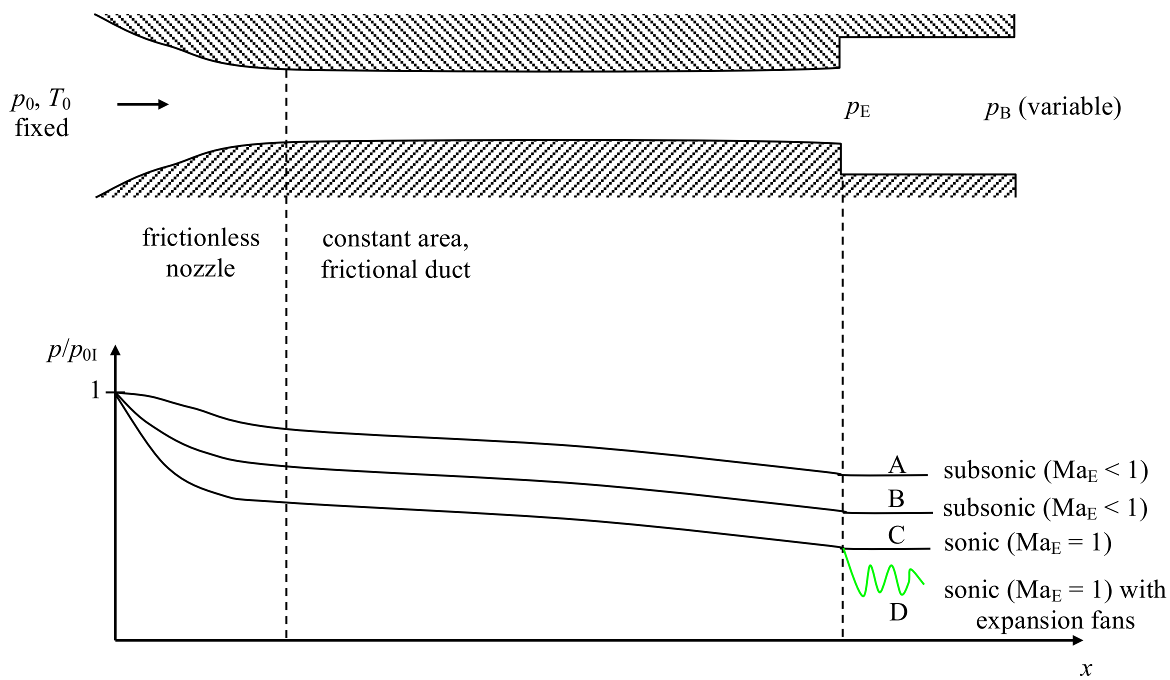


Performance of Long Ducts at Various Pressure Ratios

Now let's consider the flow in long ducts with differing inlet nozzle conditions and varying back pressures.

Ducts Fed by Converging Nozzles

Consider the flow through a long duct fed by a converging nozzle as shown in the figure below. The stagnation conditions are assumed fixed while the back pressure can be varied. For a high enough back pressure (cases A and B shown in the plots below), the flow through the entire device, including the exit, will remain subsonic and the exit pressure will equal the back pressure. If the back pressure is decreased to precisely the sonic pressure (case C), the flow throughout the device will be subsonic except at the exit where sonic conditions exist. For this particular case the exit pressure is equal to the back pressure. If the back pressure is decreased further, the flow conditions within the device will remain the same as in case C since the flow is choked at the exit; however, upon leaving the exit the flow will expand through expansion fans to reach the back pressure.

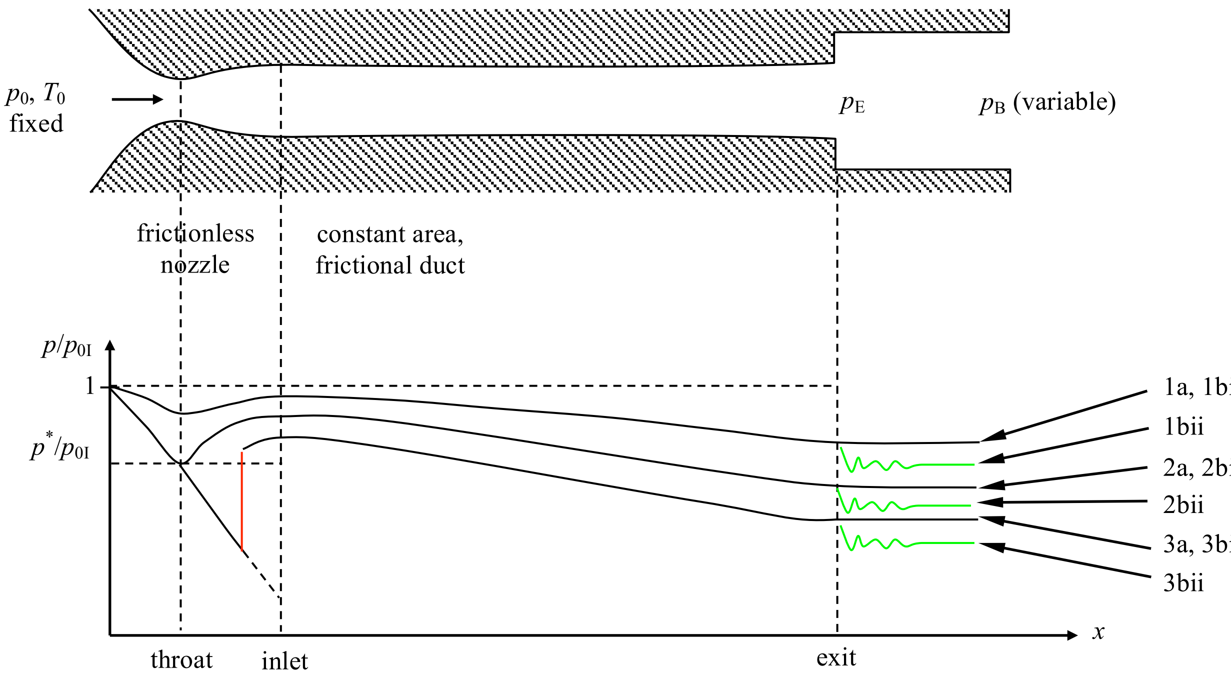


Ducts Fed by Converging-Diverging Nozzles

Now consider flow through a duct with a converging-diverging inlet nozzle. The flow through this device is considerably more complicated than through the duct fed by a converging nozzle. In the following discussion we'll assume that the stagnation conditions feeding the nozzle are constant and that the ratio of the nozzle exit/duct inlet area to the throat area is fixed, i.e., A_i/A_t = constant.

Subsonic flow at the nozzle exit / duct inlet:

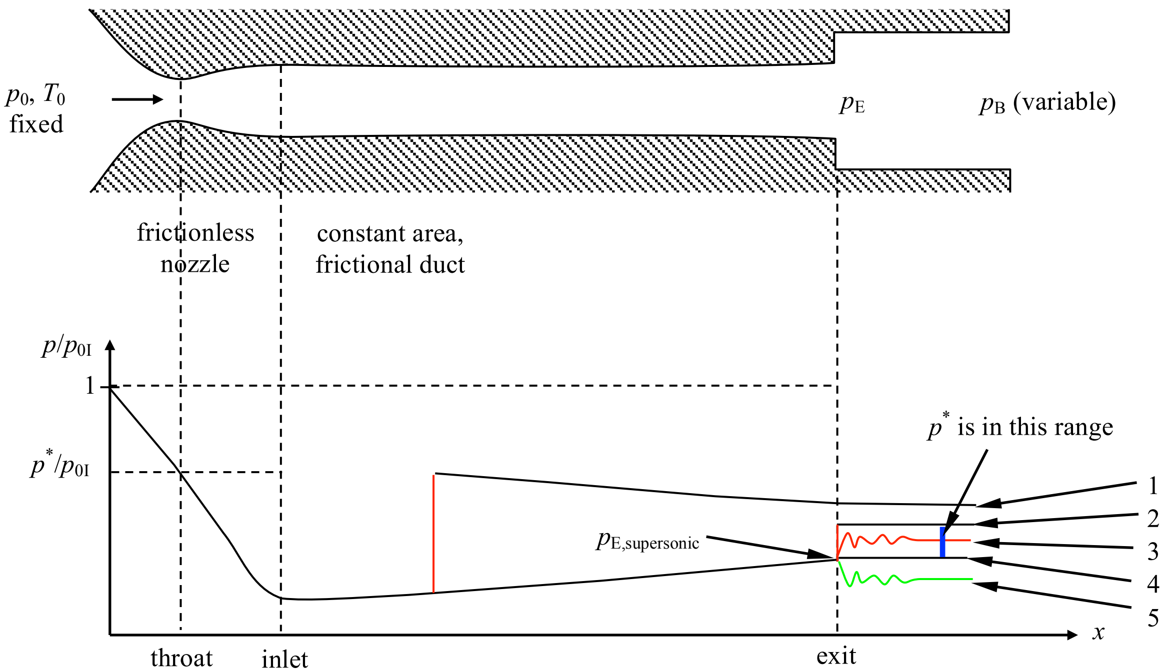
We'll first consider the cases where the Mach number at the nozzle exit/duct inlet is subsonic, i.e., $Ma_I < 1$. Refer to the following figure for the various cases described below.



As the back pressure decreases:

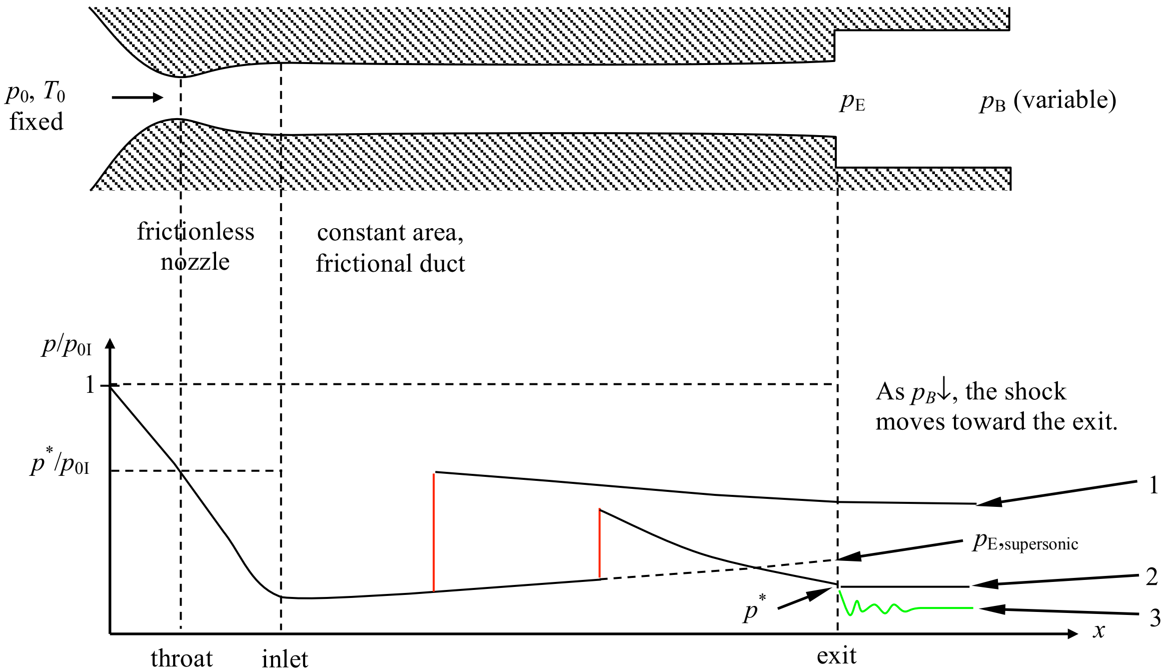
1. The flow in the nozzle and duct are subsonic. Consider cases (a) and (b) described below.
2. The flow in the nozzle is subsonic except at the nozzle throat where sonic conditions occur: $Ma_T = 1$, $p_T = p^*$. The flow is now choked at the nozzle throat. Further decreases in back pressure will not affect conditions upstream of the throat. Note that the Mach number at the duct inlet, Ma_I, for this case is determined by the area ratio, $A_I/A^* = A_I/A_T$. Consider cases (a) and (b) described below.
3. The flow downstream of the throat is supersonic for some distance then passes through a normal shock wave and becomes subsonic for the remainder of the nozzle. Hence, the flow into the duct is subsonic. Consider cases (a) and (b) described below.
   a. If the back pressure is greater than the sonic pressure for the inlet Mach number, then the exit Mach number will be subsonic and the back pressure will equal the exit pressure: $Ma_E < 1 \Rightarrow p_E = p_B > p^*$.
   b. If the back pressure is less than or equal to the sonic pressure, then the exit will be at sonic conditions: $Ma_E = 1 \Rightarrow p_E = p^* \geq p_B$. Note that the flow is now choked at the duct exit. Further decreases in the back pressure will not affect the flow within the duct or nozzle.
      i. If $p_B = p^*$ then the flow equals the back pressure when exiting the duct.
      ii. If $p_B < p^*$ then expansion fans will form outside of the duct through which the flow pressure will equilibrate with the back pressure.

Supersonic flow at the nozzle exit / duct inlet:
Now consider the cases when the Mach number at the nozzle exit/duct inlet is supersonic, i.e., $Ma_I > 1$. Refer to the following figure for the various cases described below. Note that the Mach number at the nozzle entrance is determined solely by the area ratio, A_I/A^*. Furthermore, since the Mach number is fixed, the sonic length is also fixed.



First consider the cases where the duct length is less than the sonic length, $L < L^*_I$, i.e., the exit pressure, <u>assuming supersonic flow throughout the duct</u>, is less than the sonic pressure, $p_{E,\text{supersonic}} < p^*$. (Recall that the pressure in a supersonic Fanno flow *increases* with increasing distance down the duct).

1. The back pressure is sufficiently large that the flow cannot leave the duct under supersonic conditions and come into equilibrium with the back pressure. Instead, a normal shock wave forms within the duct causing a transition to subsonic flow. The shock wave forms at a location such that the exit pressure equals the back pressure: $Ma_E < 1 \Rightarrow p_E = p_B$. Decreasing the back pressure causes the shock wave to move closer to the duct exit.
2. The entire duct flow is supersonic and the flow pressure equilibrates with the back pressure through a normal shock wave at the duct exit. The flow downstream of the shock is subsonic so the pressure just downstream of the shock is equal to the back pressure.
3. The flow within the duct is supersonic but $p_E < p_B$ (also note that $p_E < p^*$ as stated in the paragraph above). Oblique shock waves form outside the duct through which the flow pressure equilibrates with the back pressure.
4. The flow within the duct is supersonic and the pressure at the duct exit is equal to the back pressure, $p_E = p_B$. Note that $p_E < p^*$.
5. The flow within the duct is supersonic but $p_E > p_B$ (also note that $p_E < p^*$). Expansion fans form outside the duct through which the flow pressure equilibrates with the back pressure.



Now consider the cases where the duct length is greater than or equal to the sonic length, $L \geq L^*_I$, i.e. the exit pressure, assuming supersonic flow throughout the duct, is greater than or equal to the sonic pressure, $p_{\text{E,supersonic}} \geq p^*$. (Recall that the pressure in a supersonic Fanno flow *increases* with increasing distance down the duct).

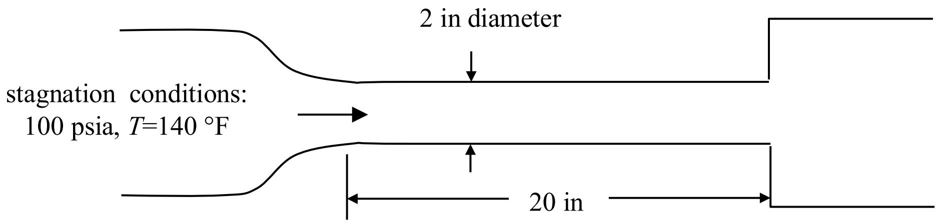
1. For the case where the back pressure is greater than the sonic pressure, i.e., $p_B > p^*$, a normal shock wave will form at a location within the duct such that the exit flow (which is subsonic) will equal the back pressure: $\text{Ma}_E < 1 \Rightarrow p_E = p_B > p^*$.
2. For the case where the back pressure is equal to the sonic pressure, i.e., $p_B = p^*$, a normal shock wave will form at a location within the duct such that the exit flow is sonic: $\text{Ma}_E = 1 \Rightarrow p_E = p_B = p^*$.
3. For the case where the back pressure is less than the sonic pressure, i.e., $p_B < p^*$, a normal shock wave will form at a location within the duct such that the exit flow is sonic: $\text{Ma}_E = 1 \Rightarrow p_E = p^* > p_B$. Expansion fans will form outside the duct through which the flow pressure equilibrates with the back pressure.

Notes:

1. Once the flow chokes at either the nozzle throat or duct exit, the mass flow rate through the system will no longer be a function of the downstream conditions and will depend only on the upstream stagnation conditions.

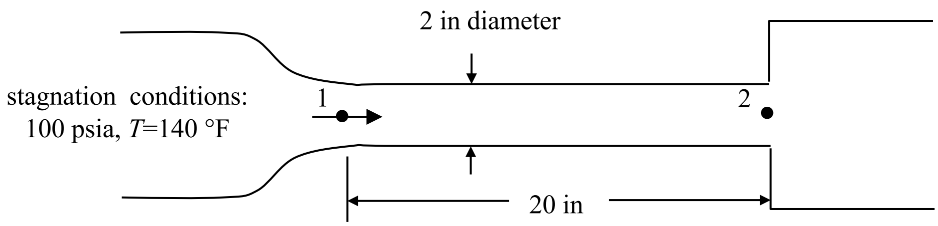
2. Shock waves within the duct will not be sharp discontinuities as idealized here. Instead, boundary layer effects will tend to smear the transition from supersonic to subsonic flow over a considerable distance. Similar shock smearing is observed when shock waves form in the diverging section of a converging-diverging nozzle.

Estimate the maximum flow rate of air (lb$_m$/s) which can flow through the passage shown, assuming that the (Fanning) friction coefficient of the duct is 0.005. For what range of back pressures will this maximum flow rate be achieved?



SOLUTION:

The maximum flow rate will occur when the flow is choked.



Determine Ma$_1$ using the adiabatic, frictional flow relations and:

$$\frac{4f_F L_1^*}{D_H} = \frac{4(0.005)(20\text{ in})}{(2\text{ in})} = 0.2 \tag{1}$$

$$\Rightarrow \ \text{Ma}_1 = 0.7043 \tag{2}$$

(The flow at 1 will be subsonic since the flow starts from stagnation conditions and does not pass through a throat before reaching the duct inlet.)

$$\Rightarrow \ \frac{p_1}{p^*} = 1.4836 \tag{3}$$

Determine the mass flow rate through the duct using the conditions at 1. The properties at 1 can be found by applying the isentropic flow relations from the stagnation conditions to 1 (flow through the converging duct is assumed isentropic).

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \quad \Rightarrow \quad p_1 \quad = \quad 10340 \text{ lb}_f/\text{ft}^2 \tag{4}$$

where p_{01} = 100 psia = 14400 lb$_f$/ft^2, γ = 1.4, Ma$_1$ = 0.7043

$$\frac{T_1}{T_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{-1} \quad \Rightarrow \quad T_1 \quad = \quad 545.8\ ^\circ\text{R} \tag{5}$$

where T_{01} = 140 °F = 600 °R

$$\frac{\rho_1}{\rho_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{1}{1-\gamma}} \quad \Rightarrow \quad \rho_1 \quad = \quad 0.3555 \text{ lb}_m/\text{ft}^3 \tag{6}$$

where $\rho_{01} = p_{01}/(RT_{01})$ = 0.4503 lb$_m$/ft^3, R = 53.3 (lb$_f$·ft)/(lb$_m$·°R)

$$V_1 = c_1\text{Ma}_1 = \sqrt{\gamma R T_1}\,\text{Ma}_1 \quad \Rightarrow \quad V_1 \quad = \quad 806.6 \text{ ft/s} \tag{7}$$

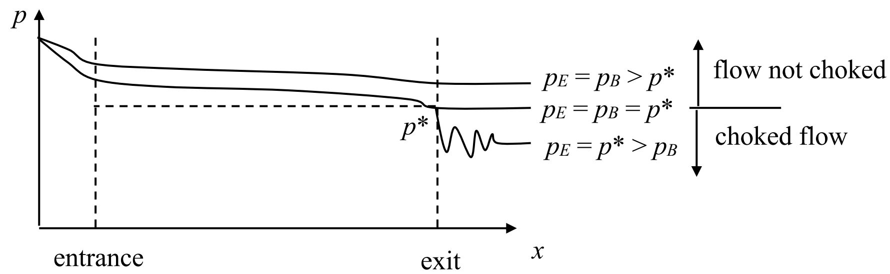
$$\dot{m} = \rho_1 V_1 A_1 \quad \Rightarrow \quad \boxed{\dot{m} \;=\; 6.26 \text{ lb}_\text{m}/\text{s}} \tag{8}$$

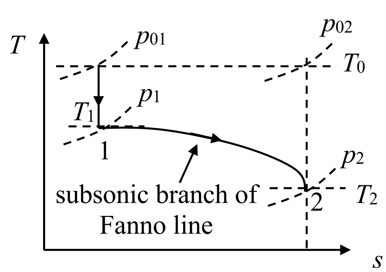
where $A_1 = \pi D^2/4 = 2.182\text{e-}2 \text{ ft}^2$

The back pressure at which choking just occurs is given by:

$$p_B = p_2 = p^* = p_1\left(\frac{p^*}{p_1}\right) \;\Rightarrow\; p_B \;=\; 6970 \text{ lb}_\text{f}/\text{ft}^2 = 48.4 \text{ psia} \tag{9}$$

Hence, choking will occur for back pressures less than 48.4 psia. When the flow is choked, it has the maximum flow rate of $\dot{m}$ = 6.26 lb$_\text{m}$/s.

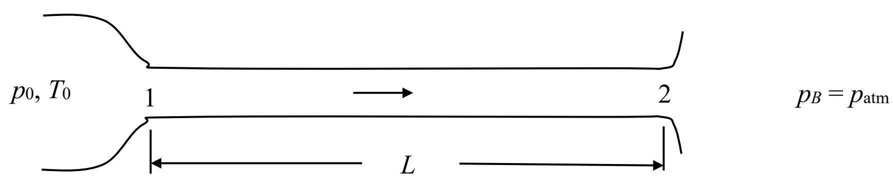




Air is drawn from a large storage tank (400 kPa (abs), 320 K) through a pipe that is 200 m long with an internal diameter of 50 mm and is discharged into the atmosphere (p_{atm} = 100 kPa (abs)). The Darcy pipe friction factor is 0.0114. Assume the flow is adiabatic. Find:

a. the mass flow rate through the pipe
b. the Mach number of the flow at the pipe entrance and exit
c. sketch the process on a *T-s* diagram.

SOLUTION:



First check to see if the flow is choked. Assume that the flow is choked so that:

$$\frac{4 f_F L^*}{D_H} = \frac{4\left(\frac{1}{4}\cdot 0.0114\right)(200 \text{ m})}{(50e\text{-}3 \text{ m})} = 45.6 \quad \text{(Note that } f_F = \tfrac{1}{4} f_D\text{.)} \tag{1}$$

$$\Rightarrow \quad \text{Ma}_1 = 0.12 \quad \text{(from the adiabatic, frictional flow relations)} \tag{2}$$

$$\Rightarrow \quad \frac{p_{01}}{p_0^*} = 4.8643 \quad \text{(from the adiabatic, frictional flow relations)} \tag{3}$$

Now calculate the pressure at location 2 and see how it compares with the back pressure.

$$p_2 = p_{01}\left(\frac{p_0^*}{p_{01}}\right)\left(\frac{p^*}{p_0^*}\right) = (400 \text{ kPa})\left(\frac{1}{4.8643}\right)(0.5283) = 43.44 \text{ kPa} \tag{4}$$

(Note that since we're assuming a choked flow, location 2 is at sonic conditions, i.e., $p_2 = p^*$.)

Since p_2 = 53.44 kPa < p_B = 100 kPa, the flow must not be choked as assumed. Hence, the flow at location 2 must be subsonic.

To solve for the flow properties within the pipe, use the following iterative approach.

1. Assume a value for Ma_2 ($\mathrm{Ma}_2 < 1$).
2. Calculate $\frac{4f_F L_2^*}{D_H}$ and $\frac{p_{02}}{p_0^*}$ using Ma_2 and the adiabatic, frictional flow relations.
3. Calculate $\frac{4f_F L_1^*}{D_H} = \frac{4f_F L_2^*}{D_H} + \frac{4f_F \Delta L}{D_H}$.
4. Calculate Ma_1 and $\frac{p_{01}}{p_0^*}$ using $\frac{4f_F L_1^*}{D_H}$ and the adiabatic, frictional flow relations.
5. Calculate $\frac{p_{02}}{p_2} = \left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{\gamma-1}}$.
6. Calculate $\frac{p_{01}}{p_2} = \left(\frac{p_{02}}{p_2}\right)\left(\frac{p_0^*}{p_{02}}\right)\left(\frac{p_{01}}{p_0^*}\right)$.
7. Check to see if $\frac{p_{01}}{p_2} = \frac{p_{01}}{p_B}$ since $p_2 = p_B$ for a subsonic exit Mach number. If $\frac{p_{01}}{p_2} > \frac{p_{01}}{p_B}$ then $p_2 < p_B$ so choose a smaller Ma_2 (otherwise choose a larger Ma_2) and repeat steps 2-7 until a converged result is achieved.

The converged results are: $\boxed{\mathrm{Ma}_1 = 0.1183 \text{ and } \mathrm{Ma}_2 = 0.46}$.

The mass flow rate can be found using the conditions at the pipe entrance.

$$\dot{m} = \rho_1 V_1 A_1 \tag{5}$$

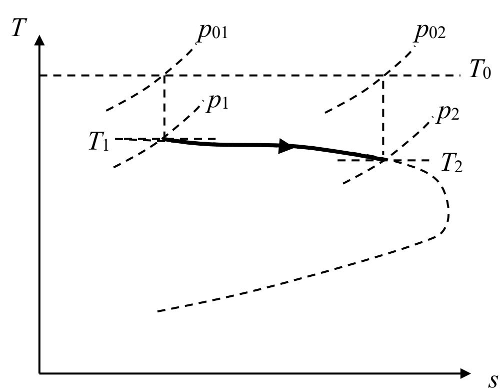
where

$$\rho_1 = \rho_0\left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{1}{1-\gamma}} \text{ and } \rho_0 = \frac{p_0}{RT_0} \tag{6}$$

$$V_1 = c_1\mathrm{Ma}_1 = \sqrt{\gamma R T_1} \text{ and } T_1 = T_0\left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-1} \tag{7}$$

Using the following data:

p_0 = 400 kPa
T_0 = 320 K
γ = 1.4
R = 287 kJ/(kg·K)
A_1 = 1.963e-3 m^2
Ma_1 = 0.1183
$\Rightarrow \boxed{\dot{m} = 0.360 \text{ kg/s}}$

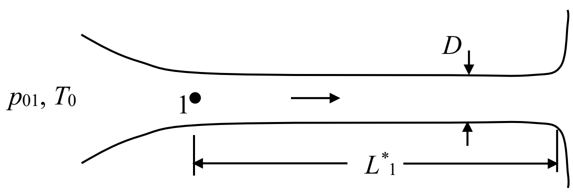


(8)

Air from the laboratory (14.5 psia, 70 °F) is drawn through a long smooth, insulated tube that has a ¼ in. inside diameter. When the flow chokes, the pressure at the tube entrance (station 1) is 7.11 inches of water vacuum (below atmospheric). Determine for the choked flow conditions:

a. the Mach number at station 1,
b. the length of the tube (in feet),
c. the mass flow rate (in lb_m/s),
d. the pressure at the tube exit (in psia), and
e. show the process on a *T*-*s* diagram.

SOLUTION:



p_{01} = 14.5 psia = 2088 lb_f/ft^2
T_0 = 70 °F = 530 °R
D = 0.25 in = $2.083*10^{-2}$ ft
γ = 1.4
R = 53.3 ($ft \cdot lb_f$)/($lb_m \cdot$°R)
p_1 = 7.11 in. H_2O vaccum = $p_{atm} - \rho_{H2O}gh$
= (2088 lb_f/ft^2) – (62.4 lb_m/ft^3)(32.2 ft/s^2)(7.11 in)(1/12 in/ft)(lb_f/32.2 $lb_m \cdot ft/s^2$) = 2051 lb_f/ft^2

Determine Ma_1 using the isentropic flow relations.

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \mathrm{Ma}_1 = 0.16 \tag{1}$$

To determine the length of the tube, we use the adiabatic, frictional flow relations (Fanno flow).

$$\mathrm{Ma}_1 = 0.16 \Rightarrow \frac{4 f_F L_1^*}{D} = 24.2 \tag{2}$$

The friction factor for a "smooth" tube can be found using a Moody chart. First estimate the Reynolds number for the flow using the conditions at 1.

$$\mathrm{Re}_1 = \frac{V_1 D_1}{\nu_1} \approx 23{,}400 \tag{3}$$

where

$$V_1 = \mathrm{Ma}_1 c_1 = \mathrm{Ma}_1\sqrt{\gamma R T_1} \Rightarrow V_1 = 180 \text{ ft/s} \tag{4}$$

$$\frac{T_1}{T_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow T_1 = 527.3 \text{ °R} \tag{5}$$

$D_1 = 2.083*10^{-2}$ ft
$\nu_1 = 1.6*10^{-4}$ ft^2/s

From the Moody chart for the "smooth" curve with Re = 23,400, $f_D = 0.025 \Rightarrow \underline{f_F = f_D/4 = 6.25*10^{-3}}$.

Solving Eqn. (2) with the calculated f_F and given pipe diameter, $\boxed{L^*_1 = 20.2 \text{ ft}}$.

Note that at the pipe exit:

$$\mathrm{Re}_2 = \frac{V_2 D_2}{\nu_2} \approx 215{,}000 \tag{6}$$

where

$$V_2 = \mathrm{Ma}_2 c_2 = \mathrm{Ma}_2\sqrt{\gamma R T_2} \Rightarrow V_1 = 1030 \text{ ft/s (Note that Ma}_2 = 1.) \tag{7}$$

$$\frac{T_2}{T_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_2^2\right)^{-1} \Rightarrow T_1 = 441.7 \text{ °R} \quad \text{(Note that the stagnation temperature remains constant.)} \tag{8}$$

$D_2 = 2.083*10^{-2}$ ft

$\nu_2 = 1.0*10^{-4}$ ft^2/s

From the Moody chart for the "smooth" curve with Re = 215,000, $f_D = 0.016$. Thus, a more accurate value of the D'Arcy friction factor would be:

$$f_D = \tfrac{1}{2}(0.025 + 0.016) = 0.021 \tag{9}$$

Following the same procedures as before, but using this new value for f_D, the sonic length comes to L^*_1 = 24.4 ft.

The mass flow rate can be determined from the conditions at state 1.

$$\dot{m} = \rho_1 V_1 A_1 \Rightarrow \boxed{\dot{m} = 6.06*10^{-2}\ \text{lb}_\text{m}/\text{s}} \tag{10}$$

where

$$\frac{\rho_1}{\rho_{01}} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right)^{\frac{1}{1-\gamma}} \Rightarrow \rho_1 = 9.87*10^{-1} \text{ lb}_\text{m}/\text{ft}^3 \tag{11}$$

$$\rho_{01} = \frac{p_{01}}{RT_0} \Rightarrow \rho_{01} = 7.39*10^{-2} \text{ lb}_\text{m}/\text{ft}^3 \tag{12}$$

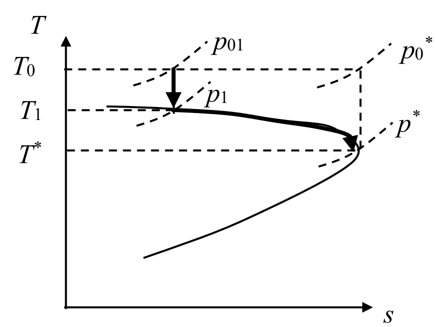
V_1 = 180 ft/s (calculated in Eqn. (4))

$A_1 = \pi D^2_1/4 = 3.408*10^{-4}$ ft^2 (13)

The pressure at the exit is found from the adiabatic, frictional flow relations (Fanno flow).

$$\text{Ma}_1 = 0.16 \Rightarrow \frac{p_1}{p^*} = 6.8291 \Rightarrow \boxed{p_e = p^* = 300 \text{ lb}_\text{f}/\text{ft}^2 = 2.09 \text{ psia}} \tag{14}$$

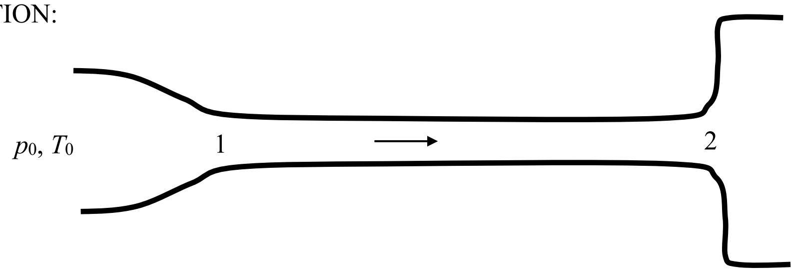
where $p_1 = 2051$ lb$_\text{f}$/ft^2.



Air, in a large reservoir, at a pressure of 200 Pa (abs) and a temperature of 30 °C is expanded through a convergent nozzle. The air then flows down a pipe with a diameter of 25 mm and a Fanning friction factor of 0.005. If the Mach number at the exit of the nozzle, i.e., at the inlet to the pipe, is 0.2 and the Mach number at the end of this pipe is 0.8, find, assuming that the flow in the nozzle is isentropic and the flow in the pipe is adiabatic,

a. the length of the pipe
b. the pressure at the exit of the pipe
c. the pressure in the reservoir into which the pipe discharges at which choking first occurs and the inlet Mach number under these conditions.
d. plot a graph of pipe inlet and outlet Mach number against discharge reservoir pressure.

SOLUTION:



Using the Fanno flow relations with $\gamma = 1.4$:

$$\frac{4 f_F L_1^*}{D_H} = 14.5322 \text{ using } \mathrm{Ma}_1 = 0.2 \tag{1}$$

$$\frac{4 f_F L_2^*}{D_H} = 0.07228 \text{ using } \mathrm{Ma}_1 = 0.8 \tag{2}$$

$$\frac{4 f_F \Delta L}{D_H} = \frac{4 f_F L_1^*}{D_H} - \frac{4 f_F L_2^*}{D_H} \Rightarrow \boxed{\Delta L = 18.1 \text{ m}} \tag{3}$$

with $f_F = 0.005$ and $D_H = 25\text{e-}3$ m.

Using the Fanno flow relations with $\gamma = 1.4$:

$$\frac{p_1}{p^*} = 5.4454 \text{ using } \mathrm{Ma}_1 = 0.2 \tag{4}$$

$$\frac{p_2}{p^*} = 1.2893 \text{ using } \mathrm{Ma}_1 = 0.8 \tag{5}$$

In addition, using the isentropic relations:

$$p_1 = p_{01}\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1 = 194.5 \text{ Pa.} \tag{6}$$

using $p_{01} = 200$ Pa and $\mathrm{Ma}_1 = 0.2$.

The pressure at station 2 can be found by combining ratios.

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right) p_1 \Rightarrow \boxed{p_2 = 46.0 \text{ Pa}}. \tag{7}$$

Choking first occurs when $Ma_2 = 1$ and $p_2 = p_B$ so that:

$$\frac{4 f_F L_1^*}{D_H} = \frac{4 f_F \Delta L}{D_H} = 14.4609 \tag{8}$$

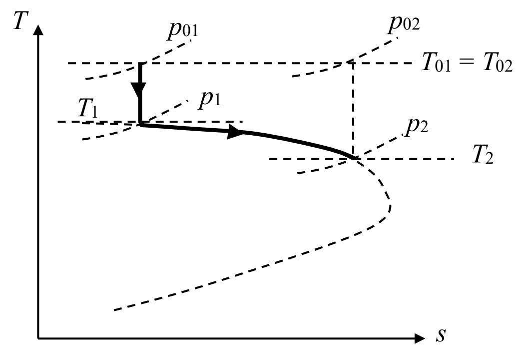
where ΔL = 18.1 m (found from Eq. (3)). Using the Fanno flow relations with the sonic length ratio given in Eq. (8):

$$\boxed{\mathrm{Ma}_1 = 0.2005} \text{ and } \frac{p_1}{p^*} = 5.455 \tag{9}$$

Combining ratios:

$$p_2 = p^* = \left(\frac{p^*}{p_1}\right)\left(\frac{p_1}{p_{01}}\right) p_{01} \Rightarrow \boxed{p_2 = p^* = p_B = 35.7 \text{ Pa.}} \tag{10}$$

where p_1/p^* is given in Eqn. (9), $p_1/p_{01} = 0.9724$ (using Eq. (6) with $Ma_1 = 0.2005$) and $p_{01} = 200$ Pa.



Use the following approach to plot Ma_1 and Ma_2 as a function of p_B.

1. Choose a value of Ma_1. Only use subsonic values since the flow starts from stagnation conditions and goes through a converging nozzle.

2. Calculate $4f_F L_1^*/D_H$ using the Fanno flow relations and Ma_1.

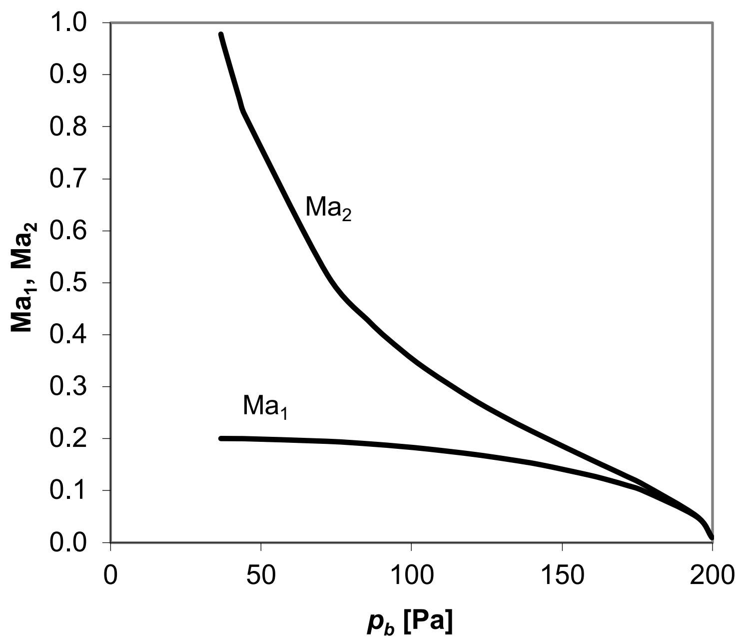
3. Calculate $4f_F L_2^*/D_H$ using:
$$\frac{4f_F L_2^*}{D_H} = \frac{4f_F L_1^*}{D_H} - \frac{4f_F \Delta L}{D_H} \tag{11}$$
with $\Delta L = 18.1$ m, $f_F = 0.005$, and $D_H = 25\text{e-}3$ m.

4. Calculate Ma_2 using the Fanno flow relations and the value of $4f_F L_2^*/D_H$ found in step 3.

5. Calculate p_1/p^* and p_2/p^* using the Fanno flow relations and Ma_1 (step 1) and Ma_2 (step 4).

6. Calculate p_1 using the isentropic flow relation:
$$p_1 = p_{01}\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \tag{12}$$
with $p_{01} = 200$ Pa.

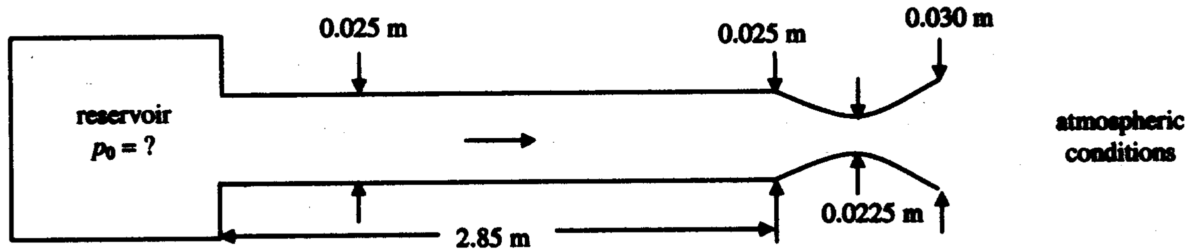
7. Calculate p_B using:
$$p_B = p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1 \tag{13}$$
Note that $p_B = p_2$ when $\mathrm{Ma}_2 < 1$ and $p_2 = p^*$ when $\mathrm{Ma}_2 = 1$ ($p_B \le p^*$). When the back pressure is less than the sonic pressure, the flow in the pipe will remain unchanged, but the flow outside the pipe will exhibit expansion fans.



A converging-diverging nozzle is connected to a large air reservoir by means of a well-insulated pipe with an average (Fanning) friction coefficient of 0.005. The inside dimensions of the pipe and nozzle are given in the diagram. The air is to exit the device at supersonic design conditions (*i.e.* the pressure at the exit is equal to the surrounding conditions).

a. Determine the stagnation pressure that must be maintained in the reservoir.

b. Sketch the flow process on a *T-s* diagram.



SOLUTION:

• Determine the exit Mach # using the area ratio relation for isentropic flow:

$$\frac{A_e}{A^*} = \frac{1}{Ma_e}\left(\frac{1+\frac{\gamma-1}{2}Ma_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

where $\frac{A_e}{A^*} = \frac{A_e}{A_t}$ since the flow must have $Ma_t = 1$ to go from subsonic to supersonic conditions

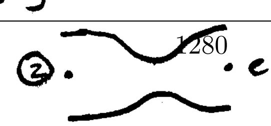
Here, $\frac{A_e}{A_t} = \frac{(0.030\text{ m})^2}{(0.0225\text{ m})^2} = 1.778$ and $\gamma = 1.4$

Solving for Ma_e: $\underline{Ma_e = 2.062}$ • use supersonic Ma # since it's given that the flow should be supersonic at the exit

$$\underline{\frac{p_e}{p_{0e}} = 0.1160}$$

But $p_e = p_{atm} \Rightarrow p_{0e} = \frac{p_{atm}}{0.1160} = \frac{101\times10^3\text{ Pa}}{0.1160} = \underline{871\times10^3\text{ Pa}}$

• Assuming the flow is isentropic throughout the converging/diverging nozzle: $p_{02} = p_{0e}$



SOLUTION...

- At ②:

$$\frac{A_2}{A^*} = \frac{(0.025)^2}{(0.0225)^2} = 1.235 \quad \text{where } A^* = A_t$$

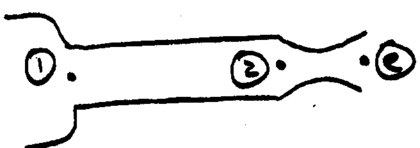
– Solving for Ma_2 using $\frac{A_2}{A^*} = \frac{1}{Ma_2}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$

$$\Rightarrow \underline{Ma_2 = 0.564}$$

$$\frac{p_2}{p_{02}} = 0.8060 \Rightarrow p_2 = (0.8060)\,p_{02}$$

$$\Rightarrow \underline{p_2 = 702 \times 10^3\,\text{Pa}} \quad \text{where } p_{02} = p_{0e} = 871 \times 10^3\,\text{Pa}$$

- Now examine the adiabatic, frictional flow in the pipe.



For $Ma_2 = 0.564 \Rightarrow \frac{4fL_2^*}{D} = 0.6529$, $\frac{p_{02}}{p_0^*} = 1.2346$

- using the adiabatic, frictional flow relations

- But $\frac{4f\Delta L}{D} = \frac{4fL_1^*}{D} - \frac{4fL_2^*}{D}$

$$\Rightarrow \underline{\frac{4fL_1^*}{D}} = \frac{4f\Delta L}{D} + \frac{4fL_2^*}{D} = \frac{4(0.005)(2.85\,\text{m})}{(0.025\,\text{m})} + 0.6529$$

$$= \underline{2.9329}$$

- Using the adiabatic, frictional flow relations:

$$\underline{Ma_1 = 0.370} \quad \text{for } \frac{4fL_1^*}{D} = 2.9329$$

$$\underline{\frac{p_{01}}{p_0^*} = 1.6962}$$

2 of 3

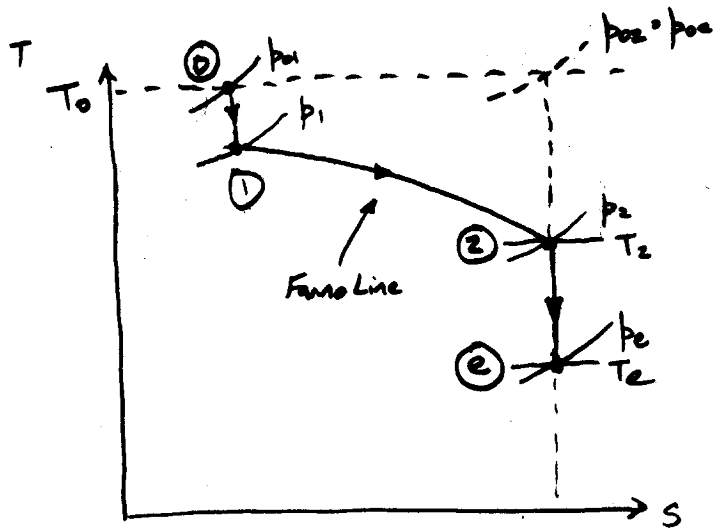
Solution...

- We can determine p_{01} by multiplying ratios:

$$p_{01} = \left(\frac{p_{01}}{p_0^*}\right)\left(\frac{p_0^*}{p_{02}}\right)p_{02}$$

$$= (1.6962)\left(\frac{1}{1.2346}\right)(871 \times 10^3\ \text{Pa})$$

$$\boxed{\therefore\ p_{01} = 1.197 \times 10^6\ \text{Pa}}$$



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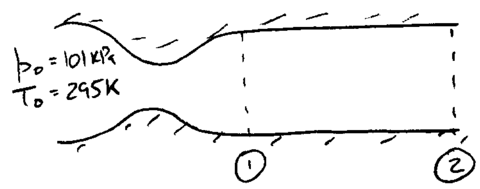
We wish to build a supersonic wind tunnel using an insulated nozzle and constant-area duct assembly. Shock-free operation is desired with a Mach number of 2.1 at the test section inlet and a Mach number of 1.1 at the test section outlet. The stagnation temperature and pressure are 295K and 101 kPa (abs), respectively. Calculate the outlet pressure and temperature and the entropy change through the test section.

SOLUTION:



$Ma_1 = 2.1 \qquad Ma_2 = 1.1$

- Calculate conditions at section 1: (assume isentropic flow through nozzle)

$$p_1/p_0 = \left(1 + \frac{\gamma-1}{2} Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1 = 11.0\text{ kPa} \text{ using } p_0 = 101\text{ kPa},\ Ma_1 = 2.1,\ \gamma = 1.4$$

$$T_1/T_0 = \left(1 + \frac{\gamma-1}{2} Ma_1^2\right)^{-1} \Rightarrow T_1 = 157\text{ K} \text{ using } T_0 = 295\text{ K},\ Ma_1 = 2.1,\ \gamma = 1.4$$

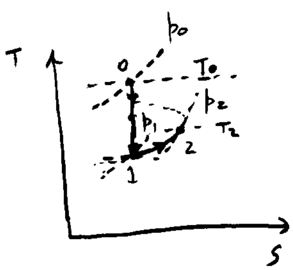
- Calculate conditions at section 2: (use Fanno flow relations)

$$p_2 = \left(p_2/p^*\right)\left(\frac{p^*}{p_1}\right) p_1$$

$\boxed{\therefore p_2 = 25.9\text{ kPa}}$ using
$p_2/p^* = 0.89358$ (using $Ma_2 = 1.1$)
$p_1/p^* = 0.38024$ (using $Ma_1 = 2.1$)
$p_1 = 11.0$ kPa

$$T_2 = \left(T_2/T^*\right)\left(T^*/T_1\right) T_1$$ using
$T_2/T^* = 0.9668$ (using $Ma_2 = 1.1$)
$T_1/T^* = 0.6376$ (using $Ma_1 = 2.1$)
$T_1 = 157$ K

$\boxed{\therefore T_2 = 238\text{ K}}$



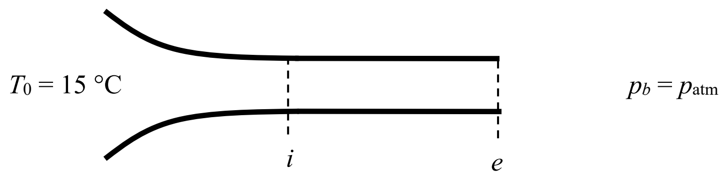
$$s_2 - s_1 = R\left(\frac{s_2 - s^*}{R} - \frac{s_1 - s^*}{R}\right)$$

$\boxed{\therefore s_2 - s_1 = 172\text{ J/kg}\cdot\text{K}}$ using
$R = 287 \frac{\text{J}}{\text{kg}\cdot\text{K}}$
$\frac{s_2 - s^*}{R} = -0.0078$ (using $Ma_2 = 1.1$)
$\frac{s_1 - s^*}{R} = -0.6081$ (using $Ma_1 = 2.1$)

1 of 1

Air flows through a converging nozzle and then a length of insulated duct. The air is supplied from a tank where the temperature is constant at 58 °F and the pressure is variable. The outlet end of the duct exhausts to atmosphere. When the exit flow is just choked, pressure measurements show the duct inlet pressure and Mach number are 53.2 psia and 0.30. Determine the pressure in the tank and the temperature, stagnation pressure, and mass flow rate of the outlet flow, if the tube diameter is 0.249 in. Show on a *Ts* diagram the effect of raising the tank pressure to 100 psia. Sketch the pressure distribution versus distance along the channel for this new flow condition.

SOLUTION:



$T_0 = (58 + 460) = 518$ °R
$D = 0.249$ in.

When the exit flow is just choked, the duct inlet pressure and Mach number are known.

$$\text{Ma}_e = 1,\ p_e = p_{atm} = p^* \ \Rightarrow\ p_i = 53.2 \text{ psia and } \text{Ma}_i = 0.30 \tag{1}$$

Assume isentropic flow in the nozzle.

$$\frac{p_i}{p_{0i}} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_i^2\right)^{\frac{\gamma}{1-\gamma}} \ \Rightarrow\ \boxed{p_{0i} = p_{tank} = 56.6 \text{ psia}} \tag{2}$$

$$\frac{T_i}{T_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_i^2\right)^{-1} \ \Rightarrow\ T_i = 509 \text{ °R} \tag{3}$$

$$\rho_i = \frac{p_i}{RT_i} \ \Rightarrow\ \rho_i = 0.282 \text{ lb}_m/\text{ft}^3 \tag{4}$$

$$V_i = \text{Ma}_i\sqrt{\gamma R T_i} \ \Rightarrow\ V_i = 332 \text{ ft/s} \tag{5}$$

$$\dot{m} = \rho_i V_i A_i \ \Rightarrow\ \boxed{\dot{m} = 3.16 \times 10^{-2} \text{ lb}_m/\text{s}} \text{ where } A_i = \pi D^2/4 \tag{6}$$

The flow is at sonic conditions at the outlet so:

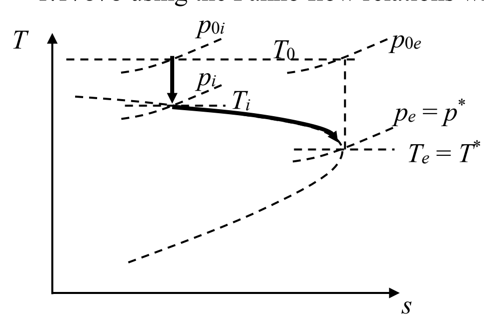
$$p_{0e} = p_0^* = \left(\frac{p_0^*}{p_{0i}}\right)p_{0i} \ \Rightarrow\ \boxed{p_{0e} = 27.8 \text{ psia}} \tag{7}$$

where $p_{0i}/p_0^* = 2.03506$ using the Fanno flow relations with $\text{Ma}_i = 0.3$.

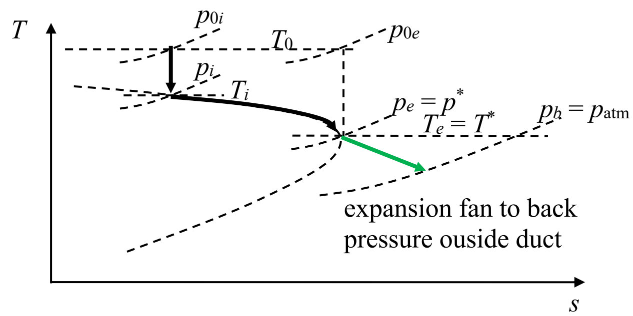
The temperature at the exit is:

$$T_e = T^* = \left(\frac{T^*}{T_i}\right)T_i \ \Rightarrow\ T_e = 432 \text{ °R} \tag{8}$$

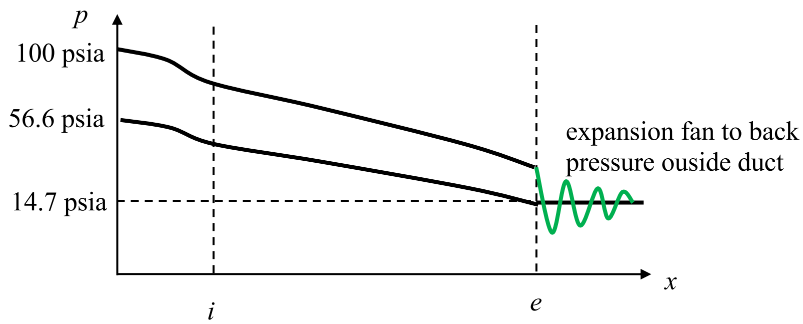
where $T_i/T^* = 1.17878$ using the Fanno flow relations with $\text{Ma}_i = 0.3$.



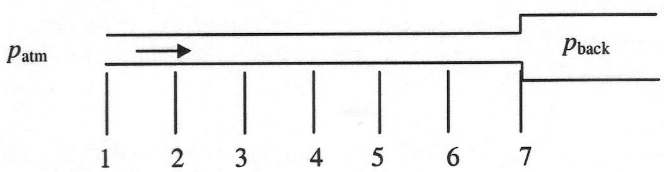
If the tank pressure is increased to p_{0i} = 100 psia, then the Ts diagram looks like the following.



Note that T^* remains the same when the tank pressure increases since T_0 remains the same and the exit flow remains at sonic conditions.



A schematic of the Fanno flow experiment in the ME309 laboratory is shown below.



p_{atm} = 29.38 in Hg
p_{back} = 14.4 in Hg
7.16 mm diameter copper tubing
distance between stations = 711 mm

For the given conditions, predict the ratio of the pressure in the pipe at each station to the inlet stagnation pressure as a function of position downstream of the inlet (non-dimensionalized by pipe diameter). Compare your predicted results to those measured in the laboratory which are given in the table below.

station	1	2	3	4	5	6	7
$p_{station}/p_{atm}$	0.971	0.915	0.847	0.775	0.695	0.602	0.488

SOLUTION:

- Since the flow starts at stagnation conditions and does not go through a C-D nozzle, assume the flow remains subsonic throughout the duct.

  $\Rightarrow p_e = p_{back}$ (NOTE: Assuming the flow is not choked.)

- Use the following iterative approach:

  1) Assume a value for Ma_e

  2) Determine p^* and $\frac{4fL_e^*}{D}$ using Fanno flow relations and Ma_e:

  $$p^* = p_e / \left(\frac{p_e}{p^*}\right)$$

  3) Determine $\frac{4fL_i^*}{D}$ using:

  $$\frac{4fL_i^*}{D} = \frac{4f\Delta L_{ie}}{D} + \frac{4fL_e^*}{D}$$

  4) Calculate inlet conditions using Fanno flow relations:

  Ma_i, p_i/p^* ← found in step 2

  Also, $p_i = (p_i/p^*)\,p^*$

1 of 4

SOLUTION...

5) Determine p_i' using the isentropic relations:

$$p_i' = p_{0i}\left(1 + \frac{\gamma-1}{2} Ma_i^2\right)^{\frac{\gamma}{1-\gamma}}$$

6) Are $\underbrace{p_i'}_{\text{found in step 5}}$ and $\underbrace{p_i}_{\text{found in step 4}}$ equal?

Yes — then finished

No — go back to step 1 and repeat

NOTES:

- $p_{atm} = 29.38 \text{ in Hg} = 99.2 \text{ KPa}$
- $p_{back} = 14.4 \text{ in Hg} = 48.6 \text{ KPa}$
- For copper tubing (assuming drawn tubing), $\varepsilon = 0.0015$
- From the Moody chart at very high Re with

  $\varepsilon/D = 0.0002 \Rightarrow f_{D'Arcy} = 0.0135$

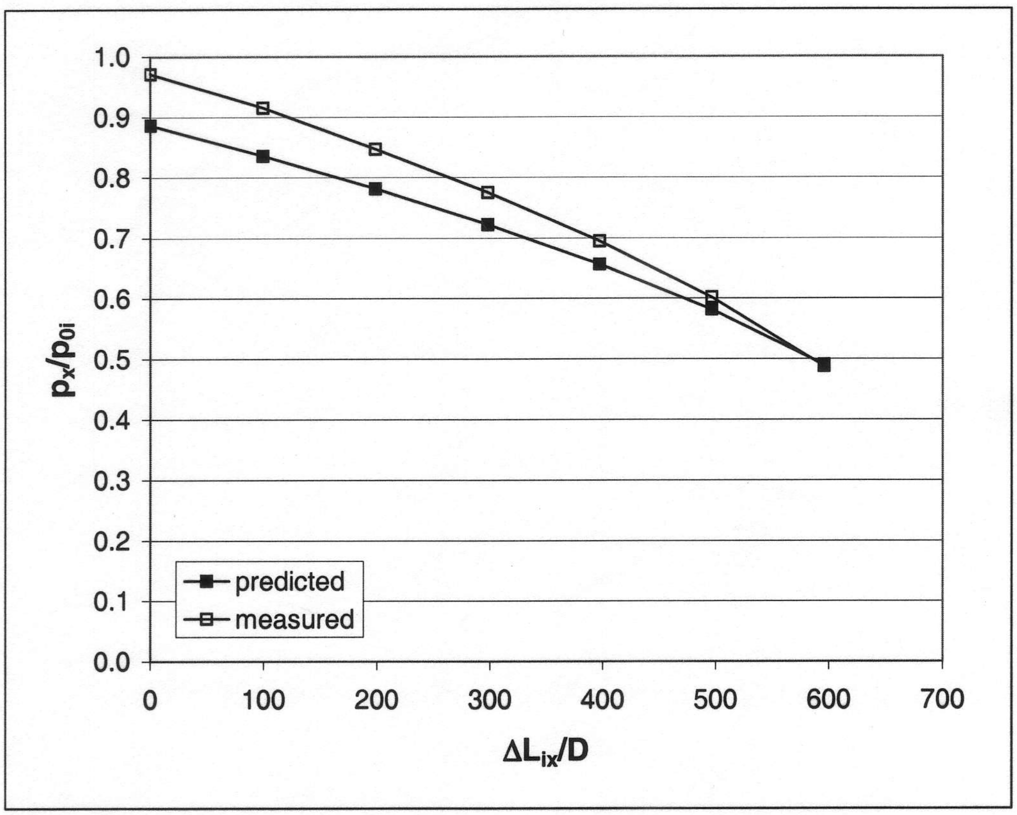
  $\Rightarrow f_{Fanno} = \frac{1}{4} f_{D'Arcy} = 0.0034$

- Solve the iterative equations using a computer spreadsheet.

γ =	1.4	
p_{0i} =	99195.9	Pa
p_b =	48618.8	Pa
ΔL_{ie} =	4.266	m
D =	0.00716	m
ε/D =	0.0002095	(assuming drawn tubing)
f =	0.003375	
$4f\Delta L_{ie}/D$ =	8.0434	

Ma_e	p_e/p^*	p^*	$4fL^*_e/D$	$4fL^*_i/D$	Ma_i	p_i/p^*	p_i	p'_i
0.1	10.9435	4442.7	66.9216	74.9650	0.09474	11.5523	51323.41	98504.7
0.5	2.1381	22739.4	1.0691	9.1125	0.24289	4.4837	101955.90	83624.1
0.4	2.6958	18034.9	2.3085	10.3519	0.23058	4.7258	85228.70	88841.3
0.45	2.3865	20372.6	1.5664	9.6099	0.23771	4.5825	93357.74	86327.0
0.42	2.5634	18966.7	1.9744	10.0178	0.23371	4.6618	88419.11	87860.0
0.41	2.6280	18500.3	2.1344	10.1778	0.23219	4.6927	86815.34	88354.9
0.415	2.5953	18733.4	2.0527	10.0961	0.23296	4.6770	87615.45	88108.5
0.417	2.5824	18826.7	2.0210	10.0644	0.23326	4.6709	87937.40	88009.4

					predicted	measured
station	$\Delta L_{ix}/D$	$4f\Delta L^*_x/D$	Ma_x	p_x/p^*	p_x/p_{0i}	p_x/p_{0i}
1	0.0	10.0644	0.23326	4.6709	0.886502	0.971
2	99.3	8.7238	0.2472	4.4046	0.835957	0.915
3	198.6	7.3833	0.26417	4.1181	0.781586	0.847
4	297.9	6.0427	0.28549	3.8062	0.722384	0.775
5	397.2	4.7021	0.31355	3.4598	0.656652	0.695
6	496.5	3.3615	0.35323	3.0632	0.581381	0.602
7	595.8	2.0210	0.41699	2.5825	0.490141	0.488



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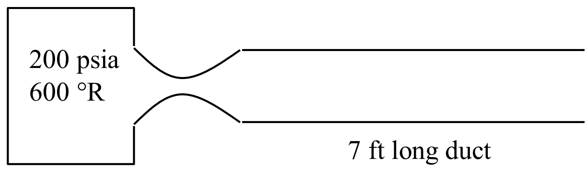
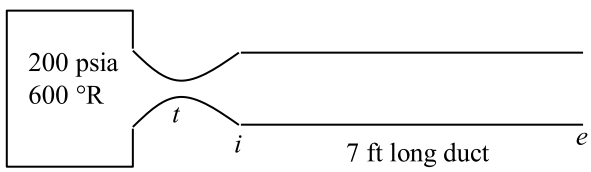
A converging-diverging nozzle with an area ratio of 3.3 is connected to a tank containing air at 200 psia and 600 °R. The throat area is 8 in^2. A 7 ft long insulated duct with a (Fanning) friction factor estimated at 0.005 is connected to the nozzle.

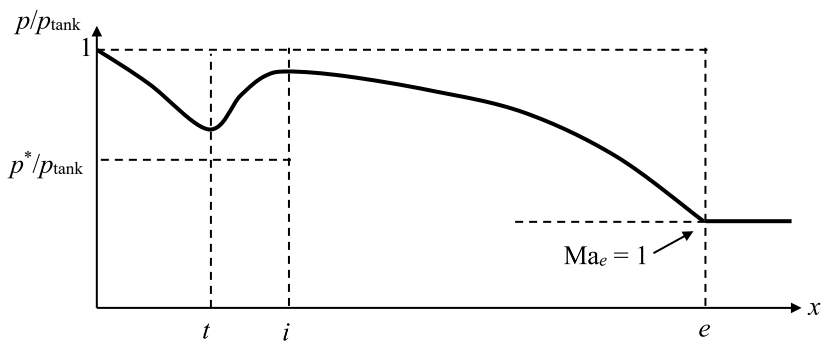
a. Find the maximum flow rate through the device.
b. What range of back pressures will produce the mass flow rate found in part (a)?
c. Sketch the process on a *T*-*s* diagram.



SOLUTION:



The maximum flow rate will occur when the flow through the system is choked. The question is, where does the flow choke first? Does it choke first at the duct exit or the converging-diverging nozzle throat? If it chokes first at the duct exit, then the pressure ratio plot will look something like the following.



Note: Since $p_{0e} < p_{0i}$,

$$\frac{p_e^*}{p_{\text{tank}}} = \underbrace{\left(\frac{p_e^*}{p_{0e}}\right)}_{=0.5283}\underbrace{\left(\frac{p_{0e}}{p_{\text{tank}}}\right)}_{<1} < 0.5283$$

For this case, we assume that the exit Mach number is one:

$$\Rightarrow \text{Ma}_e = 1,\ p_e = p^*,\ L_{ie} = L^*_i \tag{1}$$

Now determine the Mach number at the duct inlet using the Fanno flow relations.

$$\frac{4 f_F L_i^*}{D} = \frac{4 f_F L_{ie}}{D} = \frac{4(0.005)(7\text{ ft})}{(0.483\text{ ft})} = 0.2898 \ \Rightarrow\ \text{Ma}_i = 0.6632 \tag{2}$$

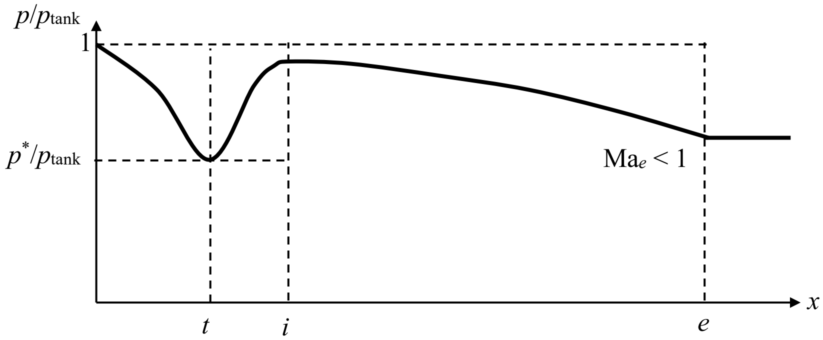
Note that the inlet Mach number will be subsonic since we're assuming that the flow chokes first at the duct exit and not at the converging-diverging nozzle throat.

With the given inlet Mach number, calculate the sonic area A^* using the isentropic relations (*IR*) and Ma_i = 0.6632 (Eqn. (2)).

$$\frac{A_i}{A^*} \underset{IR}{=} fcn\left(\mathrm{Ma}_i\right) \;\Rightarrow\; A_i/A^* = 1.12372 \tag{3}$$

Since the calculated A_i/A^* is smaller than the given $A_i/A_t = 3.3 \Rightarrow A_t < A^*$, we conclude that the assumption that the flow chokes first at the duct exit must be incorrect.

Now assume that the system chokes first at the converging-diverging nozzle throat. For this case, the pressure profile looks something like the following.



From the isentropic relations:

$$\frac{A_i}{A_t} = \frac{A_i}{A^*} \underset{IR}{=} fcn\left(\mathrm{Ma}_i\right) \;\Rightarrow\; \mathrm{Ma}_i = 0.17874 \tag{4}$$

where $A_i/A_t = 3.3$ and the "*IR*" underneath the "=" sign signifies the isentropic relations.

Using the given inlet Mach number and the Fanno flow relations (*FFR*):

$$\mathrm{Ma}_i = 0.17874 \Rightarrow \frac{4 f_F L_i^*}{D} = 18.8426 \;\text{ and } p_i/p^* = 6.10922 \tag{5}$$

Using the Fanno flow relations, the Mach number at the duct exit is then:

$$\frac{4 f_F L_e^*}{D} = \frac{4 f_F L_{ie}}{D} - \frac{4 f_F L_i^*}{D} = \frac{4\left(0.005\right)\left(7 \text{ ft}\right)}{\left(0.483 \text{ ft}\right)} + 18.8426 = 18.5528 \;\Rightarrow \mathrm{Ma}_e = 0.1800,\; p_e/p^* = 6.0676 \tag{6}$$

Since the exit Mach number is subsonic, there is no inconsistency with the assumption of choking first at the nozzle throat. Our assumption that choking occurs first at the nozzle throat is ok.

The maximum mass flow rate through the device will simply be the choked mass flow rate:

$$\dot{m}_{\max} = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \;\Rightarrow\; \boxed{\dot{m}_{\max} = 34.8 \text{ lb}_\text{m}/\text{s}} \tag{7}$$

where p_0 = 200 psia = 28800 $\text{lb}_\text{f}/\text{ft}^2$, T = 600 °R, γ = 1.4, R = 53.3 $(\text{ft}\cdot\text{lb}_\text{f})/(\text{lb}_\text{m}\cdot{}^\circ\text{R})$, and $A^* = A_t = 8 \text{ in}^2/(144 \text{ in}^2/\text{ft}^2) = 0.056 \text{ ft}^2$.

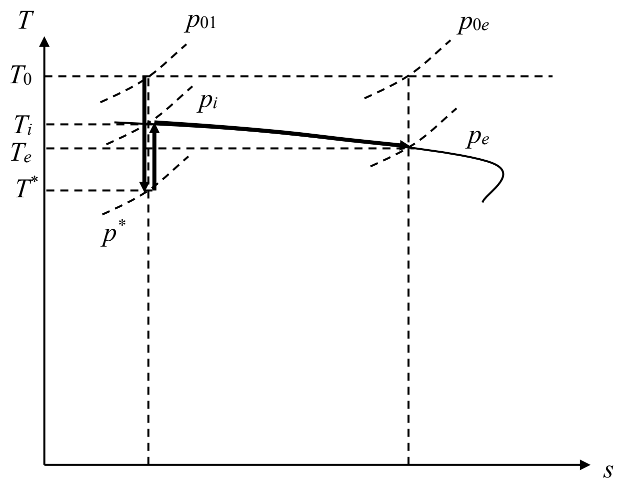
The back pressure corresponding to when the flow just becomes choked may be found using the results in Eqs. (5) and (6) coupled with the isentropic relations.

$$\mathrm{Ma}_i = 0.17874 \underset{IR}{\Rightarrow} \frac{p_i}{p_0} = 0.97795 \tag{8}$$

$$p_b = p_e = \left(\frac{p_e}{p^*}\right)\left(\frac{p^*}{p_i}\right)\left(\frac{p_i}{p_0}\right) p_0 = \left(6.0676\right)\left(\frac{1}{6.10922}\right)\left(0.97795\right)\left(200 \text{ psia}\right) = 194 \text{ psia} \tag{9}$$

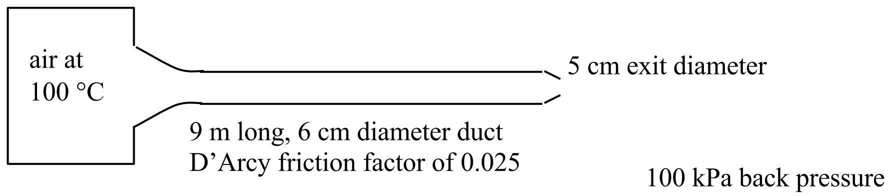
Since the flow will remain choked for back pressures lower than this value, even if the flow chokes again at the duct exit, the range of back pressures for which the maximum flow rate in Eqn. (7) will occur is:

$\boxed{p_b \leq 194 \text{ psia}}$ (10)

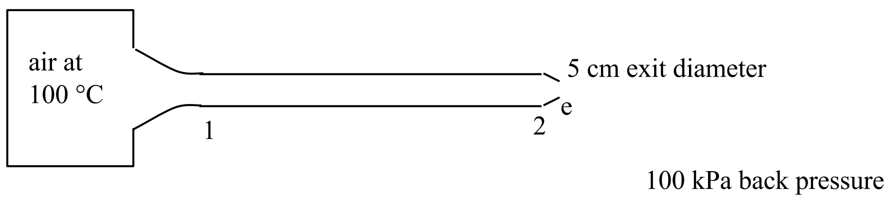


Air flows steadily from a tank through the pipe shown in the figure. There is a converging nozzle on the end. If the mass flow rate is 3 kg/s and the nozzle is choked, estimate:

a. the Mach number at duct inlet, and
b. the pressure inside the tank.
c. Sketch the process on a *T*-*s* diagram.



SOLUTION:



If the flow is choked, it will be choked at the exit ⇒ $\text{Ma}_e = 1$ and $A_e = A^*$. (1)

Use the isentropic flow relations to determine the conditions at station 2.

$$\frac{A_2}{A^*} = \frac{1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_2^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_2 = 0.45 \quad (2)$$

where $A_2/A^* = (D_2/D_e)^2 = (6\text{ cm}/5\text{ cm})^2 = 1.44$.

Use the Fanno flow relations and $\text{Ma}_2 = 0.45$ to determine the sonic length ratio at 2.

$$\frac{4 f_F L_2^*}{D} = 1.5664 \quad (3)$$

Determine the sonic length ratio at 1 and then use the Fanno flow relations to determine the Mach number at 1.

$$\frac{4 f_F L_1^*}{D} = \frac{4 f_F L_{12}}{D} + \frac{4 f_F L_2^*}{D} = 5.3164 \Rightarrow \boxed{\text{Ma}_1 = 0.30} \quad (4)$$

where $4f_F L^*_2/D = 1.5664$ (found above), $f_F = f_D/4 = 0.00625$, $L_{12} = 9$ m, $D = 0.06$ m.

The pressure within the tank can be found using a combination of the isentropic relations and the mass flow rate.

$$\dot{m} = \rho_1 V_1 A_1 \Rightarrow \rho_1 = \frac{\dot{m}}{V_1 A_1} \Rightarrow \rho_1 = 9.22\text{ kg/m}^3 \quad (5)$$

where

$$V_1 = \text{Ma}_1 c_1 = \text{Ma}_1\sqrt{\gamma R T_1} \quad (6)$$

$$T_1 = T_0\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-1} \tag{7}$$

$$A_1 = \frac{\pi D_1^2}{4} \tag{8}$$

and $\dot{m} = 3\ \mathrm{kg/s}$, $\gamma = 1.4$, $\mathrm{Ma}_1 = 0.30$, $R = 287$ J/(kg·K), $T_0 = 100$ °C = 373 K, $D_1 = 0.06$ m.

The stagnation density is found using the isentropic relations and the ideal gas law is used to determine the stagnation pressure (the pressure in the tank).

$$\rho_1 = \rho_{01}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \rho_{01} = 9.64\ \mathrm{kg/m^3} \tag{9}$$

$$p_{01} = \rho_{01}RT_0 \Rightarrow \boxed{p_{01} = 1.03\ \mathrm{MPa}} \tag{10}$$

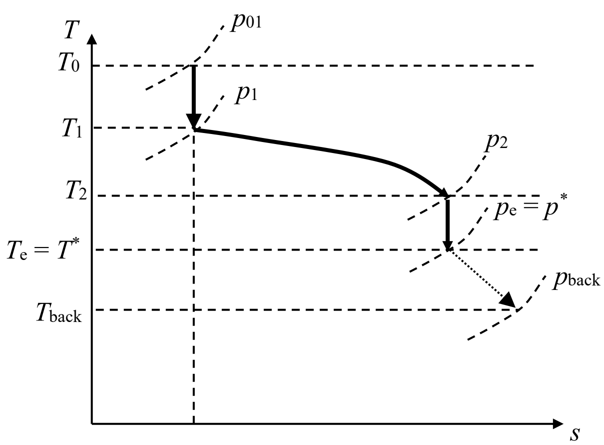
Note that the conditions at the exit may be found in the following manner.

$$\dot{m} = \rho_e V_e A_e = \left(\frac{p_e}{RT_e}\right)\left(\mathrm{Ma}_e\sqrt{\gamma RT_e}\right)A_e = p_e\left(\mathrm{Ma}_e\sqrt{\frac{\gamma}{RT_0}}\right)\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{1}{2}}\left(\frac{\pi D_e^2}{4}\right) \tag{11}$$

$$p_e = \frac{\dot{m}}{\mathrm{Ma}_e\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{1}{2}}\left(\sqrt{\frac{\gamma}{RT_0}}\right)\left(\frac{\pi D_e^2}{4}\right)} \Rightarrow p_e = 386\ \mathrm{kPa} \tag{12}$$

$$T_e = T_0\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{-1} \Rightarrow T_e = 311\ \mathrm{K} = 38\ °\mathrm{C} \tag{13}$$

where $\dot{m} = 3\ \mathrm{kg/s}$, $\mathrm{Ma}_e = 1$, $\gamma = 1.4$, $R = 287$ J/(kg·K), $T_0 = 100$ °C = 373 K, $D_e = 0.05$ m.

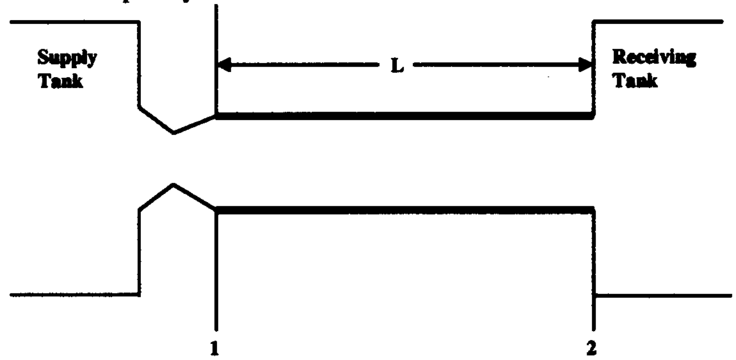


(Assuming $T_{back} < T_e$.)

Consider the following system in which a supply tank and receiving tank are connected with an insulated pipe of length, *L*. Assume that isentropic flow exists in the converging-diverging nozzle.

Calculate the following:

a. **The pipe internal diameter.**
b. **The length of the pipe.**
c. **Is this design possible? Explain your answer.**



Supply tank temperature	**37 °C**
Supply tank pressure	**866 kPa (abs)**
Receiving tank pressure	**100 kPa (abs)**
Mass flow rate	**0.482 kg/s**
Pipe inlet (Station 1) Mach number	**2.0**
Pipe outlet (Station 2) Mach number	**1.2**
Average friction factor	**0.012 (D'Arcy)**

SOLUTION:

- Determine the pipe diameter from the mass flow rate and the isentropic relations:

$$\dot{m} = \rho_1 V_1 A_1$$

where $\rho_1 = \rho_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1}{1-\gamma}} = \frac{p_{01}}{RT_{01}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1}{1-\gamma}}$

$$V_1 = Ma_1\sqrt{\gamma R T_1} = Ma_1\sqrt{\gamma R T_{01}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{-1/2}$$

$$A_1 = \frac{\pi D^2}{4}$$

$\therefore$ $\rho_{01} = 2.24\ \text{kg/m}^3$

$V_1 = 526\ \text{m/s}$

using $p_{01} = 866\ \text{kPa}$, $R = 287\ \text{J/kg}\cdot\text{K}$, $\gamma = 1.4$, $T_{01} = 37°\text{C} = 310\text{K}$, $Ma_1 = 2.0$, $\dot{m} = 0.482\ \text{kg/s}$

$$\boxed{\therefore\ D = 2.28\times10^{-2}\ \text{m} = 2.28\ \text{cm}}$$

1 of 2

SOLUTION...

- Use the Fanno Flow relations to determine the pipe length:

$$\frac{4fL_1^*}{D} = 0.30499 \quad \text{using } Ma_1 = 2.0$$

$$\frac{4fL_2^*}{D} = 0.03363 \quad \text{using } Ma_2 = 1.2$$

$$\frac{4f\Delta L_{12}}{D} = \frac{4fL_1^*}{D} - \frac{4fL_2^*}{D}$$

$$\boxed{\therefore \Delta L_{12} = 0.52\,\text{m}} \quad \text{using } \frac{4fL_1^*}{D} = 0.30499,\ \frac{4fL_2^*}{D} = 0.03363,\ f = \tfrac{1}{4}(0.012),\ D = 2.28\times10^{-2}\,\text{m}$$

- Consider the pressure at location 2:

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1$$

$$\therefore p_2 = 219\,\text{KPa} \quad \text{using } p_2/p^* = 0.80436 \text{ using } Ma_2 = 1.2$$

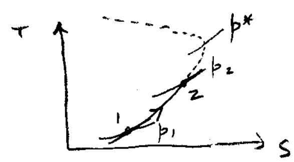
$$p_1/p^* = 0.40824 \text{ using } Ma_1 = 2.0$$

$$p_1 = p_{01}\left(1 + \frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} = 111\,\text{KPa} \quad \text{using } p_{01} = 866\,\text{KPa},\ Ma_1 = 2.0,\ \gamma = 1.4$$

$p_2 > p_{back} = 100\,\text{KPa}$ OK

$$p^* = \left(\frac{p^*}{p_1}\right)p_1$$

$$\therefore p^* = 272\,\text{KPa}$$



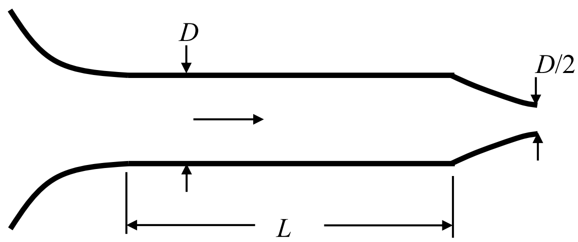
We observe that $p_2 < p^*$. This is ok since for supersonic Fanno flow the pressure increases as we head downstream. Thus, there are no pressure inconsistencies.

2 of 2

⇒ The flow is possible.

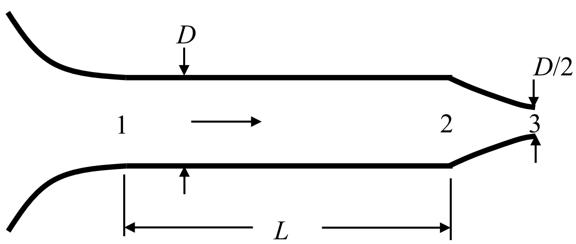
Compressed air is supplied from a reservoir to a pipe 1 cm in diameter and 5 m long. It is estimated that the average (Fanning) friction factor of the flow in the pipe is 0.02. At the end of this long pipe is a short, converging nozzle whose opening to the atmosphere has one-half the diameter of the pipe. Assuming that frictional effects in the nozzle can be neglected, find the following information pertaining to conditions when the flow through the pipe/nozzle combination is choked:

1. the Mach number of the flow entering the pipe,
2. the ratio of the pressure in the reservoir to the pressure in the exit from the nozzle, and
3. sketch the entire process on a *T*-*s* diagram.



D = 0.01 m
L = 5 m
f_F = 0.02

SOLUTION:



The flow will choke at the converging nozzle exit (location 3). Determine the Mach number at the pipe exit (location 2) taking into account the fact that the converging nozzle flow is isentropic.

$$\frac{A_2}{A_3} = \frac{A_2}{A^*} = \left(\frac{D}{\frac{1}{2}D}\right)^2 = 4.0 = \frac{1}{\mathrm{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_2 = 0.1465 \tag{1}$$

Use the isentropic relations to determine the location 3-to-location 2 pressure ratio.

$$\frac{p_3}{p_2} = \frac{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_3^2\right)^{\frac{\gamma}{1-\gamma}}}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}}} \Rightarrow p_3/p_2 = 0.5283/0.9851 = 0.5363 \tag{2}$$

Note that $\mathrm{Ma}_3 = 1$ and $p_3 = p^*$.

Use the Fanno flow relations to determine the dimensionless sonic length at location 2, then use this information in addition to the pipe length to determine the dimensionless sonic length at location 1. Once this quantity is known, use the Fanno flow relations once more to determine the Mach number at the pipe inlet (location 1).

$$\mathrm{Ma}_2 = 0.1465 \Rightarrow 4f_F L_2^*/D = 29.4266 \tag{3}$$

$$\frac{4f_F L_1^*}{D} = \frac{4f_F L_2^*}{D} + \frac{4f_F L}{D} = 29.4266 + 40.0000 \Rightarrow 4f_F L_1^*/D = 69.4266 \tag{4}$$

$$4f_F L_1^*/D = 69.4266 \Rightarrow \boxed{\mathrm{Ma}_1 = 0.09827} \tag{5}$$

Now find the location 2-to-location 1 pressure ratio using the Fanno flow relations.

$\text{Ma}_2 = 0.1465 \Rightarrow p_2/p^* = 7.4614$ (6)

$\text{Ma}_1 = 0.09827 \Rightarrow p_1/p^* = 11.1365$ (7)

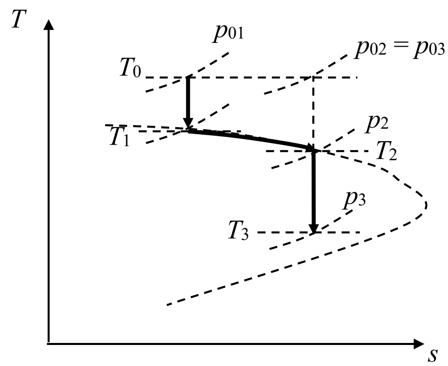
$p_2/p_1 = (p_2/p^*)/(p_1/p^*) \Rightarrow p_2/p_1 = 0.6700$ (8)

Use the isentropic relations to determine the location 1-to-reservoir pressure ratio.

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1/p_{01} = 0.9933 \quad (9)$$

Now find the ratio of the pressure in the reservoir to the pressure at the nozzle exit by multiplying the pressure ratios.

$$\frac{p_{01}}{p_3} = \left(\frac{p_{01}}{p_1}\right)\left(\frac{p_1}{p_2}\right)\left(\frac{p_2}{p_3}\right) \Rightarrow \boxed{p_{01}/p_3 = 2.8018} \quad (10)$$



How do we know that the flow chokes first at the nozzle exit as opposed to the pipe exit? Because the flow in going from 2 to 3 is isentropic, and the Mach number can only equal one at a minimum area, i.e., at location 3, in an isentropic flow. There is a contradiction if the Mach number is one at location 2.

Isothermal Flow with Friction

In long pipelines there is too much surface area for the flow to be considered adiabatic. Instead a more reasonable assumption is that the flow is isothermal. The Mach numbers in such flows tend to be small but the pressure change, Δp, can be large $\Rightarrow$ we can't treat the flow as being incompressible.

Consider isothermal (dT=0), 1D, steady flow of an ideal gas in a constant area (dA=0) duct. Using the same approach as the differential analysis for adiabatic frictional flow, we determine the following relations:

COM: $$\boxed{\frac{d\rho}{\rho}+\frac{dV}{V}=0} \tag{103}$$
Note: dA=0

LME: $$\boxed{dp+\rho V dV+\frac{4\tau_{\mathrm{w}}}{D_{\mathrm{H}}}dx=0} \tag{104}$$

COE: $$\boxed{c_p dT_0 = V dV = \delta q_{\text{into}}} \tag{105}$$
Notes:
1. dh=0 since dT=0.
2. $dh_0 = c_p dT + V dV = \delta q_{\text{into}} = \dot{Q}_{\text{into}}/\dot{m}$
3. We are not assuming adiabatic conditions so heat transfer must be included.

2nd Law: $$\boxed{ds > \frac{\delta q_{\text{into}}}{T}} \tag{106}$$
Since the flow is frictional, it is also irreversible.

Ideal Gas Law: $$\boxed{\frac{dp}{p}=\frac{d\rho}{\rho}} \tag{107}$$
Note: dT=0

Mach # relation: $$\boxed{\frac{d(\mathrm{Ma})}{\mathrm{Ma}}=\frac{dV}{V}} \tag{108}$$
Note: dT=0

Entropy relation: $$\boxed{ds=-R\frac{dp}{p}} \tag{109}$$
Note: dT=0

The local adiabatic stagnation temperature is given by:
$$T_0 = T\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)$$
so that
$$\frac{dT_0}{T_0}=\frac{(\gamma-1)\mathrm{Ma}\,d(\mathrm{Ma})T}{T\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)} \qquad \text{where } dT=0$$
$$\boxed{\frac{dT_0}{T_0}=\frac{(\gamma-1)\mathrm{Ma}^2}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)}\frac{d(\mathrm{Ma})}{\mathrm{Ma}}} \tag{110}$$

The local stagnation pressure is given by:

$$p_0 = p\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{\gamma/1-\gamma}$$

so that

$$\frac{dp_0}{p_0} = \frac{dp\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{\gamma/1-\gamma} + (\gamma-1)\mathrm{Ma}d(\mathrm{Ma})\,p\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{2\gamma-1/1-\gamma}}{p\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{\gamma/1-\gamma}}$$

$$\frac{dp_0}{p_0} = \frac{dp}{p} + (\gamma-1)\mathrm{Ma}d(\mathrm{Ma})\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{-1}$$

Combine Eqs. (103) through (110) leaving ($4f_F dx/D$) as the independent variable:

$$\boxed{\frac{dp}{p} = \frac{d\rho}{\rho} = -\frac{dV}{V} = -\frac{1}{2}\frac{d\left(\mathrm{Ma}^2\right)}{\mathrm{Ma}^2} = -\frac{ds}{R} = \frac{-\gamma\mathrm{Ma}^2}{2\left(1-\gamma\mathrm{Ma}^2\right)}\left(\frac{4f_F}{D_H}dx\right)} \tag{111}$$

$$\boxed{\frac{dp_0}{p_0} = \frac{-\gamma\mathrm{Ma}^2\left(1-\frac{\gamma+1}{2}\mathrm{Ma}^2\right)}{2\left(1-\gamma\mathrm{Ma}^2\right)\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)}\left(\frac{4f_F}{D_H}dx\right)} \tag{112}$$

$$\boxed{\frac{dT_0}{T_0} = \frac{\gamma(\gamma-1)\mathrm{Ma}^4}{2\left(1-\gamma\mathrm{Ma}^2\right)\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)}\left(\frac{4f_F}{D_H}dx\right)} \tag{113}$$

$\mathrm{Ma}^2 < 1/\gamma$ (subsonic)		$\mathrm{Ma}^2 > 1/\gamma$ (subsonic/supersonic)
$d(\mathrm{Ma}) > 0$	Eq. (111)	$d(\mathrm{Ma}) < 0$
$dV > 0$	Eq. (111)	$dV < 0$
$dp < 0$	Eq. (111)	$dp > 0$
$dT = 0$	isothermal	$dT = 0$
$d\rho < 0$	Eq. (111)	$d\rho > 0$
$ds > 0$	Eq. (111)	$ds < 0$ (Note: $ds > \delta q_{\mathrm{into}}/T$)
$dT_0 > 0$ (heat added)	Eq. (113)	$dT_0 < 0$ (heat removed
$dp_0 < 0$	Eq. (112)	$dp_0 > 0$ for $\mathrm{Ma}^2 < 2/(\gamma+1)$
		$dp_0 < 0$ for $\mathrm{Ma}^2 > 2/(\gamma+1)$

Working equations can be found by integrating Eqs. (111) through (113) using the location where $\text{Ma}^2=1/\gamma$ as a reference point (indicated by the superscript "$*t$" to signify limiting conditions for isothermal, frictional flow).

$$\boxed{\frac{4f_F L^{*t}}{D_\text{H}} = \frac{1-\gamma\text{Ma}^2}{\gamma\text{Ma}^2} + \ln\left(\gamma\text{Ma}^2\right)} \tag{114}$$

$$\boxed{\frac{V}{V^{*t}} = \frac{\rho^{*t}}{\rho} = \frac{p^{*t}}{p} = \sqrt{\gamma}\text{Ma}} \tag{115}$$

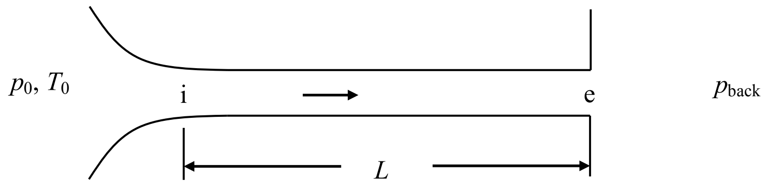
$$\boxed{\frac{p_0}{p_0^{*t}} = \frac{1}{\sqrt{\gamma}}\left(\frac{2\gamma}{3\gamma-1}\right)^{\gamma/\gamma-1}\frac{\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{\gamma/\gamma-1}}{\text{Ma}}} \tag{116}$$

$$\boxed{\frac{T_0}{T_0^{*t}} = \frac{2\gamma}{3\gamma-1}\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)} \tag{117}$$

Notes:

1. Let's determine the amount of heat that must be added (in the case of subsonic flow) or removed (in the case of supersonic flow) for the flow to go from a given Mach number, Ma, to the choked flow Mach number, $\text{Ma}^{*t} = 1/\gamma^{1/2}$. From Eq. (105) we have:
   $$c_p dT_0 = \delta q_\text{into}$$
   and from Eq. (113) we have
   $$\frac{dT_0}{T_0} = \frac{\gamma(\gamma-1)\text{Ma}^4}{2\left(1-\gamma\text{Ma}^2\right)\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)}\left(\frac{4f_F}{D_H}dx\right)$$
   Combining these two relations gives:
   $$\delta q_\text{into} = \frac{c_p T_0 \gamma(\gamma-1)\text{Ma}^4}{2\left(1-\gamma\text{Ma}^2\right)\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)}\left(\frac{4f_F}{D_H}dx\right) \tag{118}$$
   From Eq. (118) we see that as the Mach number approaches its limiting value of $1/\gamma^{1/2}$, the local heat transfer required to maintain isothermal conditions becomes very large. Hence, the assumption of isothermal flow for Mach numbers in the neighborhood of the limiting Mach number may not be a good one since there may not be sufficient heat transfer to maintain isothermal conditions.

2. Consider the following flow situation.

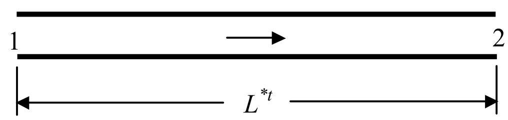


Assume that for the given back pressure, p_{back}, the duct length is the sonic length, i.e., $L=L^{*t}$ and $\text{Ma}_e=1/\gamma^{1/2}$. Note that since the exit Mach number is subsonic, the exit pressure will equal the back pressure, i.e., $p_e = p^{*t} = p_{\text{back}}$. What happens if we now increase the duct length or drop the back pressure? It can be shown that there is no solution using the isothermal, frictional flow equations for the new conditions. The isothermal flow assumption breaks down since we would need to supply an infinite amount of heat transfer to get to the new conditions (refer to Eq. (118)).

Natural gas (γ=1.3, R=75 (ft·lb$_f$)/(lb$_m$·°R)) flows through a pipeline of diameter 3 ft and (Fanning) friction factor 0.001. At a particular point in the pipeline the pressure is 200 psia, the temperature is 500 °R, and the velocity is 50 ft/s. The pipeline is kept at constant temperature.

a. Determine the maximum length of the pipe that one could have from this point given these conditions.
b. What would be the pressure at the maximum length?

SOLUTION:



D = 3 ft
f_F = 0.001
p_1 = 200 psia
T_1 = 500 °R
V_1 = 50 ft/s

First find the Mach number at 1.

$$\mathrm{Ma}_1 = \frac{V_1}{c_1} = \frac{V_1}{\sqrt{\gamma R T_1}} \Rightarrow \mathrm{Ma}_1 = 0.0399 \tag{1}$$

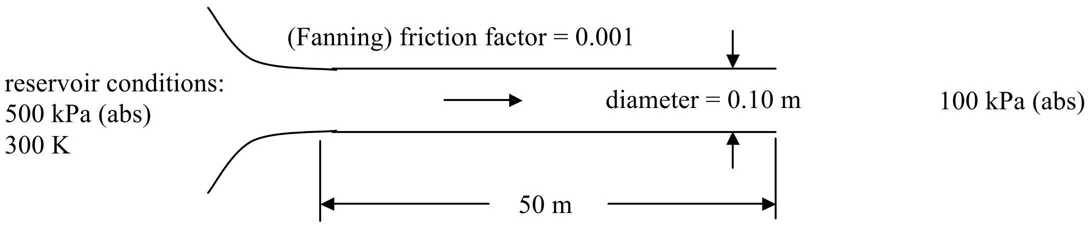
Now find L^{*t} using the isothermal, frictional flow relations.

$$\frac{4 f_F L_1^{*t}}{D_H} = \frac{1-\gamma \mathrm{Ma}_1^2}{\gamma \mathrm{Ma}_1^2} + \ln\left(\gamma \mathrm{Ma}_1^2\right) \Rightarrow \boxed{L_1^{*t} = 67.6 \text{ miles}} \tag{2}$$

Now find p^{*t} using the isothermal, frictional flow relations.

$$\frac{p^{*t}}{p_1} = \sqrt{\gamma}\mathrm{Ma}_1 \Rightarrow \boxed{p^{*t} = 9.10 \text{ psia}} \tag{3}$$

Consider the air pipeline attached to a converging nozzle as shown below.



Determine the mass flow rate through the pipeline:
a. if the pipeline is well-insulated, and
b. if the pipeline is held at constant temperature.

SOLUTION:

For part (a), use the (adiabatic) Fanno flow relations. First check if the flow is choked or not at the pipe exit. Assume that the flow is choked so that $L_{ie} = L^*_i$, $\text{Ma}_e = 1$, and $p_e = p^* \geq p_b$.

$$\frac{4 f_F L_i^*}{D} = \frac{4(0.001)(50\text{ m})}{(0.10\text{ m})} = 2.000 \;\Rightarrow \text{Ma}_i = 0.4183 \;\text{(using the Fanno flow relations)} \tag{1}$$

$$\Rightarrow\; p_i/p^* = 2.5739 \text{ (using the Fanno flow relations)} \tag{2}$$

Calculate the inlet stagnation pressure ratio using the isentropic relations.

$$\text{Ma}_i = 0.4183 \Rightarrow\; p_i/p_{0i} = 0.8866 \;\text{(using the isentropic flow relations)} \tag{3}$$

Now calculate the exit pressure using the various pressure ratios in order to verify that the choked flow assumption is correct.

$$p_e = \left(\frac{p_e}{p^*}\right)\left(\frac{p^*}{p_i}\right)\left(\frac{p_i}{p_{0i}}\right)p_{0i} = (1)\left(\frac{1}{2.5739}\right)(0.8866)(500\text{ kPa}) = 172\text{ kPa} \tag{4}$$

Note that $p_e > p_b = 100$ kPa, which is consistent with the assumption of choked flow.

Now calculate the mass flow rate through the pipe using the conditions at the pipe inlet.

$$\dot{m} = \rho_i V_i A_i = \left(\frac{p_i}{RT_i}\right)\left(\text{Ma}_i\sqrt{\gamma RT_i}\right)\left(\tfrac{\pi}{4}D^2\right) = \left(p_i \text{Ma}_i\sqrt{\frac{\gamma}{RT_i}}\right)\left(\tfrac{\pi}{4}D^2\right) \tag{5}$$

$$\dot{m} = p_{0i}\left(1+\frac{\gamma-1}{2}\text{Ma}_i^2\right)^{\frac{\gamma}{1-\gamma}} \text{Ma}_i\sqrt{\frac{\gamma}{RT_{0i}}}\left(1+\frac{\gamma-1}{2}\text{Ma}_i^2\right)^{\frac{1}{2}}\left(\tfrac{\pi}{4}D^2\right) \tag{6}$$

$$\dot{m} = \text{Ma}_i\left(1+\frac{\gamma-1}{2}\text{Ma}_i^2\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_{0i}\sqrt{\frac{\gamma}{RT_{0i}}}\left(\tfrac{\pi}{4}D^2\right) \tag{7}$$

$$\boxed{\dot{m} = 5.97\text{ kg/s}} \tag{8}$$

For part (b), use the iso-thermal with friction relations. Follow the same approach as described above. First assume that the flow is choked at the outlet. If this is the case, then $L_{ie} = L^{*t}_i$, $\mathrm{Ma}_e = (1/\gamma)^{1/2}$, and $p_e = p^{*t} = p_b$ (since the exit Mach number is subsonic).

$$\frac{4 f_F L_i^{*t}}{D} = \frac{4(0.001)(50\text{ m})}{(0.10\text{ m})} = 2.000 \Rightarrow \mathrm{Ma}_i = 0.3982 \text{ (using the isothermal w/ friction relations)} \tag{9}$$

$$\Rightarrow p_i/p^{*t} = 2.1225 \text{ (using the isothermal w/ friction relations)} \tag{10}$$

Calculate the inlet stagnation pressure ratio using the isentropic relations.

$$\mathrm{Ma}_i = 0.3982 \Rightarrow p_i/p_{0i} = 0.8965 \text{ (using the isentropic flow relations)} \tag{11}$$

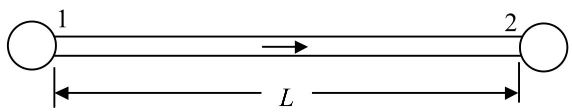
Now calculate the exit pressure using the various pressure ratios in order to verify that the choked flow assumption is correct.

$$p_e = \left(\frac{p_e}{p^{*t}}\right)\left(\frac{p^{*t}}{p_i}\right)\left(\frac{p_i}{p_{0i}}\right)p_{0i} = (1)\left(\frac{1}{2.1225}\right)(0.8965)(500\text{ kPa}) = 211\text{ kPa} \tag{12}$$

However, for choked isothermal flow with friction, the exit Mach number is subsonic so we would need $p_e = p_b = 100$ kPa, which is not the case here. Furthermore, the pressure found in Eq. (12) is the lowest exit pressure that can be generated given the iso-thermal flow with friction assumptions and starting from subsonic conditions (note that the flow starts from stagnation conditions and goes through a converging nozzle). Hence, <u>we must conclude that it is not possible to maintain the isothermal flow with friction assumptions for the given flow conditions.</u>

Natural gas (R=75 (ft·lb$_f$)/(lb$_m$·°R), γ=1.3) is to be pumped through a pipe of 36 inches I.D. connecting two compressor stations forty miles apart. At the upstream station the pressure is not to exceed 90 psig and at the downstream station it is to be at least 10 psig. Calculate the maximum allowable mass flow rate, assuming that there is sufficient heat transfer through the pipe wall to maintain the gas at 70 °F.

SOLUTION:



The maximum flow rate will occur when the flow is choked. For an isothermal flow with friction, this will occur when $\text{Ma}_2 = 1/\gamma^{1/2}$. The mass flow rate for choked flow conditions is given by:

$$\dot{m}_{\text{choked}} = \rho^{*t} V^{*t} A = \left(\frac{p^{*t}}{RT}\right)\left(\frac{\sqrt{\gamma RT}}{\sqrt{\gamma}}\right) A = \frac{p^{*t}}{\sqrt{RT}} A \tag{1}$$

where $V^{*t} = \text{Ma}^{*t}(\gamma RT)^{1/2}$, $\text{Ma}^{*t} = 1/\gamma^{0.5}$, A = constant (constant area duct), and T = constant (isothermal conditions). The sonic pressure can be determined in terms of the conditions at station 1 using the isothermal, frictional flow relations:

$$\frac{p_1}{p^{*t}} = \frac{p_1}{p_2} = \frac{1}{\text{Ma}_1\sqrt{\gamma}} \Rightarrow p^{*t} = p_1\sqrt{\gamma}\text{Ma}_1 \tag{2}$$

Keeping in mind the constraint that:

$$\left.\frac{p_1}{p^{*t}}\right|_{\max} = \left.\frac{p_1}{p_2}\right|_{\max} = \frac{(90+14.7)\text{ psia}}{(10+14.7)\text{ psia}} = \frac{104.7\text{ psia}}{24\text{ psia}} = 4.2389 \tag{3}$$

Substitute Eq. (2) into Eq. (1) to get:

$$\dot{m}_{\text{choked}} = p_1\sqrt{\frac{\gamma}{RT}}\text{Ma}_1 A \tag{4}$$

Hence, to maximize the mass flow rate, the pressure at 1 should be maximized (= 90 psig = 104.7 psia) and Ma_1 should be as large as possible. From Eq. (2) we observe that as Ma_1 increases, p^{*t} also increases (hence we needn't worry about dropping below the minimum allowable pressure). The maximum mass flow rate will occur when $p^{*t} = p_1$ ($\Rightarrow \text{Ma}_1 = 1/\gamma^{0.5}$) with $\boxed{\dot{m}_{\max} = 3033\text{ lb}_m/\text{s}}$. The (Fanning) friction factor corresponding to this case is zero, i.e., $\underline{f_F = 0}$.

The largest allowable friction factor will correspond to:

$$\left.\frac{p_1}{p^{*t}}\right|_{\max} = 4.2389 = \frac{1}{\text{Ma}_1\sqrt{\gamma}} \Rightarrow \text{Ma}_1 = 0.2069 \tag{5}$$

$$\Rightarrow \frac{4f_F L^{*t}}{D_H} = \frac{1-\gamma\text{Ma}_1^2}{\gamma\text{Ma}_1^2} + \ln\left(\gamma\text{Ma}_1^2\right) = 14.081 \Rightarrow \underline{f_F = 5\text{e-}5} \tag{6}$$

The mass flow rate corresponding to this condition is $\underline{\dot{m}_{\max f_F} = 715.6\text{ lb}_m/\text{s}}$.

Larger friction factors are not possible without going outside of the range of allowable pressures. For example, if f_F = 1e-4:

$$\frac{4f_F L^{*t}}{D_H} = 28.16 = \frac{1-\gamma\text{Ma}_1^2}{\gamma\text{Ma}_1^2} + \ln\left(\gamma\text{Ma}_1^2\right) \Rightarrow \text{Ma}_1 = 0.1535 \text{ and } p_1/p^{*t} = 5.7137 \tag{7}$$

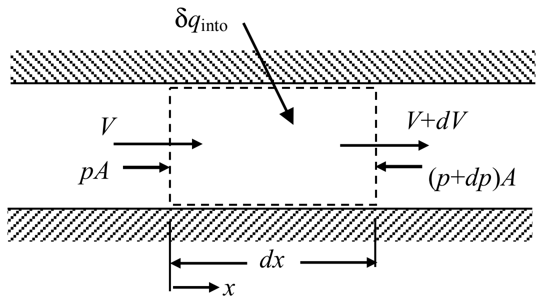
This pressure ratio is greater than the allowable pressure ratio given in Eq. (3).

Friction factors smaller than 5e-5 will result in flow rates between 715.6 lb$_m$/s and 3033 lb$_m$/s.

8. Flow with Heat Transfer (Rayleigh Flow)

Now let's consider compressible flows where heat transfer is significant. Examples of such flows include those in which combustion, evaporation, condensation, or wall heat exchange occurs. Note that we won't worry about how the heat gets into (or out of) the flow. We'll just assume that we know its value, otherwise we would need to include heat transfer analyses into our discussions.

To analyze the effects of heat transfer on a compressible flow, consider frictionless, 1D, steady flow of an ideal gas in a constant area (dA=0) duct.



Using the same approach as the differential analysis for frictional flow gives:

COM: $$\boxed{\frac{d\rho}{\rho}+\frac{dV}{V}=0} \tag{119}$$

Note: dA=0

LME: $$\boxed{dp+\rho V dV=0} \tag{120}$$

COE: $$\boxed{dh+VdV=\delta q_{\text{into}}} \tag{121}$$

Note: $\delta q = \delta \dot{Q}/\dot{m}$ (heat addition per unit mass flow rate)

2nd Law: $$\boxed{ds \geq \frac{\delta q_{\text{into}}}{T}} \tag{122}$$

Note: The flow may be reversible if the temperature gradients are very small.

Ideal Gas Law: $$\boxed{\frac{dp}{p}=\frac{d\rho}{\rho}+\frac{dT}{T}} \tag{123}$$

caloric Eq of state: $$\boxed{dh=c_p dT} \tag{124}$$

$$c_p dT_0 = c_p dT + VdV = \delta q_{\text{into}}$$

Mach # relation: $$\boxed{\frac{d(\text{Ma})}{\text{Ma}}=\frac{dV}{V}-\frac{dT}{2T}} \tag{125}$$

Entropy relation: $$\boxed{ds=c_p\frac{dT}{T}-R\frac{dp}{p}} \tag{126}$$

Local stagnation pressure and temperature:

$$\frac{T}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)^{-1} \tag{127}$$

$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)^{\gamma/1-\gamma} \tag{128}$$

Combining these equations gives and using dT_0 (= $\delta q_{\text{into}}/c_P$) as the independent variable:

$$\boxed{\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \frac{\left(1 + \gamma\mathrm{Ma}^2\right)\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)}{2\left(1 - \mathrm{Ma}^2\right)}\frac{dT_0}{T_0}} \tag{129}$$

$$\boxed{\frac{dp}{p} = -\frac{\gamma\mathrm{Ma}^2\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)}{1 - \mathrm{Ma}^2}\left(\frac{dT_0}{T_0}\right)} \tag{130}$$

$$\boxed{\frac{dT}{T} = \frac{\left(1 - \gamma\mathrm{Ma}^2\right)\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)}{\left(1 - \mathrm{Ma}^2\right)}\left(\frac{dT_0}{T_0}\right)} \tag{131}$$

$$\boxed{\frac{dV}{V} = -\frac{d\rho}{\rho} = \frac{\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)}{1 - \mathrm{Ma}^2}\left(\frac{dT_0}{T_0}\right)} \tag{132}$$

$$\boxed{\frac{dp_0}{p_0} = -\frac{\gamma\mathrm{Ma}^2}{2}\left(\frac{dT_0}{T_0}\right)} \tag{133}$$

$$\boxed{\frac{ds}{c_p} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)\left(\frac{dT_0}{T_0}\right)} \tag{134}$$

Note: for $\delta q_{\text{into}} > 0 \Rightarrow dT_0 > 0$
for $\delta q_{\text{into}} < 0 \Rightarrow dT_0 < 0$

Ma < 1		Ma > 1	
heat addition	heat removal	heat addition	heat removal
$dT_0 > 0$	$dT_0 < 0$	$dT_0 > 0$	$dT_0 < 0$
$d(\text{Ma}) > 0$	$d(\text{Ma}) < 0$	$d(\text{Ma}) < 0$	$d(\text{Ma}) > 0$
$dV > 0$	$dV < 0$	$dV < 0$	$dV > 0$
$dp < 0$	$dp > 0$	$dp > 0$	$dp < 0$
$dT > 0$ for $\text{Ma}^2<1/\gamma$	$dT < 0$ for $\text{Ma}^2<1/\gamma$	$dT > 0$	$dT < 0$
$dT < 0$ for $\text{Ma}^2>1/\gamma$	$dT > 0$ for $\text{Ma}^2>1/\gamma$		
$d\rho < 0$	$d\rho > 0$	$d\rho > 0$	$d\rho < 0$
$dp_0 < 0$	$dp_0 > 0$	$dp_0 < 0$	$dp_0 > 0$
$ds > 0$	$ds < 0$	$ds > 0$	$ds < 0$

Notes:
1. $dT_0 = \delta q_{into}/c_p$
2. Heating always decreases the stagnation pressure $\Rightarrow$ a loss in pressure recovery and thus efficiency. Removing heat results in an increasing stagnation pressure; however, in practice other effects act to decrease the stagnation pressure.
3. For $1/\gamma^{0.5}<\text{Ma}<1$, heat addition *decreases* the flow temperature while heat removal *increases* the flow temperature. The energy input from the heat goes into the kinetic energy of the flow rather than the thermal energy for this range of Mach numbers.
4. Mathematically, it appears that we could transition from subsonic to supersonic flow by controlling the heat transfer rate. In practice, transitioning from subsonic to supersonic flow using heat transfer has not been observed.

Integrating Eqs. (129) through (134) using sonic conditions as a reference point gives the following working equations:

$$\boxed{\frac{p}{p^*} = \frac{\gamma+1}{1+\gamma \text{Ma}^2}} \tag{135}$$

$$\boxed{\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1+\gamma \text{Ma}^2}{(\gamma+1)\text{Ma}^2}} \tag{136}$$

$$\boxed{\frac{T}{T^*} = \frac{(\gamma+1)^2 \text{Ma}^2}{\left(1+\gamma \text{Ma}^2\right)^2}} \tag{137}$$

$$\boxed{\frac{p_0}{p_0^*} = \frac{\gamma+1}{1+\gamma \text{Ma}^2}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)\right]^{\gamma/\gamma-1}} \tag{138}$$

$$\boxed{\frac{T_0}{T_0^*} = \frac{2(\gamma+1)\text{Ma}^2}{\left(1+\gamma \text{Ma}^2\right)^2}\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)} \tag{139}$$

$$\boxed{\frac{s-s^*}{c_p} = \ln\left[\text{Ma}^2\left(\frac{\gamma+1}{1+\gamma \text{Ma}^2}\right)^{\gamma+1/\gamma}\right]} \tag{140}$$

$$\boxed{q_{12} = c_p T_{01}\left(\frac{T_{02}}{T_{01}} - 1\right)} \tag{141}$$

Note that the maximum amount of heat that can be added to a flow for a given initial state is:

$$q_{12,\max} = c_p T_{01}\left(\frac{T_0^*}{T_{01}} - 1\right)$$

where, using Eq. (139),

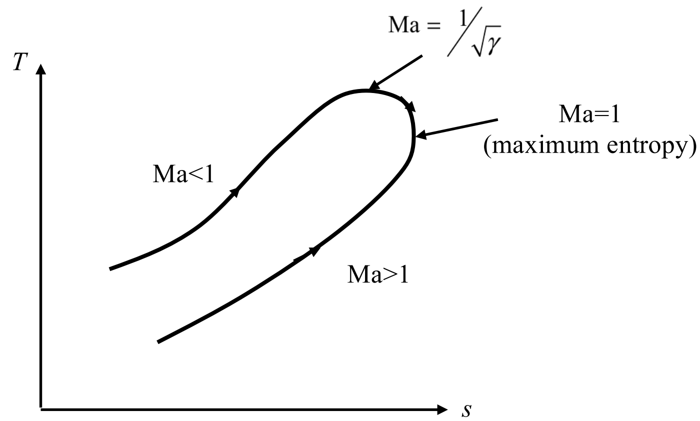
$$\frac{T_0^*}{T_{01}} = \left[\frac{2(\gamma+1)\text{Ma}^2}{\left(1+\gamma \text{Ma}^2\right)^2}\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)\right]^{-1}$$

so that

$$q_{12,\max} = c_p T_{01}\left\{\left[\frac{2(\gamma+1)\text{Ma}^2}{\left(1+\gamma \text{Ma}^2\right)^2}\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)\right]^{-1} - 1\right\} \tag{142}$$

Plotting the locus of all possible states for Rayleigh flow on a *T*-*s* diagram gives the Rayleigh Line:

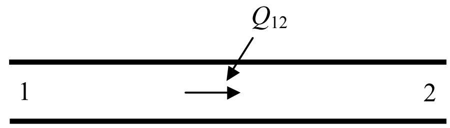
$$\frac{s^* - s_1}{c_p} = \ln \frac{\left(T^* / T_1 \right)}{\left(p^* / p_1 \right)^{\gamma-1/\gamma}}$$



The arrows are drawn for heat addition.
Reverse the arrows for heat removal.

Air flows with negligible friction through a duct of area 0.25 ft^2. At section 1, flow properties are T_1 = 600 °R, p_1 = 20 psia, and V_1 = 360 ft/s. At section 2, p_2 = 10 psia. The flow is heated between sections 1 and 2. Determine the properties at section 2, the energy added, and the entropy change. Finally, plot the process on a *T*-*s* diagram.

SOLUTION:



V_1 = 360 ft/s
p_1 = 20 psia
T_1 = 600 °R

p_2 = 10 psia

γ = 1.4
R = 53.3 (ft.lb$_f$)/(lb$_m$.°R)
c_P = 0.240 Btu/(lb$_m$.°R)

At section 1:

$$\mathrm{Ma}_1 = \frac{V_1}{\sqrt{\gamma R T_1}} \Rightarrow \mathrm{Ma}_1 = 0.30 \tag{1}$$

Using the Rayleigh flow relations and the calculated Ma_1:

$$\mathrm{Ma}_1 = 0.30 \Rightarrow p_1/p^* = 2.1314 \tag{2}$$

$$\mathrm{Ma}_1 = 0.30 \Rightarrow T_1/T^* = 0.4089 \tag{3}$$

Combine with the condition at 2:

$$\frac{p_2}{p^*} = \left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p^*}\right) \Rightarrow p_2/p^* = 1.0657 \tag{4}$$

Use the Rayleigh flow relations to determine the remainder of the conditions at 2.

$$p_2/p^* = 1.0657 \Rightarrow \boxed{\mathrm{Ma}_2 = 0.947} \tag{5}$$

$$\mathrm{Ma}_2 = 0.947 \Rightarrow T_2/T^* = 1.015 \tag{6}$$

$$T_2 = \left(\frac{T_2}{T^*}\right)\left(\frac{T^*}{T_1}\right)T_1 \Rightarrow \boxed{T_2 = 1490\ \text{°R}} \tag{7}$$

$$\rho_2 = \frac{p_2}{RT_2} \Rightarrow \boxed{\rho_2 = 1.81*10^{-2}\ \mathrm{lb_m/ft^3}} \tag{8}$$

$$\frac{p_2}{p_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_{02} = 17.8\ \text{psia}} \tag{9}$$

$$\frac{T_2}{T_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{-1} \Rightarrow \boxed{T_{02} = 1760\ \text{°R}} \tag{10}$$

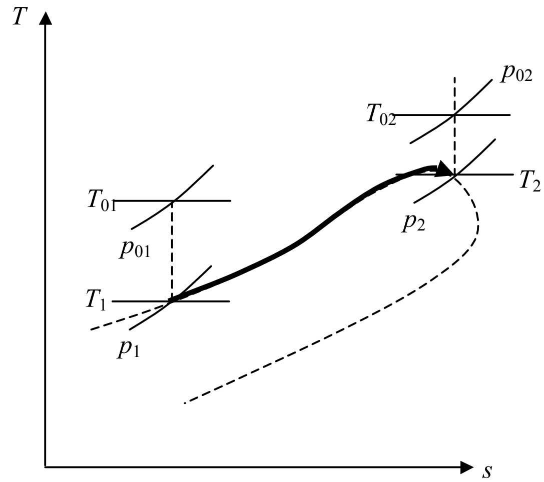
$$V_2 = \mathrm{Ma}_2\sqrt{\gamma R T_2} \Rightarrow \boxed{V_2 = 1792\ \text{ft/s}} \tag{11}$$

The heat added per unit mass between 1 and 2 may be found from Conservation of Energy.

$$q_{12} = c_P\left(T_{02} - T_{01}\right) \Rightarrow \boxed{q_{12} = 276\ \mathrm{Btu/(lb_m.°R)}} \tag{12}$$

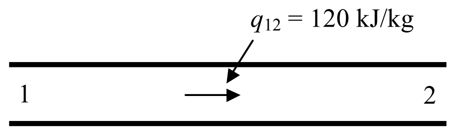
The change in entropy between 1 and 2 may be found from the *Tds* equation.

$$s_2 - s_1 = c_P \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \Rightarrow \boxed{s_2 - s_1 = 0.266\ \mathrm{Btu/(lb_m.°R)}} \tag{13}$$


T
p_{02}
T_{02}
T_2
T_{01}
p_2
p_{01}
T_1
p_1
s

Air flows in a constant-area duct. At the inlet the Mach number is 0.2, the static pressure is 90 kPa (abs), and the static temperature is 27 °C. Heat is added at a rate of 120 kJ/(kg of air). Assuming a perfect gas with constant specific heats, determine the properties of the air at the end of the duct. Assume also that the flow is frictionless and that c_p = 1000 J/(kg·K).

SOLUTION:



γ = 1.4
R = 287 J/(kg.K)
c_P = 1000 J/(kg.K)

Ma_1 = 0.2
p_1 = 90 kPa (abs)
T_1 = (27+293) = 300 K

From the isentropic relations:

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_{01} = 92.5 \text{ kPa (abs)} \tag{1}$$

$$\frac{T_1}{T_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow T_{01} = 302 \text{ K} \tag{2}$$

$$V_1 = \mathrm{Ma}_1\sqrt{\gamma R T_1} \Rightarrow V_1 = 69.4 \text{ m/s} \tag{3}$$

Determine the stagnation temperature at the end of the duct using Conservation of Energy.

$$q_{12} = c_P\left(T_{02}-T_{01}\right) \Rightarrow T_{02} = 422 \text{ K} \tag{4}$$

Use the Rayleigh flow relations and the Mach number at the state 1.

$$\mathrm{Ma}_1 = 0.2 \Rightarrow T_{01}/T_0^* = 0.1736 \tag{5}$$
$$p_{01}/p_0^* = 1.2346 \tag{6}$$
$$T_1/T^* = 0.2066 \tag{7}$$
$$p_1/p^* = 2.2727 \tag{8}$$

Now determine the conditions at state 2 using the Rayleigh flow relations.

$$\frac{T_{02}}{T_0^*} = \left(\frac{T_{02}}{T_{01}}\right)\left(\frac{T_{01}}{T_0^*}\right) \Rightarrow T_{02}/T_0^* = 0.2426 \Rightarrow \boxed{\mathrm{Ma}_2 = 0.24} \tag{9}$$

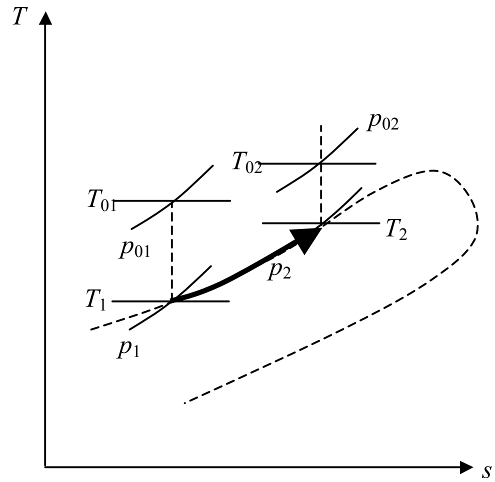
$$\mathrm{Ma}_2 = 0.24 \Rightarrow p_{02}/p_0^* = 1.2213 \tag{10}$$
$$p_2/p^* = 2.2209 \tag{11}$$
$$T_2/T^* = 0.2841 \tag{12}$$

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1 \Rightarrow \boxed{p_2 = 87.9 \text{ kPa (abs)}} \tag{13}$$

$$T_2 = \left(\frac{T_2}{T^*}\right)\left(\frac{T^*}{T_1}\right)T_1 \Rightarrow \boxed{T_2 = 413 \text{ K}} \tag{14}$$

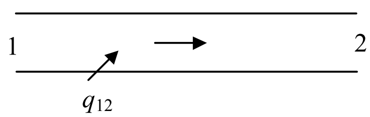
$$p_{02} = \left(\frac{p_{02}}{p_0^*}\right)\left(\frac{p_0^*}{p_{01}}\right)p_{01} \Rightarrow \boxed{p_{02} = 91.5 \text{ kPa (abs)}} \tag{15}$$

$$V_2 = \mathrm{Ma}_2\sqrt{\gamma R T_2} \Rightarrow \boxed{V_2 = 97.8 \text{ m/s}} \tag{16}$$


T
p02
T02
T01
T2
p01
p2
T1
p1
s

A gaseous mixture of air and fuel enters a ram-jet combustion chamber with a velocity of 200 ft/s, at a temperature of 120 °F, and at a pressure of 5 psia. The heat of reaction, Δh_0, of the mixture for the particular fuel-air ratio employed is 500 Btu/(lb$_m$ of the mixture). It is desired to find the stream properties at the exit of the combustion chamber. What is the maximum heat of reaction that can be accommodated for the given inlet conditions? It will be assumed that friction is negligible, that the cross-sectional area is constant, and that the properties of both the reactants and the products are equivalent to air in respect to molecular weight and specific heat.

SOLUTION:



$V_1 = 200$ ft/s
$T_1 = (120 + 460)$ °R = 580 °R
$p_1 = 5$ psia

$\Delta h_0 = 500$ BTU/lb$_m$

Determine the Mach number at 1.

$$\mathrm{Ma}_1 = \frac{V_1}{\sqrt{\gamma R T_1}} \Rightarrow \mathrm{Ma}_1 = 0.169 \tag{1}$$

Determine the stagnation properties at 1.

$$\frac{p_1}{p_{01}} = \left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_{01} = 5.101 \text{ psia} \tag{2}$$

$$\frac{T_1}{T_{01}} = \left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow T_{01} = 583.3 \text{ °R} \tag{3}$$

The stagnation temperature at station 2 may be found knowing the total specific enthalpy change during the reaction.

$$\Delta h_0 = c_p\left(T_{02} - T_{01}\right) \Rightarrow \boxed{T_{02} = 2667 \text{ °R}} \tag{4}$$

The Mach number at station 2 may be found from the sonic stagnation temperature ratio there and the Rayleigh flow relations.

$$\frac{T_{02}}{T_0^*} = \left(\frac{T_{02}}{T_{01}}\right)\left(\frac{T_{01}}{T_0^*}\right) \Rightarrow T_{02}/T_0^* = 0.585 \Rightarrow \boxed{\mathrm{Ma}_2 = 0.433} \tag{5}$$

where $T_{02}/T_{01} = 4.572$ and $T_{01}/T_0^* = 0.128$ (from the Rayleigh flow relations).

The remainder of the flow properties may be found using the Rayleigh flow relations.

$$\mathrm{Ma}_1 = 0.169 \Rightarrow p_1/p^* = 2.308,\ T_1/T^* = 0.153,\ p_{01}/p_0^* = 1.244 \tag{6}$$

$$\mathrm{Ma}_2 = 0.433 \Rightarrow p_2/p^* = 1.901,\ T_2/T^* = 0.677,\ p_{02}/p_0^* = 1.143 \tag{7}$$

$$T_2 = \left(\frac{T_2}{T^*}\right)\left(\frac{T^*}{T_1}\right)T_1 \Rightarrow \boxed{T_2 = 2570 \text{ °R}} \tag{8}$$

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1 \Rightarrow \boxed{p_2 = 4.12 \text{ psia}} \tag{9}$$

$$V_2 = \mathrm{Ma}_2\sqrt{\gamma R T_2} \Rightarrow \boxed{V_2 = 1075 \text{ ft/s}} \tag{10}$$

Note that the maximum heat of reaction ($\Delta h'_0$) for the given inlet conditions may be found when the exit conditions are sonic ($T'_{02} = T_0^*$).

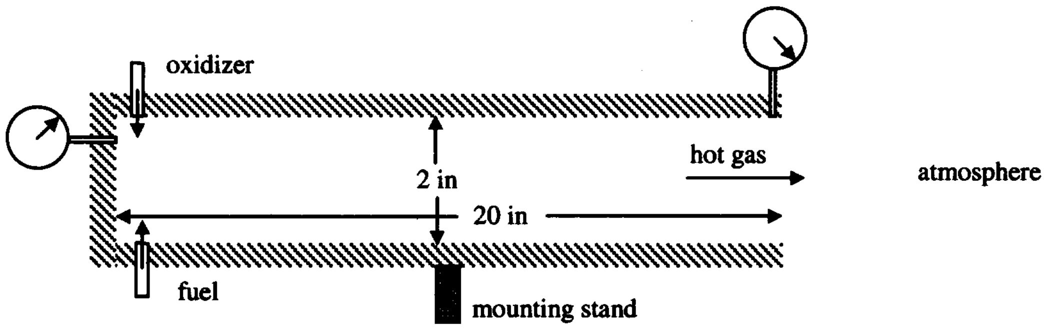
$$\Delta h'_0 = c_p\left(T_{02'} - T_{01}\right) \tag{11}$$

where

$$T_{02'} = \underbrace{\left(\frac{T_{02'}}{T_0^*}\right)}_{=1}\left(\frac{T_0^*}{T_{01}}\right)T_{01} \;\Rightarrow\; T_{02'} = 4550\ °\text{R} \;\Rightarrow\; \boxed{\Delta h'_0 = 952\ \text{BTU/lb}_\text{m}} \tag{12}$$

If a larger heat of reaction occurs, then the inlet conditions must change.

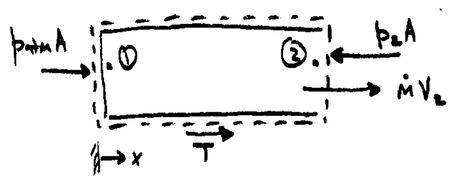
The sketch shows a tube rocket motor, comprised of a long tube which is fed with liquid fuel and liquid oxidant at one end, with hot gases leaving the open end. In one test of such a motor, the measured thrust was 676 lb_f; the pressure on the back face of the motor was 265 psia; and atmospheric pressure was 15 psia. With the fuel-oxidant combination used, the gases are characterized by a specific heat ratio of 1.3 and a gas constant of 30 ($ft \cdot lb_f$)/($lb_m \cdot °R$). Because of uncertainties as to whether combustion is completed and as to the degree of dissociation, the stagnation temperature of the leaving gases is not known, but it is suspected to be between 3000 °F and 5000 °F.



a. **What is the pressure in the exit plane (in psia)?**
b. **What is the average shearing stress acting on the inside walls of the motor (in psi)?**

SOLUTION:

- Apply COLM in x-dir to the following CV:



$$\frac{d}{dt}\int_{CV} u_x \rho \, dV + \int_{CS} u_x \rho (\underline{u}_{rel} \cdot \hat{n}) \, dS = F_{x,\text{body on CV}} + F_{x,\text{surface on CV}}$$

where $\frac{d}{dt}\int_{CV} u_x \rho \, dV = 0$ (steady flow)

$$\int_{CS} u_x \rho (\underline{u}_{rel} \cdot \hat{n}) \, dS = \rho_2 V_2^2 A$$

$$F_{x,\text{body on CV}} = 0$$

$$F_{x,\text{surface on CV}} = T + (p_{atm} - p_2) A$$

$$\Rightarrow \underline{\rho_2 V_2^2 A = T + (p_{atm} - p_2) A}$$

Solution...

- Assume the flow is choked so that $Ma_2 = 1$

$$\Rightarrow \quad V_2^2 = \cancel{Ma_2^2}^{\,1}(\gamma R T_2)$$

- Also, $\quad \rho_2 = \dfrac{p_2}{RT_2}$

$$\Rightarrow \rho_2 V_2^2 A = \left(\frac{p_2}{RT_2}\right)(\gamma R T_2) A = \gamma p_2 A$$

- Substitute into COLM eqn:

$$\rho_2 V_2^2 A = T + (p_{atm} - p_2) A$$

$$\Rightarrow \quad \gamma p_2 A = T + (p_{atm} - p_2) A$$

$$\Rightarrow \quad p_2 (1+\gamma) A = T + p_{atm} A$$

$$\therefore \quad p_2 = \frac{T/A + p_{atm}}{1+\gamma}$$

$$\boxed{\therefore \; p_2 = 100\,\text{psia}}$$

using $T = 676\ \text{lb}_f$

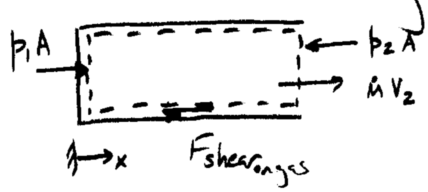
$A = \dfrac{\pi (2\,\text{in})^2}{4} = 3.142\ \text{in}^2$

$p_{atm} = 15\ \text{psia}$

$\gamma = 1.3$

Note: Since $p_2 > p_1$, the assumption of choked flow is validated.

- Apply COLM to the following CV:



$$\frac{d}{dt}\int_{CV} u_x \rho \, d\forall + \int_{CS} u_x \rho (\underline{u}_{rel} \cdot \hat{n}) \, dS = F_{x,\,\text{body on CV}} + F_{x,\,\text{surface on CV}}$$

where $\quad \dfrac{d}{dt}\displaystyle\int_{CV} u_x \rho \, d\forall = 0$

$$\int_{CS} u_x \rho (\underline{u}_{rel} \cdot \hat{n}) \, dS = \rho_2 V_2^2 A$$

$$F_{x,\,\text{body on CV}} = 0$$

$$F_{x,\,\text{surface on CV}} = -F_{shear} + (p_1 - p_2) A$$

SOLUTION...

$$\Rightarrow \quad \rho_2 V_2^2 A = -F_{shear} + (p_1 - p_2)A$$

$$\Rightarrow \quad F_{shear} = (p_1 - p_2)A - \rho_2 V_2^2 A$$

but $\rho_2 V_2^2 A = T + (p_{atm} - p_2)A$ (from previous CV analysis)

$$\Rightarrow \quad F_{shear} = (p_1 - p_2)A - T - (p_{atm} - p_2)A$$

$$\therefore F_{shear} = (p_1 - p_{atm})A - T$$

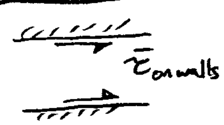
$\therefore F_{shear_{on\ gas}} = 109\ lb_f$ using $T = 676\ lb_f$, $p_{atm} = 15\ psia$, $p_1 = 265\ psia$, $A = 3.142\ in^2$

$\therefore F_{shear_{on\ walls}} = 109\ lb_f$ (in opposite direction)

- Determine the avg. shear stress:

$$\bar{\tau}_{on\ walls} = \frac{F_{shear\ on\ walls}}{\pi D L}$$

$\boxed{\therefore \bar{\tau}_{on\ walls} = 0.87\ psi}$ using $F_{shear} = 109\ lb_f$, $D = 2\ in$, $L = 20\ in$



NOTE: • A Rayleigh flow analysis from station 1 to station 2 is not appropriate since a strict Rayleigh flow analysis assumes frictionless flow. Here, friction effects are not negligible. If a Rayleigh flow is assumed, then the pressure at 2 is given by:

$\frac{F_{shear}}{T} = 0.16$

$\Rightarrow$ shear force is 16% of the rocket thrust

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1 \quad \text{where } \frac{p_2}{p^*} = 1 \quad (\text{since } Ma_2 = 1)$$

$\frac{p_1}{p^*} = 2.3$ (using frictionless flow w/ heat transfer relations and $Ma_1 = 0$, $\gamma = 1.3$)

$p_1 = 265\ psia$

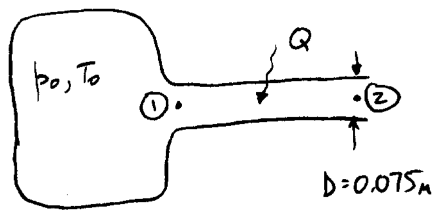
$\therefore p_2 = 115\ psia$

$\Rightarrow$ an error of 15% when compared to the momentum (correct) analysis

Air flows from a large reservoir where the static pressure is 480 kPa and the static temperature is 320 K through a 0.075 m internal diameter pipe and exhausts into the atmosphere (at 1 atm). Heat in the amount of 335 kJ/kg is transferred to the gas. Calculate:

a. **the duct inlet and exit Mach numbers**
b. **the inlet and exit values of temperature, pressure, density and velocity**
c. **the mass flow rate**

SOLUTION:



$p_{atm} = 101\text{ kPa}$

$\gamma = 1.4$
$c_p = 1005 \frac{J}{kg \cdot K}$
$R = 287 \frac{J}{kg \cdot K}$

$p_{01} = 480\text{ kPa}$
$T_{01} = 320\text{ K}$

${}_1q_2 = 335 \frac{kJ}{kg}$

• Assume isentropic flow from reservoir to station 1.

• Assume flow is choked.

$$\Rightarrow Ma_2 = 1 \quad \Rightarrow \frac{p_{02}}{p_0^*} = 1 \qquad \frac{T_{02}}{T_0^*} = 1$$

• Flow from 1 to 2:

$${}_1q_2 = c_p\left(T_{02} - T_{01}\right) = c_p T_{01}\left(\frac{T_{02}}{T_{01}} - 1\right)$$

$$\text{but} \quad \frac{T_{02}}{T_{01}} = \left(\frac{T_{02}}{T_0^*}\right)\left(\frac{T_0^*}{T_{01}}\right)$$

$$\Rightarrow \left(\frac{T_{02}}{T_0^*}\right)\left(\frac{T_0^*}{T_{01}}\right) = 1 + \frac{{}_1q_2}{c_p T_{01}}$$

$$\therefore \frac{T_{01}}{T_0^*} = \left(\frac{T_{02}}{T_0^*}\right)\left(1 + \frac{{}_1q_2}{c_p T_{01}}\right)^{-1}$$

$$\therefore \underline{\frac{T_{01}}{T_0^*} = 0.4898} \qquad \text{using} \quad \frac{T_{02}}{T_0^*} = 1 \quad (Ma_2 = 1)$$

${}_1q_2 = 335 \frac{kJ}{kg}$
$c_p = 1.005 \frac{kJ}{kg \cdot K}$
$T_{01} = 320\text{ K}$

SOLUTION...

$\Rightarrow$ $Ma_1 = 0.378$ using frictionless flow w/ heat transfer relations and $\frac{T_0}{T_0^*} = 0.4898$, $\gamma = 1.4$

- Determine the pressure at station 2:

$$p_2 = \left(\frac{p_2}{p_{02}}\right)\left(\frac{p_{02}}{p_0^*}\right)\left(\frac{p_0^*}{p_{01}}\right)p_{01} \quad \text{where} \quad \frac{p_2}{p_{02}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{\gamma}{1-\gamma}} \quad (Ma_2 = 1)$$

$\therefore$ $p_2 = 217\text{ KPa}$ using
$\frac{p_2}{p_{02}} = 0.5283$
$\frac{p_{02}}{p_0^*} = 1$ $(Ma_2 = 1)$
$\frac{p_{01}}{p_0^*} = 1.1660$ using frictionless flow w/ heat transfer relations and $Ma_1 = 0.378$, $\gamma = 1.4$
$p_{01} = 480\text{KPa}$

- Since $p_2 > p_{atm}$, the assumption of choked flow is valid.

$$\boxed{\therefore \quad Ma_1 = 0.378 \quad Ma_2 = 1}$$

- At station 1:

$$T_1 = T_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{-1}$$

$$p_1 = p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\rho_1 = \frac{p_1}{RT_1}$$

$$V_1 = Ma_1\sqrt{\gamma R T_1}$$

$\Rightarrow$ $\boxed{T_1 = 311\text{ K},\ p_1 = 435\text{ KPa},\ \rho_1 = 4.87\text{ kg/m}^3,\ V_1 = 134\text{ m/s}}$ using
$T_{01} = 320\text{K}$
$p_{01} = 480\text{KPa}$
$Ma_1 = 0.378$
$\gamma = 1.4$
$R = 287\frac{J}{kg \cdot K}$

Solution...

- At station 2:

$$T_2 = T_{02}\left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{-1}$$

$$p_2 = p_{02}\left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\rho_2 = \frac{p_2}{RT_2}$$

$$V_2 = Ma_2\sqrt{\gamma RT_2}$$

where $T_{02} = \left(\frac{T_{02}}{T_0^*}\right)\left(\frac{T_0^*}{T_{01}}\right)T_{01}$

$$= (1)\left(\frac{1}{0.4898}\right)(320\text{K})$$

$\therefore$ $T_{02} = 653\text{K}$

$$\Rightarrow \boxed{\begin{aligned} T_2 &= 544\text{K} \\ p_2 &= 217\text{ kPa} \\ \rho_2 &= 1.39\ \text{kg/m}^3 \\ V_2 &= 468\ \text{m/s} \end{aligned}}$$

using $T_{02} = 653\text{K}$
$Ma_2 = 1$
$\gamma = 1.4$
$R = 287\ \frac{\text{J}}{\text{kg}\cdot\text{K}}$

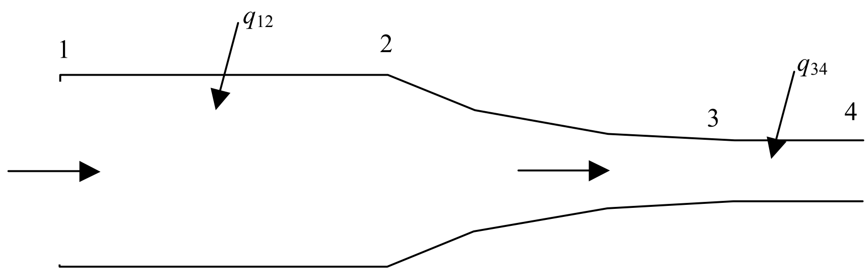
- Mass flow rate:

$$\dot{m} = \rho_1 V_1 A_1$$

$$\boxed{\therefore\ \dot{m} = 2.88\ \text{kg/s}}$$

using $\rho_1 = 4.87\ \text{kg/m}^3$
$V_1 = 134\ \text{m/s}$
$A_1 = \frac{\pi D^2}{4} = \frac{\pi(0.075\text{m})^2}{4} = 4.418\times10^{-3}\ \text{m}^2$

A ram jet device has air entering it at station 1 with a pressure of 10 psia, temperature of 500 °R, and velocity of 250 ft/sec. The inlet area is 3 ft^2. In the combustion chamber (from 1 to 2), 50 Btu/lb$_m$ of heat is added to the flow (it can be assumed that this is done without friction). At the end of the combustion chamber (station 2) the flow passes through a nozzle with an exit area of 2.5 ft^2 (the exit is at station 3). Assume that the flow is isentropic in this part. Finally, more heat is added in a constant area afterburner (from station 3 to 4) to raise the exit Mach number to 1. How much heat must be added in the afterburner to achieve this?



SOLUTION:

At station 1:

$$\mathrm{Ma}_1 = \frac{V_1}{\sqrt{\gamma R T_1}} \Rightarrow \underline{\mathrm{Ma}_1 = 0.228} \quad (1)$$

$$T_{01} = T_1\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right) \Rightarrow \underline{T_{01} = 505\ °\mathrm{R}} \quad (2)$$

using V_1 = 250 ft/s, γ= 1.4, R = 53.3 (ft·lb$_f$)/(lb$_m$·°R) and T_1 = 500 °R.

At station 2:

$$T_{02} = T_{01} + \frac{q_{12}}{c_P} \Rightarrow \underline{T_{02} = 713\ °\mathrm{R}} \quad (3)$$

where T_{01} = 505 °R, q_{12} = 50 Btu/lb$_m$, and c_P = 0.240 Btu/(lb$_m$·°R). In addition:

$$\frac{T_{02}}{T_0^*} = \left(\frac{T_{02}}{T_{01}}\right)\left(\frac{T_{01}}{T_0^*}\right) \Rightarrow \underline{\frac{T_{02}}{T_0^*} = 0.3093} \quad (4)$$

where T_{02} and T_{01} are given above and T_{01}/T_0^* = 0.2919 (found using the Rayleigh flow relations with Ma$_1$ = 0.228 and γ= 1.4). Using the Rayleigh flow relations with the given T_{02}/T_0^*, $\underline{\mathrm{Ma}_2 = 0.280}$. (5)

Using the isentropic flow relations, the sonic area at 2 can also be found:

$$\frac{A_2}{A^*} = \frac{1}{\mathrm{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \underline{A^* = 1.385\ \mathrm{ft}^2} \quad (6)$$

where A_2 = 3 ft^2, γ= 1.4, and Ma$_2$ = 0.280.

At station 3:

$$\frac{A_3}{A^*} = \frac{1}{\mathrm{Ma}_3}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_3^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \underline{\mathrm{Ma}_3 = 0.343} \quad (7)$$

using A^* = 1.385 ft^2, A_3 = 2.5 ft^2, and noting that the flow from 2 to 3 is isentropic. Also at station 3, $\underline{T_{03} = T_{02}}$ since the flow is isentropic from 2 to 3, and $\underline{T_{03}/T_0^* = 0.4261}$ (using the Rayleigh flow relations with the calculated Ma$_3$).

At station 4:

$$\frac{T_{04}}{T_0^*} = 1 \tag{8}$$

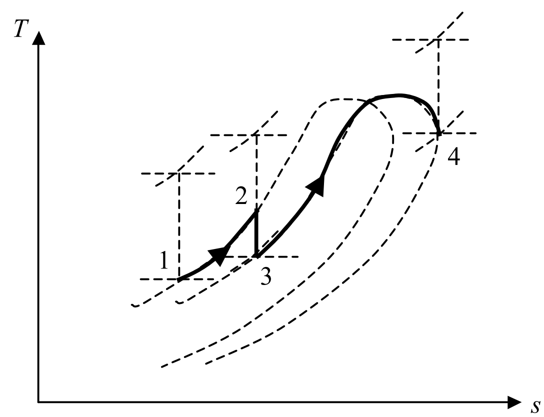
since $Ma_4 = 1$. The heat addition between 3 and 4 can be found from the stagnation temperatures at those stations.

$$q_{34} = c_P\left(T_{04} - T_{03}\right) = c_P T_{03}\left(\frac{T_{04}}{T_{03}} - 1\right) \tag{9}$$

where

$$\frac{T_{04}}{T_{03}} = \left(\frac{T_{04}}{T_0^*}\right)\left(\frac{T_0^*}{T_{03}}\right) \tag{10}$$

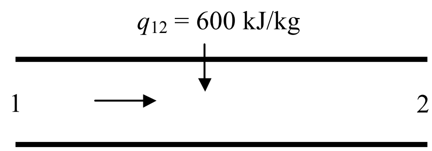
Using $c_P = 0.240$ Btu/(lb$_m$·°R), $T_{03} = 713$ °R, $T_{04}/T_0^* = 1$, and $T_{03}/T_0^* = 0.4261 \Rightarrow \boxed{q_{34} = 230 \text{ Btu/lb}_m}$.



A constant area combustor is to pass 215 kg/sec of air (stagnation pressure and temperature of 800 kPa (abs) and 400 K, respectively) through a 5 m long chamber with a flow area of 0.25 m^2. The combustion process adds heat at 600 kJ/kg. Assume ideal gas behavior with γ=1.4 and c_p=1.004 kJ/(kg·K). Find:

a. the Mach number of the flow at the combustor entrance,
b. the stagnation temperature of the flow leaving the combustor,
c. the Mach number of the flow leaving the combustor, and
d. sketch the process on a *T*-*s* diagram.

SOLUTION:



$\dot{m}$ = 215 kg/s
A = 0.25 m^2
p_{01} = 800 kPa (abs)
T_{01} = 400 K

At station 2:

$$q_{12} = c_P\left(T_{02} - T_{01}\right) \Rightarrow T_{02} = T_{01} + \frac{q_{12}}{c_P} \;\Rightarrow\; \boxed{T_{02} = 998\text{ K}} \tag{1}$$

At station 1:

$$\begin{aligned}
\dot{m} = \rho_1 V_1 A_1 &= \left[\rho_0\left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{1}{1-\gamma}}\right]\left(\text{Ma}_1\sqrt{\gamma R T_1}\right)A_1 \\
&= \left[\frac{p_0}{RT_0}\left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{1}{1-\gamma}}\right]\left(\text{Ma}_1\sqrt{\gamma R T_{01}\left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{-1}}\right)A_1 \qquad (2)\\
&= p_0\left(\sqrt{\frac{\gamma}{RT_{01}}}\right)\text{Ma}_1\left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{1+\gamma}{2(1-\gamma)}}A_1
\end{aligned}$$

Iterate to find that $\boxed{\text{Ma}_1 = 0.328}$.

At station 2:

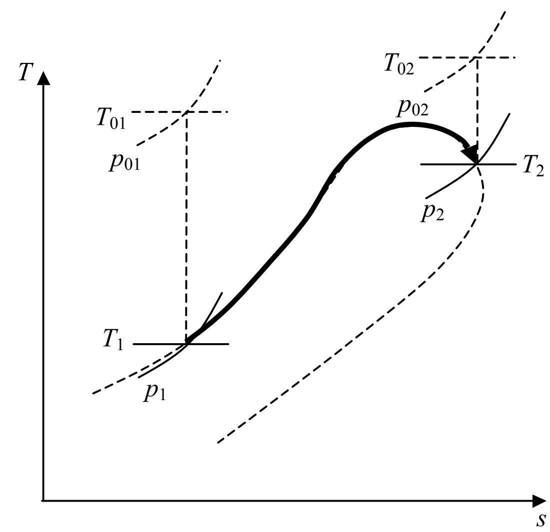
$$T_{02} = \left(\frac{T_{01}}{T_0^*}\right)\left(\frac{T_{02}}{T_{01}}\right) \;\Rightarrow\; T_{02}/T_0^* = 0.9940 \tag{3}$$

using

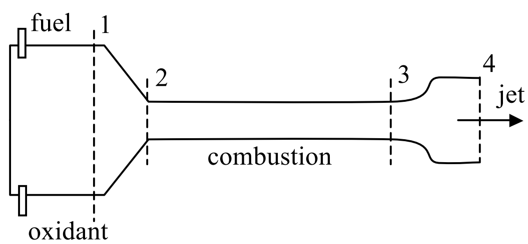
$\text{Ma}_1 = 0.328 \;\Rightarrow\; T_{01}/T_0^* = 0.3984$ from the Rayleigh flow relations. (4)

Again, from the Rayleigh flow relations,

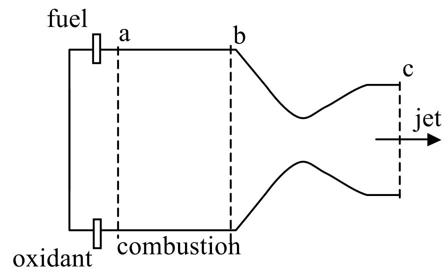
$T_{02}/T_0^* = 0.9940 \;\Rightarrow\; \boxed{\text{Ma}_2 = 0.912}$ (5)


T
s
T_{01}
p_{01}
T_{02}
p_{02}
T_1
p_1
T_2
p_2

It has been suggested that the combustion in rocket motors should be carried out at high speed in a tube of constant cross-section, as indicated in the sketch. It is desired to compare the specific thrust (thrust force per unit mass rate of flow, in lb_f-sec/slug) of such a rocket motor with the specific thrust of the conventional motor, also shown in the sketch.



proposed design



conventional design

To simplify the calculations it will be assumed that friction is negligible, that the jet leaves the nozzle at the pressure of the ambient atmosphere (say 5 psia), that the fuel and oxidant are supplied as gases, and that both rockets are supplied at the same pressure ($p_{01} = p_{0a} = 500$ psia). Also, all processes will be considered adiabatic, except that the combustion will be replaced by an equivalent heat process with $T_{01}=T_{0a} = 500$ °R and $T_{03}=T_{0b} = 5000$ °R. As an approximation, take γ=1.4 and use a molecular weight of 20 lb_m/mol.

For the proposed design, assume that the velocity is negligible at section 1, that all combustion occurs between 2 and 3, and that the Mach number is unity at section 3. For the conventional design assume negligible velocity at *a* and complete combustion between *a* and *b*.

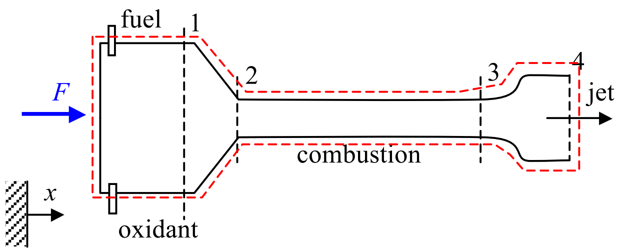
Calculate:

a. the specific impulse for both motors (in lb_f-sec/slug) [the specific impulse, I_{sp} is defined as the thrust, T, developed by consuming a unit mass of the propellants in unit time, i.e., $I_{sp} = T/\dot{m}$]

b. the throat area per unit of thrust (ft^2/lb_f).

SOLUTION:

First analyze the proposed design. Note that $R = R_U/M = 77.25$ (ft·lb$_f$)/(lb$_m$·°R) where R_U is the universal gas constant (= 1545 (ft·lb$_f$)/(mol·°R)) and M is the molecular weight (= 20 lb$_m$/mol for the current problem).



$p_{01} = p_{02} = 500$ psia
$T_{01} = T_{02} = 500$ °R
$T_{03} = T_{04} = 5000$ °R
$\mathrm{Ma}_1 \approx 0$

$p_4 = 5$ psia
$\mathrm{Ma}_3 = 1$

At station 3:

$$\mathrm{Ma}_3 = 1 \Rightarrow \frac{p_{03}}{p_0^*} = 1 \text{ and } \frac{T_{03}}{T_0^*} = 1 \quad \text{(using the Rayleigh flow relations)} \tag{1}$$

At station 2:

$$\frac{T_{02}}{T_0^*} = \left(\frac{T_{02}}{T_{03}}\right)\left(\frac{T_{03}}{T_0^*}\right) = \left(\frac{500\ ^{\text{o}}\text{R}}{5000\ ^{\text{o}}\text{R}}\right)(1) = 0.1000 \tag{2}$$

$$\Rightarrow \mathrm{Ma}_2 = 0.148 \text{ and } \frac{p_{02}}{p_0^*} = 1.2490 \quad \text{(using the Rayleigh flow relations)} \tag{3}$$

Back to station 3:

$$p_{03} = \left(\frac{p_{03}}{p_0^*}\right)\left(\frac{p_0^*}{p_{02}}\right)p_{02} = (1)\left(\frac{1}{1.2490}\right)(500 \text{ psia}) = 400 \text{ psia} = p_{04} \tag{4}$$

At station 4:

$$\frac{p_4}{p_{04}} = \left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_4^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \mathrm{Ma}_4 = 3.534 \tag{5}$$

$$\frac{T_4}{T_{04}} = \left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_4^2\right)^{-1} \Rightarrow T_4 = 1430 \text{ °R} \tag{6}$$

$$V_4 = \mathrm{Ma}_4\sqrt{\gamma R T_4} \Rightarrow V_4 = 7890 \text{ ft/s} \tag{7}$$

The rocket thrust, F, can be found using the linear momentum equation in the x-direction applied to the control volume shown in the previous figure. Note that the pressure forces cancel since the pressure acting everywhere on the control volume is 5 psia.

$$F = \dot{m}V_4 \tag{8}$$

The specific impulse, I_{sp}, is the thrust per mass flow rate:

$$I_{sp} = \frac{F}{\dot{m}} = V_4 \;\Rightarrow\; \boxed{I_{sp} = 7890 \text{ lb}_f\text{-s/slug}} \tag{9}$$

The throat area per unit thrust is:

$$\frac{A_T}{F} = \frac{A_3}{F} = \frac{\frac{\dot{m}}{\rho_3 V_3}}{\dot{m} V_4} = \frac{1}{\rho_3 V_3 V_4} \tag{10}$$

where

$$\rho_3 = \frac{p_3}{RT_3} \;\Rightarrow\; \rho_3 = 9.439\text{e-2 lb}_m/\text{ft}^3 \tag{11}$$

with

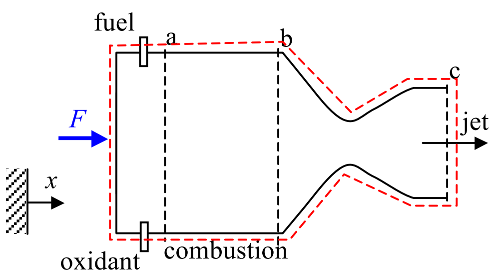
$$p_3 = p_{03}\left(1+\frac{\gamma-1}{2}\text{Ma}_3^2\right)^{\frac{\gamma}{1-\gamma}} \;\Rightarrow\; p_3 = 211 \text{ psia} \tag{12}$$

$$T_3 = T_{03}\left(1+\frac{\gamma-1}{2}\text{Ma}_3^2\right)^{-1} \;\Rightarrow\; T_3 = 4167\ °\text{R} \tag{13}$$

$$V_3 = \text{Ma}_3\sqrt{\gamma R T_3} \;\Rightarrow\; V_3 = 3810 \text{ ft/s} \tag{14}$$

$$\boxed{\therefore A_T/F = 1.135\text{e-5 ft}^2/\text{lb}_f} \tag{15}$$

Now analyze the conventional design.



$p_{0a} = 500$ psia
$T_{0a} = 500$ °R
$T_{0b} = T_{0c} = 5000$ °R
$\text{Ma}_a \approx 0$

$p_c = 5$ psia
$p_{0c} = p_{0b}$

At station a:

$$\text{Ma}_a = 0 \;\Rightarrow\; \frac{T_{0a}}{T_0^*} = 0 \;\text{ and }\; \frac{p_{0a}}{p_0^*} = 1.2679 \;\;\text{(using the Rayleigh relations)} \tag{16}$$

At station b:

$$\frac{T_{0b}}{T_0^*} = \left(\frac{T_{0a}}{T_0^*}\right)\left(\frac{T_{0b}}{T_{0a}}\right) = (0)\left(\frac{5000\ ^{\text{o}}\text{R}}{500\ ^{\text{o}}\text{R}}\right) = 0 \tag{17}$$

$$\Rightarrow\; \text{Ma}_b = 0 \;\Rightarrow\; \frac{p_{0b}}{p_0^*} = 1.2679 \;\;\text{(using the Rayleigh relations)} \tag{18}$$

$$p_{0b} = \left(\frac{p_{0b}}{p_0^*}\right)\left(\frac{p_0^*}{p_{0a}}\right)p_{0a} = (1.2679)\left(\frac{1}{1.2679}\right)(500 \text{ psia}) = 500 \text{ psia} \tag{19}$$

At station c:

$$\frac{p_c}{p_{0c}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_c^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \mathrm{Ma}_c = 3.693 \tag{20}$$

$$\frac{T_c}{T_{0c}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_c^2\right)^{-1} \Rightarrow T_c = 1340\ °\mathrm{R} \tag{21}$$

$$\rho_c = \frac{p_c}{RT_c} \Rightarrow \rho_c = 6.96\text{e-3 lb}_\mathrm{m}/\mathrm{ft}^3 \tag{22}$$

$$V_c = \mathrm{Ma}_c\sqrt{\gamma R T_c} \Rightarrow V_c = 7980\ \mathrm{ft/s} \tag{23}$$

The specific impulse can be found following the analysis for deriving Eqs. (8) and (9).

$$I_{sp} = \frac{F}{\dot{m}} = V_c \Rightarrow \boxed{I_{sp} = 7980\ \mathrm{lb_f\,s/slug}} \tag{24}$$

The throat area per unit thrust is given by:

$$\frac{A_T}{F} = \frac{A_T}{\dot{m}V_c} = \frac{A_T}{\rho_c V_c^2 A_c} = \frac{A_T}{A_c}\frac{1}{\rho_c V_c^2} \tag{25}$$

where

$$\frac{A_c}{A_T} = \frac{A_c}{A^*} = \frac{1}{\mathrm{Ma}_c}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_c^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow A_c/A_T = 8.1168 \tag{26}$$

$$\boxed{\therefore \frac{A_T}{F} = 5.897\text{e-4 ft}^2/\mathrm{lb_f}} \tag{27}$$

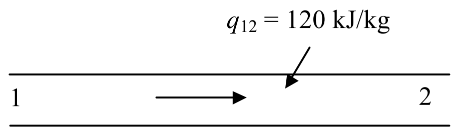
To summarize:

	Conventional Design	Proposed Design
I_{sp} [lb$_f$ s/slug]	7980	7890
A_T/F [ft^2/lb$_f$]	5.897e-4	1.135e-5

Both designs have similar specific impulses, with the conventional design being slightly more efficient (more thrust per mass flow rate). The proposed design has a smaller throat area to thrust ratio, which may impact the structural (and, hence, mass) aspects of the design.

Air flows in a constant-area duct. At the inlet the Mach number is 0.2, the static pressure is 90 kPa (abs), and the static temperature is 27 °C. Heat is added at a rate of 120 kJ/(kg of air). Assuming a perfect gas, determine the Mach number and static pressure of the air at the end of the duct. Sketch the process on a *T-s* diagram.

SOLUTION:



$\mathrm{Ma}_1 = 0.2$
$p_1 = 90$ kPa (abs)
$T_1 = (27 + 273) = 300$ K

Determine the stagnation temperature at 1.

$$\frac{T_1}{T_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow T_1/T_{01} = 0.9921 \Rightarrow T_{01} = 302.4\text{ K} \quad (1)$$

Determine the sonic stagnation temperature using the Rayleigh flow relations.

$$\frac{T_{01}}{T_0^*} = \frac{2(\gamma+1)\mathrm{Ma}_1^2}{\left(1+\gamma \mathrm{Ma}_1^2\right)^2}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right) \Rightarrow T_{01}/T_0^* = 0.1736 \quad (2)$$

Determine the stagnation temperature ratio at 2.

$$q_{12} = c_P T_{01}\left(\frac{T_{02}}{T_{01}}-1\right) \Rightarrow T_{02}/T_{01} = 1.3952 \quad (3)$$

Determine the Mach number at the end of the duct using the Rayleigh flow relations and the sonic stagnation temperature ratio.

$$\frac{T_{02}}{T_0^*} = \left(\frac{T_{02}}{T_{01}}\right)\left(\frac{T_{01}}{T_0^*}\right) \Rightarrow T_{02}/T_0^* = 0.2421 \quad (4)$$

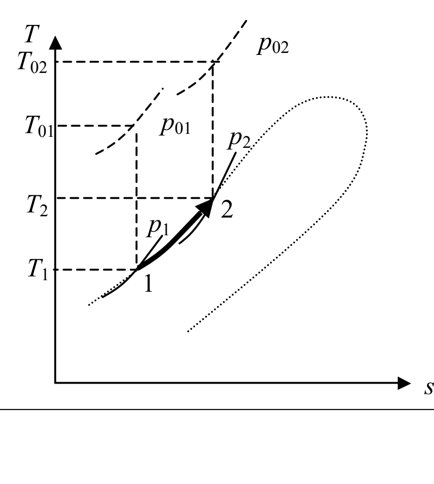
$$\frac{T_{02}}{T_0^*} = \frac{2(\gamma+1)\mathrm{Ma}_2^2}{\left(1+\gamma \mathrm{Ma}_2^2\right)^2}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right) \Rightarrow \boxed{\mathrm{Ma}_2 = 0.24} \quad (5)$$

Determine the sonic pressure ratio at the duct inlet and outlet given the Mach numbers there.

$$\frac{p_1}{p^*} = \frac{\gamma+1}{1+\gamma \mathrm{Ma}_1^2} \Rightarrow p_1/p^* = 2.2727 \quad (6)$$

$$\frac{p_2}{p^*} = \frac{\gamma+1}{1+\gamma \mathrm{Ma}_2^2} \Rightarrow p_2/p^* = 2.2188 \quad (7)$$

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1 \Rightarrow \boxed{p_2 = 87.9\text{ kPa (abs)}} \quad (8)$$

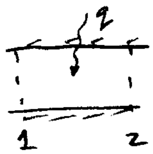


It is desired to achieve a stagnation temperature ratio of 4.0 in a ramjet combustor by the addition of heat, and not exceed a Mach number of 0.9 at its exit. The initial stagnation temperature is 350 K. Determine:

a. **the initial Mach number, and**
b. **the amount of heat which is to be added.**

Assume a specific heat ratio of 1.3 and a gas constant of 287.04 J/(kg·K).

SOLUTION:



$$\frac{T_{02}}{T_{01}} = 4.0 \qquad \gamma = 1.3 \qquad R = 287.04\ \frac{\text{J}}{\text{kg}\cdot\text{K}}$$

$T_{01} = 350\text{K} \qquad Ma_2 = 0.9$

$$\frac{T_{01}}{T_0^*} = \left(\frac{T_{01}}{T_{02}}\right)\left(\frac{T_{02}}{T_0^*}\right)$$

$$\therefore \frac{T_{01}}{T_0^*} = 0.2479 \qquad \text{using} \quad \frac{T_{02}}{T_{01}} = 4.0$$

$$\frac{T_{02}}{T_0^*} = 0.99143 \quad \text{using} \quad Ma_2 = 0.9,\ \gamma = 1.3$$

$$\Rightarrow \boxed{Ma_1 = 0.25}$$ using Rayleigh Flow relations w/ $\gamma = 1.3$

$$q_{12} = c_p\left(T_{02} - T_{01}\right)$$
$$= c_p T_{01}\left(T_{02}/T_{01} - 1\right)$$

$$\boxed{\therefore q_{12} = 1.31\ \frac{\text{MJ}}{\text{kg}}}$$

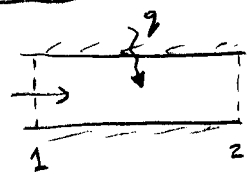
using $c_p = \frac{\gamma R}{\gamma - 1} = 1244\ \frac{\text{J}}{\text{kg}\cdot\text{K}}$

$T_{01} = 350\text{K}$

$T_{02}/T_{01} = 4.0$

Air flows through a constant area duct. The air has a temperature of 20 °C and a Mach number of 0.5 at the entrance to the duct. It is desired to transfer heat to the duct such that at the exit of the duct the stagnation temperature is 1180 °C. Is this possible? If not, what adjustment must be made to the Mach number at the entrance in order to give a discharge stagnation temperature of 1180 °C? Ignore the effects of friction.

SOLUTION:



$T_1 = 20°C = 293K \quad T_2 = 1180°C = 1453K$

$Ma_1 = 0.5$

- Use the Rayleigh flow relations:

NOTE: T_{MAX} will occur at $Ma_2 = 1/\sqrt{\gamma}$ for subsonic flow.

$\therefore Ma_2 = 0.85$ for $\gamma = 1.4$

$$T_{MAX} = \left(\frac{T_2}{T^*}\right)\left(\frac{T^*}{T_1}\right)T_1$$

$\therefore \underline{T_{MAX} = 381K}$ using $T_1 = 293K$, $T_1/T^* = 0.7902$ using $Ma_1 = 0.5$, $T_2/T^* = 1.02853$ using $Ma_2 = 0.85$

Thus, we see that for the given inlet conditions the largest temperature we can reach is 381K. Hence the design is not possible.

- Use the Rayleigh relations to determine inlet conditions to reach the desired exit temperature.

$T_2 = 1453K = T_{MAX} \quad (Ma_2 = 1/\sqrt{\gamma} = 0.85)$

$T_1 = 293K$

$$\Rightarrow \frac{T_1}{T^*} = \left(\frac{T_2}{T^*}\right)\left(\frac{T_1}{T_2}\right)$$

$\therefore T_1/T^* = 0.2074$ using $\frac{T_2}{T^*} = 1.02853$, $T_1 = 293K$, $T_2 = 1453K$

$\Rightarrow \boxed{Ma_1 = 0.20}$

SOLUTION...

- Calculate the amount of heat the must be added to the flow.

$$q_{12} = c_p \left(T_{02} - T_{01}\right)$$

$$\therefore \underline{q_{12} = 1.37 \tfrac{\text{MJ}}{\text{kg}}}$$

where

$c_p = 1005 \frac{\text{J}}{\text{kg}\cdot\text{K}}$

$T_{01} = T_1 \left(1 + \frac{\gamma - 1}{2} Ma_1^2\right)$ using $T_1 = 293\text{K}$, $Ma_1 = 0.20$

$\therefore \underline{T_{01} = 295\text{K}}$

$T_{02} = T_2 \left(1 + \frac{\gamma - 1}{2} Ma_2^2\right)$ using $T_2 = 1453\text{K}$, $Ma_2 = 0.85$

$\therefore \underline{T_{02} = 1663\text{K}}$

Air enters a combustion chamber at a velocity of 80 m/s with a pressure and temperature of 180 kPa (abs) and 120 °C. Find the maximum amount of heat that can be generated in the combustion chamber per unit mass of air for these conditions. If the fuel has a heating value of 45 MJ/kg, find the ~~air-fuel~~ fuel to air ratio. If the ~~air-fuel~~ fuel to air ratio is adjusted until it is 90% of this value, find the reduction in the mass flow rate through the combustion chamber that must occur if the inlet stagnation pressure and stagnation temperature remain the same. Assume that the flow is steady, that the effects of wall friction can be neglected, that the effects of the mass of the fuel can be neglected and that the air behaves as a perfect gas.

SOLUTION:

- First determine the inlet conditions:

$$Ma_1 = \frac{V_1}{\sqrt{\gamma R T_1}}$$

$$\therefore Ma_1 = 0.20 \quad \text{using} \quad V_1 = 80\,\text{m/s},\ \gamma = 1.4,\ R = 287\,\frac{\text{J}}{\text{kg}\cdot\text{K}},\ T_1 = 120\,^\circ\text{C} = 393\,\text{K}$$

$$T_{01} = T_1\left(1+\frac{\gamma-1}{2}Ma_1^2\right)$$

$$\therefore T_{01} = 396\,\text{K}$$

$$p_{01} = p_1\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\therefore p_{01} = 185\,\text{kPa} \quad \text{using} \quad p_1 = 180\,\text{kPa}$$

- Use the Rayleigh Flow relations:

$$q_{12}\big|_{max} = c_p\left(T_{02}-T_{01}\right)$$

where $T_{02} = T_0^* = \left(\frac{T_0^*}{T_{01}}\right)T_{01}$

$$\Rightarrow q_{12}\big|_{max} = c_p T_{01}\left(\frac{T_0^*}{T_{01}} - 1\right)$$

$$\boxed{\therefore q_{12}\big|_{max} = 1.90\,\frac{\text{MJ}}{\text{kg}}} \quad \text{using} \quad c_p = 1005\,\frac{\text{J}}{\text{kg}\cdot\text{K}},\ T_{01} = 396\,\text{K},\ T_{01}/T_0^* = 0.17355 \text{ using } Ma_1 = 0.20$$

- $$\text{fuel to air ratio} \equiv \frac{\text{heat generated per unit mass of fuel}}{\text{heat generated per unit mass of air}}$$

$$\therefore \boxed{\text{fuel to air ratio} = \frac{45\,\text{MJ/kg}}{1.90\,\text{MJ/kg}} = 23.7}$$

Solution...

- Now let the fuel-to-air ratio be 90% of this value:

$$\left.\begin{matrix}\text{fuel-to-air}\\ \text{ratio}\end{matrix}\right|_{new} = 0.900\left(\left.\begin{matrix}\text{fuel-to-air}\\ \text{ratio}\end{matrix}\right|_{old}\right) = 0.900(23.7)$$

$$= 21.3$$

$$\Rightarrow \begin{matrix}\text{heat generated per}\\ \text{unit mass of air}\end{matrix} = q_{12} = \frac{45\ \text{MJ/kg}}{21.3} = 2.11\ \text{MJ/kg}$$

But $\quad q_{12} = c_p T_{01}\left(\frac{T_0^*}{T_{01}} - 1\right)$

$$\Rightarrow \frac{T_{01}}{T_0^*} = 0.1587 \quad \text{using} \quad \begin{matrix} c_p = 1005\ \text{J/kg·K} \\ T_{01} = 396\ \text{K} \\ q_{12} = 2.11\ \text{MJ/kg}\end{matrix}$$

$$\Rightarrow \underline{Ma_1 = 0.19}$$

$$\frac{\dot{m}_{new}}{\dot{m}_{old}} = \frac{(\rho_1 V_1)_{new} A}{(\rho_1 V_1)_{old} A} = \frac{\left[(\rho_1/\rho_{01})(Ma_1\ c_1/c_{01})\right]_{new} \cancel{(\rho_{01} c_{01} A)}}{\left[(\rho_1/\rho_{01})(Ma_1\ c_1/c_{01})\right]_{old} \cancel{(\rho_{01} c_{01} A)}}$$

Note: stagnation conditions remain constant

$$= \frac{\left[\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1}{1-\gamma}} Ma_1 \left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{-1/2}\right]_{new}}{\left[\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1}{1-\gamma}} Ma_1 \left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{-1/2}\right]_{old}}$$

$$\boxed{\therefore \frac{\dot{m}_{new}}{\dot{m}_{old}} = 0.95} \quad \text{using} \quad \begin{matrix} Ma_{1new} = 0.19 \\ Ma_{1old} = 0.20 \end{matrix}$$

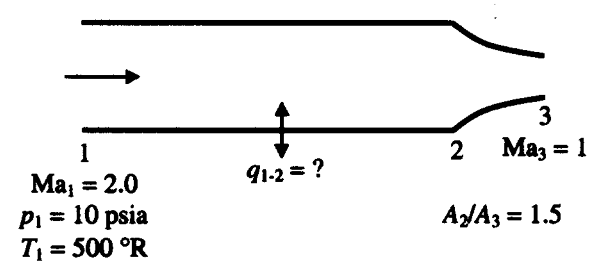
$\therefore$ The mass flow rate will reduce by ~ 5%.

As shown in the figure, air initially at a Mach number of 2, a static pressure of 10 psia, and a static temperature of 500 °R enters a constant-area duct that is followed by a converging segment with an area ratio of 1.5. At the minimum are of the converging segment the Mach number is 1.

a. Determine the amount and direction (addition or removal) of the heat transfer.

b. Sketch the process on a *T-s* diagram.

c. If a normal shock wave stands in the duct, are the previous answers still correct? Explain.



SOLUTION:

- Determine the Mach # at station 2 assuming isentropic flow from 2→3:

$$\frac{A_2}{A_3} = \frac{A_2}{A^*} = \frac{1}{Ma_2}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}}\right)$$

NOTE: $A_3 = A^*$ since $Ma_3 = 1$

$$\therefore \quad Ma_2 = 1.85 \quad \text{using} \quad \frac{A_2}{A_3} = 1.5, \quad \gamma = 1.4$$

NOTE: Choose the supersonic Mach # since the flow at station 1 is supersonic.

- Since $Ma_2 = 1.85$ and $Ma_1 = 2.0$ we have $Ma_2 < Ma_1$

  ⇒ we have $\boxed{\text{heat addition from 1 to 2.}}$

- To determine the amount of heat transfer from 1 to 2 we need to determine the change in stagnation temperature from station 1 to station 2:

$$q_{1\text{-}2} = c_p\left(T_{02} - T_{01}\right)$$

$$= c_p T_{01}\left(\frac{T_{02}}{T_{01}} - 1\right)$$

$$= c_p T_{01}\left[\left(\frac{T_{02}}{T_0^*}\right)\left(\frac{T_0^*}{T_{01}}\right) - 1\right]$$

Solution...

$$q_{1\text{-}2} = c_p T_{01}\left[\left(\frac{T_{02}}{T_0^*}\right)\left(\frac{T_0^*}{T_{01}}\right) - 1\right]$$

$$\boxed{\therefore\ q_{1\text{-}2} = 6700\ \frac{\text{ft}\cdot\text{lb}_f}{\text{lb}_m} = 8.6\ \frac{\text{Btu}}{\text{lb}_m}}$$

using

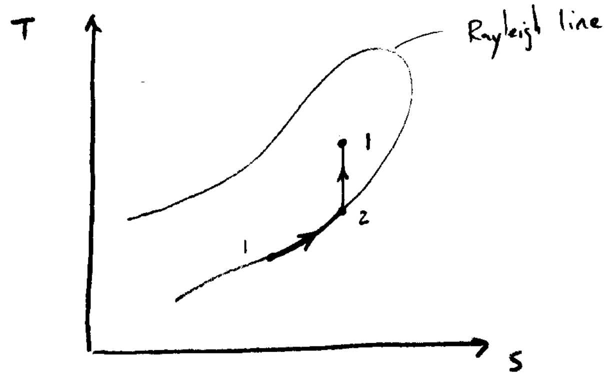
$c_p = 187\ \frac{\text{ft}\cdot\text{lb}_f}{\text{lb}_m\cdot{}^\circ\text{R}}$

$T_{01} = T_1\left(1 + \frac{\gamma-1}{2}Ma_1^2\right) = 900^\circ\text{R}$ using $T_1 = 500^\circ\text{R}$, $Ma_1 = 2.0$, $\gamma = 1.4$

$\frac{T_{02}}{T_0^*} = 0.8250$ using $Ma_2 = 1.85$

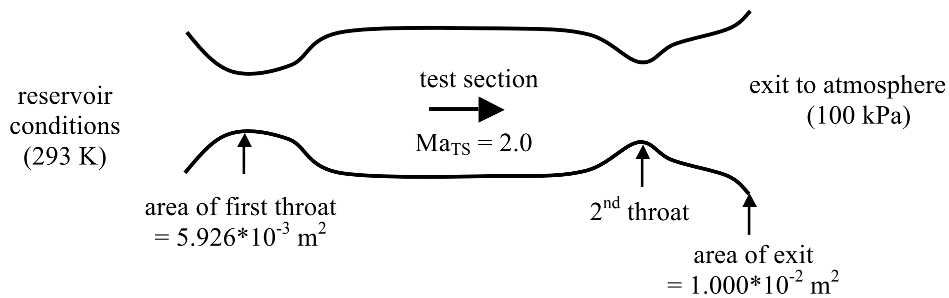
$\frac{T_{01}}{T_0^*} = 0.7934$ using $Ma_1 = 2.0$

using Rayleigh flow relations



- Across a normal shock wave the stagnation temperature will remain constant. Thus, the appearance of a normal shock will not affect the amount of heat transfer (although the T-s diagram will look different).

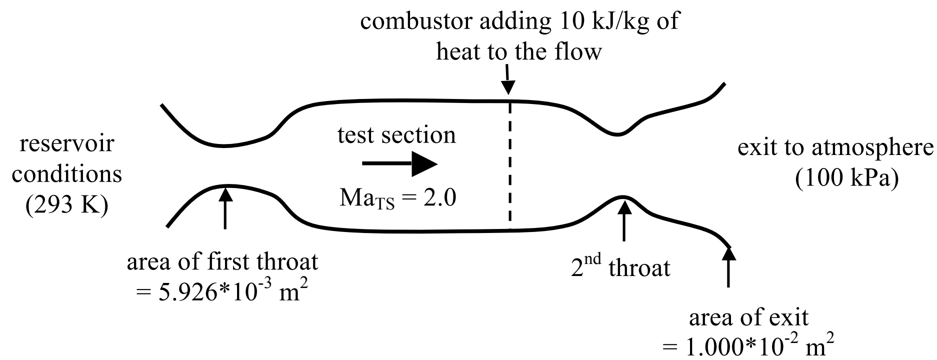
Consider the supersonic wind tunnel shown below.



Assume the working fluid is air.

a. Determine the area of the test section.
b. Determine the minimum area of the second throat in order to start the wind tunnel.
c. What minimum reservoir pressure is required to start the wind tunnel?

Now consider the case where the wind tunnel is driven by igniting fuel in a combustor downstream of the test section. Assume the combustor adds heat over a short distance so that the area downstream of the combustor is equal to the test section area.



d. What is the sonic area downstream of the combustion region?

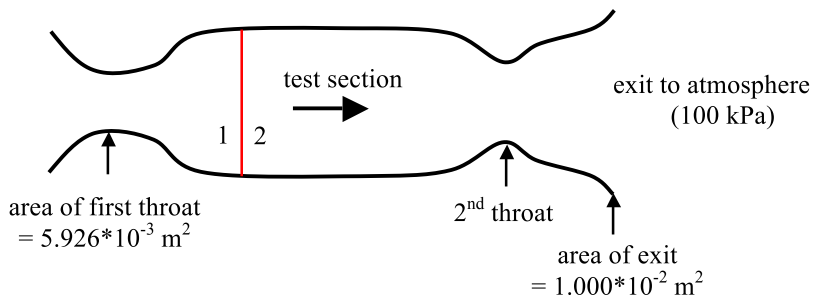
SOLUTION:

The test section area can be determined using the isentropic sonic area ratio.

$$\frac{A_{TS}}{A_1^*} = \frac{1}{\mathrm{Ma}_{TS}}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_{TS}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{A_{TS} = 1.000\text{e-2 m}^2} \tag{1}$$

using $A_1^* = A_{T1} = 5.926\text{e-3 m}^2$ (since the flow must be sonic at the throat in order to reach supersonic conditions), $\gamma = 1.4$ (air), and $\mathrm{Ma}_{TS} = 2.0$.

In order to start the wind tunnel, the minimum area for the 2nd throat must be large enough to swallow a shock located at the entrance to the test section.



The sonic area ratio across the normal shock is given by:

$$\frac{p_{02}}{p_{01}} = \frac{\rho_{02}}{\rho_{01}} = \frac{A_1^*}{A_2^*} = \left[\frac{\frac{\gamma+1}{2}\mathrm{Ma}_1^2}{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}\right]^{\gamma/\gamma-1}\left[\frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{1/1-\gamma} \Rightarrow \frac{p_{01}}{p_{02}} = \frac{\rho_{01}}{\rho_{02}} = \frac{A_2^*}{A_1^*} = 1.3872 \tag{2}$$

where $\mathrm{Ma}_1 = 2.0$ and $\gamma = 1.4$. The 2nd throat area should be greater than or equal to the new sonic area. Hence:

$$A_{T2,\min} = A_2^* = \left(\frac{A_2^*}{A_1^*}\right)A_1^* = \left(\frac{A_2^*}{A_1^*}\right)A_{T1} = (1.3872)\left(5.926\text{e-3 m}^2\right) \tag{3}$$

$$\boxed{\therefore A_{T2,\min} = 8.221\text{e-3 m}^2}$$

The minimum reservoir pressure to start the wind tunnel will be the stagnation pressure (p_{01}) corresponding to when the shock stands at the entrance of the test section.

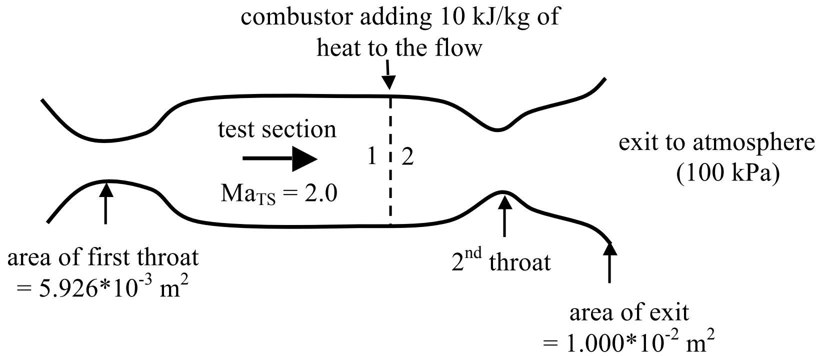
$$\frac{A_E}{A_2^*} = \frac{1.000\text{e-2 m}^2}{8.221\text{e-3 m}^2} = 1.2164 = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_E = 0.5774 \text{ (subsonic at exit)} \tag{4}$$

$$\frac{p_E}{p_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \frac{p_E}{p_{02}} = 0.7978 \tag{5}$$

$$p_{01} = \left(\frac{p_{01}}{p_{02}}\right)\left(\frac{p_{02}}{p_E}\right)p_E = (1.3872)\left(\frac{1}{0.7978}\right)(100\text{ kPa}) \Rightarrow \boxed{p_{01,\min} = 173.9\text{ kPa}} \tag{6}$$

where $p_E = p_{atm} = 100$ kPa (the exit Mach number is subsonic), p_{01}/p_{02} is found from Eq. (2), and p_E/p_{02} is found from Eq. (5).

Now consider the case where heat is added just downstream of the test section.



Determine the Mach number just downstream of the heat addition using the Rayleigh flow relations given the amount of heat addition per unit mass.

$$q_{12} = c_P\left(T_{02} - T_{01}\right) = c_P T_{01}\left(\frac{T_{02}}{T_{01}} - 1\right) \Rightarrow \frac{T_{02}}{T_{01}} = \frac{q_{12}}{c_P T_{01}} + 1 \;\Rightarrow\; \frac{T_{02}}{T_{01}} = 1.034 \tag{7}$$

using q_{12} = 10 kJ/kg, c_P = 1004 J/(kg·K), and T_{01} = 293 K.

In addition, from the Rayleigh flow relations:

$$\frac{T_{01}}{T_0^*} = \frac{2(\gamma+1)\mathrm{Ma}_1^2}{\left(1+\gamma \mathrm{Ma}_1^2\right)^2}\left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2\right) \;\Rightarrow\; \frac{T_{01}}{T_0^*} = 0.7934 \tag{8}$$

using γ= 1.4 and Ma$_1$ = Ma$_{TS}$ = 2.0.

The sonic stagnation temperature ratio just downstream of the heat addition is:

$$\frac{T_{02}}{T_0^*} = \left(\frac{T_{02}}{T_{01}}\right)\left(\frac{T_{01}}{T_0^*}\right) = (1.034)(0.7934) = 0.8204 \tag{9}$$

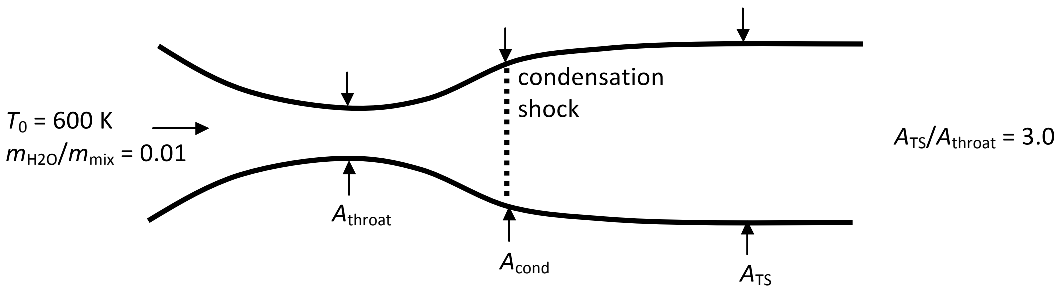
$$\frac{T_{02}}{T_0^*} = 0.8204 = \frac{2(\gamma+1)\mathrm{Ma}_2^2}{\left(1+\gamma \mathrm{Ma}_2^2\right)^2}\left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_2^2\right) \;\Rightarrow\; \mathrm{Ma}_2 = 1.871 \;\text{ (Rayleigh flow relations)} \tag{10}$$

The sonic area downstream of the heat addition is found from the isentropic relations.

$$\frac{A_2}{A_2^*} = \frac{1}{\mathrm{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \;\Rightarrow\; \frac{A_2}{A_2^*} = 1.5199 \;\Rightarrow\; \boxed{A_2^* = 6.579\text{e-3 m}^2} \tag{11}$$

using Ma$_2$ = 1.871, γ= 1.4, and $A_2 = A_{TS}$ = 1.000e-2 m^2 (Eq. (1)).

A converging-diverging nozzle with a test section-to-throat area ratio of 3.0 supplies air to a supersonic wind tunnel. If there is moisture in the air, it is possible for the water vapor to condense during the expansion process if the local static temperature drops below the saturation temperature. In practice, this condensation process occurs very rapidly, leading to an almost discontinuous change in the flow properties (and thus is referred to as a "condensation shock"). Assume that the stagnation temperature of the air/water vapor mixture entering the nozzle is 600 K and that the mass fraction of water vapor in the stream is $m_{H2O}/m_{mix} = 0.01$ (the ratio of the mass of water vapor to the mass of the vapor air mixture). The saturation temperature for the air/water vapor mixture is 14 °C and the heat of vaporization of water is 2470 kJ/kg (i.e., the heat released **per unit mass of water** when water vapor condenses to liquid water). You may assume that the air/vapor mixture behaves as a perfect gas and has the same flow properties as air ($\gamma = 1.4$, $R = 287$ J/(kg.K)).



a. Determine the area ratio, A_{cond}/A_{throat}, where the condensation shock occurs, *i.e.* the area ratio where the static temperature of the flow first drops below the saturation temperature of 14 °C.
b. Determine the test section Mach number when no condensation shock is present.
c. Determine the test section Mach number when the condensation shock is present. [Hint: Be careful differentiating between m_{H20} and m_{mix}.]
d. Sketch the process with the condensation shock on a *T-s* diagram.

SOLUTION:

Assume the flow is isentropic up to the point of the condensation shock. The Mach number at the saturation temperature of T_{cond} = (14 + 273) = 287 K may be found using:

$$\frac{T_{cond}}{T_0} = \left(1 + \frac{\gamma+1}{2}\text{Ma}_{cond}^2\right)^{-1} \Rightarrow \underline{\text{Ma}_{cond} = 2.335} \quad \text{where } T_{cond} = 287 \text{ K and } T_0 = 600 \text{ K } (\Rightarrow T_{cond}/T_0 = 0.4783) \tag{1}$$

The area ratio at this Mach number may be found using:

$$\frac{A_{cond}}{A_{throat}} = \frac{A_{cond}}{A^*} = \frac{1}{\text{Ma}_{cond}}\left(\frac{1+\frac{\gamma+1}{2}\text{Ma}_{cond}^2}{1+\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{A_{cond}/A_{throat} = 2.264} \tag{2}$$

Note that in the previous equation $A_{throat} = A^*$.

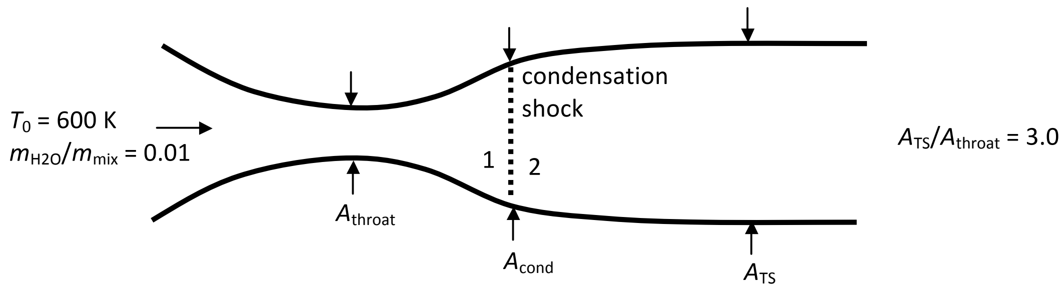
The Mach number in the test section when no condensation shock is present may be found from the area ratio $A_{TS}/A_{throat} = A_{TS}/A^* = 3.0$:

$$\frac{A_{TS}}{A_{TS}} = \frac{A_{TS}}{A^*} = \frac{1}{\text{Ma}_{TS}}\left(\frac{1+\frac{\gamma+1}{2}\text{Ma}_{TS}^2}{1+\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{\text{Ma}_{TS} = 2.637} \tag{3}$$

When the condensation shock is present, we must account for the heat released by the flow as the water vapor condenses from vapor to liquid. The rate at which heat is released into the flow is the mass flow rate of water multiplied by the heat of vaporization, h_{fg}:

$$\dot{q} = \dot{m}_{H2O} h_{fg} \tag{4}$$

Thus, the flow through the condensation shock may be modeled as a Rayleigh flow.



The stagnation temperature change through the condensation shock is given from conservation of energy as:

$$q_{12} = c_P \left(T_{02} - T_{01}\right) \tag{5}$$

where

$$q_{12} = \frac{\dot{q}}{\dot{m}_{\text{mix}}} = \frac{\dot{m}_{H2O} h_{fg}}{\dot{m}_{\text{mix}}} \tag{6}$$

Combine the previous two equations to get:

$$\frac{\dot{m}_{H2O} h_{fg}}{\dot{m}_{\text{mix}}} = c_P\left(T_{02} - T_{01}\right) \Rightarrow T_{02} = T_{01} + \frac{\dot{m}_{H2O}}{\dot{m}_{\text{mix}}}\frac{h_{fg}}{c_P} \;\Rightarrow\; \underline{T_{02} = 624.1\text{ K}} \quad (\text{or } T_{02}/T_{01} = 1.041) \tag{7}$$

The Mach number just downstream of the condensation shock may be found using T_{02}/T^*:

$$\frac{T_{02}}{T_0^*} = \left(\frac{T_{02}}{T_{01}}\right)\left(\frac{T_{01}}{T_0^*}\right) \Rightarrow \underline{T_{02}/T_0^* = 0.7641} \Rightarrow \underline{\text{Ma}_{\text{cond},2} = 2.154} \quad \text{(from the Rayleigh flow relations)} \tag{8}$$

where

$T_{01}/T_0^* = 0.7340$ using $\text{Ma}_{\text{cond},1} = 2.335$ and the Rayleigh flow relations. (9)

The sonic area ratio corresponding to the downstream Mach number is:

$$\frac{A_{\text{cond}}}{A_2^*} = \frac{1}{\text{Ma}_{\text{cond},2}}\left(\frac{1+\frac{\gamma+1}{2}\text{Ma}_{\text{cond},2}^2}{1+\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \underline{A_{\text{cond}}/A_2^* = 1.926} \tag{10}$$

The sonic area ratio for the test section is:

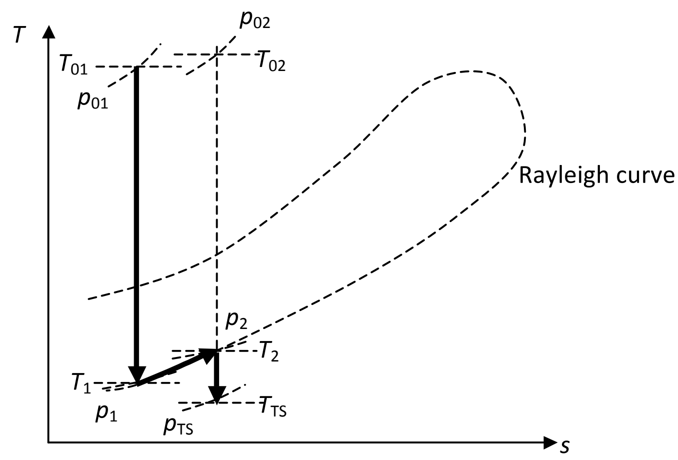
$$\frac{A_{TS}}{A_2^*} = \left(\frac{A_{TS}}{A_{\text{throat}}}\right)\left(\frac{A_{\text{throat}}}{A_{\text{cond}}}\right)\left(\frac{A_{\text{cond}}}{A_2^*}\right) \Rightarrow \underline{A_{TS}/A_2^* = 2.552} \tag{11}$$

The Mach number in the test section may be found from the sonic area ratio.

$$\frac{A_{TS}}{A_2^*} = \frac{1}{\text{Ma}_{TS}}\left(\frac{1+\frac{\gamma+1}{2}\text{Ma}_{TS}^2}{1+\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{\text{Ma}_{TS} = 2.465} \tag{12}$$

Thus, we see that the Mach number in the test section decreases when there is a condensation front. It would be a good idea to de-humidify the air before sending it through the wind tunnel.

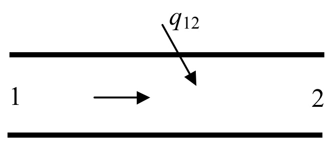
Sketch the process on a *T*-*s* diagram.



In a gas turbine, the combustor increases the thermal energy of the gas by burning fuel. A basic model for this process is a simple heat addition flow. Consider the one-dimensional flow of air, a perfect gas, in a frictionless, constant-area duct with energy added by heat addition. The air enters the duct with a total temperature of 450 K, a total pressure of 8 atm (abs), and a Mach number of 0.3.

a. Determine the heat addition per unit mass that chokes the flow at the exit and the exit total temperature and total pressure.
b. For an exit Mach number of 0.6, determine the heat addition per unit mass and the exit total temperature and pressure.
c. Sketch the *T*-*s* diagram for the case described in (b).

SOLUTION:



$T_{01} = 450$ K
$p_{01} = 8$ atm
$\mathrm{Ma}_1 = 0.3$

a. Use the Rayleigh flow relations to determine the state at 2. Note that state 2 is choked.

$\mathrm{Ma}_1 = 0.3 \Rightarrow T_{01}/T_0^* = 0.3469,\ p_{01}/p_0^* = 1.1985$ (1)

$$T_0^* = \left(\frac{T_0^*}{T_{01}}\right)T_{01} = \left(\frac{1}{0.3469}\right)(450\text{ K}) \Rightarrow \boxed{\therefore T_{02} = T_0^* = 1300\text{ K}} \tag{2}$$

$$p_0^* = \left(\frac{p_0^*}{p_{01}}\right)p_{01} = \left(\frac{1}{1.1985}\right)(8\text{ atm}) \Rightarrow \boxed{\therefore p_{02} = p_0^* = 6.7\text{ atm}} \tag{3}$$

$$q_{12} = c_p\left(T_{02} - T_{01}\right) = \left(1000\ \frac{\text{J}}{\text{kg}\cdot\text{K}}\right)(1300\text{ K} - 450\text{ K}) \Rightarrow \boxed{\therefore q_{12} = 850\ \frac{\text{kJ}}{\text{kg}}} \tag{4}$$

b. Use the Rayleigh flow relations to determine the state at 2.

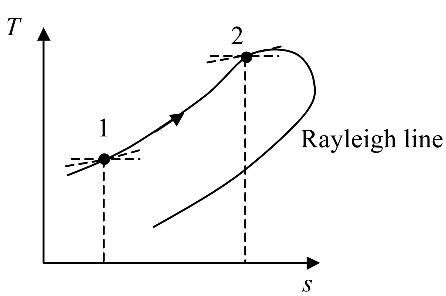
$\mathrm{Ma}_2 = 0.6 \Rightarrow T_{02}/T_0^* = 0.8189,\ p_{02}/p_0^* = 1.0753$ (5)

$$T_{02} = \left(\frac{T_{02}}{T_0^*}\right)\left(\frac{T_0^*}{T_{01}}\right)T_{01} = (0.8189)\left(\frac{1}{0.3469}\right)(450\text{ K}) \Rightarrow \boxed{\therefore T_{02} = 1100\text{ K}} \tag{6}$$

$$p_{02} = \left(\frac{p_{02}}{p_0^*}\right)\left(\frac{p_0^*}{p_{01}}\right)p_{01} = (1.0753)\left(\frac{1}{1.1985}\right)(8\text{ atm}) \Rightarrow \boxed{\therefore p_{02} = 7.2\text{ atm}} \tag{7}$$

$$q_{12} = c_p\left(T_{02} - T_{01}\right) = \left(1000\ \frac{\text{J}}{\text{kg}\cdot\text{K}}\right)(1100\text{ K} - 450\text{ K}) \Rightarrow \boxed{\therefore q_{12} = 650\ \frac{\text{kJ}}{\text{kg}}} \tag{8}$$

c.



13.17. Normal Shock Waves

Consider the movement of a piston in a cylinder, as shown in Figure 13.25. When we first move the piston, an infinitesimal (compression) pressure wave travels down the cylinder at the sonic speed. Behind the wave, the pressure, temperature, density, and increase slightly and the fluid has a small velocity following the wave (refer to Section 13.4).

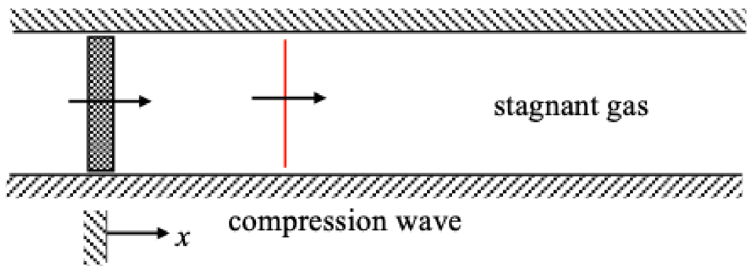


Figure 13.25. A single compression wave caused by a moving piston and traveling down the length of a cylinder into stagnant gas.

If we continue to increase the piston velocity, additional pressure waves will propagate down the cylinder (Figure 13.26). However, these waves travel at a slightly increased speed relative to a fixed observer due to the increased fluid temperature and fluid movement. The result is that the waves formed later catch up to the previous waves. When the waves catch up to the first wave, their effects add together so that the small changes across the individual waves now become a sudden and finite change called a shock wave (Figure 13.27).

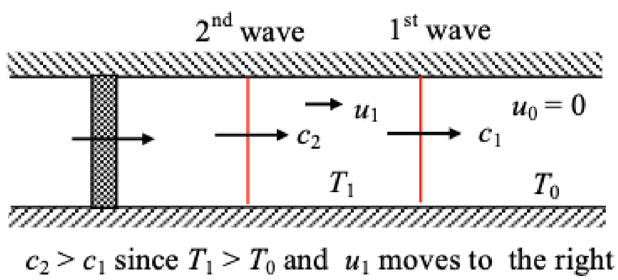


Figure 13.26. Multiple compression waves caused by an accelerating piston and traveling down the length of a cylinder.

Notes:

(1) The velocity of a shock wave is greater than the speed of sound. From the analysis used to determine the speed of a pressure wave Eq. (13.51),

$$c^2 = \frac{\Delta p}{\Delta \rho}\left(1 + \frac{\Delta \rho}{\rho}\right). \tag{13.138}$$

For a sound wave, $\Delta\rho \to d\rho \implies \Delta\rho/\rho \to 0$. For a shock wave, however, $\Delta\rho > 0$ so that $c_{\text{shock wave}} > c_{\text{sound wave}}$.

(2) A shock wave is a pressure wave across which there is a finite change in the flow properties.

(3) Shock waves only occur in supersonic flows. This fact is proven later in this section using the Second Law of Thermodynamics.

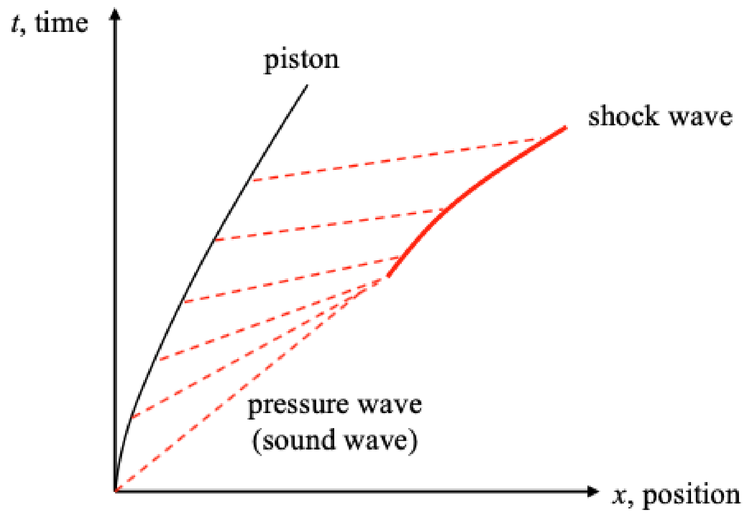


Figure 13.27. A plot illustrating the paths of an accelerating piston, weak compression waves, and the formation of a shock wave. The shock wave is defined as occurring where the compression waves first intersect.

(4) Shock waves are typically very thin, with thicknesses on the order of 1 µm. Thus, we consider the changes in the flow properties across the wave to be discontinuous.
(5) The sudden change in flow properties across the shock wave occurs non-isentropically since the thermal and velocity gradients are large within the shock wave itself.

To analyze a shock wave, we'll use an approach similar to that used to examine a sound wave. Let's consider a fixed shock wave across which flow properties change. A thin control volume of cross-sectional area A encompasses the wave as shown in Figure 13.28.

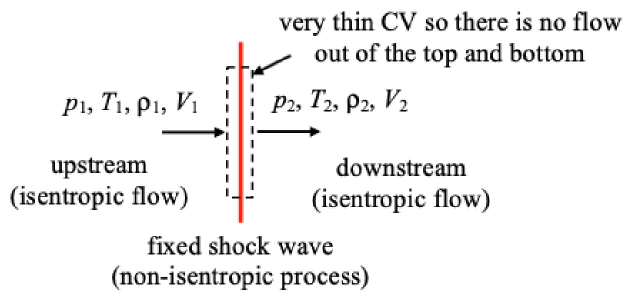


Figure 13.28. The control volume used to analyze changes in properties across a normal shock wave. Note that there is no heat transfer into the control volume.

From Conservation of Mass,

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2, \tag{13.139}$$

$$\rho_1 V_1 = \rho_2 V_2. \tag{13.140}$$

From the Linear Momentum Equation,

$$\dot{m}V_2 - \dot{m}V_1 = p_1 A - p_2 A, \tag{13.141}$$

$$\rho_1 V_1 (V_2 - V_1) = \rho_2 V_2 (V_2 - V_1) = p_1 - p_2. \tag{13.142}$$

From the First Law of Thermodynamics,

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2, \tag{13.143}$$

$$h_{01} = h_{02}. \tag{13.144}$$

Note that no heat is transferred into the control volume and, thus, the process is adiabatic.

From the Second Law of Thermodynamics,

$$\dot{m}s_2 - \dot{m}s_1 = \dot{\sigma}, \tag{13.145}$$

$$s_2 - s_1 = \sigma > 0. \tag{13.146}$$

The entropy production is greater than zero since *within the shock wave* there are internal irreversibilities. The thermal gradient and velocity gradient are enormous since the temperature and velocity have finite changes within the shock and the shock thickness is very small.

Since we're assuming we're working with a perfect gas,

$$\frac{p_1}{\rho_1 T_1} = \frac{p_1}{\rho_1 T_1} = R \quad \text{(ideal gas law)}, \tag{13.147}$$

$$\Delta h = c_p \Delta T. \tag{13.148}$$

Combining Eqs. (13.140) and (13.142),

$$\frac{p_1}{\rho_1 V_1} - \frac{p_2}{\rho_2 V_2} = V_2 - V_1. \tag{13.149}$$

Substituting Eq. (13.147),

$$\frac{RT_1}{V_1} - \frac{RT_2}{V_2} = V_2 - V_1. \tag{13.150}$$

Substituting Eqs. (13.144) and (13.148),

$$\frac{R}{V_1}\left(T_0 - \frac{V_1^2}{2c_p}\right) - \frac{R}{V_2}\left(T_0 - \frac{V_2^2}{2c_p}\right) = V_2 - V_1, \tag{13.151}$$

$$RV_2T_0 - \frac{RV_2V_1^2}{2c_p} - RV_1T_0 - \frac{RV_1V_2^2}{2c_p} = V_1V_2(V_2 - V_1), \tag{13.152}$$

$$V_1V_2(V_2 - V_1) = R\left[(V_2 - V_1)T_0 + (V_2 - V_1)\frac{V_1V_2}{2c_p}\right], \tag{13.153}$$

$$V_1V_2(V_2 - V_1) = R(V_2 - V_1)\left(T_0 + \frac{V_1V_2}{2c_p}\right), \tag{13.154}$$

$$V_1V_2 = R\left(T_0 + \frac{V_1V_2}{2c_p}\right), \tag{13.155}$$

$$V_1V_2\left(1 - \frac{R}{2c_p}\right) = RT_0, \tag{13.156}$$

$$V_1V_2 = \frac{RT_0}{1 - \frac{R}{2c_p}}. \tag{13.157}$$

Finally, substituting the ideal gas relation,

$$\frac{R}{c_p} = \frac{c_p - c_v}{c_p} = \frac{k-1}{k}, \tag{13.158}$$

and re-arranging gives,

$$\boxed{V_1V_2 = \frac{2kRT_0}{k+1}} \quad \underline{\text{Prandtl's Equation}}. \tag{13.159}$$

Dividing both sides of Prandtl's equation by the sound speed on either side of the shock wave and utilizing the definition for the Mach number for an ideal gas,

$$\frac{V_1}{\sqrt{kRT_1}}\frac{V_2}{\sqrt{kRT_2}} = \frac{2}{k+1}\frac{\sqrt{kRT_0}}{\sqrt{kRT_1}}\frac{\sqrt{kRT_0}}{\sqrt{kRT_2}}, \tag{13.160}$$

$$\mathrm{Ma}_1\mathrm{Ma}_2 = \frac{2}{k+1}\sqrt{\frac{T_0}{T_1}}\sqrt{\frac{T_0}{T_2}}. \tag{13.161}$$

Recall that for the adiabatic flow of a perfect gas,

$$\frac{T}{T_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{-1}, \tag{13.162}$$

so that,

$$\mathrm{Ma}_1\mathrm{Ma}_2 = \frac{2}{k+1}\sqrt{\frac{T_0}{T_1}}\sqrt{\frac{T_0}{T_2}}, \tag{13.163}$$

$$= \frac{2}{k+1}\left(1 + \frac{k-1}{2}\mathrm{Ma}_1^2\right)^{\frac{1}{2}}\left(1 + \frac{k-1}{2}\mathrm{Ma}_2^2\right)^{\frac{1}{2}}. \tag{13.164}$$

After additional algebra, we can reduce this equation to,

$$\boxed{\mathrm{Ma}_2^2 = \frac{(k-1)\mathrm{Ma}_1^2 + 2}{2k\mathrm{Ma}_1^2 - (k-1)}}. \tag{13.165}$$

This equation relates the upstream and downstream Mach numbers across a normal shock wave.

Notes:

(1) When $\mathrm{Ma}_1 > 1$, then $\mathrm{Ma}_2 < 1$ (supersonic to subsonic flow) and when $\mathrm{Ma}_1 < 1$, then $\mathrm{Ma}_2 > 1$ (subsonic to supersonic flow).
(2) From experiments, we observe that shock waves never form in subsonic flows ($\mathrm{Ma}_1 < 1$) even though Eq. (13.165) doesn't give any indication that this would be the case. We will use the Second Law in a moment to show that shock waves can only form in supersonic flows ($\mathrm{Ma}_1 > 1$).

The temperature ratio across the shock wave can be determined using the adiabatic stagnation temperature relation for a perfect gas and noting that the stagnation temperature remains constant across a shock,

$$\frac{T_2/T_0}{T_1/T_0} = \frac{\left(1 + \frac{k-1}{2}\mathrm{Ma}_2^2\right)^{-1}}{\left(1 + \frac{k-1}{2}\mathrm{Ma}_1^2\right)^{-1}}, \tag{13.166}$$

$$\frac{T_2}{T_1} = \frac{\left(1 + \frac{k-1}{2}\mathrm{Ma}_1^2\right)}{\left(1 + \frac{k-1}{2}\mathrm{Ma}_2^2\right)}. \tag{13.167}$$

The pressure ratio across the shock can be determined by combining Eqs. (13.167), (13.147), and (13.140),

$$\frac{p_2}{p_1} = \frac{\rho_2 T_2}{\rho_1 T_1} = \frac{V_1 T_2}{V_2 T_1} = \frac{(\sqrt{kRT_1}\mathrm{Ma}_1)T_2}{(\sqrt{kRT_2}\mathrm{Ma}_2)T_1}, \tag{13.168}$$

$$= \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\sqrt{\frac{T_2}{T_1}}, \tag{13.169}$$

$$\frac{p_2}{p_1} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1 + \frac{k-1}{2}\mathrm{Ma}_1^2}{1 + \frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1}{2}}. \tag{13.170}$$

The density ratio across the shock is,

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{p_2 T_1}{p_1 T_2} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1}{2}}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right), \tag{13.171}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{3}{2}}. \tag{13.172}$$

We can also determine the ratio of the isentropic stagnation pressures and densities across the shock wave,

$$\frac{p_1/p_{01}}{p_2/p_{02}} = \left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k}{1-k}}, \tag{13.173}$$

$$\frac{p_{02}}{p_{01}} = \left(\frac{p_2}{p_1}\right)\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k}{1-k}}, \tag{13.174}$$

$$\frac{p_{02}}{p_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1}{2}}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k}{1-k}}, \tag{13.175}$$

$$\frac{p_{02}}{p_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1+k}{2(1-k)}}. \tag{13.176}$$

For the stagnation density,

$$\frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}}\frac{T_{01}}{T_{02}}, \tag{13.177}$$

but since $T_{01} = T_{02}$ (refer to Eq. (13.148)),

$$\frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1+k}{2(1-k)}}. \tag{13.178}$$

The sonic area ratio across the shock can be determined from the fact that the mass flow rate across the shock must remain constant,

$$\dot{m}_1 = \dot{m}_2, \tag{13.179}$$

$$\rho_1^* V_1^* A_1^* = \rho_2^* V_2^* A_2^*, \tag{13.180}$$

$$\frac{A_2^*}{A_1^*} = \frac{\rho_1^*}{\rho_2^*}\frac{V_1^*}{v_2^*}. \tag{13.181}$$

The sonic ratios can be determined from the following analyses,

$$\frac{\rho_1^*}{\rho_{01}} = \frac{\rho_2^*}{\rho_{02}} = \left(1+\frac{k-1}{2}\right)^{\frac{1}{1-k}}, \tag{13.182}$$

$$\frac{\rho_1^*}{\rho_2^*} = \frac{\rho_{01}}{\rho_{02}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{1+k}{2(1-k)}}, \tag{13.183}$$

and,

$$\frac{V_1^*}{V_2^*} = \frac{c_1^*}{c_2^*} = \sqrt{\frac{T_1^*}{T_2^*}} = \sqrt{\frac{T_1^*/T_0}{T_2^*/T_0}} = 1. \tag{13.184}$$

Note that $T_{01} = T_{02}$ has been used in the previous equation. Substituting these two sonic ratios and simplifying gives,

$$\frac{A_2^*}{A_1^*} = \frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k+1}{2(k-1)}}. \tag{13.185}$$

Note that we could have also used the isentropic area ratios on either side of the shock wave to determine the sonic area ratio across the shock,

$$\frac{A_1}{A_1^*} = \frac{1}{\mathrm{Ma}_1}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{and} \quad \frac{A_2}{A_2^*} = \frac{1}{\mathrm{Ma}_2}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_2^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}}, \tag{13.186}$$

so that,

$$\frac{A_2^*}{A_1^*} = \frac{A_1/A_1^*}{A_2/A_2^*} = \frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{k+1}{2(k-1)}}, \tag{13.187}$$

where $A_1 = A_2$.

Notes:

(1) The previous equations may be written only in terms of Ma_1 by substituting in Eq. (13.165). The resulting equations (after much algebra) are,

$$\boxed{\mathrm{Ma}_2^2 = \frac{(k-1)\mathrm{Ma}_1^2+2}{2k\mathrm{Ma}_1^2-(k-1)}}, \tag{13.188}$$

$$\boxed{\frac{T_2}{T_1} = \left[2+(k-1)\mathrm{Ma}_1^2\right]\left[\frac{2k\mathrm{Ma}_1^2-(k-1)}{(k+1)^2\mathrm{Ma}_1^2}\right]}, \tag{13.189}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(k+1)\mathrm{Ma}_1^2}{(k-1)\mathrm{Ma}_1^2+2}}, \tag{13.190}$$

$$\boxed{\frac{p_2}{p_1} = \frac{2k}{k+1}\mathrm{Ma}_1^2 - \frac{k-1}{k+1}}, \tag{13.191}$$

$$\boxed{\frac{T_{02}}{T_{01}} = 1}, \tag{13.192}$$

$$\boxed{\frac{p_{02}}{p_{01}} = \frac{\rho_{02}}{\rho_{01}} = \frac{A_1^*}{A_2^*} = \left(\frac{\frac{k+1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_1^2}\right)^{\frac{k}{k-1}}\left(\frac{2k}{k+1}\mathrm{Ma}_1^2-\frac{k-1}{k+1}\right)^{\frac{1}{1-k}}}. \tag{13.193}$$

(2) Note let's examine the change in specific entropy across the shock using the following expression for a perfect gas,

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right). \tag{13.194}$$

Substituting Eqs. (13.189) and (13.191) into this equation and plotting we obtain Figure 13.29. We observe that for $\mathrm{Ma}_1 < 1$ the entropy decreases across the shock. The Second Law, however, states that the entropy must increase across the shock (refer to Eq. (13.146)). Thus, shock waves can only form when $\mathrm{Ma}_1 > 1$.

Also note that as the upstream Mach number approaches one ($\mathrm{Ma}_1 \to 1$), the flow through the shock approaches an isentropic process. An infinitesimally weak shock wave, one occurring when $\mathrm{Ma}_1 = 1$, results in an isentropic process. This type of shock is, in fact, just a sound wave!

(3) Plots of the various property ratios are shown in Figure 13.30 as functions of the upstream Mach number. The temperature, pressure, density, and sonic area increase across the shock, with the ratios increasing as the Mach number increases. The stagnation pressure, stagnation density, and velocity decrease across the shock, with the ratios decreasing as the Mach number increases. The downstream Mach number also decreases across the shock and becomes smaller as the upstream Mach number increases. The stagnation temperature remains constant across the shock. Normal shock tables with the numerical values for these ratios can be found in the appendices of most compressible flow textbooks.

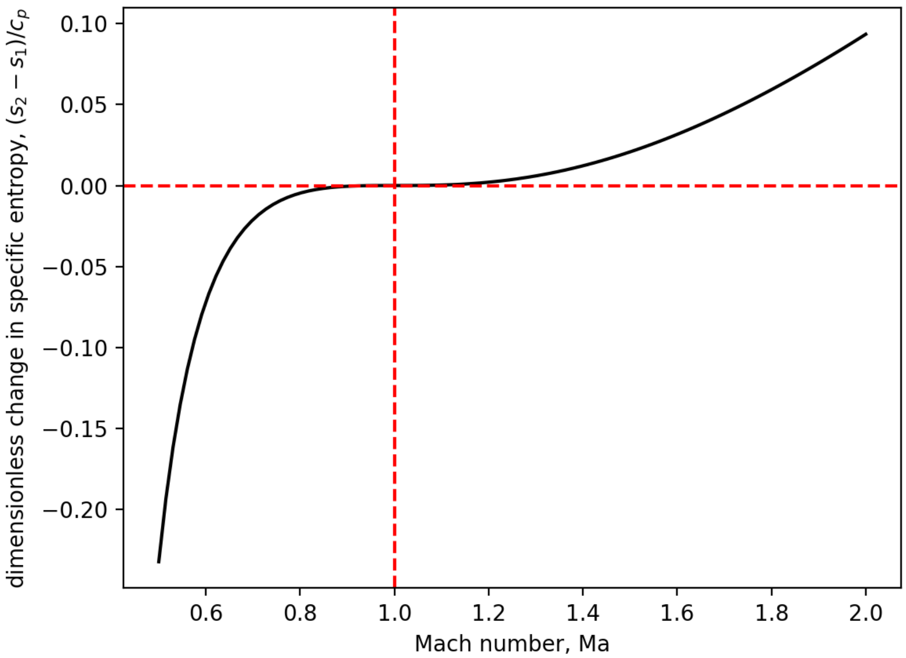


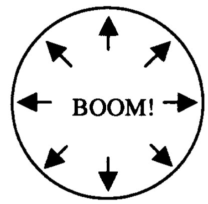
Figure 13.29. A plot of the dimensionless specific entropy change across a normal shock wave, $\Delta s/c_p$ as a function of the upstream Mach number, Ma_1, for $k = 1.4$.

(4) The shock strength is defined as the change in pressure across the shock wave relative to the upstream pressure, i.e., $\Delta p/p_1 = p_2/p_1 - 1$. Viewing the trends shown in Figure 13.30, the larger the incoming Mach number the stronger the shock wave.

(5) On a T-s diagram, the states across a shock wave correspond to the intersection of the Fanno and Rayleigh lines for the flow, as shown in Figure 13.31. The reason is because the flow across the shock satisfies the Fanno relations for Conservation of Mass (Eq. (13.140)), the First Law (Eq. (13.144)), and the ideal gas relations (Eqs. (13.147) and (13.148)). The shock also satisfies the Rayleigh relations for Conservation of Mass, the Linear Momentum Equation (Eq. (13.142)), and the ideal gas relations. The shock states must, therefore, occur at the intersection of the Fanno and Rayleigh lines in order for the shock to satisfy all of the basic relations simultaneously. Furthermore, state 2 lies to the right of state 1 in the T-s diagram since entropy must increase across the shock (from the Second Law).

An explosion creates a spherical shock wave propagating radially into still air at standard conditions. A recording instrument registers a maximum pressure of 200 psig as the shock wave passes by. Estimate:

a. **the speed of the shock wave with respect to a fixed observer in ft/sec**
b. **the wind speed following the shock with respect to a fixed observer in ft/sec**



SOLUTION:

- Note that although the shock wave is spherical, the flow across the shock can be considered as flow across a normal shock wave (assuming that the radius of curvature of the shock is large).

$Ma_1, V_1, p_1, T_1, \rho_1 \longrightarrow$ | $Ma_2, V_2, p_2, T_2, \rho_2 \longrightarrow$

$p_1 = 14.7$ psia
$T_1 = 70°F = 530°R$

change frame of reference to move w/ the shock wave

$\gamma = 1.4$
$R = 53.3 \ \frac{ft \cdot lb_f}{lb_m \cdot °R}$

$$\frac{p_2}{p_1} = \frac{(200 + 14.7)\text{ psia}}{14.7\text{ psia}} = 14.6$$

- Use normal shock relations to find:

$Ma_1 = 3.56$ using $\frac{p_2}{p_1} = 14.6$, $\gamma = 1.4$

$Ma_2 = 0.45$

- Upstream of the shock:

$V_1 = Ma_1 \sqrt{\gamma R T_1}$

$\therefore V_1 = 4020 \text{ ft/s}$ using $Ma_1 = 3.56$, $\gamma = 1.4$, $R = 53.3 \frac{ft \cdot lb_f}{lb_m \cdot °R}$, $T_1 = 530°R$

$\therefore$ The shock propagates into the still air at a speed of 4020 ft/s

1 of 2

SOLUTION...

- Also from the normal shock relations:

$\frac{T_2}{T_1} = 3.398$ using $Ma_1 = 3.56$, $\gamma = 1.4$

$\Rightarrow \underline{T_2 = 1800\,°R}$ using $\frac{T_2}{T_1} = 3.398$, $T_1 = 530\,°R$

NOTE: Could have also found T_2 using:

$$\frac{T_2}{T_{02}} = \left(1 + \frac{\gamma-1}{2} Ma_2^2\right)^{-1}$$

and

$$\frac{T_1}{T_{01}} = \left(1 + \frac{\gamma-1}{2} Ma_1^2\right)^{-1}$$

where $T_{02} = T_{01}$

$\Rightarrow V_2 = Ma_2\sqrt{\gamma R T_2}$

$\therefore \underline{V_2 = 936\ \text{ft/s}}$ using $Ma_2 = 0.45$, $\gamma = 1.4$, $R = 53.3\ \frac{\text{ft}\cdot\text{lb}_f}{\text{lb}_m\cdot°R}$, $T_2 = 1800\,°R$

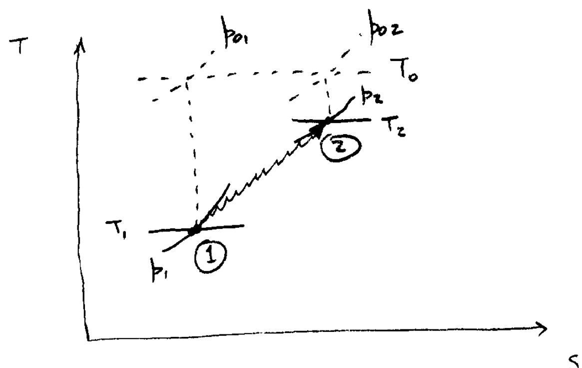
Note that this is the downstream air velocity <u>relative to the shock</u>.

NOTE: Could have also found V_2 using:

$$c_p T_1 + \frac{1}{2}V_1^2 = c_p T_2 + \frac{1}{2}V_2^2$$

$\therefore$ air velocity relative to a fixed observer

$$\boxed{V_{\text{downstream air, relative to fixed observer}} = V_{\text{shock}} - V_2 = 4020\ \text{ft/s} - 936\ \text{ft/s} = 3080\ \text{ft/s}}$$



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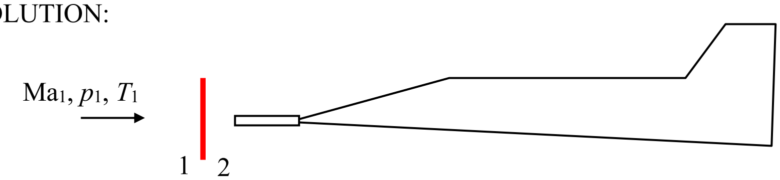
Stagnation pressure and temperature probes are located on the nose of a supersonic aircraft at 35,000 ft altitude. A normal shock stands in front of the probes. The temperature probe indicates $T_0 = 420$ °F behind the shock.

a. Calculate the Mach number and airspeed of the plane.
b. Find the static and stagnation pressures behind the shock.
c. Show the process and the static and stagnation points on a *T-s* diagram.

SOLUTION:



The pressure and temperature at an altitude of 35,000 ft using a U.S. Standard Atmospheric table (e.g., Table C.5 in Zucrow and Hoffman or using an online calculator such as http://www.digitaldutch.com/atmoscalc/) are:

$$p_1 = 3.458 \text{ psia} \tag{1}$$
$$T_1 = 393.9 \text{ °R} \tag{2}$$

The Mach number of the aircraft may be found by noting that the stagnation temperature remains constant across the shock wave ($T_{01} = T_{02} = 879$ °R) and using the adiabatic stagnation temperature ratio:

$$\frac{T_1}{T_{01}} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right)^{-1} \Rightarrow \boxed{\text{Ma}_1 = 2.48} \tag{3}$$

where, for air, $\gamma = 1.4$. The velocity is found from the Mach number and speed of sound:

$$V_1 = \text{Ma}_1 c_1 = \text{Ma}_1\sqrt{\gamma R T_1} \Rightarrow \boxed{V_1 = 2410 \text{ ft/s}} \tag{4}$$

where $R = 53.3$ (lbf·ft)/(lbm·°R).

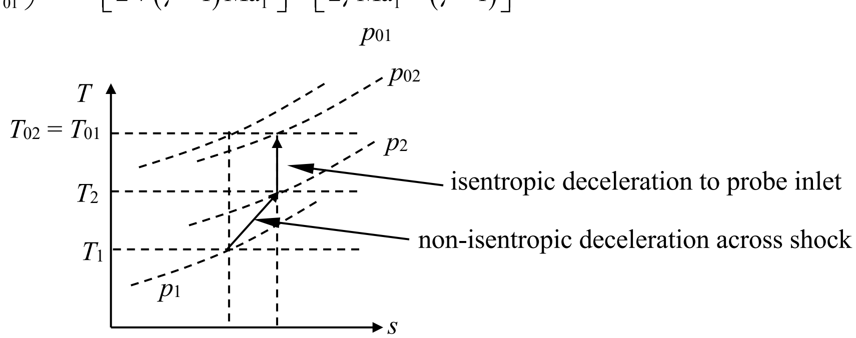
The static pressure downstream of the shock, p_2, may be found from the normal shock relations.

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\text{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \Rightarrow \boxed{p_2 = 24.2 \text{ psia}} \tag{5}$$

The stagnation pressure may be found by combining the stagnation pressure upstream of the shock with the stagnation pressure ratio across the shock.

$$\frac{p_1}{p_{01}} = \left(1 + \frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_{01} = 57.4 \text{ psia} \tag{6}$$

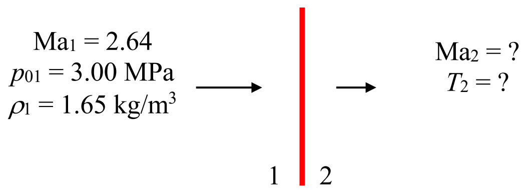
$$p_{02} = p_{01}\left(\frac{p_{02}}{p_{01}}\right) = p_{01}\left[\frac{(\gamma+1)\text{Ma}_1^2}{2+(\gamma-1)\text{Ma}_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \Rightarrow \boxed{p_{02} = 29.1 \text{ psia}} \tag{7}$$



Video solution: https://www.youtube.com/watch?v=pex-DiC_OnA

An air stream approaches a normal shock at $\mathrm{Ma}_1 = 2.64$. Upstream, $p_{01} = 3.00$ MPa (abs) and $\rho_1 = 1.65$ kg/m^3. Determine the downstream Mach number and temperature.

SOLUTION:



The downstream Mach number may be found from the normal shock relations:

$$\mathrm{Ma}_2^2 = \frac{(\gamma-1)\mathrm{Ma}_1^2+2}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)} \tag{1}$$

where $\mathrm{Ma}_1 = 2.64$ and $\gamma = 1.4$.

$$\boxed{\therefore \mathrm{Ma}_2 = 0.50}$$

One method of finding the downstream temperature is to determine the upstream stagnation temperature and then use the downstream Mach number and the adiabatic stagnation temperature ratio along with the fact that the stagnation temperature remains constant across the shock wave to determine the downstream static temperature.

$$\frac{\rho_1}{\rho_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{1}{1-\gamma}} \Rightarrow \rho_{01} = 14.6 \text{ kg/m}^3 \text{ (where } \rho_1 = 1.65 \text{ kg/m}^3\text{)} \tag{2}$$

$$T_{01} = \frac{p_{01}}{\rho_{01}R} \Rightarrow T_{01} = 714 \text{ K} \quad \text{(where } R = 287 \text{ J/(kg·K))} \tag{3}$$

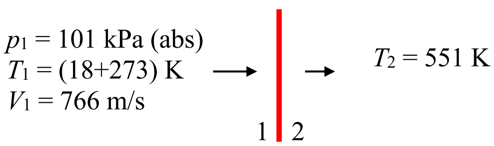
$$T_{02} = T_{01} \Rightarrow T_{02} = 714 \text{ K} \tag{4}$$

$$\frac{T_2}{T_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{-1} \Rightarrow \boxed{T_2 = 680 \text{ K}} \text{ (where } \mathrm{Ma}_2 = 0.50\text{)} \tag{5}$$

Air approaches a normal shock with $T_1 = 18$ °C, $p_1 = 101$ kPa (abs), and $V_1 = 766$ m/s. The temperature immediately downstream from the shock is $T_2 = 551$ K.

1. Determine the velocity immediately downstream from the shock.
2. Determine the pressure change across the shock.
3. Calculate the corresponding pressure change for a frictionless, shockless deceleration between the same speeds and temperatures.

SOLUTION:



The velocity downstream of the shock may be found from conservation of energy.

$$c_P T_1 + \tfrac{1}{2}V_1^2 = c_P T_2 + \tfrac{1}{2}V_2^2 \tag{1}$$

$$V_2 = \sqrt{V_1^2 + 2c_P\left(T_1 - T_2\right)} \;\Rightarrow\; \boxed{V_2 = 254.3\text{ m/s}} \tag{2}$$

using $c_P = 1004$ J/(kg·K).

The pressure change across the shock may be found using the normal shock relations.

$$\Delta p = p_2 - p_1 = p_1\left(\frac{p_2}{p_1} - 1\right) \;\Rightarrow\; \boxed{\Delta p = 4.73*10^5\text{ Pa}} \tag{3}$$

where

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\text{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \;\Rightarrow\; p_2/p_1 = 5.6880 \tag{4}$$

and

$$\text{Ma}_1 = \frac{V_1}{c_1} = \frac{V_1}{\sqrt{\gamma R T_1}} \;\Rightarrow\; \text{Ma}_1 = 2.24 \tag{5}$$

In addition,

$$\text{Ma}_2^2 = \frac{\left(\gamma-1\right)\text{Ma}_1^2 + 2}{2\gamma\text{Ma}_1^2 - \left(\gamma-1\right)} \;\Rightarrow\; \text{Ma}_2 = 0.54 \tag{6}$$

Note that we could have also simply used:

$$\text{Ma}_2 = \frac{V_2}{c_2} = \frac{V_2}{\sqrt{\gamma R T_2}} \;\Rightarrow\; \text{Ma}_2 = 0.54 \text{ (Same result as the previous one!)} \tag{7}$$

The corresponding pressure change for an isentropic deceleration between the same speeds may be found by combining isentropic stagnation pressure ratios,

$$\Delta p = p_2 - p_1 = p_1\left(\frac{p_2}{p_1} - 1\right) \;\Rightarrow\; \boxed{\Delta p_{\text{isentropic}} = 8.41*10^5\text{ Pa}} \tag{8}$$

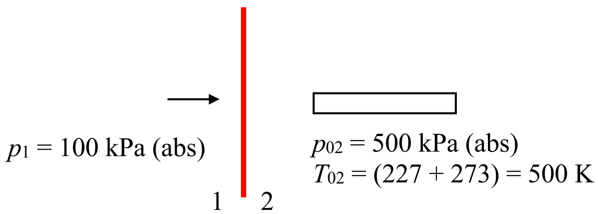
where,

$$\frac{p_2}{p_1} = \frac{\left(p_2/p_0\right)}{\left(p_1/p_0\right)} = \frac{\left(1 + \frac{\gamma-1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}}}{\left(1 + \frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}}} \;\Rightarrow\; p_2/p_1 = 9.3253 \tag{9}$$

A total pressure probe is inserted into a supersonic air flow. A shock wave forms just upstream of the impact hole. The probe measures a total pressure of 500 kPa (abs) and the stagnation temperature at the probe head is 227 °C. The static pressure upstream of the shock is measured with a wall tap to be 100 kPa (abs).

a. Determine the Mach number of the incoming flow.
b. Determine the velocity of the incoming flow.
c. Sketch the process on a *T*-*s* diagram.

SOLUTION:



Determine the upstream Mach number by combining the isentropic pressure ratio and the stagnation pressure ratio across a normal shock.

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \tag{1}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\text{Ma}_1^2}{2+(\gamma-1)\text{Ma}_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \tag{2}$$

$$\frac{p_1}{p_{02}} = \left(\frac{p_1}{p_{01}}\right)\left(\frac{p_{01}}{p_{02}}\right) = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}}\left[\frac{(\gamma+1)\text{Ma}_1^2}{2+(\gamma-1)\text{Ma}_1^2}\right]^{\frac{\gamma}{1-\gamma}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{1-\gamma}} \tag{3}$$

Solve Eqn. (3) numerically for Ma_1 given that p_1 = 100 kPa and p_{02} = 500 kPa (and γ = 1.4).

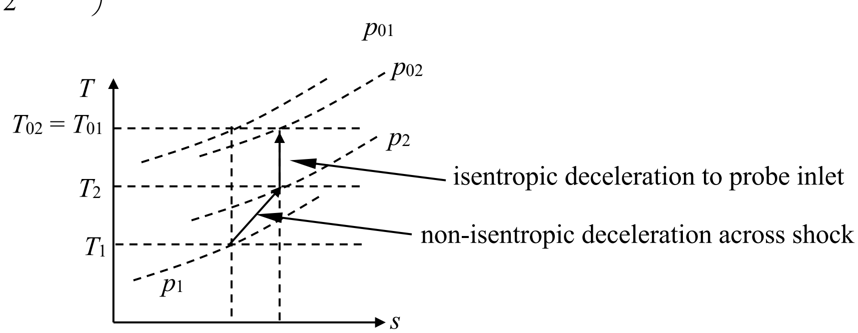
$$\boxed{\text{Ma}_1 = 1.87} \tag{4}$$

The velocity may be found from the Mach number and speed of sound on the upstream side of the shock wave.

$$V_1 = c_1\text{Ma}_1 = \sqrt{\gamma RT_1}\,\text{Ma}_1 \Rightarrow \boxed{V_1 = 643.1\text{ m/s}} \tag{5}$$

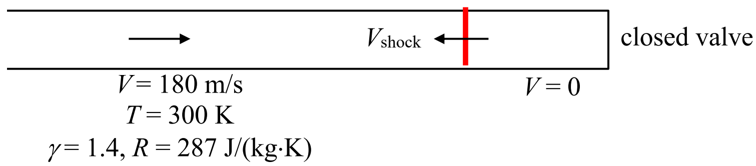
where the upstream static temperature is found from the adiabatic stagnation temperature ratio and noting that $T_{01} = T_{02}$.

$$\frac{T_1}{T_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{-1} \Rightarrow T_1 = 294.1\text{ K} \tag{6}$$

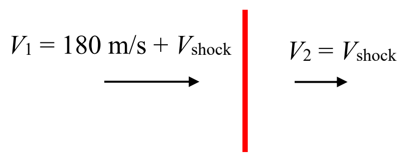


Air, with a temperature of 300 K, is flowing at 180 m/s through a constant-area, 30 m long pipe. A valve at the end of the pipe is suddenly closed and a normal shock wave propagates back into the pipe starting from the valve. How long will it be before the effect of closing the valve is felt at the pipe inlet?

SOLUTION:



Change the frame of reference shown in the figure (which is with respect to the ground), to a frame of reference that is fixed to the shock wave so that the normal shock relations may be used.



Use an iterative procedure for determining V_{shock}.

1. Assume a value for Ma_1.
2. Calculate Ma_2 using the normal shock relations.
$$\mathrm{Ma}_2^2 = \frac{(\gamma-1)\mathrm{Ma}_1^2+2}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)} \tag{1}$$
3. Determine T_2 using the normal shock relations.
$$T_2 = \left(\frac{T_2}{T_1}\right)T_1 \tag{2}$$
where
$$\frac{T_2}{T_1} = \left[2+(\gamma-1)\mathrm{Ma}_1^2\right]\left[\frac{2\gamma\mathrm{Ma}_1^2-(\gamma-1)}{(\gamma+1)^2\mathrm{Ma}_1^2}\right] \tag{3}$$
4. Determine V_{shock} using:
$$V_{\text{shock}} = V_2 = \mathrm{Ma}_2\sqrt{\gamma R T_2} \tag{4}$$
5. Determine V_1.
$$V_1 = \mathrm{Ma}_1\sqrt{\gamma R T_1} \tag{5}$$
6. Determine V_1'.
$$V_1' = 180\text{ m/s} + V_{\text{shock}} \tag{6}$$
7. If V_1' is greater than V_1, then the assumed value for Ma_1 was too small and a larger value for Ma_1 should be used. If $V_1' < V_1$, then choose a smaller value for Ma_1. Repeat steps 2 – 7 until a converged solution is obtained.

Using the given iterative procedure:

$$\mathrm{Ma}_1 = 1.36 \Rightarrow V_{\text{shock}} = 291\text{ m/s} \tag{7}$$

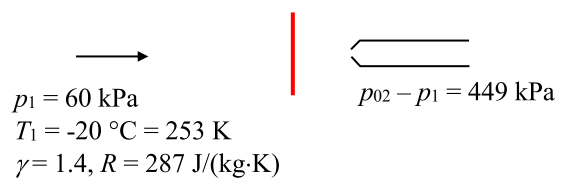
The valve closing will be felt 30 m upstream in time:

$$\boxed{T = L/V_{\text{shock}} = 0.10\text{ s}} \tag{8}$$

A stagnation tube is placed in a supersonic flow in which the static pressure and temperature far upstream are 60 kPa (abs) and -20 °C. The difference between the stagnation pressure measured by the stagnation tube and the upstream static pressure is 449 kPa. Determine the upstream Mach number and velocity of the flow.

SOLUTION:

Since there is no throat upstream of the stagnation tube, there must be a shock wave that forms in order to slow the flow from supersonic to subsonic conditions, and eventually stagnation conditions at the inlet to the stagnation tube.



Re-arrange the given conditions in order to solve for the upstream Mach number.

$$p_{02} - p_1 = \left(\frac{p_{02}}{p_1} - 1\right)p_1 = \left(\frac{p_{02}}{p_2}\frac{p_2}{p_1} - 1\right)p_1 = 449 \text{ kPa} \quad (1)$$

where

$$\text{Ma}_2^2 = \frac{(\gamma-1)\text{Ma}_1^2 + 2}{2\gamma \text{Ma}_1^2 - (\gamma-1)} \quad (2)$$

$$\frac{p_2}{p_{02}} = \left(1 + \frac{\gamma-1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \quad (3)$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\text{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \quad (4)$$

Iterate to a converged solution using the following approach.

1. Assume a value for Ma_1.
2. Determine Ma_2 using Eq. (2).
3. Determine p_2/p_{02} using Eq. (3).
4. Determine p_2/p_1 using Eq. (4).
5. Substitute the values calculated in the previous steps into the left-hand side of Eq. (1), along with p_1 = 60 kPa.
6. Check to see if the calculation from step 5 equals the right-hand side of Eq. (1). If the calculation is smaller than the right-hand side of Eq. (1) then the assumed Ma_1 was too small and a larger Ma_1 should be chosen. If the calculation is larger than the right-hand side of Eq. (1) then the assumed Ma_1 was too large and a smaller Ma_1 should be chosen. Steps 2 through 6 should be repeated until a converged solution results.

Following the previous iterative procedure:

$$\boxed{\text{Ma}_1 = 2.493}$$

and

$$\boxed{V_1 = \text{Ma}_1\sqrt{\gamma R T_1} = 795 \text{ m/s}} \quad (5)$$

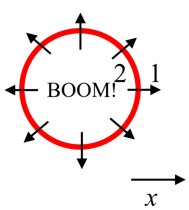
Video solution: https://www.youtube.com/watch?v=ljtlZpx4lnw

According to a newspaper article, at the center of a 12,600 lb_m "Daisy-Cutter" bomb explosion the overpressure in the air is approximately 1000 psi. Estimate:

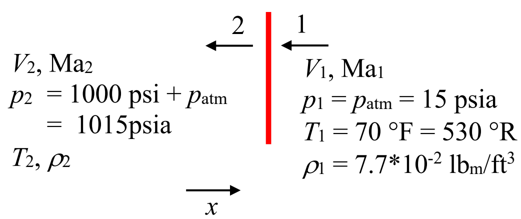
a. the speed of the resulting shock wave into the surrounding air,
b. the wind speed following the shock wave,
c. the temperature after the shock wave has passed, and
d. the air density after the shock wave has passed.

SOLUTION:

Change the frame of reference from one that is fixed to the ground to one that is fixed to the wave as shown in the schematic below. Treat the explosion shock wave as a normal shock.



Change the frame of reference so the wave appears stationary (substract the velocity of the wave to all velocities.)



The pressure ratio across the wave is:

$$\frac{p_2}{p_1} = \frac{1015 \text{ psia}}{15 \text{ psia}} = 69$$

Using the normal shock relations:

$$\begin{aligned} p_2/p_1 = 69 \Rightarrow\ & Ma_1 = 7.7 \\ \Rightarrow\ & Ma_2 = 0.4 \\ \Rightarrow\ & T_2/T_1 = 12.5 \\ \Rightarrow\ & \rho_2/\rho_1 = 5.5 \end{aligned}$$

Now determine the unknown quantities.

$$V_1 = \mathrm{Ma}_1\sqrt{\gamma R T_1} = (7.7)\sqrt{(1.4)\left(53.3\ \tfrac{\text{ft}\cdot\text{lb}_\text{f}}{\text{lb}_\text{m}\cdot{}^\circ\text{R}}\right)(530\ {}^\circ\text{R})}$$

$$\therefore V_1 = 8700\ \text{ft/s} \quad \text{(Note that is the velocity w/r/t the wave.)}$$

$$T_2 = \left(\frac{T_2}{T_1}\right)T_1 = (12.5)(530\ {}^\circ\text{R})$$

$$\boxed{\therefore T_2 = 6600\ {}^\circ\text{R} = 6100\ {}^\circ\text{F}}$$

$$\rho_2 = \left(\frac{\rho_2}{\rho_1}\right)\rho_1 = (5.5)\left(7.7*10^{-2}\ \text{lb}_\text{m}/\text{ft}^3\right)$$

$$\boxed{\therefore \rho_2 = 0.42\ \text{lb}_\text{m}/\text{ft}^3}$$

$$V_2 = \mathrm{Ma}_2\sqrt{\gamma R T_2} = (0.4)\sqrt{(1.4)\left(53.3\ \tfrac{\text{ft}\cdot\text{lb}_\text{f}}{\text{lb}_\text{m}\cdot{}^\circ\text{R}}\right)(6600\ {}^\circ\text{R})}$$

$$\therefore V_2 = 1600\ \text{ft/s} \quad \text{(Note that is the velocity w/r/t the wave.)}$$

To determine the shock and downstream wind speed with respect to the ground, we must change back to our original frame of reference.

$$V_{\substack{\text{shock,}\\ \text{w/r/t ground}}} = V_1$$

$$\boxed{\therefore V_{\substack{\text{shock,}\\ \text{w/r/t ground}}} = 8700\ \text{ft/s}}$$

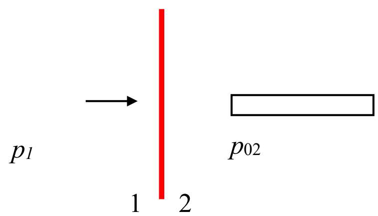
$$V_{\substack{\text{downstream wind,}\\ \text{w/r/t ground}}} = \underbrace{V_{\substack{\text{downstream wind,}\\ \text{w/r/t shock}}}}_{=-V_2} + V_{\substack{\text{shock,}\\ \text{w/r/t ground}}}$$

$$\boxed{\therefore V_{\substack{\text{downstream wind,}\\ \text{w/r/t ground}}} = 7100\ \text{ft/s}}$$

A Pitot tube which senses the stagnation pressure at its mouth, p_0, is often used to measure the speed of an airplane. Such a device is incorporated into the nose of a supersonic airplane for the purpose of measuring the Mach number, Ma, at which the airplane is traveling (Ma > 1).

Assume that the ambient pressure of the air, p_1, through which the airplane is traveling, is known. If a bow shock forms ahead of the Pitot tube, find the relation between the measured quantity, p_1/p_0, and the required quantity, Ma. The relation also involves the specific heat ratio. Note that the answer cannot be written explicitly as Ma = fcn(p_1/p_0), but can be written as p_1/p_0 = fcn(Ma).

SOLUTION:



Determine the upstream Mach number by combining the isentropic pressure ratio and the stagnation pressure ratio across a normal shock.

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \tag{1}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\mathrm{Ma}_1^2}{2+(\gamma-1)\mathrm{Ma}_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \tag{2}$$

$$\boxed{\frac{p_1}{p_{02}} = \left(\frac{p_1}{p_{01}}\right)\left(\frac{p_{01}}{p_{02}}\right) = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}}\left[\frac{(\gamma+1)\mathrm{Ma}_1^2}{2+(\gamma-1)\mathrm{Ma}_1^2}\right]^{\frac{\gamma}{1-\gamma}}\left[\frac{\gamma+1}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{1-\gamma}}} \tag{3}$$

Solve Eq. (3) numerically for Ma given that p_1 and p_{02} (and γ).

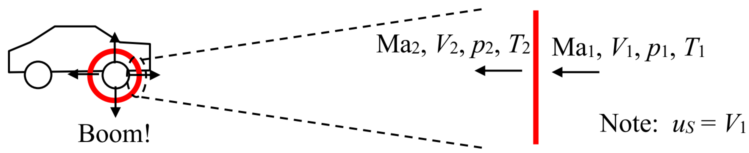
An automobile tire bursts sending a shock wave (assume this is a normal shock wave) propagating into the ambient air that has a pressure of p_1, sonic speed, c_1, and specific heat ratio, γ. If the pressure behind the shock is p_2 (roughly the inflated tire pressure), show that the speed of propagation of the shock, u_S, is given by:

$$u_S = c_1\sqrt{\frac{\gamma-1}{2\gamma}+\frac{p_2}{p_1}\frac{\gamma+1}{2\gamma}}$$

Calculate this speed if the temperature of the ambient air is 30 °C and the pressure ratio is $p_2/p_1 = 3.0$ (*e.g.* $p_1 = 14.7$ psia and $p_2 = 44.1$ psia).

SOLUTION:

Put our frame of reference on the shock wave.



Write the normal shock relation for the pressure rise across the shock.

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \tag{1}$$

Re-arrange and express the Mach number in terms of the velocity and speed of sound.

$$\boxed{V_1 = u_S = c_1\sqrt{\frac{\gamma-1}{2\gamma}+\frac{p_2}{p_1}\frac{\gamma+1}{2\gamma}}} \quad \text{where } \mathrm{Ma}_1 = V_1/c_1 \tag{2}$$

For $T_1 = 30 + 273 = 303$ K, $\gamma = 1.4$, $R = 287$ J/(kg.K), and $p_2/p_1 = 3.0$, $\underline{c_1 = 348.9 \text{ m/s}}$, and $\boxed{u_S = 574.8 \text{ m/s}}$.

The Mach number and temperature upstream of a shock wave are 2 and 7 °C, respectively. What is the air speed, relative to the shock wave, downstream of the shock wave?

SOLUTION:

Use the normal shock relations to determine the downstream Mach number.

$$\mathrm{Ma}_2^2 = \frac{(k-1)\mathrm{Ma}_1^2 + 2}{2k\mathrm{Ma}_1^2 - (k-1)} \Rightarrow \underline{\mathrm{Ma}_2 = 0.58} \tag{1}$$

where k = 1.4 and Ma_1 = 2.

Determine the stagnation temperature upstream of the shock wave.

$$\frac{T_1}{T_{01}} = \left(1 + \frac{k-1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow \underline{T_{01} = 504\ \mathrm{K}} \tag{2}$$

where T_1 = (273 + 7) K = 280 K.

Note that the stagnation temperature remains constant across a shock wave, so $\underline{T_{02} = T_{01}}$. Use the downstream stagnation temperature and downstream Mach number to determine the downstream static temperature:

$$\frac{T_2}{T_{02}} = \left(1 + \frac{k-1}{2}\mathrm{Ma}_2^2\right)^{-1} \Rightarrow \underline{T_2 = 473\ \mathrm{K}} \tag{3}$$

Use the definition of the Mach number and the speed of sound for an ideal gas to determine the air speed downstream of the shock wave:

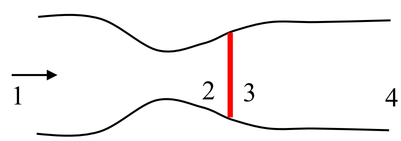
$$V_2 = \mathrm{Ma}_2 c_2 \Rightarrow V_2 = \mathrm{Ma}_2\sqrt{kRT_2} \Rightarrow \boxed{V_2 = 252\ \mathrm{m/s}} \tag{4}$$

where R_{air} = 287 J/(kg.K).

A supersonic aircraft flies at a Mach number of 2.7 at an altitude of 20 km. Air enters the engine inlet and is slowed isentropically to a Mach number of 1.3. A normal shock occurs at that location. The resulting flow is decelerated adiabatically, but not isentropically, further to a Mach number of 0.4. The final static pressure is 104 kPa (abs). Evaluate:

a. the stagnation temperature for the flow,
b. the pressure change, Δp, across the shock,
c. the final stagnation pressure, and
d. the total entropy change throughout the entire process.
e. Sketch the process on a Ts diagram.

SOLUTION:



The static temperature and pressure at an altitude of 20 km is, using the U.S. Standard Atmosphere, are T_1 = 216.65 K and p_1 = 5474.9 Pa (abs) (using http://www.digitaldutch.com/atmoscalc/ for example). The stagnation temperature is then:

$$\frac{T_1}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow T_1/T_0 = 0.4068 \Rightarrow \boxed{T_0 = 533\text{ K}} \quad (1)$$

Note that the stagnation temperature will remain constant throughout the entire process since there is no heat transfer.

The pressure ratio across the shock wave is may be found using the normal shock relations and noting that Ma_1 = 2.7 and Ma_2 = 1.3.

$$\frac{p_3}{p_2} = \frac{2\gamma}{\gamma + 1}\mathrm{Ma}_2^2 - \frac{\gamma - 1}{\gamma + 1} \Rightarrow p_3/p_2 = 1.8050 \quad (2)$$

$$\frac{p_1}{p_{01}} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1/p_{01} = 0.0430 \Rightarrow p_{01} = 127\text{ kPa (abs)} \quad (3)$$

$$\frac{p_2}{p_{01}} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_2/p_{01} = 0.3609 \Rightarrow p_2 = 46.0\text{ kPa (abs)} \quad (4)$$

$$\Delta p = p_3 - p_2 = p_2\left(\frac{p_3}{p_2} - 1\right) \Rightarrow \boxed{\Delta p = 37.0\text{ kPa}} \quad (5)$$

The stagnation pressure at station 4 may be found from the isentropic stagnation pressure ratio and knowing that Ma_4 = 0.4 and p_4 = 104 kPa (abs).

$$\frac{p_4}{p_{04}} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_4^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_4/p_{04} = 0.8956 \Rightarrow \boxed{p_{04} = 116\text{ kPa (abs)}} \quad (6)$$

The total entropy change throughout the process may be found using the perfect gas, entropy relation:

$$\Delta s = s_4 - s_1 = c_P \ln\frac{T_4}{T_1} - R\ln\frac{p_4}{p_1} \;\Rightarrow\; \boxed{\Delta s = 26.3 \text{ J/(kg·K)}} \tag{7}$$

where p_1 and T_1 are 5474.9 Pa and 216.65 K, respectively (from a U.S. Standard Atmosphere), p_4 = 104 kPa (given), c_P = 1004 J/(kg·K), and R = 287 J/(kg·K). The temperature T_4 may be found from:

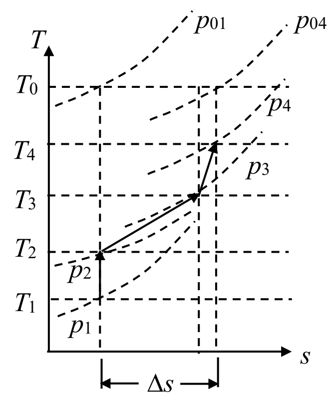
$$\frac{T_4}{T_{04}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_4^2\right)^{-1} \;\Rightarrow T_4/T_{04} = 0.9690 \;\Rightarrow\; T_4 = 516 \text{ K} \tag{8}$$

where $T_{04} = T_{01}$ = 533 K (from Eq. (1)).

Note that we could have also found Δs using stagnation conditions (refer to the Ts diagram below).

$$\Delta s = s_{04} - s_{01} = c_P \ln\frac{T_{04}}{T_{01}} - R\ln\frac{p_{04}}{p_{01}} \;\Rightarrow\; \Delta s = 26.4 \text{ J/(kg·K)} \quad \text{(Same as before, within error!)} \tag{9}$$

where $T_{04} = T_{01}$, p_{04} = 116 kPa, and p_{01} = 127 kPa.



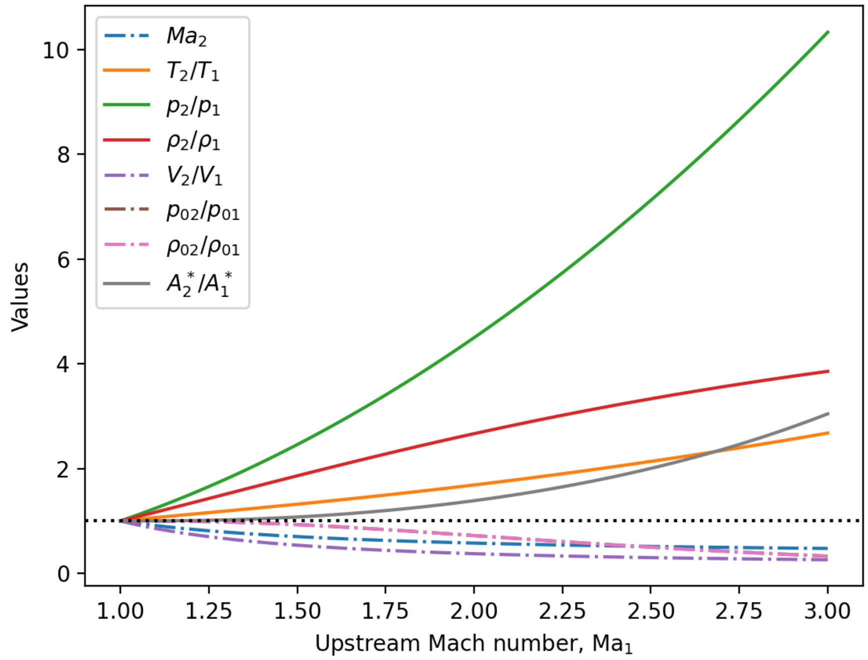


Figure 13.30. Plots of the various property ratios as functions of the upstream Mach number, Ma_1, for $k = 1.4$. Note that $T_{02}/T_{01} = 1$ and $p_{02}/p_{01} = \rho_{02}/\rho_{01}$.

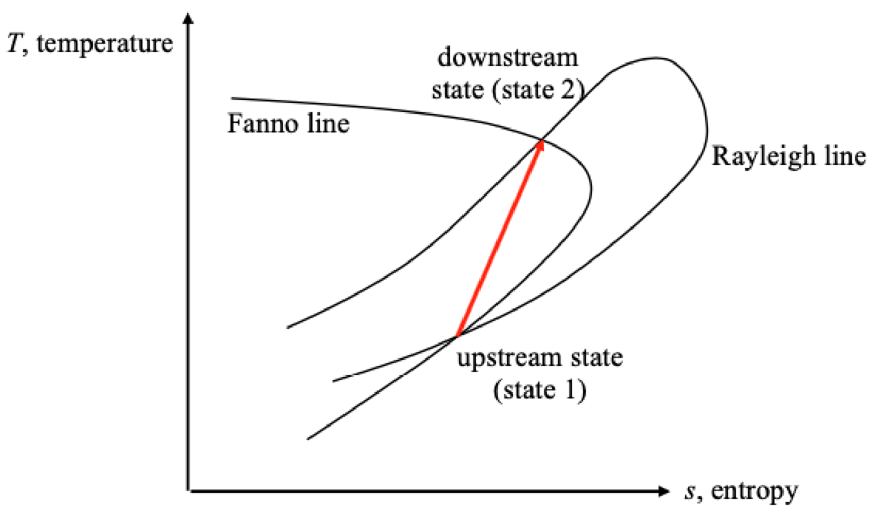


Figure 13.31. On a T-s plot, the states across a normal shock occur at the intersection of the Fanno and Rayleigh lines.

13.18. Flow through Converging-Diverging Nozzles

Consider flow through a converging-diverging nozzle (aka a de Laval nozzle) as shown in Figure 13.32. Let's hold the stagnation pressure, p_0, fixed and vary the back pressure, p_B. The plot shown in Figure 13.33 shows

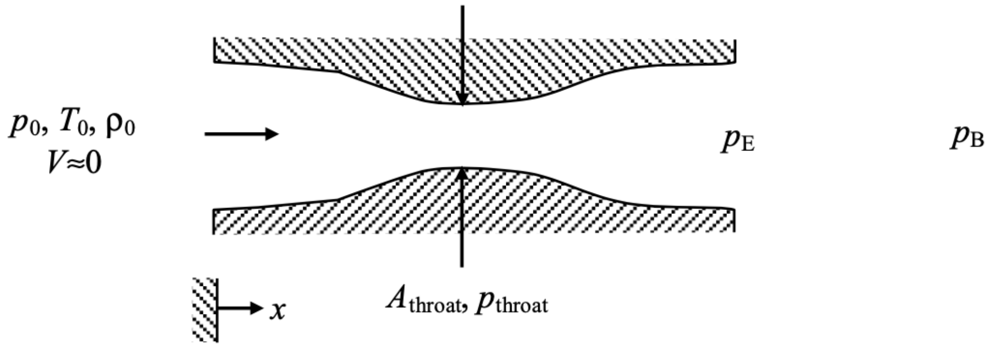


Figure 13.32. A schematic of a converging-diverging nozzle.

how the pressure ratio, p/p_0, varies with the location, x, in the nozzle for various values of the back pressure ratio, p_B/p_0.

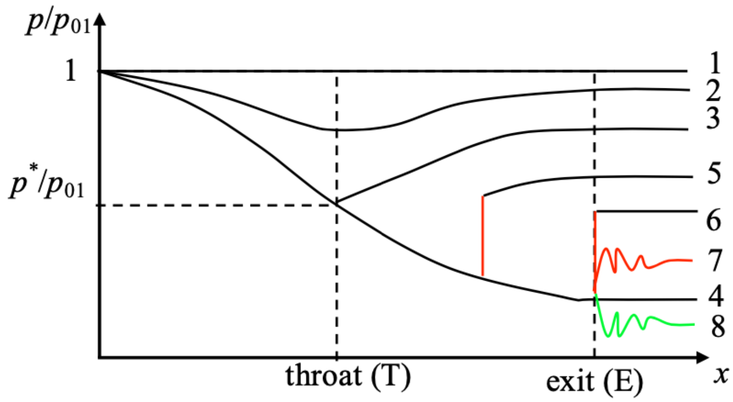


Figure 13.33. The pressure ratio p/p_0 plotted as a function of location x in a converging-diverging nozzle for different back pressure ratios p_B/p_0. The different cases, identified by the numbers on the right-side of the plot, are described in the text.

Cases (identified by the numbers on the right side in the plot):

(1) There is no flow through the device since $p_B = p_0$.

(2) There is subsonic flow throughout the device. The exit pressure equals the back pressure, i.e., $p_E = p_B$, since the exit Mach number is subsonic. Also, $\mathrm{Ma}_T < 1, A_T > A^*, \mathrm{Ma}_E < 1, \dot{m} < \dot{m}_{\text{choked}}$. The flow everywhere is isentropic.

(3) There is subsonic flow throughout the device except at the throat where $p_T = p^*$ ($\mathrm{Ma}_T = 1, A_T = A^*$). The flow is now choked since downstream pressure changes won't make it upstream of the throat. The mass flow rate is now $\dot{m} = \dot{m}_{\text{choked}}$. Further decreases in p_B will not affect the flow upstream of the throat and the mass flow rate will remain at the choked mass flow rate value. The exit pressure equals the back pressure for this case $p_E = p_B$, since the flow is subsonic at the exit ($\mathrm{Ma}_E < 1$). The flow everywhere is isentropic.

(4) Subsonic flow will occur in the converging section, sonic flow will occur at the throat ($p_T = p^*, \mathrm{Ma}_T = 1, A_T = A^*$), and supersonic flow will occur in the diverging section ($\mathrm{Ma}_E > 1$). This type of flow is called correctly expanded, perfectly expanded, or design flow since no shock waves form anywhere in the device and $p_E = p_B$. Note that p_E does not equal p_B because the flow is subsonic at the exit, but it's because the flow is at design conditions (a special case).

(5) Subsonic flow will occur in the converging section and sonic flow will occur at the throat ($p_T = p^*, \mathrm{Ma}_T = 1, A_T = A_1^*$). A portion of the diverging section will be supersonic with a normal shock wave occurring at a location such that the subsonic flow downstream of the shock will have an exit pressure equal to the back pressure: $p_E = p_B$ since $\mathrm{Ma}_E < 1$. As the back pressure decreases, the shock wave moves downstream of the throat and toward the exit. The pressure rise across the shock wave also increases as the back pressure decreases. There is isentropic flow upstream of the shock and downstream of it, but across the shock the flow is non-isentropic.

(6) This case is similar to Case 5 except that the shock wave is precisely at the nozzle exit. The pressure just downstream of the shock wave equals the back pressure since the flow is subsonic there ($p_{E2} = p_B, \mathrm{Ma}_{E2} < 1$). The flow everywhere within the converging-diverging nozzle is isentropic except right at the exit.

(7) The flow within the converging-diverging nozzle (and the exit) is isentropic ($\mathrm{Ma}_E > 1$). The normal shock that was located at the exit for Case 6 has moved outside the device to form a complicated sequence of oblique shock waves alternating with expansion fans. These are two-dimensional phenomena to be discussed in a following section of notes. This case is called the over-expanded case since the diverging section of the device has an area that over-expands the flow to a pressure that is lower than the back pressure ($p_E < p_B$). External shock waves are required to compress the flow to match the back pressure.

(8) This case is similar to Case 7 except that the flow outside of the device forms a sequence of expansion fans alternating with oblique shock waves (a sequence out of phase with the sequence mentioned in Case 7). This case is called the under-expanded case since the diverging section of the device has an area that is not large enough to drop the exit pressure to the back pressure ($p_E > p_B, \mathrm{Ma}_E > 1$). External expansion waves are required expand the flow to match the back pressure.

Notes:

(1) The critical back pressure ratio corresponding to Case 3 can be found from the isentropic relations (the flow throughout the entire device is isentropic). Assume that the geometry, and hence the exit-to-throat area ratio, A_E/A_T, is given. Since for Case 3 the flow is choked we know that $A_T = A^*$. Furthermore, since the exit flow is subsonic we also know that $p_E = p_B$. From the area ratio we can determine the exit Mach number, Ma_E,

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{(where the subsonic } \mathrm{Ma}_E \text{ is found).} \tag{13.195}$$

The back pressure ratio, p_B/p_0, for Case 3 can be determined given the exit Mach number,

$$\frac{p_B}{p_0} = \frac{p_E}{p_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}_E^2\right)^{\frac{k}{1-k}}. \tag{13.196}$$

(2) The critical back pressure ratio corresponding to Case 4 can be determined in a manner similar to that described previously in Note 1. For Case 4 however, the supersonic value for Ma_E should be used when determining the exit Mach number from the area ratio.

(3) The critical back pressure ratio corresponding to Case 6 can be found by combining the isentropic relations with the normal shock wave relations. When the shock wave occurs right at the exit of the device, the flow just upstream of the exit can be found from the isentropic relations,

$$\frac{A_E}{A_1^*} = \frac{A_E}{A_T} = \frac{1}{\mathrm{Ma}_{E1}}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_{E1}^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{(where the supersonic } \mathrm{Ma}_{E1} \text{ is found),} \tag{13.197}$$

$$\frac{p_E}{p_{01}} = \left(1 + \frac{k-1}{2}\mathrm{Ma}_{E1}^2\right)^{\frac{k}{1-k}}. \tag{13.198}$$

Note that the subscript "1" denotes the conditions just upstream of the shock wave. To determine the conditions just downstream of the shock we use the normal shock wave relations,

$$\frac{p_{E2}}{p_{E1}} = \frac{2k}{k+1}\mathrm{Ma}_{E1}^2 - \frac{k-1}{k+1}, \tag{13.199}$$

where p_{E2} is the pressure just downstream of the shock. Since the downstream flow is subsonic and because we're at the exit of the device, the downstream pressure, p_{E2}, must also equal the back pressure, p_B. Thus,

$$\frac{p_B}{p_{01}} = \frac{p_{E2}}{p_{01}} = \frac{p_{E2}}{p_{E1}}\frac{p_{E1}}{p_{01}}, \tag{13.200}$$

where the different ratios are given by Eqs. (13.198) and (13.199).

(4) The location of a shock wave for a back pressure in the range between Case 3 and Case 5 can be determined through iteration. For example:
   (a) Assume a location for the shock wave, e.g., pick a value for A/A_T since the geometry is known.
   (b) Determine the Mach number and pressure just upstream of the shock, Ma_1 and p_1, using the isentropic relations as discussed in Note 3,

$$\frac{A}{A_1^*} = \frac{A}{A_T} = \frac{1}{\mathrm{Ma}_1}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{(where the supersonic } \mathrm{Ma}_1 \text{ is found)}, \tag{13.201}$$

$$\frac{p_1}{p_{01}} = \left(1+\frac{k-1}{2}\mathrm{Ma}_1^2\right)^{\frac{k}{1-k}}. \tag{13.202}$$

   (c) Calculate the stagnation pressure ratio and sonic area ratio across the shock using the normal shock relations,

$$\frac{p_{02}}{p_{01}} = \frac{A_1^*}{A_2^*} = \left(\frac{\frac{k+1}{2}\mathrm{Ma}_1^2}{1+\frac{k-1}{2}\mathrm{Ma}_1^2}\right)^{\frac{k}{k-1}}\left(\frac{2k}{k+1}\mathrm{Ma}_1^2 - \frac{k-1}{k+1}\right)^{\frac{1}{1-k}}. \tag{13.203}$$

   (d) Determine the exit Mach number and exit pressure ratio using the isentropic relations and the downstream sonic area and stagnation pressure,

$$\frac{A_E}{A_2^*} = \frac{A_E}{A_T}\frac{A_1^*}{A_2^*} = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \quad \text{(where the subsonic } \mathrm{Ma}_E \text{ is chosen)}. \tag{13.204}$$

   Note that since the flow is choked, the throat area is equal to the upstream sonic area, i.e., $A_T = A_1^*$. The exit pressure ratio is found from the isentropic relations,

$$\frac{p_E}{p_{02}} = \left(1+\frac{k-1}{2}\mathrm{Ma}_E^2\right)^{\frac{k}{1-k}}. \tag{13.205}$$

   Note that since the exit Mach number is subsonic, the exit pressure will equal the back pressure, i.e., $p_E = p_B$.
   (e) Calculate the ratio of the back pressure to the upstream stagnation pressure,

$$\frac{p_B}{p_{01}} = \frac{p_E}{p_{02}}\frac{p_{02}}{p_{01}}. \tag{13.206}$$

   (f) Check to see if the back pressure ratio calculated in Step (e) matches with the given back pressure ratio. If so, then the assumed location of the shock is correct. If not, then the go back to Step (a) and repeat. If the back pressure ratio calculated in Step (e) is less than the given back pressure ratio, then the assumed shock location is too far downstream. If the back pressure ratio calculated in Step (e) is greater than the given back pressure ratio, then the assumed shock location is too far upstream. The logic for this step is illustrated in Figure 13.34.
   (g) Photographs for the various converging-diverging nozzle cases are shown in Figure 13.35.

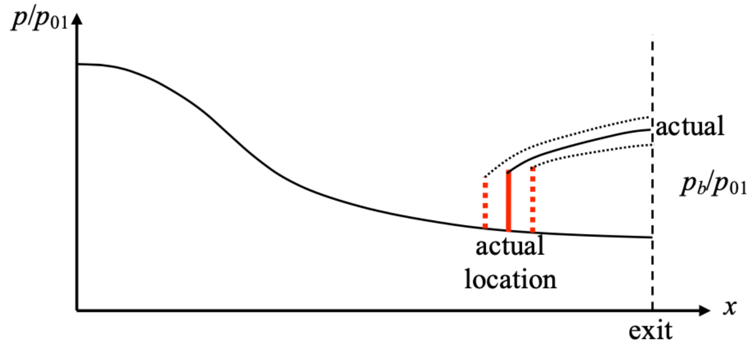


Figure 13.34. A plot illustrating in what direction to change the shock location during iteration. The calculated back pressure is compared to the actual back pressure. If the calculated back pressure is larger than the actual back pressure, then the shock should be moved further downstream. Alternately, if the calculated back pressure is smaller than the actual back pressure, then the shock should be moved further upstream.

(h) In real nozzles flows, the flow will typically separate from the nozzle walls as a result of the large adverse pressure gradient occurring across a shock wave. Interaction of the shock with the separated boundary layer results in a more gradual pressure rise than what is expected for the ideal, normal shock analysis.
It is also possible that downstream pressure information can propagate upstream in the diverging section even when the core flow is supersonic. In a real flow, a boundary layer will form along the wall with the flow in part of this boundary layer being subsonic. Thus, pressure information can propagate upstream within the subsonic part of the boundary layer and affect the flow in the diverging section. When the back pressure is in the range corresponding to Case 7 (back pressure less than the exit pressure when a shock stands at the exit, and greater than the isentropic case corresponding to supersonic diverging section flow), oblique shocks will typically form within the diverging section and flow separation occurs as shown in Figure 13.36. The exact pressure and location of the separation point are dependent on the boundary layer flow.

(i) Experimental pressure measurement data within a converging-diverging nozzle are shown in Figure 13.37. Also included in the plot are predictions using the analysis described in this section (a combination of isentropic flow relations and normal shock wave relations). As can be observed in the plot, the real data are predicted well by the models.

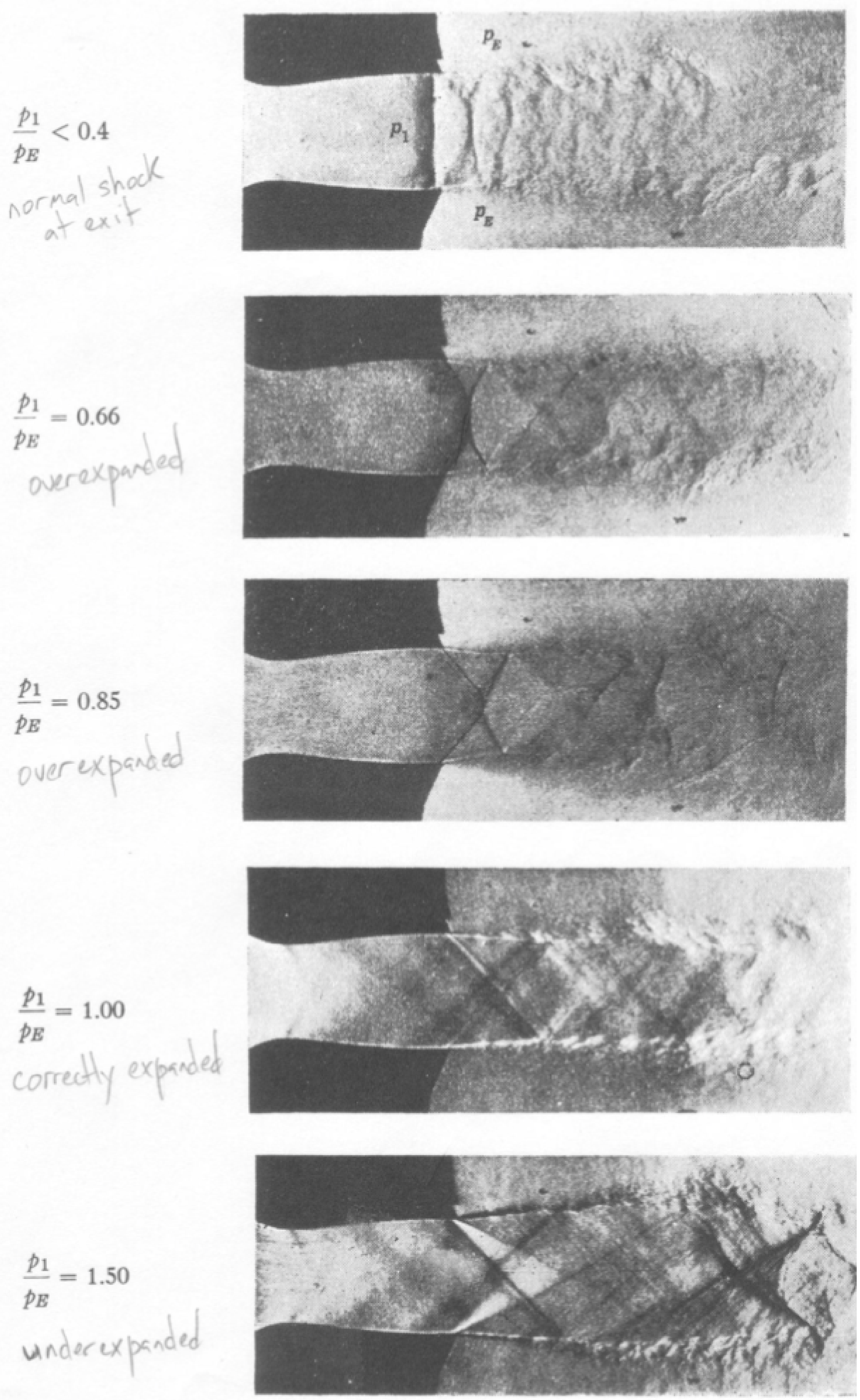


Fig. 5·4 Schlieren photographs of flow from a supersonic nozzle at different back pressures. (Figure from: Liepmann, H.W. and Roshko, A., *Elements of Gasdynamics*, Wiley.)

Figure 13.35. Photographs corresponding to the different converging-diverging nozzle cases shown in Figure 13.33.

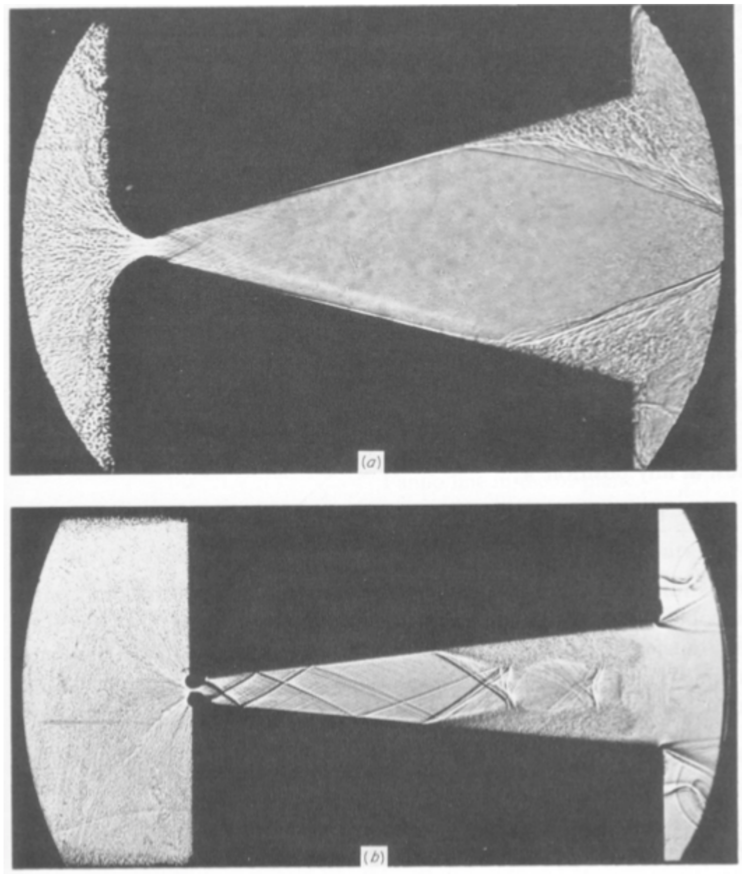
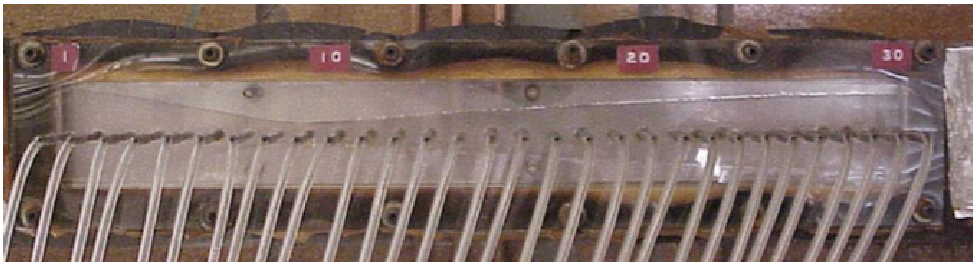


Figure 6.16
Shadowgraphs showing flow separation in supersonic nozzles. (*Courtesy of H. O. Amann, Ernst Mach Institut.*)

Figure 13.36. Photographs showing separated flow in supersonic flow in the diverging section of a converging-diverging nozzle.

A converging-diverging nozzle with pressure taps along the length of the device. The flow is from left to right.



The pressure ratio as a function of the axial distance in the CD nozzle for various back pressures.

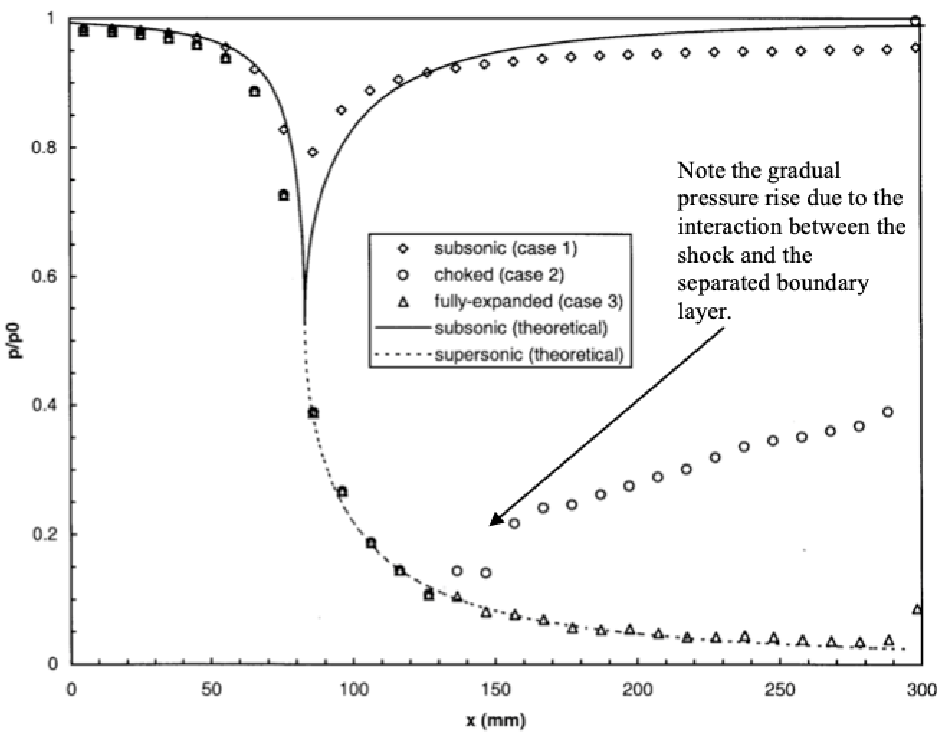
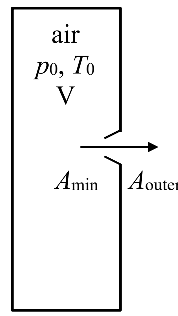


Figure 13.37. A photograph of a converging-diverging nozzle and corresponding pressure data shown in a plot.

During a test docking of the Progress M-34 supply ship with the Mir space station in 1997, a collision occurred which punctures the hull of Spektr Module of Mir. Assume the puncture hole had a minimum area of 1.0 cm^2 and an outer area of 1.5 cm^2 (the size of the hole was not directly measured). The volume of the Spektr module was 61.9 m^3 and had an initial interior pressure of 100 kPa (abs) and temperature of 34 °C.

1. Determine the mass flow rate of air from the capsule when the hole initially occurred.
2. Write an equation relating how the mass of air inside the module changed with time. You may assume that the air behaved as a perfect gas throughout the entire discharge process and that the temperature remained constant inside the space station (thanks to the small discharge rate and onboard heaters).
3. Calculate the thrust acting the space station for the initial conditions.



SOLUTION:

Since the air in the space station is discharging into space, the back pressure is essentially zero and the flow will always be choked with a mass flow rate of,

$$\dot{m} = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} \rho_0 \sqrt{\gamma R T_0} A^* \tag{1}$$

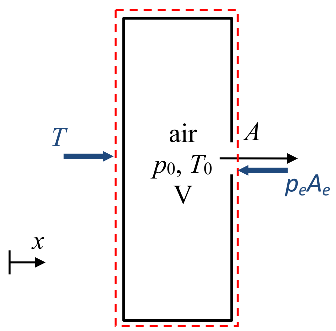
where,

$$\rho_0 = \frac{p_0}{R T_0} = \frac{M}{V} \tag{2}$$

where M is the mass of air within the space station and V is the interior volume of the station. Using the given data:

γ = 1.4
R = 287 J/(kg.K)
$p_{0,t=0}$ = 100*10^3 Pa (abs)
T_0 = 34 + 273 = 307 K
$A^* = A_{\text{min}}$ = 1 cm^2 = 1*10^{-4} m^2
V = 61.9 m^3
$\Rightarrow \rho_0$ = 1.135 kg/m^3
$\Rightarrow M_{t=0}$ = 70.25 kg
$\Rightarrow \boxed{\dot{m}_{t=0} = 2.90 \times 10^{-2} \text{ kg/s}}$

The mass in the space station may be found as a function of time by applying conservation of mass to a control volume surrounding the station as shown in the figure below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0 \tag{3}$$

where,

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{dM}{dt} \tag{4}$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = \dot{m} \tag{5}$$

Note that since the back pressure is always zero, the mass flow rate out of the space station will always be choked. Substitute and simplify.

$$\frac{dM}{dt} = -\dot{m} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\rho_0\sqrt{\gamma RT_0}\,A^* = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{M}{V}\right)\sqrt{\gamma RT_0}\,A^* \tag{6}$$

where Eqs. (1) and (2) have been used. Solve the differential equation given in Eq. (6).

$$\int_{M=M_{t=0}}^{M=M}\frac{dM}{M} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{\sqrt{\gamma RT_0}\,A^*}{V}\right)\int_{t=0}^{t=t}dt \quad \text{(Note that } T_0 = \text{constant.)} \tag{7}$$

$$\ln\left(\frac{M}{M_{t=0}}\right) = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{\sqrt{\gamma RT_0}\,A^*}{V}\right)t \tag{8}$$

$$\boxed{M = M_{t=0}\exp\left[-\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(\frac{\sqrt{\gamma RT_0}\,A_{\min}}{V}\right)t\right]} \quad \text{where } A^* = A_{\min} \tag{9}$$

The thrust acting on the space station may be found by applying the Linear Momentum Equation to the same control volume,

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{10}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(The thrust is the force required to hold Mir stationary.)} \tag{11}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{m} V_e \tag{12}$$

$$F_{B,x} = 0 \tag{13}$$

$$F_{S,x} = T - p_e A_e \quad \text{(where } A_e = A_{\text{outer}}\text{)} \tag{14}$$

Substitute and simplify,

$$T = \dot{m} V_e + p_e A_e \tag{15}$$

The exit conditions may be found using isentropic relations since the flow through the hole is under-expanded.

$$\frac{A_e}{A^*} = \frac{A_{\text{outer}}}{A_{\min}} = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_e = 1.8541 \tag{16}$$

$$\frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_e/p_0 = 0.1602 \Rightarrow p_e = 16.02\text{ kPa (abs)} \quad (p_0 = 100\text{ kPa abs}) \tag{17}$$

$$\frac{T_e}{T_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{-1} \Rightarrow T_e/T_0 = 0.5926 \Rightarrow T_e = 181.9\text{ K} \quad (T_0 = 307\text{ K}) \tag{18}$$

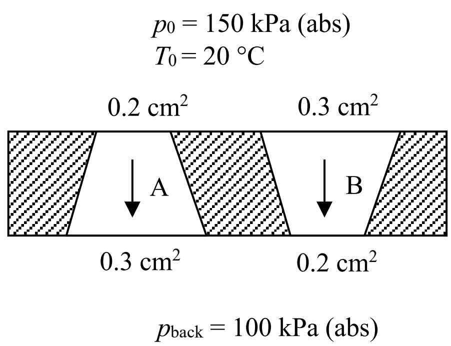
$$V_e = \text{Ma}_e\sqrt{\gamma R T_e} \Rightarrow V_e = 501.3\text{ m/s} \tag{19}$$

Now calculate the thrust using Eq. (15) and the mass flow rate found in the first part of this problem.

$$\boxed{T_{t=0} = 16.94\text{ N}} \tag{20}$$

Note that this is the thrust at $t = 0$. The thrust will vary with time since the stagnation pressure, and thus exit pressure, will vary with time as mass discharges from the space station.

The orientation of a hole can make a difference. Consider holes A and B in the figure below which are identical but reversed. For the given air properties on either side, compute the mass flow rate through each hole and explain why they are different.



SOLUTION:

First consider flow through hole B which can be considered a converging nozzle. First check to see if the flow is choked.

$$\frac{p_B}{p_0} = \frac{100\text{ kPa}}{150\text{ kPa}} = 0.6667 > \frac{p^*}{p_0} = 0.5283 \Rightarrow \text{The flow is not choked. (Note that } \gamma_{air} = 1.4.) \tag{1}$$

The mass flow rate can be found from the conditions at the hole exit.

$$\dot{m} = \rho_E V_E A_E \tag{2}$$

where,

$$\rho_E = \rho_0\left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{1}{1-\gamma}} \tag{3}$$

$$\rho_0 = \frac{p_0}{RT_0} \tag{4}$$

$$\frac{p_E}{p_0} = \frac{p_B}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \tag{5}$$

$$V_E = \text{Ma}_E\sqrt{\gamma R T_E} \tag{6}$$

$$T_E = T_0\left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{-1} \tag{7}$$

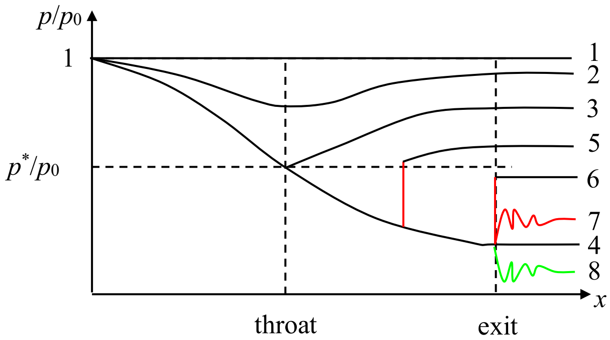
Using the given data:

γ = 1.4
R = 287 J/(kg·K)
p_0 = 150e3 Pa (abs)
T_0 = 20 °C = 293 K
p_E = 100e3 Pa (abs) (Note that since the exit flow is subsonic, $p_E = p_B$.)
A_E = 0.2 cm² = 2.0e-5 m²
Ma_E = 0.784
ρ_0 = 1.784 kg/m³
ρ_E = 1.335 kg/m³
T_E = 261 K
V_E = 254 m/s

$$\boxed{\therefore\ \dot{m}_B = 6.78\text{e-3 kg/s}}$$

Video solution: https://www.youtube.com/watch?v=nD-3DbTDYqg

Now consider hole A which can be modeled as a converging-diverging nozzle. Check to see what p_B/p_0 ratio will result in choked flow (case 3 in the figure below).



$$\frac{A_{E,\text{crit}}}{A^*} = \frac{A_E}{A_T} = \frac{1}{\text{Ma}_{E,\text{crit}}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_{E,\text{crit}}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_{E,\text{crit}} = 0.43 \tag{8}$$

using $A_E = 0.3\text{ cm}^2$ and $A_T = 0.2\text{ cm}^2$.

$$\frac{p_{E,\text{crit}}}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_{E,\text{crit}}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \frac{p_{E,\text{crit}}}{p_0} = 0.8805 \tag{9}$$

For the given situation, $p_B/p_0 = 0.6667$ (refer to Eq. (1)) $< p_{E,\text{crit}}/p_0 = 0.8805$ so the flow for hole A must be choked! The mass flow rate through the hole can be found using the (sonic) conditions at the throat.

$$\dot{m} = \rho_T V_T A_T \tag{10}$$

where

$$\rho_T = \rho^* = \rho_0\left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{1-\gamma}} = 0.6339 \quad (\text{using } \gamma_{\text{air}} = 1.4) \tag{11}$$

$$\rho_0 = \frac{p_0}{RT_0} \tag{12}$$

$$\frac{p_E}{p_0} = \frac{p_B}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \tag{13}$$

$$V_T = c^* = \sqrt{\gamma R T^*} \tag{14}$$

$$T^* = T_0\left(1+\frac{\gamma-1}{2}\right)^{-1} = 0.8333 \tag{15}$$

Using the given data:

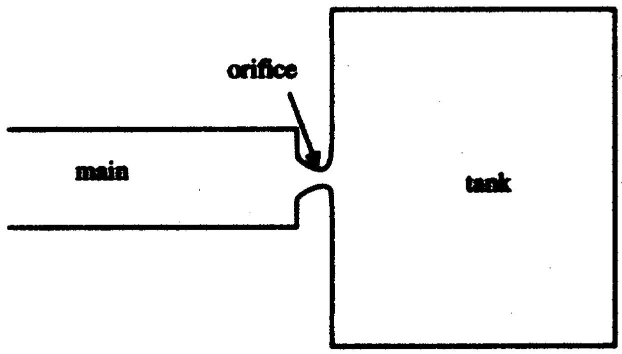
A_T = $0.2\text{ cm}^2 = 2.0\text{e-}5\text{ m}^2$
ρ^* = 1.131 kg/m^3
T^* = 244.2 K
V_T = 313.2 m/s

$$\boxed{\therefore\ \dot{m}_A = 7.08\text{e-}3\text{ kg/s}}$$

The different mass flow rates through holes A and B are because the flow through hole A is choked (the hole acts as a converging-diverging nozzle) while through hole B the flow is not choked (the hole acts as a converging nozzle).

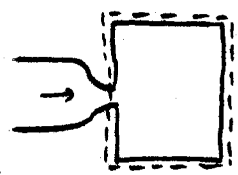
A large main is connected to an evacuated tank with a volume of 10 ft^3 by means of a rounded-entrance, converging nozzle having a diameter of 0.01 in. Initially, a diaphragm over the orifice seals the tank from the main. The air in the main is at 100 psia and 70 °F. The diaphragm is suddenly broken and air rushes into the tank. Estimate the time required for the pressure in the tank to reach 25psia, based on the following assumptions:

a. **The flow is quasi-static.**
b. **There is no heat conduction from the tank to the air.**
c. **The pressure and temperature in the main remain constant.**



SOLUTION:

- Apply COE to the following CV:



$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h+\tfrac{1}{2}V^2\right)\rho\left(\underline{u}_{rel}\cdot\hat{n}\right)dS = \dot{Q}_{\substack{into\\CV}} + \dot{W}_{\substack{other\\on\,CV}}$$

where $\frac{d}{dt}\int_{CV} e\rho\, dV = \frac{d}{dt}\left(Mu\right)_{tank}$ (specific internal energy)

$$\int_{CS}\left(h+\tfrac{1}{2}V^2\right)\rho\left(\underline{u}_{rel}\cdot\hat{n}\right)dS = -\dot{m}\left(h+\tfrac{1}{2}V^2\right)$$

$$\dot{Q}_{\substack{into\\CV}} = 0$$

$$\dot{W}_{\substack{other\\on\,CV}} = 0$$

- Also treat the air as a perfect gas:

$$u = c_v T \qquad p = \rho R T$$
$$h = c_p T$$

$$\Rightarrow \frac{d}{dt}\left(c_v M T\right)_{tank} - \dot{m}\left(c_p T_e + \tfrac{1}{2}V_e^2\right) = 0$$

- the subscript "e" refers to the conditions at the orifice exit

$$\Rightarrow M_{tank}\left.\frac{dT}{dt}\right|_{tank} + T_{tank}\left.\frac{dM}{dt}\right|_{tank} = \dot{m}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)$$

1 of 3

SOLUTION...

- Now apply COM to the same CV:

$$\frac{d}{dt}\int_{CV}\rho\, d\forall + \int_{CS}\rho(\underline{u}_{rel}\cdot\hat{n})dS = 0$$

where $\frac{d}{dt}\int_{CV}\rho\, d\forall = \left.\frac{dM}{dt}\right|_{tank}$

$$\int_{CS}\rho(\underline{u}_{rel}\cdot\hat{n})dS = -\dot{m}$$

$$\Rightarrow \left.\frac{dM}{dt}\right|_{tank} = -\dot{m}$$

Also,

$\dot{m} = \rho_e V_e A_e$

where the subscript "e" refers to the orifice exit

- Substitute into the COE eqn:

$$M_{tank}\left.\frac{dT}{dt}\right|_{tank} - \dot{m}T_{tank} = \dot{m}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)$$

- Now consider the conditions within the tank:

$$(p = \rho RT)_{tank} \text{ where } \rho_{tank} = \frac{M_{tank}}{\forall_{tank}}$$

$$\Rightarrow T_{tank} = \frac{p\forall_{tank}}{M_{tank}R} \Rightarrow \frac{dT_{tank}}{dt} = \frac{dp_{tank}}{dt}\left(\frac{\forall_{tank}}{M_{tank}R}\right) - \underbrace{\frac{dM_{tank}}{dt}}_{=-\dot{m}}\left(\frac{p\forall_{tank}}{RM_{tank}^2}\right)$$

- Substitute and simplify:

$$\frac{dp_{tank}}{dt}\frac{\forall_{tank}}{R} + \dot{m}\cancel{\frac{p_{tank}\forall_{tank}}{RM_{tank}}} - \dot{m}\cancel{\frac{p_{tank}\forall_{tank}}{RM_{tank}}} = \dot{m}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)$$

$$\Rightarrow \frac{dp_{tank}}{dt} = \frac{\rho_e V_e A_e R}{\forall_{tank}}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)$$

- Now determine the conditions at the orifice exit. Note that the flow will be choked until the tank pressure rises above:

$$\frac{p_{tank}}{p_0} = \frac{p_e}{p_0} = \frac{p^*}{p_0} = 0.5283 \Rightarrow p_{tank} = 52.83 \text{ psia}$$

where $p_0 = 100$ psia

2 of 3

SOLUTION...

- Thus, the flow into the tank for tank pressures less than 52.83 psia will be choked. Since we're only interested in the pressure up to 25 psia, the flow will remain choked throughout the entire filling process.

$$\Rightarrow Ma_e = 1 \Rightarrow \begin{array}{l} \rho_e = \rho^* \\ T_e = T^* \\ V_e = c^* = \sqrt{\gamma R T^*} \end{array}$$

where
$$\rho^* = \rho_0 \left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{1-\gamma}} = \frac{p_0}{RT_0}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{1-\gamma}}$$
$$T^* = T_0\left(1+\frac{\gamma-1}{2}\right)^{-1}$$

- For
$\gamma = 1.4$
$R = 1716 \frac{ft^2}{s^2 \cdot {}^\circ R}$
$p_0 = 100\text{ psia} = 1.44\times10^4 \frac{lb_f}{ft^2}$
$T_0 = 70^\circ F = 529.67\ {}^\circ R$

$$\Rightarrow \begin{array}{l} \rho^* = 1.00\times10^{-2} \frac{slug}{ft^3} \\ T^* = 440\ {}^\circ R \\ V^* = 1030\ ft/s \end{array}$$

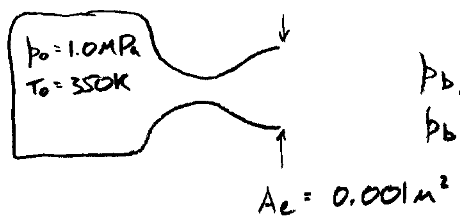
- Additionally,
$\mathcal{V}_{tank} = 10\ ft^3$
$c_v = 4280 \frac{ft^2}{s^2 \cdot {}^\circ R}$
$A_e = \frac{\pi (0.01\ in)^2}{4} = 7.85\times10^{-5}\ in^2 = 5.45\times10^{-7}\ ft^2$

$$\Rightarrow p_{tank} = \int \frac{dp_{tank}}{dt}\,dt = \frac{\rho_e V_e A_e R}{\mathcal{V}_{tank}}\left(\gamma T_e + \frac{V_e^2}{2c_v}\right)\Delta t$$

- For $p_{tank} = 25\text{ psia} = 3600 \frac{lb_f}{ft^2}$, $\boxed{\Delta t = 5050\text{ s} = 1.4\text{ hrs}}$

Air flows isentropically from a large reservoir where the pressure and temperature are 1.0 MPa (abs) and 350 K, respectively, through a converging-diverging nozzle, with exit area of 0.001 m^2. The design back pressure is 87.5 kPa (abs) but the nozzle operates at a back pressure of 50.0 kPa (abs). Determine the exit Mach number and mass flow rate.

SOLUTION:



$p_{b,design} = 87.5\,kPa$
$p_b = 50.0\,kPa$

- Since the nozzle operates at a pressure below the design pressure, the flow within the nozzle will remain isentropic and choked.

$$\underline{p_e = p_{b,design} = 87.5\,kPa}$$

$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma-1}{2} Ma_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\Rightarrow \boxed{Ma_e = 2.24}$$ using $p_e = 87.5\,kPa$, $p_0 = 1.0\,MPa$, $\gamma = 1.4$

$$\frac{A_e}{A^*} = \frac{1}{Ma_e}\left(\frac{1 + \frac{\gamma-1}{2} Ma_e^2}{1 + \frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\Rightarrow \underline{A^* = 4.80 \times 10^{-4}\,m^2}$$ using $Ma_e = 2.24$, $A_e = 0.001\,m^2$

$$\dot{m} = \left(1 + \frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}}\, A^*$$

$$\therefore \boxed{\dot{m} = 1.04\,kg/s}$$ using
$\gamma = 1.4$
$p_0 = 1.0 \times 10^6\,Pa$
$R = 287\,J/kg\cdot K$
$T_0 = 350\,K$
$A^* = 4.80 \times 10^{-4}\,m^2$

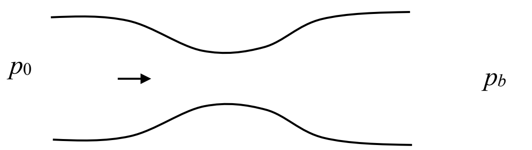
Video solution: https://www.youtube.com/watch?v=vqQQseeENYU

A rocket engine is designed to operate at a pressure ratio (inlet reservoir pressure/back pressure) of 37. Find:

a. the ratio of the exit area to the throat area which is necessary for the supersonic exhaust to be correctly expanded,
b. the Mach number of the exit flow under correctly expanded conditions,
c. the lowest pressure ratio (p_0/p_b) at which the same nozzle would be choked, and
d. the pressure ratio (p_0/p_b) at which there would be a normal shock wave at the exit.

Assume the specific heat ratio of the gas is 1.4.

SOLUTION:



$p_0/p_e = 37$

The area ratio may be found from the isentropic sonic area ratio and the isentropic pressure ratio.

$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{\mathrm{Ma}_e = 3.0} \text{ (since at design conditions, the flow is isentropic)} \tag{1}$$

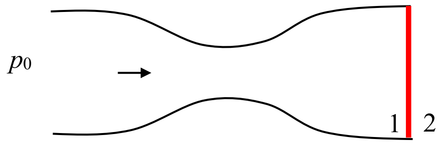
$$\frac{A_e}{A^*} = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{A_e/A_t = 4.3} \text{ (since } A_t = A^*\text{)} \tag{2}$$

The lowest pressure ratio for which the nozzle will be choked may be found Eqns. (2) and (1), but using the subsonic Mach number.

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_e = 0.14 \tag{3}$$

$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_0/p_b = 1.01} \text{ Note that } p_e = p_b \text{ when the flow just becomes choked.} \tag{4}$$

Now consider a case where a shock wave occurs at the exit of the device.



From Eqn. (1), $\text{Ma}_{e1} = 3.0$ and $p_{01}/p_{e1} = 37$. From the normal shock relations,

$$\text{Ma}_2^2 = \frac{(\gamma-1)\text{Ma}_1^2+2}{2\gamma\text{Ma}_1^2-(\gamma-1)} \Rightarrow \text{Ma}_2 = 0.475 \tag{5}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\text{Ma}_1^2}{2+(\gamma-1)\text{Ma}_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_1^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \Rightarrow p_{02}/p_{01} = 0.327 \tag{6}$$

and the isentropic relations:

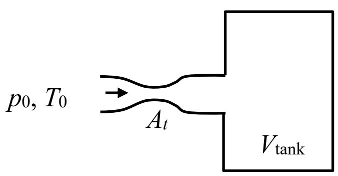
$$\frac{p_2}{p_{02}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_2/p_{02} = 0.857 \tag{7}$$

Since the flow downstream of the shock is subsonic, $p_2 = p_b$. Thus,

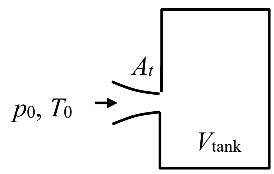
$$\frac{p_{01}}{p_b} = \left(\frac{p_{01}}{p_{02}}\right)\left(\frac{p_{02}}{p_2}\right) \Rightarrow \boxed{p_{01}/p_b = 3.6} \tag{8}$$

Video solution: https://www.youtube.com/watch?v=rWfR3HLPiag

Which nozzle will fill the tank faster (or will they fill at the same rate), assuming that the tank is initially evacuated? Justify your answer. The upstream stagnation properties, throat areas, and tank volumes are identical in both cases.



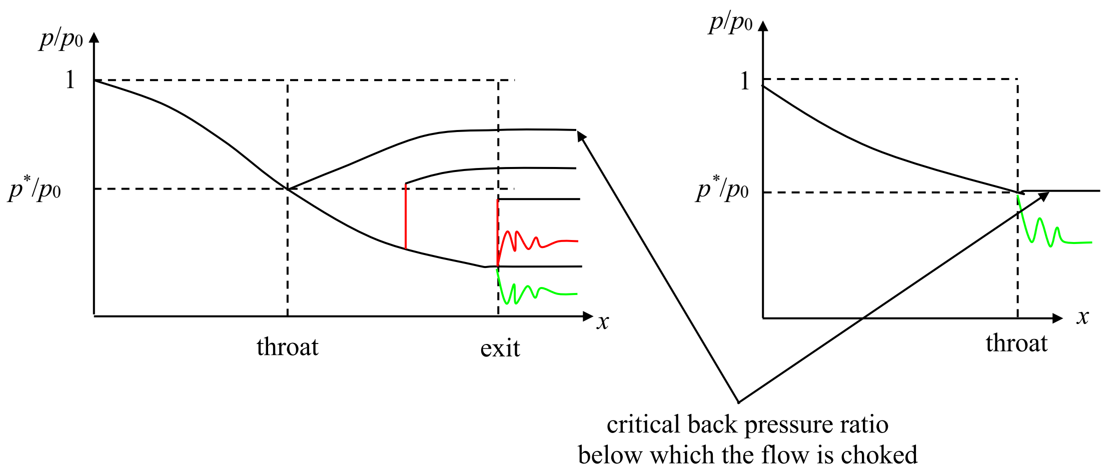
converging-diverging nozzle



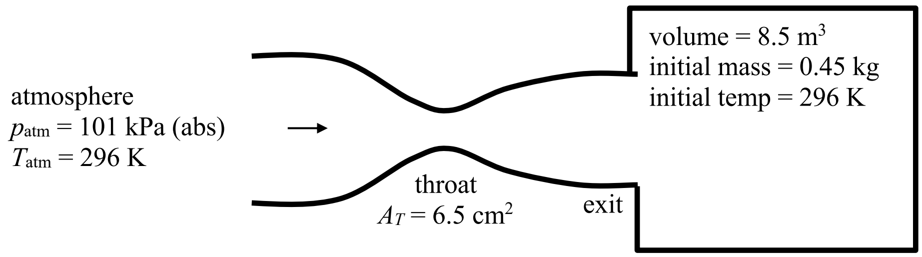
converging nozzle

SOLUTION:

The converging-diverging nozzle will fill the tank faster. Since the tank is initially evacuated, the flow will start at choked conditions in each case. Hence, the mass flow rate into each tank will be the choked flow mass flow rate (i.e., the maximum mass flow rate), which will be identical in both cases since the throat areas and stagnation properties are identical. However, the converging-diverging nozzle will remain choked for a wider range of back pressure ratios than the converging nozzle. Hence, converging-diverging nozzle will fill the tank more rapidly.



An 8.5 m^3 vacuum tank is to be used to create a flow at an exit Mach number of $\text{Ma}_E = 2.0$ (refer to the figure below). A plug is put into the nozzle and the tank is evacuated until it contains 0.45 kg of air at a temperature of 296 K. When the plug is removed, air flows from the atmosphere into the tank through the converging-diverging nozzle. The throat area is $A_T = 6.5$ cm^2.



a. Determine the design exit area.
b. Determine the initial pressure in the tank.
c. Determine the initial mass flow rate through the nozzle.
d. Determine the exit pressure, p_E, immediately after the flow begins.
e. Determine the tank pressure at which a normal shock wave will stand in the nozzle exit plane.

SOLUTION:

The design exit area may be found from the design exit Mach number, $\text{Ma}_{E,d} = 2.0$, and the isentropic flow relations.

$$\frac{A_{E,d}}{A^*} = \frac{1}{\text{Ma}_{E,d}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_{E,d}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow A_{E,d}/A^* = 1.6875 \Rightarrow \boxed{A_{E,d} = 11.0\text{ cm}^2} \tag{1}$$

where the sonic area is equal to the throat area, $A^* = A_T = 6.5$ cm^2, since the flow goes from stagnation conditions to supersonic conditions.

The initial pressure in the tank may be found using the ideal gas law,

$$p = \rho RT = \left(\frac{M}{V}\right)RT \Rightarrow \boxed{p_{\text{tank}}(t=0) = 4.50\text{ kPa (abs)}} \tag{2}$$

where $M = 0.45$ kg, $V = 8.5$ m^3, $R = 287$ J/(kg.K), and $T = 296$ K.

To determine the exit plane pressure and initial mass flow rate through the nozzle, first determine whether or not the flow is choked. Determine the pressure at the exit plane when the flow first becomes choked (*i.e.*, $\text{Ma}_T = 1$) by first determining the exit Mach number when the flow first becomes choked, then using this Mach number and the isentropic relations to determine the exit pressure ratio.

$$\frac{A_E}{A^*} = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_E^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_E = 0.372 \tag{3}$$

where $A_E/A^* = 1.6875$ from Eq. (1) (note that when the flow is choked, $A^* = A_T$). The pressure at the exit for this condition is found from the isentropic flow relation.

$$\frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_E/p_0 = 0.9088 \Rightarrow p_E = 91.8\text{ kPa (abs)} \tag{4}$$

where $p_0 = p_{\text{atm}} = 101$ kPa (abs). Since this exit pressure is larger than the initial tank pressure, the flow must be choked and the mass flow rate is then,

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \Rightarrow \boxed{\dot{m}_{\text{choked}} = 0.154\text{ kg/s}} \tag{5}$$

where p_0 = 101 kPa (abs), T_0 = 296 K, R = 287 J/(kg.K), γ = 1.4, and $A^* = A_T$ = 6.5 cm^2.

The design pressure for the nozzle is found using the isentropic relations and the design Mach number.

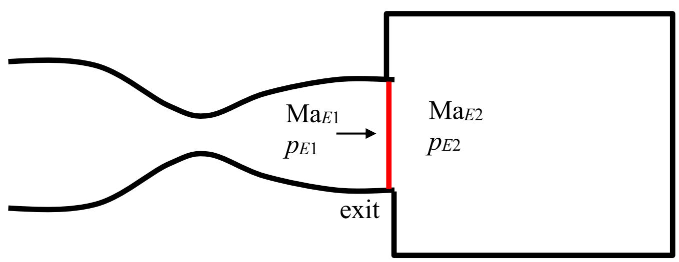
$$\frac{p_{E,d}}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_{E,d}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_{E,d}/p_0 = 0.1278 \Rightarrow p_{E,d} = 12.9\text{ kPa (abs)} \tag{6}$$

Since the exit pressure at design is larger than the initial tank pressure, the flow must be underexpanded and the exit pressure will be equal to the design exit pressure of $\boxed{p_{E,d} = 12.9\text{ kPa (abs)}}$.

The tank pressure at which a normal shock stands in the exit plane is found by using the design Mach number and exit pressure found in Eq. (6) just upstream of the shock, then applying the normal shock relations across the exit shock wave.

$$\text{Ma}_{E1} = 2.0 \Rightarrow p_{E2}/p_{E1} = 4.500 \text{ (from the normal shock relations)} \Rightarrow p_{E2} = 58.1\text{ kPa (abs)} \tag{7}$$

where p_{E1} = 12.9 kPa (abs) from Eq. (6). Since the flow just downstream of the shock is subsonic, the downstream exit pressure will equal the back pressure. Thus, the tank pressure at which a normal shock just stands at the exit is $\boxed{58.1\text{ kPa (abs)}}$.



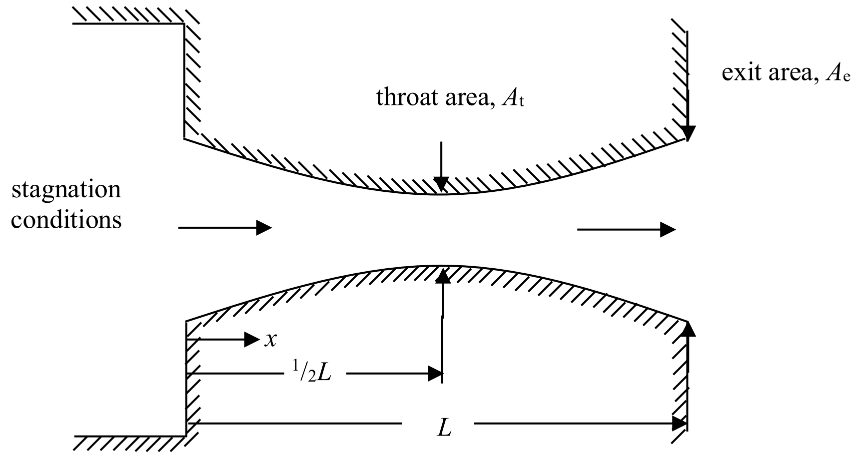
A converging-diverging nozzle, with an exit to throat area ratio, A_e/A_t, of 1.633, is designed to operate with atmospheric pressure at the exit plane, $p_e = p_{atm}$.

a. Determine the range(s) of stagnation pressures for which the nozzle will be free from normal shocks.

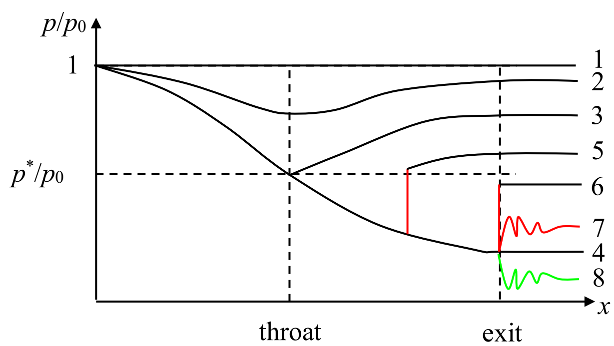
b. If the stagnation pressure is $1.5p_{atm}$, at what position, x, will the normal shock occur?

The converging-diverging nozzle area, A, varies with position, x, as:

$$\frac{A(x)}{A_E} = \left(\frac{A_E}{A_T} - 1\right)\left(2\frac{x}{L} - 1\right)^2 + 1$$



SOLUTION:



If there are no shocks, then the flow is assumed to remain isentropic. Determine the back pressure corresponding to isentropic sonic area ratio. Consider, for the moment, only the subsonic condition (case 3 shown in the figure above).

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = 1.633 = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_E^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_E = 0.39 \quad \text{(isentropic flow relations)} \tag{1}$$

$$\Rightarrow \frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} = 0.9016 \quad \text{(isentropic flow relations)} \tag{2}$$

Hence, for $p_{atm} \le p_0 \le p_{atm}/0.9016 = 1.11\, p_{atm}$ the flow throughout the nozzle will be subsonic and, as a result, there will be no shocks within the nozzle.

It's also possible to have isentropic flow within the nozzle, yet have a shock wave at the nozzle exit (case 6 in the figure). For the case when a normal shock wave is stationed at the nozzle exit:

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = 1.633 = \frac{1}{\mathrm{Ma}_{E1}}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_{E1}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_E = 1.96 \text{ (isentropic flow relations)} \tag{3}$$

$$\Rightarrow \frac{p_{E1}}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_{E1}^2\right)^{\frac{\gamma}{1-\gamma}} = 0.1359 \text{ (isentropic flow relations)} \tag{4}$$

$$\text{and } \frac{p_{E2}}{p_{E1}} = \frac{p_{\text{atm}}}{p_{E1}} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_{E1}^2 - \frac{\gamma-1}{\gamma+1} = 4.3152 \text{ (normal shock relations)} \tag{5}$$

Note that since downstream of the shock the flow is subsonic and at the exit, $p_{E2} = p_{\text{atm}}$.

Now determine the upstream stagnation pressure corresponding to the given conditions.

$$\frac{p_{E2}}{p_{01}} = \frac{p_{\text{atm}}}{p_{01}} = \left(\frac{p_{E2}}{p_{E1}}\right)\left(\frac{p_{E1}}{p_{01}}\right) = (4.3152)(0.1359) = 0.5864 \tag{6}$$

$$\therefore p_{01} = \frac{p_{\text{atm}}}{0.5864} = 1.7052 p_{\text{atm}} \tag{7}$$

Therefore, the device will not contain normal shocks for the following range of stagnation conditions:

$$\boxed{1 \le \frac{p_0}{p_{\text{atm}}} \le 1.11 \text{ and } \frac{p_0}{p_{\text{atm}}} > 1.71} \tag{8}$$

Based on Eq. (8), a normal shock will occur somewhere within the diverging portion of the nozzle if the stagnation pressure is $p_{01} = 1.5 p_{\text{atm}}$. Use an iterative approach to determine the location of the shock as given below.

a. Assume a location for the shock wave (e.g., pick a value for A/A_t since the geometry is known).
b. Determine the Mach number and pressure just upstream of the shock, Ma_1 and p_1, using the isentropic relations as discussed in Note 2.

$$\frac{A}{A_1^*} = \frac{A}{A_t} = \frac{1}{\mathrm{Ma}_1}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \text{ (where the supersonic } \mathrm{Ma}_1 \text{ is chosen)} \tag{9}$$

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \tag{10}$$

c. Calculate the stagnation pressure ratio and sonic area ratio across the shock using the normal shock relations:

$$\frac{p_{02}}{p_{01}} = \frac{A_1^*}{A_2^*} = \left[\frac{\frac{\gamma+1}{2}\mathrm{Ma}_1^2}{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}\right]^{\gamma/\gamma-1}\left[\frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{1/1-\gamma} \tag{11}$$

d. Determine the exit Mach number and exit pressure ratio using the isentropic relations and the downstream sonic area and stagnation pressure:

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_t}\frac{A_1^*}{A_2^*} = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \text{ (where the subsonic } \mathrm{Ma}_e \text{ is chosen)} \tag{12}$$

Note that since the flow is choked, the throat area is equal to the upstream sonic area, i.e., $A_t = A_1^*$.

$$\frac{p_e}{p_{02}} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \tag{13}$$

Note that since the exit Mach number is subsonic, the exit pressure will equal the back pressure, i.e., $p_e = p_b$.

e. Calculate the ratio of the back pressure to the upstream stagnation pressure:

$$\frac{p_b}{p_{01}} = \frac{p_e}{p_{02}}\frac{p_{02}}{p_{01}} \tag{14}$$

f. Check to see if the back pressure ratio calculated in step (e) matches with the given back pressure ratio. If so, then the assumed location of the shock is correct. If not, then the go back to step (a) and repeat. If the back pressure ratio calculated in part (e) is less than the given back pressure ratio, then the assumed shock location is too far upstream. If the back pressure ratio calculated in part (e) is greater than the given back pressure ratio, then the assumed shock location is too far downstream.

Apply this algorithm using the given data and summarize in the following table. (Note that the correct position is found manually in this case, but the method could easily be made into a computer program and the correct position could be found using an approach such as a bisection method.)

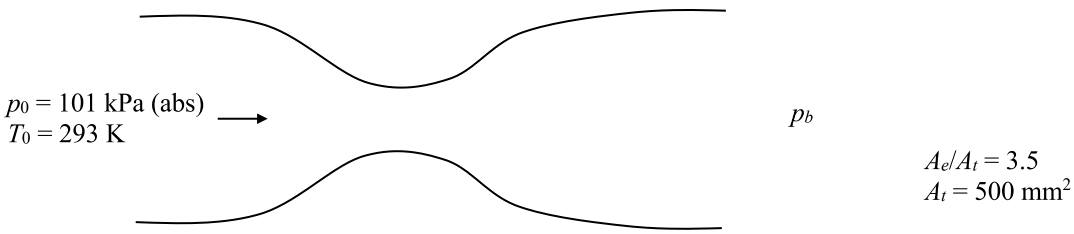
γ =	1.4
A_E/A_T =	1.633
p_{01}/p_{atm} =	1.5

x/L	**$A(x)/A_T$**	**Ma_1**	**p_1/p_{01}**	**p_{02}/p_{01}**	**A_1^*/A_2^***	**A_E/A_2^***	**Ma_E**	**p_E/p_{02}**	**p_B/p_{01}**	**p_{01}/p_B**	**comment**
0.8000	1.2279	1.5716	0.2454	0.9056	0.9056	1.4788	0.4382	0.8764	0.7937	1.2600	shock too far upstream
0.9000	1.4051	1.7681	0.1827	0.8267	0.8267	1.3500	0.4948	0.8459	0.6993	1.4299	shock too far upstream
0.9500	1.5127	1.8649	0.1575	0.7835	0.7835	1.2794	0.5343	0.8234	0.6451	1.5502	shock too far downstream
0.9250	1.4573	1.8167	0.1696	0.8052	0.8052	1.3150	0.5134	0.8354	0.6727	1.4865	shock too far upstream
0.9375	1.4846	1.8408	0.1635	0.7944	0.7944	1.2973	0.5235	0.8296	0.6591	1.5173	shock too far downstream
0.9313	1.4709	1.8288	0.1665	0.7998	0.7998	1.3061	0.5184	0.8326	0.6659	1.5017	shock too far downstream
0.9281	1.4641	1.8228	0.1681	0.8025	0.8025	1.3105	0.5159	0.8340	0.6693	1.4941	shock too far upstream
0.9297	1.4675	1.8258	0.1673	0.8012	0.8012	1.3083	0.5172	0.8333	0.6676	1.4980	shock too far upstream
0.9305	1.4692	1.8273	0.1669	0.8005	0.8005	1.3072	0.5178	0.8329	0.6667	1.4998	shock too far upstream
0.9309	1.4700	1.8280	0.1667	0.8002	0.8002	1.3067	0.5181	0.8327	0.6663	1.5007	shock too far downstream
0.9307	1.4696	1.8276	0.1668	0.8004	0.8004	1.3070	0.5179	0.8329	0.6666	1.5002	shock too far downstream
0.9306	**1.46947**	**1.82753**	**0.1669**	**0.8004**	**0.8004**	**1.30702**	**0.51791**	**0.8329**	**0.6666**	**1.5001**	**just about right!**

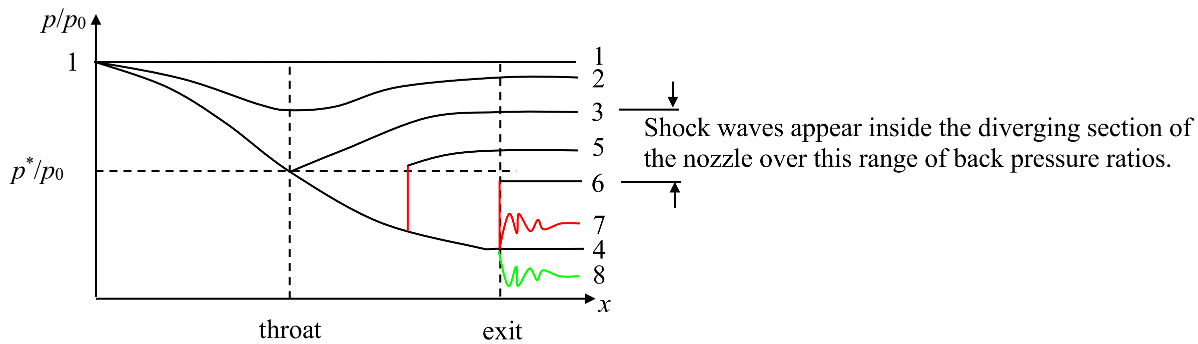
Thus, the shock is located at $\boxed{x/L = 0.9306}$.

Air flows through a converging-diverging nozzle, with $A_e/A_t = 3.5$ where $A_t = 500\ \text{mm}^2$. The upstream stagnation conditions are atmospheric; the back pressure is maintained by a vacuum system. Determine the range of back pressures for which a normal shock will occur within the nozzle and the corresponding mass flow rate.

SOLUTION:



A shock wave will appear within the nozzle for the range of back pressures indicated in the figure shown below.



The back pressure ratio corresponding to case 3 may be found from the isentropic relations:

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_e = 0.168 \tag{1}$$

(Note that for case 3, $A_t = A^*$ since the flow is choked.)

$$\frac{p_b}{p_0} = \frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_0 = 0.980 \tag{2}$$

(Note that since the flow is subsonic at the exit, $p_e = p_b$.)

The back pressure ratio corresponding to case 6 may be found by combining the isentropic relations with the normal shock relations.

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\text{Ma}_{e1}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_{e1}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_{e1} = 2.80 \tag{3}$$

$$\text{Ma}_{e2}^2 = \frac{(\gamma-1)\text{Ma}_{e1}^2+2}{2\gamma\text{Ma}_{e1}^2-(\gamma-1)} \Rightarrow \text{Ma}_{e2} = 0.488 \tag{4}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\text{Ma}_{e1}^2}{2+(\gamma-1)\text{Ma}_{e1}^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_{e1}^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \Rightarrow p_{02}/p_{01} = 0.389 \tag{5}$$

$$\frac{p_b}{p_{02}} = \frac{p_{e2}}{p_{02}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_{e2}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_{02} = 0.850 \tag{6}$$

$$\frac{p_b}{p_{01}} = \left(\frac{p_b}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right) \Rightarrow p_b/p_{01} = 0.331 \tag{7}$$

Thus, a normal shock wave will appear in the diverging portion of the converging-diverging nozzle over the range:

$$\boxed{0.331 < p_b/p_0 < 0.980} \tag{8}$$

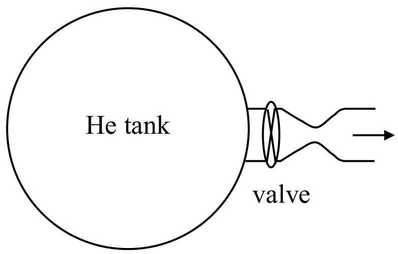
where p_0 = 1 atm = 101 kPa (abs).

The mass flow rate when the flow is choked is:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \Rightarrow \boxed{\dot{m}_{\text{choked}} = 0.119\ \text{kg/s}} \tag{9}$$

where $\gamma = 1.4$, p_0 = 101 kPa, R = 287 J/(kg·K), T_0 = 293 K, and $A^* = A_t$ = 500 mm^2.

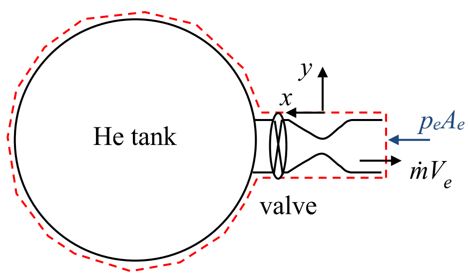
A satellite includes a correctional propulsive unit consisting of a tank that is 3 ft^3 in volume and contains helium initially at 2000 psia. Heaters on the satellite maintain the tank temperature at 0 °F. The tank is connected to a short, insulated, convergent-divergent nozzle having a throat area of 1 in^2 and an exit area of 3 in^2. The mass of the satellite, exclusive of the helium, is 50 lb_m. Plot the acceleration of the satellite as a function of time if the valve is left open.



SOLUTION:

The pressure in space is nearly zero so the flow from the nozzle will always be underexpanded.

Apply the LME in the x-direction to the CV shown below. Use a frame of reference fixed to the satellite.



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{CV} a_x \rho dV \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(The velocity of He in tank is zero in the given FOR.)} \tag{2}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}V_e \tag{3}$$

$$F_{B,x} = 0 \tag{4}$$

$$F_{S,x} = p_e A_e \tag{5}$$

$$\int_{CV} a_x \rho dV = M_{\text{sat}} a_{\text{sat}} \tag{6}$$

Substitute and simplify.

$$M_{\text{sat}} a_{\text{sat}} = \dot{m}V_e + p_e A_e \tag{7}$$

Since the flow within the nozzle will always be choked and isentropic (the back pressure is zero), the mass flow rate is:

$$\dot{m} = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \quad \text{where } A^* = A_{\text{throat}} \tag{8}$$

The exit velocity and pressure may be expressed in terms of the exit Mach number:

$$V_e = \mathrm{Ma}_e\left(\frac{c_e}{c_0}\right)c_0 = \mathrm{Ma}_e\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{-1/2}\sqrt{\gamma R T_0} \tag{9}$$

$$p_e = p_0\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \tag{10}$$

Substitute and simplify.

$$M_{\text{sat}}a_{\text{sat}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\gamma p_0 A_t \mathrm{Ma}_e\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{-1/2} + p_0\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}}A_e \tag{11}$$

The Mach number at the exit is found using the given area ratio:

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{3\text{ in.}^2}{1\text{ in.}^2} = 3.0 = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_e = 3.0 \quad (\text{Note: } \gamma_{\text{He}} = 1.66) \tag{12}$$

Using the given data:

p_0 = 2000 psia
A_t = 1 in^2
A_e = 3 in^2
Ma$_e$ = 3.0
γ_{He} = 1.66
R_{He} = 386.1 (ft.lb$_f$)/(lb$_m$.°R)
V_{tank} = 3 ft^3
T_0 = 460 °R

$$\Rightarrow\ M_{\text{sat}}(t=0) = 50\text{ lb}_\text{m} + \underbrace{\left(\frac{p_0 V_{\text{tank}}}{RT_0}\right)}_{\text{initial mass of He}} = 54.9\text{ lb}_\text{m} \tag{13}$$

$$\Rightarrow\ a_{\text{sat}}(t=0) = 54.7\text{ ft/s}^2 = 1.7g \quad (\text{where } g \text{ is } 32.2\text{ ft/s}^2) \tag{14}$$

Note that the pressure within the tank will decrease with time as helium is discharged from the tank (the tank temperature remains constant due to the heaters).

$$p_0 = \rho_0 R T_0 = \left(\frac{M_{\text{He}}}{V_{\text{tank}}}\right)RT_0 \tag{15}$$

$$\frac{dp_0}{dt} = \frac{dM_{\text{He}}}{dt}\left(\frac{RT_0}{V_{\text{tank}}}\right) = -\dot{m}\left(\frac{RT_0}{V_{\text{tank}}}\right) \quad \text{from conservation of mass} \tag{16}$$

Substitute Eqn. (8) and simplify.

$$\frac{dp_0}{dt} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}p_0\sqrt{\frac{\gamma}{RT_0}}A_t\left(\frac{RT_0}{V_{\text{tank}}}\right) = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\frac{p_0\sqrt{\gamma RT_0}A_t}{V_{\text{tank}}} \tag{17}$$

$$\int_{p_0(t=0)}^{p_0(t)}\frac{dp_0}{p_0} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\frac{\sqrt{\gamma RT_0}A_t}{V_{\text{tank}}}\int_{t=0}^{t=t}dt \tag{18}$$

$$p_0 = p_{0,t=0}\exp\left[-\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\frac{\sqrt{\gamma RT_0}A_t t}{V_{\text{tank}}}\right] \tag{19}$$

Substitute Eq. (19) into Eq. (13) to determine how the satellite mass changes with time.

$$M_{\text{sat}}(t) = 50 \text{ lb}_{\text{m}} + \frac{p_{0,t=0} V_{\text{tank}}}{RT_0} \exp\left[-\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} \frac{\sqrt{\gamma RT_0} A_t t}{V_{\text{tank}}} \right] \tag{20}$$

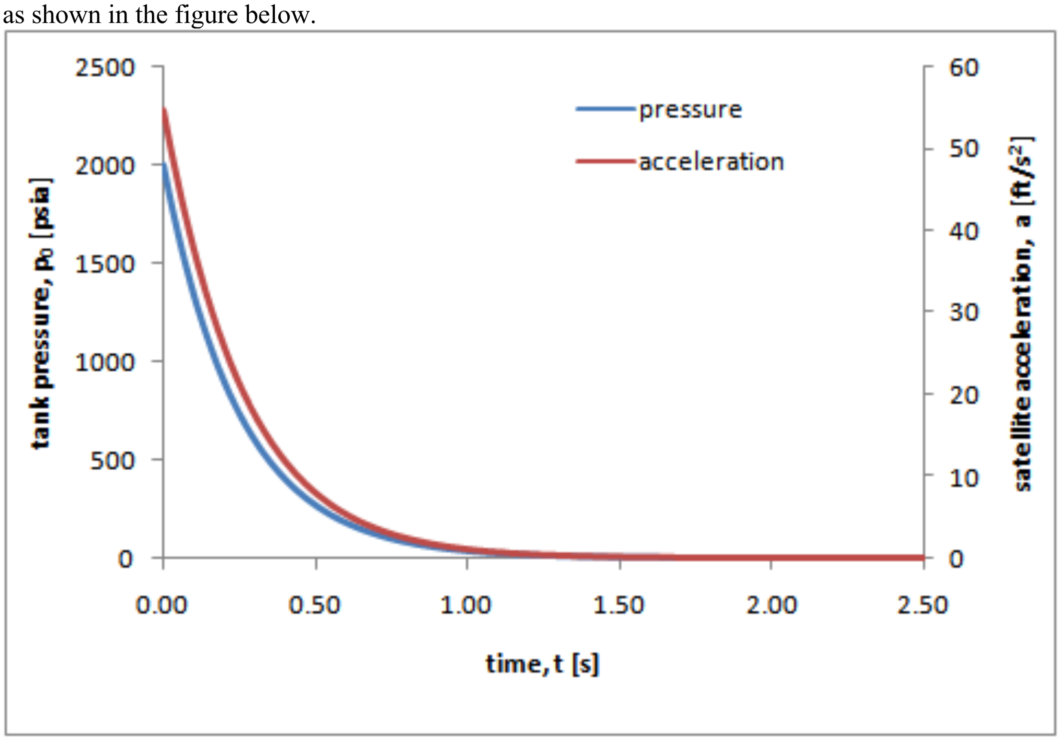
Summarizing:

$$p_0 = p_{0,t=0} \exp\left[-\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} \frac{\sqrt{\gamma RT_0} A_t t}{V_{\text{tank}}} \right] \tag{21}$$

$$M_{\text{sat}} = 50 \text{ lb}_{\text{m}} + \frac{p_0 V_{\text{tank}}}{RT_0} \tag{22}$$

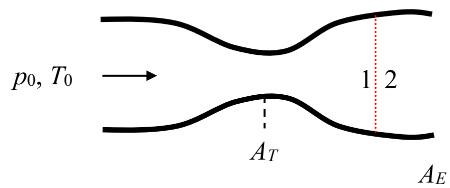
$$M_{\text{sat}} a_{\text{sat}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} \gamma p_0 A_t \text{Ma}_e \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{-1/2} + p_0 \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} A_e \tag{23}$$

Using the given initial data, the satellite tank pressure and acceleration may be plotted as a function of time as shown in the figure below.



Air flows through a frictionless, adiabatic converging-diverging nozzle. The air in the reservoir feeding the nozzle has a pressure and temperature of 700 kPa (abs) and 500 K, respectively. The ratio of the nozzle exit to throat area is 11.91. A normal shock wave stands where the upstream Mach number is 3.0. Calculate the Mach number, the static temperature, and static pressure at the nozzle exit plane.

SOLUTION:



p_0 = 700 kPa (abs)
T_0 = 500 K
A_E/A_T = 11.91
Ma_1 = 3.0

Using the normal shock relations:

$$\mathrm{Ma}_1 = 3.0 \Rightarrow \quad \mathrm{Ma}_2 = 0.4752 \tag{1}$$
$$T_{02}/T_{01} = 1 \tag{2}$$
$$p_{02}/p_{01} = 0.3283 \tag{3}$$

The flow is isentropic from the reservoir to just upstream of the shock (location 1) so that:

$$p_{01} = p_0 \tag{4}$$
$$T_{01} = T_0 \tag{5}$$
$$A_1/A_1^* = 4.2346 \text{ (using } \mathrm{Ma}_1 = 3.0) \tag{6}$$

The flow is also isentropic from just downstream of the shock (location 2) to the exit so that:

$$p_{0E} = p_{02} \tag{7}$$
$$T_{0E} = T_{02} \tag{8}$$
$$A_2/A_2^* = 1.390 \text{ (using } \mathrm{Ma}_2 = 0.4752) \tag{9}$$

Combine the previous equations to get the exit stagnation conditions.

$$p_{0E} = p_{02} = \left(\frac{p_{02}}{p_{01}}\right)p_{01} = \left(\frac{p_{02}}{p_{01}}\right)p_0 = (0.3283)(700 \text{ kPa}) = 229.8 \text{ kPa} \tag{10}$$

$$T_{0E} = T_{02} = \left(\frac{T_{02}}{T_{01}}\right)T_{01} = \left(\frac{T_{02}}{T_{01}}\right)T_0 = (1)(500 \text{ K}) = 500 \text{ K} \tag{11}$$

Now determine the exit sonic area ratio (A_E/A^*) so that it can be used to determine the exit Mach number.

$$\frac{A_E}{A_2^*} = \left(\frac{A_E}{A_T}\right)\left(\frac{A_1^*}{A_1}\right)\left(\frac{A_2}{A_2^*}\right) = (11.91)\left(\frac{1}{4.2346}\right)(1.390) = 3.9094 \quad \text{(Note that } A_T = A_1^*.) \tag{12}$$

Use this area ratio and the isentropic flow sonic area relation to determine the exit Mach number. Note that the exit Mach number will be subsonic since the flow downstream of the shock wave is subsonic.

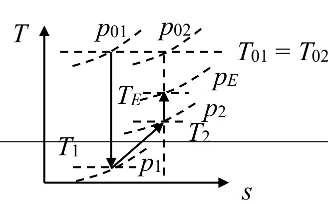
$$\frac{A_E}{A_2^*} = 3.9094 \Rightarrow \boxed{\mathrm{Ma}_E = 0.15} \tag{13}$$

Use the isentropic flow relations with the exit Mach number to determine the stagnation temperature and pressure ratios.

$$\frac{T_E}{T_{0E}} = 0.9955 \quad \text{and} \quad \frac{p_E}{p_{0E}} = 0.9844 \tag{14}$$

Combine Eqns. (14) with Eqs. (10) and (11) to determine the exit static temperature and pressure.

$\boxed{T_E = 498 \text{ K}}$ and $\boxed{p_E = 226 \text{ kPa (abs)}}$



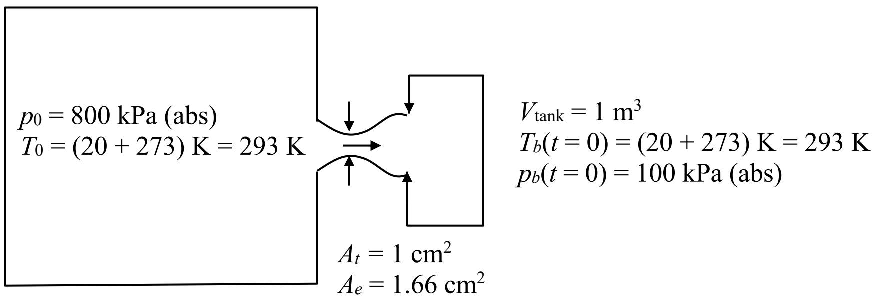
A large reservoir at 20 °C and 800 kPa (abs) is used to fill a small tank through a converging-diverging nozzle with 1 cm^2 throat area and 1.66 cm^2 exit area. The small tank has a volume of 1 m^3 and is initially at 20 °C and 100 kPa (abs). Estimate the elapsed time when:

a. shock waves begin to appear inside the nozzle, and
b. the mass flow rate begins to drop below its maximum value.

You may assume that the tank filling process occurs isothermally.

c. Describe (but you need not work out) how your solution approach would change if the tank is well insulated so that the filling process occurs adiabatically.

SOLUTION:



First check to see where the flow is on the diagram below. At $t = 0$:

$$p_b/p_0 = (100\text{ kPa})/(800\text{ kPa}) = 0.125 \tag{1}$$

The back pressure ratios corresponding to cases 3 (onset of choked flow) and 4 (design conditions) – refer to the plot below – may be found from the isentropic relations.

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_e = 0.3796,\ \mathrm{Ma}_e = 1.9802 \tag{2}$$

using A_e = 1.66 cm^2 and $A^* = A_t$ = 1 cm^2 ($\Rightarrow A_e/A^* = 1.66$).

$$\frac{p_b}{p_0} = \frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_0 = 0.9053,\ 0.1318 \tag{3}$$

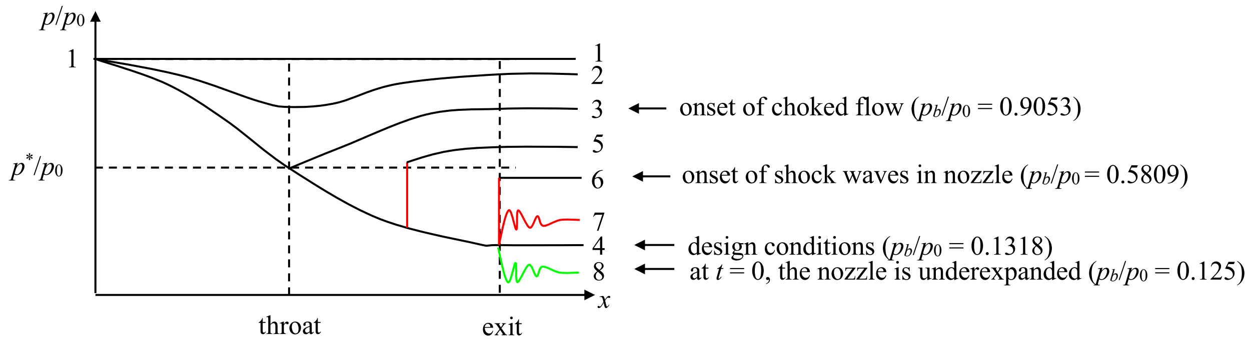
The back pressure ratio when a normal shock wave stands at the nozzle exit may be found by combining the isentropic and normal shock wave relations.

$$\mathrm{Ma}_{e2}^2 = \frac{(\gamma-1)\mathrm{Ma}_{e1}^2+2}{2\gamma\mathrm{Ma}_{e1}^2-(\gamma-1)} \Rightarrow \mathrm{Ma}_{e2} = 0.5808\ \ \text{(using } \mathrm{Ma}_{e1} = 1.9802 \text{ – from Eq. (2))} \tag{4}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\mathrm{Ma}_{e1}^2}{2+(\gamma-1)\mathrm{Ma}_{e1}^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\mathrm{Ma}_{e1}^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \Rightarrow p_{02}/p_{01} = 0.7301 \tag{5}$$

$$\frac{p_b}{p_{02}} = \frac{p_{e2}}{p_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_{e2}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_{02} = 0.7957 \tag{6}$$

$$\frac{p_b}{p_{01}} = \left(\frac{p_b}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right) \Rightarrow p_b/p_{01} = 0.5809 \tag{7}$$



The mass flow rate into the tank will be choked until $p_b/p_0 \geq 0.9053$. The choked mass flow rate into the tank is given by:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \;\Rightarrow\; \dot{m}_{\text{choked}} = 0.1889\ \text{kg/s} \tag{8}$$

The (back) pressure in the tank may be found by applying conservation of mass to a control volume surrounding the tank and making use of the ideal gas law.

$$M_{\text{tank}} = \rho_b V_{\text{tank}} = \frac{p_b V_{\text{tank}}}{RT_b} = \dot{m}t + M_0 \;\Rightarrow\; p_b = \frac{RT_b}{V_{\text{tank}}}\left(\dot{m}t + M_0\right) \tag{9}$$

Note that the mass flow rate into the tank is the choked mass flow rate (Eq. (8), which remains constant up until case 3 is reached), and M_0 is the mass inside the tank at $t = 0$.

$$M_0 = M_{\text{tank},t=0} = \left(\frac{p_b V_{\text{tank}}}{RT_b}\right)_{t=0} \;\Rightarrow\; \underline{M_0 = 1.189\ \text{kg}} \tag{10}$$

where $p_{b,t=0} = 100$ kPa, $T_{b,\,t=0} = 293$ K, $R = 287$ J/(kg·K), and $V_{\text{tank}} = 1\ \text{m}^3$.

Thus, the time for the onset of shock waves in the nozzle (case 6) is:

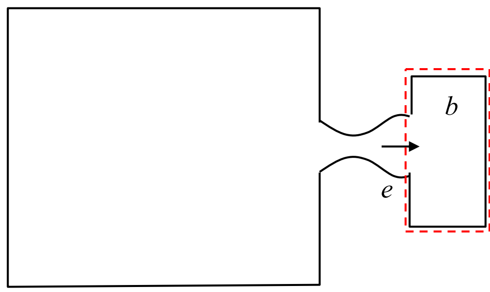
$$\frac{p_b}{p_0} = 0.5809 = \frac{RT_b}{p_0 V_{\text{tank}}}\left(\dot{m}t_{\text{shocks}} + M_0\right) \tag{11}$$

$$t_{\text{shocks}} = \frac{1}{\dot{m}}\left[0.5809\left(\frac{p_0 V_{\text{tank}}}{RT_b}\right) - M_0\right] \;\Rightarrow\; \boxed{t_{\text{shocks}} = 23.0\ \text{s}} \tag{12}$$

The time for when the flow is no longer choked (case 3) is:

$$t_{\text{unchoked}} = \frac{1}{\dot{m}}\left[0.9053\left(\frac{p_0 V_{\text{tank}}}{RT_b}\right) - M_0\right] \;\Rightarrow\; \boxed{t_{\text{unchoked}} = 39.3\ \text{s}} \tag{13}$$

If we assume that the tank fills adiabatically (likely a more realistic scenario), then the calculations become much more involved since the temperature in the tank will also vary as the pressure varies. The (back) pressure in the tank will increase as additional mass enters the tank. We can determine how the pressure varies by applying conservation of energy and conservation of mass to a control volume surrounding the tank as shown below.



Applying conservation of energy to a control volume surrounding the tank gives:

$$\frac{d}{dt}\left(M_{\text{tank}} c_V T_b\right) - \dot{m}_e\left(c_P T_e + \tfrac{1}{2}V_e^2\right) = 0 \tag{14}$$

$$c_V\left(T_b \frac{dM_{\text{tank}}}{dt} + M_{\text{tank}}\frac{dT_b}{dt}\right) - \dot{m}_e\left(c_P T_e + \tfrac{1}{2}V_e^2\right) = 0 \tag{15}$$

$$c_V\left[\dot{m}_e T_b + \left(\dot{m}_e t + M_0\right)\frac{dT_b}{dt}\right] - \dot{m}_e\left(c_P T_e + \tfrac{1}{2}V_e^2\right) = 0 \quad \text{(where conservation of mass has been used)} \tag{16}$$

$$c_V\left(\dot{m}_e t + M_0\right)\frac{dT_b}{dt} + \dot{m}_e c_V T_b = \dot{m}_e\left(c_P T_e + \tfrac{1}{2}V_e^2\right) \tag{17}$$

where the mass flow rate entering the tank is given by Eq. (8) (the flow is choked for the conditions we're interested in so that mass flow rate will remain constant). Note that we are assuming that the tank is well insulated indicating that the filling process occurs adiabatically ($\dot{Q}_{\text{into tank}} = 0$). The temperature and velocity of the air entering the tank (T_e and V_e) may be found following an approach similar to the ones used previously to determine the exit pressure. For back pressures less than the value corresponding to case 6 (normal shock at the exit plane), the exit temperature and velocity are given by:

$$\frac{T_e}{T_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_e^2\right)^{-1} \Rightarrow T_e/T_0 = 0.5605 \Rightarrow \underline{T_e = 164.2\text{ K}} \tag{18}$$

$$V_e = \text{Ma}_e c_e = \text{Ma}_e\sqrt{\gamma R T_e} \Rightarrow \underline{V_e = 508.7\text{ m/s}} \tag{19}$$

The (back) pressure in the tank may be found by applying conservation of mass to the same control volume and making use of the ideal gas law.

$$M_{\text{tank}} = \rho_b V_{\text{tank}} = \frac{p_b V_{\text{tank}}}{R T_b} = \dot{m}t + M_0 \Rightarrow p_b = \frac{R T_b}{V_{\text{tank}}}\left(\dot{m}t + M_0\right) \tag{20}$$

where T_b is found from the (numerical) solution of Eq. (17). Note that the mass flow rate into the tank is the choked mass flow rate (Eqn. (8), which remains constant up until case 3 is reached), and M_0 is the mass inside the tank at $t = 0$ (Eq. (10)).

When $p_b/p_0 \geq 0.5806$ (corresponding to case 6), then the temperature and velocity of the air entering the tank (T_e and V_e) must be found by taking into consideration a normal shock wave located somewhere within the diverging portion of the nozzle. The exit temperature and velocity will depend upon the location of the shock wave, which in turn will depend upon the back pressure. Hence, the shock finding algorithm described in the course notes (it won't be repeated here) must be used for a given back pressure to determine the location of the normal shock wave. Once this location has been found (and hence, Ma_1 is known) , the exit temperature and velocity may be found by combining the isentropic and normal shock relations:

$$\frac{T_e}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_e^2\right)^{-1} \quad \text{(Note that } T_0 = T_{01} = T_{02}\text{)} \tag{21}$$

$$V_e = \mathrm{Ma}_e c_e = \mathrm{Ma}_e \sqrt{\gamma R T_e} \tag{22}$$

where the exit Mach number is found from the area ratio:

$$\frac{A_e}{A_2^*} = \left(\frac{A_e}{A_1^*}\right)\left(\frac{A_1^*}{A_2^*}\right) = \left(\frac{A_e}{A_t}\right)\left(\frac{A_1^*}{A_2^*}\right) = \frac{1}{\mathrm{Ma}_e}\left(\frac{1 + \frac{\gamma - 1}{2}\mathrm{Ma}_e^2}{1 + \frac{\gamma - 1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{23}$$

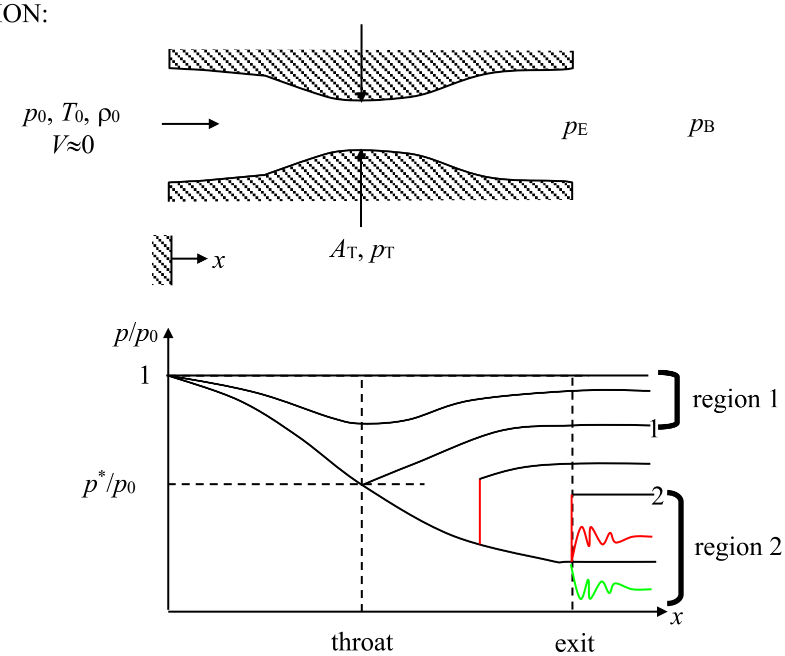
where

$$\frac{A_1^*}{A_2^*} = \left[\frac{(\gamma + 1)\mathrm{Ma}_1^2}{2 + (\gamma - 1)\mathrm{Ma}_1^2}\right]^{\gamma/(\gamma-1)}\left[\frac{\gamma + 1}{2\gamma \mathrm{Ma}_1^2 - (\gamma - 1)}\right]^{1/(\gamma-1)} \tag{24}$$

where Ma_1 is the Mach number just upstream of the shock wave. With the calculated T_e and V_e, Eqn. (17) may be solved numerically simultaneously with Eq. (20) so that p_b may be determined.

A converging-diverging nozzle, with $A_e/A_t = 1.633$, is designed to operate with atmospheric pressure at the exit plane. Determine the range(s) of stagnation pressures for which the nozzle will be free from normal shocks.

SOLUTION:



There will be two ranges of back pressures that will not produce shock waves *within* the C-D nozzle. In region 1 shown above, the entire flow remains subsonic (with possible sonic flow at the throat). In region 2 the flow is subsonic in the converging section, sonic at the throat, then subsonic throughout the diverging section. Shock waves and expansion fans may occur *outside* of the C-D nozzle in region 2.

Consider pressure curve 1 indicated in the figure above. For this case the exit Mach number is given by:

$$\frac{A_E}{A_T} = \frac{A_E}{A^*} = 1.633 = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{1}$$

Solve for the subsonic exit Mach number to get:
$\mathrm{Ma}_E = 0.387$

Now use the isentropic stagnation pressure ratio to determine the reservoir stagnation pressure for these conditions.

$$\frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_0 = 1.11 p_E = 112\text{ kPa} \ \ (\text{where } p_E = p_{atm} = 101\text{ kPa}) \tag{2}$$

Hence, the nozzle will be shock free for:

$\boxed{p_{atm} \le p_0 \le 1.11 p_{atm} \ \text{ or } \ 101\text{ kPa} \le p_0 \le 112\text{ kPa}}$

Now consider pressure curve 2 indicated in the figure above. For this case a normal shock wave occurs at the nozzle exit plane. Just upstream of the shock wave the Mach number can be found using the sonic area ratio.

$$\frac{A_{E1}}{A^*} = \frac{A_E}{A_T} = 1.633 \quad \Rightarrow \quad \text{Ma}_{E1} = 1.96 \text{ (using the isentropic flow relations)}$$

$\Rightarrow$ $p_{E2}/p_{E1} = 4.3152$ (using the normal shock relations with $\text{Ma}_{E1} = 1.96$)
$\Rightarrow$ $p_{E1}/p_{01} = 0.1359$ (using the isentropic flow relations with $\text{Ma}_{E1} = 1.96$)

Now solve for p_{E2}/p_{01}.

$$\frac{p_{E2}}{p_{01}} = \left(\frac{p_{E2}}{p_{E1}}\right)\left(\frac{p_{E1}}{p_{01}}\right) = (4.3152)(0.1359) = 0.5864$$

Note that $p_{01} = p_0$ (the reservoir pressure) and $p_{E2} = p_{\text{atm}}$ (since the flow downstream of the shock is subsonic).

$\Rightarrow$ $p_0 = 1.7052\, p_{\text{atm}}$

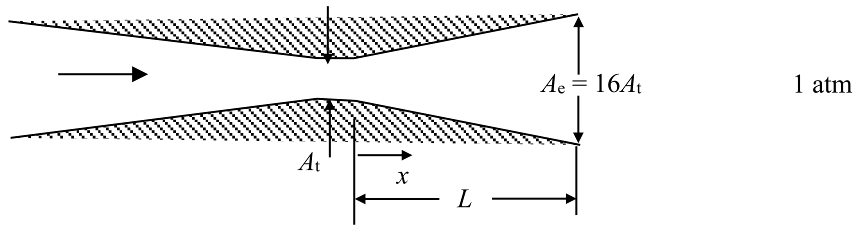
Thus, normal shocks will not form in the C-D nozzle when:

$\boxed{p_0 > 1.71\, p_{\text{atm}} \text{ or } p_0 > 172 \text{ kPa}}$

To summarize, the C-D nozzle will remain shock-free for the following range of stagnation pressures:

$\boxed{p_{\text{atm}} \le p_0 \le 1.11 p_{\text{atm}} \text{ and } p_0 > 1.71\, p_{\text{atm}}}$

A crude converging-diverging nozzle with an exit-to-throat area ratio of $A_e/A_t = 16$ is built using a straight-sided conical diffuser as shown in the figure below.



The nozzle is supplied by an air reservoir of pressure, p_{res}, and temperature, T_{res}. The nozzle discharges into atmospheric conditions (p_{atm} = 1 atm).

a. If a shock wave forms half-way along the diffuser, i.e., $x/L = 0.5$, determine the reservoir pressure, p_{res}.
b. Determine over what range of reservoir pressures the flow will be choked.

SOLUTION:

First determine the area in the straight-sided nozzle as a function of position in the nozzle.

$$r = \left(r_e - r_t\right)\left(\frac{x}{L}\right) + r_t \tag{1}$$

$$A = \pi r^2 \tag{2}$$

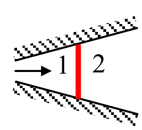
$$\frac{A}{A_t} = \left(\frac{r}{r_t}\right)^2 = \left[\left(\frac{r_e}{r_t} - 1\right)\left(\frac{x}{L}\right) + 1\right]^2 \quad \text{where} \quad \frac{r_e}{r_t} = \sqrt{\frac{A_e}{A_t}} \tag{3}$$

$$\therefore \frac{A}{A_t} = \left[\left(\sqrt{\frac{A_e}{A_t}} - 1\right)\left(\frac{x}{L}\right) + 1\right]^2 \tag{4}$$

For $x/L = 1/2$ and $A_e/A_t = 16$, $A/A_t = 6.25$. (5)

Using the isentropic flow relations (or tables) for air ($\gamma = 1.4$) and noting that the throat is also the sonic area since there is a shock wave in the diverging section:

$$\frac{A_1}{A^*} = 6.25 \Rightarrow \text{Ma}_1 = 3.411 \text{ and } \frac{p_1}{p_{01}} = 0.0149 \tag{6}$$



Using the normal shock relations (or tables) for air:

$$\text{Ma}_1 = 3.411 \Rightarrow \text{Ma}_2 = 0.4547, \frac{p_{02}}{p_{01}} = 0.2300, \frac{A_2^*}{A_1^*} = 4.3474 \tag{7}$$

Now determine the sonic area ratio at the exit, downstream of the shock wave.

$$\frac{A_e}{A_2^*} = \left(\frac{A_e}{A_1^*}\right)\left(\frac{A_1^*}{A_2^*}\right) = \left(\frac{16}{1}\right)\left(\frac{1}{4.3474}\right) = 3.6804 \tag{8}$$

Using the isentropic flow relations (or tables) for air:

$$\frac{A_e}{A_2^*} = 3.6804 \Rightarrow \text{Ma}_e = 0.1597, \frac{p_e}{p_{02}} = 0.9824 \tag{9}$$

Now determine the upstream stagnation pressure using the pressure ratios. Note that $p_e = p_{atm} = 1$ atm since the exit Mach number is subsonic.

$$p_{01} = \left(\frac{p_{01}}{p_{02}}\right)\left(\frac{p_{02}}{p_e}\right)p_e = \left(\frac{1}{0.2300}\right)\left(\frac{1}{0.9824}\right)(1\text{ atm}) \tag{10}$$

$$\boxed{\therefore p_{01} = 4.43\text{ atm}} \tag{11}$$

For a flow that just becomes choked:

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = 16 \Rightarrow \text{Ma}_e = 0.0362, \frac{p_e}{p_0} = 0.9991 \tag{12}$$

$$p_0 = \left(\frac{p_0}{p_e}\right)p_e = \left(\frac{1}{0.9991}\right)(1\text{ atm}) \tag{13}$$

$$\therefore p_0 = 1.001\text{ atm} \tag{14}$$

Therefore, $\boxed{\text{the flow will be choked for } p_0 \geq 1.001\text{ atm}}$.

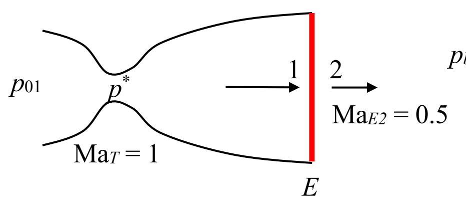
For the purposes of an experiment, we wish to design a de Laval nozzle which will be supplied from a compressed air reservoir (specific heat ratio of 1.4). It is required that:

1. there is a normal shock across the exit of the diffuser, and
2. the jet emerging downstream of the shock should have a Mach number of 0.5.

Find:

a. the ratio of the cross-sectional area at the diffuser exit to the cross-sectional area of the throat,
b. the ratio of the ambient pressure downstream of the shock to the pressure in the compressed air reservoir, and
c. the ratio of the ambient pressure downstream of the shock to the throat pressure.

SOLUTION:



The Mach number just upstream of the shock wave at the exit may be found using the normal shock relations,

$$\text{Ma}_{E2}^2 = \frac{(k-1)\text{Ma}_{E1}^2 + 2}{2k\text{Ma}_{E1}^2 - (k-1)} \Rightarrow \underline{\text{Ma}_{E1} = 2.6457} \tag{1}$$

The ratio of the cross-sectional area at the diffuser exit to the cross-sectional area of the throat may be found using the isentropic sonic area ratio and the Mach number just upstream of the shock,

$$\frac{A_E}{A_T} = \frac{A_E}{A^*} = \frac{1}{\text{Ma}_{E1}}\left(\frac{1+\frac{k-1}{2}\text{Ma}_{E1}^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \boxed{A_E/A_T = 3.0236} \tag{2}$$

Note that since the flow at the exit is supersonic, the throat must be at a sonic Mach number.

The pressure ratio, p_b/p_{01}, is given by,

$$\frac{p_b}{p_{01}} = \left(\frac{p_b}{p_{E2}}\right)\left(\frac{p_{E2}}{p_{E1}}\right)\left(\frac{p_{E1}}{p_{01}}\right) \Rightarrow \boxed{p_b/p_{01} = 0.3736} \tag{3}$$

where

$$\frac{p_b}{p_{E2}} = 1 \text{ (since Ma}_{E2} < 1) \tag{4}$$

$$\frac{p_{E2}}{p_{E1}} = \frac{2k}{k+1}\text{Ma}_{E1}^2 - \frac{k-1}{k+1} \Rightarrow p_{E2}/p_{E1} = 7.9997 \text{ (normal shock relations)} \tag{5}$$

$$\frac{p_{E1}}{p_{01}} = \left(1+\frac{k-1}{2}\text{Ma}_{E1}^2\right)^{\frac{k}{1-k}} \Rightarrow p_{E1}/p_{01} = 0.0467 \text{ (isentropic stagnation pressure ratio)} \tag{6}$$

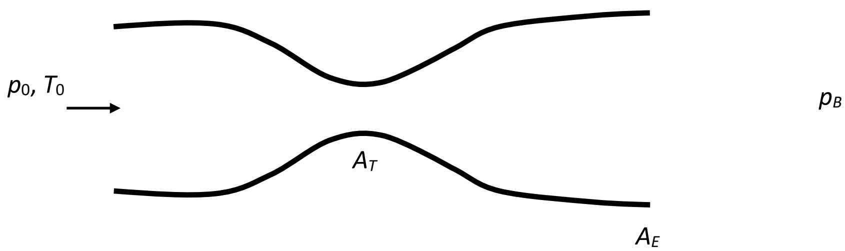
The pressure ratio, p_b/p^*, is given by,

$$\frac{p_b}{p^*} = \left(\frac{p_b}{p_{01}}\right)\left(\frac{p_{01}}{p^*}\right) \Rightarrow \boxed{p_b/p^* = 0.7071} \tag{7}$$

where

$$\frac{p^*}{p_{01}} = \left(1+\frac{k-1}{2}\right)^{\frac{k}{1-k}} \Rightarrow p^*/p_{01} = 0.5283 \text{ (isentropic stagnation pressure ratio)} \tag{8}$$

Consider the flow of air through the converging-diverging nozzle shown in the figure below. The flow begins at stagnation conditions with p_0 = 100 kPa (abs) and T_0 = 300 K. The nozzle exit-to-throat area ratio is A_E/A_T = 1.688 with a throat area of $A_T = 1.0*10^{-4}$ m^2.



a. Determine the back pressure at which the flow first becomes choked.
b. Determine the *range* of back pressures at which the flow at the exit is supersonic.
c. Determine the mass flow rate through the nozzle when the exit Mach number is 0.2.

SOLUTION:
The flow first becomes choked when the Mach number at the throat is equal to one ($A_T = A^*$) then goes back to subsonic conditions. The area ratio for this case is

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = 1.688 = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k-1}{2(k+1)}} \underset{\mathrm{Ma}_E<1}{\Rightarrow} \mathrm{Ma}_E = 0.3721\,. \tag{1}$$

where k = 1.4. Since the flow is entirely isentropic, the back pressure ratio corresponding to this Mach number may be found using,

$$\frac{p_E}{p_0} = \left(1+\tfrac{k-1}{2}\mathrm{Ma}_E^2\right)^{\frac{k}{1-k}} \Rightarrow p_E/p_0 = 0.9088 \tag{2}$$

Since the exit Mach number is subsonic, the exit pressure and back pressure are the same. Hence,

$$p_B = p_E. \tag{3}$$

Using the given inlet stagnation pressure and Eqs. (2) and (3),

$$\boxed{p_B = 90.8 \text{ kPa (abs)}} \tag{4}$$

The flow at the exit will be supersonic for back pressures less than the case when a normal shock wave stands at the nozzle exit. The back pressure at which a normal shock stands at the exit may be found by noting that the flow upstream of the exit will be entirely isentropic (and choked), with the Mach number just upstream of the shock at the exit being supersonic. Hence,

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = 1.688 = \frac{1}{\mathrm{Ma}_{E1}}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_{E1}^2}{1+\frac{k-1}{2}}\right)^{\frac{k-1}{2(k+1)}} \underset{\mathrm{Ma}_{E1}>1}{\Rightarrow} \mathrm{Ma}_{E1} = 2.000\,. \tag{5}$$

The pressure ratio just upstream of the shock at the exit may be found from the isentropic relations,

$$\frac{p_{E1}}{p_{01}} = \left(1+\tfrac{k-1}{2}\mathrm{Ma}_{E1}^2\right)^{\frac{k}{1-k}} \Rightarrow p_{E1}/p_{01} = 0.1278 \Rightarrow p_{E1} = 12.78 \text{ kPa (abs)} \tag{6}$$

where p_{01} is the upstream stagnation pressure (the stagnation pressure decreases across the shock).

The static pressure ratio across the shock may be found using the normal shock relations,

$$\frac{p_{E2}}{p_{E1}} = \frac{2k}{k+1}\mathrm{Ma}_{E1}^2 - \frac{k-1}{k+1} \Rightarrow p_{E2}/p_{E1} = 4.500 \tag{7}$$

so that the pressure just downstream of the shock is,

$$p_{E2} = \left(\frac{p_{E2}}{p_{E1}}\right)\left(\frac{p_{E1}}{p_{01}}\right)p_{01} \Rightarrow p_{E2} = 57.51 \text{ kPa (abs)}. \tag{8}$$

Note that the Mach number just downstream of the exit will be subsonic, so the downstream pressure will be equal to the back pressure. Hence,

$$p_B = p_{E2} \,. \tag{9}$$

Thus, the range of back pressures for which the exit Mach number will be supersonic is,

$$\boxed{p_B < 57.51 \text{ kPa (abs).}} \tag{10}$$

Note that the Mach number downstream of the shock wave is,

$$\mathrm{Ma}_{E2}^2 = \frac{(k-1)\mathrm{Ma}_{E1}^2 + 2}{2k\mathrm{Ma}_{E1}^2 - (k-1)} \Rightarrow \mathrm{Ma}_{E2} = 0.5774 \tag{11}$$

and the stagnation pressure ratio across the shock wave is,

$$\frac{p_{02}}{p_{01}} = \left[\frac{(k+1)\mathrm{Ma}_{E1}^2}{2+(k-1)\mathrm{Ma}_{E1}^2}\right]^{k/(k-1)} \left[\frac{k+1}{2k\mathrm{Ma}_{E1}^2 - (k-1)}\right]^{1/(k-1)} \Rightarrow p_{02}/p_{01} = 0.7209 \Rightarrow p_{02} = 72.1 \text{ kPa (abs)} \tag{12}$$

The isentropic stagnation pressure ratio at the downstream side of the shock is,

$$\frac{p_{E2}}{p_{02}} = \left(1 + \tfrac{k-1}{2}\mathrm{Ma}_{E2}^2\right)^{\frac{k}{1-k}} \Rightarrow p_{E2}/p_{02} = 0.7978, \tag{13}$$

The back pressure for this case can be found by combining relations in the following manner,

$$p_b = p_{E2} = \left(\frac{p_{E2}}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right)p_{01} = (0.7978)(0.7209)(100 \text{ kPa}) = 57.51 \text{ kPa}\,, \tag{14}$$

which is exactly the same result found in Eqs. (8) and (9).

Given that the flow chokes at an exit Mach number of 0.3721 (found from Eq. (1)), the flow in the device must be entirely subsonic when the exit Mach number is 0.2. Thus, the mass flow rate may be found from the isentropic relations evaluated at the exit,

$$\dot{m} = \rho_E V_E A_E \,, \tag{15}$$

where,

$$\rho_E = \rho_0\left(1 + \tfrac{k-1}{2}\mathrm{Ma}_E^2\right)^{\frac{1}{1-k}} \text{ and } \rho_0 = \frac{p_0}{RT_0}\,, \tag{16}$$

$$V_E = c_E \mathrm{Ma}_E = \sqrt{kRT_E}\,\mathrm{Ma}_E\,, \tag{17}$$

$$T_E = T_0\left(1 + \tfrac{k-1}{2}\mathrm{Ma}_E^2\right)^{-1}, \tag{18}$$

$$A_E = \left(\frac{A_E}{A_T}\right)A_T\,. \tag{19}$$

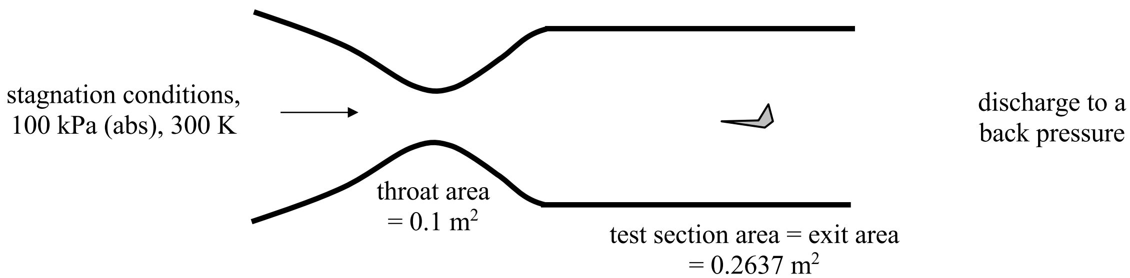
Using the given data,

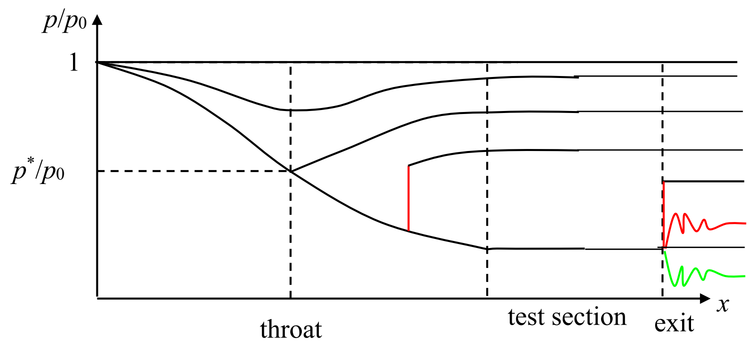
$\Rightarrow$ ρ_0 = 1.161 kg/m^3
ρ_E = 1.139 kg/m^3
T_E = 297.6 K
c_E = 345.8 m/s
V_E = 69.16 m/s
A_E = 1.69*10^{-4} m^2

$$\Rightarrow \boxed{\dot{m} = 1.33 * 10^{-2} \text{ kg/s}} \tag{20}$$

Note that this mass flow rate is less than the choked flow mass flow rate (since the flow isn't choked).

Consider the supersonic wind tunnel shown in the following schematic. Air is the working fluid and the test section area is constant.





a. What is the design Mach number of the test section?

SOLUTION:

The test section design Mach number may be found using the isentropic sonic area ratio and choosing the supersonic test section Mach number (case 4 in the diagram above),

$$\frac{A_{TS}}{A_T} = \frac{0.2637\ \text{m}^2}{0.1\ \text{m}^2} = 2.637 = \frac{A_{TS}}{A^*} = \frac{1}{\text{Ma}_{TS}}\left(\frac{1+\frac{k-1}{2}\text{Ma}_{TS}^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \boxed{\text{Ma}_{TS} = 2.50}. \tag{1}$$

Note that at design conditions, the throat Mach number is one.

b. What is the mass flow rate through the wind tunnel at design conditions?

SOLUTION:

The flow through the wind tunnel will be choked at design conditions, with a mass flow rate of,

$$\dot{m}_{\text{choked}} = \left(1+\frac{k-1}{2}\right)^{\frac{k+1}{2(1-k)}} p_0\sqrt{\frac{k}{RT_0}}A^* \Rightarrow \boxed{\dot{m} = 23.3\ \text{kg/s}}, \tag{2}$$

where $A^* = A_T$.

c. What is the maximum back pressure at which the throat will reach sonic conditions?

SOLUTION:

When the throat just reaches sonic conditions (case 3 in the diagram above), the throat area will equal the sonic area ($A^* = A_T$) and the exit Mach number may be found using the isentropic sonic area ratio since the flow through the entire converging-diverging nozzle will be subsonic (no shock waves),

$$\frac{A_E}{A_T} = \frac{0.2637\text{ m}^2}{0.1\text{ m}^2} = 2.637 = \frac{A_E}{A^*} = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{k-1}{2}\text{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \text{Ma}_E = 0.2263. \quad (3)$$

The exit pressure may be found from this Mach number using the isentropic stagnation pressure ratio,

$$\frac{p_E}{p_0} = \left(1+\frac{k-1}{2}\text{Ma}_E^2\right)^{\frac{k}{1-k}} \Rightarrow p_E/p_0 = 0.9650 \Rightarrow p_E = 96.5\text{ kPa (abs)}, \quad (4)$$

using p_0 = 100 kPa (abs). Since the exit Mach number is subsonic, the exit and back pressures are equal. Hence,

$$\boxed{p_B = p_E = 96.5\text{ kPa (abs)}}. \quad (5)$$

d. Assume a shock wave stands in the diverging section where the area is 0.1688 m^2. What is the back pressure at these conditions?

SOLUTION:

The Mach number just upstream of the shock wave may be found using the isentropic sonic area ratio since the flow leading up to the shock wave is isentropic and the throat area is at sonic conditions (since shock waves only form in supersonic flows, case 5 in the diagram shown above),

$$\frac{A_1}{A_T} = \frac{0.1688\text{ m}^2}{0.1\text{ m}^2} = 1.688 = \frac{A_1}{A^*} = \frac{1}{\text{Ma}_1}\left(\frac{1+\frac{k-1}{2}\text{Ma}_1^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \text{Ma}_1 = 2.00. \quad (6)$$

The stagnation pressure ratio and sonic area ratio across the shock are,

$$\frac{p_{02}}{p_{01}} = \frac{A_1^*}{A_2^*} = \left[\frac{(k+1)\text{Ma}_1^2}{2+(k-1)\text{Ma}_1^2}\right]^{k/(k-1)}\left[\frac{k+1}{2k\text{Ma}_1^2-(k-1)}\right]^{1/(k-1)} \Rightarrow p_{02}/p_{01} = A^*_1/A^*_2 = 0.7209. \quad (7)$$

The flow downstream of the shock wave is isentropic and subsonic. Thus, the pressure at the exit may be found

$$\frac{A_E}{A_2^*} = \frac{A_E}{A_1^*}\frac{A_1^*}{A_2^*} = \frac{A_E}{A_T}\frac{A_1^*}{A_2^*} = \left(\frac{0.2637\text{ m}^2}{0.1\text{ m}^2}\right)(0.7209) = 1.901 = \frac{1}{\text{Ma}_E}\left(\frac{1+\frac{k-1}{2}\text{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \text{Ma}_E = 0.3240. \quad (8)$$

The exit pressure may be found from the isentropic stagnation pressure ratio downstream of the shock and the exit Mach number,

$$\frac{p_{TS}}{p_{02}} = \left(1+\frac{k-1}{2}\text{Ma}_{TS}^2\right)^{\frac{k}{1-k}} \Rightarrow p_{TS}/p_{02} = 0.9299. \quad (9)$$

Accounting for the change in stagnation pressure ratio across the shock,

$$p_{TS} = \left(\frac{p_{TS}}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right)p_{01} = (0.9299)(0.7209)(100\text{ kPa}) \Rightarrow p_{TS} = 67.03\text{ kPa (abs)} \quad (10)$$

Since the exit is at a subsonic Mach number, the exit and back pressures are equal,

$$\boxed{p_B = p_E = 67.0\text{ kPa (abs)}}. \quad (11)$$

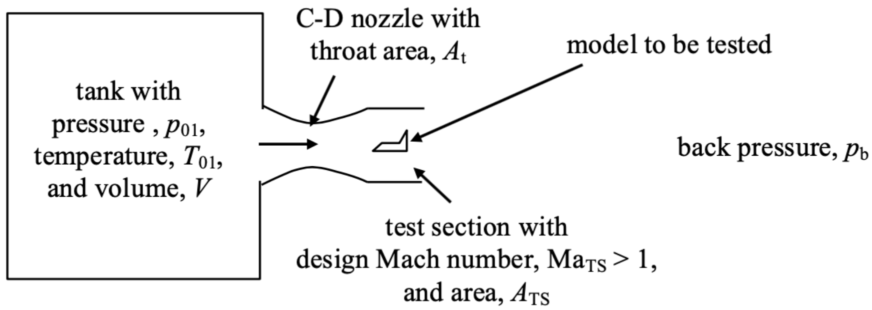


Figure 13.38. A schematic of a blowdown-style wind tunnel.

13.19. Supersonic Wind Tunnel Design

There are three common designs for supersonic wind tunnels:

(1) high-pressure gas storage tanks (and/or vacuum tanks) for blowdown wind tunnels,
(2) a compressor and diffuser for continuous-duty wind tunnels, and
(3) shock tubes for high-enthalpy wind tunnels.

Here we'll study only the first two categories: blowdown and continuous-duty wind tunnels.

13.19.1. Blowdown Wind Tunnels

A schematic for a typical blowdown wind tunnel is shown in Figure 13.38 (Figure 13.39 shows a more detailed example). Another possible design would be to use atmospheric conditions at the inlet and use a vacuum tank at the exit (such a design is effectively the same as the one shown in the figure).

The wind tunnel will have supersonic flow in the test section as long as the back-to-tank pressure ratio, p_B/p_{01}, is less than the back pressure ratio corresponding to case 6 shown in Figure 13.40 (no shock waves anywhere within the device). If the back pressure ratio becomes too large, then a shock wave will form in the diverging section of the nozzle and there will be subsonic flow in the test section.

Notes:

(1) There is a fixed amount of time for which the device will operate at the design test section Mach number, Ma_{TS}, since the tank mass will decrease with time. To extend the duration of the test, a diverging section can be added to the exit of the device as shown in Figure 13.41. The presence of the diverging section allows the tank to drop to a lower pressure before a shock wave appears ahead of the test section.

13.19.2. Continuous-duty Wind Tunnels

Continuous-duty wind tunnels utilize a compressor to produce the driving pressure gradient for the flow. In order to minimize the required compressor power, the wind tunnel should operate as efficiently as possible, i.e., as close to isentropic conditions as possible. Continuous-duty wind tunnels can be either open-circuit where air is drawn in and exhausted to the surroundings, or closed-circuit where the working gas is recycled through the system. The schematics shown in Figures 13.42 and 13.43 provide examples of both types of systems.

Again, in order to minimize the compressor power requirements, the losses in the system should be minimized. The ideal case (shown in Figure 13.44) is to have an isentropic deceleration from supersonic to subsonic speeds.

Consider what happens if we design the wind tunnel such that the downstream throat has a smaller area than the upstream throat, i.e., $A_{t2} < A_{t1}$. As we decrease the back pressure ratio, p_{back}/p_{01}, then we will have

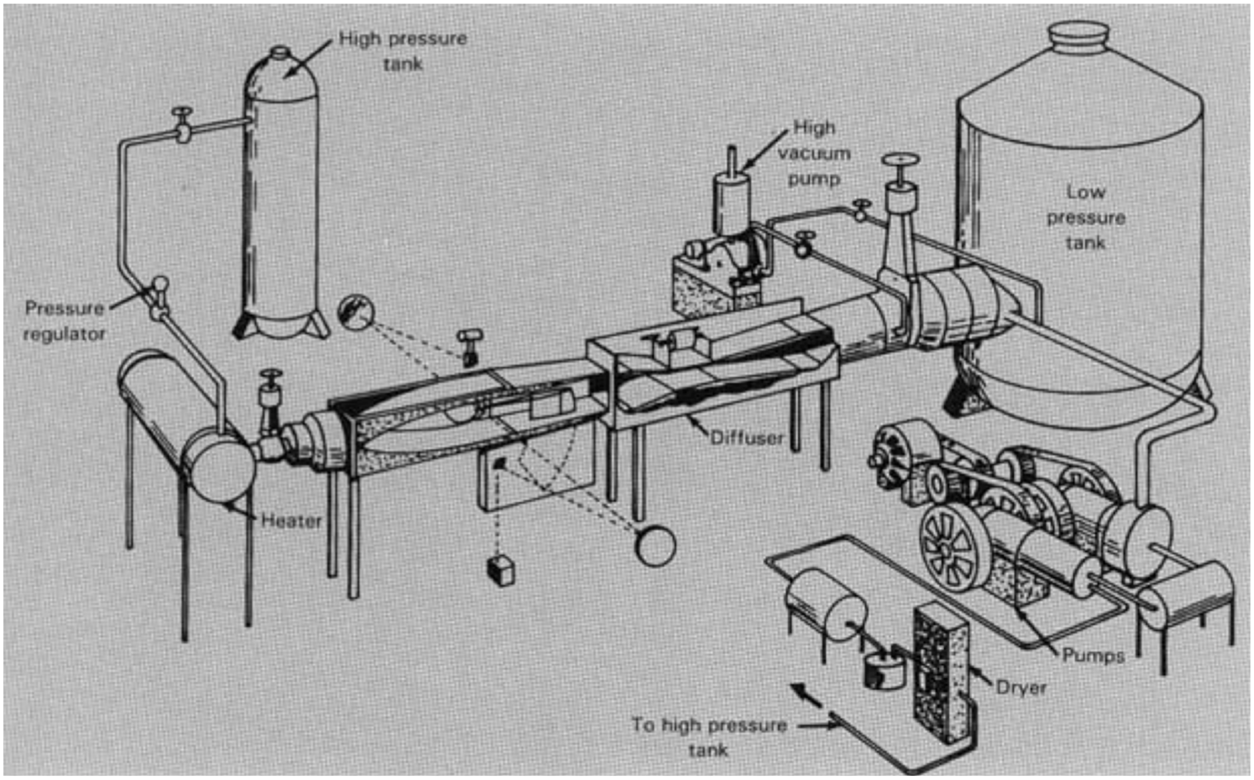


Figure 13.39. A schematic of a blowdown-style wind tunnel used at NASA. Image from https://history.nasa.gov/SP-440/ch5-6.htm.

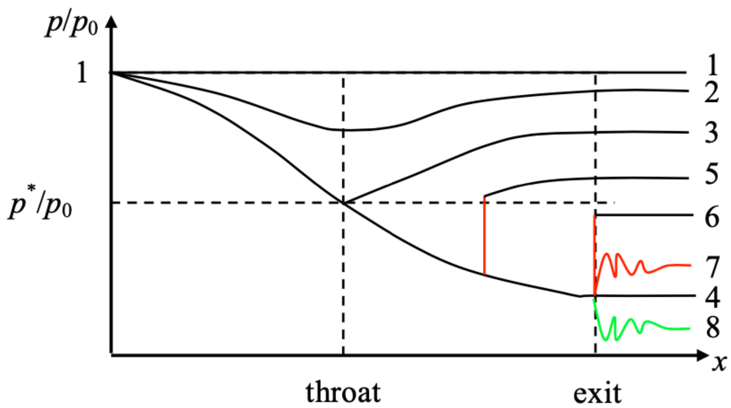


Figure 13.40. Stagnation pressure ratio plotted as a function of position in a converging-diverging nozzle.

subsonic flow throughout the device until at a critical back pressure the flow through the second throat will become choked. As we decrease the back pressure further, a shock wave will form in the diverging section of the downstream diffuser (refer to Figure 13.45). The test section is considered blocked since further reductions in the back pressure will not cause any changes upstream of the second throat. Since the test section was subsonic before blocking occurred, it will remain subsonic after blocking.

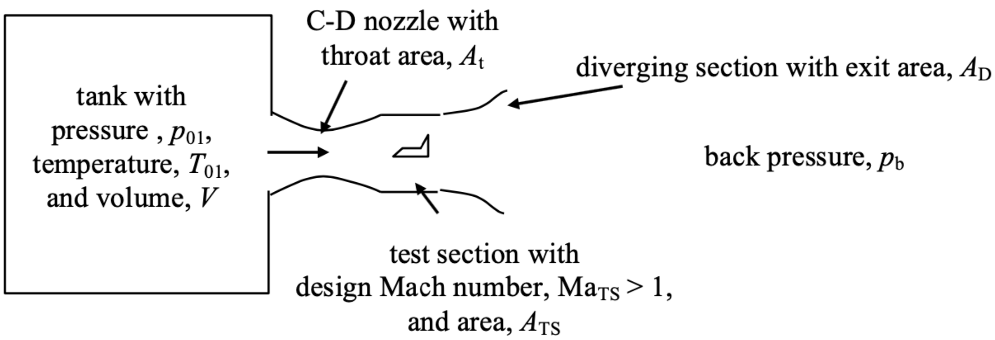


Figure 13.41. A schematic of a blowdown-style wind tunnel with a diffuser at the exit.

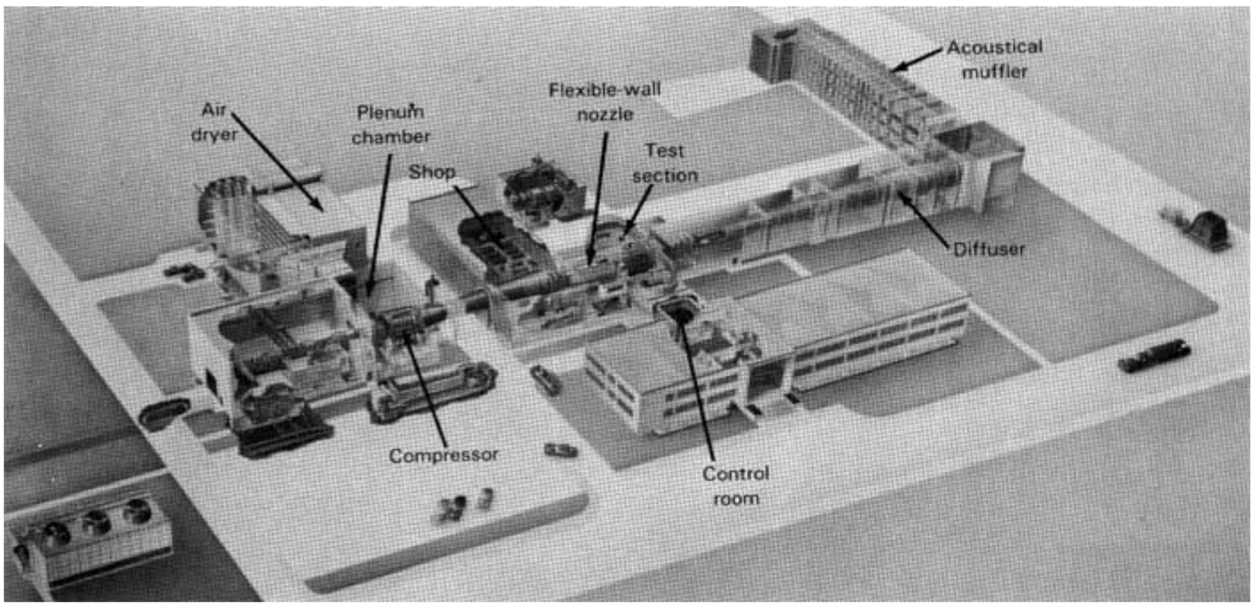


Figure 13.42. An example of an open-circuit, continuous-duty wind tunnel. Figure from: http://history.nasa.gov/SP-440/ch5-4.htm

Now consider what happens if we make the second throat just a little bit larger than the first throat. As we decrease the back pressure we will reach a case where the flow in the first throat becomes choked and a shock wave forms in the diverging section of the first throat. The flow in the test section will be subsonic. As the back pressure decreases, the shock wave in the first throat moves further downstream and becomes stronger. Recall that as the shock becomes stronger, A_2^*/A_1^* increases. If the second throat area is smaller than the A_2^* for the strongest shock wave, i.e., one that stands at the test section entrance, then a second shock will appear downstream of the second throat and the flow is once again blocked. Figure 13.46 illustrates this condition.

If the downstream throat has an area greater than the sonic area downstream of the shock when the shock wave stands at the entrance of the test section (Figure 13.47), and we decrease the back pressure further, then the shock will be swallowed by the second throat and the flow within the test section will, at last, be supersonic (Figure 13.48).

Now let's get back to the original discussion. Once the shock has been swallowed by the second throat, the shock will stand in the diverging section of the downstream diffuser. The wind tunnel is now considered

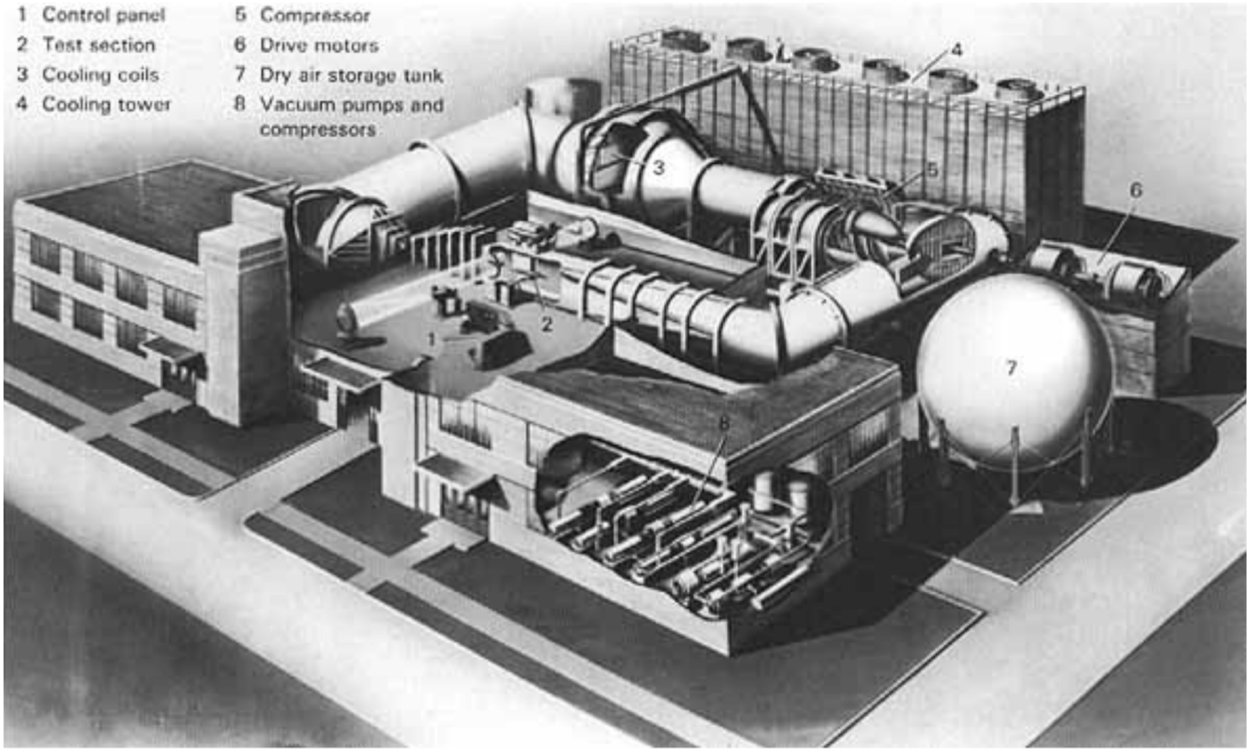
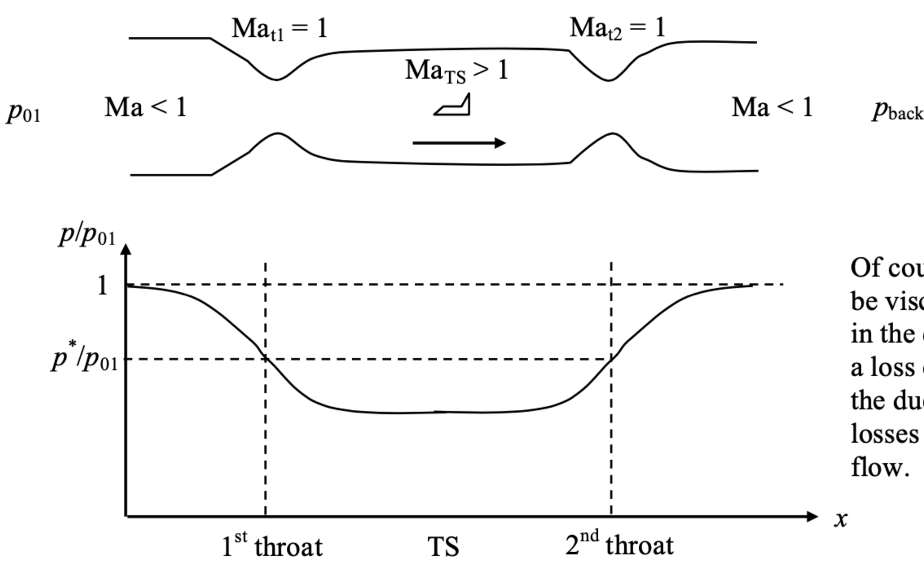


Figure 13.43. An example of a closed-circuit, continuous-duty wind tunnel. Figure from: http://history.nasa.gov/SP-440/ch5-3.htm



Of course in reality there would be viscous boundary layer losses in the duct, which would result in a loss of stagnation pressure in the duct. We'll ignore those losses here and assume isentropic flow.

Figure 13.44. The ideal case for a supersonic, continuous-duty wind tunnel.

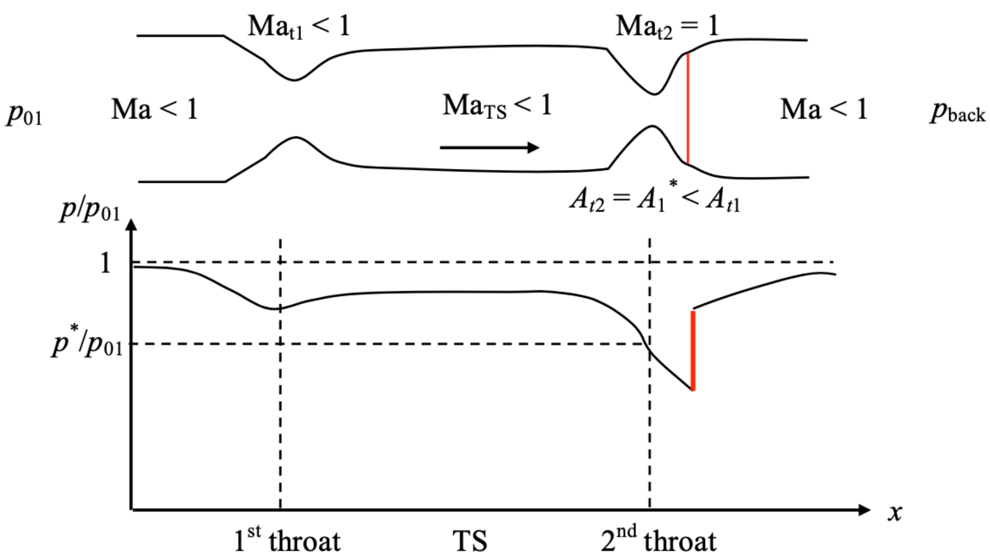


Figure 13.45. A wind tunnel blocked downstream of the second throat. The flow in the test section is subsonic.

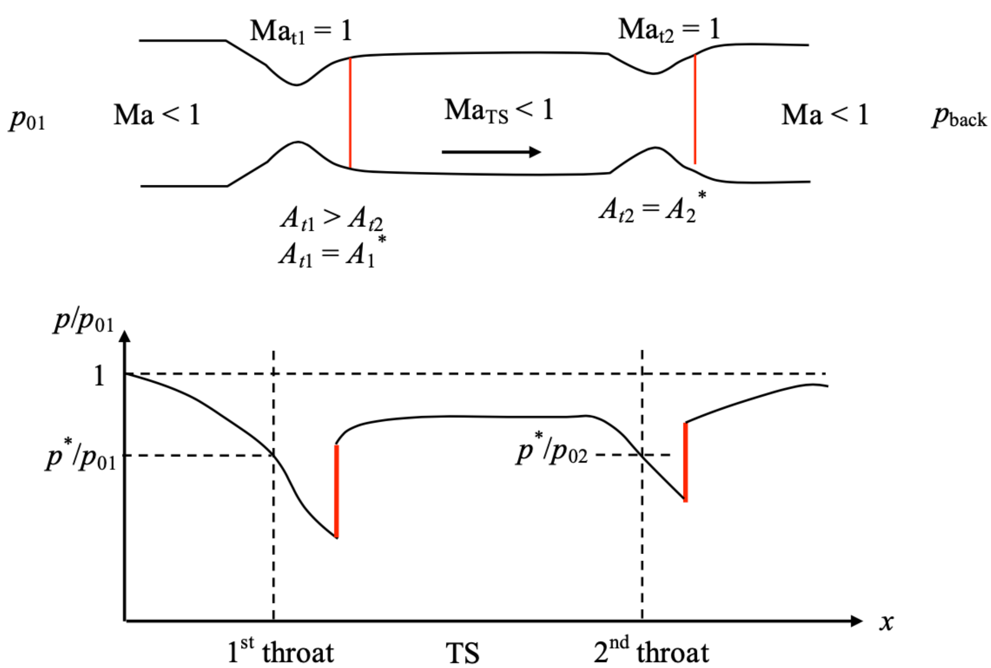


Figure 13.46. A wind tunnel blocked at both throats.

running or started. In order to isentropically decelerate the flow we should now decrease the area of the second throat so that it is approximately the same as the upstream throat area (Figure 13.49). Since there are no longer any shock waves upstream of the second throat, we theoretically could approach the first throat area; however, due to boundary layer effects we will always have to make the second throat slightly larger

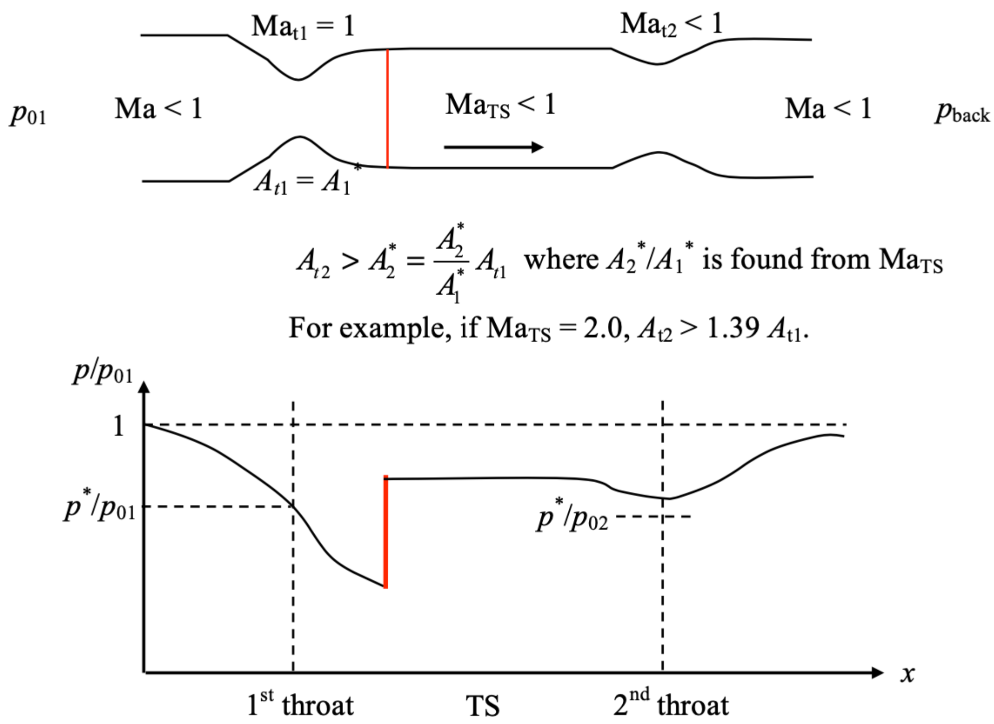


Figure 13.47. A wind tunnel with the downstream throat having an area larger than the sonic area downstream of the upstream shock.

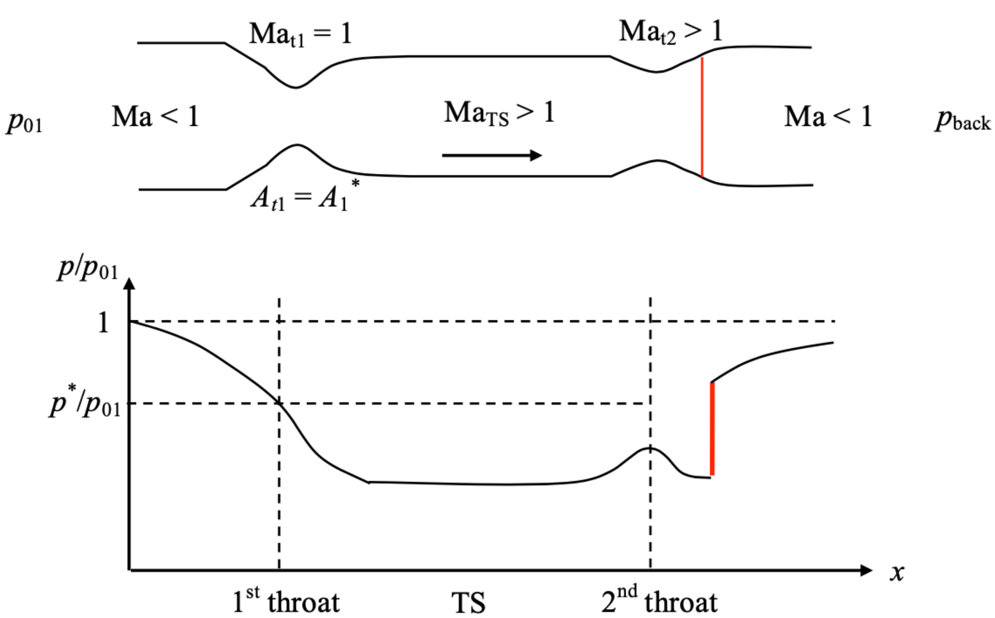


Figure 13.48. A wind tunnel in which the upstream shock wave has been swallowed. The shock now stands downstream of the second throat while the flow in the test section is supersonic.

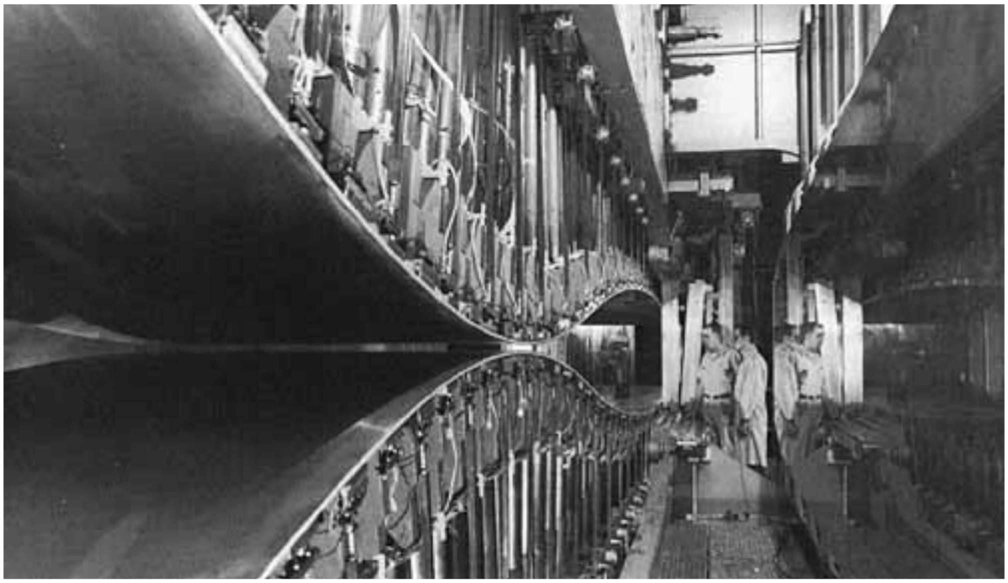


Figure 13.49. A wind tunnel with a variable throat area. Figure from: http://history.nasa.gov/SP-440/ch5-5.htm

than the first. Decreasing the second throat area too much results in shock waves in the diverging sections of both the first and second throats (discussed previously) and the wind tunnel is once again blocked.

Once the wind tunnel is running and we've decreased the second throat area, we should try to minimize the stagnation pressure loss through the shock wave in the second diverging section (and, hence, increase the tunnel efficiency). To do this we increase the back pressure, p_{back}, so that the shock wave will move further toward the second throat thus decreasing the shock's strength (Figure 13.50). The ideal case is to have the shock positioned exactly at the second throat. In practice, however, a shock standing exactly at the second throat is unstable and could disgorge, i.e., move back into the diverging section of the first throat, and block the test section once again.

Notes:

(1) An excellent reference on the history of wind tunnel development at NACA/NASA can be found at: http://www.hq.nasa.gov/office/pao/History/SP-440/contents.htm

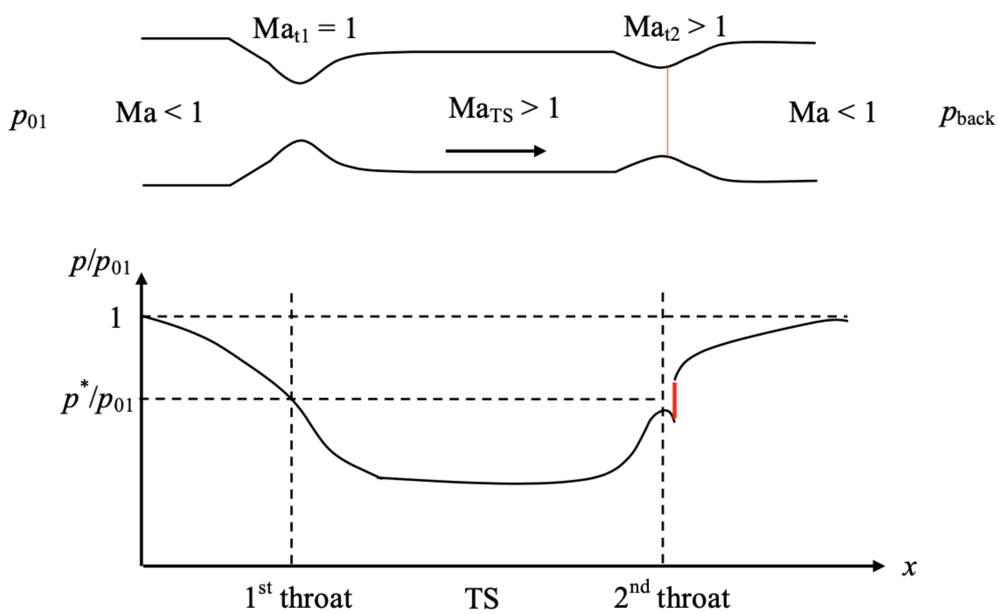


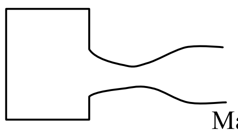
Figure 13.50. A running supersonic wind tunnel with a weak normal shock downstream of the second throat. The weak second shock results in a small amount of entropy generation in the flow.

A well insulated, blowdown wind tunnel exhausting to atmospheric pressure (14.7 psia) is to be designed. The test section cross-sectional area is specified to be 1 ft^2, and the desired test section Mach number is 2.0. The supply tank can be pressurized to 150 psia and heated to 150 °F (at least initially). Determine the throat area and supply tank volume required for a testing time of 30.0 s.

If a diverging section with an area ratio equal to 3.375 times that of the throat is added downstream of the test section, what is the new testing time?

SOLUTION:

p_0 = 150 psia
T_0 = 150 °F



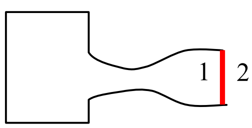
p_b = 14.7 psia

$\mathrm{Ma}_{\mathrm{TS}}$ = 2.0
A_{TS} = 1 ft^2

First determine the throat area required for a test section Mach number of $\mathrm{Ma}_{\mathrm{TS}}$ = 2.0 and test section area of A_{TS} = 1 ft^2:

$$\frac{A_{TS}}{A_t} = \frac{A_{TS}}{A^*} = \frac{1}{\mathrm{Ma}_{\mathrm{TS}}}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_{\mathrm{TS}}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{A_t = A^* = 0.59\text{ ft}^2} \tag{1}$$

The tank will operate at the design Mach number until the tank pressure drops enough so that a normal shock wave stands at the nozzle exit.



For an upstream Mach number of Ma_1 = 2.0, the downstream Mach number, Ma_2, and pressure ratio across the shock may be found using the normal shock relations.

$$\mathrm{Ma}_2^2 = \frac{(\gamma-1)\mathrm{Ma}_1^2+2}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)} \Rightarrow \mathrm{Ma}_2 = 0.5774 \tag{2}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \Rightarrow p_2/p_1 = 4.500 \tag{3}$$

Note that the pressure just downstream of the shock wave, p_2, is also equal to the back pressure, p_b, since the downstream Mach number is subsonic and the shock is located at the nozzle exit. The pressure just upstream of the shock may be related to the tank pressure using the isentropic relations.

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1/p_{01} = 0.1278 \tag{4}$$

Combine Eqs. (3) and (4) to determine the critical tank pressure for when a normal shock is located at the nozzle exit plane.

$$p_{01} = \left(\frac{p_{01}}{p_1}\right)\left(\frac{p_1}{p_2}\right)p_b \Rightarrow \underline{p_{01} = 25.6\text{ psia}} \tag{5}$$

To determine the required tank volume, we need to know how the pressure in the tank varies with time. Apply conservation of mass to a control volume surrounding the tank. Note that the mass flow rate from the tank will be choked during operation of the wind tunnel.

$$\frac{dM}{dt} = -\dot{m} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} \rho_{01}\sqrt{\gamma RT_0}\,A^* \tag{6}$$

where $M = \rho_{01}V$ and $\rho_{01} = p_{01}/(RT_0)$.

$$\frac{V}{R}\frac{d}{dt}\left(\frac{p_{01}}{T_0}\right) = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} p_{01}\sqrt{\frac{\gamma}{RT_0}}\,A^* \tag{7}$$

Since the tank is well insulated, assume that the discharge process occurs isentropically so that:

$$\frac{p_{01}}{p_{01,i}} = \left(\frac{T_0}{T_{0,i}}\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow T_0 = T_{0,i}\left(\frac{p_{01}}{p_{01,i}}\right)^{\frac{\gamma-1}{\gamma}} \tag{8}$$

$$\frac{d}{dt}\left(\frac{p_{01}}{T_0}\right) = \frac{d}{dt}\left[\frac{p_{01}}{T_{0,i}}\left(\frac{p_{01}}{p_{01,i}}\right)^{\frac{1-\gamma}{\gamma}}\right] = \frac{\left(p_{01,i}\right)^{\frac{\gamma-1}{\gamma}}}{T_{0,i}}\frac{d}{dt}\left[\left(p_{01}\right)^{\frac{1}{\gamma}}\right] = \frac{1}{\gamma}\frac{\left(p_{01,i}\right)^{\frac{\gamma-1}{\gamma}}}{T_{0,i}}\left(p_{01}\right)^{\frac{1-\gamma}{\gamma}}\frac{dp_{01}}{dt} \tag{9}$$

$$\frac{p_{01}}{\sqrt{T_0}} = \frac{\left(p_{01,i}\right)^{\frac{\gamma-1}{2\gamma}}}{\sqrt{T_{0,i}}}\left(p_{01}\right)^{\frac{1+\gamma}{2\gamma}} \tag{10}$$

Substitute Eqs. (9) and (10) into Eq. (7) and simplify.

$$\frac{V}{R}\frac{1}{\gamma}\frac{\left(p_{01,i}\right)^{\frac{\gamma-1}{\gamma}}}{T_{0,i}}\left(p_{01}\right)^{\frac{1-\gamma}{\gamma}}\frac{dp_{01}}{dt} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\frac{\left(p_{01,i}\right)^{\frac{\gamma-1}{2\gamma}}}{\sqrt{T_{0,i}}}\left(p_{01}\right)^{\frac{1+\gamma}{2\gamma}}\sqrt{\frac{\gamma}{R}}\,A^* \tag{11}$$

$$\int_{p_{01,i}}^{p_{01,f}}\left(p_{01}\right)^{\frac{1-3\gamma}{2\gamma}}dp_{01} = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(p_{01,i}\right)^{\frac{1-\gamma}{2\gamma}}\gamma\sqrt{\gamma RT_{0,i}}\,\frac{A^*}{V}\int_0^{\Delta t}dt \tag{12}$$

where $p_{01,f}$ is the pressure corresponding to when a normal shock appears at the nozzle exit (Eqn. (5)).

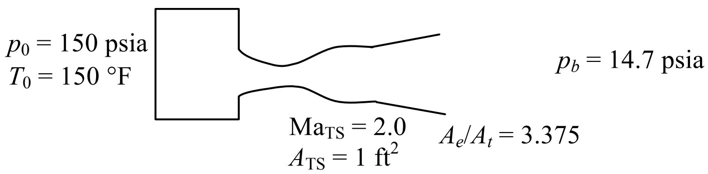
$$\frac{2\gamma}{1-\gamma}\left[\left(p_{01,f}\right)^{\frac{1-\gamma}{2\gamma}} - \left(p_{01,i}\right)^{\frac{1-\gamma}{2\gamma}}\right] = -\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left(p_{01,i}\right)^{\frac{1-\gamma}{2\gamma}}\gamma\sqrt{\gamma RT_{0,i}}\,\frac{A^*}{V}\Delta t \tag{13}$$

$$\boxed{V = \frac{\gamma-1}{2}\left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}\left[\left(\frac{p_{01,f}}{p_{01,i}}\right)^{\frac{1-\gamma}{2\gamma}} - 1\right]^{-1}\sqrt{\gamma RT_{0,i}}\,A^*\Delta t} \tag{14}$$

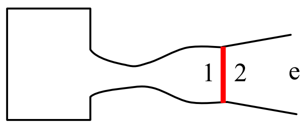
Using the given data:

$\gamma = 1.4$
$R = 287$ J/(kg.K)
$p_{01,f} = 25.6$ psia (Eq. (5))
$p_{01,i} = 150$ psia
$T_{0,i} = (150 + 460)$ °R = 610 °R
$A^* = A_t = 0.59$ ft^2 (Eq. (1))
$\Delta t = 30$ s
$\Rightarrow \boxed{V = 8670 \text{ ft}^3}$

Now attach a diverging section downstream of the test section.



Testing can continue until a shock wave forms at the test section exit.



The conditions at location 1 just upstream of the shock were found previously in Eqs. (2) and (3) using the normal shock relations. These quantities are repeated here for convenience.

$\mathrm{Ma}_1 = 2.0 \quad \Rightarrow \quad \mathrm{Ma}_2 = 0.5774$
$\Rightarrow \quad p_2/p_1 = 4.500$
$\Rightarrow \quad p_{01}/p_{02} = A^*_2/A^*_1 = 1.3872$

The Mach number at the exit may be found using the sonic area ratio:

$$\frac{A_e}{A_2^*} = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_e = 0.2466 \tag{15}$$

using

$$\frac{A_e}{A_2^*} = \left(\frac{A_e}{A_t}\right)\left(\frac{A_1^*}{A_2^*}\right) = 2.4330 \quad \text{where } A_e/A_t = 3.375 \tag{16}$$

The stagnation pressure in the tank for this case is given by:

$$p_{01} = \left(\frac{p_{01}}{p_{02}}\right)\left(\frac{p_{02}}{p_e}\right)p_b \tag{17}$$

where p_{01}/p_{02} = 1.3872 from the normal shock relations, $p_e = p_b$ = 14.7 psia since the exit Mach number is subsonic, and

$$\frac{p_e}{p_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_e/p_{02} = 0.9586 \tag{18}$$

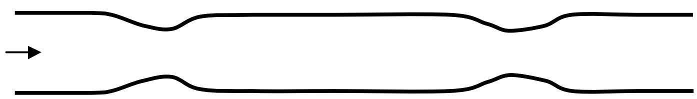
Combining, we find that $\underline{p_{01} = 21.3\text{ psia}}$.

Using Eq. (14) with $p_{01,f}$ = 21.3 psia and V = 8670 ft^3 and solving for Δt gives: $\boxed{\Delta t = 33.6\text{ s}}$. Thus, adding the diffuser extends the useful run time by approximately 12%.

A continuous-duty supersonic wind tunnel (air is the working fluid) is to be designed. The test section specifications are a Mach number of 2, a static pressure of 40 kPa (abs), a static temperature of 250 K, and an area of 0.5 m^2.

a. Determine the stagnation conditions required upstream of the test section, the mass flow rate required, and the throat area required.
b. During the startup process, what is the maximum stagnation pressure loss across the shock system?
c. If a fixed-area diffuser is used (the throat area cannot be adjusted during operation), what is the minimum diffuser throat area?
d. If a variable-area diffuser is used, explain the startup sequence required to achieve shock-free operation.

SOLUTION:



$\text{Ma}_{\text{TS}} = 2$
$p_{\text{TS}} = 40$ kPa (abs)
$T_{\text{TS}} = 250$ K
$A_{\text{TS}} = 0.5$ m^2

Determine the upstream conditions given the test section Mach number.

$$\frac{p_{TS}}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_{TS}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_0 = 313 \text{ kPa (abs)}} \quad (1)$$

using $p_{\text{TS}} = 40$ kPa (abs), $\text{Ma}_{\text{TS}} = 2$, and $\gamma = 1.4$.

$$\frac{T_{TS}}{T_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_{TS}^2\right)^{-1} \Rightarrow \boxed{T_0 = 450 \text{ K}} \quad (2)$$

using $T_{\text{TS}} = 250$ K.

$$\rho_0 = \frac{p_0}{RT_0} \Rightarrow \boxed{\rho_0 = 2.42 \text{ kg/m}^3} \quad (3)$$

The mass flow rate is found using the conditions in the test section.

$$\dot{m} = \rho_{TS} V_{TS} A_{TS} \Rightarrow \boxed{\dot{m} = 176 \text{ kg/s}} \quad (4)$$

where

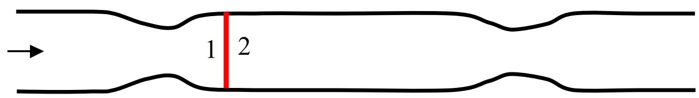
$$\rho_{TS} = \rho_0\left(1+\frac{\gamma-1}{2}\text{Ma}_{TS}^2\right)^{\frac{1}{1-\gamma}} \Rightarrow \rho_{\text{TS}} = 0.557 \text{ kg/m}^3 \quad (5)$$

$$V_{TS} = \text{Ma}_{TS}\sqrt{\gamma R T_{TS}} \Rightarrow V_{\text{TS}} = 634 \text{ m/s} \quad (6)$$

The required throat area is found from the area ratio.

$$\frac{A_{TS}}{A_T} = \frac{A_{TS}}{A^*} = \frac{1}{\text{Ma}_{TS}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_{TS}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{A_T = 0.2963 \text{ m}^2} \quad (7)$$

The maximum stagnation pressure loss during start-up will occur when a shock stands at the inlet of the test section (this is when the Mach number upstream of the shock is the greatest).



$$\frac{p_{02}}{p_{01}} = \left[\frac{\frac{\gamma+1}{2}\mathrm{Ma}_1^2}{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}\right]^{\gamma/\gamma-1}\left[\frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2-\frac{\gamma-1}{\gamma+1}\right]^{1/1-\gamma} \Rightarrow \frac{p_{02}}{p_{01}} = 0.7208 \quad (8)$$

using $\mathrm{Ma}_1 = \mathrm{Ma}_{TS} = 2$.

$$\Delta p_0 = p_{01} - p_{02} = p_{01}\left(\frac{p_{02}}{p_{01}} - 1\right) \Rightarrow \boxed{\Delta p_0 = -87.4 \text{ kPa}} \quad (9)$$

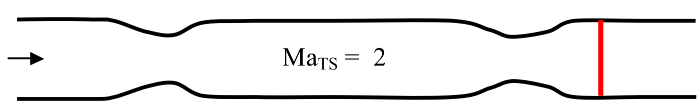
using $p_{01} = 313$ kPa (refer to Eq. (1)).

The minimum diffuser area must be able to accommodate the mass flow rate when the shock wave stands at the entrance to the test section.

$$\frac{A_2^*}{A_1^*} = \frac{p_{01}}{p_{02}} = 1.387 \quad \text{(using the result from Eq. (8))} \quad (10)$$

$$A_{T2} = A_2^* = \left(\frac{A_2^*}{A_1^*}\right)A_1^* \Rightarrow \boxed{A_{T2} = 0.411 \text{ m}^2} \quad (11)$$

A sketch of the steady state operation of the tunnel for a fixed area diffuser is shown below.



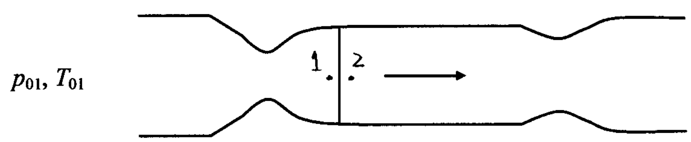
Note that once the shock has been swallowed, the Mach number at the downstream throat will be:

$$\frac{A_{T2}}{A_{T1}} = \frac{A_{T2}}{A^*} = \frac{1}{\mathrm{Ma}_{T2}}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_{T2}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_{T2} = 1.75 \quad (12)$$

using $A_{T2} = 0.411$ m^2 and $A_{T1} = 0.2963$ m^2.

Using a variable area diffuser we could eliminate the shock wave using the following procedure:
1. Maintain a downstream throat area of at least 0.411 m^2 during start-up so that the upstream normal shock can be swallowed.
2. Once the shock has been swallowed, the downstream throat area can be decreased to a value just greater than the upstream throat area of 0.296 m^2.
3. Simultaneously, increase the back pressure so that the downstream shock moves toward the downstream throat (and, hence, becomes weaker).

Consider a supersonic wind tunnel starting as shown in the figure below. The upstream nozzle throat area is 1.25 ft^2, and the test section design Mach number is 2.50. As the tunnel starts, a normal shock stands in the divergence of the nozzle where the area is 3.05 ft^2. Upstream stagnation conditions are T_{01} = 1080 °R and p_{01} = 115 psia. Find the minimum possible diffuser throat area at this instant. Calculate the entropy increase across the shock. What would be the minimum possible diffuser throat area to start this wind tunnel?



SOLUTION:

$$\frac{A_1}{A_1^*} = \frac{3.05\text{ ft}^2}{1.25\text{ ft}^2} = 2.44 \xrightarrow{\text{isentropic}} \underline{Ma_1 = 2.42}$$

$$Ma_1 = 2.42 \xrightarrow{\text{normal shock}} \underline{\frac{A_2^*}{A_1^*} = 1.8754}$$

$$\underline{A_2^*} = \left(\frac{A_2^*}{A_1^*}\right)\underbrace{A_1^*}_{=A_{t1}} = (1.8754)(1.25\text{ ft}^2) = \underline{2.34\text{ ft}^2}$$

$$\boxed{\therefore\ A_{t2} \geq 2.34\text{ ft}^2 \text{ at this instant}}$$

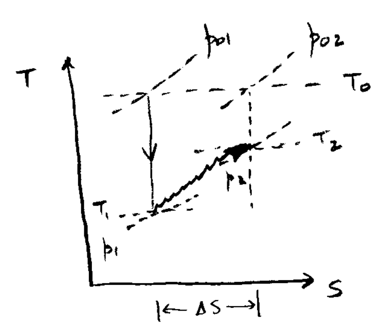
$$s_2 - s_1 = c_p \ln\frac{T_{02}}{T_{01}} - R\ln\frac{p_{02}}{p_{01}}$$

NOTE: $T_{02} = T_{01}$

$p_{02}/p_{01} = \dfrac{A_1^*}{A_2^*}$

$$s_2 - s_1 = -\left(0.0686\ \frac{\text{Btu}}{\text{lbm}\cdot{}^\circ\text{R}}\right)\ln\left(\frac{1}{1.8754}\right)$$

$$\boxed{\therefore\ \Delta s = 4.31\times10^{-2}\ \frac{\text{Btu}}{\text{lbm}\cdot{}^\circ\text{R}}}$$



$$Ma_1 = 2.50 \xrightarrow{\text{normal shock}} \underline{\frac{A_2^*}{A_1^*} = 2.0039}$$

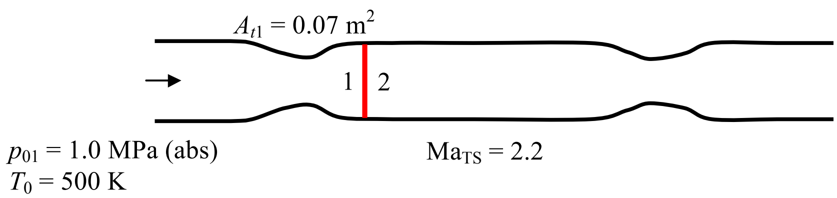
$$A_2^* = \left(\frac{A_2^*}{A_1^*}\right)\underbrace{A_1^*}_{=A_{t1}} = (2.0039)(1.25\text{ ft}^2)$$

$$\boxed{\therefore\ A_{t2} \geq 2.50\text{ ft}^2}$$

A supersonic wind tunnel is to be operated at a Mach number of 2.2 in the test section. Upstream from the test section, the nozzle throat area is 0.07 m^2. Air is supplied at stagnation conditions of 500 K and 1.0 MPa (abs). At one flow condition, while the tunnel is being brought up to speed, a normal shock stands at the nozzle exit plane. The flow is steady. For this starting condition, immediately downstream from the shock, find:

a. the Mach number,
b. the static pressure,
c. the stagnation pressure, and
d. the minimum area theoretically possible for the second throat downstream from the test section.
e. On a *Ts* diagram, show static and stagnation state points and the process path.

SOLUTION:



The Mach number just upstream of the shock wave will be the test section design Mach number, *i.e.* $\mathrm{Ma}_1 = 2.2$. The conditions just downstream of the shock wave may be found using the normal shock relations.

$$\mathrm{Ma}_2^2 = \frac{(\gamma-1)\mathrm{Ma}_1^2+2}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)} \Rightarrow \boxed{\mathrm{Ma}_2 = 0.5471} \tag{1}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \Rightarrow p_2/p_1 = 5.4800 \tag{2}$$

$$\frac{p_{02}}{p_{01}} = \frac{A_1^*}{A_2^*} = \left[\frac{(\gamma+1)\mathrm{Ma}_1^2}{2+(\gamma-1)\mathrm{Ma}_1^2}\right]^{\gamma/(\gamma-1)}\left[\frac{\gamma+1}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)}\right]^{1/(\gamma-1)} \Rightarrow p_{02}/p_{01} = A_1^*/A_2^* = 0.6281 \tag{3}$$

The stagnation pressure just downstream of the shock is:

$$p_{02} = \left(\frac{p_{02}}{p_{01}}\right)p_{01} = (0.6281)(1.0\text{ MPa}) \Rightarrow \boxed{p_{02} = 0.6281\text{ MPa (abs)}} \tag{4}$$

The static-to-stagnation pressure ratio just upstream of the shock may be found using the isentropic relations.

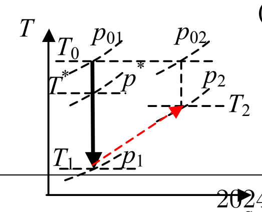
$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1/p_{01} = 0.0935 \tag{5}$$

The pressure just downstream of the shock is then:

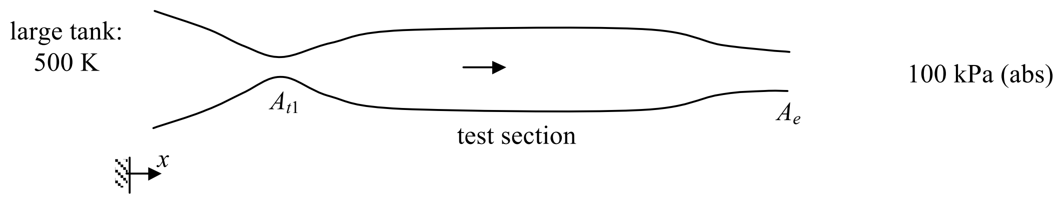
$$p_2 = \left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_{01}}\right)p_{01} = (5.4800)(0.0935)(1.0\text{ MPa}) \Rightarrow \boxed{p_2 = 0.5124\text{ MPa (abs)}} \tag{6}$$

The minimum second throat area will be A_2^*.

$$A_2^* = \left(\frac{A_2^*}{A_1^*}\right)A_{t1} = \left(\frac{1}{0.6281}\right)(0.07\text{ m}^2) \Rightarrow \boxed{A_2^* = 0.11\text{ m}^2} \quad (\text{where } A_1^* = A_{t1}) \tag{7}$$



Consider the supersonic wind tunnel design shown in the figure below. The system consists of a converging-diverging nozzle located downstream from a large tank of air held at a constant pressure and a constant temperature of 500 K. Downstream from the converging-diverging nozzle is the test section, and downstream of the test section is a *converging* nozzle. The converging nozzle discharges to the atmosphere which is at 100 kPa (abs).



If the test section is to operate at a Mach number of 1.5 and have a cross-sectional area of 0.01 m^2, determine:

a. the converging-diverging nozzle throat area, A_{t1}, and
b. the minimum area of the downstream converging nozzle, A_e, in order to start the wind tunnel.
c. Given the areas calculated in parts (a) and (b), sketch how the pressure ratio, p/p_{tank}, varies with position in the tunnel, x, just prior to when the tunnel is started (just before the shock is swallowed). Clearly indicate the location of the upstream throat, the test section region, the exit plane and p^*/p_{01} and p^*/p_{02}.
d. What is the minimum tank pressure required to start the wind tunnel using the areas given in parts (a) and (b)?
e. What is the minimum tank pressure that results in choked flow within the wind tunnel using the areas found in parts (a) and (b)?
f. Given the areas calculated in parts (a) and (b) and the tank pressure in part (d), sketch how the pressure ratio, p/p_{tank}, varies with position in the tunnel, x, just *after* the tunnel is started. Clearly indicate the location of the upstream throat, the test section region, the exit plane and p^*/p_{01}.

SOLUTION:

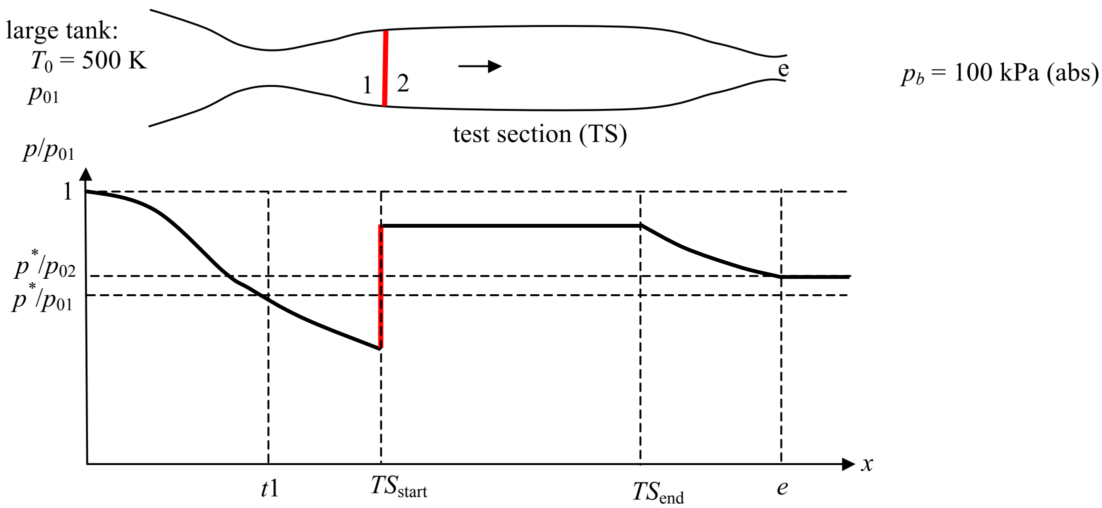
Determine the upstream sonic area, A^*, given that $\text{Ma}_{\text{TS}} = 1.5$ and $A_{\text{TS}} = 0.01$ m^2. Note that the upstream throat area, A_{t1}, will be equal to the sonic area, A^*.

$$\frac{A_{TS}}{A^*} = \frac{1}{\text{Ma}_{\text{TS}}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_{\text{TS}}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow A_{TS}/A^* = 1.1762 \Rightarrow \boxed{A_{t1} = A^* = 8.5*10^{-3}\text{ m}^2} \tag{1}$$

The minimum area of the downstream converging nozzle will be equal to the sonic area following a normal shock wave located at the entrance to the test section.

$$\frac{A_2^*}{A_1^*} = \frac{p_{01}}{p_{02}} = \left[\frac{(\gamma+1)\text{Ma}_1^2}{2+(\gamma-1)\text{Ma}_1^2}\right]^{\frac{\gamma}{1-\gamma}}\left[\frac{\gamma+1}{2\gamma\text{Ma}_1^2-(\gamma-1)}\right]^{\frac{\gamma}{1-\gamma}} \Rightarrow A^*_2/A^*_1 = 1.0755 \Rightarrow \boxed{A_e = A^*_2 = 9.1*10^{-3}\text{ m}^2} \tag{2}$$

where $\text{Ma}_1 = \text{Ma}_{\text{TS}}$ and A^*_1 was found in Eq. (1).



The smallest tank pressure required to start the wind tunnel will occur when a normal shock occurs at the entrance to the wind tunnel. For this case, $\mathrm{Ma}_e = 1$ since $A_e = A^*_2$. Furthermore,

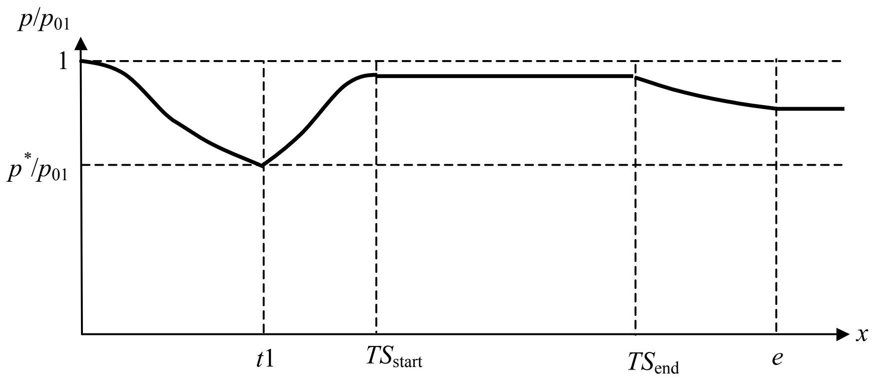
$$\frac{p_e}{p_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_e/p_{02} = p^*/p_{02} = 0.5283 \tag{3}$$

The tank pressure is found by multiplying pressure ratios.

$$p_{01,\min} = \left(\frac{p_{01}}{p_{02}}\right)\left(\frac{p_{02}}{p_e}\right)p_e \Rightarrow \boxed{p_{01,\min} = 204 \text{ kPa (abs)}} \tag{4}$$

where $p_e = p_b = 100$ kPa (abs) and p_{02}/p_{01} was found in Eq. (2).

The tank pressure corresponding to when the flow through the device just becomes choked may be found using the isentropic relations and noting that $p_e = p_b$.

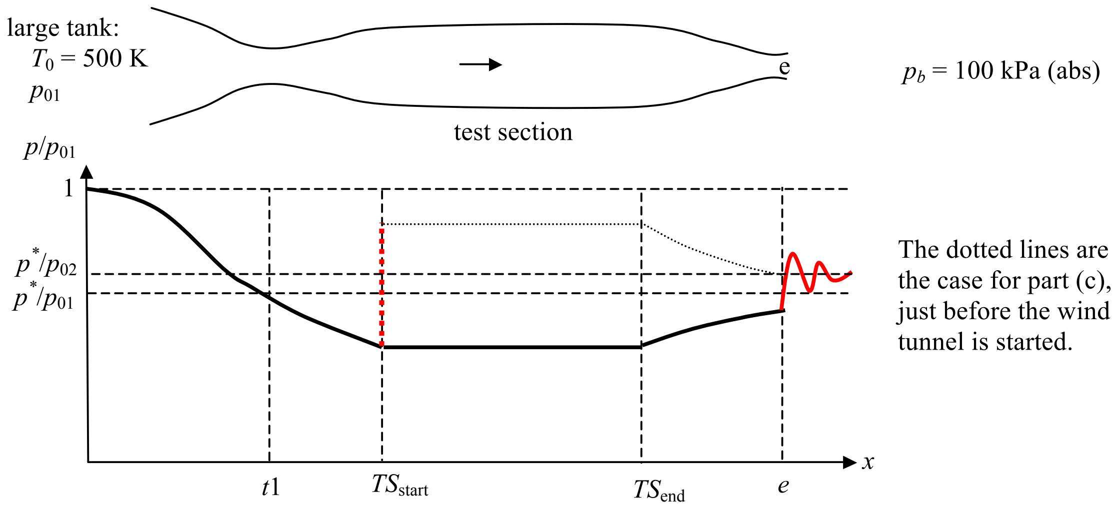


$$\frac{A_e}{A^*} = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_e = 0.7279 \tag{5}$$

where $A_e = A^*_2$ and $A^* = A^*_1 \Rightarrow A^*_2/A^*_1 = 1.0755$ (from Eq. (2)). Now, using the isentropic stagnation pressure ratio relation:

$$\frac{p_b}{p_0} = \frac{p_e}{p_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_0 = 0.7029 \Rightarrow \boxed{p_0 = 142\text{ kPa (abs)}} \tag{6}$$

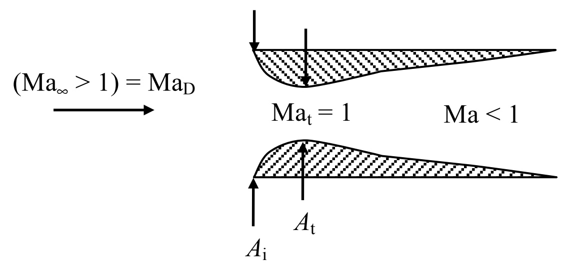
where $p_b = 100$ kPa.



Supersonic Diffuser Design

Another application where the efficient deceleration of a supersonic flow is of interest is a supersonic diffuser at the inlet of aircraft jet engines. The flow entering a jet engine typically needs to be subsonic in order to avoid shocks in the compressor section and give efficient combustion in the combustor. Obviously the most efficient deceleration of the incoming flow is desired since the thrust out of the device will decrease if the upstream stagnation pressure decreases.

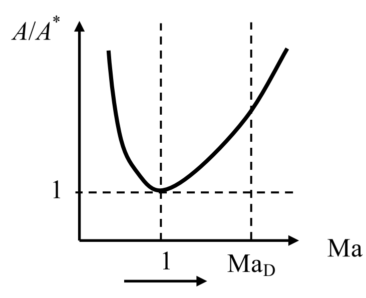
Many of the same ideas discussed previously for the design of supersonic wind tunnels are pertinent here as well. Consider a diffuser with a fixed inlet and throat area, A_i and A_t, respectively, in a supersonic flow with an upstream Mach number of $\text{Ma}_\infty > 1$. At design conditions, i.e., $\text{Ma}_\infty = \text{Ma}_\text{D}$, the flow through the diffuser will be shockless (isentropic) as shown in the figure below.



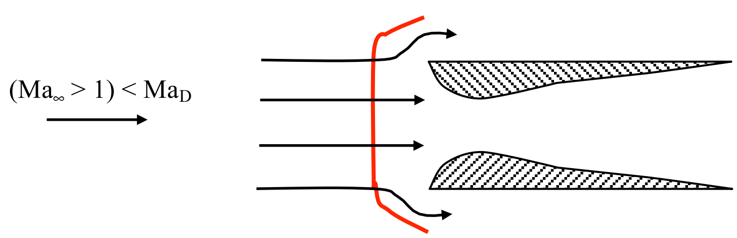
In the ideal case the inlet-to-throat area ratio will be related to the design Mach number, Ma_D, by the isentropic relation for the sonic area ratio, i.e.:

$$\frac{A_i}{A^*} = \frac{A_i}{A_t} = \frac{1}{\text{Ma}_\text{D}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_\text{D}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{165}$$

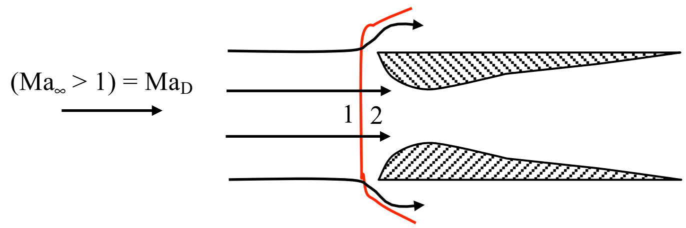
However, we need to consider what happens as the aircraft comes up to the design Mach number from rest ($\text{Ma}_\infty = 0$). Consider the plot shown below:



For $\text{Ma}_\infty < \text{Ma}_\text{D}$, $(A_\text{i}/A^*) < (A_\text{i}/A^*)_\text{D}$ so that $A_\text{t} = A^*_D < A^*_{\text{Ma} < \text{MaD}}$. Thus, the diffuser cannot "swallow" all of the air flowing toward the inlet. A shock wave forms in front of the inlet to produce subsonic flow so that some of the air can spill over the inlet as shown in the figure below.



As the upstream Mach number increases, the sonic area approaches the throat area, i.e., $A^* \to A_t$, and the shock moves closer to the inlet (the shock gets weaker and less flow needs to be diverted around the diffuser). Eventually we'll reach design conditions but a normal shock will still appear ahead of the inlet since the sonic area after the shock, A_2^*, will be greater than the throat area, A_t, at design conditions.

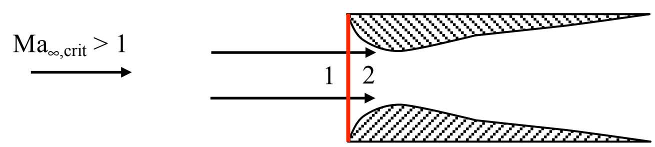


$$\frac{A_i}{A_2^*} = \frac{A_i}{A_1^*}\frac{A_1^*}{A_2^*} = \frac{A_i}{A_t}\underbrace{\frac{A_1^*}{A_2^*}}_{<1} \Rightarrow A_2^* > A_t$$

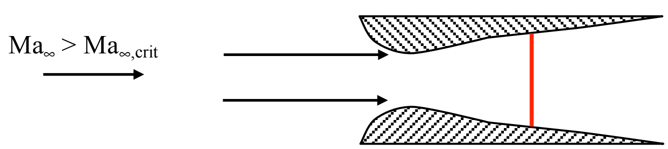
If we continue to increase Ma_∞ the shock wave will move closer to the inlet until at a critical upstream Mach number, $\text{Ma}_{\infty,\text{ crit}}$, the shock will be positioned exactly at the inlet such that the diffuser can accommodate all of the mass flow heading toward it (no spill-over). This occurs when $A_2^* = A_t$:

$$\frac{A_i}{A_t} = \frac{A_i}{A_2^*} = \frac{A_i}{A_1^*}\frac{A_1^*}{A_2^*} = fcn\left(\text{Ma}_{\infty,\text{ crit}}\right)$$

where A_i/A_1^* is found from the isentropic relations and A_1^*/A_2^* is found from the normal shock relations.



A further increase in the upstream Mach number will cause the shock wave to be **swallowed** by the diffuser where it will come to a steady state position within the diverging section.



Now the upstream Mach number can be decreased back down to the design Mach number so that the shock wave will travel back upstream toward the throat and, hence, become weaker (less stagnation pressure drop across the shock). The ideal case is to bring the Mach number down exactly to the design Mach number so that the shock occurs exactly at the throat and has zero strength.

Notes:

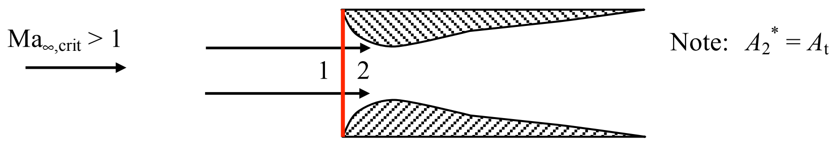
1. In practice we don't want to operate the diffuser exactly at design conditions since any small decrease in the upstream Mach number will cause the shock wave to **disgorge** from the diffuser and the entire process for swallowing the shock must be repeated again.

2. One measure of the diffuser performance is the **stagnation pressure recovery** coefficient defined as:

$$\eta \equiv \frac{p_{0,\text{exit of diffuser}}}{p_{0\infty}}$$

At design conditions under ideal conditions this coefficient should be unity.

3. Over-speeding the diffuser is often impractical. For example, consider a diffuser designed to operate at a Mach number of 1.7 ($A_i/A_\infty^* = A_i/A_t = 1.338$). The critical Mach number for swallowing the shock will be:

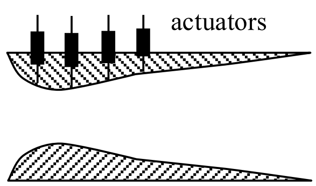


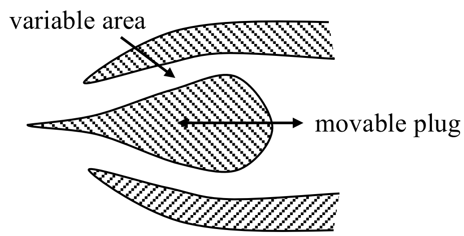
$$\left.\frac{A_i}{A_2^*}\right|_{\text{sub}} = \frac{A_i}{A_t} = 1.338 = \frac{1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_2^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \text{Ma}_2 = 0.5 \Rightarrow \text{Ma}_1 = 2.65$$

$$\therefore \text{Ma}_{\infty,\text{crit}} = 2.65$$

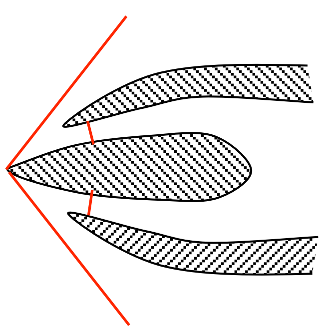
Thus, to achieve isentropic flow through the diffuser, we would need to operate our diffuser at a Mach number just greater than 2.65 then decrease the Mach number down to just greater than 1.7. Designing an aircraft to achieve this over-speed Mach number is often impractical.

4. Since fixed geometry diffusers are often impractical, other diffuser types have been designed. These include designs that have variable areas so that the throat area can be increased to swallow the shock, then decreased again to the design conditions (very similar to what was discussed for supersonic wind tunnels).





Oblique shock wave diffusers are also often used. The stagnation pressure loss across an oblique shock is less than that across a normal shock wave. The weaker the oblique shock wave, the smaller the stagnation pressure loss. Normal shock waves may still appear in the device but they'll be weaker than if there wasn't an oblique shock since the oblique shock helps to decelerate the flow.



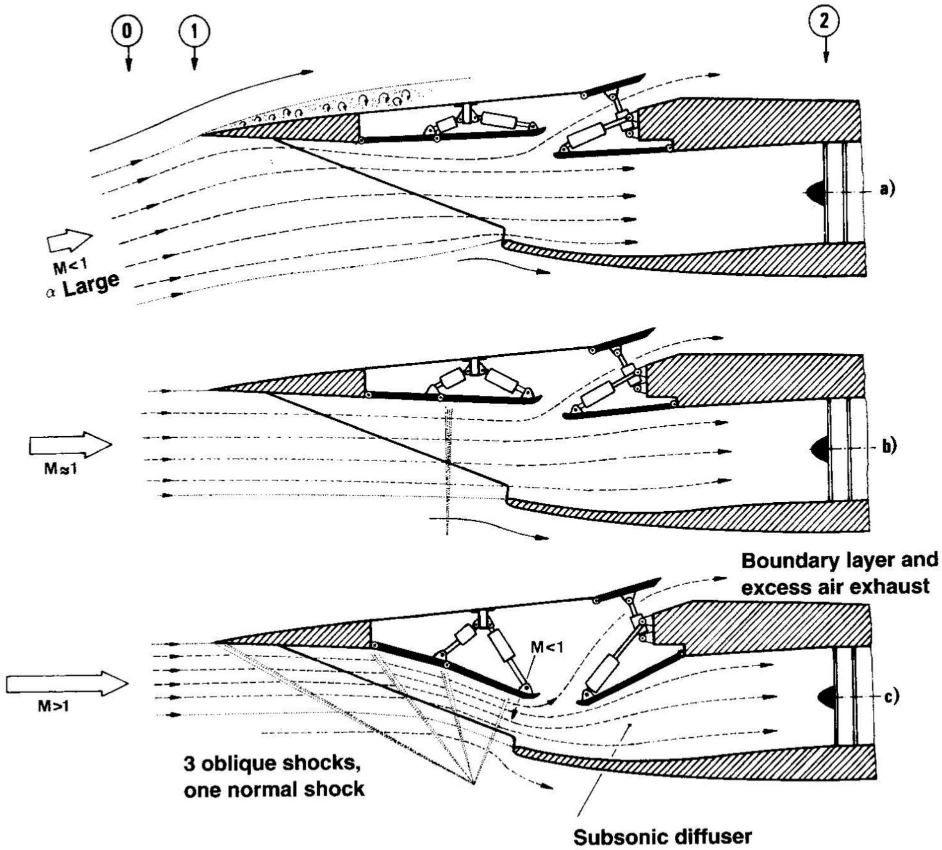
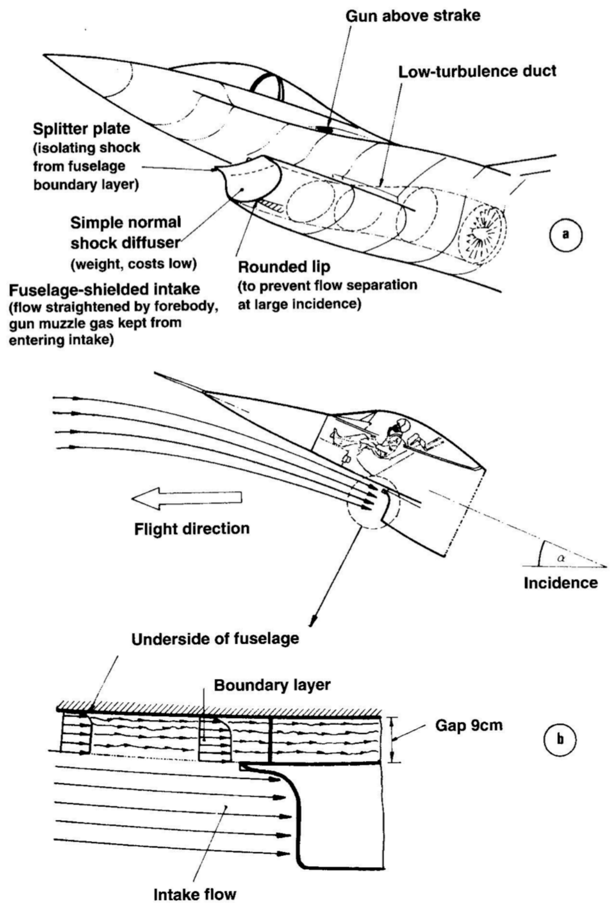


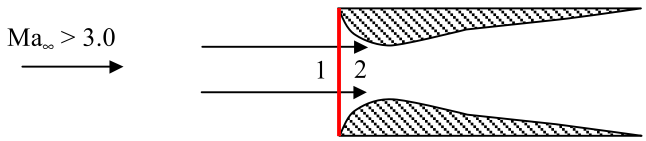
Fig 3-25 F-14 intake characteristics
a) subsonic flow at high angle-of-attack (manoeuvring flight)
b) transonic flow with normal shock emerging on ramp
c) External compression of supersonic flow through four shockwaves (3 oblique shock plus terminating normal shock)

Note that engine inlets typically "bleed off" or remove boundary layers as shown in the F-16 engine inlet design. This is done in order to avoid exposing engine components to the unsteady conditions resulting from wakes formed by separated boundary layers.



A convergent-divergent supersonic diffuser is to be used at Mach 3.0. The diffuser is to use a variable throat area so as to swallow the starting shock. What percentage increase in throat area will be necessary?

SOLUTION:



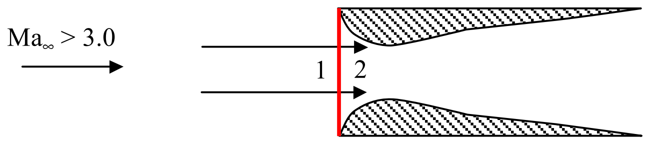
If there is no shock in front of the diffuser, then the throat area will be $A_t = A^*$ at design conditions (Ma_∞ = 3.0). However, during start up conditions a normal shock wave will appear in front of the diffuser. The largest required throat area will occur when the normal shock wave stands at the entrance of the diffuser. For these conditions, the minimum throat area will be equal to the sonic area downstream of the shock wave, *i.e.* $A_t = A^*_2$. Hence, the ratio of the throat area required to swallow the shock (= A^*_2) to the throat area without the shock (= A^*_1) at the design Mach number (Ma_∞ = 3.0) is:

$Ma_\infty = 3.0 \Rightarrow$ (normal shock relations) $\boxed{A^*_2/A^*_1 = 3.0456}$. (1)

Thus, the throat area must increase by approximately 305% in order to swallow the shock wave.

A convergent-divergent supersonic diffuser is to be used at Mach 3.0. The diffuser is to use a variable throat area so as to swallow the starting shock. What percentage increase in throat area will be necessary?

SOLUTION:



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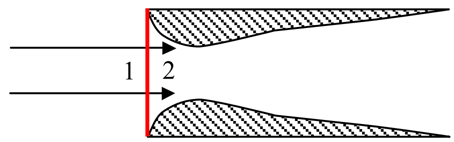
$\text{Ma}_\infty = 3.0 \Rightarrow$ (normal shock relations) $\boxed{A^*_2/A^*_1 = 3.0456}$. (1)

Thus, the throat area must increase by approximately 305% in order to swallow the shock wave.

A small jet aircraft that is designed to cruise at a Mach number of 1.7 has a convergent-divergent intake diffuser with a fixed area ratio. Find the ideal area ratio for this diffuser and the Mach number to which the aircraft must be taken in order to swallow the normal shock wave if the diffuser has this ideal area ratio.

SOLUTION:

Just before the shock is swallowed, the flow is as shown below.



The area ratio corresponding to the ideal Mach number of 1.7 (with no shock wave) is:

$$\frac{A_i}{A_t} = \frac{A_i}{A^*} = \frac{1}{\mathrm{Ma}}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{A_i/A_t = 1.3376} \quad \text{(where Ma = 1.7 and } A_t = A^*\text{)} \tag{1}$$

With the shock located at the inlet plane, the downstream area ratio required to swallow the shock is:

$A_i/A_2^* = 1.3376$

which corresponds to a subsonic Mach number downstream of the shock at the inlet of:

$$\frac{A_i}{A_2^*} = \frac{1}{\mathrm{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_2 = 0.5012 \tag{2}$$

The corresponding upstream Mach number is:

$$\mathrm{Ma}_2^2 = \frac{(\gamma-1)\mathrm{Ma}_1^2+2}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)} \Rightarrow \boxed{\mathrm{Ma}_1 = 2.6313} \tag{3}$$

A small jet aircraft designed to cruise at a Mach number of 2.5 has an intake diffuser with a variable area ratio. Find the ratio of the throat area under these cruise conditions to the throat area required when the aircraft is flying at a Mach number of 1.3. Assume the diffuser intake area does not change. When flying at cruise conditions, if the aircraft is suddenly slowed down without altering the diffuser area ratio, sketch the diffuser flow pattern that will exist.

SOLUTION:

The inlet-to-throat area ratio for cruise conditions is:

$$\frac{A_i}{A_t} = \frac{A_i}{A^*} = \frac{1}{\text{Ma}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow (A_i/A_t)_{\text{Ma}=2.5} = 2.6367 \tag{1}$$

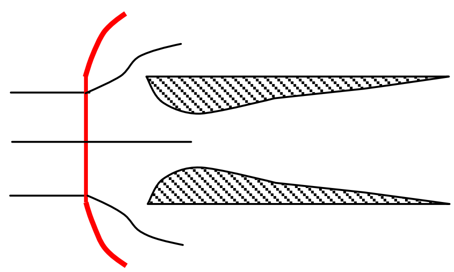
The inlet-to-throat area ratio for a Mach number of 1.3 is:

$$\frac{A_i}{A_t} = \frac{A_i}{A^*} = \frac{1}{\text{Ma}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow (A_i/A_t)_{\text{Ma}=1.3} = 1.0663 \tag{2}$$

Thus, the ratio of the throat area at cruise to the throat area at Ma = 1.3 is:

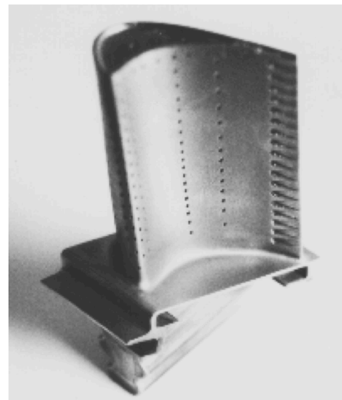
$$\frac{A_{t,\text{Ma}=2.5}}{A_{t,\text{Ma}=1.3}} = \frac{\left(A_i/A_t\right)_{\text{Ma}=1.3}}{\left(A_i/A_t\right)_{\text{Ma}=2.5}} = \frac{1.0663}{2.6367} \Rightarrow \boxed{A_{t,\text{Ma}=2.5}/A_{t,\text{Ma}=1.3} = 0.4044} \tag{3}$$

If the aircraft is slowed without increasing the throat area, the engine will not be able to accommodate the flow rate and a shock will form upstream of the inlet in order to divert some of the flow.

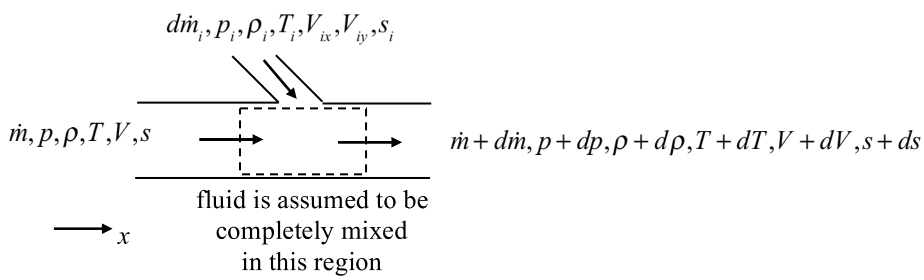


11. Flows with Mass Addition

Now let's consider compressible flows where mass addition (or removal) occurs. Examples of such flows include those in which solid rocket propellant is burned, a gas coolant is added to the flow such as in film cooling of turbine blades, or the boundary layer flow is removed such as in some wind tunnels.



We'll make the following assumptions for our analysis of flow with mass addition:
- no heat transfer
- no other work
- no friction
- no area change
- no significant elevation changes
- steady flow
- 1D flow (This isn't a very good assumption in general, but we'll make the assumption here for simplicity.)



Using the same control volume approach as in previous analyses gives:
COM:

$$\left(\dot{m}+d\dot{m}\right)-\dot{m}-d\dot{m}_i=0$$

$$d\dot{m}=d\left(\rho VA\right)=d\dot{m}_i$$

$$\boxed{\frac{d\rho}{\rho}+\frac{dV}{V}=\frac{d\dot{m}}{\dot{m}}} \qquad (166)$$

Note: dA=0

LME in x-direction:

$$\left[\dot{m}V + d\left(\dot{m}V\right)\right] - \dot{m}V - d\dot{m}_i V_{ix} = pA - \left[pA + d\left(pA\right)\right]$$

$$d\left(\dot{m}V\right) - d\dot{m}_i V_{ix} = -dpA$$

$$Vd\dot{m} + \dot{m}dV - d\dot{m}_i V_{ix} = -dpA$$

$$dp + \frac{\rho V\left(V - V_{ix}\right)d\dot{m}}{\rho VA} + \frac{\rho V\dot{m}dV}{\rho VA} = 0$$

$$\boxed{dp + \rho VdV + \rho V^2\left(1 - y\right)\frac{d\dot{m}}{\dot{m}} = 0} \tag{167}$$

Note: $y \equiv V_{ix}/V$, and $d\dot{m} = d\dot{m}_i$

COE:

$$\left[\dot{m}h_0 + d\left(\dot{m}h_0\right)\right] - \dot{m}h_0 - d\dot{m}_i h_{0i} = 0$$

$$dh_0 + \left(h_0 - h_{0i}\right)\frac{d\dot{m}}{\dot{m}} = 0$$

$$\boxed{dh_0 = 0} \tag{168}$$

Note: For <u>simple</u> mass addition we assume that the inlet fluid has the same composition and stagnation enthalpy as the main stream, i.e., $h_{0i} = h_0$. In general, however, the addition stream may have a different stagnation enthalpy than the main flow. The general case will be considered when we investigate general 1D flows.

2nd Law:

$$\left[\dot{m}s + d\left(\dot{m}s\right)\right] - \dot{m}s - d\dot{m}_i s_i > 0$$

$$\boxed{ds + \left(s - s_i\right)\frac{d\dot{m}}{\dot{m}} > 0} \tag{169}$$

Note: The mass addition stream may have, in general, a different entropy than the main stream. Moreover, mixing processes are generally irreversible.

Ideal Gas Law:

$$\boxed{\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}} \tag{170}$$

caloric Eq of state

$$\boxed{dh = c_p dT} \tag{171}$$

When combined with COE (Eq. (168)):

$$c_p dT_0 = 0 \Rightarrow dT_0 = 0 \tag{172}$$

Mach # relation:

$$\boxed{\frac{d\left(\text{Ma}\right)}{\text{Ma}} = \frac{dV}{V} - \frac{dT}{2T}} \tag{173}$$

Gibbs Eq:

$$\boxed{ds = c_p\frac{dT}{T} - R\frac{dp}{p} = -R\frac{dp_0}{p_0}} \tag{174}$$

Note: $dT_0 = 0$ from Eq. (172).

Local stagnation pressure and temperature:

$$T_0 = T\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right) \tag{175}$$

$$p_0 = p\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{\gamma/\gamma-1} \tag{176}$$

Combining these equations so that $d\dot{m}$ is the driving potential, and using the substitution, $y \equiv V_{ix}/V$, gives:

$$\boxed{\frac{d\left(\mathrm{Ma}\right)}{\mathrm{Ma}} = \left[\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right]\left[\left(1+\gamma\mathrm{Ma}^2\right)-y\gamma\mathrm{Ma}^2\right]\frac{d\dot{m}}{\dot{m}}} \tag{177}$$

$$\boxed{\frac{dp}{p} = \frac{-\gamma\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\left[2\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)\left(1-y\right)+y\right]\frac{d\dot{m}}{\dot{m}}} \tag{178}$$

$$\boxed{\frac{d\rho}{\rho} = \frac{-1}{1-\mathrm{Ma}^2}\left[\left(\gamma+1\right)\mathrm{Ma}^2-y\gamma\mathrm{Ma}^2\right]\frac{d\dot{m}}{\dot{m}}} \tag{179}$$

$$\boxed{\frac{dT}{T} = \frac{-\left(\gamma-1\right)\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\left[\left(1+\gamma\mathrm{Ma}^2\right)-y\gamma\mathrm{Ma}^2\right]\frac{d\dot{m}}{\dot{m}}} \tag{180}$$

$$\boxed{\frac{dV}{V} = \frac{1}{1-\mathrm{Ma}^2}\left[\left(1+\gamma\mathrm{Ma}^2\right)-y\gamma\mathrm{Ma}^2\right]\frac{d\dot{m}}{\dot{m}}} \tag{181}$$

$$\boxed{\frac{dp_0}{p_0} = -\gamma\mathrm{Ma}^2\left(1-y\right)\frac{d\dot{m}}{\dot{m}}} \tag{182}$$

$$\boxed{dT_0 = 0} \tag{183}$$

$$\boxed{\frac{ds}{c_p} = \left(\gamma-1\right)\mathrm{Ma}^2\left(1-y\right)\frac{d\dot{m}}{\dot{m}}} \tag{184}$$

Notes:

1. The trends in Eqs. (177)-(184) will depend on $d\dot{m}$, y, and Ma. Equation (182) shows that for mass addition, the stagnation pressure will decrease if $y < 1$ and will increase if $y > 1$. Equation (184) shows that entropy has the opposite trend.

2. For $y < 1$, all factors that involve y are positive so that for mass addition ($d\dot{m} > 0$) we have:

Ma < 1	Ma > 1
$d(\text{Ma}) > 0$	$d(\text{Ma}) < 0$
$dp < 0$	$dp > 0$
$d\rho < 0$	$d\rho > 0$
$dT < 0$	$dT > 0$
$dV > 0$	$dV < 0$
$dp_0 < 0$	$dp_0 < 0$
$ds > 0$	$ds > 0$

   The opposite trends occur for mass removal ($d\dot{m} < 0$). Note that choking is possible with mass addition since the Mach number approaches unity for both subsonic and supersonic flows.

3. We cannot simply add mass to transition from a subsonic to a supersonic flow. Equation (177) indicates that as the flow approaches a sonic Mach number, the addition stream must have a large mass flow rate to continue approaching the sonic Mach number. In the limit as Ma → 1, $d\dot{m} \to \infty$.

4. For $y = 0$ (i.e., the flow comes in normal to the stream), Eqs. (177)-(184) become exact differentials. Integrating using sonic conditions (denoted by the superscript "*") as a reference gives:

$$\boxed{\frac{\dot{m}}{\dot{m}^*} = \frac{\text{Ma}}{1+\gamma\text{Ma}^2}\left[2(\gamma+1)\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)\right]^{1/2}} \tag{185}$$

$$\boxed{\frac{p}{p^*} = \frac{\gamma+1}{1+\gamma\text{Ma}^2}} \tag{186}$$

$$\boxed{\frac{\rho}{\rho^*} = \frac{2}{1+\gamma\text{Ma}^2}\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)} \tag{187}$$

$$\boxed{\frac{T}{T^*} = \left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)\right]^{-1}} \tag{188}$$

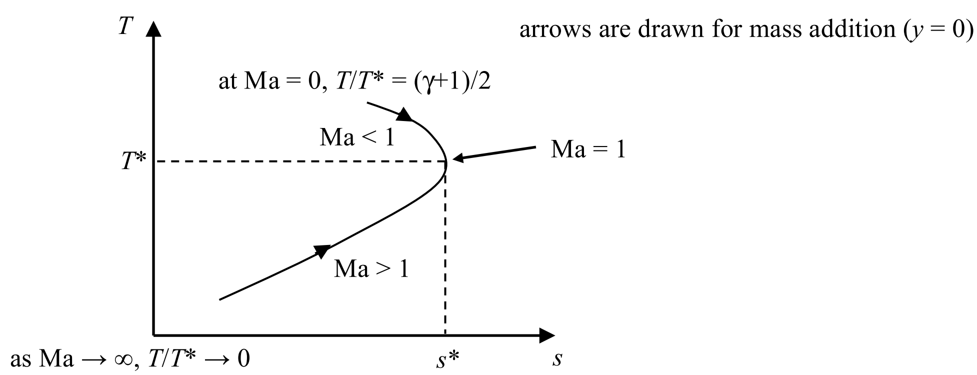
$$\boxed{\frac{V}{V^*} = \text{Ma}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)\right]^{-1/2}} \tag{189}$$

$$\boxed{\frac{p_0}{p_0^*} = \frac{\gamma+1}{1+\gamma\text{Ma}^2}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)\right]^{\gamma/(\gamma-1)}} \tag{190}$$

$$\boxed{\frac{T_0}{T_0^*} = 1} \tag{191}$$

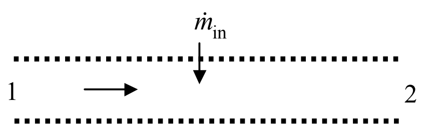
$$\boxed{\frac{s-s^*}{R} = -\ln\frac{p_0}{p_0^*}} \tag{192}$$

5. The relationship between T and s may be found by considering Eqs. (188), (190), and (192). Below is a T-s diagram for a flow with mass addition.



Air with initial stagnation conditions of 600 K and 1 MPa (abs) flows at a Mach number of 0.3 at the entrance to a constant-area, porous-walled duct. During passage through the duct, the mass flow rate is increased by 50%. Find the exit conditions and sketch the process on a *T*-*s* diagram.

SOLUTION:



$\text{Ma}_1 = 0.3$ $\qquad$ $\dot{m}_2 = 1.5\dot{m}_1$
$T_{01} = 600$ K
$p_{01} = 1$ MPa (abs)

Since the wall is porous, assume the incoming air is oriented normal to the main flow, i.e., $y = V_{ix}/V = 0$.

Use the simple mass addition relations at state 1.

$$\text{Ma}_1 = 0.3 \Rightarrow \frac{\dot{m}_1}{\dot{m}^*} = 0.5889 \tag{1}$$

$$\frac{p_1}{p^*} = 2.1314 \tag{2}$$

$$\frac{T_1}{T^*} = 1.1788 \tag{3}$$

$$\frac{p_{01}}{p_0^*} = 1.1985 \tag{4}$$

At state 2:

$$\dot{m}_2 = 1.5\dot{m}_1 \quad \Rightarrow \quad \frac{\dot{m}_2}{\dot{m}^*} = \left(\frac{\dot{m}_2}{\dot{m}_1}\right)\left(\frac{\dot{m}_1}{\dot{m}^*}\right) = (1.5)(0.5889) = 0.8834 \tag{5}$$

Now use the simple mass addition relations to determine conditions at state 2.

$$\frac{\dot{m}_2}{\dot{m}^*} = 0.8834 \quad \Rightarrow \quad \text{Ma}_2 = 0.5663 \tag{6}$$

$$\frac{p_2}{p^*} = 1.6563 \tag{7}$$

$$\frac{T_2}{T^*} = 1.1277 \tag{8}$$

$$\frac{p_{02}}{p_0^*} = 1.0877 \tag{9}$$

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1 \Rightarrow \boxed{p_2 = 0.7301 \text{ MPa (abs)}} \tag{10}$$

where

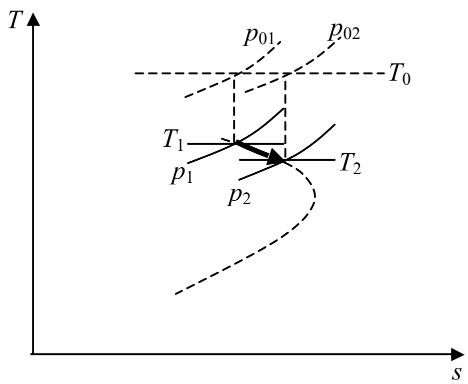
$$\frac{p_1}{p_{01}} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1 = 0.9395 \text{ MPa (abs)} \tag{11}$$

$$T_2 = \left(\frac{T_2}{T^*}\right)\left(\frac{T^*}{T_1}\right)T_1 \Rightarrow \boxed{T_2 = 563.9\text{ K}} \tag{12}$$

where

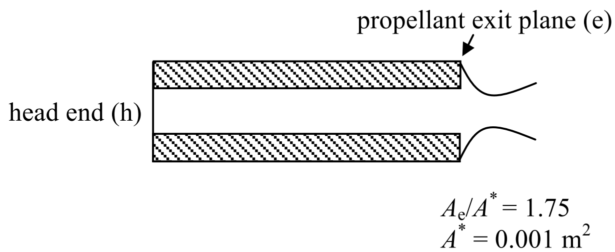
$$\frac{T_1}{T_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow T_1 = 589.4\text{ K} \tag{13}$$

$$p_{02} = \left(\frac{p_{02}}{p_0^*}\right)\left(\frac{p_0^*}{p_{01}}\right)p_{01} \Rightarrow \boxed{p_{02} = 0.9076\text{ MPa (abs)}} \tag{14}$$



A solid rocket configuration is shown in the figure below. With the configuration as given and with a mass flow rate of 5 kg/s, find:

a. the Mach number at the propellant exit plane,
b. the head-end stagnation pressure, and
c. the change in static pressure from the head-end to the propellant exit plane.



$\gamma =$ 1.26
$T_0 =$ 4000.0 K
$R =$ 377.9 J/(kg·K)

SOLUTION:

Use the isentropic sonic area ratio to determine the Mach number at the exit plane. Note that the Mach number should be subsonic since the flow in the mass addition region starts at stagnation conditions.

$$\frac{A_e}{A^*} = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{\text{Ma}_e = 0.3615} \quad \text{(using } A_e/A^* = 1.75 \text{ and } \gamma = 1.26\text{)} \tag{1}$$

Use the simple mass addition relations to determine the stagnation pressure at the head end. The mass flow rate at the head end will be zero and, hence, from the simple mass addition relations:

$$\frac{\dot{m}_h}{\dot{m}^*} = 0 \Rightarrow \frac{p_{0h}}{p_0^*} = (\gamma+1)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow \underline{\frac{p_{0h}}{p_0^*} = 1.2499} \tag{2}$$

The sonic stagnation pressure may be found using:

$$p_0^* = \left(\frac{p_0^*}{p_{0e}}\right)p_{0e} \tag{3}$$

where, from the simple mass addition relations:

$$\text{Ma}_e = 0.3615 \Rightarrow \boxed{p_{0e}/p_0^* = 1.1645} \tag{4}$$

and, from the ideal gas law:

$$\underline{p_{0e} = \rho_{0e}RT_{0e}} \tag{5}$$

The stagnation temperature at the exit plane is given in the problem statement and the stagnation density at the exit plane may be found from the mass flow rate at the exit.

$$\dot{m}_e = \rho_e V_e A_e = \left[\rho_{0e}\left(1+\tfrac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{1}{1-\gamma}}\right]\left[\text{Ma}_e\sqrt{\gamma RT_{0e}\left(1+\tfrac{\gamma-1}{2}\text{Ma}_e^2\right)^{-1}}\right]\left[\left(\frac{A_e}{A^*}\right)A^*\right] \tag{6}$$

$$\underline{\rho_{0e} = \frac{\dot{m}_e}{\left[\left(1+\tfrac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{1}{1-\gamma}}\right]\left[\text{Ma}_e\sqrt{\gamma RT_{0e}\left(1+\tfrac{\gamma-1}{2}\text{Ma}_e^2\right)^{-1}}\right]\left[\left(\frac{A_e}{A^*}\right)A^*\right]}} \tag{7}$$

Use the given data:

γ = 1.26
R = 377.9 J/(kg·K)
Ma_e = 0.3615
T_{0e} = 4000 K
A_e/A^* = 1.75
A^* = 0.001 m^2
$\dot{m}_e$ = 5 kg/s

$\Rightarrow$ ρ_{0e} = 6.162 kg/m^3 (from Eq. (7))
$\Rightarrow$ p_{0e} = 9.314 MPa (from Eq. (5))
$\Rightarrow$ p_0^* = 7.999 MPa (from Eq. (3))
$\Rightarrow$ $\boxed{p_{0h} = 9.998 \text{ MPa}}$ (from Eq. (2))

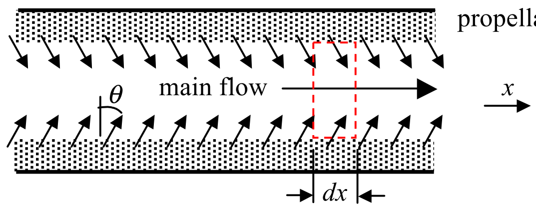
Since the Mach number at the head end is zero, the static pressure there will be equal to the stagnation pressure, i.e., $p_h = p_{0h}$. The pressure at the exit plane is given by:

$$p_e = p_{0e}\left(1+\tfrac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \quad \Rightarrow \quad p_e = 8.584 \text{ MPa} \tag{8}$$

Hence:

$$\boxed{\Delta p = p_e - p_h = -1.414 \text{ MPa} \ \ (\Delta p/p_h = -14.1\%)} \tag{9}$$

You are asked to model the main flow inside a solid rocket motor. A schematic of the flow is shown in the diagram below.



To simplify your model, you may assume that the flow area remains constant and that the flow is 1D, steady, frictionless, and adiabatic. You should, however, include the effects of mass addition to the flow. You should also assume that the mass addition stream enters the flow at an angle of θ with respect to the vertical, and that the mass addition stream has different flow properties than the main stream.

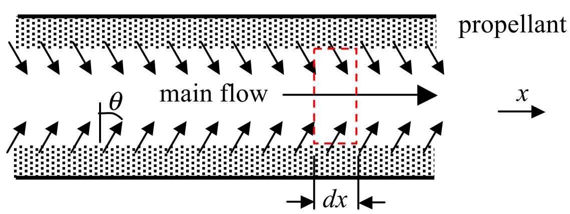
Using the differential control volume shown in the diagram, write the equations for:

a. conservation of mass,
b. the linear momentum principle,
c. conservation of energy, and
d. the second law of thermodynamics.

for the given flow conditions. Clearly state any additional assumptions you make in deriving your relations.

SOLUTION:

Consider the differential control volume shown below.



Conservation of Mass:

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -\left[(\rho VA) + \frac{d(\rho VA)}{dx}\left(-\tfrac{1}{2}dx\right)\right] + \left[(\rho VA) + \frac{d(\rho VA)}{dx}\left(\tfrac{1}{2}dx\right)\right] - \underbrace{d\dot{m}}_{\substack{\text{mass added by} \\ \text{mass addition stream}}} \tag{1}$$

$$= \frac{d(\rho VA)}{dx}dx - d\dot{m}$$

Substitute and simplify.

$$\frac{d(\rho VA)}{dx}dx - d\dot{m} = 0 \tag{2}$$

$$AVd\rho + A\rho dV = d\dot{m} \quad \text{(Note that } A = \text{constant.)}$$

$$\boxed{\frac{d\rho}{\rho} + \frac{dV}{V} = \frac{d\dot{m}}{\dot{m}}} \quad \text{where } \dot{m} = \rho VA \tag{3}$$

Linear Momentum Equation in the *x*-direction

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\begin{aligned}\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) &= -\left[V\dot{m} + \frac{d\left(V\dot{m}\right)}{dx}\left(-\tfrac{1}{2}dx\right)\right] + \left[V\dot{m} + \frac{d\left(V\dot{m}\right)}{dx}\left(\tfrac{1}{2}dx\right)\right] - V_{ix}d\dot{m} \\ &= \frac{d\left(V\dot{m}\right)}{dx}dx - V_{ix}d\dot{m}\end{aligned} \tag{4}$$

(where V_{ix} is the x-component of the velocity of the mass addition (incoming) stream)

$F_{B,x} = 0$ (neglect body forces since the fluid is a gas)

$$F_{S,x} = \left[\left(pA\right) + \frac{d\left(pA\right)}{dx}\left(-\tfrac{1}{2}dx\right)\right] - \left[\left(pA\right) + \frac{d\left(pA\right)}{dx}\left(\tfrac{1}{2}dx\right)\right] = -\frac{d\left(pA\right)}{dx}dx \tag{5}$$

(Note that the flow is assumed frictionless and 1D.)

Substitute and simplify.

$$\frac{d\left(V\dot{m}\right)}{dx}dx - V_{ix}d\dot{m} = -\frac{d\left(pA\right)}{dx}dx \tag{6}$$

$$d\left(V\dot{m}\right) - V_{ix}d\dot{m} = -Adp \quad (A = \text{constant})$$

$$Adp + \dot{m}dV + Vd\dot{m} - V_{ix}d\dot{m} = 0$$

$$dp + \frac{\dot{m}}{A}dV + \left(V - V_{ix}\right)\frac{d\dot{m}}{A} = 0 \quad \text{(Note: } \dot{m} = \rho VA\text{.)}$$

$$\boxed{dp + \rho V dV + \rho V^2\left(1 - \frac{V_{ix}}{V}\right)\frac{d\dot{m}}{\dot{m}}} \tag{7}$$

Conservation of Energy

$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{on CV}}$$

where

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady flow)}$$

$$\begin{aligned}\int_{CS}\left(h+\tfrac{1}{2}V^2+gz\right)\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) &= -\left[\dot{m}h_T + \frac{d\left(\dot{m}h_T\right)}{dx}\left(-\tfrac{1}{2}dx\right)\right] + \left[\dot{m}h_T + \frac{d\left(\dot{m}h_T\right)}{dx}\left(\tfrac{1}{2}dx\right)\right] - h_{iT}d\dot{m} \\ &= \frac{d\left(\dot{m}h_T\right)}{dx}dx - h_{iT}d\dot{m}\end{aligned} \tag{8}$$

(where h_T is the total specific enthalpy of the main flow and h_{iT} is the total specific enthalpy of the mass addition (incoming) stream)

$$\dot{Q}_{\text{into CV}} = 0 \quad \text{(the flow is assumed adiabatic)}$$

$$\dot{W}_{\text{on CV}} = 0 \quad \text{(no work other than pressure is performed on the control volume)} \tag{9}$$

Substitute and simplify.

$$\frac{d\left(\dot{m}h_T\right)}{dx}dx - h_{iT}d\dot{m} = 0 \tag{10}$$

$$d\left(\dot{m}h_T\right) - h_{iT}d\dot{m} = 0$$

$$\dot{m}dh_T + h_T d\dot{m} - h_{iT}d\dot{m} = 0$$

$$\boxed{dh_T + \left(h_T - h_{iT}\right)\frac{d\dot{m}}{\dot{m}} = 0} \tag{11}$$

2nd Law of Thermodynamics

$$\frac{d}{dt}\int_{CV} s\rho\, dV + \int_{CS} s\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) \geq \int_{CV}\frac{\delta\dot{q}_{\text{into CV}}}{T}$$

where

$$\frac{d}{dt}\int_{CV} s\rho dV = 0 \quad \text{(steady flow)}$$

$$\begin{aligned}\int_{CS} s\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) &= -\left[\dot{m}s + \frac{d\left(\dot{m}s\right)}{dx}\left(-\tfrac{1}{2}dx\right)\right] + \left[\dot{m}s + \frac{d\left(\dot{m}s\right)}{dx}\left(\tfrac{1}{2}dx\right)\right] - s_i d\dot{m} \\ &= \frac{d\left(\dot{m}s\right)}{dx}dx - s_i d\dot{m}\end{aligned} \tag{12}$$

(where s_i is the total specific entropy of the mass addition (incoming) stream)

$$\int_{CV}\frac{\delta\dot{q}_{\text{into CV}}}{T} = 0 \quad \text{(the flow is assumed adiabatic)}$$

Substitute and simplify.

$$\frac{d\left(\dot{m}s\right)}{dx}dx - s_i d\dot{m} > 0 \quad \text{(The “>” is used since the streams are mixing, which is irreversible.)} \tag{13}$$

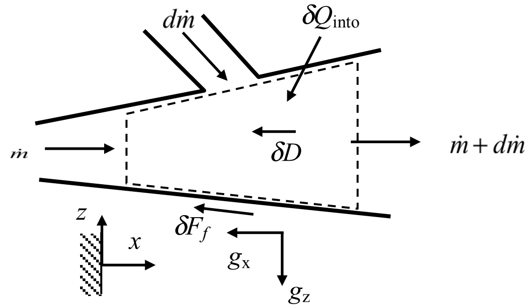
$$d\left(\dot{m}s\right) - s_i d\dot{m} > 0$$

$$\dot{m}ds + sd\dot{m} - s_i d\dot{m} > 0$$

$$\boxed{ds + \left(s - s_i\right)\frac{d\dot{m}}{\dot{m}} > 0} \tag{14}$$

12. Generalized Steady, One-Dimensional Flow

In our previous analyses we have only considered simple flows where there is only one potential affecting the flow (e.g., area change, friction, heat transfer, or mass addition). Now we'll discuss how to analyze flows where multiple effects are considered. Our approach is very similar to the approach we have used many times before: we'll draw a control volume, apply our conservation laws, utilize definitions, and simplify our resulting equations.



In our analysis, we'll include the effects of:
- area change, dA
- friction, δF_f
- heat transfer, δQ
- mass addition, $d\dot{m}$
- "other" work, δW
- other forces, δD
- gravity body force, $\rho\mathbf{g}Adx$

We'll further assume that the flow is steady, one-dimensional, and that the fluid is well-mixed within the control volume.

Now let's apply our conservation laws:

COM:

$$\left(\dot{m}+d\dot{m}\right)-\dot{m}-d\dot{m}_i=0$$

$$d\dot{m}=d\left(\rho VA\right)=d\dot{m}_i$$

$$\boxed{\frac{d\dot{m}}{\dot{m}}=\frac{d\rho}{\rho}+\frac{dV}{V}+\frac{dA}{A}} \tag{193}$$

LME (in x-direction):

$$\left[\dot{m}V + d\left(\dot{m}V\right)\right] - \dot{m}V - d\dot{m}_i V_{ix} =$$

$$pA - \left[pA + d\left(pA\right)\right] + \left(p + \tfrac{1}{2}dp\right)dA - \delta D - \delta F_f - \rho g_x \left(A + \tfrac{1}{2}dA\right)dx$$

$$d\dot{m}V + \dot{m}dV - d\dot{m}_i V_{ix} = -dpA - \delta D - \delta F_f - \rho g_x A dx$$

$$dpA + \dot{m}dV + \rho g_x A dx + \delta F_f + \delta D + d\dot{m}\left(V - V_{ix}\right) = 0$$

Substituting the following:

$\dot{m} = \rho V A$, $y \equiv V_{ix}/V$, and $\delta F_f = \tau_w P dx = f_F \tfrac{1}{2}\rho V^2 P dx = \tfrac{1}{2}\rho V^2 \left(4 A f_F / D_H\right)$:

$$\boxed{dp + \rho V dV + \rho g_x dx + \tfrac{1}{2}\rho V^2 \left(\frac{4 f_F dx}{D_H}\right) + \frac{\delta D}{A} + \rho V^2 \left(1 - y\right)\frac{d\dot{m}}{\dot{m}} = 0} \tag{194}$$

COE:

$$\left[\dot{m}h_0 + d\left(\dot{m}h_0\right)\right] - \dot{m}h_0 - d\dot{m}_i h_{0i} = \delta \dot{Q}_{\text{into CV}} + \delta \dot{W}_{\text{on CV}}$$

$$\boxed{dh_0 + \left(h_0 - h_{0i}\right)\frac{d\dot{m}}{\dot{m}} = \delta q_{\text{into CV}} + \delta w_{\text{on CV}}} \tag{195}$$

The term in parentheses represents the difference in the stagnation enthalpy of the main stream and the incoming flow. Note that: $h_0 = h + \frac{1}{2}V^2 + g_z z$.

2nd Law:

$$\left[\dot{m}s + d\left(\dot{m}s\right)\right] - \dot{m}s - d\dot{m}_i s_i \geq \int_{\text{CV}} \frac{\delta \dot{Q}_{\text{into CV}}}{T}$$

$$\boxed{ds + \left(s - s_i\right)\frac{d\dot{m}}{\dot{m}} \geq \int_{\text{CV}} \frac{\delta q_{\text{into CV}}}{T}} \tag{196}$$

Now let's specify that we're dealing with an ideal gas:

$$\boxed{p = \rho R T} \tag{197}$$

$$\boxed{dh = c_p dT} \tag{198}$$

$$\boxed{ds = c_p \frac{dT}{T} - R\frac{dp}{p}} \tag{199}$$

$$\boxed{\text{Ma} = \frac{V}{\sqrt{\gamma R T}}} \tag{200}$$

Since we're concerned with a gas, we'll also assume that gravitational effects are negligible compared to the other terms in the equations (i.e., $\mathbf{g} \approx \mathbf{0}$).

We'll also utilize the definitions of the isentropic stagnation pressure and the adiabatic stagnation temperature:

$$\boxed{\frac{p}{p_0}=\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{\gamma/1-\gamma}} \tag{201}$$

$$\boxed{\frac{T}{T_0}=\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{-1}} \tag{202}$$

Equations (193)-(202) are a system of equations that can be combined and solved[1] for the dependent variables:

$$\frac{d(\mathrm{Ma})}{\mathrm{Ma}},\frac{dp}{p},\frac{d\rho}{\rho},\frac{dT}{T},\frac{dV}{V},\frac{dp_0}{p_0},\frac{ds}{c_p}$$

in terms of the independent variables, or driving potentials:

$$\frac{dA}{A},\left[\left(\frac{4f_F dx}{D_H}+\frac{2\gamma\delta D}{\gamma\mathrm{Ma}^2 pA}\right)\right],\frac{dT_0}{T_0},\frac{d\dot{m}}{\dot{m}}$$

The following table summarizes the resulting equations.

[1] For details, refer to Zucrow and Hoffman, *Gas Dynamics: Volume I*, Wiley.

Table 1. Change in flow properties in terms of driving potentials.

Change in Flow Property	Driving Potentials			
	$\frac{dA}{A}$	$\left[\left(\frac{4 f_F dx}{D_H}\right)+\frac{2\delta D}{\gamma \mathrm{Ma}^2 pA}\right]$	$\frac{dT_0}{T_0}$	$\frac{d\dot{m}}{\dot{m}}$
$\frac{d(\mathrm{Ma})}{\mathrm{Ma}}$	$-\frac{\Psi}{1-\mathrm{Ma}^2}$	$\frac{\gamma \mathrm{Ma}^2\Psi}{2\left(1-\mathrm{Ma}^2\right)}$	$\frac{\left(1+\gamma \mathrm{Ma}^2\right)\Psi}{2\left(1-\mathrm{Ma}^2\right)}$	$\frac{\Psi\left[\left(1+\gamma \mathrm{Ma}^2\right)-y\gamma \mathrm{Ma}^2\right]}{1-\mathrm{Ma}^2}$
$\frac{dp}{p}$	$\frac{\gamma \mathrm{Ma}^2}{1-\mathrm{Ma}^2}$	$-\frac{\gamma \mathrm{Ma}^2\left[1+(\gamma-1)\mathrm{Ma}^2\right]}{2\left(1-\mathrm{Ma}^2\right)}$	$-\frac{\gamma \mathrm{Ma}^2\Psi}{1-\mathrm{Ma}^2}$	$-\frac{\gamma \mathrm{Ma}^2\left[2\Psi(1-y)+y\right]}{1-\mathrm{Ma}^2}$
$\frac{d\rho}{\rho}$	$\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}$	$-\frac{\gamma \mathrm{Ma}^2}{2\left(1-\mathrm{Ma}^2\right)}$	$-\frac{\Psi}{1-\mathrm{Ma}^2}$	$-\frac{\left[(\gamma+1)\mathrm{Ma}^2-y\gamma \mathrm{Ma}^2\right]}{1-\mathrm{Ma}^2}$
$\frac{dT}{T}$	$\frac{(\gamma-1)\mathrm{Ma}^2}{1-\mathrm{Ma}^2}$	$-\frac{\gamma(\gamma-1)\mathrm{Ma}^4}{2\left(1-\mathrm{Ma}^2\right)}$	$\frac{\left(1-\gamma \mathrm{Ma}^2\right)\Psi}{1-\mathrm{Ma}^2}$	$-\frac{(\gamma-1)\mathrm{Ma}^2\left[\left(1+\gamma \mathrm{Ma}^2\right)-y\gamma \mathrm{Ma}^2\right]}{1-\mathrm{Ma}^2}$
$\frac{dV}{V}$	$-\frac{1}{1-\mathrm{Ma}^2}$	$\frac{\gamma \mathrm{Ma}^2}{2\left(1-\mathrm{Ma}^2\right)}$	$\frac{\Psi}{1-\mathrm{Ma}^2}$	$\frac{\left[\left(1+\gamma \mathrm{Ma}^2\right)-y\gamma \mathrm{Ma}^2\right]}{1-\mathrm{Ma}^2}$
$\frac{dp_0}{p_0}$	0	$-\frac{\gamma \mathrm{Ma}^2}{2}$	$-\frac{\gamma \mathrm{Ma}^2}{2}$	$-\gamma \mathrm{Ma}^2(1-y)$
$\frac{dF}{F}$	$\frac{1}{1+\gamma \mathrm{Ma}^2}$	$-\frac{\gamma \mathrm{Ma}^2}{2\left(1+\gamma \mathrm{Ma}^2\right)}$	0	$\frac{y\gamma \mathrm{Ma}^2}{1+\gamma \mathrm{Ma}^2}$
$\frac{ds}{c_p}$	0	$\frac{(\gamma-1)\mathrm{Ma}^2}{2}$	Ψ	$(\gamma-1)\mathrm{Ma}^2(1-y)$

where $\Psi = 1+\frac{\gamma-1}{2}\mathrm{Ma}^2$

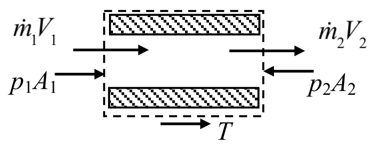
$y = \frac{V_{ix}}{V}$

$F \equiv$ impulse function $= pA + \dot{m}V$

Notes:

1. The terms in the table are often referred to as **influence coefficients**.

2. The **impulse function**, $F \equiv pA + \dot{m}V$, is a convenient definition that is helpful in determining the reaction force for one-dimensional steady flows. For example, the thrust on the jet engine shown below can be determined from the difference between the outgoing and incoming impulse functions:



$$\begin{aligned}
&\dot{m}_2 V_2 - \dot{m}_1 V_1 = T + p_1 A_1 - p_2 A_2 \\
&T = \left(\dot{m}_2 V_2 + p_2 A_2\right) - \left(\dot{m}_1 V_1 + p_1 A_1\right) \\
&\therefore T = F_2 - F_1
\end{aligned}$$

3. As can be seen from Eqs. (195) and (198), the effects of heat transfer, other work, and the difference in the main and incoming stream stagnation enthalpies are all included in the change in the stagnation temperature, dT_0.

4. How one uses the table is best shown by example. Let's say we're interested in determining how the Mach number varies as a function of the driving potentials. From the table we see that the Mach number variation is given by:

$$\begin{aligned}
\frac{d\left(\mathrm{Ma}\right)}{\mathrm{Ma}} &= -\frac{\Psi}{1-\mathrm{Ma}^2}\frac{dA}{A} + \frac{\gamma \mathrm{Ma}^2 \Psi}{2\left(1-\mathrm{Ma}^2\right)}\left[\left(\frac{4 f_F dx}{D_H}\right) + \frac{2\delta D}{\gamma \mathrm{Ma}^2 pA}\right] \\
&\quad + \frac{\left(1+\gamma \mathrm{Ma}^2\right)\Psi}{2\left(1-\mathrm{Ma}^2\right)}\frac{dT_0}{T_0} + \frac{\Psi\left[\left(1+\gamma \mathrm{Ma}^2\right) - y\gamma \mathrm{Ma}^2\right]}{1-\mathrm{Ma}^2}\frac{d\dot{m}}{\dot{m}}
\end{aligned} \tag{203}$$

   We would now need to know how to model the driving potentials as we move downstream in flow (e.g., How does the area vary as we move downstream in the duct?) Finally, we would integrate the resulting equation (numerically if necessary) to solve for the Mach number variation.

5. We can also extract trends from the influence coefficients. For example, Eq. (203) may be written as:

$$\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \frac{\Psi}{1-\mathrm{Ma}^2}\underbrace{\left\{\begin{array}{l} -\dfrac{dA}{A} + \dfrac{\gamma \mathrm{Ma}^2}{2}\left[\left(\dfrac{4f_F dx}{D_H}\right) + \dfrac{2\delta D}{\gamma \mathrm{Ma}^2 pA}\right] \\ +\dfrac{\left(1+\gamma \mathrm{Ma}^2\right)}{2}\dfrac{dT_0}{T_0} + \left[\left(1+\gamma \mathrm{Ma}^2\right) - y\gamma \mathrm{Ma}^2\right]\dfrac{d\dot{m}}{\dot{m}} \end{array}\right\}}_{\equiv \Lambda}$$

Note that Ψ is always positive.

a. If $\Lambda < 0$, then the Mach number will decrease for subsonic flow and increase for supersonic flow, *i.e.* the Mach number diverges from one.

b. If $\Lambda > 0$, then the Mach number will increase toward one for subsonic Mach numbers while the Mach number will decrease toward one for supersonic Mach numbers. Hence, choking is possible if $\Lambda > 0$.

c. If $\Lambda = 0$, then the Mach number does not change since $d(\mathrm{Ma}) = 0$ (the flow is at an inflection point).

It is possible that a flow may have changes in the sign of Λ as the flow moves downstream. For example, simple isentropic flow in a converging-diverging nozzle has $\Lambda > 0$ in the converging section, $\Lambda = 0$ at the throat, and $\Lambda < 0$ in the diverging section.

Note that sonic conditions (Ma = 1) can only occur where $\Lambda = 0$ otherwise $d(\mathrm{Ma})$ would be infinite. For example, consider the special case where only friction and area changes are present ($\delta D = dT_0 = d\dot{m} = 0$) so that:

$$\Lambda = -\frac{dA}{A} + \frac{\gamma \mathrm{Ma}^2}{2}\left(\frac{4f_F dx}{D_H}\right)$$

Since the friction term will always be positive, and since sonic conditions must occur when $\Lambda = 0$, the sonic point must occur in a diverging section ($dA > 0$). The exact location of the sonic point can be determined since we know how the area varies with x:

$$0 = -\underbrace{\frac{1}{A}\frac{dA}{dx}dx}_{=\frac{dA}{A}} + \frac{\gamma(1)^2}{2}\left(\frac{4f_F dx}{D_H}\right)$$

$$\frac{1}{A}\frac{dA}{dx} = \frac{\gamma}{2}\left(\frac{4f_F}{D_H}\right)$$ (Since $A(x)$, f_F, and D_H are given, one can solve for x.)

6. Note that many of the relations in Table 1 have a $(1 - \text{Ma}^2)$ in the denominator. Hence, one must proceed with care when integrating the relations near the sonic point since the gradients become very large there. At the sonic point in particular, terms with $(1 - \text{Ma}^2)$ in the denominator are undefined so they cannot be integrated directly. If the independent variables can be expressed explicitly as a function of x, gradients at the sonic point can be resolved using l'Hopital's rule. For example, let's again consider the case for a frictional flow with area change ($\delta D = dT_0 = d\dot{m} = 0$) so that the Mach number relation is given by:

$$\frac{d(\text{Ma})}{dx} = \frac{\text{Ma}\Psi}{1-\text{Ma}^2}\underbrace{\left[-\frac{1}{A}\frac{dA}{dx} + \frac{\gamma \text{Ma}^2}{2}\left(\frac{4f_F}{D_H}\right)\right]}_{\equiv\lambda} = \frac{\text{Ma}\Psi\lambda(x)}{1-\text{Ma}^2}$$

$$\left(\frac{d(\text{Ma})}{dx}\right)^* = \lim_{\text{Ma}\to 1}\frac{d(\text{Ma})}{dx} = \left.\frac{\frac{d}{dx}(\text{Ma}\Psi\lambda)}{-2\text{Ma}\frac{d}{dx}(\text{Ma})}\right|_{\substack{x=x^*\\ \text{Ma}=1}}$$

$$\left[\left(\frac{d(\text{Ma})}{dx}\right)^*\right]^2 = -\frac{1}{2}\left.\frac{d(\text{Ma}\Psi\lambda)}{dx}\right|_{\substack{x=x^*\\ \text{Ma}=1}}$$

$$= -\frac{(\gamma+1)}{4}\left.\frac{d\lambda}{dx}\right|_{\substack{x=x^*\\ \text{Ma}=1}} - \left(\frac{3\gamma-1}{4}\right)\lambda\Big|_{\substack{x=x^*\\ \text{Ma}=1}}\left(\frac{d(\text{Ma})}{dx}\right)^*$$

$$\left[\left(\frac{d(\text{Ma})}{dx}\right)^*\right]^2 + \left(\frac{3\gamma-1}{4}\right)\lambda\Big|_{\substack{x=x^*\\ \text{Ma}=1}}\left(\frac{d(\text{Ma})}{dx}\right)^* + \frac{(\gamma+1)}{4}\left.\frac{d\lambda}{dx}\right|_{\substack{x=x^*\\ \text{Ma}=1}} = 0 \tag{204}$$

where

$$\lambda\Big|_{\substack{x=x^*\\ \text{Ma}=1}} = -\left(\frac{1}{A}\frac{dA}{dx}\right)_{x=x^*} + \frac{\gamma}{2}\left(\frac{4f_F}{D_H}\right) \tag{205}$$

$$\left.\frac{d\lambda}{dx}\right|_{\substack{x=x^*\\ \text{Ma}=1}} = -\frac{d}{dx}\left(\frac{1}{A}\frac{dA}{dx}\right)_{x=x^*} + \gamma\left(\frac{4f_F}{D_H}\right)\left(\frac{d(\text{Ma})}{dx}\right)^* \tag{206}$$

Equations (204), (205), and (206) can be combined to give a single quadratic equation in terms of the unknown sonic Mach number gradient. Two solutions can be found with the appropriate one being determined by the downstream boundary conditions (Recall our previous discussions concerning isentropic flow through a C-D nozzle. Whether the flow remains subsonic or supersonic after the throat depends on the back pressure.)

7. The previous generalized flow table (Table 1) can be simplified to give our previous results for simple flows as shown in the following table.

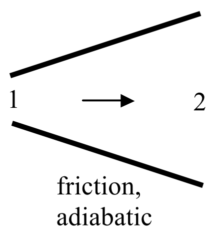
	Simple Area Change	Simple Frictional Flow	Simple Diabatic	Simple Mass Addition, $y=0$
$\frac{A}{A^*}$	$\frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{(\gamma+1)/2(\gamma-1)}$	1	1	1
$\frac{T_0}{T_0^*}$	1	1	$\frac{2(\gamma+1)\mathrm{Ma}^2}{\left(1+\gamma\mathrm{Ma}^2\right)^2}$	1
$\frac{4f_F L^*}{D_H}$	0	$\frac{1-\mathrm{Ma}^2}{\gamma\mathrm{Ma}^2}+\left(\frac{\gamma+1}{2\gamma}\right)\ln\left\{\mathrm{Ma}^2\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-1}\right\}$	0	0
$\frac{\dot{m}}{\dot{m}^*}$	1	1	1	$\frac{\mathrm{Ma}\sqrt{2(\gamma+1)\Psi}}{1+\gamma\mathrm{Ma}^2}$
$\frac{p}{p^*}$	$\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{\gamma/(1-\gamma)}$	$\frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-1/2}$	$\frac{\gamma+1}{1+\gamma\mathrm{Ma}^2}$	$\frac{\gamma+1}{1+\gamma\mathrm{Ma}^2}$
$\frac{\rho}{\rho^*}$	$\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{1/(1-\gamma)}$	$\frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{1/2}$	$\frac{1+\gamma\mathrm{Ma}^2}{(\gamma+1)\mathrm{Ma}^2}$	$\frac{2\Psi}{1+\gamma\mathrm{Ma}^2}$
$\frac{T}{T^*}$	$\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-1}$	$\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-1}$	$\frac{(\gamma+1)^2\mathrm{Ma}^2}{\left(1+\gamma\mathrm{Ma}^2\right)^2}$	$\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-1}$
$\frac{V}{V^*}$	$\mathrm{Ma}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-1/2}$	$\mathrm{Ma}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-1/2}$	$\frac{(\gamma+1)\mathrm{Ma}^2}{1+\gamma\mathrm{Ma}^2}$	$\mathrm{Ma}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-1/2}$
$\frac{p_0}{p_0^*}$	1	$\frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{(\gamma+1)/2(\gamma-1)}$	$\frac{\gamma+1}{1+\gamma\mathrm{Ma}^2}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{\gamma/(\gamma-1)}$	$\frac{\gamma+1}{1+\gamma\mathrm{Ma}^2}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{\gamma/(\gamma-1)}$
$\frac{F}{F^*}$	$\frac{1+\gamma\mathrm{Ma}^2}{\mathrm{Ma}\sqrt{2(\gamma+1)\Psi}}$	$\frac{1+\gamma\mathrm{Ma}^2}{\mathrm{Ma}\sqrt{2(\gamma+1)\Psi}}$	1	1
$\frac{s-s^*}{c_p}$	0	$\ln\left\{\mathrm{Ma}^{(\gamma-1)/\gamma}\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-(\gamma+1)/2\gamma}\right\}$	$\ln\left\{\mathrm{Ma}^2\left(\frac{\gamma+1}{1+\gamma\mathrm{Ma}^2}\right)^{(\gamma+1)/\gamma}\right\}$	$\ln\left\{\left[\left(\frac{2}{\gamma+1}\right)\Psi\right]^{-1}\left(\frac{1+\gamma\mathrm{Ma}^2}{\gamma+1}\right)^{(\gamma-1)/\gamma}\right\}$

Table 2. Change in flow properties for simple flows.

A stream flowing in an insulated, circular duct with friction is to be maintained at constant Mach Number through suitable changes in duct area. Assume that a 1D treatment is acceptable and that at section 1 of the duct the properties are Ma_1, p_1, A_1, etc.

a. Show that the product of the area and pressure is the same for all cross sections.
b. Find an expression for the area at a point downstream of section 1 in terms of the area at 1, the Mach number, the (Fanning) friction coefficient, and the number of length-to-diameter ratios (based on D_1) between section 1 and the downstream section.
c. If $\mathrm{Ma}_1 = 0.5$, $p_1 = 1$ atm (abs), and $f_F = 0.005$, compute the ratios A_2/A_1, p_2/p_1, and p_{02}/p_{01} if section 2 is 50 diameters downstream of section 1.

SOLUTION:



Apply conservation of mass between any two stations.

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \tag{1}$$

Combine with the ideal gas law and the definition of the Mach number.

$$\left(\frac{p_1}{RT_1}\right)\mathrm{Ma}_1\sqrt{\gamma R T_1}A_1 = \left(\frac{p_2}{RT_2}\right)\mathrm{Ma}_2\sqrt{\gamma R T_2}A_2 \tag{2}$$

Since the Mach number, the specific heat ratio, and gas constant are the same at stations 1 and 2, the previous equation may be simplified.

$$\left(\frac{p_1}{\sqrt{T_1}}\right)A_1 = \left(\frac{p_2}{\sqrt{T_2}}\right)A_2 \tag{3}$$

Since the duct is insulated, the stagnation temperature, T_0, will remain constant (from conservation of energy). As a result, the static temperature will also remain constant (since the Mach number is constant).

$$\frac{T}{T_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{-1} \quad \Rightarrow \text{ If Ma = constant and } T_0 = \text{constant, then } T = \text{constant.} \tag{4}$$

Thus, Eqn. (3) becomes:

$$\boxed{p_1 A_1 = p_2 A_2} \tag{5}$$

For generalized, 1D, steady flow with area change and friction:

$$\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \left(\frac{-\Psi}{1-\mathrm{Ma}^2}\right)\frac{dA}{A} + \left[\frac{\gamma \mathrm{Ma}^2 \Psi}{2\left(1-\mathrm{Ma}^2\right)}\right]\frac{4 f_F\, dx}{D_H} \tag{6}$$

But since the Mach number is constant, $d(\mathrm{Ma}) = 0$ and the previous equation becomes:

$$\frac{dA}{A} = \left(\frac{\gamma \mathrm{Ma}^2}{2}\right)\frac{4 f_F\, dx}{D_H} \tag{7}$$

Since the duct is circular,

$$\frac{dA}{A} = \frac{\frac{\pi}{2} D\,dD}{\frac{\pi}{4} D^2} = 2\frac{dD}{D} \quad \text{and } D_H = D \tag{8}$$

Substitute into Eq. (7) and simplify.

$$dD = \gamma \mathrm{Ma}^2 f_F dx \tag{9}$$

$$\int_{D_1}^{D_2} dD = \gamma \mathrm{Ma}^2 f_F \int_{x_1}^{x_2} dx \tag{10}$$

$$D_2 - D_1 = \gamma \mathrm{Ma}^2 f_F \left(x_2 - x_1\right) \tag{11}$$

$$\frac{D_2}{D_1} = 1 + \gamma \mathrm{Ma}^2 f_F \frac{L}{D_1} \quad \text{where } L = x_2 - x_1 \tag{12}$$

$$\boxed{\frac{A_2}{A_1} = \left[1 + \gamma \mathrm{Ma}^2 f_F \frac{L}{D_1}\right]^2} \tag{13}$$

If $\mathrm{Ma}_1 = 0.5$
$p_1 = 1$ atm (abs)
$f_F = 0.005$
$\gamma = 1.4$
$L/D_1 = 50$
then $\boxed{A_2/A_1 = 1.183}$

From Eq. (5),

$$\boxed{p_2/p_1 = A_1/A_2 = 0.845} \tag{14}$$

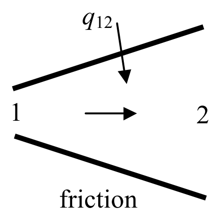
The stagnation pressure ratio is given by:

$$\frac{p_{02}}{p_{01}} = \left(\frac{p_2}{p_1}\right)\left(\frac{p_{02}}{p_2}\right)\left(\frac{p_1}{p_{01}}\right) = \left(\frac{p_2}{p_1}\right)\left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}}\left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{\gamma-1}} \tag{15}$$

$$\boxed{\therefore \frac{p_{02}}{p_{01}} = \frac{p_2}{p_1} = 0.845} \quad \text{since } \mathrm{Ma}_1 = \mathrm{Ma}_2 \tag{16}$$

Consider steady air flow through a duct that has a circular cross-sectional shape. The inlet diameter of the duct is 6 cm and the duct has a length of 1.5 m. The air enters the duct with a Mach number of 0.35 and a temperature of 40 °C. Heat is added to the flow in the duct at a uniform rate, which is such that the stagnation temperature increases by 246 K over the length of the duct. If the (Fanning) friction factor is assumed to be 0.003, determine the Mach number and temperature variations along the duct if:
a. its diameter increases linearly to a final diameter of 7.2 cm.
b. its diameter increases linearly to a final diameter of 6.12 cm.
c. its diameter remains constant.

SOLUTION:



For generalized, 1D, steady flow with area change, friction, and heat transfer:

$$\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \left(\frac{-\Psi}{1-\mathrm{Ma}^2}\right)\frac{dA}{A} + \left[\frac{\gamma \mathrm{Ma}^2}{2\left(1-\mathrm{Ma}^2\right)}\right]\frac{4 f_F dx}{D_H} + \left[\frac{\left(1+\mathrm{Ma}^2\right)\Psi}{2\left(1-\mathrm{Ma}^2\right)}\right]\frac{dT_0}{T_0} \tag{1}$$

where

$$\Psi = 1 + \frac{\gamma-1}{2}\mathrm{Ma}^2 \tag{2}$$

and

$$\frac{dA}{A} = \frac{\frac{\pi}{2} D dD}{\frac{\pi}{4} D^2} = 2\frac{dD}{D} \tag{3}$$

Since D varies linearly:

$$dD = \left(\frac{D_2 - D_1}{L}\right)dx \tag{4}$$

Also, since heat is added uniformly over the length:

$$dT_0 = \left(\frac{T_{02} - T_{01}}{L}\right)dx \tag{5}$$

Substitute Eqs. (3) – (5) into Eq. (1).

$$\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \left(\frac{-\Psi}{1-\mathrm{Ma}^2}\right)\left(\frac{D_2 - D_1}{L}\right)\frac{2dx}{D} + \left[\frac{\gamma \mathrm{Ma}^2}{2\left(1-\mathrm{Ma}^2\right)}\right]\frac{4 f_F dx}{D} + \left[\frac{\left(1+\mathrm{Ma}^2\right)\Psi}{2\left(1-\mathrm{Ma}^2\right)}\right]\left(\frac{T_{02} - T_{01}}{T_0}\right)\frac{dx}{L} \tag{6}$$

Use a simple numerical scheme to solve Eq. (1) as a function of distance moved downstream.

$$x_{n+1} = x_n + \Delta x \tag{7}$$

$$D_{n+1} = D_n + \left(\frac{D_2 - D_1}{L}\right)\Delta x \quad \text{(making use of Eq. (4))} \tag{8}$$

$$T_{0n+1} = T_{0n} + \left(\frac{T_{02} - T_{01}}{L}\right)\Delta x \quad \text{(making use of Eq. (5))} \tag{9}$$

$$\frac{\Delta(\mathrm{Ma})_n}{\mathrm{Ma}_n} = \left(\frac{-\Psi_n}{1-\mathrm{Ma}_n^2}\right)\left(\frac{D_2 - D_1}{L}\right)\frac{2\Delta x}{D_n} + \left[\frac{\gamma \mathrm{Ma}_n^2}{2\left(1-\mathrm{Ma}_n^2\right)}\right]\frac{4 f_F \Delta x}{D_n} + \left[\frac{\left(1+\mathrm{Ma}_n^2\right)\Psi_n}{2\left(1-\mathrm{Ma}_n^2\right)}\right]\left(\frac{T_{02} - T_{01}}{T_{0n}}\right)\frac{\Delta x}{L} \tag{10}$$

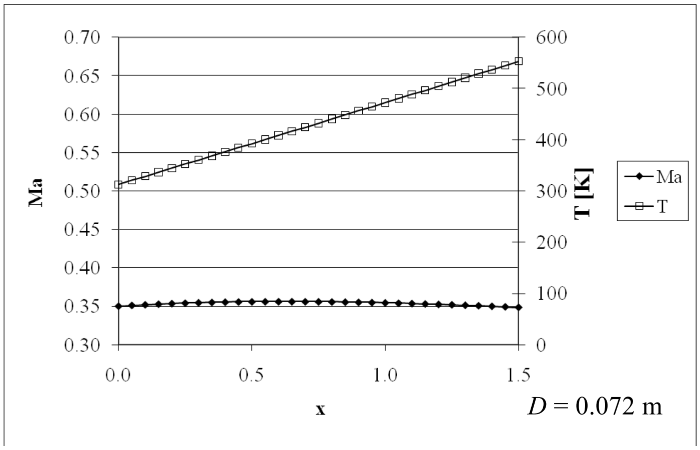
where

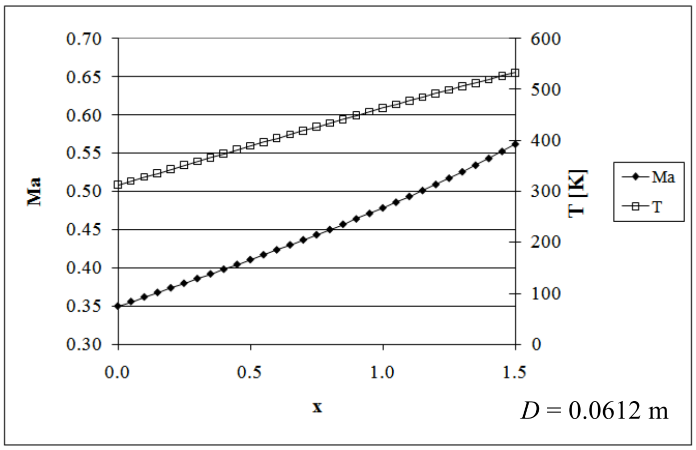
$$\Psi_n = 1 + \frac{\gamma - 1}{2}\mathrm{Ma}_n^2 \quad \text{(making use of Eqs. (2) and (6))} \tag{11}$$

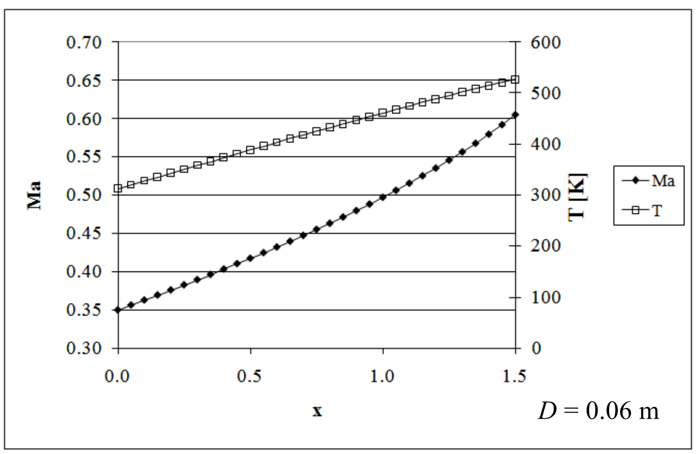
$$\mathrm{Ma}_{n+1} = \mathrm{Ma}_n + \Delta(\mathrm{Ma})_n \tag{12}$$

Use the following parameters as boundary conditions.

γ = 1.4
L = 1.5 m
D_1 = 0.06 m
Ma_1 = 0.35
f_F = 0.003
T_{01} = 320.7 K using $T_{01} = T_1\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_1^2\right)$ with T_1 = (40 + 273) K = 313 K (13)
T_{02} = 566.7 K using $T_{02} = T_{01} + 246$ K (14)
D_2 = 0.072 m (case 1); 0.0612 m (case 2); 0.06 m (case 3)


0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
Ma
600
500
400
300
200
100
0
T [K]
0.0
0.5
1.0
1.5
x
Ma
T
D = 0.072 m


0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
Ma
600
500
400
300
200
100
0
T [K]
0.0
0.5
1.0
1.5
x
Ma
T
D = 0.0612 m


0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
Ma
600
500
400
300
200
100
0
T [K]
0.0
0.5
1.0
1.5
x
Ma
T
D = 0.06 m

Air flows adiabatically in a tube of circular cross-section with an initial Mach number of 0.5, temperature of 1070 K, and pressure of 690 kPa (abs). The tube is to be changed in cross-sectional area so that, taking friction into account, there is no change in the temperature of the stream. Assuming the (Fanning) friction coefficient is 0.005 and that the exit is 100 initial tube diameters downstream of the inlet, find:

a. the final Mach number
b. the ratio of the final diameter to the initial diameter
c. the final stagnation pressure (in kPa)

SOLUTION:

For a 1D flow with area change and friction:

$$\frac{dT}{T}=\frac{(\gamma-1)\text{Ma}^2}{1-\text{Ma}^2}\left(\frac{dA}{A}\right)-\frac{\gamma(\gamma-1)\text{Ma}^4}{2\left(1-\text{Ma}^2\right)}\left(\frac{4f_F dx}{D}\right) \tag{1}$$

For an isothermal flow, $dT = 0$. Also, for a circular duct:

$$A=\frac{\pi D^2}{4}\Rightarrow dA=\frac{\pi D dD}{2}\Rightarrow\frac{dA}{A}=2\frac{dD}{D} \tag{2}$$

Substitute into Eq. (1) and simplify.

$$0=\frac{(\gamma-1)\text{Ma}^2}{1-\text{Ma}^2}\left(2\frac{dD}{D}\right)-\frac{\gamma(\gamma-1)\text{Ma}^4}{2\left(1-\text{Ma}^2\right)}\left(\frac{4f_F dx}{D}\right) \tag{3}$$

$$\underline{\therefore dx=\frac{dD}{\gamma f_F \text{Ma}^2}} \tag{4}$$

For a 1D flow with area change and friction, we also have:

$$\frac{d(\text{Ma})}{\text{Ma}}=\frac{-\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)}{1-\text{Ma}^2}\left(\frac{dA}{A}\right)+\frac{\gamma\text{Ma}^2\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)}{2\left(1-\text{Ma}^2\right)}\left(\frac{4f_F dx}{D}\right) \tag{5}$$

Substitute Eqns. (2) and (4) and simplify.

$$\frac{d(\text{Ma})}{\text{Ma}}=\frac{-\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)}{1-\text{Ma}^2}\left(2\frac{dD}{D}\right)+\frac{\gamma\text{Ma}^2\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)}{2\left(1-\text{Ma}^2\right)}\left(\frac{4f_F\frac{dD}{\gamma f_F\text{Ma}^2}}{D}\right) \tag{6}$$

$$\frac{d(\text{Ma})}{\text{Ma}}=\frac{\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)}{\left(1-\text{Ma}^2\right)}\left[-2\frac{dD}{D}+2\frac{dD}{D}\right]$$

$$d(\text{Ma})=0$$

$$\underline{\therefore \text{Ma}=\text{constant}} \tag{7}$$

Hence, since $\text{Ma}_1 = 0.5$, $\boxed{\text{Ma}_2 = 0.5}$.

The final-to-initial diameter ratio may be found from Eq. (4) utilizing the fact that the Mach number remains constant.

$$\int_{x=0}^{x=L} dx = \int_{D=D_1}^{D=D_2} \frac{dD}{\gamma f_F \mathrm{Ma}^2} \tag{8}$$

$$L = \frac{D_2 - D_1}{\gamma f_F \mathrm{Ma}^2} = \frac{D_1}{\gamma f_F \mathrm{Ma}^2}\left(\frac{D_2}{D_1} - 1\right)$$

$$\boxed{\frac{D_2}{D_1} = 1 + \gamma f_F \mathrm{Ma}^2 \frac{L}{D_1}} \tag{9}$$

For $\gamma = 1.4$, $f_F = 0.005$, Ma = 0.5, $L/D_1 = 100 \Rightarrow \boxed{D_2/D_1 = 1.175}$.

The stagnation pressure for a 1D flow with varying area and friction is given by:

$$\frac{dp_0}{p_0} = -\frac{\gamma \mathrm{Ma}^2}{2}\left(\frac{4 f_F dx}{D}\right) \tag{10}$$

Substitute Eqn. (4) and simplify.

$$\frac{dp_0}{p_0} = -2\frac{dD}{D} \tag{11}$$

$$\int_{p_0=p_{01}}^{p_0=p_{02}} \frac{dp_0}{p_0} = -2\int_{D=D_1}^{D=D_2} \frac{dD}{D}$$

$$\ln\frac{p_{02}}{p_{01}} = -2\ln\frac{D_2}{D_1}$$

$$\boxed{\frac{p_{02}}{p_{01}} = \left(\frac{D_1}{D_2}\right)^2} \tag{12}$$

where

$$p_{01} = p_1\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)^{\frac{\gamma}{\gamma-1}} \tag{13}$$

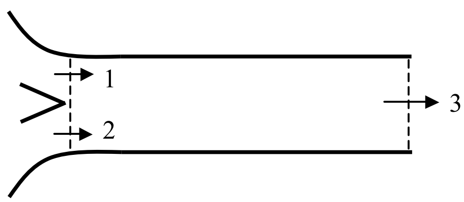
For $D_2/D_1 = 1.175$, $\gamma = 1.4$, Ma = 0.5, and $p_1 = 690$ kPa $\Rightarrow p_{01} = 818.5$ kPa and $\boxed{p_{02} = 593 \text{ kPa}}$.

Consider the constant-area mixing between two streams, as shown in the sketch. It is agreed to assume:

1. that just before mixing the individual streams have equal cross-sectional areas,
2. that wall friction in the mixing section is negligible and that the walls of the mixing section are insulated against heat transfer,
3. that the streams are completely mixed at section 3, and
4. that air behaves as a perfect gas.

Calculate:

a. the isentropic stagnation pressure (psia) for stream 1,
b. the mass flow rate per unit area ($\text{lb}_\text{m}/(\text{s}\cdot\text{ft}^2)$) for stream 1,
c. the stagnation temperature at section 3 (°R) at section 3, and
d. the isentropic stagnation pressure (psia) at section 3.



p_1 = 10 psia
T_1 = 600 °R
V_1 = 722 ft/s

p_2 = 10 psia
T_2 = 2400 °R
V_2 = 722 ft/s

SOLUTION:

First calculate the stagnation pressure at station 1.

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_{01} = 12.8 \text{ psia}} \tag{1}$$

where

$$\text{Ma}_1 = \frac{V_1}{\sqrt{\gamma R T_1}} \Rightarrow \text{Ma}_1 = 0.60 \tag{2}$$

Similarly, calculate the stagnation pressure at station 2.

$$\frac{p_2}{p_{02}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_{02} = 10.6 \text{ psia} \tag{3}$$

where

$$\text{Ma}_2 = \frac{V_2}{\sqrt{\gamma R T_2}} \Rightarrow \text{Ma}_2 = 0.30 \tag{4}$$

Now calculate the stagnation temperature at stations 1 and 2.

$$\frac{T_1}{T_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{-1} \Rightarrow T_{01} = 643 \text{ °R} \tag{5}$$

$$\frac{T_2}{T_{02}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_2^2\right)^{-1} \Rightarrow T_{02} = 2443 \text{ °R} \tag{6}$$

Now calculate the mass flow rate per unit area at stations 1 and 2.

$$\dot{m}_1 = \rho_1 V_1 A_1 \;\Rightarrow\; \boxed{\dot{m}_1 = 32 \text{ lb}_\text{m}/(\text{s.ft}^2)} \tag{7}$$

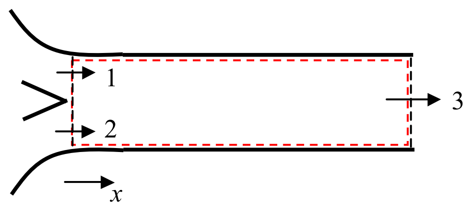
$$\dot{m}_2 = \rho_2 V_2 A_2 \;\Rightarrow\; \dot{m}_2 = 8.1 \text{ lb}_\text{m}/(\text{s.ft}^2) \tag{8}$$

where

$$\rho_1 = \frac{p_1}{RT_1} \;\Rightarrow\; \rho_1 = 4.5*10^{-2} \text{ lb}_\text{m}/\text{ft}^3 \tag{9}$$

$$\rho_2 = \frac{p_2}{RT_2} \;\Rightarrow\; \rho_2 = 1.1*10^{-2} \text{ lb}_\text{m}/\text{ft}^3 \tag{10}$$

To determine the stagnation temperature at section 3, apply conservation of energy to the following control volume.



$$\frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}} \left(h + \tfrac{1}{2}V^2\right)\left(\rho \mathbf{u}_\text{rel} \cdot d\mathbf{A}\right) = \dot{Q}_\text{into CV} + \dot{W}_\text{other, on CV} \tag{11}$$

where

$$\frac{d}{dt}\int_{\text{CV}} e\rho dV = 0 \quad \text{(steady flow)} \tag{12}$$

$$\int_{\text{CS}} \left(h + \tfrac{1}{2}V^2\right)\left(\rho \mathbf{u}_\text{rel} \cdot d\mathbf{A}\right) = \dot{m}_3 h_{03} - \dot{m}_1 h_{01} - \dot{m}_2 h_{02} \tag{13}$$

$$\dot{Q}_\text{into CV} = 0 \quad \text{(adiabatic)} \tag{14}$$

$$\dot{W}_\text{other, on CV} = 0 \quad \text{(no work other than pressure work)} \tag{15}$$

Substitute and simplify.

$$\dot{m}_3 h_{03} - \dot{m}_1 h_{01} - \dot{m}_2 h_{02} = 0 \tag{16}$$

$$T_{03} = \frac{\dot{m}_1}{\dot{m}_1 + \dot{m}_2} T_{01} + \frac{\dot{m}_2}{\dot{m}_1 + \dot{m}_2} T_{02} \tag{17}$$

$$T_{03} = \frac{1}{1+\mu} T_{01} + \frac{\mu}{1+\mu} T_{02} \tag{18}$$

where a perfect gas model has been employed ($h_0 = c_P T_0$) and conservation of mass has been applied to the same control volume ($\dot{m}_3 = \dot{m}_1 + \dot{m}_2$; $\mu = \dot{m}_2 / \dot{m}_1$). Using the previously calculated values for the stagnation temperatures and mass flow rates,

$$\boxed{T_{03} = 1003 \text{ °R}} \quad (\mu = 0.25) \tag{19}$$

To find the stagnation pressure at station 3, apply the linear momentum in the x-direction to the control volume.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_\text{rel} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} \tag{20}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)} \tag{21}$$

$$\int_{\text{CS}} u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{m}_3V_3 - \dot{m}_2V_2 - \dot{m}_1V_1 \tag{22}$$

$$F_{B,x} = 0 \;\; \text{(body forces are negligible)} \tag{23}$$

$$F_{S,x} = p_1A_1 + p_2A_2 - p_3A_3 \tag{24}$$

Substitute and simplify.

$$\dot{m}_3V_3 - \dot{m}_2V_2 - \dot{m}_1V_1 = p_1A_1 + p_2A_2 - p_3A_3 \tag{25}$$

$$\frac{\dot{m}_1+\dot{m}_2}{A}V_3 - \frac{\dot{m}_2}{A}V_2 - \frac{\dot{m}_1}{A}V_1 = p_1 + p_2 - 2p_3 \quad \text{(where } A_3 = A_1 + A_2 \text{ and } A_1 = A_2 = A\text{)} \tag{26}$$

$$\left(\frac{\dot{m}_1}{A}\right)\left[\left(1+\mu\right)V_3 - \mu V_2 - V_1\right] = p_1 + p_2 - 2p_3 \tag{27}$$

Re-write p_3.

$$p_3 = \rho_3RT_3 = \left(\frac{\dot{m}_3}{2AV_3}\right)R\left(T_{03} - \frac{V_3^2}{2c_P}\right) = \left(\frac{\dot{m}_1+\dot{m}_2}{2AV_3}\right)R\left(T_{03} - \frac{V_3^2}{2c_P}\right) = \frac{1}{2}\frac{\dot{m}_1}{A}\left(\frac{1+\mu}{V_3}\right)R\left(T_{03} - \frac{V_3^2}{2c_P}\right) \tag{28}$$

Substitute into Eq. (27) and solve for V_3.

$$\left(\frac{\dot{m}_1}{A}\right)\left[\left(1+\mu\right)V_3 - \mu V_2 - V_1\right] = p_1 + p_2 - \frac{\dot{m}_1}{A}\left(\frac{1+\mu}{V_3}\right)R\left(T_{03} - \frac{V_3^2}{2c_P}\right) \tag{29}$$

$$\left(1+\mu\right)V_3 - \left(\frac{p_1+p_2}{\dot{m}_1/A} + \mu V_2 + V_1\right) + \left(\frac{1+\mu}{V_3}\right)R\left(T_{03} - \frac{V_3^2}{2c_P}\right) = 0 \tag{30}$$

$$\left(1+\mu\right)V_3^2 - \left(\frac{p_1+p_2}{\dot{m}_1/A} + \mu V_2 + V_1\right)V_3 + \left(1+\mu\right)R\left(T_{03} - \frac{V_3^2}{2c_P}\right) = 0 \tag{31}$$

$$\left(1+\mu\right)\left(1 - \frac{R}{2c_P}\right)V_3^2 - \left(\frac{p_1+p_2}{\dot{m}_1/A} + \mu V_2 + V_1\right)V_3 + \left(1+\mu\right)RT_{03} = 0 \tag{32}$$

$$AV_3^2 + BV_3 + C = 0 \tag{33}$$

where

$$A = \left(1+\mu\right)\left(1 - \frac{R}{2c_P}\right) \tag{34}$$

$$B = -\left(\frac{p_1+p_2}{\dot{m}_1/A} + \mu V_2 + V_1\right) \tag{35}$$

$$C = \left(1+\mu\right)RT_{03} \tag{36}$$

Using the given data:

$$V_3 = 722 \text{ ft/s} \;\text{ or }\; 2781 \text{ ft/s} \tag{37}$$

Determine the properties corresponding to each of these velocities.

$$T_3 = T_{03} - \frac{V_3^2}{2c_P} \tag{38}$$

$$\rho_3 = \left(\frac{\dot{m}_3}{2A}\right)\frac{1}{V_3} = \frac{1}{2}\left(\frac{\dot{m}_1}{A}\right)\frac{1+\mu}{V_3} \tag{39}$$

$$p_3 = \rho_3 R T_3 \tag{40}$$

$$\mathrm{Ma}_3 = \frac{V_3}{\sqrt{\gamma R T_3}} \tag{41}$$

$$p_{03} = p_3\left(1 + \frac{\gamma-1}{2}\mathrm{Ma}_3^2\right)^{\frac{\gamma}{\gamma-1}} \tag{42}$$

V_3 = 722 ft/s $\Rightarrow$ T_3 = 960 °R
$\rho_3 = 2.81*10^{-2}$ lb$_m$/ft^3
p_3 = 10 psia
Ma_3 = 0.48
p_{03} = 12 psia

V_3 = 2781 ft/s $\Rightarrow$ T_3 = 361 °R
$\rho_3 = 7.31*10^{-3}$ lb$_m$/ft^3
p_3 = 1 psia
Ma_3 = 2.99
p_{03} = 35 psia

Which velocity is the correct one? Apply the 2$^{\text{nd}}$ Law of Thermodynamics to the same control volume to determine.

$$\frac{d}{dt}\int_{\mathrm{CV}} s\rho\, dV + \int_{\mathrm{CS}} s\left(\rho\mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) \geq \int_{\mathrm{CV}} \frac{\delta\dot{q}_{\mathrm{into}}}{T} \tag{43}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} s\rho dV = 0 \qquad \text{(steady flow)} \tag{44}$$

$$\int_{\mathrm{CS}} s\left(\rho\mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = \dot{m}_3 s_3 - \dot{m}_2 s_2 - \dot{m}_1 s_1 \tag{45}$$

$$\int_{\mathrm{CV}} \frac{\delta\dot{q}_{\mathrm{into}}}{T} = 0 \qquad \text{(the mixing occurs adiabatically)} \tag{46}$$

Substitute and simplify. Also note that the ">" sign should be used since mixing is an irreversible process.

$$\dot{m}_3 s_3 - \dot{m}_2 s_2 - \dot{m}_1 s_1 > 0 \tag{47}$$

$$(1+\mu)s_3 - \mu s_2 - s_1 > 0 \tag{48}$$

The specific entropies may be found using the *Tds* relation.

$$s_3 - s_{\mathrm{ref}} = c_P \ln\left(\frac{T_3}{T_{\mathrm{ref}}}\right) - R\ln\left(\frac{p_3}{p_{\mathrm{ref}}}\right) \tag{49}$$

$$s_2 - s_{\mathrm{ref}} = c_P \ln\left(\frac{T_2}{T_{\mathrm{ref}}}\right) - R\ln\left(\frac{p_2}{p_{\mathrm{ref}}}\right) \tag{50}$$

$$s_1 - s_{\mathrm{ref}} = c_P \ln\left(\frac{T_1}{T_{\mathrm{ref}}}\right) - R\ln\left(\frac{p_1}{p_{\mathrm{ref}}}\right) \tag{51}$$

Substitute and simplify.

$$\begin{aligned}&(1+\mu)\left[s_{\text{ref}}+c_P\ln\left(\frac{T_3}{T_{\text{ref}}}\right)-R\ln\left(\frac{p_3}{p_{\text{ref}}}\right)\right]-\mu\left[s_{\text{ref}}+c_P\ln\left(\frac{T_2}{T_{\text{ref}}}\right)-R\ln\left(\frac{p_2}{p_{\text{ref}}}\right)\right]\\&\qquad-\left[s_{\text{ref}}+c_P\ln\left(\frac{T_1}{T_{\text{ref}}}\right)-R\ln\left(\frac{p_1}{p_{\text{ref}}}\right)\right]>0\end{aligned} \tag{52}$$

$$\begin{aligned}&(1+\mu-\mu-1)s_{\text{ref}}+(1+\mu)\left[c_P\ln\left(\frac{T_3}{T_{\text{ref}}}\right)-R\ln\left(\frac{p_3}{p_{\text{ref}}}\right)\right]-\mu\left[c_P\ln\left(\frac{T_2}{T_{\text{ref}}}\right)-R\ln\left(\frac{p_2}{p_{\text{ref}}}\right)\right]\\&\qquad-\left[c_P\ln\left(\frac{T_1}{T_{\text{ref}}}\right)-R\ln\left(\frac{p_1}{p_{\text{ref}}}\right)\right]>0\end{aligned} \tag{53}$$

$$\ln\left(\frac{T_3}{T_{\text{ref}}}\right)^{(1+\mu)c_P}+\ln\left(\frac{p_3}{p_{\text{ref}}}\right)^{-(1+\mu)R}+\ln\left(\frac{T_2}{T_{\text{ref}}}\right)^{-\mu c_P}+\ln\left(\frac{p_2}{p_{\text{ref}}}\right)^{\mu R}+\ln\left(\frac{T_1}{T_{\text{ref}}}\right)^{-c_P}+\ln\left(\frac{p_1}{p_{\text{ref}}}\right)^{R}>0 \tag{54}$$

$$c_P\ln\frac{\left(\frac{T_3}{T_{\text{ref}}}\right)^{(1+\mu)}}{\left(\frac{T_2}{T_{\text{ref}}}\right)^{\mu}\left(\frac{T_1}{T_{\text{ref}}}\right)}-R\ln\frac{\left(\frac{p_3}{p_{\text{ref}}}\right)^{(1+\mu)}}{\left(\frac{p_2}{p_{\text{ref}}}\right)^{\mu}\left(\frac{p_1}{p_{\text{ref}}}\right)}>0 \tag{55}$$

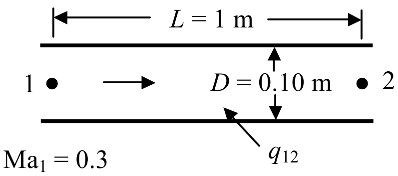
$$c_P\ln\frac{(T_3)^{(1+\mu)}}{(T_2)^{\mu}(T_1)}-R\ln\frac{(p_3)^{(1+\mu)}}{(p_2)^{\mu}(p_1)}>0 \tag{56}$$

For the V_3 = 722 ft/s data, the left hand side of Eq. (57) is 45.1 (lb_f.ft)/(lb_m.°R), which satisfies the 2nd Law. For the V_3 = 2781 ft/s data, the left hand side of Eq. (57) is -28.5 (lb_f.ft)/(lb_m.°R), which does not satisfy the 2nd Law.

Thus, the correct velocity is V_3 = 722 ft/s and the correct stagnation pressure is $\boxed{p_{03} = 12\text{ psia}}$.

Consider steady air flow through a constant area circular duct which has a diameter of 10 cm and a length of 1 m. The Mach number, pressure, and temperature at the inlet to the duct are 0.3, 200 kPa (abs), and 80 °C. Heat is added at a uniform rate to the air as it flows through the duct causing the stagnation temperature to increase by 300 K. If the (Fanning) friction factor can be assumed to be 0.003, find the Mach number, pressure, and temperature at the outlet of the duct.

SOLUTION:



$\mathrm{Ma}_1 = 0.3$
p_1 = 200 kPa (abs)
T_1 = (80 + 273) K = 353 K
$f_F = 0.003$
$T_{02} - T_{01}$ = 300 K

First determine the stagnation properties at station 1.

$$p_{01} = p_1\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow p_{01} = 212.9\text{ kPa} \tag{1}$$

$$T_{01} = T_1\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right) \Rightarrow T_{01} = 359.4\text{ K} \tag{2}$$

$$\Rightarrow T_{02} = 659.4\text{ K} \text{ since } T_{02} - T_{01} = 300\text{ K (given)} \tag{3}$$

The Mach number will change as we move downstream due to the frictional contribution and the heat transfer contribution.

$$\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \frac{\gamma \mathrm{Ma}^2\Psi}{2\left(1-\mathrm{Ma}^2\right)}\left(\frac{4f_F\,dx}{D}\right) + \frac{\left(1+\gamma\mathrm{Ma}^2\right)}{2\left(1-\mathrm{Ma}^2\right)}\frac{dT_0}{T_0} \quad \text{where} \quad \Psi = 1+\frac{\gamma-1}{2}\mathrm{Ma}^2 \tag{4}$$

Note that since the heat is added at a uniform rate, the change in stagnation temperature over the length of the duct can be written as:

$$\frac{dT_0}{dx} = \frac{T_{02}-T_{01}}{L} \Rightarrow dT_0 = \left(\frac{T_{02}-T_{01}}{L}\right)dx \tag{5}$$

Substitute Eq. (5) into Eq. (4) and simplify.

$$\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \frac{\gamma \mathrm{Ma}^2\Psi}{2\left(1-\mathrm{Ma}^2\right)}\left(\frac{4f_F\,dx}{D}\right) + \frac{\left(1+\gamma\mathrm{Ma}^2\right)}{2\left(1-\mathrm{Ma}^2\right)}\left(\frac{T_{02}-T_{01}}{T_0 L}\right)dx \tag{6}$$

$$\frac{d(\mathrm{Ma})}{dx} = \frac{\gamma \mathrm{Ma}^3\Psi}{2\left(1-\mathrm{Ma}^2\right)}\left(\frac{4f_F}{D}\right) + \frac{\mathrm{Ma}\left(1+\gamma\mathrm{Ma}^2\right)}{2\left(1-\mathrm{Ma}^2\right)}\left(\frac{T_{02}-T_{01}}{T_0 L}\right) \tag{7}$$

Solve Eq. (7) numerically. Note that the numerical approach shown below is crude, but as Δx is mad smaller, the results become more accurate.

$$\mathrm{Ma}\left(x+\Delta x\right) \approx \mathrm{Ma}\left(x\right)+\frac{d\left(\mathrm{Ma}\right)}{dx}\Delta x \tag{8}$$

A similar approach may be used to solve for the other flow properties.

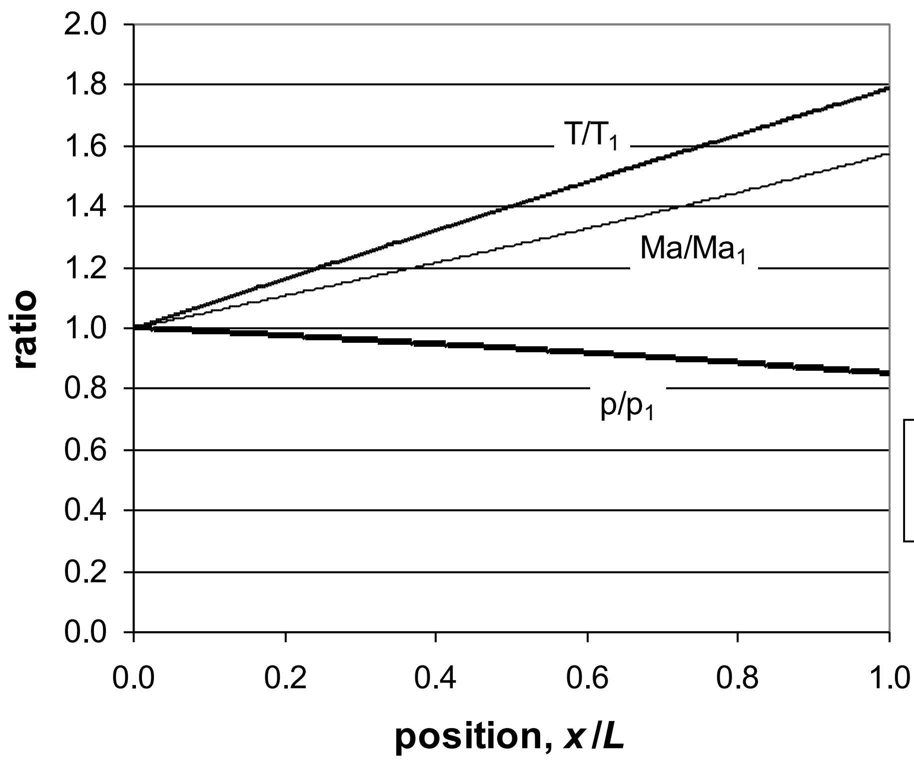
$$\frac{dp}{p}=\frac{-\gamma \mathrm{Ma}^2\left[1+\left(\gamma-1\right)\mathrm{Ma}^2\right]}{2\left(1-\mathrm{Ma}^2\right)}\left(\frac{4f_F dx}{D}\right)-\frac{\gamma \mathrm{Ma}^2 \Psi}{1-\mathrm{Ma}^2}\frac{dT_0}{T_0} \tag{9}$$

$$\frac{dT}{T}=\frac{-\gamma\left(\gamma-1\right)\mathrm{Ma}^4}{2\left(1-\mathrm{Ma}^2\right)}\left(\frac{4f_F dx}{D}\right)+\frac{\left(1-\gamma \mathrm{Ma}^2\right)\Psi}{1-\mathrm{Ma}^2}\frac{dT_0}{T_0} \tag{10}$$

Note that we could also determine T using:

$$T=T_0\left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{-1} \tag{11}$$

since the T_0 distribution is known.



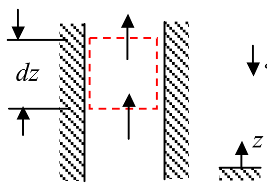
$\mathrm{Ma}_2 = 0.47$
$p_2 = 169.9$ kPa (abs)
$T_2 = 631.2$ K

Consider a perfect gas flowing in a constant-area duct adiabatically and without friction. Changes in state come about as the result of changes in elevation in the earth's gravity field. The z-direction is away from the center of the earth, and hence gravity acts in the negative z-direction.

a. Starting from first principles, determine by analysis the direction of change (increase or decrease) of the Mach #, gas speed, sound speed, density, pressure, stagnation temperature, and isentropic stagnation pressure, all for a positive increase in z.
   i. for subsonic speeds
   ii. for supersonic speeds
b. Is choking possible for this type of flow? Justify your answer.

SOLUTION:

Determine the working equations for this flow by applying the basic laws, in differential form, to the following control volume. Assume the flow is steady, 1D, adiabatic, and frictionless. Also assume that the fluid is a perfect gas and that the gravitational acceleration is a constant.



Conservation of mass:

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS}\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A} = -(\rho VA) + \left[(\rho VA) + d(\rho VA)\right] = d(\rho VA) \qquad (1)$$

Substitute and simplify.

$$d(\rho VA) = 0$$

$$\underline{\frac{d\rho}{\rho} + \frac{dV}{V} = 0} \quad \text{(Note that } A = \text{constant.)} \qquad (2)$$

Linear momentum equation in the z-direction:

$$\frac{d}{dt}\int_{CV} u_z \rho dV + \int_{CS} u_z \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,z} + F_{S,z}$$

where

$$\frac{d}{dt}\int_{CV} u_z \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\dot{m}V + \left[\dot{m}V + d\left(\dot{m}V\right)\right] = \dot{m}dV = \rho VAdV \tag{3}$$

$$F_{B,z} = -\rho Adzg \tag{4}$$

$$F_{S,z} = \left(pA\right) - \left[\left(pA\right) + d\left(pA\right)\right] = -Adp \quad \text{(Note that } A = \text{constant.)}$$

Substitute and simplify.

$$\rho VAdV = -\rho Adzg - Adp \tag{5}$$

$$\underline{dp + \rho VdV + \rho gdz = 0}$$

Conservation of energy:

$$\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS} h_0\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{on CV}}$$

where

$$\frac{d}{dt}\int_{CV} e\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} h_0\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}h_0 + \left[\dot{m}h_0 + d\left(\dot{m}h_0\right)\right] = \dot{m}dh_0 = \rho VAdh_0 \tag{6}$$

$$\dot{Q}_{\text{into CV}} = 0 \quad \text{(adiabatic)}$$

$$\dot{W}_{\text{on CV}} = 0 \quad \text{(no other work besides that due to pressure)}$$

Substitute and simplify.

$$\rho VAdh_0 = 0 \tag{7}$$

$$dh_0 = d\left(h + \tfrac{1}{2}V^2 + gz\right) = 0 \tag{8}$$

$$\underline{dh + VdV + gdz = 0} \tag{9}$$

Thermal equation of state for an ideal gas (the ideal gas law):

$$p = \rho RT \tag{10}$$

$$\underline{\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}} \tag{11}$$

Caloric equation of state for an ideal gas:

$$\underline{dh = c_P dT} \tag{12}$$

Speed of sound for an ideal gas:

$$c = \sqrt{\gamma RT} \tag{13}$$

$$\underline{\frac{dc}{c} = \frac{1}{2}\frac{dT}{T}} \tag{14}$$

Mach number for an ideal gas:

$$\text{Ma} = \frac{V}{c} = \frac{V}{\sqrt{\gamma RT}} \tag{15}$$

$$\underline{\frac{d(\text{Ma})}{\text{Ma}} = \frac{dV}{V} - \frac{1}{2}\frac{dT}{T}} \tag{16}$$

Adiabatic stagnation temperature for a perfect gas:

$$dh_0 = c_p dT_0 \quad \text{(from Eq. (12))} \tag{17}$$

$$\underline{\therefore dT_0 = 0} \quad \text{(since } dh_0 = 0 \text{ from Eq. (8))} \tag{18}$$

Note that $c_P T_0 = c_P T + \frac{1}{2}V^2$ shouldn't be used since it doesn't include elevation effects.

Isentropic stagnation pressure for a perfect gas:

$$\frac{p_{02}}{p_{01}} = \left(\frac{T_{02}}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{(the flow is isentropic)} \tag{19}$$

Since T_0 = constant, p_0 = constant and:

$$\underline{dp_0 = 0} \tag{20}$$

Note that $p_0 = p\left(1 + \frac{\gamma-1}{2}\text{Ma}^2\right)^{\frac{\gamma}{\gamma-1}}$ shouldn't be used since it doesn't include elevation effects.

Summarize the previously derived equations.

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \tag{21}$$

$$dp + \rho V dV = -\rho g dz \tag{22}$$

$$dh + V dV = -g dz \tag{23}$$

$$\frac{dp}{p} - \frac{d\rho}{\rho} - \frac{dT}{T} = 0 \tag{24}$$

$$dh - c_P dT = 0 \tag{25}$$

$$\frac{dc}{c} - \frac{1}{2}\frac{dT}{T} = 0 \tag{26}$$

$$\frac{d(\text{Ma})}{\text{Ma}} - \frac{dV}{V} + \frac{1}{2}\frac{dT}{T} = 0 \tag{27}$$

$$dT_0 = 0 \tag{28}$$

$$dp_0 = 0 \tag{29}$$

In the previous equations we observe that there are six equations with six unknowns ($d\rho$, dV, dp, dT, dh, d(Ma)). Note that dz will be considered a known quantity since it will be the quantity that we will vary. Now combine the previous equations so that they are all in terms of dz.

From Eq. (21):

$$\frac{d\rho}{\rho} = -\frac{dV}{V} \tag{30}$$

From Eq. (22):

$$\frac{dp}{p} + \frac{\rho V^2}{p}\frac{dV}{V} = -\frac{\rho g}{p}dz \quad \text{but from the ideal gas law: } p/\rho = RT$$

$$\frac{dp}{p} + \gamma\frac{V^2}{\gamma RT}\frac{dV}{V} = -\gamma\frac{V^2}{\gamma RT}\frac{gdz}{V^2} \quad \text{but Ma} = V/(\gamma RT)^{1/2} \text{ for an ideal gas}$$

$$\frac{dp}{p} = -\gamma \mathrm{Ma}^2\frac{dV}{V} - \gamma \mathrm{Ma}^2\frac{gdz}{V^2} \tag{31}$$

Combining Eqs. (23) and (25) gives:

$$c_P dT + VdV = -gdz$$

$$\frac{dT}{T} + \frac{V^2}{c_P T}\frac{dV}{V} = -\frac{1}{c_P T}gdz \quad \text{but } c_P = \gamma R/(\gamma-1) \text{ for an ideal gas}$$

$$\frac{dT}{T} = -(\gamma-1)\frac{V^2}{\gamma RT}\frac{dV}{V} - (\gamma-1)\frac{V^2}{\gamma RT}\frac{gdz}{V^2} \quad \text{but Ma} = V/(\gamma RT)^{1/2} \text{ for an ideal gas}$$

$$\frac{dT}{T} = (1-\gamma)\mathrm{Ma}^2\frac{dV}{V} + (1-\gamma)\mathrm{Ma}^2\frac{gdz}{V^2} \tag{32}$$

Substitute Eqs. (30) - (32) into Eqn. (24) and simplify.

$$\left[-\gamma \mathrm{Ma}^2\frac{dV}{V} - \gamma \mathrm{Ma}^2\frac{gdz}{V^2}\right] - \left[-\frac{dV}{V}\right] - \left[(1-\gamma)\mathrm{Ma}^2\frac{dV}{V} + (1-\gamma)\mathrm{Ma}^2\frac{gdz}{V^2}\right] = 0$$

$$\left(1-\mathrm{Ma}^2\right)\frac{dV}{V} - \mathrm{Ma}^2\frac{gdz}{V^2} = 0$$

$$\boxed{\frac{dV}{V} = \left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{33}$$

Now combine Eq. (33) with Eq. (30).

$$\boxed{\frac{d\rho}{\rho} = -\frac{dV}{V} = -\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{34}$$

Combine Eq. (33) with Eq. (31).

$$\frac{dp}{p} = -\gamma \mathrm{Ma}^2\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\frac{gdz}{V^2}\right) - \gamma \mathrm{Ma}^2\frac{gdz}{V^2}$$

$$\frac{dp}{p} = -\gamma \mathrm{Ma}^2\left[\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right)+1\right]\frac{gdz}{V^2}$$

$$\boxed{\frac{dp}{p} = -\gamma\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{35}$$

Combine Eqs. (33) and (32).

$$\frac{dT}{T} = (1-\gamma)\mathrm{Ma}^2\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\frac{gdz}{V^2}\right) + (1-\gamma)\mathrm{Ma}^2\frac{gdz}{V^2}$$

$$\frac{dT}{T} = (1-\gamma)\mathrm{Ma}^2\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}+1\right)\frac{gdz}{V^2}$$

$$\boxed{\frac{dT}{T} = (1-\gamma)\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{36}$$

Combining Eqs. (36) and (26) gives:

$$\boxed{\frac{dc}{c} = \frac{1}{2}\frac{dT}{T} = \frac{1-\gamma}{2}\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{37}$$

Combine Eqn. (33), (36), and (27).

$$\frac{d(\mathrm{Ma})}{\mathrm{Ma}} - \left[\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\frac{gdz}{V^2}\right] + \frac{1}{2}\left[(1-\gamma)\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right)\frac{gdz}{V^2}\right] = 0$$

$$\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \left[\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2} + \frac{1}{2}(\gamma-1)\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right)\right]\frac{gdz}{V^2}$$

$$\boxed{\frac{d(\mathrm{Ma})}{\mathrm{Ma}} = \frac{1+\gamma}{2}\left(\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{38}$$

Summarizing the results:

$$\boxed{\frac{dV}{V}=\left(\frac{\text{Ma}^2}{1-\text{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{39}$$

$$\boxed{\frac{d\rho}{\rho}=-\frac{dV}{V}=-\left(\frac{\text{Ma}^2}{1-\text{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{40}$$

$$\boxed{\frac{dp}{p}=-\gamma\left(\frac{\text{Ma}^2}{1-\text{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{41}$$

$$\boxed{\frac{dT}{T}=(1-\gamma)\left(\frac{\text{Ma}^2}{1-\text{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{42}$$

$$\boxed{\frac{dc}{c}=\frac{1}{2}\frac{dT}{T}=\frac{1-\gamma}{2}\left(\frac{\text{Ma}^2}{1-\text{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{43}$$

$$\boxed{\frac{d(\text{Ma})}{\text{Ma}}=\frac{1+\gamma}{2}\left(\frac{\text{Ma}^2}{1-\text{Ma}^2}\right)\left(\frac{gdz}{V^2}\right)} \tag{44}$$

$$\boxed{dT_0=0} \tag{45}$$

$$\boxed{dp_0=0} \tag{46}$$

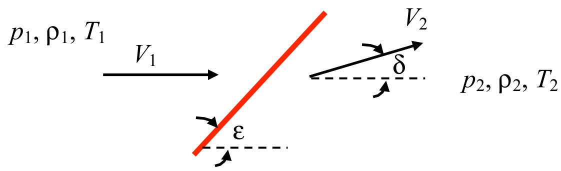
Examining Eqs. (39) - (44) (and noting that $\gamma > 1$) indicates that:

	Ma < 1		**Ma > 1**	
	***dz* < 0**	***dz* > 0**	***dz* < 0**	***dz* > 0**
$d\rho$	+	-	-	+
dV	-	+	+	-
dp	+	-	-	+
dT	+	-	-	+
dh	+	-	-	+
dc	+	-	-	+
d(Ma)	-	+	+	-
dT_0	0	0	0	0
dp_0	0	0	0	0

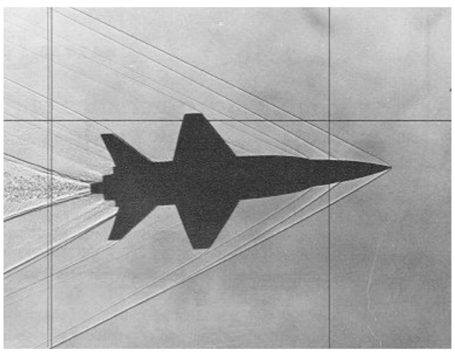
<u>Choking is possible for this flow since the Mach number approaches one for both subsonic and supersonic flows ($dz > 0$).</u>

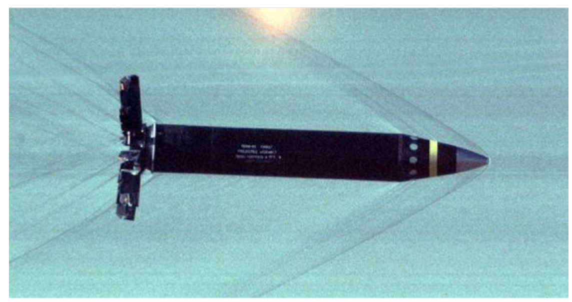
13. Oblique Shock Waves

An oblique shock wave is a shock wave that forms at an angle with respect to the incoming flow (note that normal shock waves are a special case of an oblique shock wave). As with normal shock waves, the flow properties abruptly change when passing through an oblique shock. In addition, the flow is turned through the shock as shown in the figure below.

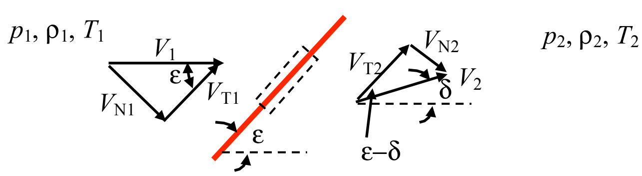


We'll refer to ε as the shock angle (the angle of the shock with respect to the upstream velocity) and δ as the flow turning (or deflection) angle.





The working equations for oblique shock waves can be determined by analyzing a very thin control volume that straddles the shock wave (an approach identical to what we used to analyze normal shock waves). We'll also break the upstream and downstream velocities into components that are normal and tangential to the shock wave.



COM:

$$\rho_1 V_{N1} = \rho_2 V_{N2} \tag{207}$$

LME:

N-dir: $$\rho_2 V_{N2}^2 - \rho_1 V_{N1}^2 = p_1 - p_2 \tag{208}$$

T-dir: $$\dot{m} V_{T2} - \dot{m} V_{T1} = 0 \quad \Rightarrow \quad V_{T2} = V_{T1} = V_T \tag{209}$$

(The tangential velocity components on either side of the wave are equal!)

COE:

$$h_1 + \tfrac{1}{2} V_1^2 = h_2 + \tfrac{1}{2} V_2^2 \quad \Rightarrow \quad h_1 + \tfrac{1}{2} V_{N1}^2 = h_2 + \tfrac{1}{2} V_{N2}^2 \quad \text{(since } V_{T1} = V_{T2}\text{)} \tag{210}$$

2nd Law:

$$s_2 > s_1 \quad \text{(The large gradients within a shock wave result in an irreversible process.)} \tag{211}$$

Equations of state for a perfect gas:

$$p = \rho RT \tag{212}$$

$$h = c_p T \tag{213}$$

For an ideal gas:

$$\mathrm{Ma} = \frac{V}{\sqrt{\gamma RT}} \tag{214}$$

$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)^{\gamma/(1-\gamma)} \tag{215}$$

$$\frac{T}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}^2\right)^{-1} \tag{216}$$

From geometry:

$$\begin{aligned} V_1^2 &= V_{N1}^2 + V_T^2 \\ V_2^2 &= V_{N2}^2 + V_T^2 \end{aligned} \tag{217}$$

$$\begin{aligned} V_{N1} &= V_1 \sin\varepsilon \\ V_{N2} &= V_2 \sin(\varepsilon - \delta) \\ V_T &= V_1 \cos\varepsilon = V_2 \cos(\varepsilon - \delta) \end{aligned} \tag{218}$$

Combining the previous equations and, after much simplifying, we have:

$$\mathrm{Ma}_{N1} = \mathrm{Ma}_1 \sin\varepsilon \tag{219}$$

$$\mathrm{Ma}_{N2} = \mathrm{Ma}_2 \sin(\varepsilon - \delta) \tag{220}$$

$$\mathrm{Ma}_{N2}^2 = \frac{(\gamma-1)\mathrm{Ma}_{N1}^2 + 2}{2\gamma \mathrm{Ma}_{N1}^2 - (\gamma-1)} \tag{221}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_{N1}^2 - \frac{\gamma-1}{\gamma+1} \tag{222}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_{N1}}{V_{N2}} = \frac{\tan\varepsilon}{\tan(\varepsilon-\delta)} = \frac{(\gamma+1)\mathrm{Ma}_{N1}^2}{2+(\gamma-1)\mathrm{Ma}_{N1}^2} \tag{223}$$

$$\frac{T_2}{T_1} = \left[2+(\gamma-1)\mathrm{Ma}_{N1}^2\right]\frac{2\gamma\mathrm{Ma}_{N1}^2 - (\gamma-1)}{\left[(\gamma+1)\mathrm{Ma}_{\mathrm{N1}}\right]^2} \tag{224}$$

$$\frac{V_2}{V_1} = \frac{\sin\varepsilon}{\sin(\varepsilon-\delta)}\left[\frac{2}{(\gamma+1)\mathrm{Ma}_{N1}^2} + \frac{\gamma-1}{\gamma+1}\right] \tag{225}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\mathrm{Ma}_{N1}^2}{2+(\gamma-1)\mathrm{Ma}_{N1}^2}\right]^{\gamma/(\gamma-1)}\left[\frac{\gamma+1}{2\gamma\mathrm{Ma}_{N1}^2 - (\gamma-1)}\right]^{1/(\gamma-1)} \tag{226}$$

$$\frac{T_{02}}{T_{01}} = 1 \tag{227}$$

$$\frac{\tan\varepsilon}{\tan(\varepsilon-\delta)} = \frac{2}{\gamma+1}\left[\frac{1}{\mathrm{Ma}_{N2}^2} + \frac{\gamma-1}{2}\right] \tag{228}$$

$$\frac{1}{\tan\delta} = \left[\frac{\gamma+1}{2}\frac{\mathrm{Ma}_1^2}{\mathrm{Ma}_{N1}^2 - 1} - 1\right]\tan\varepsilon \tag{229}$$

Notes:

1. Equations (221)-(227) are the same relations as for a normal shock wave except that the normal component of the velocity is used. Thus, we can use normal shock relations to determine flow properties across an oblique shock as long as we use the normal component of the velocity.

2. There are two free parameters in the Eqs. (221) - (229). These typically are the incoming Mach number, Ma_1, and the flow turning angle, δ. Note that the flow is always turned toward the shock wave.

3. Analysis using the 2nd Law (refer to the notes on normal shock waves) states that an oblique shock will only form if the incoming normal Mach number is greater than or equal to one, i.e., $Ma_{N1} \geq 1$. The downstream normal Mach number will be less than one, i.e., $Ma_{N2} \leq 1$. Note that the total downstream Mach number, Ma_2, may not be less than one due to the tangential velocity component.

4. Based on the previous note, the minimum value for ε (the angle of the shock to the incoming flow) is $\sin^{-1}(1/Ma_1)$ which corresponds to a Mach wave (i.e., a sound wave). The maximum value for ε is 90° corresponding to a normal shock wave. For both of these shock angle limits we find (from Eq. (229)) that the turning angle of the flow is $\delta = 0$. Note that δ will have a maximum value since it is zero for $\varepsilon_{min} = \sin^{-1}(1/Ma_1)$, is positive for larger values of ε, then returns to zero for $\varepsilon_{max} = 90°$. We will discuss this maximum value for δ in a later note.

5. Since the equations relating the incoming Mach number, Ma_1, the wave angle, ε, and the turning angle, δ, are complicated, it's often more instructive and useful to present the data in graphical form rather than in equation form.

The following figure plots the wave angle, ε, as a function of the incoming Mach number, Ma_1, for different values of the turning angle, δ, for γ=1.4 using Eq. (229) [plot from Zucrow and Hoffman, *Gas Dynamics: Vol. I*, Wiley]. The lines corresponding to the maximum turning angle, δ_{max}, and where the downstream flow is sonic, $Ma_2 = 1$, are also shown in the figure. Note that for a given Ma_1, there are two values for ε, the larger corresponding to a "strong" shock with $Ma_2 < 1$ and the smaller ε corresponding to a "weak" shock with $Ma_2 > 1$. We'll discuss these observations in greater detail in a subsequent note.

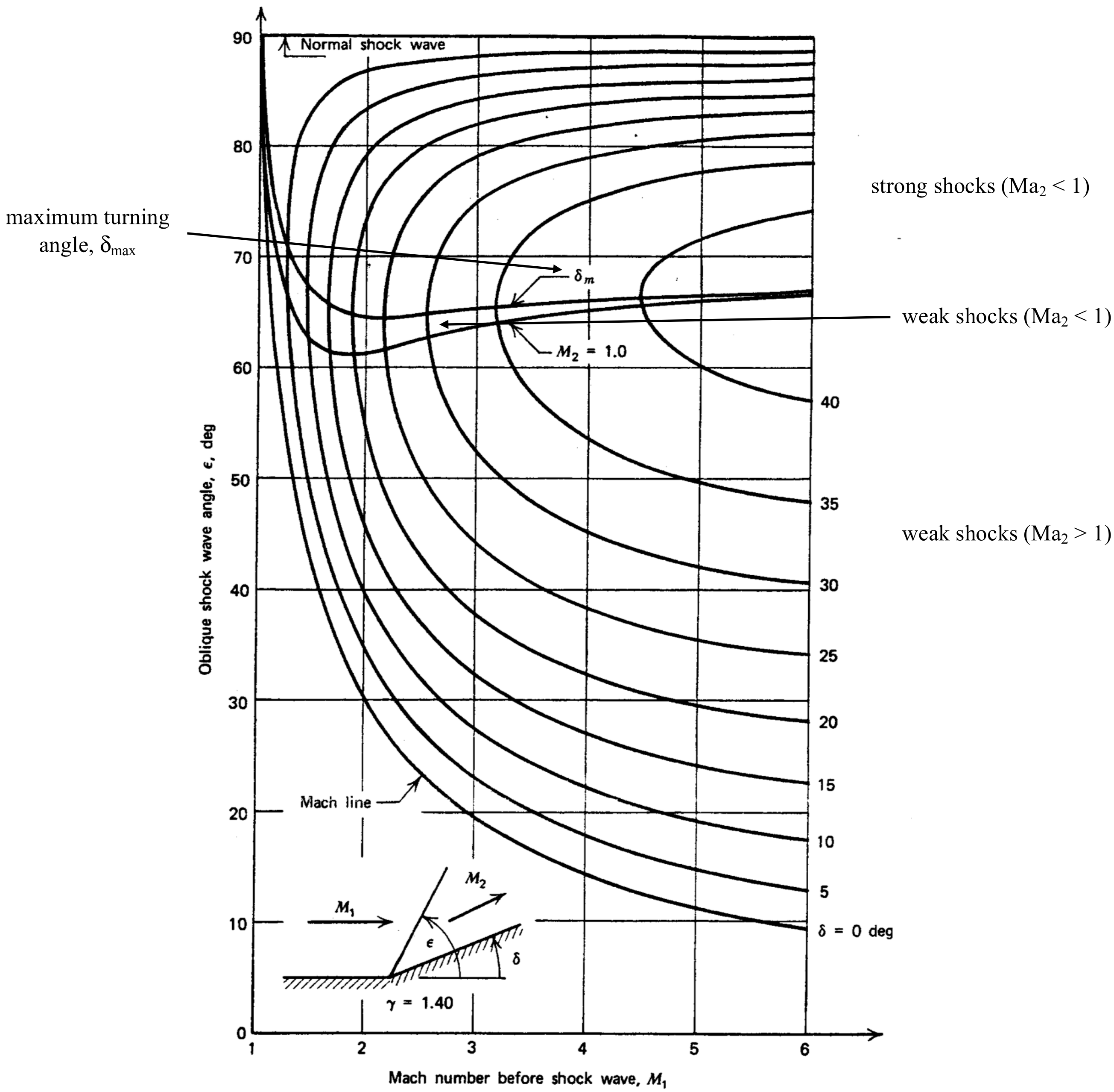


Figure 7.17 Shock wave angle ε as a function of the initial Mach number M_1 for different values of the flow deflection angle δ for γ = 1.4.

The following figure plots the downstream Mach number, Ma_2, as a function of the incoming Mach number, Ma_1, for different values of the turning angle, δ, for γ=1.4 using Eqs. (228) and (229) [plot from Zucrow and Hoffman, *Gas Dynamics: Vol. I*, Wiley]. The line corresponding to the maximum turning angle, δ_{max}, is also shown in the figure.

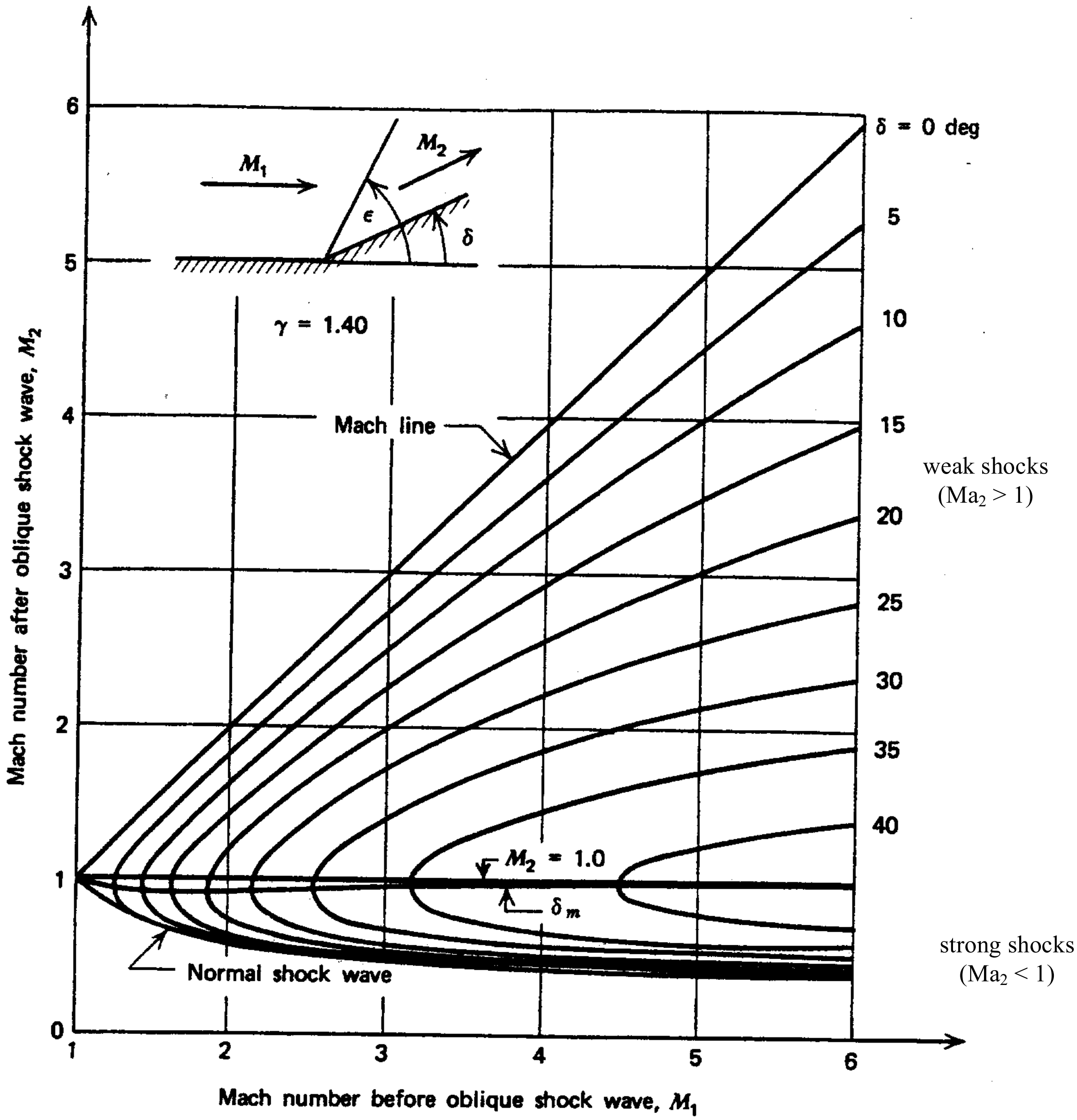
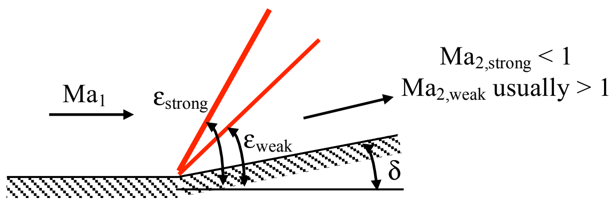


Figure 7.18 Mach number downstream of an oblique shock wave M_2 as a function of the initial Mach number M_1 for different values of the flow deflection angle δ for γ = 1.4.

6. The plots shown in Note 5 indicate that there are two possible wave angles, ε, corresponding to a particular incoming Mach number, Ma_1, and turning angle, δ. The smaller value of ε corresponds to a "weak" oblique shock and the larger value corresponds to a "strong" oblique shock. Flow downstream of the strong shock is always subsonic while the flow downstream of the weak shock is usually supersonic, except for a region where δ is close to δ_{max}.



Both types of shocks occur in practice with the weak shock being more prevalent. If there is a blockage or a high-pressure condition downstream of the shock, the strong shock solution will typically occur (e.g., at the inlet of a supersonic jet engine diffuser where the internal flow is at a high pressure). For flows occurring in the atmosphere where the pressure far downstream of the deflection can only be infinitesimally different from the pressure far upstream of the deflection, the weak shock will occur (e.g., supersonic flow over the surface of an aircraft). To further complicate matters, since the governing equations of the fluid motion are nonlinear, it is possible to have multiple, stable flow solutions. Which solution is observed will depend on the path taken to get to the solution. The hysteresis associated with the starting/overspeeding of a supersonic jet engine diffuser is a good example of a flow situation having multiple, stable solutions (*e.g.* operating at the design speed with a shock in front of the inlet, or after overspeeding where the shock has been swallowed). The solution that occurs in this flow situation depends on the path taken to get to that solution. If no other information is available, it is generally reasonable to assume that the weak shock occurs.

7. The figures in Note 5 also indicate that there is a maximum turning angle, δ_{max}, that can be achieved *through an oblique shock* (this maximum turning angle separates the weak and strong shock solutions). It can be shown that the maximum wave angle corresponding to the maximum turning angle for a given upstream Mach number is (refer to Ferri, A., *Elements of Aerodynamics of Supersonic Flow*, Macmillan, NY):

$$\sin^2 \varepsilon_{max} = \frac{1}{\gamma \mathrm{Ma}_1^2}\left\{\frac{\gamma+1}{4}\mathrm{Ma}_1^2 - 1 + \sqrt{(\gamma+1)\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2+\frac{\gamma+1}{16}\mathrm{Ma}_1^4\right)}\right\} \quad (230)$$

The corresponding maximum turning angle may be found using Eqs. (219) and (229).

The following figure plots the maximum turning angle, δ_{max}, and the wave angle corresponding to this maximum turning angle, ε_{max}, as a function of the incoming Mach number, Ma_1 [plot from Zucrow and Hoffman, *Gas Dynamics: Vol. I*, Wiley].

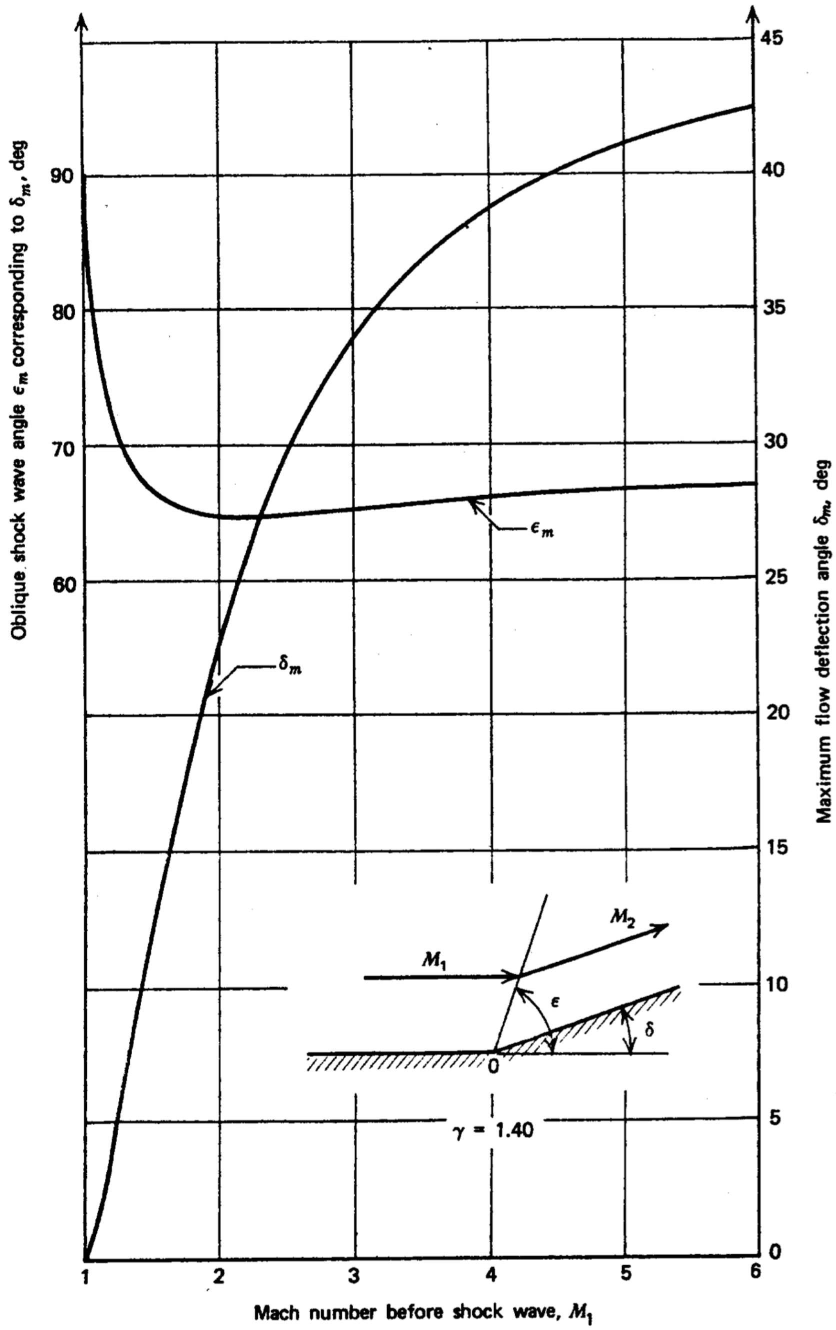
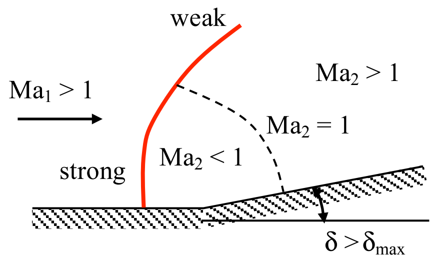
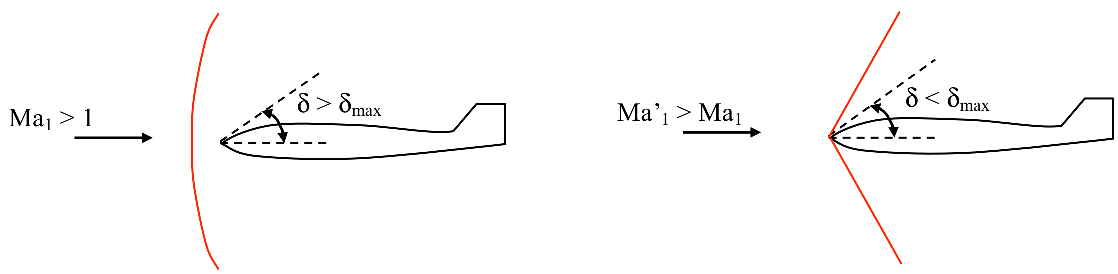


Figure 7.20 Maximum flow deflection angle δ_m and the corresponding wave angle ε_m for oblique shock waves as a function of the upstream Mach number M_1 for $\gamma = 1.40$.

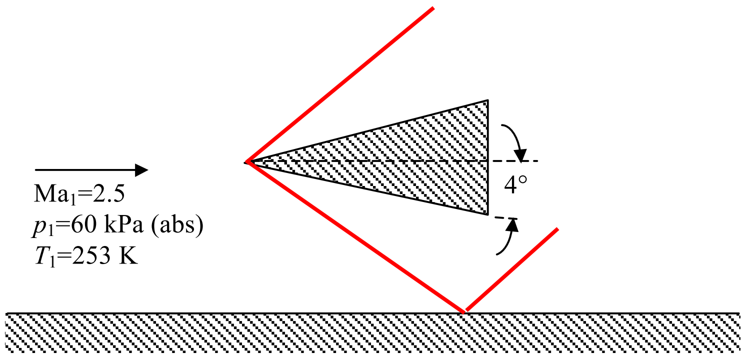
What happens if the flow is deflected by an angle larger than δ_{max}? A curved, detached shock wave occurs with a strong shock below the sonic line and a weak shock above sonic line. Analysis of curved shock waves is very difficult due to the existence of subsonic, transonic, and supersonic flow, each of which has very different governing differential equations. In addition, the flow downstream of the shock system will have curved streamlines and be irrotational (we'll discuss this in a different set of notes when investigating Crocco's Theorem).



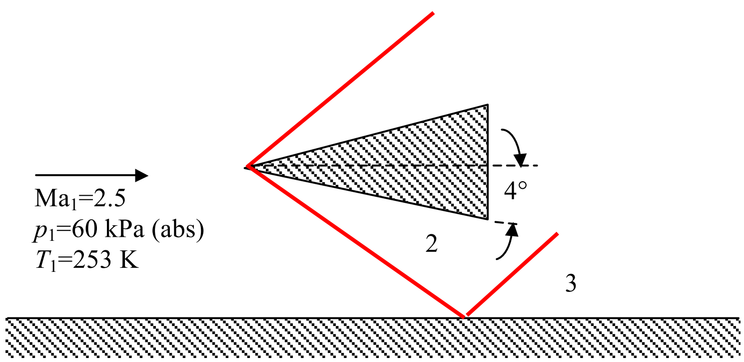
Note that δ_{max} increases with increasing upstream Mach number. Hence, it's possible that a flow with a detached, curved shock at a low Mach number may produce an attached, oblique shock at larger Mach numbers.



Air flowing with a Mach number of 2.5 with a pressure of 60 kPa (abs) and a temperature of 253 K passes over a wedge which turns the flow through an angle of 4° leading to the generation of an oblique shock wave. This oblique shock wave impinges on a flat wall, which is parallel to the flow upstream of the wedge, and is "reflected" from it. Find the pressure and velocity behind the reflected shock wave.



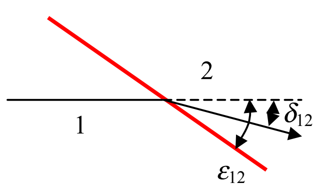
SOLUTION:



Use the oblique shock relations to determine the conditions in region 2.

$\mathrm{Ma}_1 = 2.5,\ \delta_{12} = 4° \quad \Rightarrow \quad \varepsilon_{12} = 26.6°,\ \mathrm{Ma}_2 = 2.333,\ p_2/p_1 = 1.2961,\ T_2/T_1 = 1.0775$ (1)

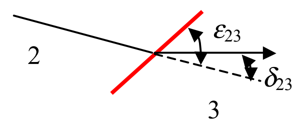
(Can use the normal shock relations with $\mathrm{Ma}_{1N} = \mathrm{Ma}_1 \sin\varepsilon_{12} = 1.12$ to determine the property ratios across the shock.)



Use the oblique shock relations to determine the conditions in region 3.

$\text{Ma}_2 = 2.333$, $\delta_{23} = 4° \Rightarrow \varepsilon_{23} = 28.5°$, $\text{Ma}_3 = 2.17$, $p_3/p_2 = 1.2787$, $T_3/T_2 = 1.0732$ (2)

($\text{Ma}_{2\text{N}} = \text{Ma}_2 \sin\varepsilon_{23} = 1.113$)



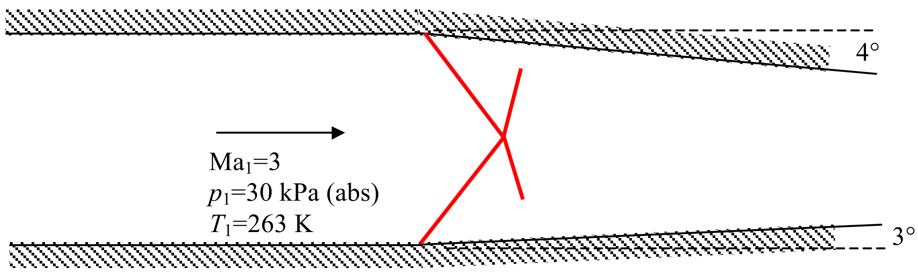
Thus,

$$p_3 = \left(\frac{p_3}{p_2}\right)\left(\frac{p_2}{p_1}\right)p_1 \Rightarrow \boxed{p_3 = 99.4\text{ kPa (abs)}} \tag{3}$$

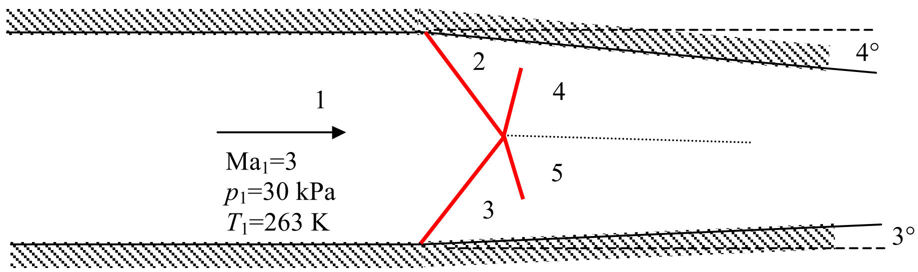
$$T_3 = \left(\frac{T_3}{T_2}\right)\left(\frac{T_2}{T_1}\right)T_1 \Rightarrow \boxed{T_3 = 293\text{ K}} \tag{4}$$

$$V_3 = \text{Ma}_3\sqrt{\gamma R T_3} \Rightarrow \boxed{V_3 = 745\text{ m/s}} \tag{5}$$

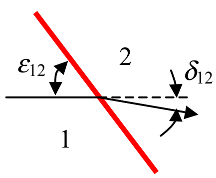
Air is flowing down a wide channel with a Mach number of 3, pressure of 30 kPa (abs), and a temperature of 263 K. The upper wall of this channel turns through an angle of 4° towards the flow while the lower wall turns through an angle of 3° towards the flow leading to the generation of two oblique shock waves which intersect each other. Find the pressure and flow direction downstream of the shock intersection.



SOLUTION:



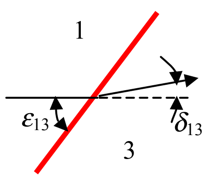
Consider the flow in region 2.



Using the oblique shock relations:

$\mathrm{Ma}_1 = 3,\ \delta_{12} = 4° \ \Rightarrow\ \varepsilon_{12} = 22.4°,\ \mathrm{Ma}_2 = 2.799,\ p_2/p_1 = 1.352$ (1)

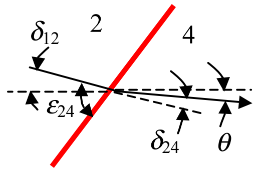
Now consider the flow in region 3.

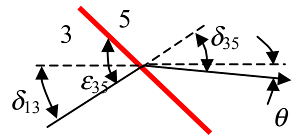


Using the oblique shock relations:

$\mathrm{Ma}_1 = 3,\ \delta_{13} = 3° \ \Rightarrow\ \varepsilon_{13} = 21.6°,\ \mathrm{Ma}_3 = 2.848,\ p_3/p_1 = 1.256$ (2)

Finally, consider flow in regions 4 and 5. Note that since the flow in regions 4 and 5 will be parallel, the flow angle from the horizontal, θ, will be the same for both regions.





From geometry:

$$\delta_{12} = \delta_{24} + \theta \;\Rightarrow\; \delta_{24} = \delta_{12} - \theta \tag{3}$$

$$\delta_{13} = \delta_{35} - \theta \;\Rightarrow\; \delta_{35} = \delta_{13} + \theta \tag{4}$$

Use an iterative approach to determine θ. Recall that the pressure in regions 4 and 5 must be identical.

1. Assume a value for θ.
2. Determine Ma_4, ε_{24}, and p_4/p_2 using the oblique shock relations and $Ma_2 = 2.799$ and $\delta_{24} = \delta_{12} - \theta$ where $\delta_{12} = 4°$.
3. Determine Ma_5, ε_{35}, and p_5/p_3 using the oblique shock relations and $Ma_3 = 2.848$ and $\delta_{35} = \delta_{13} + \theta$ where $\delta_{13} = 3°$.
4. Determine:

$$\frac{p_4}{p_1} = \left(\frac{p_4}{p_2}\right)\left(\frac{p_2}{p_1}\right) \text{ where } p_2/p_1 = 1.352 \tag{5}$$

$$\frac{p_5}{p_1} = \left(\frac{p_5}{p_3}\right)\left(\frac{p_3}{p_1}\right) \text{ where } p_3/p_1 = 1.256 \tag{6}$$

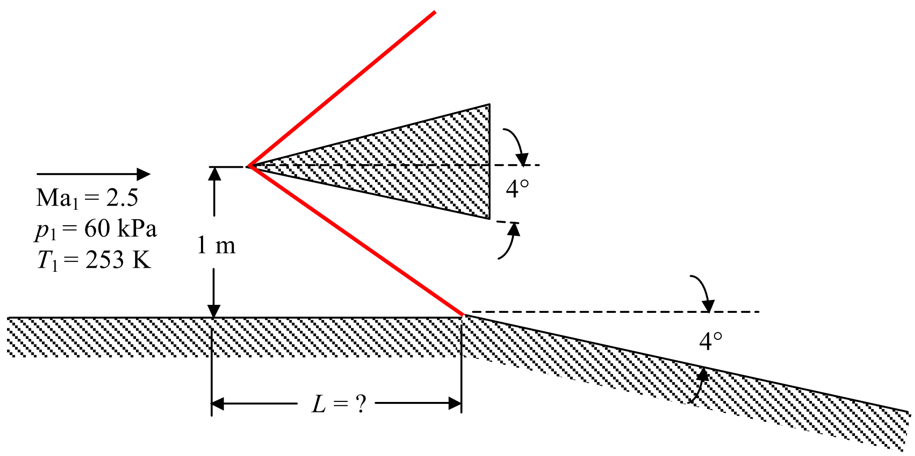
5. Is $p_4/p_1 = p_5/p_1$? Yes $\Rightarrow$ end. No $\Rightarrow$ If $p_4/p_1 > p_5/p_1$, then the assumed value for θ was too small. If $p_4/p_1 < p_5/p_1$, then the assumed value for θ was too large. Repeat the procedure starting at step 1.

The following table shows a few trials starting with $\theta = -1°$.

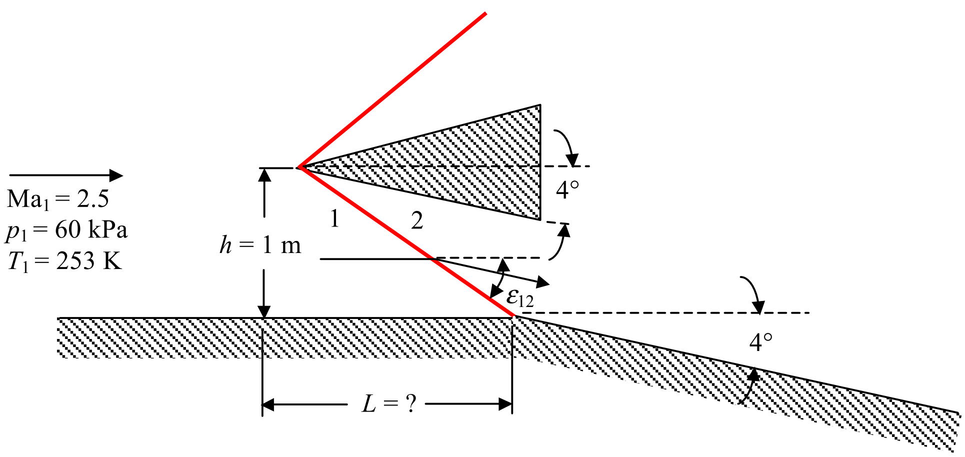
θ [deg]	p_4/p_2	p_5/p_3	p_4/p_1	p_5/p_1
-1	1.4234	1.1577	1.924	1.454
0	1.3291	1.2437	1.797	1.562
1	1.2398	1.3347	1.676	1.676

$\therefore \theta = 1°$ and $p_4 = p_5 = 50.3$ kPa (abs) (using $p_4/p_1 = p_5/p_1 = 1.676$ and $p_1 = 30$ kPa)
$Ma_4 = 2.658$, $\delta_{24} = 3°$, $\varepsilon_{24} = 23.1°$
$Ma_5 = 2.658$, $\delta_{35} = 4°$, $\varepsilon_{35} = 23.5°$

Air at a pressure of 60 kPa and a temperature of 253 K flows at a Mach number of 2.5 over a wedge, leading to the generation of an oblique shock wave. This oblique shock wave impinges on a wall which turns away from the flow by 4° exactly at the point where the oblique shock wave impinges on it. If the leading edge of the wedge is 1 m above the wall, how far behind this leading edge would the change in wall angle have to occur?



SOLUTION:



Using the oblique shock relations, determine the flow conditions in region 2.

$$\mathrm{Ma}_1 = 2.5,\ \delta_{12} = 4° \ \Rightarrow \ \mathrm{Ma}_2 = 2.333,\ \varepsilon_{12} = 26.6° \tag{1}$$

From geometry:

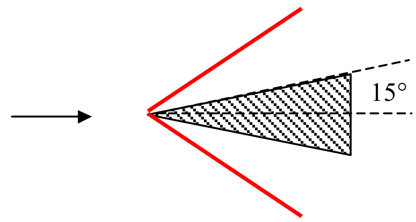
$$\tan\varepsilon_{12} = \frac{h}{L} \Rightarrow L = \frac{h}{\tan\varepsilon_{12}} \tag{2}$$

$$\boxed{\therefore L = 1.997\text{ m}}$$

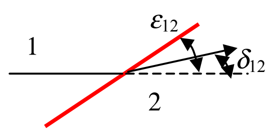
using $h = 1$ m.

Air, assumed to be a perfect gas, flows from the storage tanks of a blow-down wind tunnel where the temperature and pressure are 290 K and 7000 kPa (abs). A symmetrical wedge having a half-angle of 15° is placed in the test section where the Mach number is 3.0. Calculate the following properties on the surface of the wedge:

a. temperature,
b. pressure,
c. Mach number,
d. the oblique shock angle with respect to the incoming flow, and
e. the minimum Mach number for which the shock wave will remain attached to the wedge.



SOLUTION:



Apply the oblique shock relations across the shock wave.

$\mathrm{Ma}_1 = 3.00$, $\delta_{12} = 15°$ $\Rightarrow$ $\boxed{\varepsilon_{12} = 32.24°}$; $\boxed{\mathrm{Ma}_2 = 2.255}$; $p_2/p_1 = 2.8216$; $T_2/T_1 = 1.3883$ (1)

The pressure and temperature downstream of the shock are:

$$p_2 = \left(\frac{p_2}{p_1}\right) p_1 \Rightarrow \boxed{p_2 = 537.8\text{ kPa (abs)}} \tag{2}$$

where

$$p_1 = p_{01}\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1 = 190.6\text{ kPa (abs)} \tag{3}$$

and

$$T_2 = \left(\frac{T_2}{T_1}\right) T_1 \Rightarrow \boxed{T_2 = 143.8\text{ K}} \tag{4}$$

where

$$T_1 = T_{01}\left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_1^2\right)^{-1} \Rightarrow T_1 = 103.6\text{ K} \tag{5}$$

The minimum Mach number for which the oblique shock wave will remain attached when $\delta_{\max} = 15°$ may be found from the relation:

$$\delta_{\max} = \tan^{-1}\left(\sqrt{\frac{V_{N1}}{V_{N2}}}\right) - \tan^{-1}\left(\sqrt{\frac{V_{N2}}{V_{N1}}}\right) \tag{6}$$

or, more conveniently, but less accurately, from a plot of the maximum turning angle as a function of the upstream Mach number (e.g., from Fig. 7.20 in Zucrow and Hoffman as shown below).

$$\delta_{\max} = 15° \;\Rightarrow\; \boxed{\mathrm{Ma}_{1,\min} = 1.6} \tag{7}$$

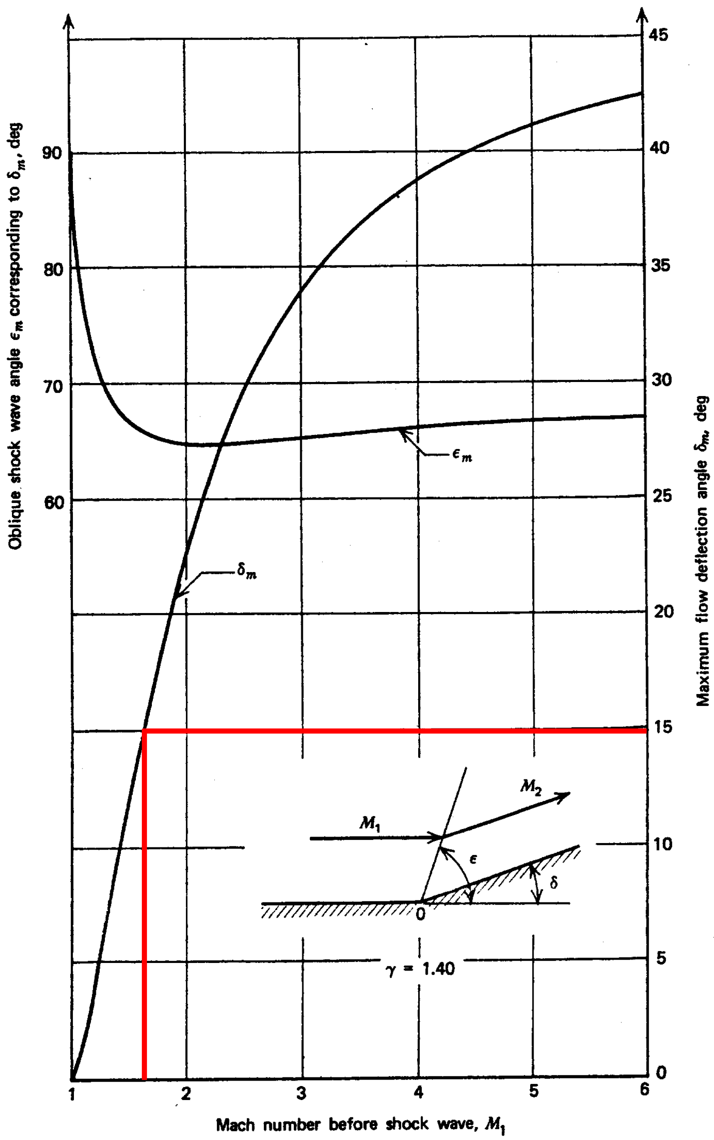
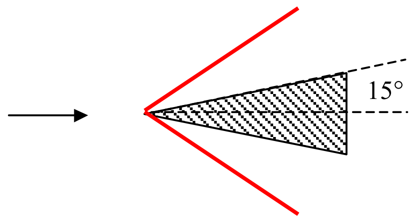
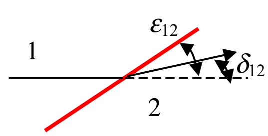


Figure 7.20 Maximum flow deflection angle δ_m and the corresponding wave angle ε_m for oblique shock waves as a function of the upstream Mach number M_1 for $\gamma = 1.40$.

Air, assumed to be a perfect gas, flows from the storage tanks of a blow-down wind tunnel where the temperature and pressure are 290 K and 7000 kPa (abs). A symmetrical wedge having a half-angle of 15° is placed in the test section. An oblique shock forms on the wedge at an angle of 30° with respect to the incoming flow. Calculate the following properties on the surface of the wedge:
a. the free-stream values of the Mach number, pressure, and temperature, and
b. the same flow properties on the surface of the wedge.



SOLUTION:



Apply the oblique shock relations across the shock wave.

$$\varepsilon_{12} = 30° \quad , \delta_{12} = 15° \quad \Rightarrow \quad \boxed{\text{Ma}_1 = 3.348};\ \boxed{\text{Ma}_2 = 2.502};\ p_2/p_1 = 3.1023;\ T_2/T_1 = 1.4397 \tag{1}$$

The pressure and temperature downstream of the shock are:

$$p_2 = \left(\frac{p_2}{p_1}\right)p_1 \Rightarrow \boxed{p_2 = 354 \text{ kPa (abs)}} \tag{2}$$

where

$$p_1 = p_{01}\left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_1 = 114.1 \text{ kPa (abs)}} \tag{3}$$

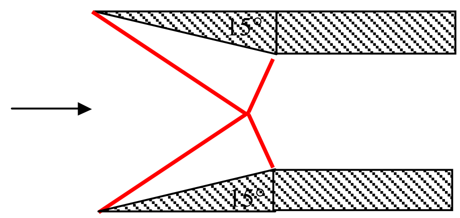
and

$$T_2 = \left(\frac{T_2}{T_1}\right)T_1 \Rightarrow \boxed{T_2 = 128.9 \text{ K}} \tag{4}$$

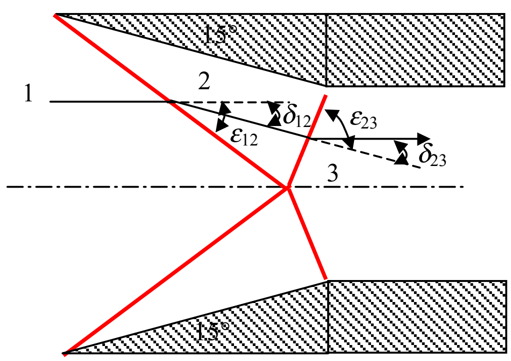
where

$$T_1 = T_{01}\left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right)^{-1} \Rightarrow \boxed{T_1 = 89.5 \text{ K}} \tag{5}$$

A two-dimensional supersonic inlet comprises two 15° wedges placed symmetrically on the top and bottom surfaces of a rectangular flow passage. An oblique shock wave propagates into the inlet from the leading edge of each wedge. Calculate the flow Mach number, pressure, and temperature behind the initial shock waves and the transmitted shock waves if the incoming Mach number is 3.0. Assume that the incoming flow is air with a pressure of 101.35 kPa (abs) and a temperature of 300 K.



SOLUTION:



Due to symmetry, the flow in the upper and lower halves will be identical.

Use the oblique shock relations to determine the conditions in each region.

$\mathrm{Ma}_1 = 3.0,\ \delta_{12} = 15°\quad \Rightarrow\quad \varepsilon_{12} = 32.24°,\ \boxed{\mathrm{Ma}_2 = 2.25};\ p_2/p_1 = 2.822;\ T_2/T_1 = 1.388$ (1)

$p_1 = 101.35\text{ kPa (abs)}\quad \Rightarrow\quad \boxed{p_2 = 286.0\text{ kPa (abs)}}$ (2)

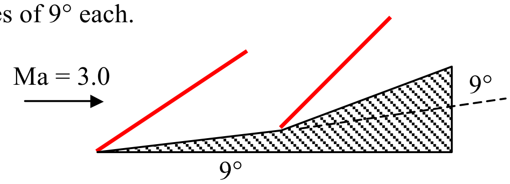
$T_1 = 300\text{ K}\quad \Rightarrow\quad \boxed{T_2 = 416.4\text{ K}}$ (3)

$\mathrm{Ma}_2 = 2.25,\ \delta_{23} = 15°\quad \Rightarrow\quad \varepsilon_{23} = 40.43°,\ \boxed{\mathrm{Ma}_3 = 1.67};\ p_3/p_2 = 2.317;\ T_3/T_2 = 1.293$ (4)

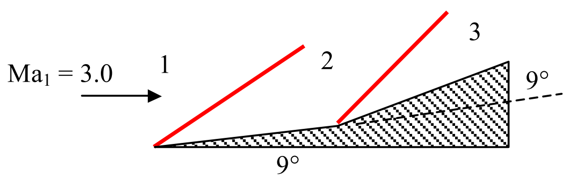
$p_2 = 286.0\text{ kPa (abs)}\quad \Rightarrow\quad \boxed{p_3 = 662.7\text{ kPa (abs)}}$ (5)

$T_2 = 416.4\text{ K}\quad \Rightarrow\quad \boxed{T_3 = 538.4\text{ K}}$ (6)

In Example 7.17 of Zucrow & Hoffman, a multiple oblique shock wave inlet is described where the total turning angle of 18° is accomplished by three flow deflection angles of 6° each, and the corresponding stagnation pressure recovery is η_p=0.97804 where $\eta_p = p_{0,\text{final}}/p_{0,\text{initial}}$. Determine the stagnation pressure recovery of a multiple oblique shock wave inlet operating at the same conditions for two equal flow deflection angles of 9° each.



SOLUTION:



Use the oblique shock relations across each shock wave.

$\text{Ma}_1 = 3.000,\ \delta_{12} = 9° \Rightarrow \varepsilon_{12} = 26.49°;\ \text{Ma}_2 = 2.554;\ p_{02}/p_{01} = 0.9722$ (1)

$\text{Ma}_2 = 2.554,\ \delta_{23} = 9° \Rightarrow \varepsilon_{23} = 30.34°;\ \text{Ma}_3 = 2.174;\ p_{03}/p_{02} = 0.9811$ (2)

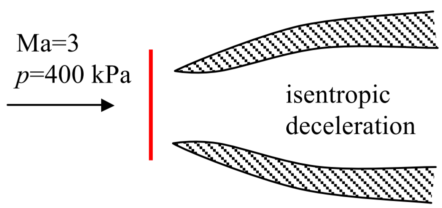
The stagnation pressure recovery is then:

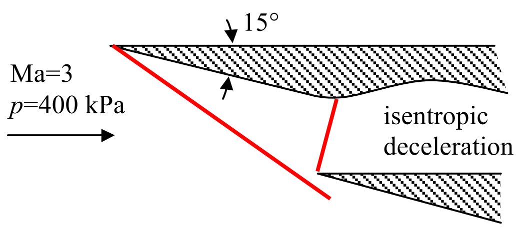
$$\eta_P = \frac{p_{03}}{p_{01}} = \left(\frac{p_{03}}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right) \Rightarrow \boxed{\eta_P = 0.9538} \quad (3)$$

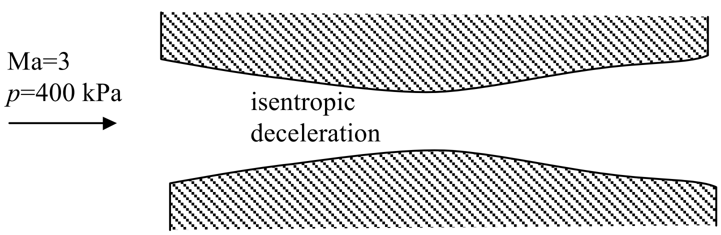
Thus, turning the flow through three 6° turns rather than two 9° turns results in a stagnation pressure recovery increase of approximately 2.5%.

An aircraft is to cruise at a Mach number of 3. The stagnation pressure in the flow ahead of the aircraft is 400 kPa. Compare the stagnation pressure recovery for three possible intake scenarios.

a. An intake that involves a normal shock in the free stream ahead of the intake followed by an isentropic deceleration of the subsonic flow behind the shock wave to an essentially zero velocity.
b. An oblique shock wave diffuser in which the air flows through an oblique shock wave and then an isentropic deceleration of the subsonic flow behind the normal shock wave to an essentially zero velocity.
c. An ideal shockless convergent-divergent diffuser in which the air is isentropically brought to an essentially zero velocity.







SOLUTION:

Consider first the normal shock wave case. Using the normal shock relations across the shock wave:

$$\text{Ma}_1 = 3.0 \quad\Rightarrow\quad \boxed{p_{02}/p_{01} = 0.3283} \tag{1}$$

Now consider the oblique / normal shock wave case. From the oblique shock relations:

$$\text{Ma}_1 = 3.0,\ \delta_{12} = 15° \quad\Rightarrow\quad \text{Ma}_2 = 2.255,\ p_{02}/p_{01} = 0.8950 \tag{2}$$

Now apply the normal shock relations:

$$\text{Ma}_2 = 2.255 \quad\Rightarrow\quad p_{03}/p_{02} = 0.6033 \tag{3}$$

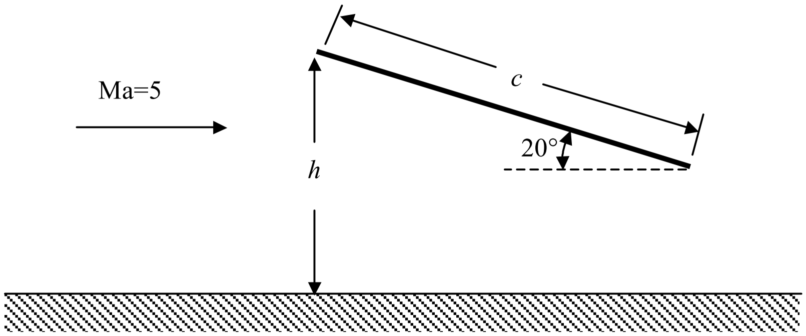
The stagnation pressure ratio drop across both shocks is:

$$\frac{p_{03}}{p_{01}} = \left(\frac{p_{03}}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right) \quad\Rightarrow\quad \boxed{p_{03}/p_{01} = 0.5400} \tag{4}$$

$\boxed{\text{For the isentropic case, the stagnation pressure doesn't change.}}$

Thus, we see that isentropic deceleration is the most efficient deceleration method, using an oblique shock with a subsequent normal shock is the next efficient method, and the least efficient is deceleration using a single normal shock.

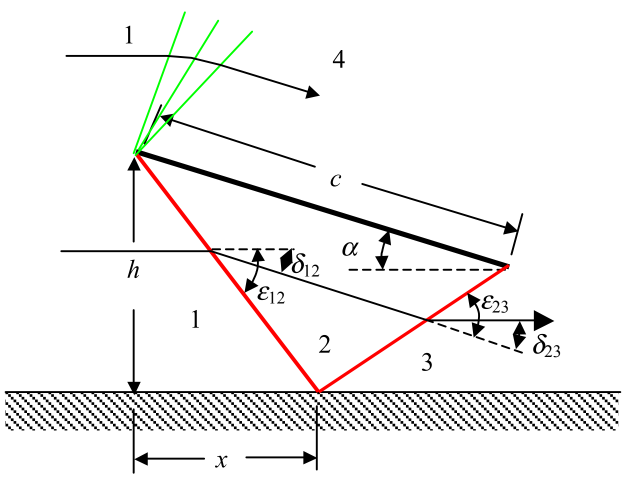
A thin, flat plate airfoil at an angle of attack of 20° is close to the ground and encounters an oncoming air stream with a Mach number of 5. The leading edge of the airfoil is a distance, *h*, from the ground and the chord length of the foil is denoted by *c*:



a. Determine the smallest value of the ratio h/c for which the lift and drag on the airfoil will not be affected by the presence of the ground.
b. What is the airfoil's lift coefficient (based on the chord length) for these conditions?
c. What is the airfoil's drag coefficient (based on the chord length) for these conditions?
d. If h/c is slightly less than the critical value found in part (a), will the drag coefficient be larger than, equal to, or less than the value found in part (c)? Explain how you arrived at your answer.

SOLUTION:

For the ground not to affect the lift and drag, the reflected oblique shock must not impinge on the airfoil.



Using the oblique shock relations:

$$\mathrm{Ma}_1 = 5.0,\ \delta_{12} = \alpha = 20° \ \Rightarrow \quad \begin{aligned} \mathrm{Ma}_2 &= 3.022 \\ \varepsilon_{12} &= 29.80° \\ p_2/p_1 &= 7.038 \end{aligned} \tag{1}$$

$$\mathrm{Ma}_2 = 3.022,\ \delta_{23} = \alpha = 20° \ \Rightarrow \quad \begin{aligned} \mathrm{Ma}_3 &= 2.008 \\ \varepsilon_{23} &= 37.59° \\ p_3/p_2 &= 3.799 \end{aligned} \tag{2}$$

From geometry:

$$\tan\varepsilon_{12} = \frac{h}{x} \Rightarrow x/c = \frac{h/c}{\tan\varepsilon_{12}} \tag{3}$$

$$\tan\left(\varepsilon_{23}-\delta_{23}\right) = \frac{h - c\sin\alpha}{c\cos\alpha - x} = \frac{h/c - \sin\alpha}{\cos\alpha - x/c} \tag{4}$$

Combining Eqs. (3) and (4) gives:

$$\tan\left(\varepsilon_{23}-\delta_{23}\right) = \frac{h/c - \sin\alpha}{\cos\alpha - \frac{h/c}{\tan\varepsilon_{12}}} \tag{5}$$

$$\boxed{\therefore \frac{h}{c} = \frac{\tan\left(\varepsilon_{23}-\delta_{23}\right)\cos\alpha + \sin\alpha}{1 + \frac{\tan\left(\varepsilon_{23}-\delta_{23}\right)}{\tan\varepsilon_{12}}}} \tag{6}$$

Using the given values, $\boxed{h/c|_{\min} = 0.412}$.

The lift and drag coefficients are given by.

$$c_L \equiv \frac{L}{\frac{1}{2}\rho_1 U_1^2 c} = \frac{p_2 c\cos\alpha - p_4 c\cos\alpha}{\frac{1}{2}\rho_1\gamma RT_1\mathrm{Ma}_1^2 c} = \frac{p_1\left(p_2/p_1 - p_4/p_1\right)\cos\alpha}{p_1\frac{1}{2}\gamma\mathrm{Ma}_1^2} \tag{7}$$

$$\boxed{\therefore c_L = \frac{\left(p_2/p_1 - p_4/p_1\right)\cos\alpha}{\frac{1}{2}\gamma\mathrm{Ma}_1^2}} \tag{8}$$

$$c_D \equiv \frac{D}{\frac{1}{2}\rho_1 U_1^2 c} = \frac{p_2 c\sin\alpha - p_4 c\sin\alpha}{\frac{1}{2}\rho_1\gamma RT_1\mathrm{Ma}_1^2 c} = \frac{p_1\left(p_2/p_1 - p_4/p_1\right)\sin\alpha}{p_1\frac{1}{2}\gamma\mathrm{Ma}_1^2} \tag{9}$$

$$\boxed{\therefore c_D = \frac{\left(p_2/p_1 - p_4/p_1\right)\sin\alpha}{\frac{1}{2}\gamma\mathrm{Ma}_1^2}} \tag{10}$$

The pressure ratio, p_4/p_1 may be found using Prandtl-Meyer angles and the stagnation pressure ratios.

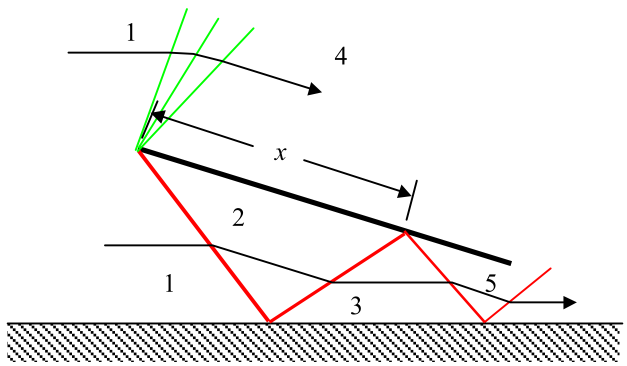
$$\mathrm{Ma}_1 = 5.0 \Rightarrow \nu_1 = -76.92° \tag{11}$$

$$\nu_4 = \nu_1 - 20° = -96.92° \Rightarrow \mathrm{Ma}_4 = 8.326 \tag{12}$$

$$\frac{p_4}{p_1} = \left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_4^2}{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \frac{p_4}{p_1} = 0.04177 \tag{13}$$

Hence, $\boxed{c_L = 0.3757,\ c_D = 0.1367}$.

If h/c is less than the value found above, the reflected shock will impinge on the airfoil surface as shown in the figure below.

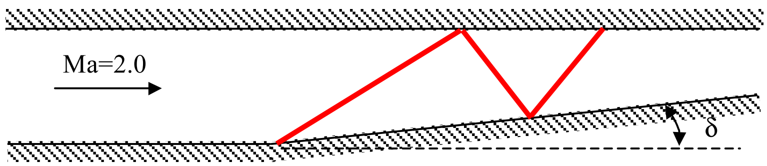


Following the same approach used to derive Eq. (10), the airfoil's drag coefficient for this case will be:

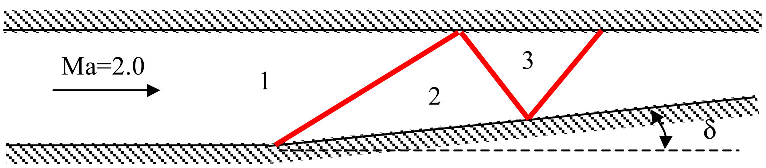
$$c_D = \frac{\left[\left(x/c\right)p_2/p_1 + \left(1-x/c\right)p_5/p_1 - p_4/p_1\right]\sin\alpha}{\frac{1}{2}\gamma \mathrm{Ma}_1^2} \quad \text{where } x \text{ is defined in the figure.} \tag{14}$$

The values for p_2/p_1 and p_4/p_1 will be the same as previously determined. The pressure ratio for region 5 will be larger than that for region 2, i.e., $p_5/p_1 > p_2/p_1$, and so the drag coefficient will be larger for $h/c < h/c|_{\min}$ than for $h/c \geq h/c|_{\min}$.

Air flows in the passage shown in the sketch with an initial Mach number of 2.0. Determine the maximum turning angle, δ, for which three oblique shock waves appear.



SOLUTION:



While there may be other more elegant approaches, the iterative approach given here is crude, but simple.

1. Assume a value of δ.
2. Determine Ma_2 using the oblique shock relations with $Ma_1 = 2.0$ and $\delta_{12} = \delta$. Note that if $\delta > \delta_{max}$ for Ma_1, then assume a new δ with $\delta_{new} < \delta_{old}$ and go to step 2.
3. Determine Ma_3 using the oblique shock relations with Ma_2 (from step 2) and $\delta_{23} = \delta$. Note that if $\delta > \delta_{max}$ for Ma_2, then assume a new δ with $\delta_{new} < \delta_{old}$ and go to step 2.
4. Determine Ma_4 using the oblique shock relations with Ma_3 (from step 3) and $\delta_{34} = \delta$.
5. Is the δ_{max} corresponding to Ma_4 (from step 3) equal to δ?
   a. If yes, then the iterations are complete.
   b. If $\delta_{max} < \delta$, then choose a new δ such that $\delta_{new} > \delta_{old}$. Go to step 2.
   c. If $\delta_{max} > \delta$, then choose a new δ such that $\delta_{new} < \delta_{old}$. Go to step 2.

After performing the iterations described above:

$\boxed{\delta = 8.8° \text{ with } Ma_1 = 2.0,\ Ma_2 = 1.68,\ Ma_3 = 1.38, \text{ and } Ma_4 = 0.94}$

```
/*
  soln_obliqueshock_10.c
  Copyright (C) Carl R. Wassgren, Jr.

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*/

# include <stdio.h>
# include <stdlib.h>
# include <math.h>
# include "ObliqueShockRelations.h"
# define PI (4.0*atan(1.0))

/* ***** */

int main(int argc, char **argv) {
  int flag = 0;
  double k, Ma1, Ma2, Ma3, Ma4, delta, deltamax;

  // Initialize the variables.
  k = 1.4;  /* air */
  Ma1 = 2.0;


  // Use a brute force approach to determine delta.  Simply march through increasing
  // values of delta until a solution is found.

  delta = 0.0*PI/180.0;  // Choose a starting point for delta.
  do {
    delta += 0.01*PI/180.0;
    Ma2 = Determine_Ma2_Given_Ma1_And_delta(k, Ma1, delta);
    if (!isnan(Ma2)) {
      Ma3 = Determine_Ma2_Given_Ma1_And_delta(k, Ma2, delta);
      if (!isnan(Ma3)) {
        deltamax = Determine_deltamax_Given_Ma1(k, Ma3);
        if (delta >= deltamax) {
          flag = 1;
        }
        Ma4 = Determine_Ma2_Given_Ma1_And_delta(k, Ma3, delta);
      }
    }
  } while (flag == 0);

  printf("(delta, deltamax) = (%.2f, %.2f) deg\n",
         delta*180.0/PI, deltamax*180.0/PI);

  Ma2 = Determine_Ma2_Given_Ma1_And_delta(k, Ma1, deltamax);
  Ma3 = Determine_Ma2_Given_Ma1_And_delta(k, Ma2, deltamax);
  Ma4 = Determine_Ma2_Given_Ma1_And_delta(k, Ma3, deltamax);
  printf("(Ma1, Ma2, Ma3, Ma4) = (%.2f, %.2f, %.2f, %.2f)\n",
         Ma1, Ma2, Ma3, Ma4);

  return 0;
}
```
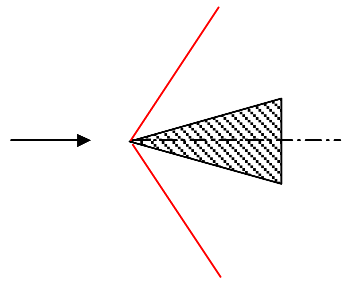
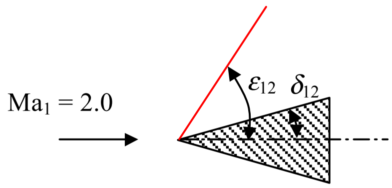

A uniform supersonic air flow traveling at Mach 2.0 passes over a wedge. An oblique shock, making an angle of 40° with the flow direction, is attached to the wedge for these flow conditions. If the static pressure and temperature in the uniform flow are 5 psia and 0° F, determine the static pressure ant temperature behind the wave, the Mach number of the flow passing over the wedge, and the wedge half angle.



SOLUTION:



$\varepsilon_{12} = 40°$
$p_1 = 5$ psia
$T_1 = 0$ °F = 460 °R

Use the oblique shock relations to determine the pressure and temperature across the shock.

$$\text{Ma}_{1N} = \text{Ma}_1 \sin\varepsilon_{12} = 1.29 \quad \Rightarrow \quad p_2/p_1 = 1.775 \quad \Rightarrow \quad \boxed{p_2 = 8.88 \text{ psia}} \tag{1}$$

$$T_2/T_1 = 1.185 \quad \Rightarrow \quad \boxed{T_2 = 545\ °\text{R}} \tag{2}$$

$$\text{Ma}_{2N} = 0.791 \tag{3}$$

$$\text{Ma}_1 = 2.0,\ \varepsilon_{12} = 40° \quad \Rightarrow \quad \boxed{\delta_{12} = 10°} \tag{4}$$

$$T_{01} = T_1\left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right) \Rightarrow T_{01} = 828\ °\text{R} \tag{5}$$

$$T_{02} = T_2\left(1 + \frac{\gamma - 1}{2}\text{Ma}_2^2\right) = T_{01} \Rightarrow \boxed{\text{Ma}_2 = 1.61} \tag{6}$$

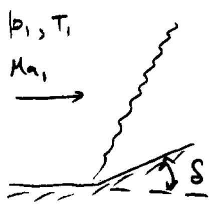
Alternately, the downstream Mach number can be found using:

$$\text{Ma}_{N2} = \text{Ma}_2 \sin\left(\varepsilon_{12} - \delta_{12}\right) \tag{7}$$

Air having an initial Mach number of 3.0, a free-stream static pressure of 1 atm, and an initial static temperature of 300 K is deflected through an angle of 15° by a frictionless surface. Assuming that a weak oblique shock wave occurs, calculate:

- **a.** **the static pressure downstream of the shock**
- **b.** **the static temperature downstream of the shock**
- **c.** **the change in the stagnation pressure across the shock**
- **d.** **the velocity downstream of the shock**
- **e.** **the Mach number downstream of the shock**

SOLUTION:



$Ma_1 = 3.0$
$p_1 = 1\text{ atm}$
$T_1 = 300\text{K}$
$\delta = 15^\circ$

- Using $Ma_1 = 3.0$ and $\delta = 15^\circ \Rightarrow \varepsilon = 32.2^\circ$ (from oblique shock relations

$\Rightarrow Ma_{1N} = Ma_1 \sin\varepsilon = 1.60$

from normal shock relations $\Rightarrow$

$$Ma_{2N} = 0.6684 = Ma_2 \sin(\overset{32.2^\circ}{\varepsilon} - \overset{15^\circ}{\delta}) \Rightarrow \boxed{Ma_2 = 2.26}$$

$$\frac{p_2}{p_1} = 2.8200 \Rightarrow \boxed{p_2 = 2.82\text{ atm}}$$

$$\frac{T_2}{T_1} = 1.3880 \Rightarrow \boxed{T_2 = 416\text{ K}}$$

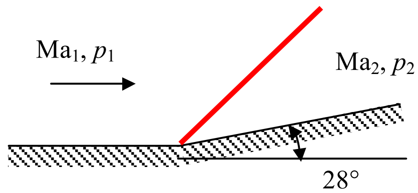
$$\frac{p_{02}}{p_{01}} = 0.8952 \Rightarrow p_{02} - p_{01} = \left(\frac{p_{02}}{p_{01}} - 1\right) p_{01}$$

$$\boxed{\therefore p_{02} - p_{01} = -3.85\text{ atm}} \quad \text{where } p_{01} = p_1\left(1 + \frac{\gamma-1}{2} Ma_1^2\right)^{\frac{\gamma}{\gamma-1}} = 36.7\text{ atm}$$

$$V_2 = Ma_2\sqrt{\gamma R T_2}$$

$$\boxed{\therefore V_2 = 924\text{ m/s}}$$

Consider a compression corner with a deflection angle of 28°. How does the pressure ratio across the oblique shock change if the incoming Mach number is doubled from 3 to 6?



SOLUTION:

$\mathrm{Ma}_1 = 3$, $\delta = 28° \Rightarrow p_2/p_1 = 5.7388$ (using the oblique shock relations) (1)
$\mathrm{Ma}_1 = 6$, $\delta = 28° \Rightarrow p_2/p_1 = 15.8361$ (using the oblique shock relations) (2)

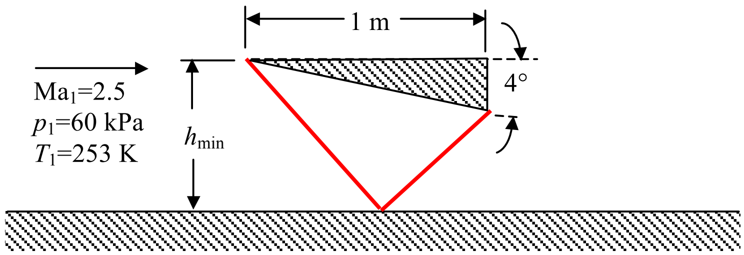
Note that:

$$\frac{\left.{p_2}/{p_1}\right|_{\mathrm{Ma}_1=6}}{\left.{p_2}/{p_1}\right|_{\mathrm{Ma}_1=3}} = \frac{15.8361}{5.7388} = 2.7595 \quad (3)$$

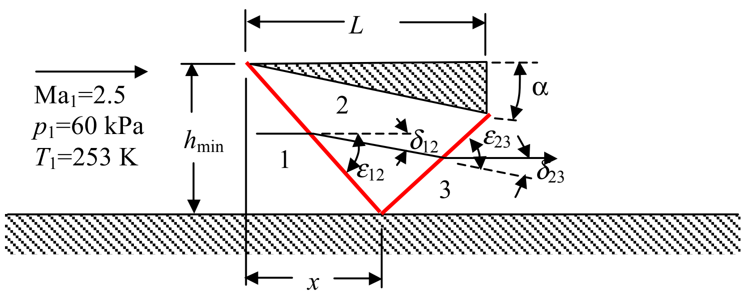
Thus, doubling the upstream Mach number more than doubles the pressure rise, i.e., the oblique shock relations are non-linear.

Air flowing with a Mach number of 2.5, a pressure of 60 kPa (abs), and a temperature of 253 K passes over a wedge which turns the flow through an angle of 4° leading to the generation of an oblique shock wave.

a. If this oblique shock wave impinges on a flat wall, which is parallel to the flow upstream of the wedge, determine the pressure and Mach number behind the reflected shock wave.
b. If the wedge is 1 m long, what is the minimum height above the wall, h_{min}, that the wedge must be in order to not intersect the reflected shock.
c. If the reflected shock did intersect with the wedge, what type of shock pattern would appear?
d. How could we avoid producing the reflected shock?



SOLUTION:



From the oblique shock relations:

$$\mathrm{Ma}_1 = 2.5,\ \delta_{12} = \alpha = 4° \ \Rightarrow \quad \boxed{\mathrm{Ma}_2 = 2.3326} \quad p_2/p_1 = 1.2961 \quad \varepsilon_{12} = 26.61° \tag{1}$$

$$p_2 = (p_2/p_1)p_1 \ \Rightarrow \ \boxed{p_2 = 77.78\text{ kPa}} \tag{2}$$

where $p_1 = 60$ kPa.

The conditions in region 3 may also be found using the oblique shock relations:

$$\mathrm{Ma}_2 = 2.3326,\ \delta_{23} = \alpha = 4° \ \Rightarrow \quad \mathrm{Ma}_3 = 2.1745 \quad p_2/p_1 = 1.2786 \quad \varepsilon_{23} = 28.50° \tag{3}$$

Trigonometry may be used to determine the minimum height so that the reflected shock does not intersect the wedge.

$$\tan\varepsilon_{12} = \frac{h_{\min}}{x} \tag{4}$$

The shock between regions 2 and 3 will have a

$$\tan(\varepsilon_{23}-\delta_{23}) = \frac{h_{\min} - L\tan\alpha}{L-x} \tag{5}$$

Substitute Eq. (4) into Eq. (5) and solve for $h_{\min}$.

$$\tan(\varepsilon_{23}-\delta_{23}) = \frac{h_{\min} - L\tan\alpha}{L - \frac{h_{\min}}{\tan\varepsilon_{12}}} = \frac{\frac{h_{\min}}{L} - \tan\alpha}{1 - \frac{1}{\tan\varepsilon_{12}}\frac{h_{\min}}{L}}$$

$$\tan(\varepsilon_{23}-\delta_{23})\left(1 - \frac{1}{\tan\varepsilon_{12}}\frac{h_{\min}}{L}\right) = \frac{h_{\min}}{L} - \tan\alpha$$

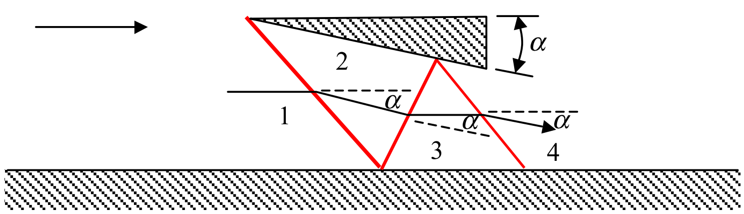
$$\frac{h_{\min}}{L}\left[1 + \frac{\tan(\varepsilon_{23}-\delta_{23})}{\tan\varepsilon_{12}}\right] = \tan(\varepsilon_{23}-\delta_{23}) + \tan\alpha$$

$$\boxed{\therefore \frac{h_{\min}}{L} = \frac{\tan(\varepsilon_{23}-\delta_{23}) + \tan\alpha}{1 + \frac{\tan(\varepsilon_{23}-\delta_{23})}{\tan\varepsilon_{12}}}} \tag{6}$$

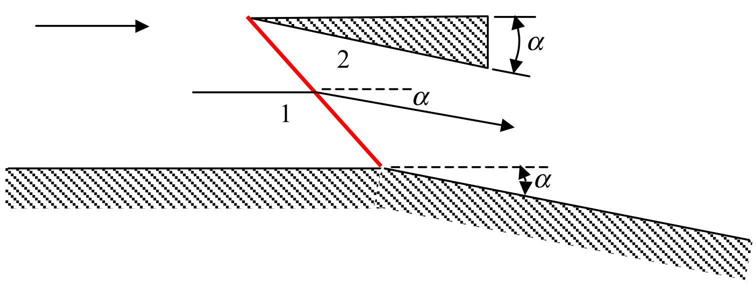
Using the following values:

ε_{23} = 28.50°
δ_{23} = 4°
α = 4°
ε_{12} = 26.61°
L = 1 m
$\Rightarrow$ $\boxed{h_{\min} = 0.28\text{ m}}$

If the height is less than $h_{\min}$, the reflected shock will reflect off the wedge as another oblique shock (refer to the figure below).



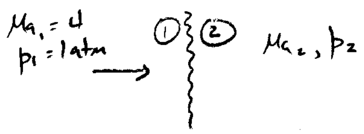
The reflected shock may be avoided by expanding the flow by the wedge angle at the intersection point of the initial shock wave as shown below.



Consider an upstream air flow with a Mach number of 4 and pressure of 1 atm. Determine the total pressure behind:

a. a single normal shock wave, and
b. an oblique shock wave with a wave angle of 40° followed by a normal shock.
c. What conclusions can you draw from this example regarding the deceleration of a supersonic flow using shock waves?

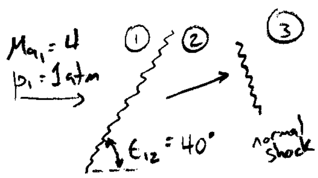
SOLUTION:



$$Ma_1 = 4 \xrightarrow{\text{normal shock relns}} \boxed{Ma_2 = 0.4350} \xrightarrow{\text{isentropic relns}} p_2/p_{02} = 0.8781$$

$$p_2/p_1 = 18.5$$

$$p_{02} = \left(\frac{p_{02}}{p_2}\right)\left(\frac{p_2}{p_1}\right)p_1 = \boxed{21.068 \text{ atm}}$$



$$Ma_1 = 4,\ \epsilon_{12} = 40° \xrightarrow{\text{oblique shock relns}} Ma_2 = 2.1230$$

$$p_2/p_1 = 7.5460$$

$$Ma_2 = 2.1230 \xrightarrow{\text{normal shock relns}} Ma_3 = 0.5579 \xrightarrow{\text{isentropic relns}} \frac{p_3}{p_{03}} = 0.8095$$

$$p_3/p_2 = 5.0917$$

$$p_{03} = \left(\frac{p_{03}}{p_3}\right)\left(\frac{p_3}{p_2}\right)\left(\frac{p_2}{p_1}\right)p_1 = \boxed{47.464 \text{ atm}}$$

NOTE:

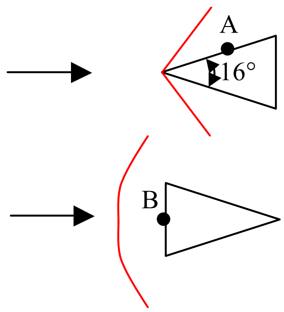
$$Ma_1 = 4 \xrightarrow{\text{isentropic relns}} \frac{p_1}{p_{01}} = 0.0066$$

$$\Rightarrow p_{01} = 151.52 \text{ atm}$$

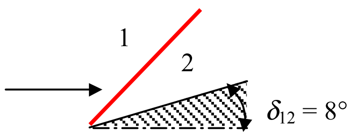
∴ Observe that a flow passing through an oblique shock first has less stagnation pressure loss than a flow passing through a single normal shock.

Air flows at an upstream Mach number of 3 and pressure of 10 psia toward a wedge of total angle 16° as shown in the figures below.

a. If the pointed edge is forward, what will be the pressure at point A?
b. If the blunt edge is forward, what will be the pressure at point B?
c. At what total wedge angle will the attached oblique shock in part (a) become a detached, curved shock?



SOLUTION:



Use the oblique shock relations to find the pressure in region 2 ($p_A = p_2$).

$$\text{Ma}_1 = 3,\ \delta_{12} = 8° \ \Rightarrow\ \text{Ma}_2 = 2.6031,\ \varepsilon_{12} = 25.6113°,\ p_2/p_1 = 1.7953 \tag{1}$$

$$p_2 = (p_2/p_1)p_1 \ \Rightarrow\ \boxed{p_2 = p_A = 17.953\text{ psia}}.\ \ \text{(Note that } p_A \text{ is } \underline{\text{not}} \text{ a stagnation pressure.)} \tag{2}$$

Use the normal shock relations to determine the pressure at B.

$$\text{Ma}_1 = 3 \ \Rightarrow\ \text{Ma}_2 = 0.4752,\ p_2/p_1 = 10.3333 \tag{3}$$

In addition,

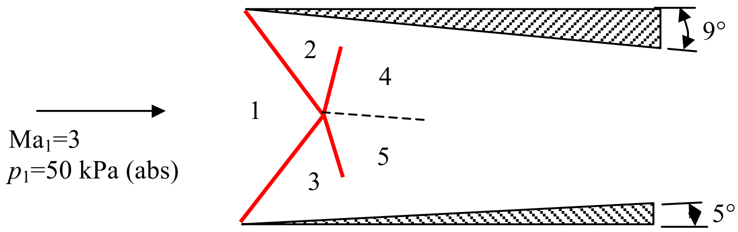
$$\frac{p_{02}}{p_2} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{\gamma-1}} \tag{4}$$

$$\underline{p_B = p_{02} = \left(\frac{p_{02}}{p_2}\right)\left(\frac{p_2}{p_1}\right)p_1} \quad \text{(Note that B is located at a stagnation point from symmetry.)} \tag{5}$$

Using the given data, $\boxed{p_B = 120.60\text{ psia}}$.

The oblique shock will become detached when $\delta_{12} \geq \delta_{max}$ for $\text{Ma}_1 = 3$. Using Fig. 7.20 from Zucrow and Hoffman, $\delta_{max} \approx 34°$. Hence, <u>the maximum total wedge angle for maintaining oblique shocks at this Mach number will be approximately 68°.</u>

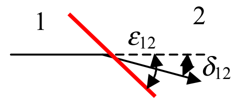
If the Mach number and pressure ahead of the oblique shock waves system shown in the figure are 3 and 50 kPa (abs), respectively, find the pressure in regions 1 through 5. Also determine the angle of the slipstream with respect to the horizontal.



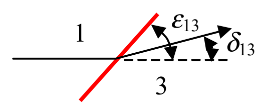
SOLUTION:

Use the oblique shock relations to determine the conditions in regions 2 and 3.

$\mathrm{Ma}_1 = 3,\ \delta_{12} = 9°$ $\Rightarrow$ $\mathrm{Ma}_2 = 2.554$, $\varepsilon_{12} = 26.49°$, $p_2/p_1 = 1.922$ (1)



$\mathrm{Ma}_1 = 3,\ \delta_{13} = 5°$ $\Rightarrow$ $\mathrm{Ma}_3 = 2.750$, $\varepsilon_{23} = 23.13°$, $p_3/p_1 = 1.454$ (2)

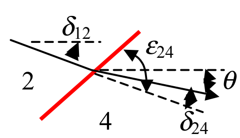


$p_2 = (p_2/p_1)p_1 \Rightarrow \boxed{p_2 = 96.08\text{ kPa}}$ (using $p_1 = 50$ kPa) (3)

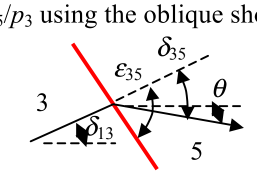
$p_3 = (p_3/p_1)p_1 \Rightarrow \boxed{p_3 = 72.70\text{ kPa}}$ (4)

Use an iterative approach to determine the conditions in regions 4 and 5.

1. Choose a value for θ (the angle of the flow in regions 4 and 5 with respect to the horizontal).
2. Determine Ma_4, ε_{24}, and p_4/p_2 using the oblique shock relations with $\mathrm{Ma}_2 = 2.554$ and $\delta_{24} = \delta_{12} - \theta$.



3. Determine Ma_5, ε_{35}, and p_5/p_3 using the oblique shock relations with $\mathrm{Ma}_3 = 2.750$ and $\delta_{35} = \delta_{13} + \theta$.



4. Determine the pressure ratios for regions 4 and 5.

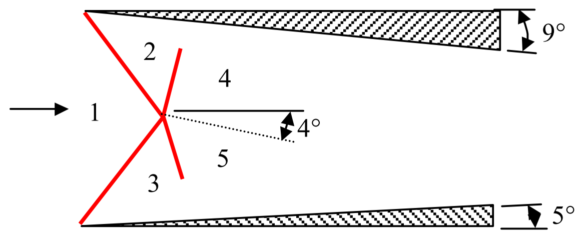
$$\frac{p_4}{p_1} = \left(\frac{p_4}{p_2}\right)\left(\frac{p_2}{p_1}\right) \tag{5}$$

$$\frac{p_5}{p_1} = \left(\frac{p_5}{p_3}\right)\left(\frac{p_3}{p_1}\right) \tag{6}$$

5. Check to see if $p_4/p_1 = p_5/p_1$. If yes, then the iterations are complete. If no, then choose a new value for θ and go to step 2.

The converged results are:

$\boxed{\theta}$	=	$\boxed{4.0°}$
Ma_4	=	2.34
Ma_5	=	2.34
$\boxed{p_4}$	=	$\boxed{133.4\text{ kPa}}$
$\boxed{p_5}$	=	$\boxed{133.4\text{ kPa}}$




```
/*
  soln_obliqueshock_17.c
  Copyright (C) Carl R. Wassgren, Jr.

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*/

# include <stdio.h>
# include <stdlib.h>
# include <math.h>
# include "ObliqueShockRelations.h"
# define PI (4.0*atan(1.0))

/* ***** */

int main(int argc, char **argv) {
  double k, Ma1, Ma2, Ma3, Ma4, Ma5,
    delta12, delta13, delta24, delta35, theta,
    epsilon12, epsilon13, epsilon24, epsilon35,
    p2p1Ratio, p3p1Ratio, p4p2Ratio, p5p3Ratio, p4p1Ratio, p5p1Ratio,
    p1, p2, p3, p4, p5, tol=1.0e-4;

  /* Initialize the variables. */
  k = 1.4;  /* air */
  Ma1 = 3.0;
  p1 = 50.0;  /* kPa */
  delta12 = 9.0*PI/180.0;  /* angle is in radians */
  delta13 = 5.0*PI/180.0;  /* angle is in radians */

  /* Determine the conditions in region 2. */
  Ma2 = Determine_Ma2_Given_Ma1_And_delta(k, Ma1, delta12);
  epsilon12 = Determine_weakepsilon_Given_Ma1_And_delta(k, Ma1, delta12);
  p2p1Ratio = Determine_p2p1Ratio_Given_Ma1_And_delta(k, Ma1, delta12);
  p2 = p2p1Ratio*p1;

  /* Determine the conditions in region 3. */
  Ma3 = Determine_Ma2_Given_Ma1_And_delta(k, Ma1, delta13);
  epsilon13 = Determine_weakepsilon_Given_Ma1_And_delta(k, Ma1, delta13);
  p3p1Ratio = Determine_p2p1Ratio_Given_Ma1_And_delta(k, Ma1, delta13);
  p3 = p3p1Ratio*p1;

  /* Use a brute force approach to determining the correct theta.
     Simply march through different values of the theta until the one
     that gives p4=p5 is found.  A more efficient approach would use a
     bisection or secant method for finding the correct value of
     theta.*/
  theta = -5.0*PI/180.0;  /* Choose a starting point for theta. */
  do {
    theta += 1.0e-4*PI/180.0;

    /* Determine the conditions in region 4. */
    delta24 = delta12-theta;
    Ma4 = Determine_Ma2_Given_Ma1_And_delta(k, Ma2, delta24);
```
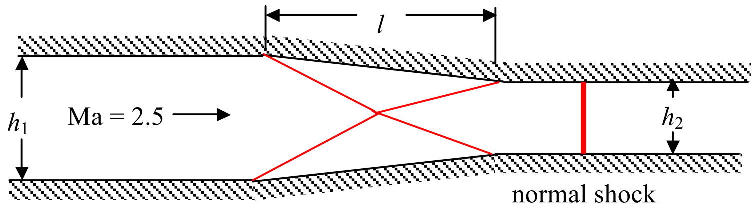

```
    epsilon24 = Determine_weakepsilon_Given_Ma1_And_delta(k, Ma2, delta24);
    p4p2Ratio = Determine_p2p1Ratio_Given_Ma1_And_delta(k, Ma2, delta24);
    p4p1Ratio = p4p2Ratio*p2p1Ratio;
    p4 = p4p1Ratio*p1;

    /* Determine the conditions in region 5. */
    delta35 = delta13+theta;
    Ma5 = Determine_Ma2_Given_Ma1_And_delta(k, Ma3, delta35);
    epsilon35 = Determine_weakepsilon_Given_Ma1_And_delta(k, Ma3, delta35);
    p5p3Ratio = Determine_p2p1Ratio_Given_Ma1_And_delta(k, Ma3, delta35);
    p5p1Ratio = p5p3Ratio*p3p1Ratio;
    p5 = p5p1Ratio*p1;

  } while (fabs((p4-p5)/p4) > tol);

  printf("theta = %.2f deg\n", theta*180.0/PI);
  printf("Ma1 = %.2f\n", Ma1);
  printf("Ma2 = %.2f\n", Ma2);
  printf("Ma3 = %.2f\n", Ma3);
  printf("Ma4 = %.2f\n", Ma4);
  printf("Ma5 = %.2f\n", Ma5);
  printf("delta12 = %.2f deg\n", delta12*180.0/PI);
  printf("delta13 = %.2f deg\n", delta13*180.0/PI);
  printf("delta24 = %.2f deg\n", delta24*180.0/PI);
  printf("delta35 = %.2f deg\n", delta35*180.0/PI);
  printf("epsilon12 = %.2f deg\n", epsilon12*180.0/PI);
  printf("epsilon13 = %.2f deg\n", epsilon13*180.0/PI);
  printf("epsilon24 = %.2f deg\n", epsilon24*180.0/PI);
  printf("epsilon35 = %.2f deg\n", epsilon35*180.0/PI);
  printf("p2p1Ratio = %.4f\n", p2p1Ratio);
  printf("p3p1Ratio = %.4f\n", p3p1Ratio);
  printf("p4p2Ratio = %.4f\n", p4p2Ratio);
  printf("p5p3Ratio = %.4f\n", p5p3Ratio);
  printf("p1 = %.2f Pa\n", p1);
  printf("p2 = %.2f Pa\n", p2);
  printf("p3 = %.2f Pa\n", p3);
  printf("p4 = %.2f Pa\n", p4);
  printf("p5 = %.2f Pa\n", p5);

  return 0;
}
```

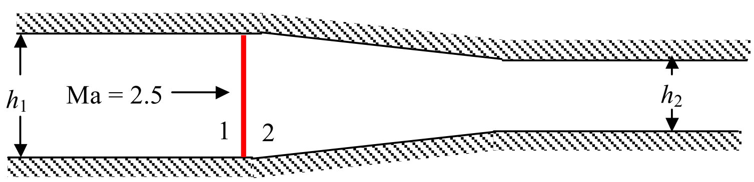
A symmetric two-dimensional supersonic diffuser, shown in the sketch, is to be designed for a Mach number of 2.5. The ratio h_2/h_1 is to be chosen so that the diffuser will barely swallow the initial shock, and the ratio l/h_1 is to be selected so as to obtain the wave pattern shown.

a. Determine h_2/h_1 and l/h_1.
b. Compare the overall stagnation pressure ratio of this diffuser with the stagnation pressure ratio of a diffuser in which a normal shock occurs at a Mach number of 2.5.



SOLUTION:

The first part of this problem is identical to the problem of starting a supersonic wind tunnel. In order to swallow the shock, the downstream area must be at least as large as the sonic area following the shock wave.



Using the normal shock relations:

$$\mathrm{Ma}_1 = 2.5 \Rightarrow A_2^*/A_1^* = 2.00394 \quad (\text{Note that } h_2 = A_2^*.) \tag{1}$$

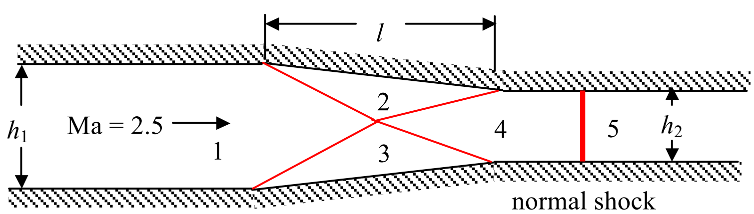
where, upstream of the shock:

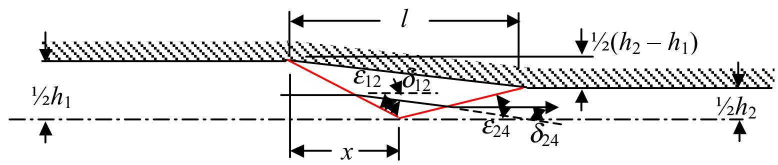
$$\frac{A_1}{A_1^*} = \frac{1}{\mathrm{Ma}_1}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow A_1/A_1^* = 2.6367 \quad \text{using Ma}_1 = 2.5 \quad (\text{Note that } h_1 = A_1.) \tag{2}$$

Hence:

$$\frac{h_2}{h_1} = \left(\frac{A_2^*}{A_1^*}\right)\left(\frac{A_1^*}{A_1}\right) \Rightarrow \boxed{h_2/h_1 = 0.7600} \tag{3}$$

For the second part of the problem, use the geometry of the oblique shock waves to determine the ratio h_1/l. Note that since the diffuser is symmetric, we need not be concerned with sliplines forming downstream of regions 2 and 3.





From geometry:

$$\tan\delta_{12} = \frac{\frac{1}{2}\left(h_1 - h_2\right)}{l} = \frac{1}{2}\frac{h_1}{l}\left(1 - \frac{h_2}{h_1}\right) \tag{4}$$

$$\tan\varepsilon_{12} = \frac{\frac{1}{2}h_1}{x} \Rightarrow x = \frac{\frac{1}{2}h_1}{\tan\varepsilon_{12}} \tag{5}$$

$$\tan\left(\varepsilon_{24} - \delta_{24}\right) = \frac{\frac{1}{2}h_2}{l - x} \tag{6}$$

Combine Eqs. (5) and (6) and make use of the fact that $\delta_{24} = \delta_{12}$ to get:

$$\tan\left(\varepsilon_{24} - \delta_{12}\right) = \frac{\frac{1}{2}h_2}{l - \frac{\frac{1}{2}h_1}{\tan\varepsilon_{12}}} = \frac{\left(\frac{h_2}{h_1}\right)}{2\left(\frac{l}{h_1}\right) - \frac{1}{\tan\varepsilon_{12}}} \tag{7}$$

Use the following iterative approach to determine the value of l/h_1 corresponding to the given geometry.

1. Choose a value for l/h_1.
2. Calculate δ_{12} using Eq. (4). Note that h_2/h_1 was determined in the previous part of the problem.
3. Calculate ε_{12} and Ma_2 using the oblique shock relations with $Ma_1 = 2.5$ and δ_{12}.
4. Calculate ε_{24} using Eq. (7).
5. Calculate ε_{24}' and Ma_4 using the oblique shock relations with Ma_2 (found in step 3) and $\delta_{24} = \delta_{12}$.
6. Check to see if $\varepsilon_{24} = \varepsilon_{24}$'. If so, then the iterations are complete. If not, then choose a new value for l/h_1 and go to step 2.

After iterating:

h_1/l = 0.54 or $\boxed{l/h_1 = 1.85}$
δ_{12} = 3.72°
ε_{12} = 26.38°
Ma_2 = 2.34
ε_{24} = 28.11°
Ma_4 = 2.20

The stagnation pressure ratio for this device is given by:

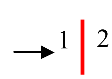
$$\frac{p_{05}}{p_{01}} = \left(\frac{p_{05}}{p_{04}}\right)\left(\frac{p_{04}}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right) \Rightarrow \boxed{p_{05}/p_{01} = 0.6280} \tag{8}$$

where

p_{05}/p_{04} = 0.6297 using the normal shock relations with $Ma_4 = 2.20$ (9)
p_{04}/p_{02} = 0.9988 using the oblique shock relations with $Ma_2 = 2.34$ and $\delta_{24} = 3.72°$ (10)
p_{02}/p_{01} = 0.9986 using the oblique shock relations with $Ma_1 = 2.50$ and $\delta_{12} = 3.72°$ (11)

The stagnation pressure ratio across a single normal shock with $Ma_1 = 2.5$ is:

$\boxed{p_{02}/p_{01} = 0.4990}$ using the normal shock relations with $Ma_1 = 2.5$ (12)



Therefore, the stagnation pressure loss is less for the oblique shock case.

```
/*
  soln_obliqueshock_18.c
  Copyright (C) Carl R. Wassgren, Jr.

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  WWW:  http://widget.ecn.purdue.edu/~wassgren
*/

# include <stdio.h>
# include <stdlib.h>
# include <math.h>
# include "ObliqueShockRelations.h"
# define PI (4.0*atan(1.0))

/* ***** */

int main(int argc, char **argv) {
  int flag = 0;
  double k, Ma1, Ma2, Ma4,
    delta12, epsilon12, delta24, epsilon24, epsilon24p,
    h1lRatio, h2h1Ratio, tol=1.0e-3,
    p02p01Ratio, p04p02Ratio, p05p04Ratio;

  /* Initialize the variables. */
  k = 1.4;  /* air */
  Ma1 = 2.5;
  h2h1Ratio = 0.7600;

  h1lRatio = 0.0;   /* Assume an initial value for h1/l. */
  do {
    h1lRatio += 0.0001;

    delta12 = atan(0.5*h1lRatio*(1.0-h2h1Ratio));
    Ma2 = Determine_Ma2_Given_Ma1_And_delta(k, Ma1, delta12);
    epsilon12 = Determine_weakepsilon_Given_Ma1_And_delta(k, Ma1, delta12);

    epsilon24 = delta12+atan(h2h1Ratio/(2.0/h1lRatio - 1.0/tan(epsilon12)));

    delta24 = delta12;
    Ma4 = Determine_Ma2_Given_Ma1_And_delta(k, Ma2, delta24);
    epsilon24p = Determine_weakepsilon_Given_Ma1_And_delta(k, Ma2, delta24);

  } while (fabs((epsilon24-epsilon24p)/epsilon24) > tol);

  p02p01Ratio = Determine_p02p01Ratio_Given_Ma1_And_delta(k,Ma1,delta12);
  p04p02Ratio = Determine_p02p01Ratio_Given_Ma1_And_delta(k,Ma2,delta24);
  p05p04Ratio = Determine_p02p01Ratio_Given_Ma1(k, Ma4);

  printf("h1lRatio = %.2f\n", h1lRatio);
  printf("(Ma1, Ma2, Ma4) = (%.2f, %.2f, %.2f)\n", Ma1, Ma2, Ma4);
  printf("(delta12, delta24) = (%.2f, %.2f) deg\n",
        delta12*180.0/PI, delta24*180.0/PI);
  printf("(epsilon12, epsilon24) = (%.2f, %.2f) deg\n",
        epsilon12*180.0/PI, epsilon24*180.0/PI);
  printf("p02p01Ratio = %.4f\n", p02p01Ratio);
  printf("p04p02Ratio = %.4f\n", p04p02Ratio);
  printf("p05p04Ratio = %.4f\n", p05p04Ratio);
  printf("p05p01Ratio = %.4f\n", p05p04Ratio*p04p02Ratio*p02p01Ratio);

  printf("Normal Shock Case with Ma1 = %.1f:", Ma1);
  printf("p02p01Ratio = %.4f\n", Determine_p02p01Ratio_Given_Ma1(k, Ma1));

  return 0;
}
```
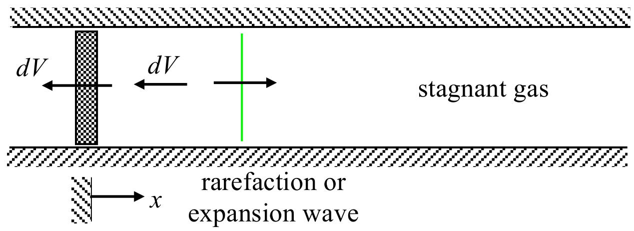
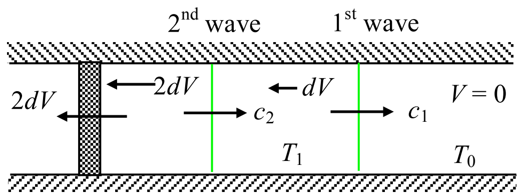

14. Expansion Waves

Recall the piston experiment in our previous discussion regarding the formation of shock waves. Now let's consider what happens when we move the piston toward the left (as shown in the figure below) so that an expansion (or rarefaction) wave propagates down the length of the tube.

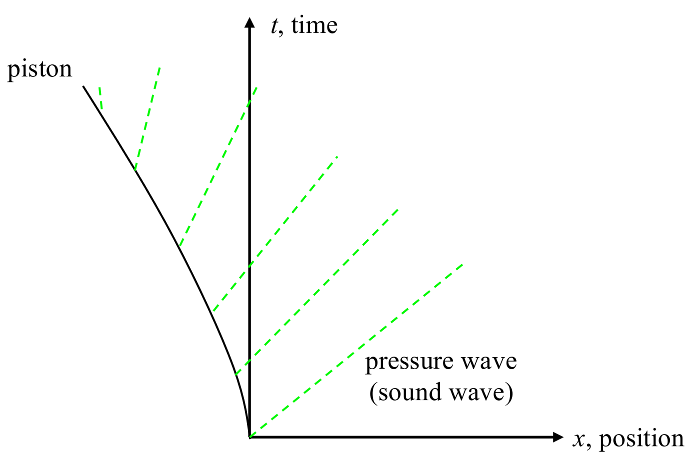


When we first move the piston, an infinitesimal strength pressure wave travels down the cylinder at the sonic speed. Behind the wave the pressure, temperature, and density decrease (refer to a previous set of notes concerning property changes across a sound wave). In addition, the flow velocity behind the wave will move in the direction of the piston (away from the wave).

If we continue to increase the piston velocity, additional pressure waves will propagate down the cylinder. However, these waves travel at a slightly lower speed relative to a fixed observer due to the decreased fluid temperature and leftward fluid velocity behind each wave. Hence, the waves start to spread out. This is the opposite of what occurred for compression waves.



$c_2 < c_1$ since $T_1 < T_0$ and dV moving to left

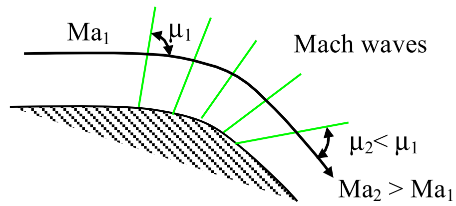


Notes:

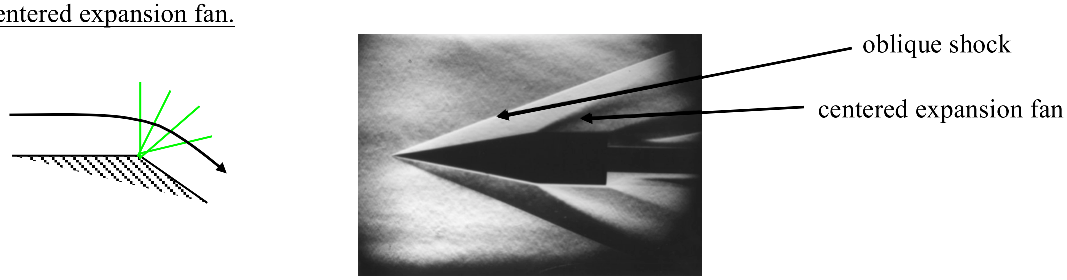
1. Since the waves do not coalesce, the change in the properties across each wave are infinitesimal. Hence, the flow through each wave is considered isentropic.
2. It is impossible to form an expansion shock wave since each subsequent wave travels slower than the previous wave. The waves will never coalesce. This can also be proven mathematically by showing that entropy would decrease across an "expansion shock."

Prandtl-Meyer Expansion Fans

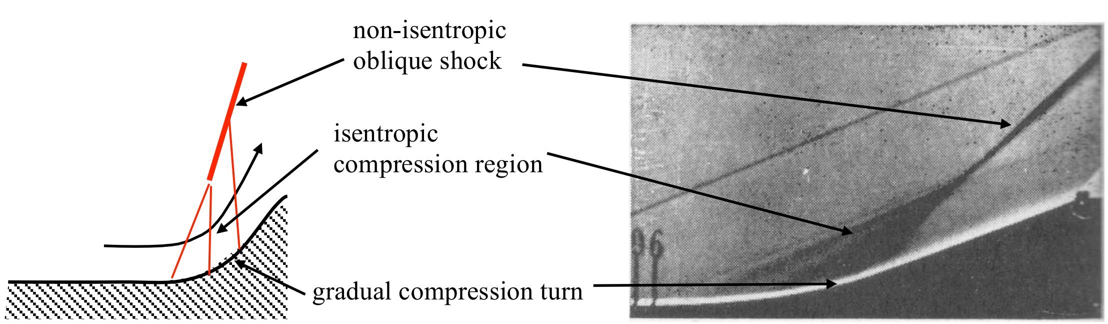
Now let's consider expanding a steady supersonic flow around a gradual, outward-turning corner as shown in the figure below.



The gradual curve can be approximated by a series of very small, discrete turns, each of which results in a small expansion Mach wave as shown in the previous figure. Recall that Mach waves exist only for supersonic flows, are infinitesimally weak pressure waves, and are inclined at an angle (known as the Mach angle), $\mu = \sin^{-1}(1/Ma)$, with respect to the flow. Note that across an expansion wave the Mach number will increase (refer to a previous set of notes concerning sound waves) so that successive expansion waves will have smaller Mach angles ($Ma_2 > Ma_1 \Rightarrow \mu_2 < \mu_1$ in the figure above). As a result, the waves diverge, remain infinitesimally weak, and thus the flow across the waves is isentropic. This type of expansion is sometimes referred to as a <u>non-centered expansion fan</u>. For a sharp corner, the waves comprising the expansion fan start at the corner point then diverge outward as shown in the figure below. This type of fan is known as a <u>centered expansion fan.</u>

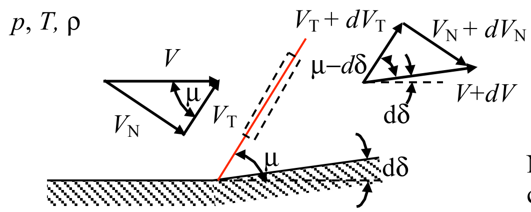


Notes:
1. The flow into and out of each Mach wave is uniform (i.e., 1D).
2. It is also possible to have isentropic compression waves resulting from gradual (infinitesimal), non-centered turns as shown in the figure below. Non-centered compression fans, however, will eventually converge to form oblique shock waves, which are non-isentropic. A centered compression fan resulting from a finite angle corner is an oblique shock wave and thus is non-isentropic.



3. A compression wave turns the flow toward the wave while an expansion wave turns the flow away from the wave.

Now let's analyze the steady, 2D flow through a single Mach wave as shown in the figure below. Our analysis will be for a compression Mach wave ($d\delta > 0$ in the figure below) but the analysis will also hold for an expansion Mach wave ($d\delta < 0$). Note that the upstream Mach number must be supersonic in order for Mach waves to exist.



Note: $d\delta$ is defined as being positive for counter-clockwise (i.e., compression) turns. For an expansion, $d\delta < 0$.

From geometry:

$$V_T = V\cos\mu \tag{231}$$

$$V_T + dV_T = (V + dV)\cos(\mu - d\delta) \tag{232}$$

From the LME in the tangential direction:

$$\dot{m}(V_T + dV_T) - \dot{m}V_T = 0 \quad \Rightarrow \quad dV_T = 0 \tag{233}$$

Combining Eqs. (231)-(233):

$$V_T = V\cos\mu = (V + dV)\cos(\mu - d\delta)$$

$$V\cos\mu = (V + dV)(\cos\mu\cos d\delta + \sin\mu\sin d\delta)$$

Since the turning angle, $d\delta$, is very small we can write:

$$V\cos\mu = (V + dV)(\cos\mu + d\delta\sin\mu)$$

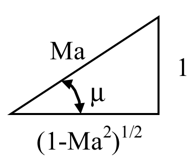
$$= V\cos\mu + Vd\delta\sin\mu + dV\cos\mu + dVd\delta\sin\mu$$

Neglecting H.O.T.s and simplifying:

$$\frac{dV}{V} = -d\delta\tan\mu \tag{234}$$

Since the expansion wave is a Mach wave (with wave angle, $\sin\mu = 1/\mathrm{Ma}$), we have:

$$\tan\mu = \frac{1}{\sqrt{\mathrm{Ma}^2 - 1}}$$



Substituting and simplifying:

$$\boxed{\frac{dV}{V} = \frac{-d\delta}{\sqrt{\mathrm{Ma}^2 - 1}}} \tag{235}$$

(Note: Since Ma > 1, $d\delta > 0 \Rightarrow dV < 0$ and $d\delta < 0 \Rightarrow dV > 0$.)

For a perfect gas:

$$V^2 = (\gamma RT)\text{Ma}^2$$
$$= \gamma RT_0 \text{Ma}^2 \left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{-1}$$

$$\frac{dV}{V} = \left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{-1}\frac{d(\text{Ma})}{\text{Ma}} \tag{236}$$

Substituting (236) into Eq. (235) gives:

$$d\delta = \frac{-\sqrt{\text{Ma}^2-1}}{1+\frac{\gamma-1}{2}\text{Ma}^2}\frac{d(\text{Ma})}{\text{Ma}} \tag{237}$$

Note: Since Ma > 1, $d\delta > 0 \Rightarrow d(\text{Ma}) < 0$ and $d\delta < 0 \Rightarrow d(\text{Ma}) > 0$.

Integrating Eq. (237) gives:

$$\delta = -\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\sqrt{\frac{\gamma-1}{\gamma+1}}\left(\text{Ma}^2-1\right)\right]+\tan^{-1}\sqrt{\text{Ma}^2-1}+\text{constant} \tag{238}$$

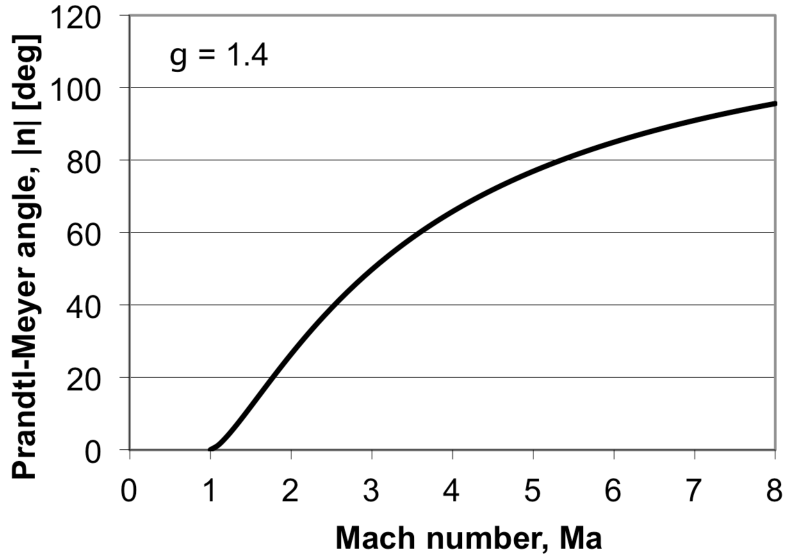
The constant of integration can be determined if the initial Mach number, Ma_1, and flow deflection angle, δ_1, are known. For convenience, we define a reference state where, Ma = 1, and $\delta = 0$. For these conditions, Eq. (238) becomes:

$$\boxed{\nu = -\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\sqrt{\frac{\gamma-1}{\gamma+1}}\left(\text{Ma}^2-1\right)\right]+\tan^{-1}\sqrt{\text{Ma}^2-1}} \qquad \textbf{Prandtl-Meyer Angle} \tag{239}$$

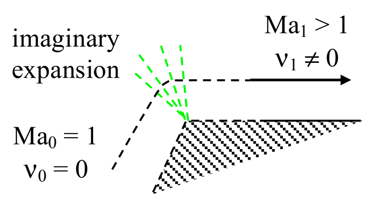
Note: The symbol "δ" has been changed to a "ν" in order to signify that this is the Prandtl-Meyer angle. The Prandtl-Meyer angle is the angle, ν, that the flow needs to be turned to go from sonic conditions to get to the new Mach number, Ma.

Notes:
1. The angle, ν, is positive for counter-clockwise (compressive) rotations and negative for clockwise (expansive) rotations. The convention, however, is to drop the negative sign when reporting Prandtl-Meyer angles.

2. The Prandtl-Meyer angle is plotted as a function of Mach number for $\gamma = 1.4$ in the figure below.

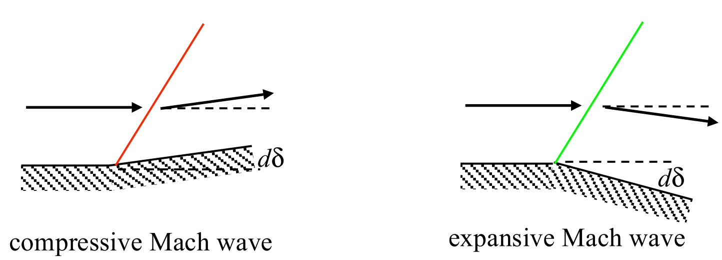


3. For an arbitrary incoming Mach number ($Ma_1 > 1$, $\nu_1 = \text{fcn}(Ma_1)$) we can imagine that there is some imaginary upstream corner that expands the flow from $Ma_0 = 1$ ($\nu_0 = 0$) to the current Mach number, Ma_1 ($\nu_1 = \text{fcn}(Ma_1)$), as shown in the figure below.



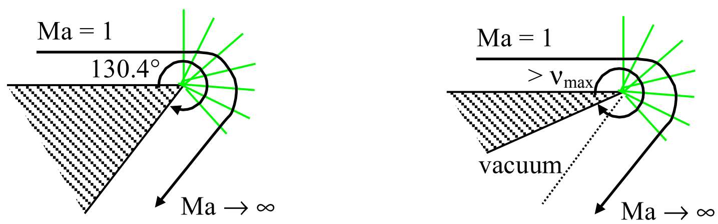
To expand a flow from Ma_1 to Ma_2 ($Ma_2 > Ma_1$), we need to turn the flow by an angle of ν_2-ν_1. For example, to get from sonic conditions ($Ma_0 = 1$) to $Ma_1 = 2.0$, we need to turn the flow by an angle of ν_1-ν_0 = -26.38° - 0 = -26.38° and to get the flow from sonic conditions to $Ma_2 = 3.0$ we need to turn the flow by an angle of ν_2-ν_0 = -49.76° - 0 = -49.76°. To go from $Ma_1 = 2.0$ to $Ma_2 = 3.0$, we need to turn the flow by an angle of ν_2-ν_1 = -49.76° - (-26.38°) = -23.38°. Note that the negative signs indicate that we need to turn the flow away from the Mach wave, i.e., expand the flow.

4. Compressive Mach waves turn the flow toward the Mach wave while expansive Mach waves turn the flow away from the Mach wave.

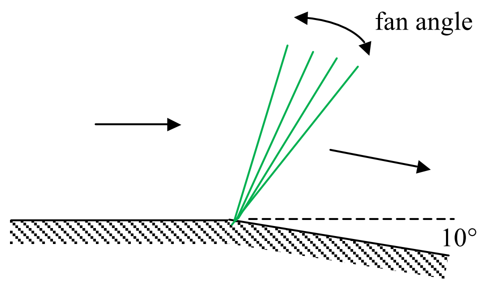


5. Since the expansion process is isentropic, the isentropic flow relations can be used throughout the expansion fan. The isentropic relations may also be used in a limited region within a non-centered compression fan where the Mach waves do not intersect. An oblique shock forms where the compression Mach waves merge.

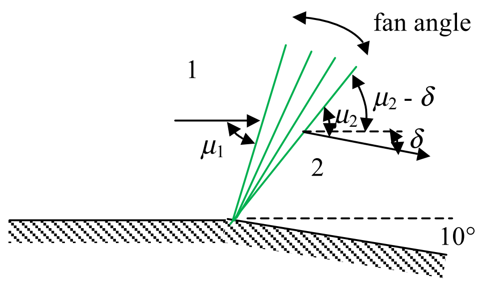
6. To expand the flow of air (γ= 1.4) from sonic conditions (Ma = 1) to an infinite Mach number, i.e., Ma $\rightarrow \infty$, we must turn the flow by ν_{max} = -130.4°.
$$\nu_{\max} = \lim_{\text{Ma}\to\infty} \nu = \frac{\pi}{2}\left(-\sqrt{\frac{\gamma+1}{\gamma-1}}+1\right) \tag{240}$$
If the corner has an angle greater than this maximum angle, then a vacuum region forms. Note that the continuum assumption would break down in the region adjacent to the vacuum.



A uniform supersonic flow at Mach 2.0, static pressure 10 psia, and temperature 400 °R expands around a 10° corner. Determine the downstream Mach number, pressure, temperature, and the fan angle.



SOLUTION:



For $Ma_1 = 2.0 \Rightarrow \nu_1 = 26.38°$ (using the Prandtl-Meyer expansion fan relation) (1)

$$\nu_2 = \nu_1 + 10^\circ = 26.38^\circ + 10^\circ = 36.38^\circ$$

$\Rightarrow$ $\boxed{Ma_2 = 2.38}$ (using the Prandtl-Meyer expansion fan relation) (2)

Since the process is isentropic:

$$p_2 = \frac{\left(p_2/p_0\right)}{\left(p_1/p_0\right)} p_1 = p_1 \left(\frac{1 + \frac{\gamma-1}{2} Ma_2^2}{1 + \frac{\gamma-1}{2} Ma_1^2} \right)^{\frac{\gamma}{1-\gamma}} \tag{3}$$

$\Rightarrow$ $\boxed{p_2 = 5.53 \text{ psia}}$ using $p_1 = 10$ psia, $Ma_1 = 2.0$, $Ma_2 = 2.38$, $\gamma = 1.4$ (4)

$$T_2 = \frac{\left(T_2/T_0\right)}{\left(T_1/T_0\right)} T_1 = T_1 \left(\frac{1 + \frac{\gamma-1}{2} Ma_2^2}{1 + \frac{\gamma-1}{2} Ma_1^2} \right)^{-1} \tag{5}$$

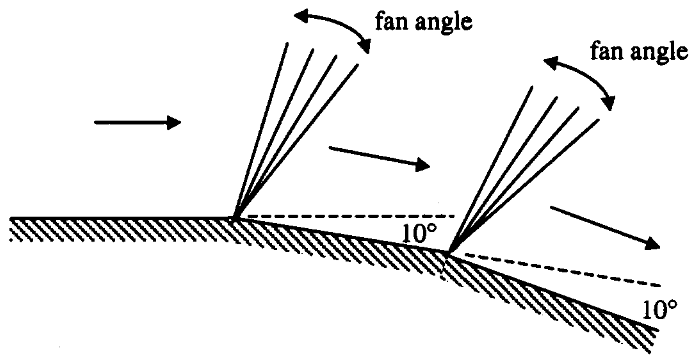
$\Rightarrow$ $\boxed{T_2 = 337 \text{ °R}}$ using $T_1 = 400$ °R, $Ma_1 = 2.0$, $Ma_2 = 2.38$, $\gamma = 1.4$ (6)

Since each fan is a Mach wave, the fan angle may be found using:

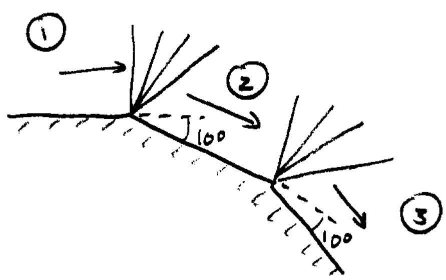
$$\mu_1 = \sin^{-1}\left(\frac{1}{Ma_1}\right) = 30^\circ \text{ and } \mu_2 = \sin^{-1}\left(\frac{1}{Ma_2}\right) = 24.85^\circ \tag{7}$$

$$\boxed{\text{fan angle} = \mu_1 - (\mu_2 - \delta) = 15.15^\circ} \tag{8}$$

A uniform supersonic flow at Mach 2.0, static pressure 10 psia, and temperature 400 °R expands around two 10° corners as shown in the figure below. Determine the downstream Mach number, pressure, temperature, and the fan angle for the second fan.



SOLUTION:



- For $Ma_1 = 2.0 \Rightarrow \underline{\nu_1 = 26.38^\circ}$

$\underline{\nu_2} = \nu_1 + 10^\circ = \underline{36.38^\circ}$

$\Rightarrow \underline{Ma_2 = 2.38}$

$\underline{\nu_3} = \nu_2 + 10^\circ = \underline{46.38^\circ}$

$\Rightarrow \boxed{Ma_3 = 2.83}$

using Prandtl-Meyer expansion fan relation

NOTE: Since the angle of the last Mach wave in the first fan is equal to the angle of the first wave in the second fan, we could have treated the flow through both fans as flow through one single fan with a larger turning angle.

$\nu_3 = \nu_1 + 10^\circ + 10^\circ = 46.38^\circ \Rightarrow Ma_3 = 2.83$

SOLUTION...

- Since the flow through the fans is isentropic:

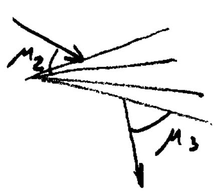
$$p_3 = \frac{(p_3/p_0)}{(p_1/p_0)} p_1 = p_1 \left(\frac{1+\frac{\gamma-1}{2} Ma_3^2}{1+\frac{\gamma-1}{2} Ma_1^2} \right)^{\frac{\gamma}{1-\gamma}}$$

$\therefore p_3 =$ using $Ma_1 = 2.0$, $Ma_3 = 2.83$, $p_1 = 10\text{psia}$, $\gamma = 1.4$

$$T_3 = \frac{(T_3/T_0)}{(T_1/T_0)} T_1 = T_1 \left(\frac{1+\frac{\gamma-1}{2} Ma_3^2}{1+\frac{\gamma-1}{2} Ma_1^2} \right)^{-1}$$

$\therefore T_3 =$ using $Ma_1 = 2.0$, $Ma_2 = 2.83$, $T_1 = 400°R$, $\gamma = 1.4$

- Fan angle (2nd fan):



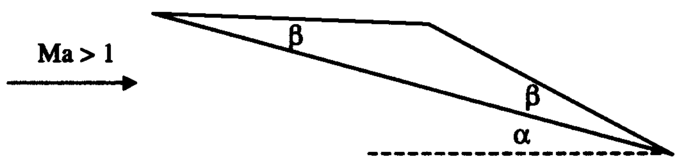
Mach waves: $\mu_2 = \sin^{-1}\left(\frac{1}{Ma_2}\right)$

$\mu_3 = \sin^{-1}\left(\frac{1}{Ma_3}\right)$

fan angle $= \mu_2 - (\mu_3 + \delta)$ using $Ma_2 = 2.38$, $Ma_3 = 2.83$, $\mu_2 = 24.85°$, $\mu_3 = 20.69°$, $\delta = 10°$

$$\boxed{\therefore \text{fan angle} = 14.16°}$$

Find the lift and drag coefficients for supersonic flow past a thin airfoil of the following shape:

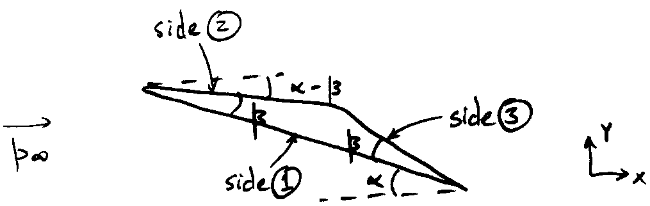


Assume that the angles α and β are very small. Your answers should be given in terms of the incoming Mach number, Ma, the angle of attack, α, and the angle, β. Comment briefly on the effect of the airfoil thickness (as given by β) on the performance of the airfoil. For a fixed β, at what angle of attack will the lift/drag ratio be a maximum?

SOLUTION:

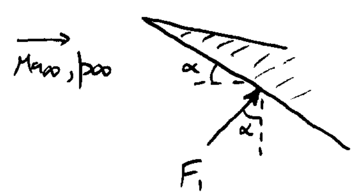
- For very small turning angles:

$$\frac{dp}{p} = \frac{\gamma Ma^2 (d\delta)}{\sqrt{Ma^2 - 1}}$$



- On side ①:

$$\frac{p_1 - p_\infty}{p_\infty} = \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}}(\alpha)$$



$$\Rightarrow F_1 = p_1 A_1 = \left[p_\infty + \frac{\gamma Ma_\infty^2 \alpha p_\infty}{\sqrt{Ma_\infty^2 - 1}}\right] A_1$$

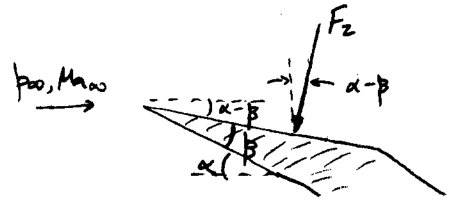
$$F_{1x} = F_1 \sin\alpha \approx F_1 \alpha \qquad \left(\begin{array}{l}\text{since } \alpha \text{ is a very} \\ \text{small angle}\end{array}\right)$$

$$F_{1y} = F_1 \cos\alpha \approx F_1$$

Solution...

- on side 2:

$$\frac{p_2 - p_\infty}{p_\infty} = \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}}\left[-(\alpha-\beta)\right]$$



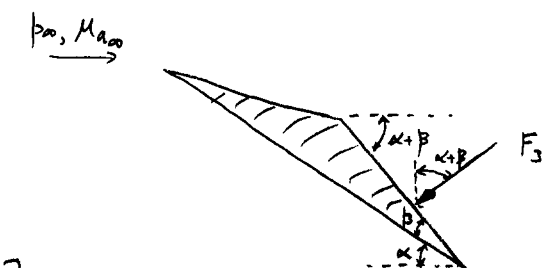
$$\Rightarrow F_2 = p_2 A_2 = \left[p_\infty - \frac{\gamma Ma_\infty^2 (\alpha-\beta) p_\infty}{\sqrt{Ma_\infty^2 - 1}}\right] A_2$$

$$F_{2x} = -F_2 \sin(\alpha-\beta) \approx -F_2(\alpha-\beta)$$
$$F_{2y} = -F_2 \cos(\alpha-\beta) \approx -F_2$$

(since $(\alpha-\beta)$ is a very small angle)

- on side 3:

$$\frac{p_3 - p_\infty}{p_\infty} = \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}}\left[-(\alpha+\beta)\right]$$

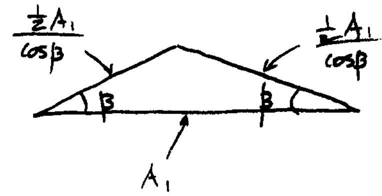


$$\Rightarrow F_3 = p_3 A_3 = \left[p_\infty - \frac{\gamma Ma_\infty^2 p_\infty}{\sqrt{Ma_\infty^2 - 1}}(\alpha+\beta)\right] A_3$$

$$F_{3x} = -F_3 \sin(\alpha+\beta) \approx -F_3(\alpha+\beta)$$
$$F_{3y} = -F_3 \cos(\alpha+\beta) \approx -F_3$$

(since $(\alpha+\beta)$ is a very small angle)

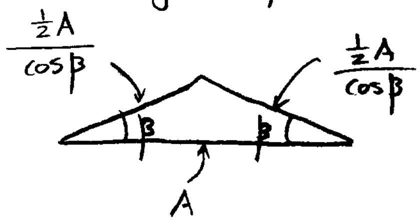
- From the geometry



Note: $\cos\beta \approx 1$ since β is very small

SOLUTION...

- From the geometry:



NOTE: $\cos\beta \approx 1$ since β is very small

$$\Rightarrow \quad F_{1x} = p_\infty\left(1+\frac{\gamma Ma_\infty^2 \alpha}{\sqrt{Ma_\infty^2-1}}\right)A\alpha$$

$$F_{1y} = p_\infty\left(1+\frac{\gamma Ma_\infty^2 \alpha}{\sqrt{Ma_\infty^2-1}}\right)A$$

$$F_{2x} = -p_\infty\left(1+\frac{\gamma Ma_\infty^2 (\alpha-\beta)}{\sqrt{Ma_\infty^2-1}}\right)\left(\frac{A}{2}\right)(\alpha-\beta)$$

$$F_{2y} = -p_\infty\left(1+\frac{\gamma Ma_\infty^2 (\alpha-\beta)}{\sqrt{Ma_\infty^2-1}}\right)\left(\frac{A}{2}\right)$$

$$F_{3x} = -p_2\left(1+\frac{2\gamma Ma_2^2 \beta}{\sqrt{Ma_2^2-1}}\right)\left(\frac{A}{2}\right)(2\beta)$$

$$F_{3y} = -p_2\left(1+\frac{2\gamma Ma_2^2 \beta}{\sqrt{Ma_2^2-1}}\right)\left(\frac{A}{2}\right)$$

where
$$p_2 = p_\infty\left(1+\frac{\gamma Ma_\infty^2 (\alpha-\beta)}{\sqrt{Ma_\infty^2-1}}\right)$$

$$Ma_2 = Ma_\infty\left(1+\frac{(1+\gamma Ma_\infty^2)(\beta-\alpha)}{2\sqrt{Ma_\infty^2-1}}\right)$$

$$\text{Drag} = F_{1x}+F_{2x}+F_{3x}$$

$$\text{Lift} = F_{1y}+F_{2y}+F_{3y}$$

Solution...

$$\Rightarrow \quad F_{1x} = p_\infty A_1 \left[1 + \frac{\gamma Ma_\infty^2 \alpha}{\sqrt{Ma_\infty^2 - 1}} \right] \alpha$$

$$F_{1y} = p_\infty A_1 \left[1 + \frac{\gamma Ma_\infty^2 \alpha}{\sqrt{Ma_\infty^2 - 1}} \right]$$

$$F_{2x} = -p_\infty A_1 \frac{1}{2} \left[1 - \frac{\gamma Ma_\infty^2 (\alpha - \beta)}{\sqrt{Ma_\infty^2 - 1}} \right] (\alpha - \beta)$$

$$F_{2y} = -p_\infty A_1 \frac{1}{2} \left[1 - \frac{\gamma Ma_\infty^2 (\alpha - \beta)}{\sqrt{Ma_\infty^2 - 1}} \right]$$

$$F_{3x} = -p_\infty A_1 \frac{1}{2} \left[1 - \frac{\gamma Ma_\infty^2 (\alpha + \beta)}{\sqrt{Ma_\infty^2 - 1}} \right] (\alpha + \beta)$$

$$F_{3y} = -p_\infty A_1 \frac{1}{2} \left[1 - \frac{\gamma Ma_\infty^2 (\alpha + \beta)}{\sqrt{Ma_\infty^2 - 1}} \right]$$

$$\text{Drag} = D = F_{1x} + F_{2x} + F_{3x}$$

$$= p_\infty A_1 \left\{ \alpha - \frac{1}{2}(\alpha - \beta) - \frac{1}{2}(\alpha + \beta) + \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}} \underbrace{\left[\alpha^2 + \frac{1}{2}(\alpha - \beta)^2 + \frac{1}{2}(\alpha + \beta)^2 \right]}_{= \alpha^2 + \frac{1}{2}\alpha^2 - \alpha\beta + \frac{1}{2}\beta^2 + \frac{1}{2}\alpha^2 + \alpha\beta + \frac{1}{2}\beta^2 \; = 2\alpha^2 + \beta^2} \right\}$$

$$= p_\infty A_1 \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}} (2\alpha^2 + \beta^2)$$

$$C_D \equiv \frac{\text{Drag}}{\frac{1}{2}\rho_\infty V_\infty^2 A_1} \qquad \text{but} \quad \rho_\infty V_\infty^2 \underbrace{=}_{\text{ideal gas law}} \left(\frac{p_\infty}{R T_\infty} \right) V_\infty^2 = \frac{\gamma p_\infty V_\infty^2}{\gamma R T_\infty} = \gamma p_\infty Ma_\infty^2$$

NOTE: lift & drag coefficients typically use the planform area instead of frontal projected area

$$\Rightarrow C_D = \frac{p_\infty A_1 \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}} (2\alpha^2 + \beta^2)}{\frac{1}{2} \gamma p_\infty Ma_\infty^2 A_1} \Rightarrow \therefore \boxed{C_D = \frac{4\alpha^2 + 2\beta^2}{\sqrt{Ma_\infty^2 - 1}}}$$

Solution...

$$\text{Lift} = F_{1y} + F_{2y} + F_{3y}$$

$$= p_\infty A_1 \left\{ 1 - \tfrac{1}{2} - \tfrac{1}{2} + \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}} \underbrace{\left[\alpha + \tfrac{1}{2}(\alpha - \beta) + \tfrac{1}{2}(\alpha + \beta) \right]}_{= 2\alpha} \right\}$$

$$= p_\infty A_1 \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}} (2\alpha)$$

- $$C_L \equiv \frac{\text{Lift}}{\frac{1}{2}\rho_\infty V_\infty^2 A_1} = \frac{\text{Lift}}{\frac{1}{2}\gamma p_\infty Ma_\infty^2 A_1} = \frac{p_\infty A_1 \gamma Ma_\infty^2 (2\alpha)}{\frac{1}{2}\gamma p_\infty Ma_\infty^2 A_1 \sqrt{Ma_\infty^2 - 1}}$$

$$\boxed{\therefore \quad C_L = \frac{4\alpha}{\sqrt{Ma_\infty^2 - 1}}}$$

- As the airfoil thickness increases ($\beta \uparrow$), the drag will increase but the lift will remain constant.

- Lift to drag ratio, $\underline{\frac{C_L}{C_D}} = \frac{4\alpha}{4\alpha^2 + 2\beta^2} = \underline{\frac{2\alpha}{2\alpha^2 + \beta^2}}$

Max $\frac{C_L}{C_D}$ occurs when

$$\frac{d\left(\frac{C_L}{C_D}\right)}{d\alpha} = \left(\frac{(2\alpha)(-1)(4\alpha)}{(2\alpha^2 + \beta^2)^2} + \frac{2}{2\alpha^2 + \beta^2} \right) = 0$$

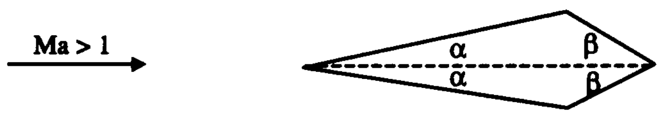
(β = constant)

$$= \frac{-8\alpha^2 + 4\alpha^2 + 2\beta^2}{(2\alpha^2 + \beta^2)^2} = 0$$

$$= \frac{2\beta^2 - 4\alpha^2}{(2\alpha^2 + \beta^2)^2} = 0$$

$$\Rightarrow \quad \alpha = \frac{\beta}{\sqrt{2}} \text{ gives } \left.\frac{C_L}{C_D}\right|_{\dots}$$

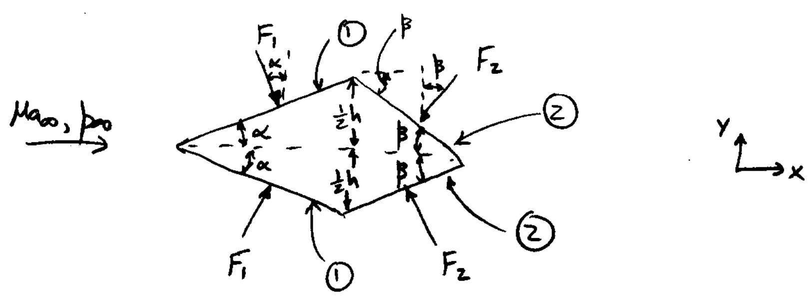
Find an expression for the drag coefficient (based on frontal projected area) of a thin diamond-shaped body in a supersonic flow in terms of Ma, α, and β:



Assume the angles of turn are very small. If α is smaller than β, how does drag differ when the body is turned around so that the end with the half angle β becomes the leading edge?

SOLUTION:

- For very small turning angles: $\dfrac{dp}{p} = \dfrac{\gamma Ma^2 (d\delta)}{\sqrt{Ma_\infty^2 - 1}}$



- On side 1:

$$\frac{p_1 - p_\infty}{p_\infty} = \frac{\gamma Ma_\infty^2 (\alpha)}{\sqrt{Ma_\infty^2 - 1}}$$

$$\rightarrow F_{1x} = 2 p_1 A_1 \underbrace{\sin\alpha}_{\approx \alpha} = 2 p_\infty A_1 \left[1 + \frac{\gamma Ma_\infty^2 \alpha}{\sqrt{Ma_\infty^2 - 1}} \right] \alpha$$

symmetry (top & bottom)

(since α is very small)

SOLUTION...

- On side 2:

$$\frac{p_2 - p_\infty}{p_\infty} = \frac{\gamma Ma_\infty^2(-\beta)}{\sqrt{Ma_\infty^2 - 1}}$$

$$\Rightarrow F_{2x} = -\underset{\substack{\uparrow \\ \text{symmetry} \\ \text{(top \& bottom)}}}{2} p_2 A_2 \underbrace{\sin\beta}_{\approx \beta} = -2p_\infty A_2\left[1 - \frac{\gamma Ma_\infty^2 \beta}{\sqrt{Ma_\infty^2 - 1}}\right]\beta$$

- From geometry:

$$A_1 \sin\alpha = \tfrac{1}{2}h \quad \Rightarrow \quad A_1 = \frac{\frac{1}{2}h}{\sin\alpha} \approx \frac{h}{2\alpha}$$

$$A_2 \sin\beta = \tfrac{1}{2}h \quad \Rightarrow \quad A_2 = \frac{\frac{1}{2}h}{\sin\beta} \approx \frac{h}{2\beta}$$

(since α, β are very small)

$$\Rightarrow \text{Drag} = F_{1x} + F_{2x}$$

$$= 2p_\infty\left(\frac{h}{2\alpha}\right)\left[1 + \frac{\gamma Ma_\infty^2 \alpha}{\sqrt{Ma_\infty^2 - 1}}\right]\alpha - 2p_\infty\left(\frac{h}{2\beta}\right)\left[1 - \frac{\gamma Ma_\infty^2 \beta}{\sqrt{Ma_\infty^2 - 1}}\right]\beta$$

$$= p_\infty h\left[1 - 1 + \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}}(\alpha + \beta)\right]$$

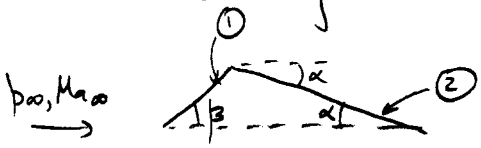
$$= p_\infty h \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}}(\alpha + \beta)$$

$$C_D \equiv \frac{\text{Drag}}{\frac{1}{2}\rho_\infty V_\infty^2 A_{\text{frontal}}} \quad \text{but} \quad \rho_\infty = \frac{p_\infty}{RT_\infty} \Rightarrow \tfrac{1}{2}\rho_\infty V_\infty^2 = \frac{\frac{1}{2}\gamma p_\infty}{\gamma R T_\infty} V_\infty^2 = \tfrac{1}{2}\gamma p_\infty Ma_\infty^2$$

$$\Rightarrow C_D = \frac{p_\infty h \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}}(\alpha + \beta)}{\frac{1}{2}\gamma p_\infty Ma_\infty^2 h} \Rightarrow \therefore \boxed{C_D = \frac{2(\alpha + \beta)}{\sqrt{Ma_\infty^2 - 1}}}$$

SOLUTION...

- If the wedge is turned around



$$\frac{p_1 - p_\infty}{p_\infty} = \frac{\gamma Ma_\infty^2 (\beta)}{\sqrt{Ma_\infty^2 - 1}} \Rightarrow F_{1x} = 2 p_1 A_1 \underbrace{\sin\beta}_{\approx \beta} = 2 p_\infty \overbrace{\left(\frac{h}{2\beta}\right)}^{A_1} \left[1 + \frac{\gamma Ma_\infty^2 \beta}{\sqrt{Ma_\infty^2 - 1}}\right]\beta$$

$$\frac{p_2 - p_\infty}{p_\infty} = \frac{\gamma Ma_\infty^2 (-\alpha)}{\sqrt{Ma_\infty^2 - 1}} \Rightarrow F_{2x} = -2 p_2 A_2 \underbrace{\sin\alpha}_{\approx \alpha} = -2 p_\infty \overbrace{\left(\frac{h}{2\alpha}\right)}^{A_2} \left[1 - \frac{\gamma Ma_\infty^2 \alpha}{\sqrt{Ma_\infty^2 - 1}}\right]\alpha$$

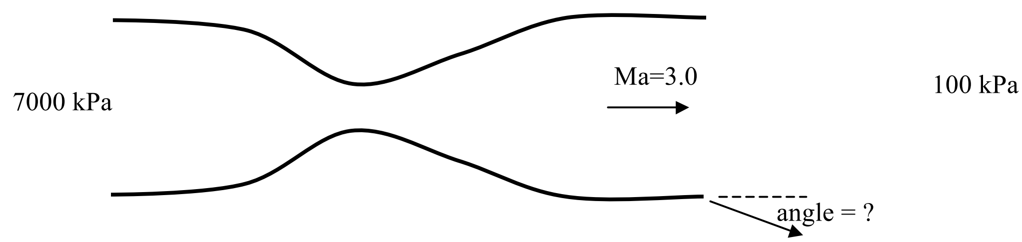
$$\Rightarrow \text{Drag} = p_\infty h \left[1 - 1 + \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}}(\beta + \alpha)\right]$$

$$= p_\infty h \frac{\gamma Ma_\infty^2}{\sqrt{Ma_\infty^2 - 1}}(\alpha + \beta)$$

This is the same result as before!

$\Rightarrow$ $\boxed{\text{$C_D$ will remain unchanged if the object were turned around}}$

A wind tunnel nozzle is designed to yield a parallel uniform flow of air with a Mach number of 3.0. The stagnation pressure of the air supply reservoir is 7000 kPa (abs), and the nozzle exhausts into the atmosphere (100 kPa). Calculate the flow angle at the exit lip of the nozzle.



SOLUTION:

Determine the Prandtl-Meyer angle for the flow leaving the nozzle exit.

$$\text{Ma}_1 = 3.0 \Rightarrow \nu_1 = 49.76° \tag{1}$$

Since the expansion fan is isentropic:

$$\frac{p_2}{p_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \text{Ma}_2 = 3.44 \quad \text{(using } p_2 = 100 \text{ kPa and } p_0 = 7000 \text{ kPa)} \tag{2}$$

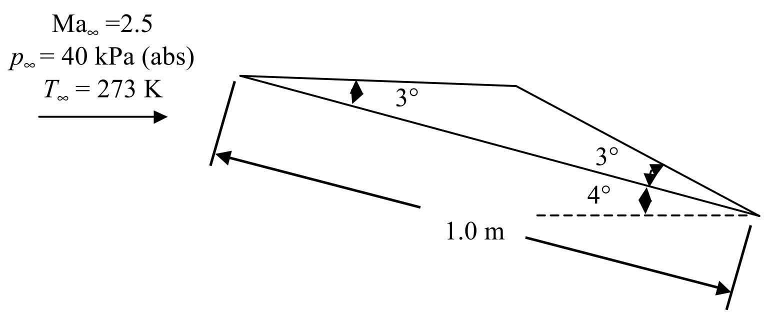
Calculate the Prandtl-Meyer angle for the flow downstream of the exit.

$$\text{Ma}_2 = 3.44 \Rightarrow \nu_2 = 57.54° \tag{3}$$

Thus, the turning angle is:

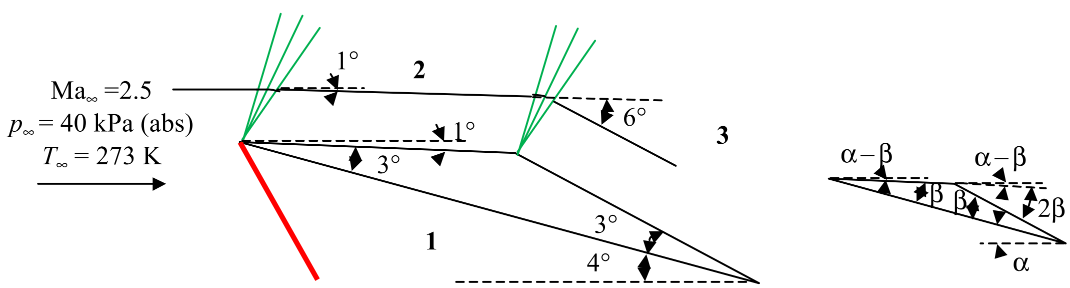
$$\delta = \nu_2 - \nu_1 \Rightarrow \boxed{\delta = 7.78°} \tag{4}$$

Find the lift and drag coefficients for the flow past the airfoil shown below. Assume the airfoil has unit length into the page. Base your drag coefficient on the chord length of the airfoil (= 1.0 m).



Do not treat the angles as being small (i.e., use the oblique shock and expansion fan relations).

SOLUTION:



Use the oblique shock relations to determine the pressure in region 1 in terms of the upstream pressure.

$\mathrm{Ma}_\infty = 2.5,\ \delta_{\infty 1} = 4° \Rightarrow \mathrm{Ma}_1 = 2.3326;\ \varepsilon_{\infty 1} = 26.6088°;\ \underline{p_1/p_\infty = 1.2961}$ (1)

Use Prandtl-Meyer angles to determine Mach numbers in regions 2 and 3.

$\mathrm{Ma}_\infty = 2.5 \Rightarrow \nu_\infty = 39.1235°$ (2)

$\nu_2 = \nu_\infty + \delta_{\infty 2} \Rightarrow \nu_2 = 40.1235°\quad (\delta_{\infty 2} = 1°)$ (3)

$\nu_2 = 40.1235° \Rightarrow \mathrm{Ma}_2 = 2.5432$ (4)

$\nu_3 = \nu_2 + \delta_{23} \Rightarrow \nu_3 = 46.1235°\quad (\delta_{23} = 6°)$ (5)

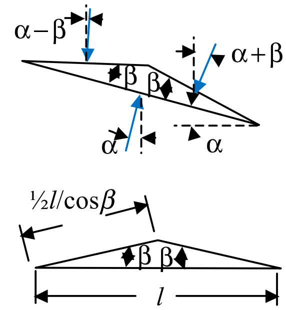
$\nu_3 = 46.1235° \Rightarrow \mathrm{Ma}_3 = 2.8182$ (6)

Use isentropic relations to determine the pressures in regions 2 and 3. Note that the stagnation pressure remains constant through the expansion fans.

$$\frac{p_2}{p_\infty} = \frac{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}}}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_\infty^2\right)^{\frac{\gamma}{1-\gamma}}} \Rightarrow \underline{p_2/p_\infty = 0.9351} \tag{7}$$

$$\frac{p_3}{p_\infty} = \frac{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_3^2\right)^{\frac{\gamma}{1-\gamma}}}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_\infty^2\right)^{\frac{\gamma}{1-\gamma}}} \Rightarrow \underline{p_3/p_\infty = 0.6124} \tag{8}$$

Now determine the lift and drag forces acting on the airfoil.



$$L = p_1 \cos\alpha(l) - p_2 \cos(\alpha-\beta)\left(\frac{\frac{1}{2}l}{\cos\beta}\right) - p_3 \cos(\alpha+\beta)\left(\frac{\frac{1}{2}l}{\cos\beta}\right) \tag{9}$$

$$D = p_1 \sin\alpha(l) - p_2 \sin(\alpha-\beta)\left(\frac{\frac{1}{2}l}{\cos\beta}\right) - p_3 \sin(\alpha+\beta)\left(\frac{\frac{1}{2}l}{\cos\beta}\right) \tag{10}$$

Form lift and drag coefficients.

$$c_L = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 l} \quad \text{and} \quad c_D = \frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 l} \tag{11}$$

where

$$\rho_\infty V_\infty^2 = \left(\frac{p_\infty}{RT_\infty}\right)\left(\mathrm{Ma}_\infty^2 \gamma R T_\infty\right) = \gamma p_\infty \mathrm{Ma}_\infty^2 \tag{12}$$

so that

$$c_L = \frac{L}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 l} \quad \text{and} \quad c_D = \frac{D}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 l} \tag{13}$$

Substitute Eqs. (9) and (10) into the lift and drag coefficients and simplify.

$$c_L = \frac{p_1 \cos\alpha(l) - p_2 \cos(\alpha-\beta)\left(\frac{\frac{1}{2}l}{\cos\beta}\right) - p_3 \cos(\alpha+\beta)\left(\frac{\frac{1}{2}l}{\cos\beta}\right)}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 l} \tag{14}$$

$$\boxed{\therefore c_L = \frac{2}{\gamma \mathrm{Ma}_\infty^2}\left[\frac{p_1}{p_\infty}\cos\alpha - \frac{1}{2}\frac{p_2}{p_\infty}\frac{\cos(\alpha-\beta)}{\cos\beta} - \frac{1}{2}\frac{p_3}{p_\infty}\frac{\cos(\alpha+\beta)}{\cos\beta}\right]} \tag{15}$$

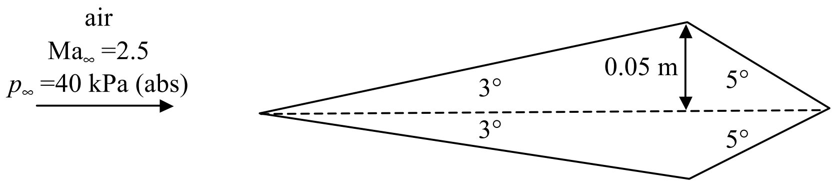
$$c_D = \frac{p_1 \sin\alpha(l) - p_2 \sin(\alpha-\beta)\left(\frac{\frac{1}{2}l}{\cos\beta}\right) - p_3 \sin(\alpha+\beta)\left(\frac{\frac{1}{2}l}{\cos\beta}\right)}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 l} \tag{16}$$

$$\boxed{\therefore c_D = \frac{2}{\gamma \mathrm{Ma}_\infty^2}\left[\frac{p_1}{p_\infty}\sin\alpha - \frac{1}{2}\frac{p_2}{p_\infty}\frac{\sin(\alpha-\beta)}{\cos\beta} - \frac{1}{2}\frac{p_3}{p_\infty}\frac{\sin(\alpha+\beta)}{\cos\beta}\right]} \tag{17}$$

Using the given data and the pressure ratio data derived previously:

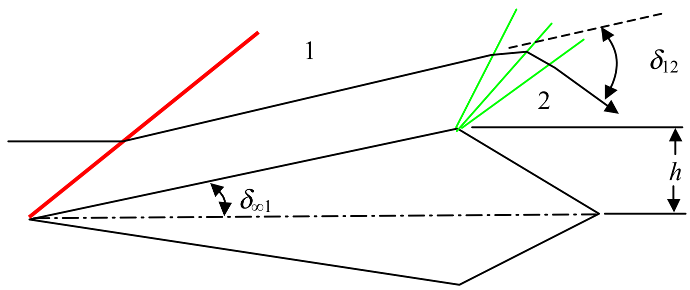
$$\boxed{c_L = 0.1190} \quad \text{and} \quad \boxed{c_D = 0.0103} \tag{18}$$

Find an expression for the drag coefficient (based on frontal projected area) of the thin diamond-shaped body shown below:



Do not assume that the angles are very small (i.e., use the oblique shock and expansion fan relations). Now calculate the drag coefficient for the object when it is turned around (the 5° angle is at the leading edge). Compare your results with those found using thin airfoil theory and explain why the results are similar or different.

SOLUTION:



Determine the conditions in regions 1 and 2. For region 1, use the oblique shock relations.

$$\mathrm{Ma}_\infty = 2.5,\ \delta_{\infty 1} = 3^\circ \ \Rightarrow\ \mathrm{Ma}_1 = 2.37,\ \varepsilon_{\infty 1} = 25.8^\circ,\ p_1/p_\infty = 1.2164 \tag{1}$$

For region 2, use Prandtl-Meyer angles.

$$\mathrm{Ma}_1 = 2.37 \ \Rightarrow\ \nu_1 = -36.02^\circ \tag{2}$$

$$\nu_2 = \nu_1 + \delta_{12} \ \Rightarrow\ \nu_2 = -44.02^\circ \ \Rightarrow\ \mathrm{Ma}_2 = 2.72 \tag{3}$$

The flow from region 2 to region 3 is isentropic, hence:

$$\frac{p_2}{p_1} = \left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2}{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}\right)^{\frac{\gamma}{1-\gamma}} \ \Rightarrow\ \frac{p_2}{p_1} = 0.5810 \tag{4}$$

$$\frac{p_2}{p_\infty} = \left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_\infty}\right) \ \Rightarrow\ \frac{p_2}{p_\infty} = 0.7067 \tag{5}$$

The drag force acting on the is:

$$F_D = p_1 h - p_2 h \tag{6}$$

$$c_D \equiv \frac{F_D}{\frac{1}{2}\rho_\infty V_\infty^2 h} = \frac{p_1(2h) - p_2(2h)}{\frac{1}{2}\rho_\infty V_\infty^2 (2h)} = \frac{p_\infty\left(\frac{p_1}{p_\infty} - \frac{p_2}{p_\infty}\right)}{\frac{1}{2}\rho_\infty \gamma R T_\infty \mathrm{Ma}_\infty^2} \tag{7}$$

$$\therefore c_D = \frac{\frac{p_1}{p_\infty} - \frac{p_2}{p_\infty}}{\frac{1}{2}\gamma \mathrm{Ma}_\infty^2} \tag{8}$$

Using the given data, $\boxed{c_D = 0.117}$.

Follow the same approach when the object is turned around. For region 1, use the oblique shock relations.

$\text{Ma}_\infty = 2.5,\ \delta_{\infty 1} = 5° \Rightarrow \text{Ma}_1 = 2.29,\ \varepsilon_{\infty 1} = 27.42°,\ p_1/p_\infty = 1.380$ (9)

For region 2, use Prandtl-Meyer angles.

$\text{Ma}_1 = 2.29 \Rightarrow \nu_1 = -34.03°$ (10)

$\nu_2 = \nu_1 + \delta_{12} \Rightarrow \nu_2 = -42.03° \Rightarrow \text{Ma}_2 = 2.63$ (11)

The flow from region 2 to region 3 is isentropic, hence:

$$\frac{p_2}{p_1} = \left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_2^2}{1+\frac{\gamma-1}{2}\text{Ma}_1^2}\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \frac{p_2}{p_1} = 0.5890 \quad (12)$$

$$\frac{p_2}{p_\infty} = \left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_\infty}\right) \Rightarrow \frac{p_2}{p_\infty} = 0.8130 \quad (13)$$

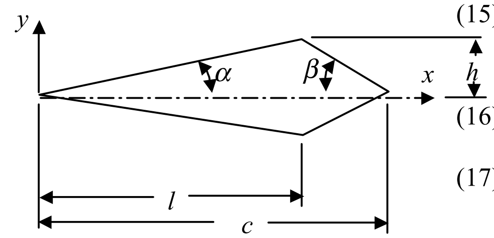
The drag coefficient is:

$$\therefore c_D = \frac{p_1/p_\infty - p_2/p_\infty}{\frac{1}{2}\gamma \text{Ma}_\infty^2} \quad (14)$$

Using the given data, $\boxed{c_D = 0.130}$.

Using Thin Airfoil Theory:

$$c_D = \frac{1}{c\sqrt{\text{Ma}_\infty^2 - 1}}\left[\int_{x=0}^{x=l} 2(\tan\alpha)^2\,dx + \int_{x=l}^{x=c} 2(-\tan\beta)^2\,dx\right] \quad (15)$$



where

$$h = l\tan\alpha = (c-l)\tan\beta \quad (16)$$

$$\frac{l}{c} = \frac{\tan\beta}{\tan\alpha + \tan\beta} \approx \frac{\beta}{\alpha+\beta} \quad (17)$$

Expanding Eqn. (15), keeping in mind that the angles are small, gives:

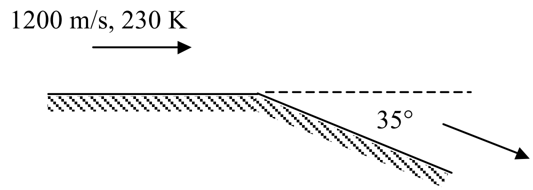
$$c_D = \frac{2\left[\alpha^2\frac{\beta}{\alpha+\beta} + \beta^2\left(1 - \frac{\beta}{\alpha+\beta}\right)\right]}{\sqrt{\text{Ma}_\infty^2 - 1}} = \frac{2\left[\frac{\alpha^2\beta + \alpha\beta^2}{\alpha+\beta}\right]}{\sqrt{\text{Ma}_\infty^2 - 1}}$$

$$\boxed{\therefore c_D = \frac{2(\alpha+\beta)}{\sqrt{\text{Ma}_\infty^2 - 1}}} \Rightarrow \boxed{\therefore c_D = 0.122} \quad (18)$$

The drag coefficient predicted using the oblique shock and expansion fan relations is nearly the same as that predicted using thin airfoil theory. This result is true because the angles through which the flow is turned are small.

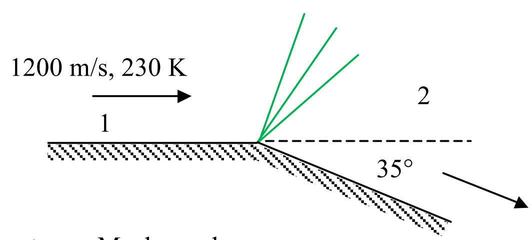
The drag coefficient is different when the object is turned around because the relations for large angles of turn (oblique shock and expansion fan relations) are non-linear. The relations for very small angles of turn are linear (thin airfoil theory) and thus are independent of the object's orientation.

A stream of air (γ= 1.4, R = 280 J/(kg·K)) with a velocity of 1200 m/s and a temperature of 230 K is turned 35° away from the flow.



a. Find the Mach number, temperature, and velocity of the flow downstream of the corner.
b. If the temperature of the upstream flow is lowered while its velocity remains at 1200 m/s, what is the theoretical minimum upstream temperature at which the flow will still be able to negotiate the turn?

SOLUTION:



First determine the upstream Mach number.

$$\mathrm{Ma}_1 = \frac{V_1}{\sqrt{\gamma R T_1}} \Rightarrow \mathrm{Ma}_1 = 4.00 \tag{1}$$

Now determine the corresponding Prandtl-Meyer angle.

$$\mathrm{Ma}_1 = 4.00 \Rightarrow \nu_1 = 65.78° \tag{2}$$

Determine the Prandtl-Meyer angle after the turn.

$$\nu_2 = \nu_1 + \delta \Rightarrow \nu_2 = 100.78° \quad \text{(where } \delta = 35°\text{)} \tag{3}$$

Using the Prandtl-Meyer angle, determine the corresponding Mach number downstream of the turn.

$$\nu_2 = 100.78° \Rightarrow \boxed{\mathrm{Ma}_2 = 9.46} \tag{4}$$

Now find the downstream temperature and velocity.

$$\frac{T_2/T_0}{T_1/T_0} = \frac{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)} \Rightarrow \boxed{T_2 = 51.1\text{ K}} \quad \text{(Note that flow through an expansion fan is isentropic.)} \tag{5}$$

$$V_2 = \mathrm{Ma}_2\sqrt{\gamma R T_2} \Rightarrow \boxed{V_2 = 1340\text{ m/s}} \tag{6}$$

Recall that as Ma→∞, the Prandtl-Meyer angle approaches 130.5° (for γ = 1.4). Thus, the maximum upstream Prandtl-Meyer angle for a 35° turn is:

$$\nu_1 = \nu_2 - \delta \Rightarrow \nu_1 = 95.4° \quad \text{where } \delta = 35° \text{ and } \nu_2 = 130.5° \tag{7}$$

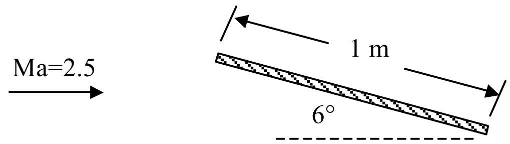
The Mach number corresponding to this Prandtl-Meyer angle is:

$$\nu_1 = 95.4° \Rightarrow \mathrm{Ma}_1 = 7.95 \tag{8}$$

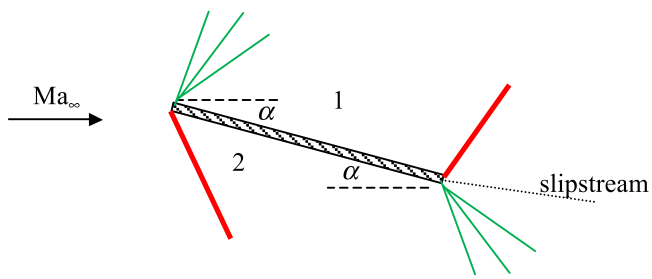
The temperature at this Mach number and V_1 = 1200 m/s is:

$$T_1 = \frac{V_1^2}{\gamma R \mathrm{Ma}_1^2} \Rightarrow \boxed{T_1 = 58.1\text{ K}} \tag{9}$$

Calculate the lift and drag coefficients (per unit depth into the page) for a flat-plate airfoil with a chord length of 1 m. The plate is at an angle of attack of 6 degree with respect to the incoming flow which has a Mach number of 2.5. Clearly sketch the wave patterns at both the leading and trailing edges of the airfoils. Note that the lift and drag coefficients are based on the planform area of the airfoil.



SOLUTION:



First consider side 1.

$\mathrm{Ma}_\infty = 2.5 \;\Rightarrow\; \nu_\infty = 39.1°$ (from the Prandtl-Meyer angle relation) (1)

$\nu_1 = \nu_\infty + \alpha \;\Rightarrow\; \nu_1 = 45.1°$ (note that $\alpha = 6°$) (2)

$\nu_1 = 45.1° \;\Rightarrow\; \mathrm{Ma}_1 = 2.77$ (from the Prandtl-Meyer angle relation) (3)

$$\frac{p_1/p_0}{p_\infty/p_0} = \frac{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}}}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}}} \Rightarrow p_1/p_\infty = 0.6591 \text{ (expansion fans are isentropic so } p_0 = \text{constant)} \quad (4)$$

Now consider side 2.

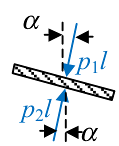
$\mathrm{Ma}_\infty = 2.5$, $\delta = \alpha = 6° \;\Rightarrow\; \varepsilon = 28.26°$, $\mathrm{Ma}_2 = 2.25$, $p_2/p_\infty = 1.4679$ (from the oblique shock relns.) (5)

The drag coefficient is defined as:

$$c_D \equiv \frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 l} = \frac{D}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 l} \quad (6)$$

where l is the chord length of the plate, and

$$\rho_\infty V_\infty^2 = \left(\frac{p_\infty}{RT_\infty}\right)\left(\mathrm{Ma}_\infty^2 \gamma R T_\infty\right) = \gamma p_\infty \mathrm{Ma}_\infty^2 \quad (7)$$



The drag force is the force component parallel to the upstream flow. Thus,

$$D = -p_1 l \sin\alpha + p_2 l \sin\alpha \quad (8)$$

Substitute Eqn. (8) into Eqn. (6) and simplify.

$$c_D = \frac{-p_1 l \sin\alpha + p_2 l \sin\alpha}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 l} = \frac{2\sin\alpha}{\gamma \mathrm{Ma}_\infty^2}\left(-\frac{p_1}{p_\infty} + \frac{p_2}{p_\infty}\right) \Rightarrow \boxed{c_D = 0.0193} \quad (9)$$

The lift coefficient is defined as:

$$c_L \equiv \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 l} = \frac{L}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 l} \tag{10}$$

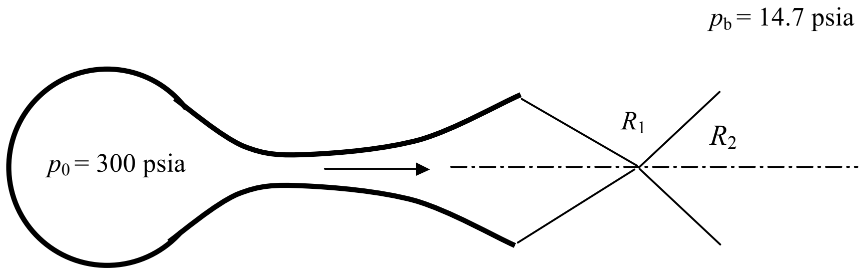
The lift force is the force component perpendicular to the upstream flow. Thus,

$$L = -p_1 l\cos\alpha + p_2 l\cos\alpha \tag{11}$$

Substitute Eq. (11) into Eq. (10) and simplify.

$$c_L = \frac{-p_1 l\cos\alpha + p_2 l\cos\alpha}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 l} = \frac{2\cos\alpha}{\gamma \mathrm{Ma}_\infty^2}\left(-\frac{p_1}{p_\infty} + \frac{p_2}{p_\infty}\right) \Rightarrow \boxed{c_L = 0.1839} \tag{12}$$

A symmetric converging-diverging nozzle is design Mach number of 2.2. With the nozzle exhausting to a back pressure of 14.7 psia and a reservoir pressure of 300 psia,



a. Determine the Mach number in regions R_1 and R_2.
b. Determine the angle of the flow with respect to the horizontal (in degrees) in regions R_1 and R_2.

SOLUTION:

Recall that at design conditions the flow is isentropic throughout the C-D nozzle.

$$\frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_E/p_0 = 0.0935 \text{ (using } \gamma = 1.4 \text{ and } \mathrm{Ma}_E = 2.2) \tag{1}$$

In region 1, the pressure must equal the surrounding air pressure, *i.e.*

$$p_1 = p_b = 14.7 \text{ psia} \tag{2}$$

Thus, the pressure ratio between region E and region 1 is:

$$\frac{p_1}{p_E} = \frac{p_1}{\left(p_E/p_0\right)p_0} = 0.5239 \text{ (using } p_1 = 14.7 \text{ psia, } p_E/p_0 = 0.0935\text{, and } p_0 = 300 \text{ psia)} \tag{3}$$

Since the pressure in region 1 is less than the pressure in the exit region, the flow must go through an expansion fan at the exit. Hence, the flow is underexpanded. The Mach number in region 1 can be found by applying the isentropic relations between regions 1 and the exit.

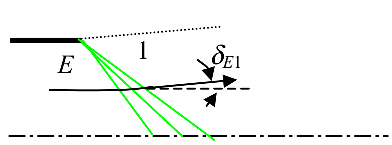
$$\frac{p_1}{p_E} = \frac{\left(p_1/p_0\right)}{\left(p_E/p_0\right)} = \frac{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}}}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}}} \Rightarrow \boxed{\mathrm{Ma}_1 = 2.6146} \tag{4}$$

The angle of the flow in region 1 may be found using Prandtl-Meyer angles for regions E and 1.

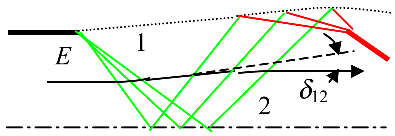
$$\mathrm{Ma}_E = 2.2 \Rightarrow \nu_E = 31.73° \tag{5}$$

$$\mathrm{Ma}_1 = 2.6146 \Rightarrow \nu_1 = 41.74° \tag{6}$$

$$\boxed{\delta_{E1} = \nu_1 - \nu_E = 10.01°} \tag{7}$$



The expansion fan reflects off the axis of symmetry as another expansion fan in order to keep the flow in region 2 parallel to the centerline.

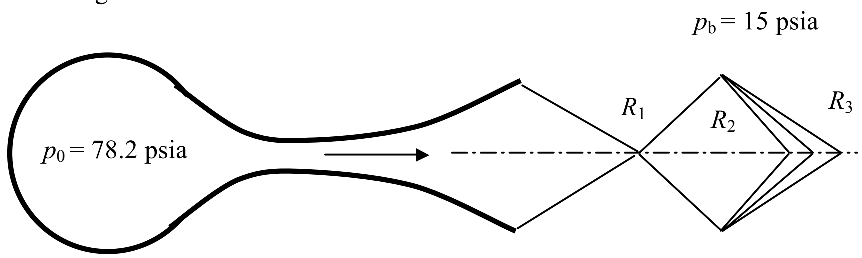


$$\delta_{12} = \delta_{E1} = 10.01° \tag{8}$$

$$\delta_{12} = \nu_2 - \nu_1 = 10.01^{\circ} \Rightarrow \nu_2 = \delta_{12} + \nu_1 = 51.75^{\circ} \tag{9}$$

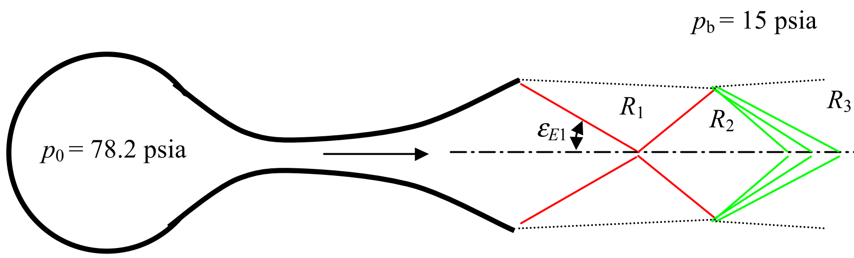
$$n_2 = 51.75° \;\Rightarrow\; \boxed{\mathrm{Ma}_2 = 3.11} \tag{10}$$

A symmetric converging-diverging nozzle is design Mach number of 2.0. With the nozzle exhausting to a back pressure of 15 psia, however, and a reservoir pressure of 78.2 psia, the nozzle is overexpanded as is shown in the figure below.



a. Determine the Mach number in regions R_1, R_2, and R_3.
b. Determine the angle of the flow with respect to the horizontal (in degrees) in regions R_1, R_2, and R_3.

SOLUTION:



Recall that at design conditions the flow is isentropic throughout the C-D nozzle.

$$\frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_E/p_0 = 0.1278 \quad (\text{using } \gamma = 1.4,\ \mathrm{Ma}_E = 2.0) \tag{1}$$

In region R_1, the pressure must equal the back pressure:

$$p_1 = p_b = 15 \text{ psia} \tag{2}$$

Thus, the pressure rise across the oblique shock (since the flow is over-expanded, oblique shocks will form at the exit) will be:

$$\frac{p_1}{p_E} = \frac{p_1}{\left(p_E/p_0\right)p_0} = 1.501 \quad (\text{using } p_1 = 15 \text{ psia},\ p_0 = 78.2 \text{ psia},\ p_E/p_0 = 0.1278) \tag{3}$$

The pressure rise across an oblique shock is determine by the normal component of the upstream Mach number:

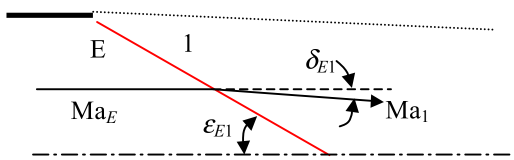
$$\frac{p_1}{p_E} = 1.501 \Rightarrow \mathrm{Ma}_{EN} = 1.195 \text{ (using the normal shock relations)} \tag{4}$$

From geometry:

$$\mathrm{Ma}_{EN} = \mathrm{Ma}_E \sin\varepsilon_{E1} \Rightarrow \underline{\varepsilon_{E1} = 36.7^\circ} \quad (\text{using } \mathrm{Ma}_{EN} = 1.195 \text{ and } \mathrm{Ma}_E = 2.0) \tag{5}$$

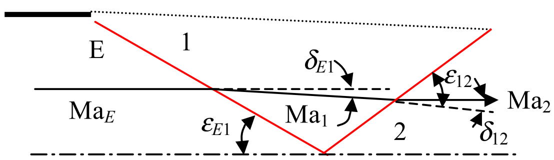
From the oblique shock relations:

$$\mathrm{Ma}_E = 2.0,\ \varepsilon_{E1} = 36.7° \ \Rightarrow\ \boxed{\mathrm{Ma}_1 = 1.73},\ \boxed{\delta_{E1} = 7.5°} \tag{6}$$



<u>In region 2, the flow must be horizontal</u> due to symmetry. Hence, the turning angle must be $\delta_{12} = \delta_{E1} = 7.5°$. Using the oblique shock relations:

$$\mathrm{Ma}_1 = 1.73,\ \delta_{12} = 7.5° \ \Rightarrow\ \boxed{\mathrm{Ma}_2 = 1.47},\ \varepsilon_{12} = 43.0°,\ p_2/p_1 = 1.4584 \tag{7}$$



In region 3 the pressure must equal the back pressure:

$$p_3 = p_b = 15 \text{ psia} \tag{8}$$

The flow from region 2 to region 3 passes through an expansion fan (since an oblique shock reflects as an expansion fan off a free pressure boundary). The Mach number in region 3 can be found from the Mach number in region 2 and the pressure ratio, p_3/p_2, using the isentropic relations.

$$\frac{p_3}{p_2} = \left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_3^2}{1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{\gamma}{1-\gamma}} \ \Rightarrow\ \boxed{\mathrm{Ma}_3 = 1.73} \tag{9}$$

where $\gamma = 1.4$, $\mathrm{Ma}_2 = 1.47$, and

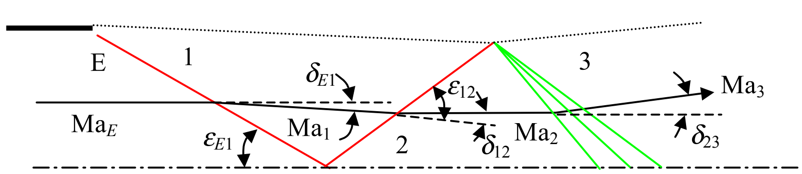
$$\frac{p_3}{p_2} = \frac{p_3}{\left(p_2/p_1\right)p_1} \quad \text{(using } p_3 = 15 \text{ psia, } p_1 = 15 \text{ psia, } p_2/p_1 = 1.4584\text{)} \tag{10}$$

The flow angle in region 3 is found using Prandtl-Meyer angles:

$$\mathrm{Ma}_2 = 1.47 \ \Rightarrow\ \nu_2 = 11.02° \tag{11}$$

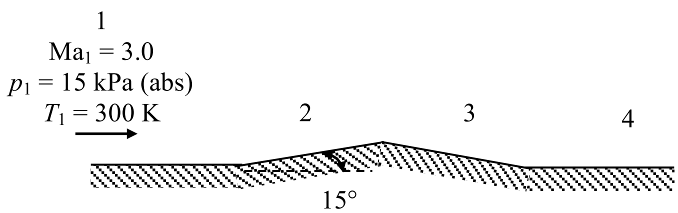
$$\mathrm{Ma}_3 = 1.73 \ \Rightarrow\ \nu_3 = 18.69° \tag{12}$$

$$\boxed{\delta_{23} = \nu_3 - \nu_2 = 7.7°} \tag{13}$$

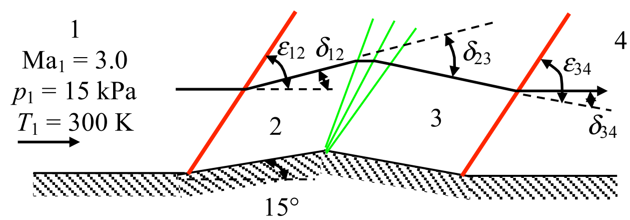


Air flowing at a Mach number of 3.0 with a static temperature of 300 K and static pressure of 15 kPa (abs) approaches a symmetric bump as shown in the figure below.

a. Determine the Mach number in region 2.
b. Determine the Mach number in region 3.
c. Determine the Mach number in region 4.
d. Determine the pressure in region 4.
e. Why are the conditions at 4 not the same as the conditions at 1 even though the bump is symmetric?



SOLUTION:



First examine region 2 using the oblique shock relations:

$$\mathrm{Ma}_1 = 3.0,\ \delta_{12} = 15° \quad \Rightarrow \quad \boxed{\mathrm{Ma}_2 = 2.25} \tag{1}$$
$$\varepsilon_{12} = 32.2°$$
$$p_2/p_1 = 2.8216$$

Now examine region 3 using Prandtl-Meyer angles:

$$\mathrm{Ma}_2 = 2.25 \quad \Rightarrow \quad \nu_2 = -33.0° \tag{2}$$
$$\nu_3 = \nu_2 - 30° = -63.0° \quad \Rightarrow \quad \boxed{\mathrm{Ma}_3 = 3.80} \tag{3}$$
$$\frac{p_3}{p_2} = \left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_3^2}{1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{\gamma}{1-\gamma}} \quad \Rightarrow \quad p_3/p_2 = 0.1000 \tag{4}$$

Lastly, examine region 4 using the oblique shock relations:

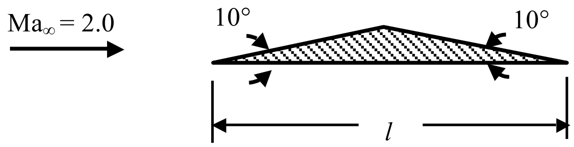
$$\mathrm{Ma}_3 = 3.80,\ \delta_{12} = 15° \quad \Rightarrow \quad \boxed{\mathrm{Ma}_4 = 2.80} \tag{5}$$
$$\varepsilon_{34} = 27.8°$$
$$p_4/p_3 = 3.5053$$

Determine the pressure in region 4 using the pressure ratios found in the previous calculations.

$$p_4 = \left(\frac{p_4}{p_3}\right)\left(\frac{p_3}{p_2}\right)\left(\frac{p_2}{p_1}\right)p_1 \quad \Rightarrow \quad \boxed{p_4 = 14.8\text{ kPa}} \quad (\text{using } p_1 = 15\text{ kPa}) \tag{6}$$

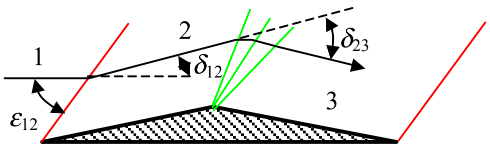
The flow is not symmetric over the bump since flow through the oblique shock waves occurs non-isentropically (i.e., irreversibly). If the bump angle was very small such that the oblique shocks could be replaced by isentropic compression waves, then the flow would be symmetric.

A two-dimensional double-wedge profile is at zero angle of attack in an air stream of Mach number 2.0.



Calculate the drag coefficient for the airfoil based on the chord length, l.

SOLUTION:



Use the oblique shock relations to determine the conditions in region 2.

$$\text{Ma}_1 = 2.0,\ \delta_{12} = 10^\circ \ \Rightarrow \text{Ma}_2 = 1.6405,\ \varepsilon_{12} = 39.31^\circ,\ p_2/p_1 = 1.7066 \tag{1}$$

Determine the Mach number in region 3 using the Prandtl-Meyer angle.

$$\text{Ma}_2 = 1.6405 \ \Rightarrow \ \nu_2 = 16.0574^\circ \tag{2}$$

$$\nu_3 = \nu_2 + \delta_{23} = 36.0574^\circ \ \text{ where } \delta_{23} = 20^\circ \tag{3}$$

$$\nu_3 = 36.0574^\circ \ \Rightarrow \ \text{Ma}_3 = 2.3717 \tag{4}$$

Determine the pressure ratio for region 3 using the isentropic relations.

$$\frac{p_3}{p_2} = \left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_3^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{\frac{\gamma}{1-\gamma}} \ \Rightarrow \ p_3/p_2 = 0.3227 \tag{5}$$

Note that:

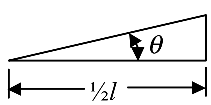
$$\frac{p_3}{p_1} = \left(\frac{p_3}{p_2}\right)\left(\frac{p_2}{p_1}\right) = 0.5507$$

The drag coefficient on the object is given by:

$$c_D = \frac{F_D}{\frac{1}{2}\rho_1 V_1^2 l} \tag{6}$$

where

$$F_D = (p_2 - p_3)\left(\tfrac{1}{2} l \tan\theta\right) \tag{7}$$



and

$$\tfrac{1}{2}\rho_1 V_1^2 = \tfrac{1}{2}\left(\frac{p_1}{RT_1}\right)\left(\gamma R T_1 \text{Ma}_1^2\right) = \tfrac{1}{2}\gamma p_1 \text{Ma}_1^2 \tag{8}$$

So that:

$$c_D = \frac{(p_2 - p_3)\left(\frac{1}{2} l \tan\theta\right)}{\frac{1}{2}\gamma p_1 \text{Ma}_1^2} = \left(\frac{p_2}{p_1} - \frac{p_3}{p_1}\right)\frac{\tan\theta}{\gamma \text{Ma}_1^2} \tag{9}$$

$$\boxed{\therefore c_D = 0.036} \ \text{ using } p_2/p_1 = 1.7066,\ p_3/p_1 = 0.5507,\ \theta = 10^\circ,\ \gamma = 1.4, \text{ and } \text{Ma}_1 = 2.0 \tag{10}$$

Now solve the problem using thin airfoil theory.

$$c_D = \frac{2}{l\sqrt{\mathrm{Ma}_\infty^2 - 1}} \int_0^l \left[\left(\left.\frac{dy}{dx}\right|_{\text{upper}} \right)^2 + \left(\left.\frac{dy}{dx}\right|_{\text{lower}} \right)^2 \right] dx \tag{11}$$

where

$$\left.\frac{dy}{dx}\right|_{\text{upper}} = \begin{cases} \tan\left(10^\circ\right) & 0 < x < \frac{1}{2}l \\ -\tan\left(10^\circ\right) & \frac{1}{2}l < x < l \end{cases} \tag{12}$$

$$\left.\frac{dy}{dx}\right|_{\text{lower}} = 0 \tag{13}$$

Substitute and simplify.

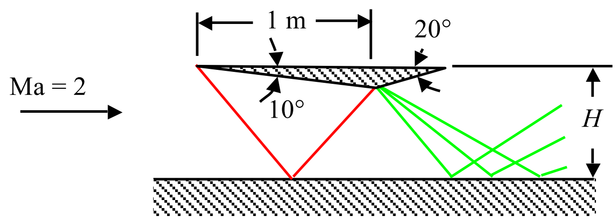
$$c_D = \frac{2}{l\sqrt{\mathrm{Ma}_\infty^2 - 1}} \left\{ \int_0^{\frac{1}{2}l} \left[\tan\left(10^\circ\right)\right]^2 dx + \int_{\frac{1}{2}l}^{l} \left[-\tan\left(10^\circ\right)\right]^2 dx \right\} \tag{14}$$

$$c_D = \frac{2l\left[\tan\left(10^\circ\right)\right]^2}{l\sqrt{\mathrm{Ma}_\infty^2 - 1}} \tag{15}$$

$$c_D = \frac{2\left[\tan\left(10^\circ\right)\right]^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}} = 0.0359 \tag{16}$$

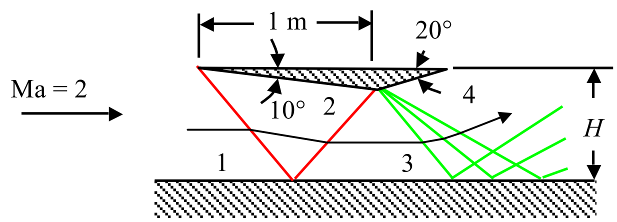
This result is nearly identical to the result found previously despite the fact that 10° is on the edge of being considered a “small” angle.

Consider the asymmetric airfoil located near a wall as shown in the figure below.



a. What must H be in order for the reflected shock to impinge on the airfoil point as shown?
b. What is the drag coefficient for the airfoil based on the <u>frontal projected area</u>?

SOLUTION:

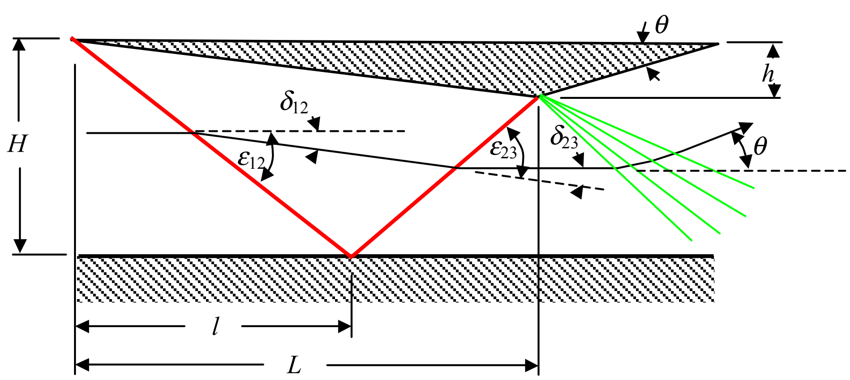


Use the oblique shock relations to determine the conditions in regions 2 and 3.

$\mathrm{Ma}_1 = 2$, $\delta_{12} = 10°$ $\Rightarrow$ $\mathrm{Ma}_2 = 1.6405$, $\varepsilon_{12} = 39.31°$, $p_2/p_1 = 1.7066$ (1)

$\mathrm{Ma}_2 = 1.6405$, $\delta_{23} = 10°$ $\Rightarrow$ $\mathrm{Ma}_3 = 1.2849$, $\varepsilon_{23} = 49.38°$, $p_3/p_2 = 1.6423$ (2)

Use geometry to determine the height H at which the reflected shock impinges on the airfoil corner.



$$\tan\varepsilon_{12} = \frac{H}{l} \Rightarrow l = \frac{H}{\tan\varepsilon_{12}} \tag{3}$$

$$\tan\left(\varepsilon_{23} - \delta_{23}\right) = \frac{H-h}{L-l} \tag{4}$$

Substitute Eq. (3) into Eq. (4) and solve for H.

$$\tan\left(\varepsilon_{23}-\delta_{23}\right)=\frac{H-h}{L-\frac{H}{\tan\varepsilon_{12}}} \tag{5}$$

$$\tan\left(\varepsilon_{23}-\delta_{23}\right)\left(L-\frac{H}{\tan\varepsilon_{12}}\right)=H-h$$

$$\boxed{\therefore H=\frac{L\tan\left(\varepsilon_{23}-\delta_{23}\right)+h}{1+\frac{\tan\left(\varepsilon_{23}-\delta_{23}\right)}{\tan\varepsilon_{12}}}} \tag{6}$$

Using δ_{23} = 10°, L = 1 m, ε_{23} = 49.38°, ε_{12} = 39.31° $\Rightarrow$ h = 0.176 m, $\boxed{H = 0.498\text{ m}}$.

The drag on the airfoil is found by summing the pressure forces.

$$F_D=p_2h-p_4h \tag{7}$$

$$c_D\equiv\frac{F_D}{\frac{1}{2}\rho_1V_1^2h}=\frac{p_2h-p_4h}{\frac{1}{2}\rho_1V_1^2h}=\frac{p_1\left(p_2/p_1-p_4/p_1\right)}{\frac{1}{2}\rho_1\gamma RT_1\mathrm{Ma}_1^2} \tag{8}$$

$$\underline{\therefore c_D=\frac{p_2/p_1-p_4/p_1}{\frac{1}{2}\gamma\mathrm{Ma}_1^2}} \tag{9}$$

The pressure in region 4 is found using Prandtl-Meyer angles.

Ma_3 = 1.2849 $\Rightarrow$ ν_3 = -5.7595° (10)

$\nu_4 = \nu_3 + \theta$ $\Rightarrow$ ν_4 = -25.7595° $\Rightarrow$ $\underline{\mathrm{Ma}_4 = 1.9776}$ where θ = 20° (11)

The flow from region 3 to region 4 is isentropic.

$$\frac{p_4}{p_3}=\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_4^2}{1+\frac{\gamma-1}{2}\mathrm{Ma}_3^2}\right)^{\frac{\gamma}{1-\gamma}}\Rightarrow\frac{p_4}{p_3}=0.3592 \tag{12}$$

$$\frac{p_4}{p_1}=\left(\frac{p_4}{p_3}\right)\left(\frac{p_3}{p_2}\right)\left(\frac{p_2}{p_1}\right)\Rightarrow\frac{p_4}{p_1}=1.0067 \tag{13}$$

Substitute values into Eq. (9).

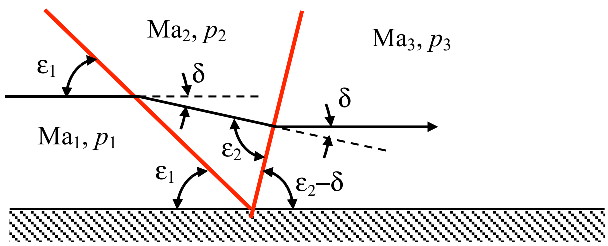
$\boxed{c_D = 0.2500}$

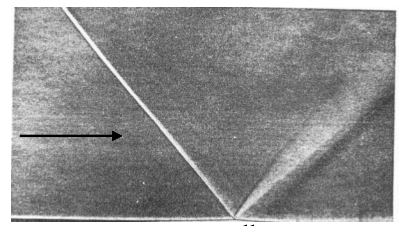
15. Reflection and Interaction of Oblique Shock Waves

In this set of notes we'll consider the interaction between oblique shocks and solid boundaries, free surfaces, and with other oblique shocks.

Reflection with a Solid Boundary

When an oblique shock intersects a solid, straight boundary, it will reflect as another oblique shock in order for the flow to remain parallel to the boundary as shown in the figure below.





wall

Image from: Shapiro, A.H., *The Dynamics and Thermodynamics of Compressible Fluid Flow Vol. I*, Wiley.

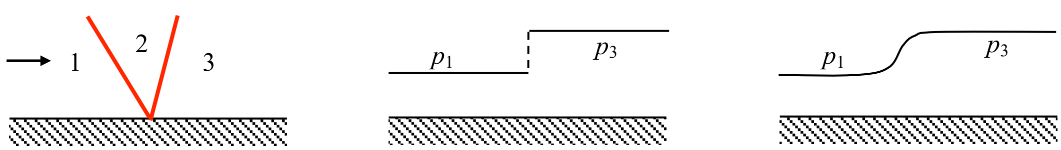
The flow properties through the incident and reflected shocks can be determined using the following procedure.

1. For the given Ma_1 and δ, determine Ma_2 and p_2/p_1.
2. For this value of Ma_2 and since the turning angle of the second wave is also δ (in order to keep the flow parallel to the flat wall), one can determine $\mathrm{Ma}_3, \varepsilon_2$, and p_3/p_2.
3. The pressure ratio across both waves, p_3/p_1, is found using:
$$\frac{p_3}{p_1} = \frac{p_3}{p_2}\frac{p_2}{p_1}$$
4. The angle that the reflected wave makes with the wall is ε_2-δ.

Notes:

1. The wave angle, ε, and the turning angle, δ, are measured with respect to the incoming flow direction.

2. Without viscous effects the pressure change is discontinuous across the shock at the wall. Due to the boundary layer, the flow drops to zero velocity at the wall and so the flow adjacent to the wall is subsonic and thus cannot sustain pressure discontinuities. Thus the boundary layer causes the pressure distribution to "spread out" as shown in the figure below.



without BL with BL

The details of the shock interaction with the wall depend on whether the boundary layer is laminar or turbulent, the thickness of the boundary layer, and the shock strength. A flow separation bubble may also occur due to an adverse pressure gradient.

3. Recall that there is a maximum possible angle through which the flow can be turned using an oblique shock. This maximum angle (δ_{max}) decreases as the incoming Mach number decreases. Thus, when the flow passes through the initial oblique shock, there exists the possibility that the reflected flow must be turned by an angle that is greater than the maximum turning angle in order to remain parallel to the wall. In this case, a **Mach reflection** appears where a curved strong shock occurs adjacent to the wall behind which there is subsonic flow. Since this subsonic flow need not be parallel to the wall, the flow above the wall shock layer also does not have to be parallel to the wall. The flow behind the curved wall shock is divided from the flow behind the "reflected" oblique shock by a **slipline** (aka contact surface, slipstream, vortex sheet). Across the slipline there are changes in velocity, temperature, and entropy. Since the flow is subsonic, the details of the flow may be affected by downstream conditions and, as a result, the analytical modeling of the Mach reflection is difficult.

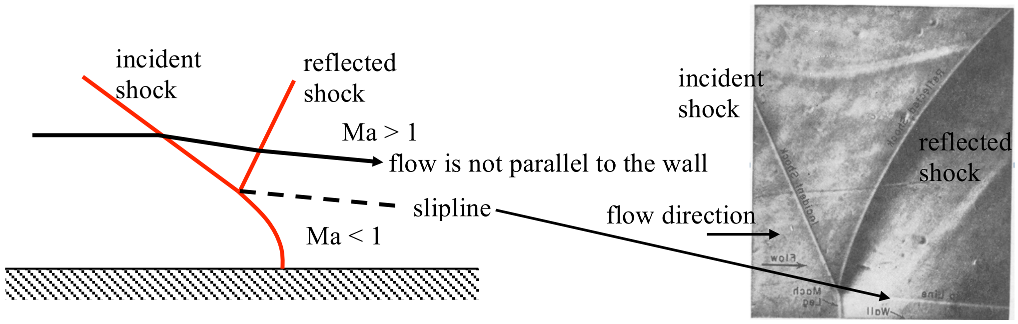
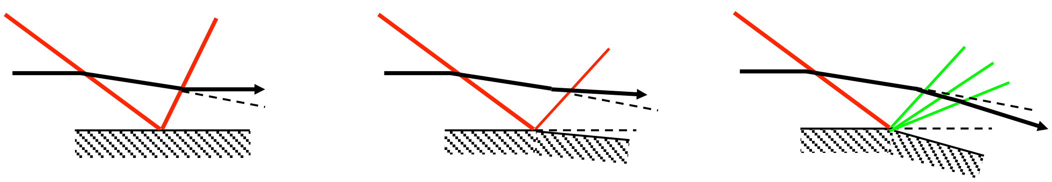


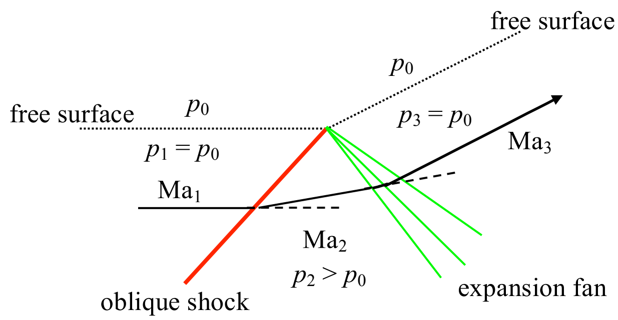
Image from: Shapiro, A.H., *The Dynamics and Thermodynamics of Compressible Fluid Flow Vol. I*, Wiley.

4. Instead of a flat wall, consider the reflection from a wall that is turned at some angle. The flow must still remain parallel to the downstream wall. The oblique shock reflected from a wall turned away from the flow will be weaker than one reflected from a flat wall. It is even possible to "cancel" the reflected oblique shock by turning the downstream shock at the angle through which the flow is turned by the initial shock. Turning the wall by an angle greater than this results in an expansion fan.



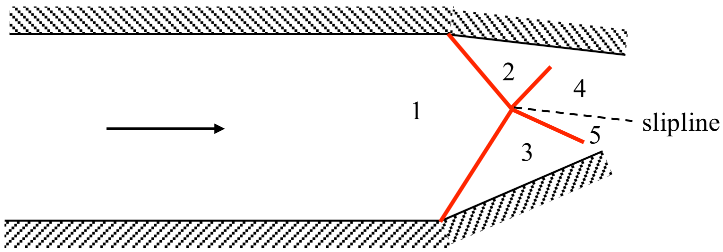
Oblique Shock Reflection from a Free Surface

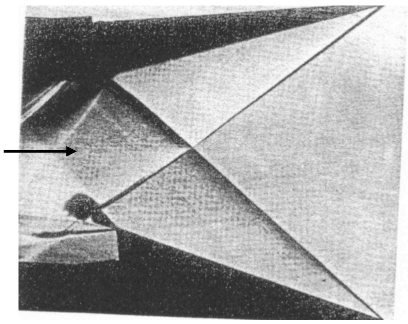
When an oblique shock intersects a free surface, the reflection must be an expansion fan so that the flow pressure remains equal to the free surface pressure as shown in the figure below.



Interaction of Oblique Shock Waves

When two oblique shock waves intersect, each wave will be transmitted through, but be affected by, the other wave. For example, consider the interaction of two oblique shocks as shown in the following diagram.

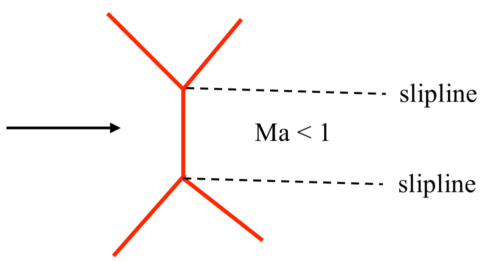




The flow in regions 2 and 3 are found given the conditions in region 1 and the wall turning angles. The flow in regions 4 and 5 must be parallel to one another and, hence, from the linear momentum equation the pressures in regions 4 and 5 must be the same. An iterative procedure can be used to solve for the flows from regions 2 and 3 into regions 4 and 5. A slipline occurs between regions 4 and 5 since these regions, in general, have different entropy, temperature, and velocity.

Notes:

1. There may be oblique shock intersection situations where no solution exists using oblique shocks. In this case, more complicated flow patterns may occur that include normal shock waves (Mach reflections).



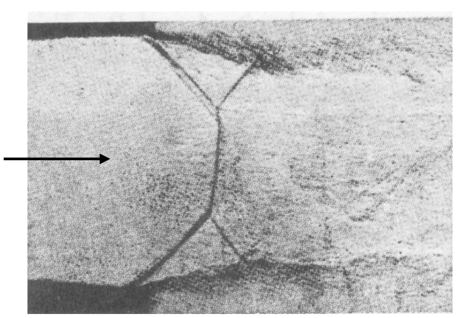
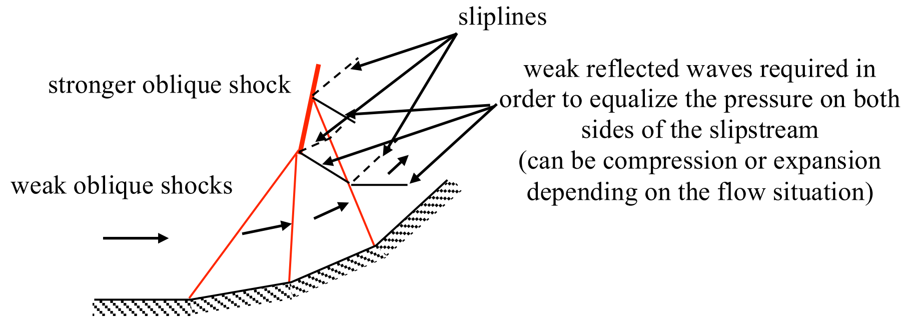


Image from: Shapiro, A.H., *The Dynamics and Thermodynamics of Compressible Fluid Flow Vol. I*, Wiley.

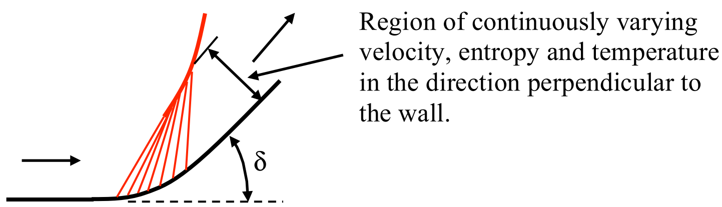
Now consider flow in a corner with several small, discrete changes in the wall angle as shown in the figure below.



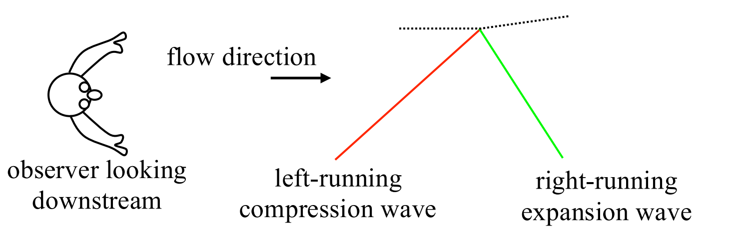
The oblique shocks will intersect and coalesce into a single oblique shock which is stronger than any of the initial oblique shock waves. Sliplines and weak reflected waves appear at the intersection of the waves.

Notes:
1. An oblique shock is generally considered to form when two or more compression waves coalesce.

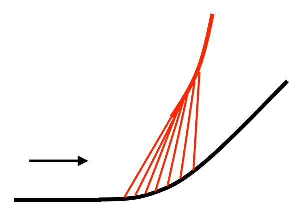
2. For a continuously inward curving wall, the slipstreams are spaced an infinitesimal distance apart so the downstream flow has continuously changing entropy, velocity, and temperature. The oblique shock that forms from the interacting compression waves will have a strength such that it turns the flow by the wall angle, δ, for the given incoming Mach number.



3. When waves travel in the same direction as viewed by an observer oriented so that they're looking downstream, the waves are said to be of the **same family**. The waves are further classified as being either **left-running** or **right-running** depending on what direction the waves are oriented. Examples of this notation are given below.



These waves are of different families since one is left-running and the other is right running.

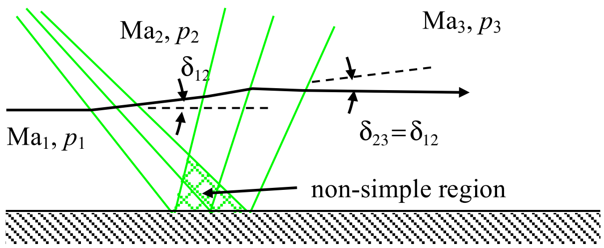


All of the compression waves shown in the figure above are of the same family since they are all left-running waves.

16. Reflection and Interaction of Expansion Waves

Reflection with a Solid Boundary

When an expansion wave intersects a solid boundary, it will reflect as another expansion fan (of opposite family) in order for the flow to remain parallel to the boundary as shown in the figure below.



The flow through the incident and reflected expansion fans can be determined using the following procedure.

1. For the given (Ma$_1$, ν_1) and δ_{12}, determine (Ma$_2$, ν_2) using $\nu_2 = \nu_1 + \delta_{12}$.
2. For this value of (Ma$_2$, ν_2) and since the turning angle of the second wave is also $\delta_{23}=\delta_{12}$ (in order to keep the flow parallel to the flat wall), one can determine (Ma$_3$, ν_3): $\nu_3 = \nu_2 + \delta_{23}$.
3. The pressure ratio across both waves, p_3/p_1, is found using the isentropic relations:

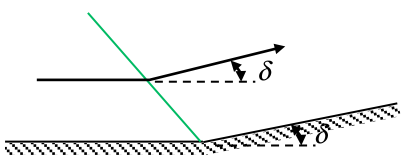
$$\frac{p_2}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{\gamma/1-\gamma} \quad \frac{p_3}{p_{01}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_3^2\right)^{\gamma/1-\gamma}$$

Notes:

1. The flow through the expansion fans is isentropic.

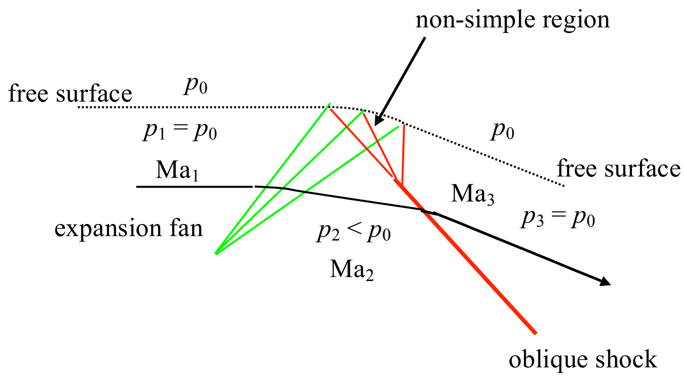
2. The region where the expansion fans interact is known as a **non-simple region**. This region is not amenable to our Prandtl-Meyer expansion fan analysis. Instead, we use can the Method of Characteristics (a topic that will be discussed in a later set of notes) to analyze the flow in this region.

3. An expansion wave may be canceled by turning the flow inward by the same amount as the flow turning angle.



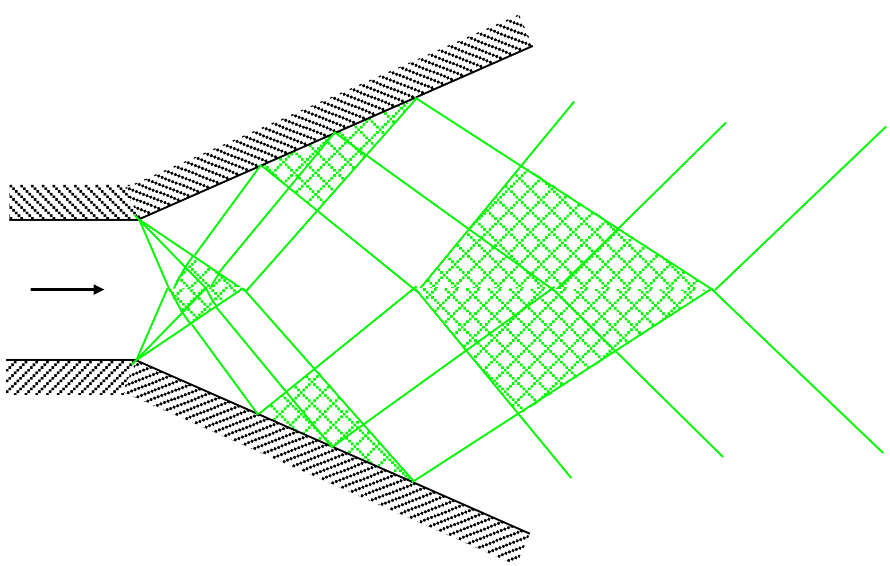
Expansion Fan Reflection from a Free Surface

When an expansion fan intersects a free surface, the reflection must be an oblique shock (of opposite family) so that the flow pressure remains equal to the free surface pressure as shown in the figure below.



Interaction of Expansion Waves

When expansion fans intersect, each expansion wave will be transmitted through, but be affected by, the other waves. There can be many regions of non-simple flow as shown in the figure below.



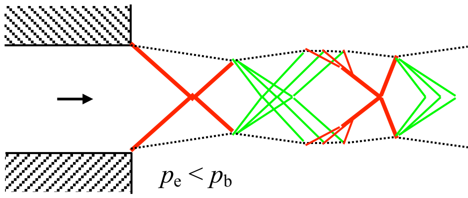
Notes:

1. No slipstreams occur with expansion fans since the flow properties change continuously through the fan.

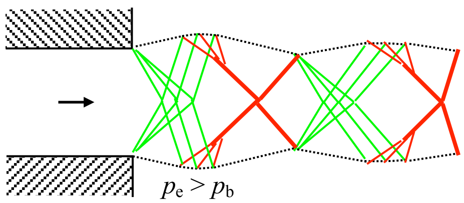
Mach Diamonds

A phenomenon known as **Mach Diamonds** (aka shock diamonds) can form when a supersonic stream exits a device and interacts with the surrounding atmosphere. Consider supersonic flow from an exit, with pressure p_e, into the surrounding atmosphere with pressure, p_b.

Over-Expanded Case



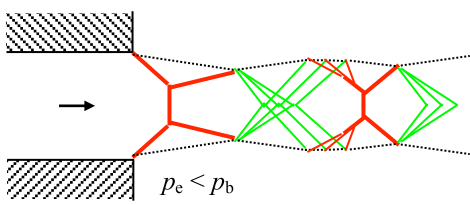
Under-Expanded Case



Dryden Flight Research Center EC92-1284 Photographed 1992
SR-71B take-off with "shock diamonds" in the exhaust. NASA photo

Notes:

1. At design conditions, $p_e = p_b$, so additional expansion or compression of the flow is not required. Hence, Mach diamonds do not appear at design conditions.
2. If the flow turning angle is sufficiently large, Mach reflections (aka Mach disks in 3D) will appear as shown in the figure below.



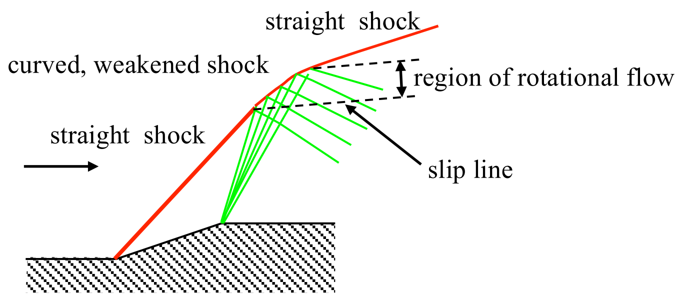
3. The bright regions in the Mach diamond shown in the photograph above are caused by heating of the gas as it passes through the oblique shocks.
4. Viscous interaction with the external fluid results in dissipation of the Mach diamond pattern.
5. Even without viscous effects, the sequence of oblique shocks and expansion fans will eventually dissipate since the stagnation pressure decreases after each shock. This can be shown by considering the Mach number in the regions downstream of an oblique shock (referred to by the subscript i) which are in contact with the free surface such that:

$$\frac{p_b}{p_{0i}} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_i^2\right)^{\frac{\gamma}{1-\gamma}}$$

Since p_{0i} decreases after passing through an oblique shock and p_b is a constant, the ratio p_b/p_{0i} increases and Ma_i decreases. Thus, the sequence of oblique shocks gets weaker.

Interaction Between An Oblique Shock and an Expansion Fan

When an oblique shock interacts with an expansion fan of the same family, the shock will be weakened and become curved. Behind the shock wave the flow becomes rotational. In addition, the incident expansion waves will be reflected.



Notes:
1. Waves will also be reflected from where waves intersect the slip line. These waves were left off the previous schematic for the sake of clarity.
2. The region of rotational flow is not isentropic (the entropy varies continuously behind this curved section of shock wave). This will be discussed in a later set of notes concerning Crocco's Theorem.

17. Equations of Motion in Terms of the Velocity Potential

Now let's consider the irrotational, isentropic flow of a compressible fluid where body forces are negligible. Recall that the momentum equations for a fluid in which viscous and body forces are negligible are given by Euler's Equations:

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p$$

We can re-write the pressure gradient term in terms of the speed of sound, c, as is shown below:

$$\nabla p\cdot d\mathbf{x}=dp=\frac{dp}{d\rho}d\rho=\frac{dp}{d\rho}\nabla\rho\cdot d\mathbf{x}$$

$$\therefore \nabla p=\frac{dp}{d\rho}\nabla\rho$$

but since the flow is isentropic, we have:

$$\nabla p=\frac{dp}{d\rho}\nabla\rho=\left.\frac{dp}{d\rho}\right|_s\nabla\rho=c^2\nabla\rho$$

Thus, the momentum equations can be written as:

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-c^2\nabla\rho$$

Now take the dot product of this equation with the velocity and re-arrange:

$$\mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\frac{c^2}{\rho}\mathbf{u}\cdot\nabla\rho$$

$$\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\frac{c^2}{\rho}\mathbf{u}\cdot\nabla\rho$$

The continuity equation can be used to re-write the RHS of the previous equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=\frac{\partial\rho}{\partial t}+\rho\nabla\cdot\mathbf{u}+\mathbf{u}\cdot\nabla\rho=0$$

$$\therefore \mathbf{u}\cdot\nabla\rho=-\frac{\partial\rho}{\partial t}-\rho\nabla\cdot\mathbf{u}$$

Thus, the momentum equations become:

$$\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=\frac{c^2}{\rho}\frac{\partial\rho}{\partial t}+c^2\nabla\cdot\mathbf{u} \tag{241}$$

The density in the previous equation may be eliminated using Bernoulli's equation. Since the flow is irrotational, we can write Bernoulli's equation as:

$$\frac{\partial\phi}{\partial t}+\int\frac{dp}{\rho}+\tfrac{1}{2}(\nabla\phi\cdot\nabla\phi)=F(t)$$

where the velocity has been written in terms of a velocity potential, ϕ.

Taking the time derivative of Bernoulli's equation gives:

$$\frac{\partial^2\phi}{\partial t^2}+\frac{\partial}{\partial t}\int\frac{dp}{\rho}+\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{u})=F'(t)$$

But,

$$\frac{\partial}{\partial t}\int\frac{dp}{\rho}=\frac{\partial}{\partial t}\int\frac{dp}{d\rho}\frac{d\rho}{\rho}=\frac{\partial}{\partial t}\int c^2\frac{d\rho}{\rho}$$

$$=\frac{\partial}{\partial\rho}\left(\int\frac{c^2}{\rho}d\rho\right)\left(\frac{\partial\rho}{\partial t}\right)$$

$$=\frac{c^2}{\rho}\left(\frac{\partial\rho}{\partial t}\right)$$

so that:

$$\frac{\partial^2\phi}{\partial t^2}+\frac{c^2}{\rho}\left(\frac{\partial\rho}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{u})=F'(t)$$

$$\frac{c^2}{\rho}\left(\frac{\partial\rho}{\partial t}\right)=F'(t)-\frac{\partial^2\phi}{\partial t^2}-\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{u})$$

Substituting into Eq. (241) gives:

$$\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=F'(t)-\frac{\partial^2\phi}{\partial t^2}-\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{u})+c^2\nabla\cdot\mathbf{u}$$

$$\nabla\cdot\mathbf{u}=\frac{1}{c^2}\left\{\frac{\partial^2\phi}{\partial t^2}+\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right]-F'(t)\right\}$$

$$=\frac{1}{c^2}\left\{\frac{\partial}{\partial t}\left[\frac{\partial\phi}{\partial t}+(\mathbf{u}\cdot\mathbf{u})-F(t)\right]+\mathbf{u}\cdot\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right]\right\}$$

Re-writing the velocities in terms of the potential function gives:

$$\boxed{\nabla^2\phi=\frac{1}{c^2}\left\{\frac{\partial}{\partial t}\left[\frac{\partial\phi}{\partial t}+(\nabla\phi\cdot\nabla\phi)-F(t)\right]+\nabla\phi\cdot\left[(\nabla\phi\cdot\nabla)\nabla\phi\right]\right\}} \tag{242}$$

Governing equation for the isentropic, irrotational flow of a compressible fluid where body forces are negligible

Notes:
1. Consider the flow of an incompressible fluid. If the flow is incompressible, then the speed of sound in the fluid will be infinite. Thus, the governing equation for the irrotational flow of an incompressible fluid becomes:

$$\boxed{\nabla^2\phi = 0} \tag{243}$$

Governing equation for the irrotational flow of an incompressible fluid in which body forces are negligible

Note that this is just Laplace's equation! A significant point regarding Eq. (243) is that it is a linear PDE which means that the principle of superposition may be used to add together "building block" solutions to form new and more complex solutions.

2. For a steady, compressible flow, Eq. (242) simplifies to:

$$\boxed{\nabla^2\phi = \frac{1}{c^2}\left\{\nabla\phi\cdot\left[\left(\nabla\phi\cdot\nabla\right)\nabla\phi\right]\right\}} \tag{244}$$

Governing equation for the steady, isentropic, irrotational flow of a compressible fluid in which body forces are negligible

3. Equations (242) and (244) are non-linear PDEs which are complex to solve by hand. There is currently no known method for analytically solving these equations in a general way (computational techniques can be used, however). Instead, we must resort to special cases for solving these equations. One of these methods, known as **small-perturbation theory**, is described in the following section.

18. Small Perturbation Theory

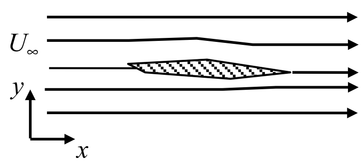
Recall that the equation of motion for an irrotational flow where body and viscous forces are negligible is:

$$\boxed{\nabla^2\phi = \frac{1}{c^2}\left\{\frac{\partial}{\partial t}\left[\frac{\partial\phi}{\partial t}+\left(\nabla\phi\cdot\nabla\phi\right)-F\left(t\right)\right]+\nabla\phi\cdot\left[\left(\nabla\phi\cdot\nabla\right)\nabla\phi\right]\right\}} \tag{245}$$

where the velocity is given in terms of a potential function, $\mathbf{u}=\nabla\phi$, and c is the speed of sound. For a steady flow, this equation simplifies to:

$$\boxed{\nabla^2\phi = \frac{1}{c^2}\left\{\nabla\phi\cdot\left[\left(\nabla\phi\cdot\nabla\right)\nabla\phi\right]\right\}} \tag{246}$$

Suppose that a uniform flow approaches an object that is sufficiently slender so that it produces only a small perturbation to the incoming stream as shown in the figure below.



The potential function for such a flow can be written as:

$$\boxed{\phi = U_\infty x + \Phi} \tag{247}$$

where U_∞ is the velocity of the incoming flow and $\mathbf{\Phi}$ is the potential function for the velocity perturbations, $\mathbf{u}$', i.e.:

$$\mathbf{u}' = \nabla\Phi$$

(Note: $\nabla\phi = \nabla\left(U_\infty x + \Phi\right) = U_\infty\hat{\mathbf{e}}_x + \underbrace{\nabla\Phi}_{=\mathbf{u}'}$)

Substituting Eq. (247) into Eq. (246) gives:

$$\nabla^2\Phi = \frac{1}{c^2}\left\{\left(U_\infty\hat{\mathbf{e}}_x+\nabla\Phi\right)\cdot\left[\left(\left(U_\infty\hat{\mathbf{e}}_x+\nabla\Phi\right)\cdot\nabla\right)\left(U_\infty\hat{\mathbf{e}}_x+\nabla\Phi\right)\right]\right\} \tag{248}$$

The speed of sound, c, should also be written in terms of a velocity perturbation. This is accomplished by recalling that for a steady, isentropic flow, the stagnation enthalpy will remain constant:

$$h_0 = h_{0,\infty} \quad\Rightarrow\quad h+\tfrac{1}{2}U^2 = h_\infty+\tfrac{1}{2}U_\infty^2$$

where h and U are the local enthalpy and velocity. If we consider the fluid to be a perfect gas, then:

$$\Delta h = c_p\Delta T = \frac{\gamma R\Delta T}{\gamma-1} = \frac{\left(\Delta c\right)^2}{\gamma-1}$$

so that:

$$\frac{c^2}{\gamma-1}+\tfrac{1}{2}U^2 = \frac{c_\infty^2}{\gamma-1}+\tfrac{1}{2}U_\infty^2$$

$$c^2 = c_\infty^2\left[1+\frac{\gamma-1}{2}\left(\mathrm{Ma}_\infty^2-\frac{U^2}{c_\infty^2}\right)\right] \tag{249}$$

where

$$U^2 = \nabla\phi\cdot\nabla\phi = \left(U_\infty\hat{\mathbf{e}}_x+\nabla\Phi\right)\cdot\left(U_\infty\hat{\mathbf{e}}_x+\nabla\Phi\right)$$

Substituting Eq. (249) into Eq. (248) and expanding (refer to the Appendix):

$$
\begin{aligned}
&\left(1-\mathrm{Ma}_\infty^2\right)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}= \\
&\quad \left(\frac{u_x'}{U_\infty}\right)\mathrm{Ma}_\infty^2\left[(\gamma+1)\frac{\partial^2\Phi}{\partial x^2}+(\gamma-1)\left(\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right)\right]+2\,\mathrm{Ma}_\infty^2\left(\frac{u_y'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial x\partial y}+2\,\mathrm{Ma}_\infty^2\left(\frac{u_z'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial x\partial z} \\
&\quad +\tfrac{1}{2}(\gamma+1)\mathrm{Ma}_\infty^2\left[\left(\frac{u_x'}{U_\infty}\right)^2\frac{\partial^2\Phi}{\partial x^2}+\left(\frac{u_y'}{U_\infty}\right)^2\frac{\partial^2\Phi}{\partial y^2}+\left(\frac{u_z'}{U_\infty}\right)^2\frac{\partial^2\Phi}{\partial z^2}\right] \\
&\quad +\tfrac{1}{2}(\gamma-1)\mathrm{Ma}_\infty^2\left\{\left(\frac{u_x'}{U_\infty}\right)^2\left[\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right]+\left(\frac{u_y'}{U_\infty}\right)^2\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\right]+\left(\frac{u_z'}{U_\infty}\right)^2\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right]\right\} \\
&\quad +2\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\left(\frac{u_y'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial y\partial x}+2\mathrm{Ma}_\infty^2\left(\frac{u_y'}{U_\infty}\right)\left(\frac{u_z'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial z\partial y}+2\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\left(\frac{u_z'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial z\partial x}
\end{aligned}
$$

Note that the <u>small</u> perturbation assumption has not been used in deriving the previous equation. Now, if we assume that the velocity perturbations are indeed small, i.e.:

$$
\begin{aligned}
&\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right),\mathrm{Ma}_\infty^2\left(\frac{u_y'}{U_\infty}\right),\mathrm{Ma}_\infty^2\left(\frac{u_z'}{U_\infty}\right)= 1 \\
&\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)^2,\mathrm{Ma}_\infty^2\left(\frac{u_y'}{U_\infty}\right)^2,\mathrm{Ma}_\infty^2\left(\frac{u_z'}{U_\infty}\right)^2= 1 \\
&\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\left(\frac{u_y'}{U_\infty}\right),\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\left(\frac{u_z'}{U_\infty}\right),\mathrm{Ma}_\infty^2\left(\frac{u_y'}{U_\infty}\right)\left(\frac{u_z'}{U_\infty}\right)= 1
\end{aligned}
\tag{250}
$$

then the PDE simplifies to:

$$
\boxed{\left(1-\mathrm{Ma}_\infty^2\right)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=(\gamma+1)\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial x^2}}
\tag{251}
$$

Eq of Motion for Small Perturbations when $\mathrm{Ma}_\infty \approx 1$

Note that the $\partial^2\Phi/\partial x^2$ term on the RHS has been retained in the previous equation. This is because when Ma_∞ is near unity, the $\partial^2\Phi/\partial x^2$ term on the LHS may be of the same order of magnitude as the RHS and thus the RHS cannot be neglected. Thus, Eq. (251) is the appropriate form for the equation of motion when the free stream Mach number is near unity.

Note that Eq. (251) is non-linear. If we can assume further that:

$$
\left|\frac{\mathrm{Ma}_\infty^2}{1-\mathrm{Ma}_\infty^2}\left(\frac{u_x'}{U_\infty}\right)\right|= 1
\tag{252}
$$

then Eq. (251) simplifies to:

$$
\boxed{\left(1-\mathrm{Ma}_\infty^2\right)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0}
\tag{253}
$$

Eq of Motion for Small Perturbations, Ma_∞ not near unity

Notes:

1. Note that the assumptions (Eq. (250)) used in deriving Eq. (253) also indicate that the Mach number cannot be too large (e.g., we can't use Eq. (253) to model hypersonic flows).

2. Equation (253) is a linear PDE. This means that we can use the principle of superposition to add together solutions of the equation to form new solutions.

3. It is instructive to examine the mathematics of Eq. (253) in greater detail. For simplicity, let's consider only 2D flows. The general form for a 2nd order, linear PDE (with x and y as the independent variables) is:

$$A\frac{\partial^2\Phi}{\partial x^2}+B\frac{\partial^2\Phi}{\partial x\partial y}+C\frac{\partial^2\Phi}{\partial y^2}+D\frac{\partial\Phi}{\partial x}+E\frac{\partial\Phi}{\partial y}+F\Phi=G$$

The behavior of the PDE will vary significantly depending on the value of its principle part, which is given by:

$$B^2-4AC\begin{cases}<0 & \text{elliptic PDE}\\ =0 & \text{parabolic PDE}\\ >0 & \text{hyperbolic PDE}\end{cases}$$

The details of the differences in behavior between the three types of PDEs will not be examined here except for dependence on boundary conditions. For an elliptic PDE, the solution at a particular point in the domain will depend on all of the boundary conditions. In the language of mathematics, this is the same as saying that there are no real characteristic directions (we will discuss characteristic curves later in the course). The prototype elliptic PDE is Laplace's equation:

$$\nabla^2\phi=0$$

For a hyperbolic PDE, the solution at a particular point in the domain will depend only on a particular region of the boundary conditions (termed the **zone of dependence**). Furthermore, the solution at that point will have an effect only on a particular region termed the **zone of influence**. Mathematically speaking, there are two real characteristic directions for hyperbolic PDEs. The prototype equation for hyperbolic PDEs is the wave equation:

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

We will not discuss parabolic PDEs here since we will only be concerned with elliptic and hyperbolic PDEs in the following analyses. In addition, we already stated that Eq. (253) is for the case when Ma_∞ is not near a value of one, and -4AC only equals zero when $\mathrm{Ma}_\infty = 1$.

How does all of this relate to the small perturbation equation of motion? Consider Eq. (253) simplified for 2D flow:

$$\left(1-\mathrm{Ma}_\infty^2\right)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0 \quad \textbf{(254)}$$

The behavior of the flow will vary significantly depending on the Mach number:

$$\mathrm{Ma}_\infty\begin{cases}<1 & \text{elliptic PDE}\\ >1 & \text{hyperbolic PDE}\end{cases} \qquad A = 1-\mathrm{Ma}_\infty^2,\ B = 0,\ C = 1 \Rightarrow B^2-4AC = 4(\mathrm{Ma}_\infty^2 - 1)$$

Recall that when $\mathrm{Ma}_\infty \approx 1$ (the transonic region), we must use Eq. (251) rather than Eq. (253). The analysis in the transonic region is complex due to the non-linearity of the governing equation.

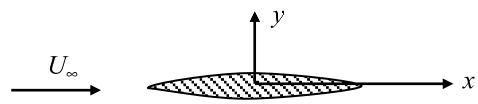
Based on our previous analyses of 1D, compressible flow, the behavior of subsonic and supersonic flows can be quite different. The same holds true here even though the governing equation (Eq. (254)) looks fairly simple. A deeper investigation of the differences between subsonic and supersonic flows will be given in a following section.

4. Since we linearized the governing PDE using the small perturbation assumption, we should also linearize the boundary conditions. The appropriate boundary condition at a solid boundary for an inviscid flow is not the no-slip condition since the fluid can slip tangent to the surface. Instead, we specify that the flow will not penetrate the solid object, i.e.:

$$\frac{\partial \phi}{\partial n} = 0$$

where n is the direction normal to the object's surface. This is the same as stating that the surface of the object is a streamline for the flow (recall that there is no flow across a streamline). Thus, we can write:

$$\left.\frac{dy}{dx}\right|_{\text{surface}} = \frac{u_y'}{U_\infty + u_x'} = \frac{u_y'/U_\infty}{1 + u_x'/U_\infty}$$



Since we're assuming that the perturbation velocities are small in comparison with the free stream velocity, we have:

$$\left.\frac{dy}{dx}\right|_{\text{surface}} \approx \frac{\left.u_y'\right|_{\text{surface}}}{U_\infty} \quad \left(\frac{u_x'}{U_\infty} = 1\right)$$

Since the object is considered to be slender, we can use a Taylor series to show that the y-perturbation velocity at y=0 can be used rather using the velocity on the surface:

$$u_y'(x_s, y_s) = u_y'(x_s, 0) + \left.\frac{\partial u_y'}{\partial y}\right|_{(x_s,0)} y_s + \mathrm{L}$$
$$\approx u_y'(x_s, 0)$$

where (x_s, y_s) are the coordinates of the object surface and y_s is assumed to be very small. Thus, the appropriate boundary condition at the object surface becomes:

$$\boxed{\left.\frac{dy}{dx}\right|_{\text{surface}} \approx \frac{\left.u_y'\right|_{(x_s,0)}}{U_\infty}} \tag{255}$$

5. An additional quantity that is often helpful when analyzing external flows is the **pressure coefficient, C_p,** which is defined as:

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}$$

This can be written for a perfect gas as:

$$C_p \equiv \frac{p/p_\infty - 1}{\frac{1}{2}\gamma \mathrm{Ma}_\infty^2} \tag{256}$$

The pressure ratio can be written in terms of the free stream and perturbation velocities by using the energy equation for an isentropic (2D) flow:

$$T + \frac{1}{2c_p}\left[\left(U_\infty + u_x'\right)^2 + \left(u_y'\right)^2\right] = T_\infty + \frac{1}{2c_p}U_\infty^2$$

where

$$c_p = \frac{\gamma R}{\gamma - 1}$$

so that

$$T + \frac{\gamma-1}{2\gamma R}\left[U_\infty^2 + 2U_\infty u_x' + \left(u_x'\right)^2 + \left(u_y'\right)^2\right] = T_\infty + \frac{\gamma-1}{2\gamma R}U_\infty^2$$

$$\frac{T}{T_\infty} = 1 - \tfrac{1}{2}(\gamma-1)\mathrm{Ma}_\infty^2\left[2\frac{u_x'}{U_\infty} + \left(\frac{u_x'}{U_\infty}\right)^2 + \left(\frac{u_y'}{U_\infty}\right)^2\right]$$

For an isentropic flow of a perfect gas:

$$\frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\gamma/(\gamma-1)} = \left\{1 - \tfrac{1}{2}(\gamma-1)\mathrm{Ma}_\infty^2\left[2\frac{u_x'}{U_\infty} + \left(\frac{u_x'}{U_\infty}\right)^2 + \left(\frac{u_y'}{U_\infty}\right)^2\right]\right\}^{\gamma/(\gamma-1)} \tag{257}$$

The previous relation may be simplified further by use of the binomial theorem, which states that for x<1:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

Since the perturbations are small:

$$\left\{1 - \tfrac{1}{2}(\gamma-1)\mathrm{Ma}_\infty^2\left[2\frac{u_x'}{U_\infty} + \left(\frac{u_x'}{U_\infty}\right)^2 + \left(\frac{u_y'}{U_\infty}\right)^2\right]\right\}^{\gamma/(\gamma-1)} =$$

$$1 - \tfrac{1}{2}\gamma\mathrm{Ma}_\infty^2\left[2\frac{u_x'}{U_\infty} + \left(\frac{u_x'}{U_\infty}\right)^2 + \left(\frac{u_y'}{U_\infty}\right)^2\right] + \cdots$$

Using the assumption that the perturbation velocities are small in comparison to the free stream velocity, the remaining terms in the series can be neglected. Substituting this result into Eq. (257) and then substituting this result back into the definition of the pressure coefficient (Eq. (256)) gives:

$$C_p = \frac{-\tfrac{1}{2}\gamma \mathrm{Ma}_\infty^2 \left[2\frac{u_x'}{U_\infty} + \left(\frac{u_x'}{U_\infty}\right)^2 + \left(\frac{u_y'}{U_\infty}\right)^2 \right]}{\tfrac{1}{2}\gamma \mathrm{Ma}_\infty^2} = -\left[2\frac{u_x'}{U_\infty} + \left(\frac{u_x'}{U_\infty}\right)^2 + \left(\frac{u_y'}{U_\infty}\right)^2 \right]$$

Again, using the small perturbation assumption we observe that the squared terms in the previous equation will be very small compared to the remaining term so that the pressure coefficient is given by:

$$\boxed{C_p = -2\left(\frac{u_x'}{U_\infty}\right)} \tag{258}$$

6. Equation (254) for a subsonic flow may be reduced to Laplace's equation using an appropriate transformation of variables. This is useful since there has been a large body of work in determining the solution to Laplace's equation for various boundary conditions. In particular, the equation of motion for an incompressible flow is Laplace's equation.

   To transform Eq. (254), define a new coordinate system using the variables (ξ, η) and a transformed perturbation potential, $\bar{\Phi}$, in the following manner:

$$\begin{aligned} &\xi = x \\ &\eta = \beta y \qquad \text{where } \beta = \sqrt{1\text{-}\mathrm{Ma}_\infty^2} \\ &\bar{\Phi} = \beta \Phi \end{aligned} \tag{259}$$

   Note that:

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\beta}\frac{\partial \bar{\Phi}}{\partial x}\right) = \left(\frac{\partial}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial}{\partial \eta}\underbrace{\frac{\partial \eta}{\partial x}}_{=0}\right)\left[\frac{1}{\beta}\left(\frac{\partial \bar{\Phi}}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial \bar{\Phi}}{\partial \eta}\underbrace{\frac{\partial \eta}{\partial x}}_{=0}\right)\right] \\ &= \frac{1}{\beta}\frac{\partial^2 \bar{\Phi}}{\partial \xi^2} \end{aligned} \tag{260}$$

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{\beta}\frac{\partial \bar{\Phi}}{\partial y}\right) = \left(\frac{\partial}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial y}}_{=0} + \frac{\partial}{\partial \eta}\underbrace{\frac{\partial \eta}{\partial y}}_{=\beta}\right)\left[\frac{1}{\beta}\left(\frac{\partial \bar{\Phi}}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial y}}_{=0} + \frac{\partial \bar{\Phi}}{\partial \eta}\underbrace{\frac{\partial \eta}{\partial y}}_{=\beta}\right)\right] \\ &= \beta\frac{\partial^2 \bar{\Phi}}{\partial \eta^2} \end{aligned} \tag{261}$$

   Substituting Eqs. (260) and (261) into Eq. (254) gives:

$$\beta^2\left(\frac{1}{\beta}\frac{\partial^2 \bar{\Phi}}{\partial \xi^2}\right) + \left(\beta\frac{\partial^2 \bar{\Phi}}{\partial \eta^2}\right) = 0$$

$$\boxed{\frac{\partial^2 \bar{\Phi}}{\partial \xi^2} + \frac{\partial^2 \bar{\Phi}}{\partial \eta^2} = 0} \tag{262}$$

   This is Laplace's Equation in the (ξ, η) plane! This is the same equation of motion as for an incompressible fluid ($\nabla^2\phi = 0$).

   Since we transformed the governing equation, we also need to transform boundary conditions. Let the shape of a boundary surface in the (x, y) plane be described by:

$$y_s = f(x_s) \tag{263}$$

   and the corresponding surface in the (ξ, η) plane be:

$$\eta_s = \bar{f}(\xi_s) \tag{264}$$

Recall from Eq. (255) that the boundary condition at a surface in the actual flow (in the (x, y) plane) is:

$$u_y'\Big|_{(x_s,0)} = \frac{\partial \Phi}{\partial y}\bigg|_{(x_s,0)} = U_\infty \frac{dy}{dx}\bigg|_{\text{surface}} \tag{265}$$

The same boundary condition expressed in the transformed flow, i.e., in the (ξ, η) plane, is:

$$u_\eta'\Big|_{(\xi_s,0)} = \frac{\partial \bar{\Phi}}{\partial \eta}\bigg|_{(\xi_s,0)} = U_\infty \frac{d\bar{f}}{d\xi}\bigg|_{\text{surface}} \tag{266}$$

Note however that:

$$\frac{\partial \Phi}{\partial y}\bigg|_{(x_s,0)} = \left(\frac{\partial}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial y}}_{=0} + \frac{\partial}{\partial \eta}\underbrace{\frac{\partial \eta}{\partial y}}_{=\beta}\right)\left(\frac{1}{\beta}\bar{\Phi}\right)\Bigg|_{(\xi_s,0)} = \frac{\partial \bar{\Phi}}{\partial \eta}\bigg|_{(\xi_s,0)} \tag{267}$$

Hence, since the left hand sides of Eqs. (265) and (266) are equal, we also have:

$$\frac{dy}{dx}\bigg|_{\text{surface}} = \frac{df}{dx}\bigg|_{\text{surface}} = \frac{d\bar{f}}{d\xi}\bigg|_{\text{surface}} \tag{268}$$

Since the slope of the actual boundary surface is $(df/dx)_{\text{surface}}$, Eq. (268) tells us that <u>the shape of the boundary surface in the transformed plane $((\xi, \eta))$ is the same as that in the real $((x, y))$ plane</u>. Furthermore, since the boundary surface is the same in the transformed plane, and since Eq. (262) is the equation of motion for an incompressible fluid, then <u>$\bar{\Phi}$ must be the perturbation velocity potential for an incompressible flow past the same boundary surface</u>. Hence, if we can determine $\bar{\Phi}$ assuming incompressible flow over the surface, the corresponding compressible flow solution can be determined using the scaling relationships given in Eq. (259).

Recall that the pressure coefficient (Eq. (258)) is given by:

$$C_p = -2\left(\frac{u_x'}{U_\infty}\right) = -\frac{2}{U_\infty}\frac{\partial \Phi}{\partial x} = -\frac{2}{U_\infty}\frac{1}{\beta}\frac{\partial \bar{\Phi}}{\partial x} = \frac{1}{\beta}\underbrace{\left(-\frac{2}{U_\infty}\frac{\partial \bar{\Phi}}{\partial \xi}\right)}_{=\bar{C}_p} \tag{269}$$

Note that the last term in parentheses in the previous equation is the pressure coefficient determined from the incompressible (i.e., transformed plane) solution, $\bar{C}_{p0}$. Hence, the pressure coefficient for the subsonic, compressible flow can be determined by scaling the incompressible pressure coefficient by:

$$\boxed{C_p = \frac{\bar{C}_p}{\sqrt{1-\text{Ma}_\infty^2}}} \tag{270}$$

This scaling relationship is known as the **Prandtl-Glauert Rule**.

Notes:

a. It can also be shown that the lift and moment coefficients are scaled in a similar manner for linearized, subsonic compressible flow (note that the drag coefficient is always zero for subsonic potential flow):

$$C_L = \frac{\bar{C}_L}{\sqrt{1-\text{Ma}_\infty^2}} \tag{271}$$

$$C_M = \frac{\bar{C}_M}{\sqrt{1-\text{Ma}_\infty^2}} \tag{272}$$

b. The effect of compressibility on the flow perturbation velocities can be determined from Eq. (259):

$$\mathbf{u}' = \nabla\Phi = \frac{1}{\beta}\underbrace{\nabla\bar{\Phi}}_{=\bar{\mathbf{u}}'} = \frac{\bar{\mathbf{u}}'}{\sqrt{1-\text{Ma}_\infty^2}} \tag{273}$$

Compressibility acts to increase the magnitude of the perturbations (since the denominator is always less than one). Hence, a disturbance to a compressible flow reaches further into the flow than for an incompressible flow.

c. The Prandtl-Glauert rule tends to underpredict the pressure coefficient for real flows. It is only reasonably accurate up to a Mach number of about 0.7. This is because the rule is based on linearization of the governing equations. Other rules have been proposed (e.g., the Karman-Tsien rule and the Laitone rule) that incorporate non-linear flow effects to give improved predictions to the pressure coefficient.

d. Recall that for an isentropic flow that

$$p = cT^{\frac{\gamma}{\gamma-1}} \Rightarrow dp = \frac{c\gamma}{\gamma-1}dT^{\frac{1}{\gamma-1}}$$

so that a decrease in pressure corresponds to a decrease in temperature. So, for subsonic flow over the suction side of an airfoil near sonic conditions, not only will the pressure drop, but the temperature will drop as well. This drop in temperature in humid conditions can result in water vapor turning to liquid water, i.e., condensation. This effect is often visible on the surface of high speed aircraft in humid environments.


(Images from: http://en.wikipedia.org/wiki/Prandtl%E2%80%93Glauert_singularity)

Appendix

$$\nabla^2\Phi = \frac{1}{c^2}\left\{\left(U_\infty \hat{\mathbf{e}}_x + \nabla\Phi\right)\cdot\left[\left(\left(U_\infty \hat{\mathbf{e}}_x + \nabla\Phi\right)\cdot\nabla\right)\left(U_\infty \hat{\mathbf{e}}_x + \nabla\Phi\right)\right]\right\}$$

$$= \frac{1}{c^2}\left[\left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\hat{\mathbf{e}}_x + \frac{\partial\Phi}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z\right]\cdot$$

$$\left[\left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial}{\partial y} + \frac{\partial\Phi}{\partial z}\frac{\partial}{\partial z}\right]\left[\left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\hat{\mathbf{e}}_x + \frac{\partial\Phi}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z\right]$$

$$= \frac{1}{c^2}\left[\left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\hat{\mathbf{e}}_x + \frac{\partial\Phi}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z\right]\cdot$$

$$\left[\begin{array}{l}\left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\frac{\partial^2\Phi}{\partial x^2}\hat{\mathbf{e}}_x + \left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\frac{\partial^2\Phi}{\partial x\partial y}\hat{\mathbf{e}}_y + \left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\frac{\partial^2\Phi}{\partial x\partial z}\hat{\mathbf{e}}_z + \\ \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial x}\hat{\mathbf{e}}_x + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y^2}\hat{\mathbf{e}}_y + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial z}\hat{\mathbf{e}}_z + \\ \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial x}\hat{\mathbf{e}}_x + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial y}\hat{\mathbf{e}}_y + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z^2}\hat{\mathbf{e}}_z\end{array}\right]$$

$$= \frac{1}{c^2}\left\{\begin{array}{l}\left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\left[\left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial x} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial x}\right] + \\ \frac{\partial\Phi}{\partial y}\left[\left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y^2} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial y}\right] + \\ \frac{\partial\Phi}{\partial z}\left[\left(U_\infty + \frac{\partial\Phi}{\partial x}\right)\frac{\partial^2\Phi}{\partial x\partial z} + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial z} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z^2}\right]\end{array}\right\}$$

$$\nabla^2\Phi = \frac{1}{c^2}\left\{\begin{array}{l}U_\infty^2\frac{\partial^2\Phi}{\partial x^2} + 2U_\infty\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2} + \\ U_\infty\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial x} + \frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial x} + U_\infty\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial x} + \frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial x} + \\ U_\infty\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x\partial y} + \left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial y^2} + \frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial y} + \\ U_\infty\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial x\partial z} + \frac{\partial\Phi}{\partial z}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x\partial z} + \frac{\partial\Phi}{\partial z}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial z} + \left(\frac{\partial\Phi}{\partial z}\right)^2\frac{\partial^2\Phi}{\partial z^2}\end{array}\right\}$$

$$= \frac{1}{c^2}\left\{\begin{array}{l}U_\infty^2\frac{\partial^2\Phi}{\partial x^2} + 2U_\infty\left[\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial x\partial z}\right] + \\ \left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial y^2} + \left(\frac{\partial\Phi}{\partial z}\right)^2\frac{\partial^2\Phi}{\partial z^2} \\ 2\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial x} + 2\frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial y} + 2\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial x}\end{array}\right\}$$

$$\nabla^2\Phi = \frac{1}{c^2}\left\{\begin{array}{l} U_\infty^2 \dfrac{\partial^2\Phi}{\partial x^2} + 2U_\infty\left[\dfrac{\partial\Phi}{\partial x}\dfrac{\partial^2\Phi}{\partial x^2} + \dfrac{\partial\Phi}{\partial y}\dfrac{\partial^2\Phi}{\partial x\partial y} + \dfrac{\partial\Phi}{\partial z}\dfrac{\partial^2\Phi}{\partial x\partial z}\right] + \\ \left(\dfrac{\partial\Phi}{\partial x}\right)^2\dfrac{\partial^2\Phi}{\partial x^2} + \left(\dfrac{\partial\Phi}{\partial y}\right)^2\dfrac{\partial^2\Phi}{\partial y^2} + \left(\dfrac{\partial\Phi}{\partial z}\right)^2\dfrac{\partial^2\Phi}{\partial z^2} \\ 2\dfrac{\partial\Phi}{\partial x}\dfrac{\partial\Phi}{\partial y}\dfrac{\partial^2\Phi}{\partial y\partial x} + 2\dfrac{\partial\Phi}{\partial y}\dfrac{\partial\Phi}{\partial z}\dfrac{\partial^2\Phi}{\partial z\partial y} + 2\dfrac{\partial\Phi}{\partial x}\dfrac{\partial\Phi}{\partial z}\dfrac{\partial^2\Phi}{\partial z\partial x} \end{array}\right\} \tag{274}$$

Recall that from the speed of sound equation:

$$\begin{aligned} c^2 &= c_\infty^2\left[1 + \frac{\gamma-1}{2}\left(\mathrm{Ma}_\infty^2 - \frac{U^2}{c_\infty^2}\right)\right] \\ &= c_\infty^2\left[1 + \frac{\gamma-1}{2}\left(\mathrm{Ma}_\infty^2 - \frac{\left|U_\infty\hat{\mathbf{e}}_x + \nabla\Phi\right|^2}{c_\infty^2}\right)\right] \\ &= c_\infty^2\left[1 + \frac{\gamma-1}{2}\left(\mathrm{Ma}_\infty^2 - \frac{U_\infty^2 + 2U_\infty\dfrac{\partial\Phi}{\partial x} + \left(\dfrac{\partial\Phi}{\partial x}\right)^2 + \left(\dfrac{\partial\Phi}{\partial y}\right)^2 + \left(\dfrac{\partial\Phi}{\partial z}\right)^2}{c_\infty^2}\right)\right] \\ &= c_\infty^2\left\{1 - \frac{\gamma-1}{2}\left(\frac{2U_\infty}{c_\infty^2}\frac{\partial\Phi}{\partial x} + \frac{1}{c_\infty^2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right]\right)\right\} \end{aligned}$$

$$c^2 = c_\infty^2\left\{1 - \frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2}\left(\frac{2}{U_\infty}\frac{\partial\Phi}{\partial x} + \frac{1}{U_\infty^2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right]\right)\right\} \tag{275}$$

Substituting (275) into Eq. (274) gives,

$$c_\infty^2\left\{1-\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2}\left(\frac{2}{U_\infty}\frac{\partial\Phi}{\partial x}+\frac{1}{U_\infty^2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]\right)\right\}\nabla^2\Phi=$$

$$\left\{\begin{array}{l} U_\infty^2\dfrac{\partial^2\Phi}{\partial x^2}+2U_\infty\left[\dfrac{\partial\Phi}{\partial x}\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial\Phi}{\partial y}\dfrac{\partial^2\Phi}{\partial x\partial y}+\dfrac{\partial\Phi}{\partial z}\dfrac{\partial^2\Phi}{\partial x\partial z}\right]+ \\ \left(\dfrac{\partial\Phi}{\partial x}\right)^2\dfrac{\partial^2\Phi}{\partial x^2}+\left(\dfrac{\partial\Phi}{\partial y}\right)^2\dfrac{\partial^2\Phi}{\partial y^2}+\left(\dfrac{\partial\Phi}{\partial z}\right)^2\dfrac{\partial^2\Phi}{\partial z^2} \\ 2\dfrac{\partial\Phi}{\partial x}\dfrac{\partial\Phi}{\partial y}\dfrac{\partial^2\Phi}{\partial y\partial x}+2\dfrac{\partial\Phi}{\partial y}\dfrac{\partial\Phi}{\partial z}\dfrac{\partial^2\Phi}{\partial z\partial y}+2\dfrac{\partial\Phi}{\partial x}\dfrac{\partial\Phi}{\partial z}\dfrac{\partial^2\Phi}{\partial z\partial x} \end{array}\right\}$$

$$\left\{1-\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2}\left(\frac{2}{U_\infty}\frac{\partial\Phi}{\partial x}+\frac{1}{U_\infty^2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]\right)\right\}\nabla^2\Phi=$$

$$\mathrm{Ma}_\infty^2\left\{\begin{array}{l} \dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{2}{U_\infty}\left[\dfrac{\partial\Phi}{\partial x}\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial\Phi}{\partial y}\dfrac{\partial^2\Phi}{\partial x\partial y}+\dfrac{\partial\Phi}{\partial z}\dfrac{\partial^2\Phi}{\partial x\partial z}\right]+ \\ \dfrac{1}{U_\infty^2}\left[\left(\dfrac{\partial\Phi}{\partial x}\right)^2\dfrac{\partial^2\Phi}{\partial x^2}+\left(\dfrac{\partial\Phi}{\partial y}\right)^2\dfrac{\partial^2\Phi}{\partial y^2}+\left(\dfrac{\partial\Phi}{\partial z}\right)^2\dfrac{\partial^2\Phi}{\partial z^2}\right]+ \\ \dfrac{2}{U_\infty^2}\left[\dfrac{\partial\Phi}{\partial x}\dfrac{\partial\Phi}{\partial y}\dfrac{\partial^2\Phi}{\partial y\partial x}+\dfrac{\partial\Phi}{\partial y}\dfrac{\partial\Phi}{\partial z}\dfrac{\partial^2\Phi}{\partial z\partial y}+\dfrac{\partial\Phi}{\partial x}\dfrac{\partial\Phi}{\partial z}\dfrac{\partial^2\Phi}{\partial z\partial x}\right] \end{array}\right\}$$

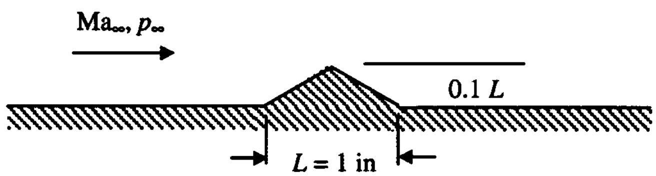
$$\begin{aligned}
\nabla^2\Phi=&\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{U_\infty}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2}+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{U_\infty}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial y^2}+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{U_\infty}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial z^2}\\
&+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2}+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial y^2}+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial z^2}\\
&+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial x^2}+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial y^2}+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial z^2}\\
&+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\left(\frac{\partial\Phi}{\partial z}\right)^2\frac{\partial^2\Phi}{\partial x^2}+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\left(\frac{\partial\Phi}{\partial z}\right)^2\frac{\partial^2\Phi}{\partial y^2}+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\left(\frac{\partial\Phi}{\partial z}\right)^2\frac{\partial^2\Phi}{\partial z^2}\\
&+\mathrm{Ma}_\infty^2\frac{\partial^2\Phi}{\partial x^2}+\frac{2\,\mathrm{Ma}_\infty^2}{U_\infty}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2}+\frac{2\,\mathrm{Ma}_\infty^2}{U_\infty}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y}+\frac{2\,\mathrm{Ma}_\infty^2}{U_\infty}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial x\partial z}\\
&+\frac{\mathrm{Ma}_\infty^2}{U_\infty^2}\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2}+\frac{\mathrm{Ma}_\infty^2}{U_\infty^2}\left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial y^2}+\frac{\mathrm{Ma}_\infty^2}{U_\infty^2}\left(\frac{\partial\Phi}{\partial z}\right)^2\frac{\partial^2\Phi}{\partial z^2}\\
&+\frac{2\mathrm{Ma}_\infty^2}{U_\infty^2}\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial x}+\frac{2\mathrm{Ma}_\infty^2}{U_\infty^2}\frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial y}+\frac{2\mathrm{Ma}_\infty^2}{U_\infty^2}\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial x}
\end{aligned}$$

$$\nabla^2\Phi - \mathrm{Ma}_\infty^2 \frac{\partial^2\Phi}{\partial x^2} = \left[\frac{\mathrm{Ma}_\infty^2(\gamma+1)}{U_\infty}\right]\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2} + \left[\frac{\mathrm{Ma}_\infty^2(\gamma+1)}{2U_\infty^2}\right]\left[\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial y^2} + \left(\frac{\partial\Phi}{\partial z}\right)^2\frac{\partial^2\Phi}{\partial z^2}\right]$$
$$+\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{U_\infty}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial y^2} + \frac{\mathrm{Ma}_\infty^2(\gamma-1)}{U_\infty}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial z^2} + \frac{2\ \mathrm{Ma}_\infty^2}{U_\infty}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y} + \frac{2\ \mathrm{Ma}_\infty^2}{U_\infty}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial x\partial z}$$
$$+\left[\frac{\mathrm{Ma}_\infty^2(\gamma-1)}{2U_\infty^2}\right]\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2\left[\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right] + \left(\frac{\partial\Phi}{\partial y}\right)^2\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\right] + \left(\frac{\partial\Phi}{\partial z}\right)^2\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right]\right\}$$
$$+\frac{2\mathrm{Ma}_\infty^2}{U_\infty^2}\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial x} + \frac{2\mathrm{Ma}_\infty^2}{U_\infty^2}\frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial y} + \frac{2\mathrm{Ma}_\infty^2}{U_\infty^2}\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z\partial x}$$

$$\left(1-\mathrm{Ma}_\infty^2\right)\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} =$$
$$\mathrm{Ma}_\infty^2(\gamma+1)\left(\frac{u_x'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial x^2} + \tfrac{1}{2}(\gamma+1)\mathrm{Ma}_\infty^2\left[\left(\frac{u_x'}{U_\infty}\right)^2\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{u_y'}{U_\infty}\right)^2\frac{\partial^2\Phi}{\partial y^2} + \left(\frac{u_z'}{U_\infty}\right)^2\frac{\partial^2\Phi}{\partial z^2}\right]$$
$$+(\gamma-1)\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\left[\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right] + 2\ \mathrm{Ma}_\infty^2\left(\frac{u_y'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial x\partial y} + 2\ \mathrm{Ma}_\infty^2\left(\frac{u_z'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial x\partial z}$$
$$+\tfrac{1}{2}(\gamma-1)\mathrm{Ma}_\infty^2\left\{\left(\frac{u_x'}{U_\infty}\right)^2\left[\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right] + \left(\frac{u_y'}{U_\infty}\right)^2\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\right] + \left(\frac{u_z'}{U_\infty}\right)^2\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right]\right\}$$
$$+2\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\left(\frac{u_y'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial y\partial x} + 2\mathrm{Ma}_\infty^2\left(\frac{u_y'}{U_\infty}\right)\left(\frac{u_z'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial z\partial y} + 2\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\left(\frac{u_z'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial z\partial x}$$

$$\boxed{\begin{aligned}&\left(1-\mathrm{Ma}_\infty^2\right)\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} =\\
&\quad\left(\frac{u_x'}{U_\infty}\right)\mathrm{Ma}_\infty^2\left[(\gamma+1)\frac{\partial^2\Phi}{\partial x^2} + (\gamma-1)\left(\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right)\right] + 2\ \mathrm{Ma}_\infty^2\left(\frac{u_y'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial x\partial y} + 2\ \mathrm{Ma}_\infty^2\left(\frac{u_z'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial x\partial z}\\
&\quad+\tfrac{1}{2}(\gamma+1)\mathrm{Ma}_\infty^2\left[\left(\frac{u_x'}{U_\infty}\right)^2\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{u_y'}{U_\infty}\right)^2\frac{\partial^2\Phi}{\partial y^2} + \left(\frac{u_z'}{U_\infty}\right)^2\frac{\partial^2\Phi}{\partial z^2}\right]\\
&\quad+\tfrac{1}{2}(\gamma-1)\mathrm{Ma}_\infty^2\left\{\left(\frac{u_x'}{U_\infty}\right)^2\left[\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right] + \left(\frac{u_y'}{U_\infty}\right)^2\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\right] + \left(\frac{u_z'}{U_\infty}\right)^2\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right]\right\}\\
&\quad+2\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\left(\frac{u_y'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial y\partial x} + 2\mathrm{Ma}_\infty^2\left(\frac{u_y'}{U_\infty}\right)\left(\frac{u_z'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial z\partial y} + 2\mathrm{Ma}_\infty^2\left(\frac{u_x'}{U_\infty}\right)\left(\frac{u_z'}{U_\infty}\right)\frac{\partial^2\Phi}{\partial z\partial x}\end{aligned}}$$

Consider uniform supersonic flow over a wall in which there exists a bump, as shown in the figure. Assuming linearized, two-dimensional potential flow, calculate the vertical and horizontal components of the force on the bump. Assume Ma$_\infty$=2.0, with p_∞=10 psia.



SOLUTION:

$$\delta = \frac{0.1L}{0.5L} = 0.2 = 11.5^\circ$$

METHOD 1: Oblique Shock & Expansion Fan Relations

region 1:

$$Ma_1 = Ma_\infty = 2.0$$
$$p_1 = p_\infty = 10 \text{ psia}$$

region 2:

$$\left.\begin{array}{l} Ma_1 = 2.0 \\ \delta_{12} = 11.5^\circ \end{array}\right\} \Rightarrow \begin{array}{l} Ma_2 = 1.58 \\ \epsilon_{12} = 40.99^\circ \\ p_2/p_1 = 1.841 \end{array}$$

using oblique shock relations

region 3:

$$Ma_2 = 1.58 \Rightarrow \nu_2 = 14.27^\circ$$

using Prandtl-Meyer expansion fan relation

$$\nu_3 = \nu_2 + \delta_{23} = 14.27^\circ + 2(11.5^\circ) = 37.27^\circ$$

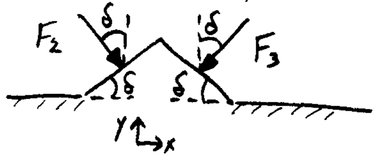
$$\Rightarrow Ma_3 = 2.42$$

$$\frac{p_3}{p_2} = \left(\frac{1+\frac{\gamma-1}{2}Ma_3^2}{1+\frac{\gamma-1}{2}Ma_2^2}\right)^{\frac{\gamma}{1-\gamma}}$$

$$\therefore \frac{p_3}{p_2} = 0.274 \quad \text{using } \begin{array}{l} Ma_2 = 1.58 \\ Ma_3 = 2.42 \end{array}$$

SOLUTION...

- Forces on bump:



$$F_{2x} = p_2(0.1L) \qquad F_{3x} = -p_3(0.1L)$$

$$F_{2y} = -p_2(0.5L) \qquad F_{3y} = -p_3(0.5L)$$

$$F_{x,net} = F_{2x} + F_{3x} = (0.1L)(p_2 - p_3)$$

$$= p_\infty(0.1L)\left(\frac{p_2}{p_\infty} - \frac{p_3}{p_\infty}\right)$$

$$\boxed{\therefore F_{x,net} = 1.34\ \frac{\text{lb}_f}{\text{in}}}$$ using $p_\infty = 10\text{ psia}$, $L = 1\text{ in}$

$$p_2/p_\infty = 1.841$$

$$p_3/p_\infty = (p_3/p_2)\left(\frac{p_2}{p_\infty}\right) = (0.274)(1.841) = 0.504$$

$$F_{y,net} = F_{2y} + F_{3y} = -(0.5L)(p_2 + p_3)$$

$$= -p_\infty(0.5L)\left(\frac{p_2}{p_\infty} + p_3/p_\infty\right)$$

$$\boxed{\therefore F_{y,net} = -11.7\ \frac{\text{lb}_f}{\text{in}}}$$ using the values given above

Solution...

METHOD 2: Small Perturbation Analysis

- Assume the angles of turn are small enough to consider the changes in the flow velocities are small perturbations.

- From the class notes:

$$C_p = \frac{2\left(\frac{dy}{dx}\right)}{\sqrt{Ma_\infty^2-1}} \quad \text{where} \quad \frac{dy}{dx} = \begin{cases} \frac{0.1L}{0.5L} = 0.2 & \text{front half of bump} \\ \frac{-0.1L}{0.5L} = -0.2 & \text{back half of bump} \end{cases}$$

$Ma_\infty = 2.0$

$$\Rightarrow C_p = \begin{cases} 0.231 & \text{front half} \\ -0.231 & \text{back half} \end{cases}$$

but $C_p = \dfrac{p-p_\infty}{\frac{1}{2}\gamma p_\infty Ma_\infty^2} \quad \Rightarrow \quad p = p_\infty\left(1+\frac{1}{2}C_p\gamma Ma_\infty^2\right)$

$\Rightarrow p_{front} = 16.47\text{ psia}$, $p_{back} = 3.532\text{ psia}$ using $p_\infty = 10\text{ psia}$, $\gamma = 1.4$, $Ma_\infty = 2.0$, $C_{p\,front} = 0.231$, $C_{p\,back} = -0.231$

$\therefore F_{x,net} = (0.1L)(p_{front} - p_{back})$

$$\boxed{\therefore F_{x,net} = 1.29\ \frac{\text{lbf}}{\text{in}}}$$

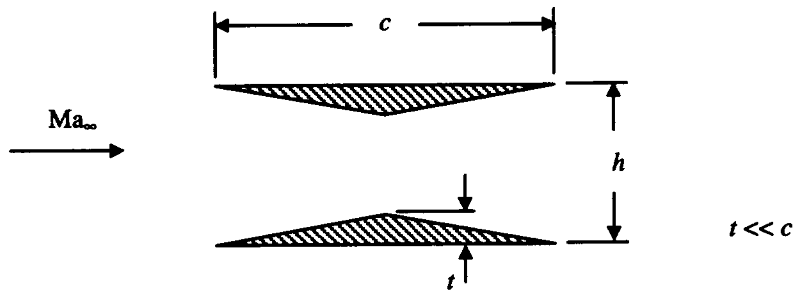
using $L = 1\text{ in}$, $p_{front} = 16.47\text{ psia}$, $p_{back} = 3.532\text{ psia}$

$F_{y,net} = -(0.5L)(p_{front} + p_{back})$

$$\boxed{\therefore F_{y,net} = -10.0\ \frac{\text{lbf}}{\text{in}}}$$

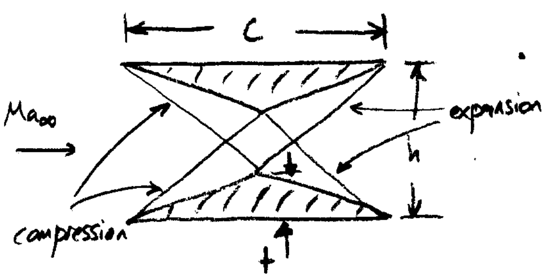
$\therefore$ Using small perturbation analysis gives a difference of 3.7% in the horz. force and 14.5% in the vertical force when compared with the oblique shock/expansion fan analysis. The horz. force error is within reasonable engineering error. The vert. force error may not be acceptable, however, depending on the application.

The Busemann biplane consists of two parallel airfoils, each of which is a half diamond as shown in the sketch. At zero angle of attack the wave drag can be made effectively zero by correctly spacing the airfoils for a given Mach number, Ma_∞. Assuming that the airfoil thickness, t, is much smaller than the chord length, c, determine the ~~correct~~ MAXIMUM spacing, h, for drag cancellation.



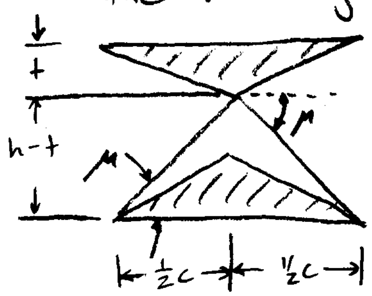
SOLUTION:

- Since $t \ll c$, use linearized flow theory.



- Since we want zero drag, we need to make sure that all of the waves are cancelled within the airfoil (so the pressure on the front and back of the airfoil is the same).

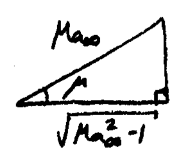
- The waves off the airfoil will be inclined at the Mach angle, μ (since we're assuming small deflections):



From the geometry:

$$\tan\mu = \frac{h-t}{\frac{1}{2}c}$$

where $\sin\mu = \frac{1}{Ma_\infty}$



$$\Rightarrow \tan\mu = \frac{1}{\sqrt{Ma_\infty^2-1}}$$

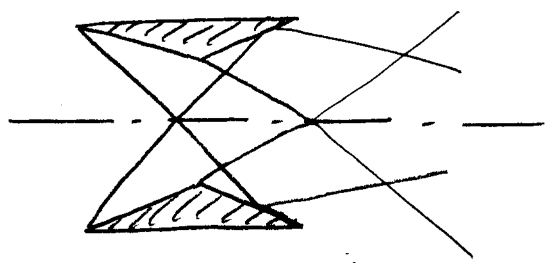
$$\Rightarrow \frac{1}{\sqrt{Ma_\infty^2-1}} = \frac{h-t}{\frac{1}{2}c}$$

$$\boxed{\therefore\ h = t + \frac{c}{2\sqrt{Ma_\infty^2-1}}} \quad \boxed{\approx \frac{c}{2\sqrt{Ma_\infty^2-1}}} \text{ since } t \ll c$$

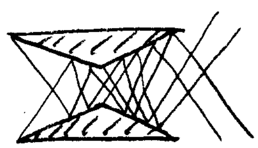
SOLUTION...

NOTE:

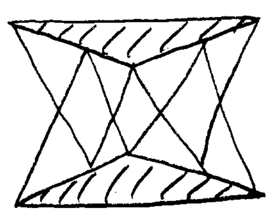
- If the spacing is too large:



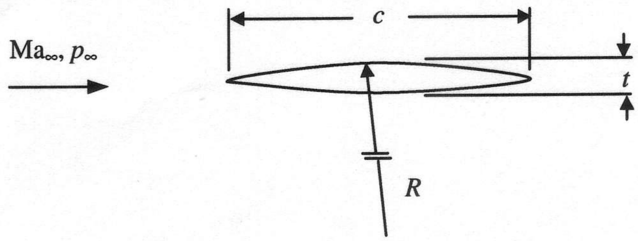
- If the spacing is too small:



- It is possible to have smaller spacings that give zero drag:

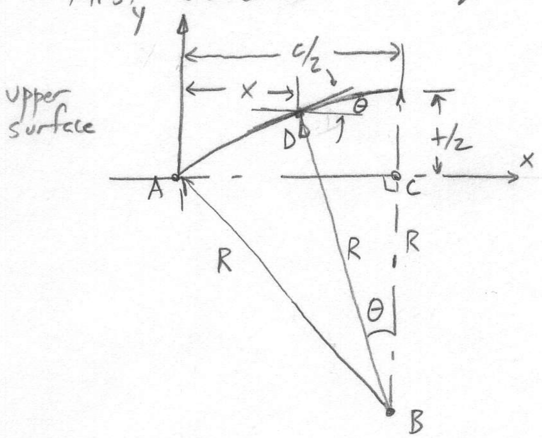


A thin symmetric airfoil of the shape shown in the figure is placed in a flow at zero angle of attack and a Mach number of 2 and a pressure of 80 kPa. Derive expressions for the pressure variations along the upper and lower surfaces. and the lift and drag coefficients.



SOLUTION:

- First determine an equation for the airfoil surface.



- From triangle ABC we find:

$$R^2 = \left(\frac{c}{2}\right)^2 + \left(R - \frac{t}{2}\right)^2$$

$$= \frac{c^2}{4} + R^2 - Rt + \frac{t^2}{4}$$

$$\Rightarrow \quad R = \frac{1}{4t}\left(c^2 + t^2\right)$$

- Now consider point D (located a distance x from the origin):

$$x = \frac{c}{2} - R\sin\theta$$

$$\Rightarrow \quad \sin\theta = \frac{1}{R}\left(\frac{c}{2} - x\right) = \frac{4t}{c^2+t^2}\left(\frac{c}{2} - x\right)$$

- Assuming that the airfoil is slender

$$\Rightarrow \quad t \ll c \quad \text{and} \quad \sin\theta \approx \theta$$

$$\Rightarrow \quad \theta \approx \left.\frac{dy}{dx}\right|_{upper} \approx \frac{4t}{c^2}\left(\frac{c}{2} - x\right) = 2\left(\frac{t}{c}\right) - 4\left(\frac{t}{c}\right)\left(\frac{x}{c}\right)$$

- The lower surface slope will have identical magnitude but opposite sign:

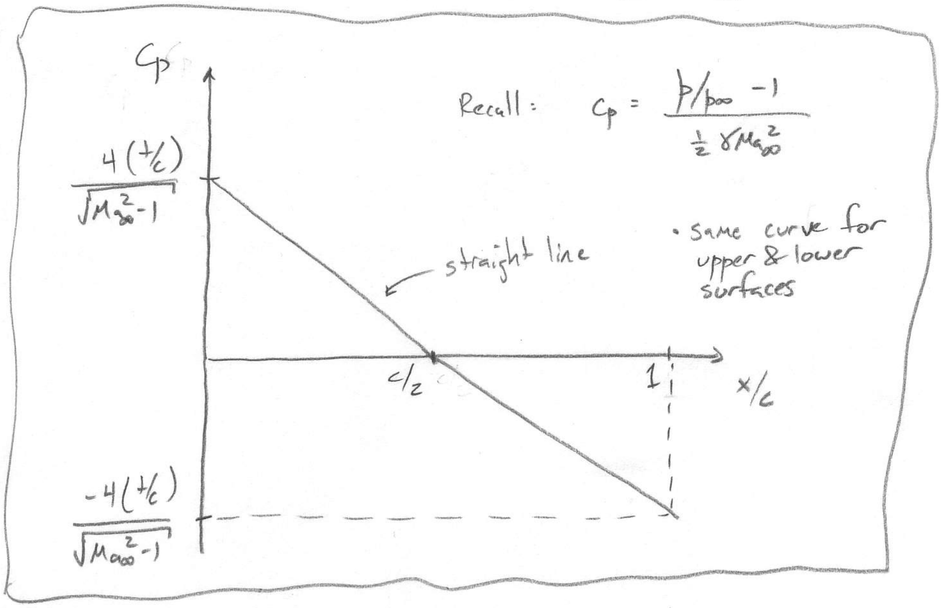
$$\left.\frac{dy}{dx}\right|_{lower} = -2\left(\frac{t}{c}\right) + 4\left(\frac{t}{c}\right)\left(\frac{x}{c}\right)$$

SOLUTION...

$$C_p\Big|_{upper} = \frac{2}{\sqrt{Ma_\infty^2 - 1}} \left.\frac{dy}{dx}\right|_{upper} = \frac{2}{\sqrt{Ma_\infty^2-1}}\left[2\left(\frac{t}{c}\right) - 4\left(\frac{t}{c}\right)\left(\frac{x}{c}\right)\right]$$

$$C_p\Big|_{lower} = \frac{-2}{\sqrt{Ma_\infty^2 - 1}} \left.\frac{dy}{dx}\right|_{lower} = \frac{2}{\sqrt{Ma_\infty^2-1}}\left[2\left(\frac{t}{c}\right) - 4\left(\frac{t}{c}\right)\left(\frac{x}{c}\right)\right]$$

same as upper surface since symmetric



• The lift coefficient on the airfoil will be zero due to symmetry.

• The drag coefficient is given by:

$$c_D = \frac{2}{c\sqrt{Ma_\infty^2-1}} \int_0^c \left[\left.\frac{dy}{dx}\right|_{upper}^2 + \left.\frac{dy}{dx}\right|_{lower}^2 \right] dx$$

where
$$\int_0^c \left.\frac{dy}{dx}\right|_{upper}^2 dx = \int_0^c \left[4\left(\frac{t}{c}\right)^2 - 16\left(\frac{t}{c}\right)^2\left(\frac{x}{c}\right) + 16\left(\frac{t}{c}\right)^2\left(\frac{x}{c}\right)^2\right] dx$$

$$= 4\left(\frac{t}{c}\right)^2 c - 8\left(\frac{t}{c}\right)^2 c + \frac{16}{3}\left(\frac{t}{c}\right)^2 c$$

$$= \frac{4}{3}\left(\frac{t}{c}\right)^2 c$$

• An identical result will be found for $\int^c \left.\frac{dy}{dx}\right|^2 dx$

Solution...

$$\Rightarrow \quad C_D = \frac{2}{c\sqrt{Ma_\infty^2 - 1}}\left\{\frac{8}{3}\left(\frac{t}{c}\right)^2 c\right\}$$

$$\boxed{\therefore \quad C_D = \frac{16}{3}\frac{(t/c)^2}{\sqrt{Ma_\infty^2 - 1}}}$$

19. Method of Characteristics

The method of characteristics is a procedure by which non-linear, hyperbolic PDEs may be solved in an algorithmic fashion.

Before beginning with the algorithm, we must first discuss a few details regarding hyperbolic PDEs. To begin, let's first define a characteristic curve. **A characteristic curve is a curve across which the value of some parameter is continuous, but the derivatives of that parameter are indeterminate**. Consider for example the flow through a Mach wave. Recall from our previous analysis that velocity changes across a Mach wave are very small, i.e., the velocity across a Mach wave goes from V to $V+dV$. This infinitesimal velocity change (dV) occurs over zero distance since the Mach wave has no thickness. Thus, although the velocity changes continuously across the Mach wave, the velocity gradient is indeterminate ($dV/0=?$). Hence, a Mach wave is a characteristic curve according to our definition. Characteristic curves have additional useful properties but we will not discuss them at this time.

Our first task will instead be to determine the slope of a characteristic curve in a general flow situation. Recall from our previous analysis that the equation of motion for a steady, irrotational, isentropic compressible flow with negligible body and viscous forces is given by:

$$\nabla^2\phi = \frac{1}{c^2}\left\{\nabla\phi\cdot\left[(\nabla\phi\cdot\nabla)\nabla\phi\right]\right\}$$

Expanding using Cartesian coordinates:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\left\{\nabla\phi\cdot\left[\left(\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}+\frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}\right)\left(\frac{\partial\phi}{\partial x}\hat{\mathbf{e}}_x+\frac{\partial\phi}{\partial y}\hat{\mathbf{e}}_y+\frac{\partial\phi}{\partial z}\hat{\mathbf{e}}_z\right)\right]\right\}$$

$$= \frac{1}{c^2}\left\{\left(\frac{\partial\phi}{\partial x}\hat{\mathbf{e}}_x+\frac{\partial\phi}{\partial y}\hat{\mathbf{e}}_y+\frac{\partial\phi}{\partial z}\hat{\mathbf{e}}_z\right)\cdot\left[\begin{array}{l}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial x}+\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z\partial x}\right)\hat{\mathbf{e}}_x+\\ \left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2}+\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z\partial y}\right)\hat{\mathbf{e}}_y+\\ \left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial z}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial z}+\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z^2}\right)\hat{\mathbf{e}}_z\end{array}\right]\right\}$$

$$= \frac{1}{c^2}\left\{\begin{array}{l}\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial x}+\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z\partial x}+\\ \frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y}+\left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial y^2}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z\partial y}+\\ \frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial z}+\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial z}+\left(\frac{\partial\phi}{\partial z}\right)^2\frac{\partial^2\phi}{\partial z^2}\end{array}\right\}$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\left\{\begin{array}{l}\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2}+\left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial y^2}+\left(\frac{\partial\phi}{\partial z}\right)^2\frac{\partial^2\phi}{\partial z^2}+\\ 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial x}+2\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z\partial y}+2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z\partial x}\end{array}\right\} \tag{276}$$

For 2D flow, the previous equation can be written as:

$$\left[c^2-\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}-2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}+\left[c^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}=0 \tag{277}$$

This non-linear equation has the form:

$$A\frac{\partial^2\Phi}{\partial x^2}+2B\frac{\partial^2\Phi}{\partial x\partial y}+C\frac{\partial^2\Phi}{\partial y^2}=0 \tag{278}$$

where A, B, and C depend on $\partial\phi/\partial x$ and $\partial\phi/\partial y$.

For an isentropic flow, the velocities, i.e., $\partial\phi/\partial x$ and $\partial\phi/\partial y$, are continuous. Since we're interested in finding the velocity characteristic curves, we should consider differential changes in the velocities:

$$du_x=d\left(\frac{\partial\phi}{\partial x}\right)=\frac{\partial^2\phi}{\partial x^2}dx+\frac{\partial^2\phi}{\partial x\partial y}dy \tag{279}$$

$$du_y=d\left(\frac{\partial\phi}{\partial y}\right)=\frac{\partial^2\phi}{\partial x\partial y}dx+\frac{\partial^2\phi}{\partial y^2}dy \tag{280}$$

Now let's rewrite Eqs. (278) through (280) as a system of linear equations where the independent variables are the velocity gradients ($\partial^2\phi/\partial x^2$, $\partial^2\phi/\partial x\partial y$, and $\partial^2\phi/\partial y^2$):

$$\begin{array}{lllll} A\dfrac{\partial^2\phi}{\partial x^2} & +2B\dfrac{\partial^2\phi}{\partial x\partial y} & +C\dfrac{\partial^2\phi}{\partial y^2} & =0 \\ dx\dfrac{\partial^2\phi}{\partial x^2} & +dy\dfrac{\partial^2\phi}{\partial x\partial y} & +0 & =d\left(\dfrac{\partial\phi}{\partial x}\right) \\ 0 & +dx\dfrac{\partial^2\phi}{\partial x\partial y} & +dy\dfrac{\partial^2\phi}{\partial y^2} & =d\left(\dfrac{\partial\phi}{\partial y}\right) \end{array} \tag{281}$$

Recall from linear algebra that Kramer's rule states that a system of equations has a unique solution if and only if the determinant of the system is not equal to zero, i.e.:

$$\begin{vmatrix} A & 2B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}=Ady^2-2Bdxdy+Cdx^2\neq 0 \tag{282}$$

Thus, if this determinant <u>is</u> equal to zero, then the velocity derivatives (or gradient) ($\partial^2\phi/\partial x^2$, $\partial^2\phi/\partial y^2$, and $\partial^2\phi/\partial x\partial y$) will be indeterminate. The slope of such a curve is given by:

$$Ady^2-2Bdxdy+Cdx^2=0$$

$$\frac{dy}{dx}=\frac{2B\pm\sqrt{4B^2-4AC}}{2A}=\frac{B\pm\sqrt{B^2-AC}}{A}$$

where

$$A=\left[c^2-\left(\frac{\partial\phi}{\partial x}\right)^2\right]=c^2-u_x^2$$

$$B=-\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}=-u_xu_y$$

$$C=\left[c^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right]=c^2-u_y^2$$

Thus,

$$\frac{dy}{dx}=\frac{-u_xu_y \pm\sqrt{u_x^2u_y^2-\left(c^2-u_x^2\right)\left(c^2-u_y^2\right)}}{c^2-u_x^2}$$

$$=\frac{-u_xu_y \pm\sqrt{u_x^2u_y^2-\left(c^4-c^2u_x^2-c^2u_y^2+u_x^2u_y^2\right)}}{c^2-u_x^2}$$

$$=\frac{-u_xu_y \pm\sqrt{c^2\left(u_x^2+u_y^2\right)-c^4}}{c^2-u_x^2}$$

$$\boxed{\frac{dy}{dx}=\frac{-\frac{u_xu_y}{c^2}\pm\sqrt{\mathrm{Ma}^2-1}}{1-\frac{u_x^2}{c^2}}} \qquad \textbf{slope of characteristic curves} \tag{283}$$

Recall from our previous discussion that a curve across which the velocity derivatives (or gradient) are indeterminate are referred to as **characteristic curves**.

Notes:

1. For Ma<1 there are no real characteristic curves (elliptic PDE). For Ma=1 there is one characteristic curve (parabolic PDE), and for Ma>1 there are two characteristic curves (hyperbolic PDE).

2. Recall from our previous, linearized (small perturbation) analysis that the governing PDE was linear:

   $$A\frac{\partial^2\Phi}{\partial x^2}+2B\frac{\partial^2\Phi}{\partial x\partial y}C\frac{\partial^2\Phi}{\partial y^2}=0$$

   where

   $$A=\left(1-\mathrm{Ma}_\infty^2\right)$$
   $$B=0$$
   $$C=1$$

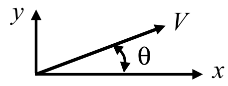
   Thus, the slope of the characteristic curves are given by:

   $$\frac{dy}{dx}=\frac{B\pm\sqrt{B^2-AC}}{A}=\frac{\pm1}{\sqrt{\mathrm{Ma}_\infty^2-1}} \qquad \text{Mach lines!}$$

   Thus, the characteristic curves for linearized flow are straight Mach lines.

3. It can be shown that the characteristic curves are inclined at the Mach angle to the local flow direction. To show this, first re-write the velocity components in terms of the velocity magnitude, V, and the local flow inclination, θ:

$$u_x = V\cos\theta \qquad u_y = V\sin\theta \tag{284}$$

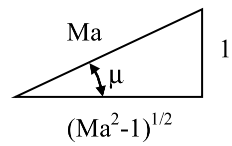


Substituting into Eq. (283) and noting that, sinμ = 1/Ma (where μ is the Mach angle), gives:

$$\frac{dy}{dx} = \frac{-\mathrm{Ma}^2\sin\theta\cos\theta \pm \sqrt{\mathrm{Ma}^2-1}}{1-\mathrm{Ma}^2\cos^2\theta}$$

$$= \frac{-\sin\theta\cos\theta \pm \frac{1}{\mathrm{Ma}^2}\sqrt{\mathrm{Ma}^2-1}}{\frac{1}{\mathrm{Ma}^2}-\cos^2\theta}$$

$$= \frac{-\sin\theta\cos\theta \pm \frac{\sin^2\mu}{\tan\mu}}{\sin^2\mu-\cos^2\theta}$$



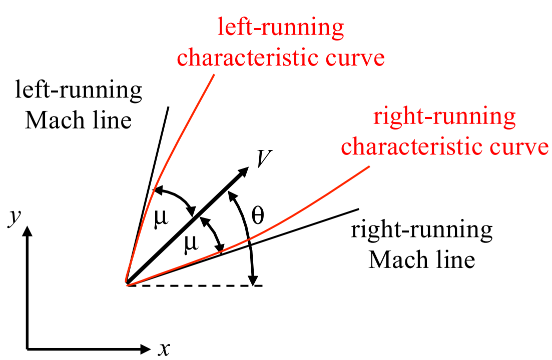
$\tan\mu = (\mathrm{Ma}^2-1)^{-1/2}$
$\sin\mu = 1/\mathrm{Ma}$

After considerable rearrangement and simplification using trig identities we find that:

$$\boxed{\frac{dy}{dx} = \tan(\theta \mp \mu)} \tag{285}$$

The "-" corresponds to the right-running characteristic while the "+" corresponds to the left-running characteristic.

Thus, **the slope of the characteristic lines will be inclined at the Mach angle to the local flow direction**.



Notes:

a. The direction of a characteristic curve or Mach line is determined by facing downstream. The left-running waves are on the LHS while the right-running waves are on the RHS.

b. The characteristic curves are the "Mach curves" for the flow.

c. Although we now have an expression for the slope of the characteristic curves (Eq. (285)), we still need to have some relation for determining how the velocities change along a characteristic curve. Recall from the system of equations from Eq. (281):

$$\begin{aligned}
&A\frac{\partial^2\phi}{\partial x^2} &&+2B\frac{\partial^2\phi}{\partial x\partial y} &&+C\frac{\partial^2\phi}{\partial y^2} &&= 0\\
&dx\frac{\partial^2\phi}{\partial x^2} &&+dy\frac{\partial^2\phi}{\partial x\partial y} &&+0 &&= d\left(\frac{\partial\phi}{\partial x}\right) = du_x\\
&0 &&+dx\frac{\partial^2\phi}{\partial x\partial y} &&+dy\frac{\partial^2\phi}{\partial y^2} &&= d\left(\frac{\partial\phi}{\partial y}\right) = du_y
\end{aligned}$$

where

$$A = \left[c^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right] = c^2 - u_x^2$$

$$B = -\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y} = -u_x u_y$$

$$C = \left[c^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right] = c^2 - u_y^2$$

The solution for the gradient of the x-velocity component in the x-direction (using Kramer's rule) is:

$$\frac{\partial u_x}{\partial x} = \frac{\partial^2\phi}{\partial x^2} = \frac{\begin{vmatrix} 0 & 2B & C \\ du_x & dy & 0 \\ du_y & dx & dy \end{vmatrix}}{\begin{vmatrix} A & 2B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}} = \frac{Cdxdu_x - Cdydu_y - 2Bdydu_x}{Ady^2 - 2Bdxdy + Cdx^2}$$

(Note: We could also consider the other velocity gradients such as $\partial u_y/\partial x$ and would get the same result given below.) Recall that along a characteristic curve the denominator in the previous expression will be zero. Thus, in order for $\partial u_x/\partial x$ to remain finite (we're assuming that the velocities in the flow will be finite so that velocity gradients will also be finite), the numerator must also be zero:

$$Cdxdu_x - Cdydu_y - 2Bdydu_x = 0$$

$$\Rightarrow \quad \frac{du_y}{du_x} = \frac{dx}{dy} - \frac{2B}{C}$$

Substituting in for the characteristic curve slope (Eq. (285)) and simplifying gives:

$$\frac{du_y}{du_x} = \frac{\frac{u_x u_y}{c^2} \mp \sqrt{\mathrm{Ma}^2 - 1}}{1 - \frac{u_y^2}{c^2}} \tag{286}$$

Re-write the velocity components in terms of the velocity magnitude, V, and direction, θ:

$$u_x = V\cos\theta \quad \Rightarrow \quad du_x = dV\cos\theta - V\sin\theta d\theta$$

$$u_y = V\sin\theta \quad \Rightarrow \quad du_y = dV\sin\theta + V\cos\theta d\theta$$

Substituting into Eq. (286) and simplifying:

$$\boxed{\frac{1}{V}\frac{dV}{d\theta} = \frac{\mp 1}{\sqrt{\text{Ma}^2 - 1}} = \mp\tan\mu} \tag{287}$$

We now need to integrate the previous expression as we move <u>along</u> a characteristic line. We've performed this same integration before when discussing Prandtl-Meyer expansion fans (refer to Eq. (236) and (237)). Instead of re-deriving the result again, we'll just copy the result here:

$$\int_a^b d\theta = \int_a^b \mp\sqrt{\text{Ma}^2 - 1}\frac{dV}{V} = \mp\left(\nu_b - \nu_a\right)$$

$$\boxed{\theta_b - \theta_a = \mp\left(\nu_b - \nu_a\right)} \tag{288}$$

where ν_a and ν_b are the Prandtl-Meyer angles for the flow at locations *a* and *b*, both of which lie on the same characteristic curve. Again, θ_a and θ_b are the flow velocity directions at points *a* and *b*. If we let the point *a* be some reference point on the characteristic curve, then:

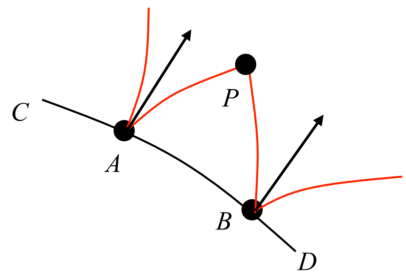
$$\boxed{\nu + \theta = \text{constant}} \quad \textbf{along a right-running characteristic curve} \tag{289}$$

$$\boxed{\nu - \theta = \text{constant}} \quad \textbf{along a left-running characteristic curve} \tag{290}$$

Equations (289), (290), and (285) can be combined together in an algorithm for solving steady, supersonic, irrotational compressible flows.

Method of Characteristics Algorithm

Suppose that the flow conditions are known along some curve *CD*:



Since the flow conditions are known on this curve, the flow angle, θ, and Prandtl-Meyer angle, ν, are known at the two points *A* and *B*. We now move along the right-running characteristic curve from point *A* until it intersects with the left-running characteristic curve from point B. Call this intersection point *P*. At point *P* we have:

$$\nu_P + \theta_P = \nu_A + \theta_A = C_R$$

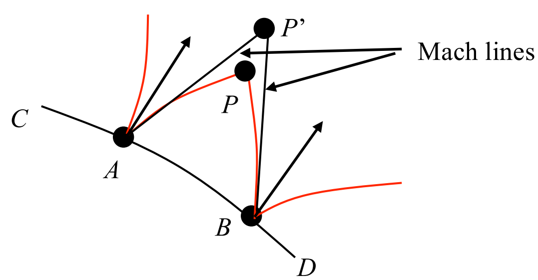
$$\nu_P - \theta_P = \nu_B - \theta_B = C_L$$

where C_R and C_L are constants. Re-arrange the previous expressions to get:

$$\nu_P = \tfrac{1}{2}\left(C_R + C_L\right)$$

$$\theta_P = \tfrac{1}{2}\left(C_R - C_L\right)$$

Since we now know the Prandtl-Meyer angle and flow direction at point *P*, we can determine the remainder of the flow properties for this isentropic flow (e.g., Mach number, pressure, temperature, velocity components, etc.) However, we still don't know the location of point *P*. Since the Mach number changes, in general, from point to point in the flow field, the characteristics curves will not necessarily be straight lines. However, if we choose the points *A* and *B* to be very close together, then we can approximate the position of point *P* by the intersection of the left- and right-running Mach lines passing through *A* and *B* (call this point *P*').



Obviously the location of point *P*' approaches *P* as the distance between *A* and *B* becomes smaller.

We can determine the conditions at other points in the flow field by repeating this procedure for other points where the flow conditions are known.

Note that along a right-running characteristic, C_R = constant so:

$$\left.\begin{aligned} \nu &= \tfrac{1}{2}\left(C_R + C_L\right) \Rightarrow \Delta\nu = \tfrac{1}{2}\left(\underbrace{\Delta C_R}_{=0} + \Delta C_L\right) = \tfrac{1}{2}\Delta C_L \\ \theta &= \tfrac{1}{2}\left(C_R - C_L\right) \Rightarrow \Delta\theta = \tfrac{1}{2}\left(\underbrace{\Delta C_R}_{=0} - \Delta C_L\right) = -\tfrac{1}{2}\Delta C_L \end{aligned}\right\} \Rightarrow \Delta\nu = -\Delta\theta \tag{291}$$

Similarly, along a left-running characteristic, C_L = constant so:

$$\left.\begin{aligned} \nu &= \tfrac{1}{2}\left(C_R + C_L\right) \Rightarrow \Delta\nu = \tfrac{1}{2}\left(\Delta C_R + \underbrace{\Delta C_L}_{=0}\right) = \tfrac{1}{2}\Delta C_R \\ \theta &= \tfrac{1}{2}\left(C_R - C_L\right) \Rightarrow \Delta\theta = \tfrac{1}{2}\left(\Delta C_R - \underbrace{\Delta C_L}_{=0}\right) = \tfrac{1}{2}\Delta C_R \end{aligned}\right\} \Rightarrow \Delta\nu = \Delta\theta \tag{292}$$

Method of Characteristics
This section remains incomplete.

Types of Points:

Interior Points:

C_L and C_R are known, determine ν and θ

Solid Boundary Points:

either C_L or C_R is known and θ is known

Free Boundary Points:

either C_L or C_R is known and ν is known
(Since p is known ⇒ can determine Ma ⇒ can determine ν)

Interaction of Waves

reflections off solid boundaries

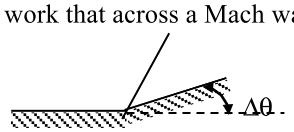
cancellation of waves

reflections off free boundaries

intersection of waves

Region-to-Region Method
Can cross characteristics too. Recall from our previous work that across a Mach wave (a characteristic curve), we have:

$$\Delta\nu = -\Delta\theta$$



where ν is the Prandtl-Meyer angle and θ is the flow orientation. These expressions can be written as:

$\nu_2 - \theta_2 = \nu_1 - \theta_1$ <u>across</u> a right-running characteristic

and

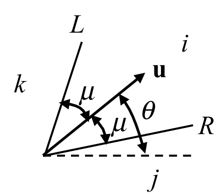
$\nu_2 + \theta_2 = \nu_1 + \theta_1$ <u>across</u> a left-running characteristic

(same as moving along the opposite characteristic)

Angle of lines separating regions

$\alpha_{ij} = \frac{1}{2}\left[\left(\theta_i - \mu_i\right) + \left(\theta_j - \mu_j\right)\right]$ (across R-running)

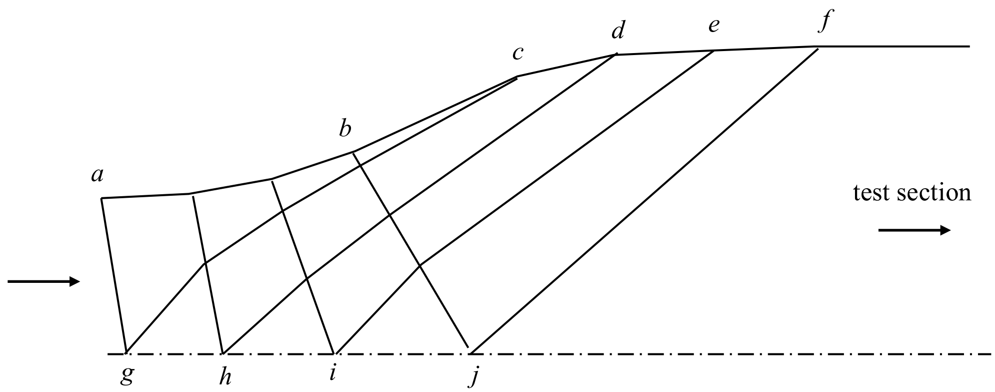
$\alpha_{ik} = \frac{1}{2}\left[\left(\theta_i + \mu_i\right) + \left(\theta_k + \mu_k\right)\right]$ (across L-running)



Design of a Supersonic Wind Tunnel Nozzle

One simple application of the Method of Characteristics is in the design of a supersonic wind tunnel nozzle. Recall that the function of a supersonic nozzle is to accelerate a flow from Ma=1 to some final supersonic Mach number. In order to simulate free flight, the flow through the test section of the wind tunnel should be parallel and uniform.

Consider a symmetric nozzle so that only the upper half of the nozzle must be considered. For simplicity, assume that the incoming flow at the throat is uniform and at Mach 1.0 (of course in a real nozzle the incoming flow would not be uniform). The flow will first expand in the region from points *a* through *b* as shown in the figure below. Four of the expansion waves (right-running characteristics) are shown in the sketch.



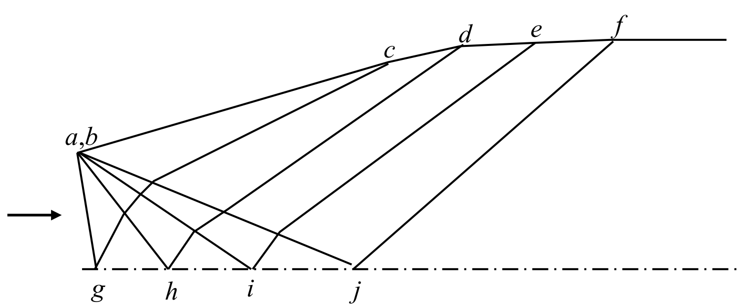
The expansion waves reflect off the centerline at points *g* through *j* as expansion waves which act to turn the flow back toward the horizontal (these are left-running characteristics). These waves impinge on the nozzle wall at points *c* through *f*. In order to avoid further reflections, the wall contour at points *c* through *f* should be sloped at the wave turning angle so that the waves are cancelled.

Notes:

1. As more waves are included in the analysis, the nozzle contour will become smoother.

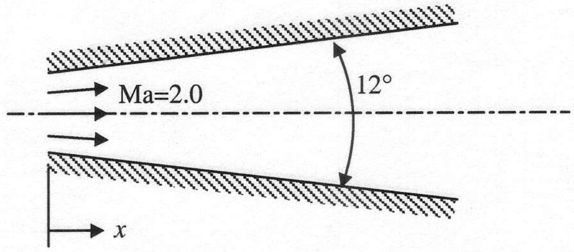
2. The initial expansion from *a* to *b* is arbitrary in the design. The critical points in the analysis are points *c* through *f* which must designed to provide wave cancellation.

3. To design a nozzle with the shortest length, the expansion from points *a* to *b* should take place as a centered Prandtl-Meyer expansion fan:



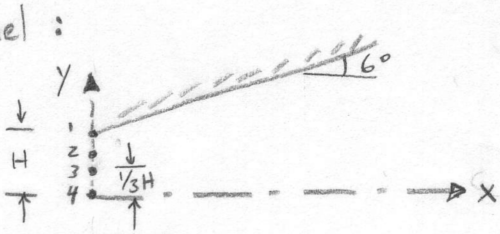
4. The design of a real nozzle should also factor in boundary layer effects in order to give the correct flow area and avoid boundary layer separation.

Uniform radial flow at Mach 2.0 enters a 2D diverging channel with straight walls. Compute the variation of Mach number in this radial flow field, assuming isentropic, steady flow. The walls are inclined at a total angle of 12°. Compare your results at the centerline with those using 1D isentropic flow analysis.



SOLUTION:

- From symmetry, we need only consider the upper half of the channel:



The conditions at $x = 0$ are known. Select four equally spaced points:

Point	1:	$Ma_1 = 2.0$	$\theta_1 = 6°$
"	2:	$Ma_2 = 2.0$	$\theta_2 = 4°$
"	3:	$Ma_3 = 2.0$	$\theta_3 = 2°$
"	4:	$Ma_4 = 2.0$	$\theta_4 = 0°$

Using the Prandtl-Meyer angle relation:

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(Ma^2-1\right)} - \tan^{-1}\sqrt{Ma^2-1}$$

And the Mach angle relation:

$$\mu = \sin^{-1}\left(\frac{1}{Ma}\right)$$

SOLUTION...

- Thus,

use Mach angle relation

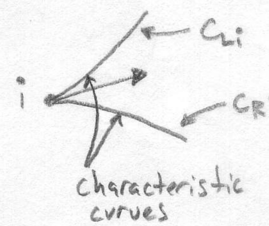
use Prandtl-Meyer angle relation

location of point

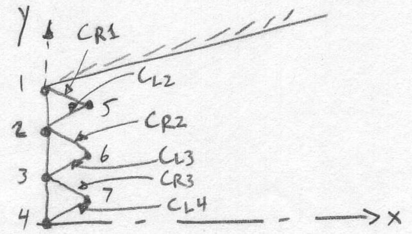
Point i	Ma_i	θ_i	μ_i	ν_i	C_{Li}	C_{Ri}	$\left(\frac{x_i}{H}, \frac{y_i}{H}\right)$
1	2.0	6°	30°	26.38°	20.38°	32.38°	(0, 1)
2	2.0	4°	30°	26.38°	22.38°	30.38°	(0, 2/3)
3	2.0	2°	30°	26.38°	24.38°	28.38°	(0, 1/3)
4	2.0	0°	30°	26.38°	26.38°	26.38°	(0, 0)

where

$$C_{Li} = \nu_i - \theta_i$$
$$C_{Ri} = \nu_i + \theta_i$$



- Now advance down the channel to points 5, 6, and 7:



use Prandtl-Meyer angle relation

use Mach angle relation

Point i	C_{Li}	C_{Ri}	ν_i	θ_i	Ma_i	μ_i	$\left(\frac{x_i}{H}, \frac{y_i}{H}\right)$
5	$= C_{L2}$ $= 22.38°$	$= C_{R1}$ $= 32.38°$	27.38°	5°	2.04	29.41°	(1.45, 1.65)
6	$= C_{L3}$ $= 24.38°$	$= C_{R2}$ $= 30.38°$	27.38°	3°	2.04	29.41°	(2.43, 1.85)
7	$= C_{L4}$ $= 26.38°$	$= C_{R3}$ $= 28.38°$	27.38°	1°	2.04	29.41°	(7.30, 4.22)

where

$$\nu_i = \frac{1}{2}\left(C_{Ri} + C_{Li}\right)$$

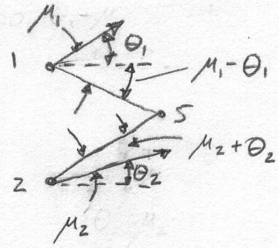
$$\theta_i = \frac{1}{2}\left(C_{Ri} - C_{Li}\right)$$

Formulas for determining point location are given on following page

SOLUTION...

- Point location:

Point 5:



$$\frac{y_5}{H} - \frac{y_1}{H} = m_1\left(\frac{x_5}{H} - \frac{x_1}{H}\right) \qquad \text{where} \qquad m_1 = \tan(\mu_1 - \theta_1)$$
$$\frac{y_5}{H} - \frac{y_2}{H} = m_2\left(\frac{x_5}{H} - \frac{x_2}{H}\right) \qquad\qquad m_2 = \tan(\mu_2 + \theta_2)$$

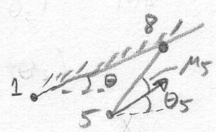
$$(m_2 - m_1)\frac{y_5}{H} = m_2\frac{y_1}{H} - m_1 m_2 \frac{x_1}{H} - m_1\frac{y_2}{H} + m_1 m_2 \frac{x_2}{H}$$

$$\rightarrow \quad (m_1 - m_2)\frac{x_5}{H} = -\frac{y_1}{H} + \frac{y_2}{H} - m_1\frac{x_1}{H} + m_2\frac{x_2}{H}$$

$$\Rightarrow \quad y_5/H = \frac{m_1(m_2 x_2/H - y_2/H) - m_2(m_1 x_1/H - y_1/H)}{m_2 - m_1} = 1.647$$

$$x_5/H = \frac{(m_2 x_2/H + y_2/H) - (m_1 x_1/H + y_1/H)}{m_1 - m_2} = 1.454$$

- A similar approach can be used to determine the location of additional points.
  - At upper boundary points (e.g. 8, 15, 22, ...) we have:



$$m_5 = \tan(\theta_5 + \mu_5)$$
$$m_B = \tan(\theta)$$

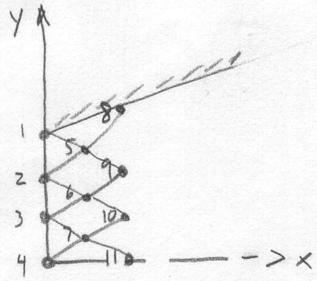
$$y_8/H - y_1/H = m_B\left(x_8/H - x_1/H\right)$$
$$y_8/H - y_5/H = m_5\left(x_8/H - x_5/H\right)$$

$$\Rightarrow \quad y_8/H = \frac{m_B(m_5 x_5/H - y_5/H) - m_5(m_B x_1/H - y_1/H)}{m_5 - m_B}$$

$$x_8/H = \frac{(m_5 x_5/H + y_5/H) - (m_B x_1/H + y_1/H)}{m_B - m_5}$$

SOLUTION...

- Now advance down the channel to points 8, 9, 10, and 11:

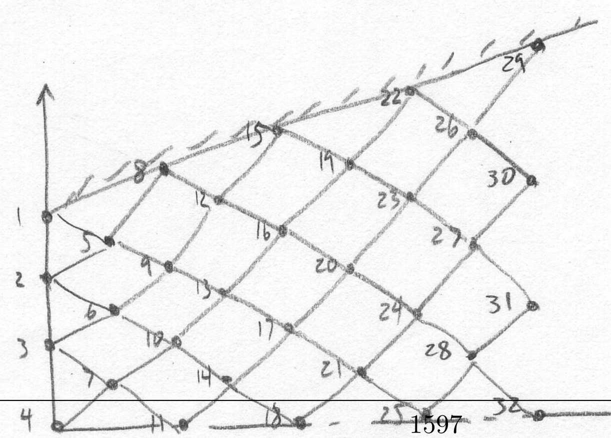


Point i	C_{Li}	C_{Ri}	ν_i	θ_i	Ma_i	μ_i	$\left(\frac{x_i}{H}, \frac{y_i}{H}\right)$
8	$= C_{L5}$ 22.38°	34.38°	28.38°	6°	2.07	28.84°	
9	$= C_{L6}$ 24.38°	$= C_{R5}$ 32.38°	28.38°	4°	2.07	28.84°	
10	$= C_{L7}$ 26.38°	$= C_{R6}$ 30.38°	28.38°	2°	2.07	28.84°	
11	28.38°	$= C_{R7}$ 28.38°	28.38°	0°	2.07	28.84°	

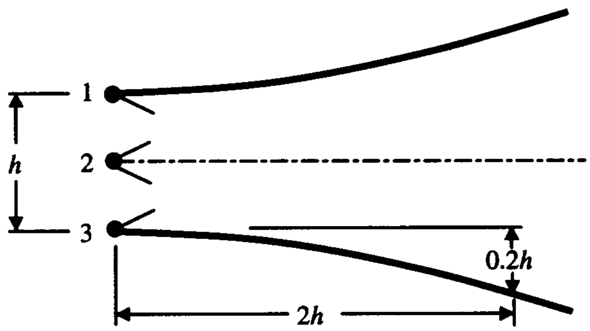
NOTE: At point 8, C_{R8} is found using C_{L8} and $\theta_8 = 6°$ (since it is a boundary point). $C_{R8} = C_{L8} + 2\theta_8$

Similarly at point 11, C_{L11} is found using C_{R11} and $\theta_{11} = 0°$. $C_{L11} = C_{R11} - 2\theta_{11}$

- Continue this approach for additional points:

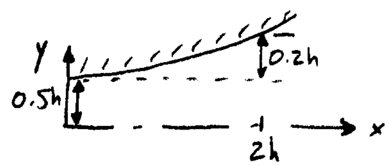


The two –dimensional passage shown in the figure has parabolically-shaped walls. At section 1-2-3 the flow is uniform and parallel with Ma=1.33. Using the coarse characteristic system suggested in the sketch, plot the pressure distribution on the walls and along the center line for a distance of 4*h* downstream of the initial section.



SOLUTION:

• Wall Profile:

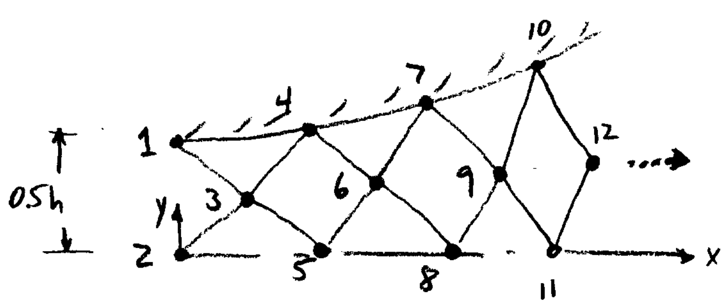


$$y_{WALL} = 0.5h + \frac{0.2h}{4h^2}x^2$$

NOTE: Assuming slope of wall at 1-2-3 is zero.

$$\therefore y_{WALL} = 0.5h + \frac{0.05}{h}x^2$$

• Apply Method of Characteristics to the upper half of the passage (only need to consider upper half due to symmetry):



Solution...

- Use the following relations to determine the properties at each point:

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}\left(Ma^2-1\right)}\right)-\tan^{-1}\left(\sqrt{Ma^2-1}\right)$$

$$\mu = \tan^{-1}\left(\frac{1}{Ma}\right)$$

$$C_L = \nu - \theta \qquad \nu = \frac{1}{2}\left(C_R + C_L\right)$$

$$C_R = \nu + \theta \qquad \theta = \frac{1}{2}\left(C_R - C_L\right)$$

$$\left.\frac{dy}{dx}\right|_L = \tan\left(\theta + \mu\right)$$

$$y - y_0 = \frac{dy}{dx}\left(x - x_0\right)$$

$$\left.\frac{dy}{dx}\right|_R = \tan\left(\theta - \mu\right)$$

Point	Ma	θ	ν	μ	C_L	C_R	$\left(\frac{x}{h}, \frac{y}{h}\right)$
1	1.33	0°	7.00°	48.75°	7.00°	7.00°	(0, 0.5)
2	1.33	0°	7.00°	48.75°	7.00°	7.00°	(0, 0)

Point	C_L	C_R	ν	θ	Ma	μ	$\left(\frac{x}{h}, \frac{y}{h}\right)$
3	$= C_{L2}$ 7.00°	$= C_{R1}$ 7.00°	7.00°	0°	1.33	48.75°	(0.22, 0.25)

$$\left.\begin{aligned} y_3 - y_1 &= \underbrace{\left.\frac{dy}{dx}\right|_{R1}}_{m_R}\left(x_3 - x_1\right) \\ y_3 - y_2 &= \underbrace{\left.\frac{dy}{dx}\right|_{L2}}_{m_L}\left(x_3 - x_2\right) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} x_3 &= \frac{\left(m_R x_1 - y_1\right) - \left(m_L x_2 - y_2\right)}{m_R - m_L} \\ y_3 &= \frac{m_L\left(m_R x_1 - y_1\right) - m_R\left(m_L x_2 - y_2\right)}{m_R - m_L} \end{aligned}$$

Here $m_R = \tan\left(\theta_1 - \mu_1\right)$

$m_L = \tan\left(\theta_2 + \mu_2\right)$

SOLUTION...

Point	C_L	C_R	ν	θ	Ma	μ	$\left(\frac{x}{h}, \frac{y}{h}\right)$
4	$=C_{L3}$ 7.00°	12.12°	9.56°	2.56°	1.42	44.77°	(0.45, 0.51)
5	7.00°	$=C_{R3}$ 7.00°	7.00°	0°	1.33	48.75°	(0.44, 0)

$$\left.\begin{aligned} y_4 &= 0.5h + \frac{0.05x_4^2}{h} \\ y_4 - y_3 &= \left.\frac{dy}{dx}\right|_{L3}(x_4 - x_3) \end{aligned}\right\} \Rightarrow \quad x_4^2 - \left(\frac{h}{0.05}\left.\frac{dy}{dx}\right|_{L3}\right)x_4 + \left(\frac{0.5h - y_3 + \left.\frac{dy}{dx}\right|_{L3} x_3}{\frac{0.05}{h}}\right) = 0$$

$$\left.\frac{dy}{dx}\right|_{L3} = \tan(\theta_3 + \mu_3)$$

$$\left.\frac{dy}{dx}\right|_{x_4} = \tan\theta_4 = \frac{0.1\,x_4}{h} \quad \text{(slope of wall)}$$

$$\nu_4 = C_{L4} + \theta_4$$

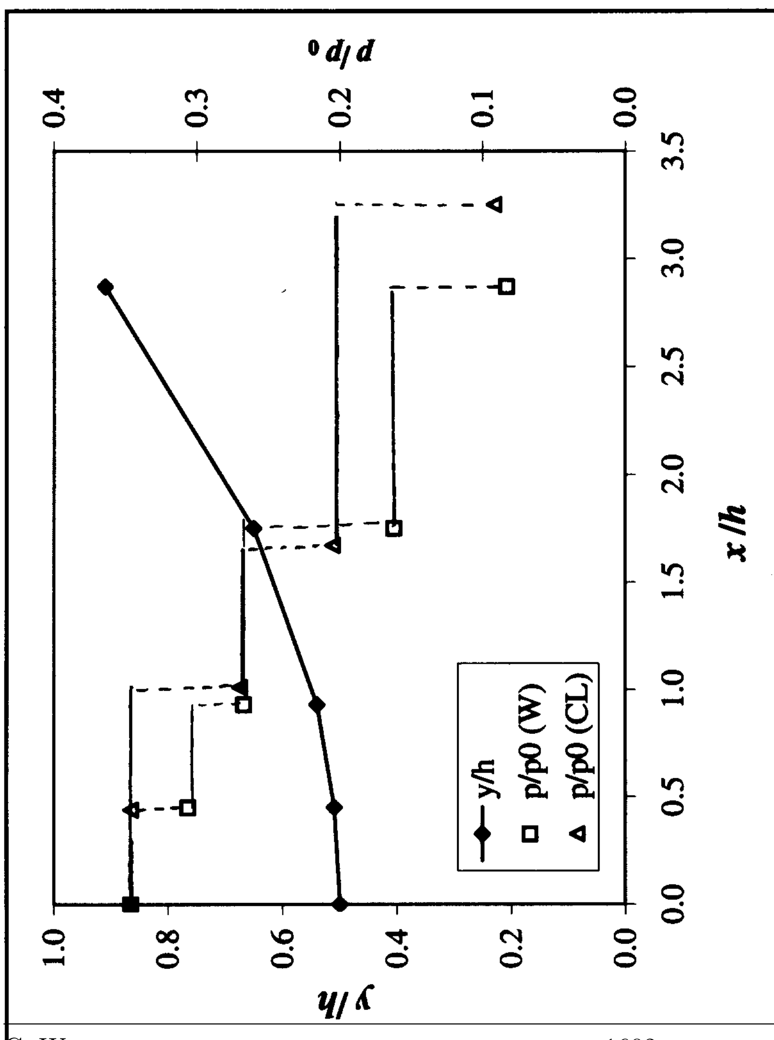
$$\nu_5 = C_{R5} - \theta_5$$

$$y_5 = 0$$

$$\cancelto{0}{y_5} - y_3 = \underbrace{\left.\frac{dy}{dx}\right|_{R3}}_{m_{R3}}(x_5 - x_3) \Rightarrow \quad x_5 = x_3 - \frac{y_3}{m_{R3}}$$

- Repeat this procedure for additional points until $\frac{x_i}{h} \geq 4.0$ (Refer to the attached program)

point	Ma	theta [deg]	mu [deg]	nu [deg]	CL [deg]	CR [deg]	x/h	y/h	p/p0
1	1.33	0.00	48.75	7.00	7.00	7.00	0.00	0.50	0.346
2	1.33	0.00	48.75	7.00	7.00	7.00	0.00	0.00	0.346
3	1.33	0.00	48.75	7.00	7.00	7.00	0.22	0.25	0.346
4	1.42	2.56	44.77	9.56	7.00	12.12	0.45	0.51	0.306
5	1.33	0.00	48.75	7.00	7.00	7.00	0.44	0.00	0.346
6	1.42	2.56	44.77	9.56	7.00	12.12	0.69	0.29	0.306
7	1.51	5.29	41.37	12.29	7.00	17.58	0.93	0.54	0.267
8	1.51	0.00	41.56	12.12	12.12	12.12	1.01	0.00	0.270
9	1.60	2.73	38.69	14.85	12.12	17.58	1.31	0.26	0.235
10	1.85	9.91	32.81	22.03	12.12	31.95	1.75	0.65	0.162
11	1.69	0.00	36.23	17.58	17.58	17.58	1.67	0.00	0.205
12	1.94	7.18	31.00	24.76	17.58	31.95	2.27	0.43	0.140
13	2.27	16.03	26.10	33.61	17.58	49.65	2.87	0.91	0.083
14	2.21	0.00	26.93	31.95	31.95	31.95	3.25	0.00	0.092
15	2.57	8.85	22.87	40.80	31.95	49.65	4.48	0.63	0.052


y/h
p/p0 (W)
p/p0 (CL)
x/h
y/h
p/p0

assn12_prob1.c

```
/*
  assn12_prob1.c

  Program to find flow properties in a symmetric channel using
  the method of characteristics.

  Carl Wassgren
  School of Mechanical Engineering
  Purdue University
  West Lafayette, IN  47907-1288
*/

# define PI (4.0*atan(1.0))
# define TWOPI (2.0*PI)
# define MAXPOINTS (20)

# include <stdio.h>
# include <stdlib.h>
# include <malloc.h>
# include <math.h>

struct state {
  double Ma, nu, theta, mu, CL, CR, x, y, pp0_ratio;
} point[MAXPOINTS+1];

/****** */

int main(int argc, char **argv) {
  double determine_Ma_given_nu(double, double);
  double determine_nu_given_Ma(double, double);
  double determine_mu_given_Ma(double);
  void determine_CL_and_CR_given_nu_and_theta(double, double,
                                              double *, double *);
  void determine_CL_given_CR_and_theta(double, double, double *);
  void determine_CR_given_CL_and_theta(double, double, double *);
  void determine_nu_and_theta_given_CL_and_CR(double, double,
                                              double *, double *);
  void determine_xy_given_pLpR(int, int, double *, double *);
  double determine_pp0_ratio_given_Ma(double, double);

  void print_data(int, FILE *);
  int i, p, flag;
  double gamma, Ma, nu, mu, theta, CL, CR, x, y, xend, A, B, C,
    h0, h1, pp0_ratio;
  FILE *outfile;

  /* Open the output file. */
  if ((outfile=fopen("methodofcharacteristics_02.out","wb"))==NULL) {
    printf("Error opening output file.\n");
    exit(1);
  }
  fprintf(outfile, "point\tMa\ttheta [deg]\tmu [deg]\tnu [deg]\tCL [deg]\tCR [deg]\tx/
h\ty/h\tp/p0\n");

  gamma = 1.4;
  h0 = 0.5;    // initial height of channel
  h1 = 0.05;   // curvature of channel
  xend = 4.0;  // maximum length of channel

  p=1;
  point[p].Ma = Ma = 1.33;
  point[p].theta = theta = 0.0;
  point[p].nu = nu = determine_nu_given_Ma(Ma, gamma);
  point[p].mu = mu = determine_mu_given_Ma(Ma);
  determine_CL_and_CR_given_nu_and_theta(nu, theta, &CL, &CR);
  point[p].CL = CL;
  point[p].CR = CR;
  point[p].x = x = 0.0;
  point[p].y = y = h0;
  point[p].pp0_ratio = determine_pp0_ratio_given_Ma(Ma, gamma);
  print_data(p, outfile);

  p++;
  point[p].Ma = Ma = 1.33;
  point[p].theta = theta = 0.0;
  point[p].nu = nu = determine_nu_given_Ma(Ma, gamma);
  point[p].mu = mu = determine_mu_given_Ma(Ma);
  determine_CL_and_CR_given_nu_and_theta(nu, theta, &CL, &CR);
  point[p].CL = CL;
  point[p].CR = CR;
  point[p].x = x = 0.0;
  point[p].y = y = 0.0;
  point[p].pp0_ratio = determine_pp0_ratio_given_Ma(Ma, gamma);
  print_data(p, outfile);

  flag = 0;
  do {
    p++;
    point[p].CL = CL = point[p-1].CL;
    point[p].CR = CR = point[p-2].CR;
    determine_nu_and_theta_given_CL_and_CR(CL, CR, &nu, &theta);
    point[p].nu = nu;
    point[p].theta = theta;
    point[p].Ma = Ma = determine_Ma_given_nu(nu, gamma);
    point[p].mu = mu = determine_mu_given_Ma(Ma);
    determine_xy_given_pLpR(p-1, p-2, &x, &y);
    point[p].x = x;
    point[p].y = y;
    point[p].pp0_ratio = determine_pp0_ratio_given_Ma(Ma, gamma);
    print_data(p, outfile);
    if (point[p].x >= xend) {
      flag = 1;
    } else {
      p++;
      A = 1.0;
      B = -tan(point[p-1].theta+point[p-1].mu)/h1;
      C = (h0-point[p-1].y+
           tan(point[p-1].theta+point[p-1].mu)*point[p-1].x)/h1;
      point[p].x = x = (-B-sqrt(B*B-4*A*C))/(2*A);
      point[p].theta = theta = atan(0.1*point[p].x);
      point[p].CL = CL = point[p-1].CL;
      determine_CR_given_CL_and_theta(CL, theta, &CR);
      point[p].CR = CR;
      determine_nu_and_theta_given_CL_and_CR(CL, CR, &nu, &theta);
      point[p].nu = nu;
      point[p].theta = theta;
      point[p].Ma = Ma = determine_Ma_given_nu(nu, gamma);
      point[p].mu = mu = determine_mu_given_Ma(Ma);
      point[p].y = y = 0.5+0.05*point[p].x*point[p].x;
      point[p].pp0_ratio = determine_pp0_ratio_given_Ma(Ma, gamma);
      print_data(p, outfile);
      if (point[p].x >= xend) {
        flag = 1;
      } else {

        p++;
        point[p].theta = theta = 0.0;
        point[p].CR = CR = point[p-2].CR;
        determine_CL_given_CR_and_theta(CR, theta, &CL);
        point[p].CL = CL;
```

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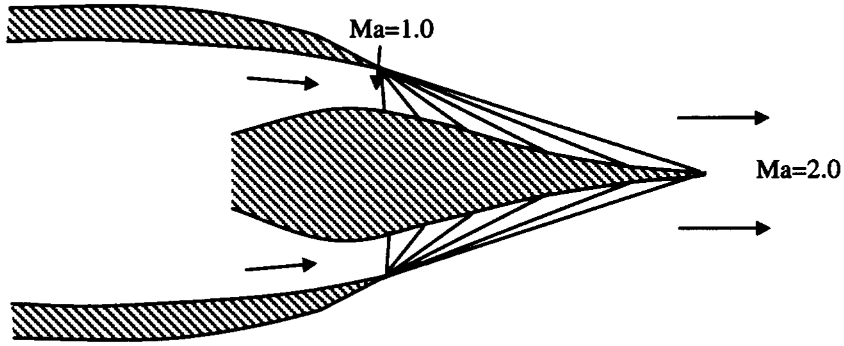
```c
      determine_nu_and_theta_given_CL_and_CR(CL, CR, &nu, &theta);
      point[p].nu = nu;
      point[p].theta = theta;
      point[p].Ma = Ma = determine_Ma_given_nu(nu, gamma);
      point[p].mu = mu = determine_mu_given_Ma(Ma);
      point[p].y = y = 0.0;
      point[p].x = x =
        point[p-2].x-point[p-2].y/tan(point[p-2].theta-point[p-2].mu);
      point[p].pp0_ratio = determine_pp0_ratio_given_Ma(Ma, gamma);
      print_data(p, outfile);
      if (point[p].x >= xend) {
        flag = 1;
      }
    }
  }
  } while (flag == 0);

  fclose(outfile);
  printf("Done.\n");
}

/* ***** */

double determine_Ma_given_nu(double nu, double gamma) {
  double c1, c2L, c2R, MaL, MaR, fL, fR, deltaMa, MaLold, MaRold, beta;

  if (nu < 0.0) {
   printf("Error!  Can't have nu<0.\n");
   exit(1);
  } else if (nu == 0) {
   return 1.0;
  } else {

   /* Use a bracketing scheme to determine the root. */
   c1 = (gamma+1)/(gamma-1);
   MaLold = MaL = 1.0;
   MaRold = MaR = 1.0e10;

   do {
     c2L = MaL*MaL-1;
     c2R = MaR*MaR-1;

     fL = sqrt(c1)*atan(sqrt(c2L/c1))-atan(sqrt(c2L))-nu;
     fR = sqrt(c1)*atan(sqrt(c2R/c1))-atan(sqrt(c2R))-nu;

     deltaMa = MaR-MaL;
     if ((fL < 0.0) && (fR > 0.0)) {
       MaLold = MaL;
       MaRold = MaR;
       MaR -= 0.5*deltaMa;
     } else if ((fL < 0.0) && (fR < 0.0)) {
       MaR = MaRold;
       MaL += 0.5*deltaMa;
     } else if ((fL > 0.0) && (fR > 0.0)) {
       MaL = MaLold;
       MaR -= 0.5*deltaMa;
     } else {
       MaL = MaLold;
       MaR = MaRold;
     }

     beta = fabs(MaR-MaL);
   } while (beta > 0.000001);

   return MaL;
  }
}

/* ***** */

double determine_nu_given_Ma(double Ma, double gamma) {
  double c1, c2;

  if (Ma < 1) {
    printf("Error!  Can't have Ma<1.\n");
    exit(1);
  } else {
    c1 = (gamma+1.0)/(gamma-1.0);
    c2 = Ma*Ma-1.0;

    return sqrt(c1)*atan(sqrt(c2/c1))-atan(sqrt(c2));
  }
}

/* ***** */

double determine_mu_given_Ma(double Ma) {

  if (Ma < 1) {
    printf("Error!  Can't have Ma<1.\n");
    exit(1);
  } else {
    return asin(1.0/Ma);
  }
}

/* ***** */

void determine_CL_and_CR_given_nu_and_theta(double nu, double theta,
                                             double *CL, double *CR) {
  *CL = nu-theta;
  *CR = nu+theta;
}

/* ***** */

void determine_CR_given_CL_and_theta(double CL, double theta, double *CR) {
  *CR = CL+2.0*theta;
}

/* ***** */

void determine_CL_given_CR_and_theta(double CR, double theta, double *CL) {
  *CL = CR-2.0*theta;
}

/* ***** */

void determine_nu_and_theta_given_CL_and_CR(double CL, double CR,
                                             double *nu, double *theta) {
  *nu = 0.5*(CR+CL);
  *theta = 0.5*(CR-CL);
}

/* ***** */

void determine_xy_given_pLpR(int pL, int pR, double *x, double *y) {
  double mR, xR, yR, mL, xL, yL;

  mL = tan(point[pL].theta+point[pL].mu);
```

assn12_prob1.c

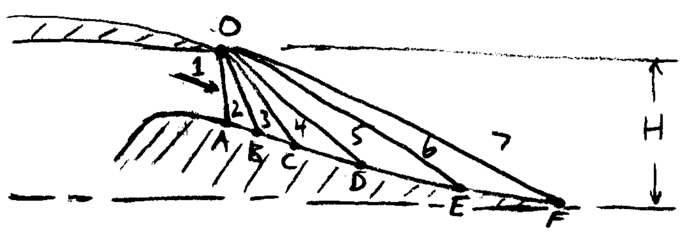
```
  xL = point[pL].x;
  yL = point[pL].y;

  mR = tan(point[pR].theta-point[pR].mu);
  xR = point[pR].x;
  yR = point[pR].y;

  *x = ((mR*xR-yR)-(mL*xL-yL))/(mR-mL);
  *y = (mL*(mR*xR-yR)-mR*(mL*xL-yL))/(mR-mL);
}

/****** */

double determine_pp0_ratio_given_Ma(double Ma, double gamma) {
  return pow(1.0+0.5*(gamma-1.0)*Ma*Ma, gamma/(1.0-gamma));
}

/****** */

void print_data(int p, FILE *outfile) {
  fprintf(outfile, "%d\t%.2f\t%.2f\t%.2f\t%.2f\t%.2f\t%.2f\t%.2f\t%.2f\t%.3f\n",
        p, point[p].Ma, point[p].theta*(180.0/PI), point[p].mu*(180.0/PI),
        point[p].nu*(180.0/PI),
        point[p].CL*(180.0/PI), point[p].CR*(180.0/PI),
        point[p].x, point[p].y, point[p].pp0_ratio);
  fflush(outfile);

  printf("p=%d\n", p);
}
```

Design the contour of a two-dimensional plug nozzle so as to produce cancellation of the waves incident on the plug. The nozzle is to provide a flow of 2 lb_m/sec of air at Mach 2.0. Assume a stagnation pressure of 100 psia and stagnation temperature of 800 °R.



SOLUTION:

- Discretize the expansion fan by using 6 waves as shown in the figure.



Region 1: $\underline{Ma_1 = 1.0} \Rightarrow \underline{\nu_1 = 0°}$

$\underline{\theta_1 = ?}$ (not yet known)

Region 7: $\underline{Ma_7 = 2.0} \Rightarrow \underline{\nu_7 = 26.38°}$

$\underline{\theta_7 = 0°}$ ↳ $\underline{\theta_1 = -26.38°}$ so that flow will be horizontal at plug tip

- Since 6 waves are used,

$$\Delta\nu = \frac{\nu_7 - \nu_1}{6} = \frac{26.38° - 0°}{6}$$

$$\therefore \underline{\Delta\nu = 4.40°}$$

SOLUTION...

- Region 2:

$$\nu_2 = \nu_1 + \Delta\nu \Rightarrow \underline{\nu_2 = 4.40^\circ} \Rightarrow \underline{Ma_2 = 1.23}$$

Across R-running characteristic from 1 to 2: (along a L-running characteristic

$$\cancelto{4.40^\circ}{\nu_2} - \theta_2 = \cancelto{0^\circ}{\nu_1} - \cancelto{-26.38^\circ}{\theta_1}$$

$$\Rightarrow \underline{\theta_2 = -21.98^\circ}$$

- Region 3:

$$\nu_3 = \nu_2 + \Delta\nu \Rightarrow \underline{\nu_3 = 8.80^\circ} \Rightarrow Ma_3 =$$

$$\cancelto{8.80^\circ}{\nu_3} - \theta_3 = \cancelto{4.40^\circ}{\nu_2} - \cancelto{-21.98^\circ}{\theta_2} \Rightarrow \underline{\theta_3 = -17.58^\circ}$$

- Use a similar approach for regions 5, 6, and 7:

Region	ν	Ma	θ	μ
1	0°	1.00	-26.38°	90.00°
2	4.40°	1.23	-21.98°	54.17°
3	8.79°	1.39	-17.59°	45.87°
4	13.19°	1.54	-13.19°	40.38°
5	17.59°	1.69	-8.79°	36.22°
6	21.98°	1.84	-4.40°	32.85°
7	26.38°	2.00	0.00°	30.00°

NOTE: $\sin\mu = \frac{1}{Ma}$

- To avoid reflections of the waves impinging on the plug, the lines segments between the waves must be inclined at the flow angle:

segment	angle w/r/t horizontal	
Before A	-26.38°	$(=\theta_1)$
AB	-21.98°	$(=\theta_2)$
BC	-17.58°	$(=\theta_3)$
CD	-13.19°	$(=\theta_4)$
DE	-8.79°	$(=\theta_5)$
EF	-4.40°	$(=\theta_6)$

SOLUTION...

- Angle of waves dividing regions
  - Since all of the waves are R-running waves:

$$\alpha_{ij} = \frac{1}{2}\left[(\theta_i - \mu_i) + (\theta_j - \mu_j)\right]$$

Wave separating Region i and j	Wave Angle, α_{ij}
1/2	$-96.27°$
2/3	$-69.81°$
3/4	$-58.51°$
4/5	$-49.29°$
5/6	$-41.13°$
6/7	$-33.62°$

- Location of points A through G w/r/t point O
  - First, note that to have $Ma_1 = 1.0$, the area of region 1 must be the sonic area, A^*.

$$A_1 = A^*$$

  - Recall that the choked mass flow rate is given by:

$$\dot{m}_{choked} = \left(1 + \frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}}\, A^*$$

$$\Rightarrow A^* = \frac{\sqrt{\frac{RT_0}{\gamma}}\,\dot{m}_{choked}}{p_0\left(1 + \frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}}$$

$$\therefore \underline{A^* = 7.38\times10^{-3}\,ft^2} \quad \text{using}$$

$$\underline{= 1.063\,in^2}$$

$\gamma = 1.4$

$R = 53.3\ \frac{ft\cdot lbf}{lbm\cdot °R} = 1716.3\ \frac{ft^2}{s^2\cdot °R}$

$T_0 = 800\,°R$

$\dot{m}_{choked} = 2\ \frac{lbm}{sec} = 6.211\times10^{-2}\ \frac{lbf\cdot s}{ft}$

$p_0 = 100\ psia = 14400\ \frac{lbf}{ft^2}$

  - Since the flow in Region 7 is horizontal, we can write:

$$A_7 = \left(\frac{A_7}{A^*}\right)A^* \quad \text{where} \quad \frac{A_7}{A^*} = 1.6875 \text{ using } Ma_7 = 2.0$$

$$A^* = 1.063\,in^2$$

$$\therefore \underline{A_7 = 1.7938\,in^2}$$

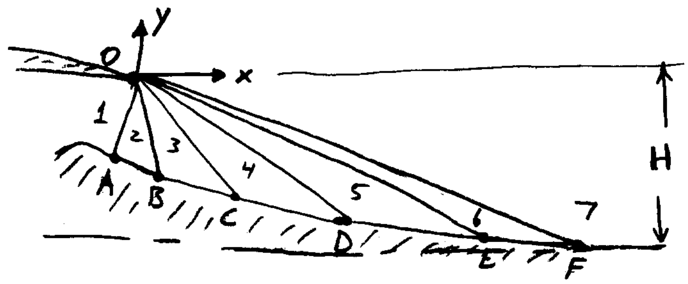
SOLUTION...

- Again, $A_1 = A^* = 1.063\,\text{in}^2$

  $A_7 = 1.7938\,\text{in}^2 = 2H$ (assume unit depth into page)

  $\Rightarrow \quad H = 0.8969\,\text{in}$

- Now find location of points w/r/t point O:



$(x_o, y_o) = (0, 0)$

point F
$$\left\{\begin{array}{l} y_F - \cancel{y_o}^{\,0} = \tan\alpha_{67}\,(x_F - \cancel{x_o}^{\,0}) \\ y_F = -0.8969\,\text{in} \;\; (= -H) \\ \Rightarrow x_F = 1.3488\,\text{in} \end{array}\right\} \Rightarrow (x_F, y_F) = (1.3488\,\text{in}, -0.8969\,\text{in})$$

point E
$$\left\{\begin{array}{l} y_E - \cancel{y_o}^{\,0} = \tan\alpha_{56}\,(x_E - \cancel{x_o}^{\,0}) \\ y_F - y_E = \tan(-4.40^\circ)\,(x_F - x_E) \end{array}\right\} \Rightarrow (x_E, y_E) = (0.9961\,\text{in}, -0.8698\,\text{in})$$

point D
$$\left\{\begin{array}{l} y_D - y_o = \tan\alpha_{45}\,(x_D - x_o) \\ y_E - y_D = \tan(-8.79^\circ)\,(x_E - x_D) \end{array}\right\} \Rightarrow (x_D, y_D) = (0.7103\,\text{in}, -0.8256\,\text{in})$$

point C
$$\left\{\begin{array}{l} y_C - y_o = \tan\alpha_{34}\,(x_C - x_o) \\ y_D - y_C = \tan(-13.19^\circ)\,(x_D - x_C) \end{array}\right\} \Rightarrow (x_C, y_C) = (0.4714\,\text{in}, -0.7696\,\text{in})$$

point B
$$\left\{\begin{array}{l} y_B - y_o = \tan\alpha_{23}\,(x_B - y_o) \\ y_C - y_B = \tan(-17.59^\circ)\,(x_C - x_B) \end{array}\right\} \Rightarrow (x_B, y_B) = (0.2582\,\text{in}, -0.7020\,\text{in})$$

SOLUTION...

point A
$$\left.\begin{cases} y_A - y_0 = \tan\alpha_{12}(x_A - x_0) \\ y_B - y_A = \tan(-21.98^\circ)(x_B - x_A) \end{cases}\right\} \Rightarrow (x_A, y_A) = (-0.0629\text{ in}, -0.5724\text{ in})$$

- The data for the design is summarized on the following two pages

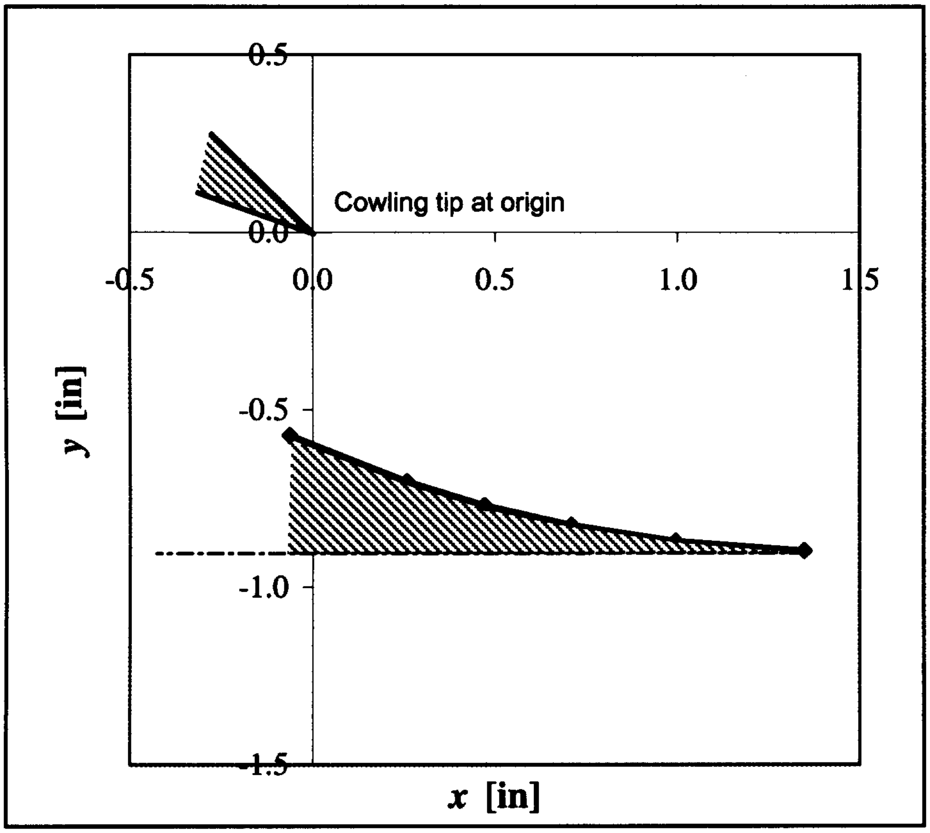
region	Ma	theta [deg]	mu [deg]	nu [deg]
1	1.00	-26.38	90.00	0.00
2	1.23	-21.98	54.18	4.40
3	1.39	-17.59	45.87	8.79
4	1.54	-13.19	40.38	13.19
5	1.69	-8.79	36.22	17.59
6	1.84	-4.40	32.85	21.98
7	2.00	0.00	30.00	26.38

WAVE ANGLES

regions	angle [deg]
1/2	-96.27
2/3	-69.81
3/4	-58.51
4/5	-49.29
5/6	-41.13
6/7	-33.62

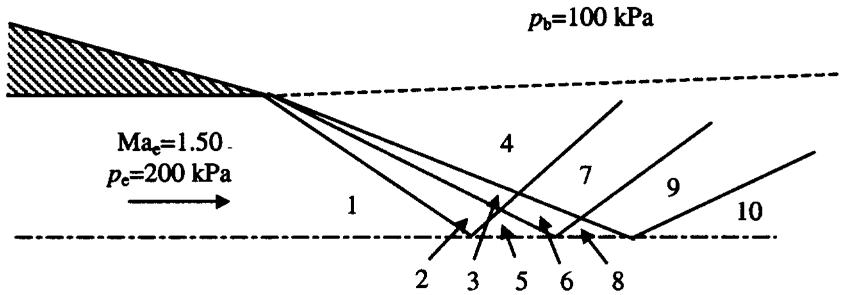
POINT LOCATIONS W/R/T COWLING TIP

point	x [in]	y [in]
A	-0.0629	-0.5724
B	0.2582	-0.7020
C	0.4714	-0.7696
D	0.7103	-0.8256
E	0.9961	-0.8698
F	1.3488	-0.8969



Cowling tip at origin
0.5
0.0
-0.5
-1.0
-1.5
-0.5
0.0
0.5
1.0
1.5
y [in]
x [in]

Flow at the exit of a Mach 1.5 supersonic nozzle is expanded from an exit-plane pressure of 200 kPa to a back pressure of 100 kPa. Determine the flow just downstream of the nozzle exit in the regions indicated.



SOLUTION:

- In region 4: $p_4 = p_b$

$$\Rightarrow \frac{p_4}{p_0} = \left(\frac{p_4}{p_e}\right)\left(\frac{p_e}{p_0}\right) \quad \text{where} \quad \frac{p_e}{p_0} = \left(1 + \frac{\gamma-1}{2}Ma_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\therefore \frac{p_4}{p_0} = 0.1362 \quad \text{using} \quad \begin{aligned} p_4 &= 100\text{ kPa} \\ p_e &= 200\text{ kPa} \\ \gamma &= 1.4 \\ Ma_e &= 1.50 \end{aligned}$$

$$\Rightarrow \underline{Ma_4 = 1.959} \quad \text{using} \quad \frac{p_4}{p_0} = \left(1 + \frac{\gamma-1}{2}Ma_4^2\right)^{\frac{\gamma}{1-\gamma}}$$

with $p_4/p_0 = 0.1362$ and $\gamma = 1.4$

$$\Rightarrow \underline{\nu_4 = 25.243^\circ}$$

using Prandtl-Meyer angle relation and $Ma_4 = 1.959$

- At exit: $Ma_e = 1.50 \Rightarrow \underline{\nu_e = 11.905^\circ}$ (using Prandtl-Meyer angle relation)

$$\therefore \overbrace{\nu_4}^{25.243^\circ} - \theta_4 = \overbrace{\nu_1}^{11.905^\circ} - \overbrace{\theta_1}^{0^\circ}$$

NOTE: Moving along a left-running characteristic from region 1 to 4

$$\therefore \underline{\theta_4 = 13.338^\circ}$$

Solution...

- Since three equally spaced Mach lines are used to approximate the flow:

$$\underline{\theta_2 = 4.446^\circ} \qquad \text{Note: } \Delta\theta = \frac{\theta_4 - \theta_1}{3} = \frac{13.338^\circ - 0^\circ}{3} = 4.446^\circ$$

$$\underline{\theta_3 = 8.892^\circ}$$

- Since $\nu_2 - \cancelto{4.446^\circ}{\theta_2} = \cancelto{11.905^\circ}{\nu_1} - \cancelto{0^\circ}{\theta_1}$

$$\Rightarrow \underline{\nu_2 = 16.351^\circ} \Rightarrow \underline{Ma_2 = 1.650}$$

$$\nu_3 - \cancelto{8.892^\circ}{\theta_3} = \cancelto{16.351^\circ}{\nu_2} - \cancelto{4.446^\circ}{\theta_2}$$

$$\underline{\nu_3 = 20.797^\circ} \Rightarrow \underline{Ma_3 = 1.802}$$

Note: Moving along left-running characteristic from 1 to 2 and from 2 to 3

- In region 5:

$\underline{\theta_5 = 0^\circ}$ (due to horizontal flow requirement)

$$\nu_5 + \cancelto{0^\circ}{\theta_5} = \cancelto{16.351^\circ}{\nu_2} + \cancelto{4.446^\circ}{\theta_2}$$

$$\Rightarrow \underline{\nu_5 = 20.797^\circ} \Rightarrow \underline{Ma_5 = 1.802}$$

Note: Moving along a right-running characteristic from 2 to 5

- In region 6:

Along left-running characteristic from region 5 to region 6:

$$\nu_6 - \theta_6 = \nu_5 - \theta_5$$

Along right-running characteristic from region 3 to region 6:

$$\nu_6 + \theta_6 = \nu_3 + \theta_3$$

$$\Rightarrow \nu_6 = \frac{1}{2}\left[\left(\cancelto{20.797^\circ}{\nu_5} - \cancelto{0^\circ}{\theta_5}\right) + \left(\cancelto{20.797^\circ}{\nu_3} + \cancelto{8.892^\circ}{\theta_3}\right)\right]$$

$$\theta_6 = \frac{1}{2}\left[\left(\cancelto{20.797^\circ}{\nu_3} + \cancelto{8.892^\circ}{\theta_3}\right) - \left(\cancelto{20.797^\circ}{\nu_5} - \cancelto{0^\circ}{\theta_5}\right)\right]$$

$$\therefore \underline{\nu_6 = 25.243^\circ} \Rightarrow \underline{Ma_6 = 1.959}$$

$$\underline{\theta_6 = 4.446^\circ}$$

- Repeat process for remaining regions noting that:

  $\theta_8 = 0°$

  $\theta_{10} = 0°$

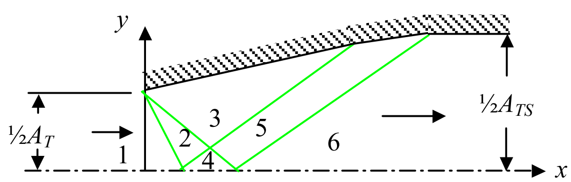
- Summarizing results:

Region	θ	ν	Ma	μ
1	0°	11.905°	1.500	41.81°
2	4.446°	16.351°	1.650	37.31°
3	8.892°	20.797°	1.802	33.71°
4	13.338°	25.243°	1.959	30.70°
5	0°	20.797°	1.802	33.71°
6	4.446°	25.243°	1.959	30.70°
7	8.892°	29.689°	2.122	28.11°
8	0°	29.689°	2.122	28.11°
9	4.446°	34.135°	2.294	25.84°
10	0°	38.581°	2.477	23.81°

Design the diverging section of a supersonic nozzle to produce uniform Mach 1.8 flow. Assume that the length of the nozzle should be kept to a minimum.

SOLUTION:

For the minimum length nozzle, a centered expansion fan should immediately follow the throat.



Assume that sonic flow occurs at the throat so that $A_T = A^*$ (and $\text{Ma}_1 = 1$). The test section area ratio may be found from the test section Mach number.

$$\frac{A_{TS}}{A^*} = \frac{1}{\text{Ma}_{TS}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_{TS}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \frac{A_{TS}}{A^*} = 1.4390 \tag{1}$$

For simplicity, consider only two (equally spaced) waves in the expansion fan so that the flow will travel through a total of four expansion waves between the throat and the test section. The total change in Prandtl-Meyer angle will be:

$$\text{Ma}_1 = 1.0 \Rightarrow \nu_1 = 0° \quad \text{and} \quad \text{Ma}_6 = \text{Ma}_{TS} = 1.8 \Rightarrow \nu_6 = 20.73° \tag{2}$$

$$\therefore \Delta\nu = (20.73° - 0°)/4 = 5.18° \tag{3}$$

Hence, through each wave the Prandtl-Meyer angle will change by 5.18°.

Note that when crossing left-running characteristic lines:

$$\Delta\theta = -\Delta\nu \tag{4}$$

and when crossing right-running characteristic lines:

$$\Delta\theta = \Delta\nu \tag{5}$$

The corresponding Mach numbers (and Prandtl-Meyer and Mach angles) are summarized in the table below.

Region	ν [deg]	Ma	μ [deg]	θ [deg]
1	0	1	90	0
2	5.18	1.26	52.33	5.18
3	10.37	1.45	43.69	10.37
4	10.37	1.45	43.69	0
5	15.55	1.62	38.03	5.18
6	20.73	1.80	33.75	0

The angle of the characteristic lines separating the regions with respect to the flow in the upstream region may be found by taking the average of the Mach angles in the neighboring regions. For example, the angle of the characteristic line separating regions 1 and 2 is:

$$\alpha_{12} \approx \tfrac{1}{2}(\mu_1 + \mu_2) = 71.17^{\text{o}} \tag{6}$$

Similarly for the other regions:

$$\alpha_{23} \approx \tfrac{1}{2}(\mu_2 + \mu_3) = 48.01^{\text{o}} \tag{7}$$

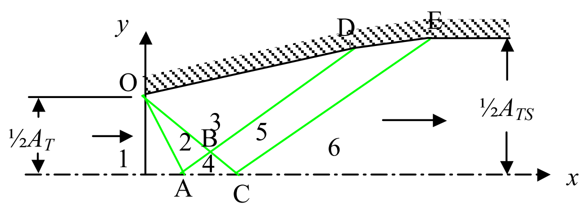
$$\alpha_{24} \approx \tfrac{1}{2}(\mu_2 + \mu_4) = 48.01^{\text{o}} \tag{8}$$

$$\alpha_{35} \approx \tfrac{1}{2}(\mu_3 + \mu_5) = 40.86^{\circ} \tag{9}$$

$$\alpha_{45} \approx \tfrac{1}{2}(\mu_4 + \mu_5) = 40.86^{\circ} \tag{10}$$

$$\alpha_{56} \approx \tfrac{1}{2}(\mu_5 + \mu_6) = 35.89^{\circ} \tag{11}$$

Note again that these are the average angles separating the various regions with respect to the upstream region.



The location of point A may be found as:

$$y_A - y_O = \tan(\theta_1 - \alpha_{12})(x_A - x_O) \tag{12}$$

where $(x_O, y_O) = (0, \tfrac{1}{2}A_T)$, $y_A = 0$, $\theta_1 = 0°$, and $\alpha_{12} = 71.17°$.

The location of point B is found by the intersection of two lines:

$$y_B - y_A = \tan(\theta_2 + \alpha_{24})(x_B - x_A) \tag{13}$$

$$y_B - y_O = \tan(\theta_2 - \alpha_{23})(x_B - x_O) \tag{14}$$

The location of point C is found using:

$$y_C - y_B = \tan(\theta_4 - \alpha_{45})(x_C - x_B) \tag{15}$$

where $y_C = 0$.

The location of point D is found from the intersection of two lines:

$$y_D - y_B = \tan(\theta_3 + \alpha_{35})(x_D - x_B) \tag{16}$$

$$y_D - y_O = \tan(\theta_3)(x_D - x_O) \quad \text{(The wall is parallel to the flow in Region 3.)} \tag{17}$$

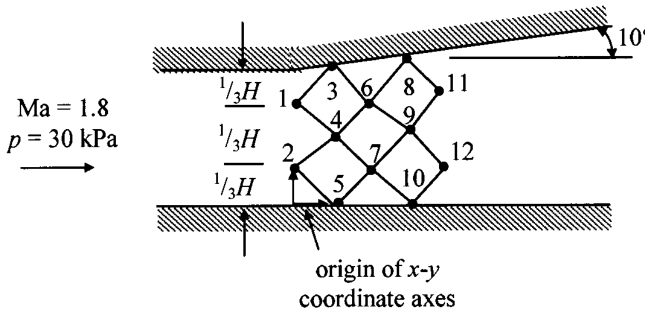
The location of point E is found from the intersection of two lines:

$$y_E - y_C = \tan(\theta_5 + \alpha_{56})(x_E - x_C) \tag{18}$$

$$y_E - y_D = \tan(\theta_5)(x_E - x_D) \quad \text{(The wall is parallel to the flow in Region 5.)} \tag{19}$$

A steady, 2D, supersonic flow at Mach 1.8 enters the channel shown in the figure. Using the method of characteristics (lattice approach), determine:

a. the Mach number at the points 3 through 7,
b. the flow angle at points 3 through 7,
c. the (x, y) location of points 3, 4, and 5, and
d. the pressure at points 3, 4, and 5.



$Ma_1 = Ma_2 = 1.8$
$\theta_1 = {}^2/_3(10°) = 6.67°$
$\theta_2 = {}^1/_3(10°) = 3.33°$

SOLUTION:

Working Eqns:

Along a R-running characteristic: $\nu + \theta = C_R$
Along a L-running characteristic: $\nu - \theta = C_L$
$\Rightarrow \nu = \frac{1}{2}(C_R + C_L)$, $\theta = \frac{1}{2}(C_R - C_L)$

$m_R = \left.\frac{dy}{dx}\right|_R = \tan(\theta - \mu)$ where $\mu = \sin^{-1}\left(\frac{1}{Ma}\right)$

$m_L = \left.\frac{dy}{dx}\right|_L = \tan(\theta + \mu)$

$y - y_0 = m(x - x_0)$

Points	Ma	ν	θ	C_L	C_R	$\left(\frac{x}{H}, \frac{y}{H}\right)$	μ
1	1.8	20.73°	6.67°	14.06°	27.40°	(0, 2/3)	33.75°
2	1.8	20.73°	3.33°	17.40°	24.06°	(0, 1/3)	33.75°

$\theta_3 = 10°$

$\nu_3 = C_{L1} + \theta_3 = 14.06° + 10° = 24.06° \Rightarrow Ma_3 = 1.92$

$\nu_4 = \frac{1}{2}(C_{R1} + C_{L2}) = \frac{1}{2}(27.40° + 17.40°) = 22.40° \Rightarrow Ma_4 = 1.86$

$\theta_4 = \frac{1}{2}(C_{R1} - C_{L2}) = \frac{1}{2}(27.40° - 17.40°) = 5.00°$

$\theta_5 = 0°$

$\nu_5 = C_{R2} - \theta_5 = 24.06° \Rightarrow Ma_5 = 1.92$

Solution...

$$\left.\begin{aligned} y_3 &= H + \tan 10^\circ x_3 \\ y_3 - y_1 &= m_{L1}(x_3 - x_1) \end{aligned}\right\} \Rightarrow \quad y_1 = H + \tan 10^\circ x_3 - m_{L1}(x_3 - x_1)$$

$$(\tan 10^\circ - m_{L1})x_3 = y_1 - H - m_{L1}x_1$$

$$x_3 = \frac{y_1 - H - m_{L1}\cancelto{0}{x_1}}{\tan 10^\circ - m_{L1}} \quad \text{where } m_{L1} = \tan(\theta_1 + \mu_1)$$

$$\therefore \quad x_3 = 0.49H$$

$$y_3 = 1.09H$$

$$\left.\begin{aligned} y_4 - y_1 &= m_{R1}(x_4 - x_1) \\ y_4 - y_2 &= m_{L2}(x_4 - x_2) \end{aligned}\right\} \Rightarrow \quad -y_1 + y_2 = m_{R1}(x_4 - x_1) - m_{L2}(x_4 - x_2) \quad \text{where } m_{L2} = \tan(\theta_2 + \mu_2)$$

$$(m_{R1} - m_{L2})x_4 = y_2 - y_1 + m_{R1}x_1 - m_{L2}x_2 \quad m_{R1} = \tan(\theta_1 - \mu_1)$$

$$x_4 = \frac{(m_{R1}\cancelto{0}{x_1} - y_1) - (m_{L2}\cancelto{0}{x_2} - y_2)}{m_{R1} - m_{L2}}$$

$$\therefore \quad x_4 = 0.26H$$

$$y_4 = 0.53H$$

$$\left.\begin{aligned} y_5 &= 0 \\ y_5 - y_2 &= m_{R2}(x_5 - x_2) \end{aligned}\right\} \Rightarrow \quad x_5 = \cancelto{0}{x_2} - \frac{y_2}{m_{R2}} \quad \text{where } m_{R2} = \tan(\theta_2 - \mu_2)$$

$$\therefore \quad x_5 = 0.57H$$

$$y_5 = 0$$

Points	Ma	ν	θ	C_L	C_R	$\left(\frac{x}{H}, \frac{y}{H}\right)$	μ
3	1.92	24.06°	10°	14.06°	34.06°	(0.49, 1.09)	31.39°
4	1.86	22.40°	5.00°	17.40°	27.40°	(0.26, 0.53)	32.52°
5	1.92	24.06°	0°	24.06°	24.06°	(0.57, 0)	31.39°

Solution...

$$\nu_6 = \frac{1}{2}\left(C_{R3} + C_{L4}\right) = \frac{1}{2}\left(34.06^\circ + 17.40^\circ\right) = 25.73^\circ$$

$$\theta_6 = \frac{1}{2}\left(C_{R3} - C_{L4}\right) = \frac{1}{2}\left(34.06^\circ - 17.40^\circ\right) = 8.33^\circ$$

$$\nu_7 = \frac{1}{2}\left(C_{R4} + C_{L5}\right) = \frac{1}{2}\left(27.40^\circ + 24.06^\circ\right) = 25.73^\circ$$

$$\theta_7 = \frac{1}{2}\left(C_{R4} - C_{L5}\right) = \frac{1}{2}\left(27.40^\circ - 24.06^\circ\right) = 1.67^\circ$$

Points	Ma_n	ν	θ
6	1.98	25.73°	8.33°
7	1.98	25.73°	1.67°

Isentropic flow throughout $\Rightarrow$ $\dfrac{p_n}{p_0} = \left(1 + \dfrac{\gamma - 1}{2} Ma_n^2\right)^{\frac{\gamma}{1-\gamma}}$

where $p_0 = (30\,\text{kPa})\left(1 + \frac{\gamma-1}{2}(1.8)^2\right)^{\frac{\gamma}{\gamma-1}} = 172.4\,\text{kPa}$

Points	Ma	p [kPa]
3	1.92	24.9
4	1.86	27.4
5	1.92	24.9

20. Flow Past a Wavy Wall Using Small Perturbation Theory

Recall that for a steady, irrotational, 2D flow with negligible body and viscous forces, the equation of motion assuming small velocity perturbations, is:

$$\left(1-\mathrm{Ma}_\infty^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{293}$$

The linearized boundary condition at the object surface is:

$$\left.\frac{dy}{dx}\right|_{\text{surface}}=\frac{\left.u_y^{'}\right|_{(x,0)}}{U_\infty} \tag{294}$$

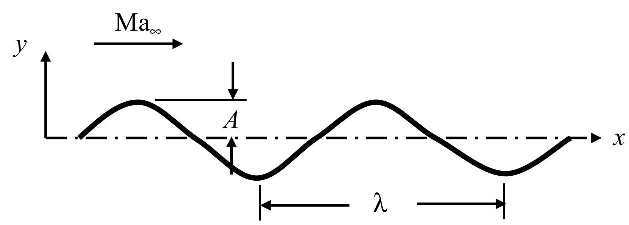
Far from the object ($y\rightarrow\infty$), the velocity perturbations must remain finite.

Recall that the pressure coefficient at the object surface is given by:

$$C_p=\frac{-2u_x^{'}}{U_\infty} \tag{295}$$

Now consider flow past a wavy wall. Let the profile of the wall be given by:

$$y_s=A\sin\left(\frac{2\pi x}{\lambda}\right)$$



We'll assume that the wall causes small perturbations in the flow, i.e.:

$A = \lambda$

Thus, the boundary condition at the wall is:

$$\left.u_y^{'}\right|_{(x,0)}=U_\infty\left.\frac{dy}{dx}\right|_{\text{surface}}=\frac{2\pi AU_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right) \tag{296}$$

Subsonic Flow

First let's examine the case when $\text{Ma}_\infty^2 < 1$ so that $(1-\text{Ma}_\infty^2) > 0$. Equation (293) will be an elliptic PDE for this case. One method for solving elliptic, linear PDEs is to use separation of variables where we assume that the solution can be written as some function of x multiplied by some function of y:

$$\Phi(x,y) = X(x)Y(y)$$

Substituting into Eq. (293) and simplifying gives:

$$\left(1-\text{Ma}_\infty^2\right)X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-1}{\left(1-\text{Ma}_\infty^2\right)}\frac{Y''}{Y} = -k^2$$

Since the LHS of the previous equation is a function only of x, and the RHS is a function only of y, then in order for the two sides to be equal, they must equal a constant, which we'll call $-k^2$. Thus, we can write the following two equations:

$$X'' + k^2 X = 0$$

$$Y'' - k^2\left(1-\text{Ma}_\infty^2\right)Y = 0$$

The solution to the first differential equation involving X is:

$$X = c_1\cos(kx) + c_2\sin(kx)$$

and the solution to the differential equation involving Y is:

$$Y = c_3\exp\left(ky\sqrt{1-\text{Ma}_\infty^2}\right) + c_4\exp\left(-ky\sqrt{1-\text{Ma}_\infty^2}\right)$$

Note that the square root term results in a real quantity since the incoming flow is subsonic. Substituting these functions into the perturbation potential and determining the corresponding velocity perturbations gives:

$$\Phi = \left[c_1\cos(kx) + c_2\sin(kx)\right]\left[c_3\exp\left(ky\sqrt{1-\text{Ma}_\infty^2}\right) + c_4\exp\left(-ky\sqrt{1-\text{Ma}_\infty^2}\right)\right]$$

$$u_x' = \frac{\partial\Phi}{\partial x} = k\left[-c_1\sin(kx) + c_2\cos(kx)\right]\left[c_3\exp\left(ky\sqrt{1-\text{Ma}_\infty^2}\right) + c_4\exp\left(-ky\sqrt{1-\text{Ma}_\infty^2}\right)\right]$$

$$u_y' = \frac{\partial\Phi}{\partial y} = k\sqrt{1-\text{Ma}_\infty^2}\left[c_1\cos(kx) + c_2\sin(kx)\right]\left[c_3\exp\left(ky\sqrt{1-\text{Ma}_\infty^2}\right) - c_4\exp\left(-ky\sqrt{1-\text{Ma}_\infty^2}\right)\right]$$

The constants c_1, c_2, c_3, and c_4 are determined from the boundary conditions at the wall and at $y\rightarrow\infty$. In order for the perturbation velocities to remain finite as $y\rightarrow\infty$ we must have $c_3=0$. At the wall ($y = 0$) we have (Eq. (296)):

$$u_y'\Big|_{(x,0)} = \frac{2\pi A U_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right) = -c_4 k\sqrt{1-\text{Ma}_\infty^2}\left[c_1\cos(kx) + c_2\sin(kx)\right]$$

Thus we see that $c_2=0$, $k=(2\pi/\lambda)$, and

$$c_1 c_4 = \frac{-U_\infty}{\sqrt{1-\text{Ma}_\infty^2}\left(\frac{2\pi}{\lambda}\right)}\left(\frac{2\pi A}{\lambda}\right) = \frac{-AU_\infty}{\sqrt{1-\text{Ma}_\infty^2}}$$

Thus, the perturbation potential and perturbation velocities become:

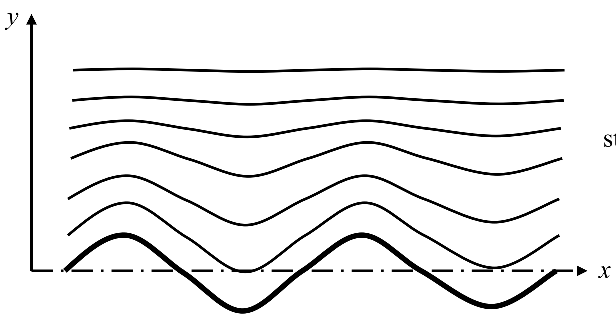
$$\boxed{\begin{aligned}\Phi &= \frac{-AU_\infty}{\sqrt{1-\mathrm{Ma}_\infty^2}}\cos\left(\frac{2\pi x}{\lambda}\right)\exp\left(-\frac{2\pi y}{\lambda}\sqrt{1-\mathrm{Ma}_\infty^2}\right)\\ u_x' &= \frac{2\pi AU_\infty}{\lambda\sqrt{1-\mathrm{Ma}_\infty^2}}\sin\left(\frac{2\pi x}{\lambda}\right)\exp\left(-\frac{2\pi y}{\lambda}\sqrt{1-\mathrm{Ma}_\infty^2}\right)\\ u_y' &= \frac{2\pi AU_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\exp\left(-\frac{2\pi y}{\lambda}\sqrt{1-\mathrm{Ma}_\infty^2}\right)\end{aligned}} \tag{297}$$

The pressure coefficient at the wall (refer to Eq. (295)) is:

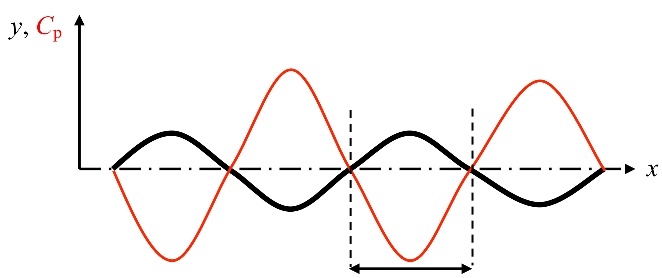
$$\boxed{C_p = \frac{-4\pi A}{\lambda\sqrt{1-\mathrm{Ma}_\infty^2}}\sin\left(\frac{2\pi x}{\lambda}\right)} \tag{298}$$

Notes:

1. The disturbance of the wall dies out as $y\to\infty$.



2. The pressure peaks occur in the troughs of the wall and vice versa, i.e., the pressure is in-phase with the wall. As a result, there will be no drag force on the wall.



pressure profile is symmetric ⇒ net horizontal force is zero

3. We can also examine this compressible subsonic flow using the coordinate transformation discussed in a previous set of notes concerning the Prandtl-Glauert rule. Recall that if we examine flow in the (ξ, η) plane where:

$$\xi = x$$
$$\eta = \beta y \quad \text{where } \beta = \sqrt{1\text{-Ma}_\infty^2} \tag{299}$$
$$\bar{\Phi} = \beta\Phi$$

then the governing equation becomes Laplace's equation, i.e.:

$$\frac{\partial^2\bar{\Phi}}{\partial\xi^2} + \frac{\partial^2\bar{\Phi}}{\partial\eta^2} = 0 \quad \text{(This is the governing equation for an incompressible flow!)} \tag{300}$$

Furthermore, the shape of the boundary surface in the (ξ, η) plane is the same as the boundary surface shape in the (x, y) plane. Solving (300) using separation of variables gives:

$$\bar{\Phi} = \left[c_1\cos(k\xi) + c_2\sin(k\xi)\right]\left[c_3\exp(k\eta) + c_4\exp(-k\eta)\right] \tag{301}$$

Since the flow must have finite velocities as $\eta \to \infty$ (this is true for both the compressible and incompressible flow cases), we can conclude that c_3 must be zero. The boundary condition at the wavy wall surface:

$$u'_\eta\Big|_{(\xi,0)} = \frac{2\pi AU_\infty}{\lambda}\cos\left(\frac{2\pi\xi}{\lambda}\right) = -c_4 k\left[c_1\cos(k\xi) + c_2\sin(k\xi)\right] \tag{302}$$

indicates that c_2 must be zero, k must be ($2\pi/\lambda$), and c_1c_4 must be $-AU_\infty$. Thus:

$$\bar{\Phi} = -AU_\infty\cos\left(\frac{2\pi\xi}{\lambda}\right)\exp\left(-\frac{2\pi\eta}{\lambda}\right) \tag{303}$$

Transforming back into the (x, y) plane using Eq. (299) gives:

$$\Phi = \frac{1}{\beta}\bar{\Phi} = \frac{-AU_\infty}{\sqrt{1-\text{Ma}_\infty^2}}\cos\left(\frac{2\pi x}{\lambda}\right)\exp\left(-\frac{2\pi y}{\lambda}\sqrt{1-\text{Ma}_\infty^2}\right) \tag{304}$$

This is precisely the same relation derived in Eq. (297). The pressure coefficient corresponding to the incompressible flow (i.e., the (ξ, η) plane) is:

$$\bar{C}_p\Big|_{\eta=0} = -\frac{2}{U_\infty}\underbrace{\frac{\partial\bar{\Phi}}{\partial\xi}}_{=u'_\xi} = -\frac{4\pi A}{\lambda}\sin\left(\frac{2\pi\xi}{\lambda}\right) \tag{305}$$

Using the Prandtl-Glauert rule, the pressure coefficient for the compressible flow (i.e., the (x, y) plane) is:

$$C_p = \frac{\bar{C}_p}{\sqrt{1-\text{Ma}_\infty^2}} = \frac{-4\pi A}{\lambda\sqrt{1-\text{Ma}_\infty^2}}\sin\left(\frac{2\pi x}{\lambda}\right) \tag{306}$$

Equation (306) is the same as Eq. (298), as expected.

Supersonic Flow

Now consider the case where $\text{Ma}_\infty^2 > 1$ so that $(1-\text{Ma}_\infty^2) < 0$ and Eq. (293) becomes a hyperbolic PDE with a form identical to that of the wave equation. Thus, we'll utilize d'Alembert's solution to the wave equation, which has the form:

$$\Phi = f_L\left(x - y\sqrt{\text{Ma}_\infty^2 - 1}\right) + f_R\left(x + y\sqrt{\text{Ma}_\infty^2 - 1}\right)$$

or

$$\Phi = f_L(\eta) + f_R(\xi) \tag{307}$$

where

$$\eta = x - y\sqrt{\text{Ma}_\infty^2 - 1}$$

$$\xi = x + y\sqrt{\text{Ma}_\infty^2 - 1}$$

Recall that the wave equation has the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and has the (d'Alembert) solution:

$$u(x,t) = f(x - ct) + g(x + ct)$$

where f and g are functions determined by the initial and boundary conditions. For our case, the governing equation is:

$$\frac{\partial^2 \Phi}{\partial y^2} = \left(\text{Ma}_\infty^2 - 1\right)^2 \frac{\partial^2 \Phi}{\partial x^2}$$

Note that the functions f_L and f_R are unknown at this point but will eventually be determined using the boundary conditions. To verify that Eq. (307) is indeed a general solution, substitute it back into Eq. (293) and simplify:

$$\left(1 - \text{Ma}_\infty^2\right)\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

where

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{d^2 f_L}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{d^2 f_L}{d\xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{d^2 f_L}{d\eta^2} + \frac{d^2 f_L}{d\xi^2}$$

$$\frac{\partial^2 \Phi}{\partial y^2} = \frac{d^2 f_L}{d\eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{d^2 f_L}{d\xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 = \left(\text{Ma}_\infty^2 - 1\right)\left[\frac{d^2 f_L}{d\eta^2} + \frac{d^2 f_L}{d\xi^2}\right]$$

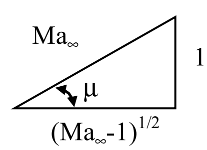
Of particular interest for this solution are the curves corresponding to f_L=constant and f_R=constant (along these curves Φ and, hence u_x' and u_y', will remain constant). Note that the form of these functions will be dictated by the boundary conditions. Thus, the value of the perturbation potential, Φ, (as well as the perturbation velocities) will be propagated from the boundary conditions into the rest of the flow along the curves where f_L and f_R are constant. The shape of these curves can be determined from:

$$f_L = \text{constant} \quad \Rightarrow \quad x - y\sqrt{\text{Ma}_\infty^2 - 1} = \text{constant} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{f_L = \text{constant}} = \frac{1}{\sqrt{\text{Ma}_\infty^2 - 1}}$$

$$f_R = \text{constant} \quad \Rightarrow \quad x + y\sqrt{\text{Ma}_\infty^2 - 1} = \text{constant} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{f_R = \text{constant}} = \frac{-1}{\sqrt{\text{Ma}_\infty^2 - 1}}$$

Thus we see that the two curves are, in fact, lines with opposite slopes. Moreover, the slope of these lines is equal to the slope of a Mach line:

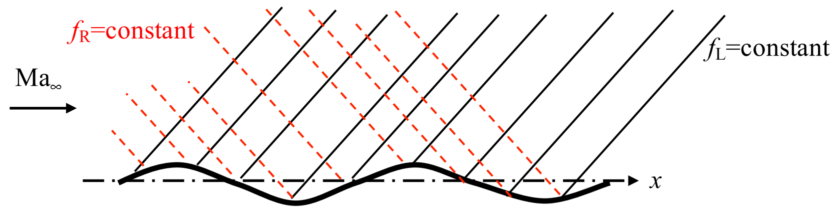
$$\sin\mu = \frac{1}{\text{Ma}_\infty} \quad \Rightarrow \quad \tan\mu = \frac{1}{\sqrt{\text{Ma}_\infty^2 - 1}}$$



It is worthwhile to re-iterate these last two important points:

1. **Information propagates along the curves where f_L and f_R are constant.**
2. **Curves along which f_L and f_R are constant correspond to Mach lines of opposite slope.**

For the problem at hand (supersonic flow over a wavy wall), the perturbation potential should only include the function f_L since if we included f_R in the solution, then information could propagate upstream along the Mach line with negative slope – an impossibility in supersonic flows.



Thus, the perturbation potential is:

$$\Phi = f_L(\eta) \quad \text{where} \quad \eta = x - y\sqrt{\mathrm{Ma}_\infty^2 - 1} \tag{308}$$

The form of f_L will be dependent on the boundary conditions. Recall that the boundary condition at the wall is given by Eq. (296):

$$u_y'\Big|_{(x,0)} = U_\infty \frac{dy}{dx}\Big|_{\text{surface}} = \frac{2\pi A U_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)$$

Substituting Eq. (308) into the boundary condition gives:

$$u_y'\Big|_{y=0} = \frac{\partial \Phi}{\partial y}\Big|_{y=0} = \left(\frac{df_L}{d\eta}\frac{\partial \eta}{\partial y}\right)\Big|_{y=0} = -\sqrt{\mathrm{Ma}_\infty^2 - 1}\,\frac{df_L}{dx} = \frac{2\pi A U_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)$$

$$\eta = x - y\sqrt{\mathrm{Ma}_\infty^2 - 1}$$

$$\frac{\partial \eta}{\partial y} = -\sqrt{\mathrm{Ma}_\infty^2 - 1}$$

$$\frac{df_L}{dx} = \frac{-2\pi A U_\infty}{\lambda\sqrt{\mathrm{Ma}_\infty^2 - 1}}\cos\left(\frac{2\pi x}{\lambda}\right)$$

Note that in the previous equations the fact that at y=0, $df_L/d\eta = df_L/dx$, has been used. Integrating gives:

$$f_L(x) = \frac{-AU_\infty}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}\sin\left(\frac{2\pi x}{\lambda}\right) + \text{constant}$$

Thus,

$$f_L(\eta) = \frac{-AU_\infty}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}\sin\left(\frac{2\pi \eta}{\lambda}\right) + \text{constant}$$

The resulting perturbation potential and corresponding perturbation velocities are:

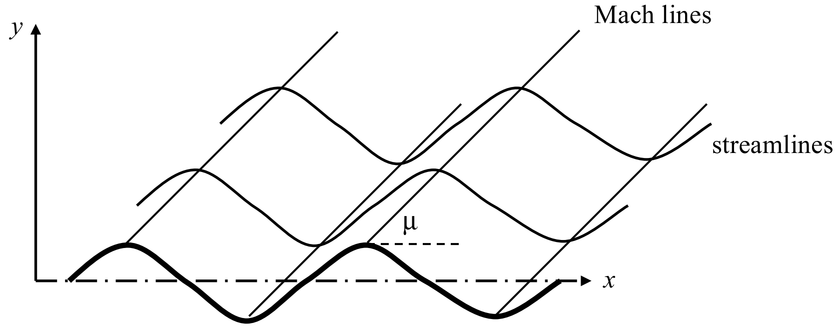
$$\boxed{\begin{aligned} \Phi &= \frac{-AU_\infty}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}\sin\left[\frac{2\pi}{\lambda}\left(x - y\sqrt{\mathrm{Ma}_\infty^2 - 1}\right)\right] + \text{constant} \\ u_x' &= \frac{-2\pi A U_\infty}{\lambda\sqrt{\mathrm{Ma}_\infty^2 - 1}}\cos\left[\frac{2\pi}{\lambda}\left(x - y\sqrt{\mathrm{Ma}_\infty^2 - 1}\right)\right] \\ u_y' &= \frac{2\pi A U_\infty}{\lambda}\cos\left[\frac{2\pi}{\lambda}\left(x - y\sqrt{\mathrm{Ma}_\infty^2 - 1}\right)\right] \end{aligned}} \tag{309}$$

The pressure coefficient at the wall (Eq. (295)) for this flow is:

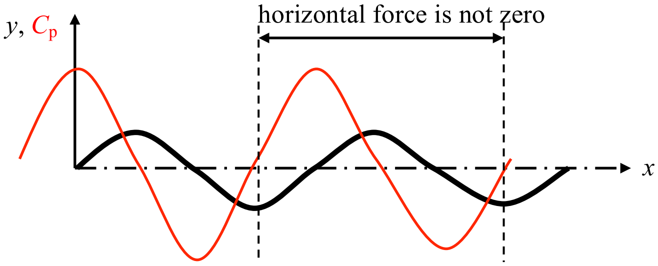
$$\boxed{C_p = -\frac{2u_x'}{U_\infty} = \frac{4\pi A}{\lambda\sqrt{\mathrm{Ma}_\infty^2 - 1}}\cos\left(\frac{2\pi x}{\lambda}\right)} \tag{310}$$

Notes:

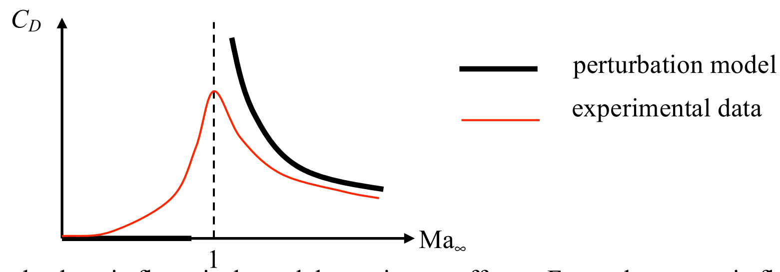
1. Along lines of x-$y(\mathrm{Ma}_\infty^2-1)^{1/2}$=constant (recall that these are the Mach lines), the perturbation potential and velocities are constant. Thus, the disturbances produced by the wall are felt equally along the same Mach line. Recall that in the subsonic flow case, the disturbances die out as $y\to\infty$.



2. In the subsonic case the drag on the wall is zero. For supersonic flow, however, the drag is not zero. This is because the pressure coefficient is out of phase with the wall shape (shown below). This type of drag is commonly referred to as supersonic **wave drag**.

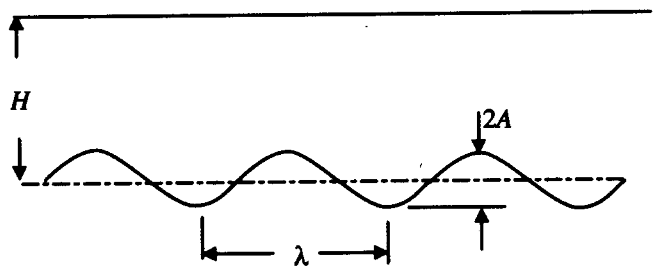


3. In an actual flow, the drag coefficient on the wall as a function of Mach number look like:



The drag occurring in real subsonic flows is due solely to viscous effects. For real supersonic flows around slender bodies, the wave drag is typically much larger than the viscous drag. Note that the assumptions for perturbation analysis do not allow for the investigation near $\mathrm{Ma}_\infty\approx1$.

A planar channel is composed of a wavy wall with waves of amplitude A and wavelength λ and a straight wall. The straight wall and the centerline of the wavy wall are separated by a distance, H. Determine the perturbation potential for a subsonic flow through this channel.



$$y = A\sin\left(\frac{2\pi x}{\lambda}\right)$$

$A << H$

$A << \lambda$

SOLUTION:

PDE: $$\left(1-Ma_\infty^2\right)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0$$

BCs: $$\left.\frac{dy}{dx}\right|_{\substack{\text{surface}\\ \text{at } y=0}} = \frac{u_y'(x,y=0)}{U_\infty} \Rightarrow u_y'(x,0)=\frac{2\pi A U_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)$$

$$\left.\frac{dy}{dx}\right|_{\substack{\text{surface}\\ \text{at } y=H}} = \frac{u_y'(x,y=H)}{U_\infty} \Rightarrow u_y'(x,H)=0$$

- Try separation of variables:

$$\Phi = X(x)\,Y(y)$$

$$\Rightarrow \left(1-Ma_\infty^2\right)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2} = \left(1-Ma_\infty^2\right)X''Y + XY'' = 0$$

$$\Rightarrow \frac{X''}{X} = \frac{-1}{1-Ma_\infty^2}\frac{Y''}{Y} = -K^2 \quad \text{(a constant)}$$

$$\Rightarrow \begin{aligned} &X'' + K^2 X = 0 \\ &Y'' + K^2\left(1-Ma_\infty^2\right)Y = 0 \end{aligned}$$

$$X(x) = c_1\cos(Kx) + c_2\sin(Kx)$$

$$Y(y) = c_3\cosh\left(K\sqrt{1-Ma_\infty^2}\,y\right) + c_4\sinh\left(K\sqrt{1-Ma_\infty^2}\,y\right)$$

Solution...

$$\Phi = \left[c_1\cos(Kx) + c_2\sin(Kx)\right]\left[c_3\cosh\left(K\sqrt{1-Ma_\infty^2}\,y\right) + c_4\sinh\left(K\sqrt{1-Ma_\infty^2}\,y\right)\right]$$

$$u_x' = \frac{\partial\Phi}{\partial x} = K\left[-c_1\sin(Kx) + c_2\cos(Kx)\right]\left[c_3\cosh(\cdots) + c_4\sinh(\cdots)\right]$$

$$u_y' = \frac{\partial\Phi}{\partial y} = K\sqrt{1-Ma_\infty^2}\left[c_1\cos(Kx)+c_2\sin(Kx)\right]\left[c_3\sinh(\cdots) + c_4\cosh(\cdots)\right]$$

- Apply BCs:

$$u_y'(x, y=0) = \frac{2\pi A U_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right) = K\sqrt{1-Ma_\infty^2}\left[c_1\cos(Kx)+c_2\sin(Kx)\right]c_4$$

$$\Rightarrow \quad c_2 c_4 = 0$$

$$K = \frac{2\pi}{\lambda}$$

$$c_1 c_4 K\sqrt{1-Ma_\infty^2} = \frac{2\pi A U_\infty}{\lambda} \Rightarrow c_1 c_4 \frac{2\pi}{\lambda}\sqrt{1-Ma_\infty^2} = \frac{2\pi A U_\infty}{\lambda}$$

$$\Rightarrow \quad c_1 c_4 = \frac{A U_\infty}{\sqrt{1-Ma_\infty^2}}$$

$$c_2 c_4 = 0 \quad \text{and} \quad c_1 c_4 = \frac{A U_\infty}{\sqrt{1-Ma_\infty^2}}$$

$$\Rightarrow \quad c_2 = 0$$

$$\Phi = \cos\left(\frac{2\pi}{\lambda}x\right)\left[c_1 c_3\cosh(\beta y) + \frac{A U_\infty}{\sqrt{1-Ma_\infty^2}}\sinh(\beta y)\right]$$

$$u_x' = \frac{2\pi}{\lambda}\sin\left(\frac{2\pi x}{\lambda}\right)\left[-c_1 c_3\cosh(\beta y) + \frac{A U_\infty}{\sqrt{1-Ma_\infty^2}}\sinh(\beta y)\right]$$

$$u_y' = \frac{2\pi}{\lambda}\sqrt{1-Ma_\infty^2}\cos\left(\frac{2\pi x}{\lambda}\right)\left[c_1 c_3\sinh(\beta y) + \frac{A U_\infty}{\sqrt{1-Ma_\infty^2}}\cosh(\beta y)\right]$$

where $\beta = \frac{2\pi}{\lambda}\sqrt{1-Ma_\infty^2}$

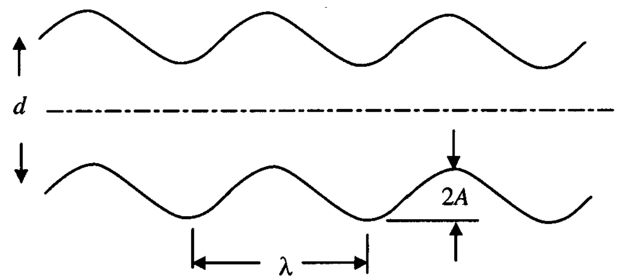
Solution...

$$u_y'(x, y=H) = 0 = \frac{2\pi}{\lambda}\sqrt{1-Ma_\infty^2}\cos\left(\frac{2\pi x}{\lambda}\right)\left[c_1 c_3 \sinh(\beta H) + \frac{AU_\infty}{\sqrt{1-Ma_\infty^2}}\cosh(\beta H)\right]$$

$$\Rightarrow \quad \underline{c_1 c_3 = \frac{-AU_\infty}{\sqrt{1-Ma_\infty^2}}\,\frac{\cosh(\beta H)}{\sinh(\beta H)}}$$

$$\boxed{\begin{aligned}
&\therefore \quad \Phi = \frac{AU_\infty}{\sqrt{1-Ma_\infty^2}}\cos\left(\frac{2\pi x}{\lambda}\right)\left[\sinh(\beta y) - \coth(\beta H)\cosh(\beta y)\right] \\
&u_x' = \frac{-2\pi AU_\infty}{\sqrt{1-Ma_\infty^2}}\sin\left(\frac{2\pi x}{\lambda}\right)\left[\sinh(\beta y) - \coth(\beta H)\cosh(\beta y)\right] \\
&u_y' = \frac{2\pi AU_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\left[\cosh(\beta y) - \coth(\beta H)\sinh(\beta y)\right] \\
&\qquad \text{where } \beta = \frac{2\pi}{\lambda}\sqrt{1-Ma_\infty^2}
\end{aligned}}$$

Two-dimensional, subsonic linearized potential flow takes place between two wavy walls as shown in the figure. Solve for Φ and determine the pressure distribution along the centerline.



$$y = A\sin\left(\frac{2\pi x}{\lambda}\right)$$

$A << d$

$A << \lambda$

SOLUTION:

- For linearized, 2D potential flow:

PDE: $$\left(1 - Ma_\infty^2\right)\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

BCs: $$\left.\frac{dy}{dx}\right|_{surface} = \frac{\left.u_y'\right|_{y=\pm d}}{U_\infty}$$

u_x', u_y' remain finite

Pressure coefficient: $$C_p = \frac{-2u_x'}{U_\infty}$$

- Here, $$\left.\frac{dy}{dx}\right|_{surface} = \frac{2\pi A}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right) = \frac{\left.u_y'\right|_{y=\pm\frac{1}{2}d}}{U_\infty}$$

$$\Rightarrow \quad u_y'(x, y=\pm\tfrac{1}{2}d) = \frac{2\pi A U_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)$$

- Try separation of variables to solve for Φ:

Assume: $\Phi = X(x)\,Y(y)$

substitute into PDE: $\left(1 - Ma_\infty^2\right) X'' Y + X Y'' = 0$

a constant

$$\Rightarrow \quad \frac{X''}{X} = \frac{-1}{\left(1 - Ma_\infty^2\right)}\frac{Y''}{Y} = -K^2$$

$$\Rightarrow \quad X'' + K^2 X = 0$$

$$Y'' - K^2\left(1 - Ma_\infty^2\right) Y = 0$$

SOLUTION...

$$\Rightarrow \quad X = c_1\cos(Kx) + c_2\sin(Kx)$$

$$Y = c_3\exp\left(K\sqrt{1-Ma_\infty^2}\,y\right) + c_4\exp\left(-K\sqrt{1-Ma_\infty^2}\,y\right)$$

$$\Rightarrow \quad \Phi = \left[c_1\cos(Kx) + c_2\sin(Kx)\right]\left[c_3\exp\left(Ky\sqrt{1-Ma_\infty^2}\right) + c_4\exp\left(-Ky\sqrt{1-Ma_\infty^2}\right)\right]$$

$$\Rightarrow \quad u_y' = \frac{\partial\Phi}{\partial y} = K\sqrt{1-Ma_\infty^2}\left[c_1\cos(Kx) + c_2\sin(Kx)\right]\left[c_3\exp\left(Ky\sqrt{1-Ma_\infty^2}\right) - c_4\exp\left(-Ky\sqrt{1-Ma_\infty^2}\right)\right]$$

- Apply BC at $y = +\frac{1}{2}d$:

$$u_y'(x, y=+\tfrac{1}{2}d) = \frac{2\pi A U_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right) = K\sqrt{1-Ma_\infty^2}\,\left[\cdots\right]\left[\cdots\right]$$

$$\Rightarrow \quad c_2 = 0, \quad K = \frac{2\pi}{\lambda}$$

$$c_1c_3\exp\left(\frac{\pi d}{\lambda}\sqrt{1-Ma_\infty^2}\right) - c_1c_4\exp\left(\frac{-\pi d}{\lambda}\sqrt{1-Ma_\infty^2}\right) = \frac{AU_\infty}{\sqrt{1-Ma_\infty^2}} \qquad (1)$$

- Apply BC at $y = -\frac{d}{2}$:

$$c_1c_3\exp\left(\frac{-2\pi d}{2\lambda}\sqrt{1-Ma_\infty^2}\right) - c_1c_4\exp\left(\frac{2\pi d}{2\lambda}\sqrt{1-Ma_\infty^2}\right) = \frac{AU_\infty}{\sqrt{1-Ma_\infty^2}} \qquad (2)$$

- Solving (1) and (2) for c_1c_3 and c_1c_4:

$$c_1c_3 = -c_1c_4 = \frac{\sinh\left(\frac{\pi d}{\lambda}\sqrt{1-Ma_\infty^2}\right)}{\sinh\left(\frac{2\pi d}{\lambda}\sqrt{1-Ma_\infty^2}\right)}\left[\frac{AU_\infty}{\sqrt{1-Ma_\infty^2}}\right]$$

SOLUTION ...

$$\boxed{\begin{aligned}
\Phi &= \frac{\sinh\left(\xi\frac{d}{2}\right)}{\sinh(\xi d)}\left[\frac{2AU_\infty}{\sqrt{1-Ma_\infty^2}}\right]\sinh(\xi y)\cos\left(\frac{2\pi x}{\lambda}\right)\\
u_x' &= \frac{\sinh\left(\xi\frac{d}{2}\right)}{\sinh(\xi d)}\left[\frac{-4\pi AU_\infty}{\lambda\sqrt{1-Ma_\infty^2}}\right]\sinh(\xi y)\sin\left(\frac{2\pi x}{\lambda}\right)\\
u_y' &= \frac{\sinh\left(\xi\frac{d}{2}\right)}{\sinh(\xi d)}\left[\frac{4\pi AU_\infty}{\lambda}\right]\cosh(\xi y)\cos\left(\frac{2\pi x}{\lambda}\right)\\
&\text{where}\quad \xi = \frac{2\pi\sqrt{1-Ma_\infty^2}}{\lambda}
\end{aligned}}$$

- Verify that Φ is a solution to the original PDE:

$$(1-Ma_\infty^2)\frac{\partial^2\Phi}{\partial x^2} = \frac{\sinh(\xi d/2)}{\sinh(\xi d)}\left[\frac{-8\pi^2 AU_\infty\sqrt{1-Ma_\infty^2}}{\lambda^2}\right]\sinh(\xi y)\cos\left(\frac{2\pi x}{\lambda}\right)$$

$$+\quad \frac{\partial^2\Phi}{\partial y^2} = \frac{\sinh(\xi d/2)}{\sinh(\xi d)}\left[\frac{8\pi^2 AU_\infty\sqrt{1-Ma_\infty^2}}{\lambda^2}\right]\sinh(\xi y)\cos\left(\frac{2\pi x}{\lambda}\right)$$

$$\Rightarrow\quad (1-Ma_\infty^2)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2} = 0 \qquad \therefore\ \underline{\Phi \text{ is a soln to the PDE!}}$$

- Now verify that the BCs are satisfied:

$$u_y'(x,\pm d/2) = \frac{\sinh(\xi d/2)}{\sinh(\xi d)}\left[\frac{4\pi AU_\infty}{\lambda}\right]\underbrace{\cosh(\pm\xi d/2)}_{=\cosh(\xi d/2)}\cos\left(\frac{2\pi x}{\lambda}\right)$$

NOTE: $\sinh(\xi d) = 2\sinh(\xi d/2)\cosh(\xi d/2)$

$$\therefore\ u_y'(x,\pm d/2) = \frac{2\pi AU_\infty}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right) \qquad \therefore\ \underline{\text{BCs are satisfied!}}$$

SOLUTION...

- The pressure coefficient along the centerline $(y=0)$ is:

$$C_p = \frac{-2u_x'}{U_\infty}$$

NOTE: $\sinh(0) = 0$

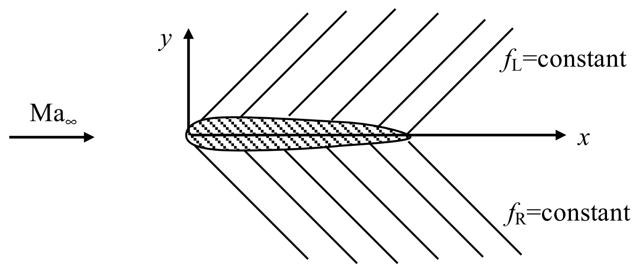
$$\Rightarrow \boxed{C_p(x, y=0) = 0}$$

21. Thin Airfoils in Supersonic Flow

Recall that for steady, irrotational, supersonic, compressible flow with negligible body and surface forces and small perturbations, the perturbation potential is given by:

$$\Phi = f_R\left(x + y\sqrt{\mathrm{Ma}_\infty^2 - 1}\right) + f_L\left(x - y\sqrt{\mathrm{Ma}_\infty^2 - 1}\right)$$

where f_L and f_R are arbitrary functions that are dependent on the boundary conditions.



Since information cannot be propagated upstream in a supersonic flow, the perturbation potential for flow over the top of the airfoil ($y>0$) should only include left-running Mach waves:

$$\Phi = f_L\left(x - y\sqrt{\mathrm{Ma}_\infty^2 - 1}\right)$$

while under the bottom of the airfoil ($y<0$) the solution should contain only right-running Mach waves:

$$\Phi = f_R\left(x + y\sqrt{\mathrm{Ma}_\infty^2 - 1}\right)$$

The boundary condition on the upper surface of the airfoil is:

$$\left.\frac{dy}{dx}\right|_{\text{upper}} = \frac{\left.u_y'\right|_{\text{upper}}}{U_\infty} = \frac{1}{U_\infty}\left.\frac{\partial \Phi}{\partial y}\right|_{\text{upper}}$$

so

$$\left.\frac{dy}{dx}\right|_{\text{upper}} = \frac{1}{U_\infty}\left.\frac{\partial \Phi}{\partial y}\right|_{\text{upper}} = \left[\frac{-\sqrt{\mathrm{Ma}_\infty^2 - 1}}{U_\infty}\right] f_L' \qquad \text{(Note: } \frac{\partial \Phi}{\partial y} = \frac{df_L}{d\eta}\frac{\partial \eta}{\partial y}\text{.)}$$

$$f_L' = \frac{-U_\infty}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}\left.\frac{dy}{dx}\right|_{\text{upper}} \tag{311}$$

The pressure coefficient on the upper surface is given by:

$$C_{p,\text{upper}} = \frac{-2\left.u_x'\right|_{y=0}}{U_\infty} = \left(\frac{-2}{U_\infty}\right)\left.\frac{\partial \Phi}{\partial x}\right|_{y=0} = \left(\frac{-2}{U_\infty}\right) f_L' \qquad \text{(Note: } \frac{\partial \Phi}{\partial x} = \frac{df_L}{d\eta}\frac{\partial \eta}{\partial x}\text{.)} \tag{312}$$

Substituting Eq. (311) into Eq. (312) gives:

$$C_{p,\text{upper}} = \left(\frac{-2}{U_\infty}\right)\left(\frac{-U_\infty}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}\left.\frac{dy}{dx}\right|_{\text{upper}}\right)$$

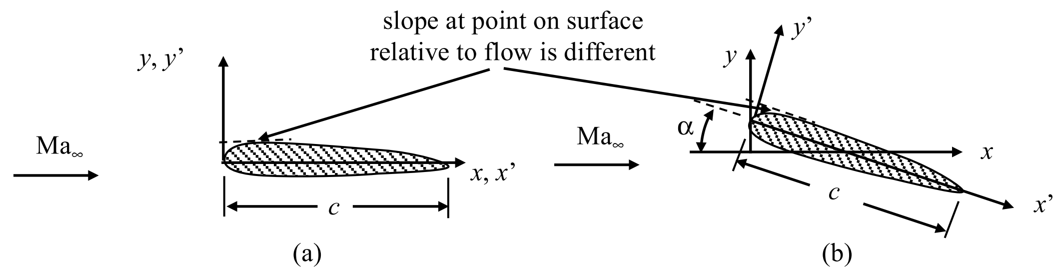
$$\boxed{C_{p,\text{upper}} = \frac{2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}\left.\frac{dy}{dx}\right|_{\text{upper}}} \tag{313}$$

A similar approach can be taken to determine that:

$$\boxed{C_{p,\text{lower}} = \frac{-2}{\sqrt{\text{Ma}_\infty^2 - 1}} \left.\frac{dy}{dx}\right|_{\text{lower}}} \tag{314}$$

Notes:

1. The slope of the upper and lower surfaces relative to the incoming flow depends not only on the airfoil shape, but also on the airfoil's angle of attack, α.



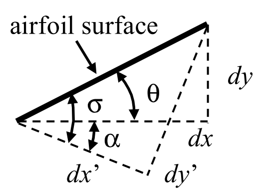
For example, the local slope of the surface relative to the incoming flow, (dy/dx), for case (a) is different than the slope of the surface at the same point for case (b). In order to isolate the effects of the airfoil shape and angle of attack, let's define the quantity, σ, as:

$$\sigma = \frac{dy'}{dx'} \tag{315}$$

so that σ represents the airfoil's surface slope relative to the airfoil's chord line, i.e., with respect to the (x', y') axes in the figures above. The slope of the surface relative to the incoming flow, θ, will be:

$$\theta = \sigma - \tan\alpha \approx \sigma - \alpha$$

Note that θ and σ are slopes, not angles.



$$\theta = \frac{dy}{dx} \qquad \sigma = \frac{dy'}{dx'} \tag{316}$$

Note that tan α≈ α since we're concerned only with small perturbations to the flow. The pressure coefficients (Eqs. (313) and (314)) may be written as:

$$C_{p,U} = \frac{2\left(\sigma_U - \alpha\right)}{\sqrt{\text{Ma}_\infty^2 - 1}} \tag{317}$$

$$C_{p,L} = \frac{-2\left(\sigma_L - \alpha\right)}{\sqrt{\text{Ma}_\infty^2 - 1}} \tag{318}$$

where the subscripts "$_U$" and "$_L$" refer to the upper and lower airfoil surfaces, respectively.

2. The airfoil's lift and drag can be determined by integrating the net pressure force acting over the entire airfoil surface. The pressures acting on the upper and lower airfoil surfaces are:

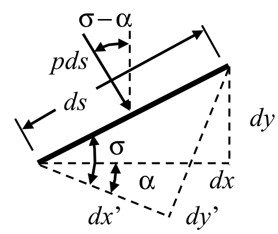
$$p_U - p_\infty = C_{p,U}\left(\tfrac{1}{2}\rho_\infty U_\infty^2\right) = C_{p,U}\left(\tfrac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2\right) = \gamma p_\infty \frac{\mathrm{Ma}_\infty^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}\left(\sigma_U - \alpha\right) \tag{319}$$

$$p_L - p_\infty = -\gamma p_\infty \frac{\mathrm{Ma}_\infty^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}\left(\sigma_L - \alpha\right) \tag{320}$$

Resolving the pressure force acting on the upper surface on a small area element, *ds* (Note that the airfoil distance into the page, also known as the span, is assumed to have unit depth.), into lift (*L*) and drag (*D*) components gives:

$$dL_U = -p_U \underbrace{ds\cos\left(\sigma_U - \alpha\right)}_{\approx dx} = -p_U dx \approx -p_U dx' \quad \text{(since the slopes are small, } dx' \approx dx\text{)}$$

$$dD_U = p_U \underbrace{ds\sin\left(\sigma_U - \alpha\right)}_{= dy} = p_U \frac{dy}{dx} dx \approx p_U \left(\sigma_U - \alpha\right) dx'$$



Similarly, for the lower surface:

$$dL_L = p_L \underbrace{ds\cos\left(\sigma_L - \alpha\right)}_{= dx} = p_L dx \approx p_L dx'$$

$$dD_L = -p_L \underbrace{ds\sin\left(\sigma_L - \alpha\right)}_{= dy} = -p_L \frac{dy}{dx} dx \approx -p_L \left(\sigma_L - \alpha\right) dx'$$

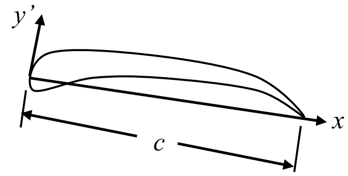
Note that the small angle approximation has been used in the expressions above. The net lift and drag are determined by integrating over the entire airfoil surface.

$$L = \int_{x'=0}^{x'=c} \left(dL_U + dL_L\right) = \int_0^c \left(-p_U + p_L\right) dx' = \gamma p_\infty \frac{\mathrm{Ma}_\infty^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}} \int_0^c \left[-\left(\sigma_U - \alpha\right) - \left(\sigma_L - \alpha\right)\right] dx'$$

$$L = \gamma p_\infty \frac{\mathrm{Ma}_\infty^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}} \int_0^c \left[-\sigma_L - \sigma_U + 2\alpha\right] dx' \tag{321}$$

$$D = \int_{x'=0}^{x'=c} \left(dD_U + dD_L\right) = \int_0^c \left[p_U\left(\sigma_U - \alpha\right) - p_L\left(\sigma_L - \alpha\right)\right] dx' = \gamma p_\infty \frac{\mathrm{Ma}_\infty^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}} \int_0^c \left[\left(\sigma_U - \alpha\right)^2 + \left(\sigma_L - \alpha\right)^2\right] dx'$$

$$D = \gamma p_\infty \frac{\mathrm{Ma}_\infty^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}} \int_0^c \left[\sigma_U^2 + \sigma_L^2 - 2\alpha\left(\sigma_U + \sigma_L\right) + 2\alpha^2\right] dx' \tag{322}$$



Note, however, that:

$$\int_0^c \sigma dx' = \int_0^c \frac{dy'}{dx'} dx' = \int_0^c dy' = 0$$

Recall that this is the chord length so that:

$$y'(x'=0) = y'(x'=c) = 0$$

so that Eqs. (321) and (322) become:

$$L = \gamma p_\infty \frac{\mathrm{Ma}_\infty^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}} 2\alpha c \tag{323}$$

$$D = \gamma p_\infty \frac{\mathrm{Ma}_\infty^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}} \left[\int_0^c \left(\sigma_U^2 + \sigma_L^2 \right) dx' + 2\alpha^2 c \right] \tag{324}$$

Written in terms of the lift and drag coefficients (based on the chord length):

$$\boxed{C_L = \frac{L}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 c} = \frac{4\alpha}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}} \tag{325}$$

$$\boxed{C_D = \frac{D}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 c} = \frac{2}{c\sqrt{\mathrm{Ma}_\infty^2 - 1}} \int_0^c \left(\sigma_U^2 + \sigma_L^2 \right) dx' + \frac{4\alpha^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}}} \tag{326}$$

3. The lift coefficient is directly proportional to the angle of attack for thin supersonic airfoils at small angles of attack. Note that there is also a component to the drag coefficient that depends only on the angle of attack. This component is often referred to as the **wave drag due to lift**:

$$C_{D,\substack{\text{wave drag}\\ \text{due to lift}}} = \frac{4\alpha^2}{\sqrt{\mathrm{Ma}_\infty^2 - 1}} \tag{327}$$

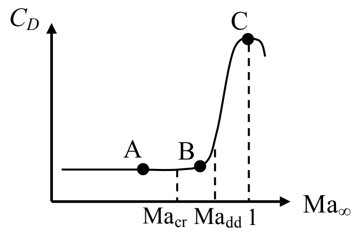
The other part of the drag coefficient depends only the shape of the airfoil and is known as the **wave drag due to thickness** since the thicker the airfoil, the larger the integral term (the slopes will be larger):

$$C_{D,\substack{\text{wave drag}\\ \text{due to thickness}}} = \frac{2}{c\sqrt{\mathrm{Ma}_\infty^2 - 1}} \int_0^c \left(\sigma_U^2 + \sigma_L^2 \right) dx' \tag{328}$$

These previous results suggest that in order to minimize the drag acting on a supersonic airfoil, the airfoil should be as thin as possible. Furthermore, the lift acting on the airfoil will be unaffected by the shape of the airfoil.

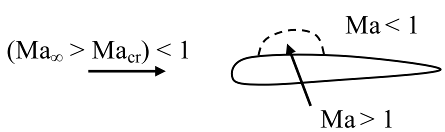
4. Let's now consider *subsonic* flow around the airfoil shown below.

For sufficiently small free stream Mach numbers, Ma_∞, the flow over the entire airfoil surface will remain subsonic and the corresponding drag coefficient for the airfoil will remain at a relatively small value (Point A on the plot shown below. Note that the total drag will be due to skin friction and form drag.).

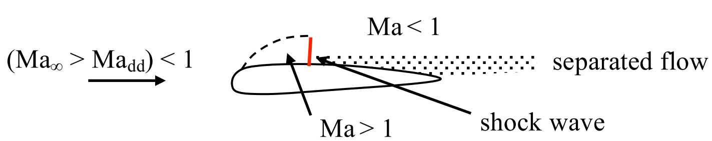


As the free stream Mach number is increased, a **critical free stream Mach number, Ma_{cr},** will be reached where a sonic Mach number will occur at the minimum pressure point on the airfoil surface.

At free stream Mach numbers slightly greater than Ma_{cr}, a small region of supersonic flow will occur near the minimum pressure point. The drag coefficient for these flow conditions (point B in the diagram above) remains close to, but only slightly greater than, the drag coefficient for purely subsonic flow.



At another critical Mach number, known as the **drag divergence Mach number, Ma_{dd}** ($Ma_{cr} < Ma_{dd} <$ 1), the drag on the airfoil increases suddenly (point C in the C_D vs. Ma_∞ plot shown above) due to the formation of a terminating shock wave. The shock wave on the airfoil surface causes the boundary layer to separate (due to the large adverse pressure gradient across the shock wave) resulting in a significant increase in the form drag.

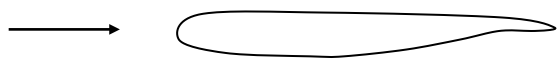


This sudden increase in the drag at Ma_{dd} is the origin of the concept of the **sound barrier**.

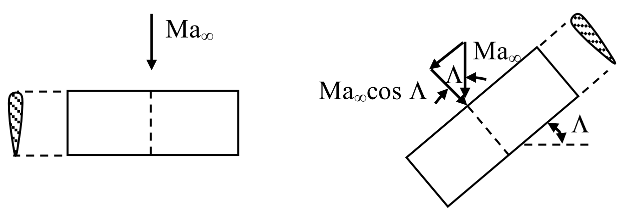
In order to minimize the drag acting on the airfoil when operating near sonic conditions, the drag divergence Mach number should be pushed as close to sonic conditions as much as possible.

a. One approach to increasing Ma_{dd} is to decrease the thickness of the airfoil. Recall that the largest Mach numbers will occur in the vicinity of the minimum pressure region. If the minimum pressure can be brought closer to the free stream pressure, then the corresponding local Mach number will deviate less from the free stream Mach number (which is subsonic). Making the airfoil thinner will result in less expansion of the flow and hence the Mach numbers over the airfoil will be smaller. As a result, the drag divergence Mach number for the thin airfoil will occur at a higher free stream Mach number than a thicker airfoil. Thinner airfoils also produce less drag for supersonic conditions as discussed previously in Note 3.

b. Another approach to reducing the drag is to use a **supercritical airfoil** (shown below), which is designed to give $Ma_{dd} \approx 1$. The airfoil shape is designed to give mostly supersonic flow and discourage the formation of shock waves.

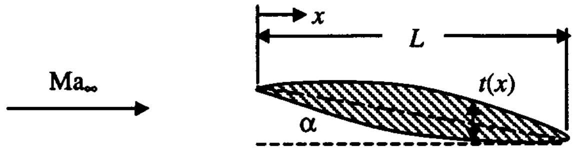


c. A third approach to delaying Ma_{dd} is to use a **swept-wing** design. Consider flow over straight and swept wings (inclined by an angle Λ from the straight wing) that have identical airfoil cross-sections. Note that only the component of the flow normal to the airfoil will be important in determining the drag divergence Mach number since the flow tangential to the surface does not "see" variations in the airfoil geometry. Hence, the effective free stream Mach number for the swept-back wing is smaller than that for the straight wing ($Ma_{\infty eff} = Ma_{\infty}\cos\Lambda$). As a result, the negative effects associated with shock formation on the airfoil can be delayed until a higher free stream Mach number is reached. The downsides of swept wings are that the wing area must be increased to generate the same lift as a straight wing (since the lift decreases as a result of the lower effective free stream Mach number), and the structural design of the wing is more complex.



Swept-wings are also advantageous in supersonic flight since the wing may be subject to a subsonic effective Mach number and, as a result, the penalty of supersonic wave drag can be avoided.

A symmetric strut of chord length, L, is placed at a small angle of attack, α, in a supersonic flow of Mach number, Ma_∞.



The geometry of the strut is such that the centerline is straight and the foil thickness, $t(x)$, is given by:

$$\frac{t(x)}{t_M} = 4\frac{x}{L}\left(1-\frac{x}{L}\right)$$

where t_M is the maximum thickness of the foil which occurs at $x/L=1/2$. Assuming that the foil is slender and that the angle of attack is small, find expressions for the lift and drag coefficients for this strut as functions of Ma_∞, α, and t_M/L.

SOLUTION:

$$\left.\frac{dy}{dx}\right|_{\text{upper surface}} = -\alpha + \frac{1}{2}\underbrace{\left[4t_M\left(\frac{1}{L} - \frac{2x}{L^2}\right)\right]}_{dt/dx} = -\alpha + 2\left(\frac{t_M}{L}\right)\left(1 - \frac{2x}{L}\right)$$

$$\left.\frac{dy}{dx}\right|_{\text{bottom surface}} = -\alpha - \frac{1}{2}\underbrace{\left[4t_M\left(\frac{1}{L} - \frac{2x}{L^2}\right)\right]}_{dt/dx} = -\alpha - 2\left(\frac{t_M}{L}\right)\left(1 - \frac{2x}{L}\right)$$

- From class notes analysis:

$$C_L = \frac{-2}{L\sqrt{Ma_\infty^2-1}}\int_0^L \left[\left.\frac{dy}{dx}\right|_L + \left.\frac{dy}{dx}\right|_u\right]dx$$

$$= \frac{-2}{L\sqrt{Ma_\infty^2-1}}\int_0^L (-2\alpha)\,dx$$

$$\therefore \boxed{C_L = \frac{4\alpha}{\sqrt{Ma_\infty^2-1}}}$$

same as lift on flat plate at angle of attack, α
thickness of airfoil does not contribute to lift

SOLUTION...

$$C_D = \frac{2}{L\sqrt{Ma_\infty^2-1}}\int_0^L\left[\left(\frac{dy}{dx}\right)_u^2+\left(\frac{dy}{dx}\right)_L^2\right]dx$$

$$= \frac{2}{L\sqrt{Ma_\infty^2-1}}\int_0^L\left\{\left[\alpha^2-2\alpha\left(\frac{t_m}{L}\right)\left(1-\frac{2x}{L}\right)+4\left(\frac{t_m}{L}\right)^2\left(1-\frac{2x}{L}\right)^2\right]+\left[\alpha^2+2\alpha\left(\frac{t_m}{L}\right)\left(1-\frac{2x}{L}\right)+4\left(\frac{t_m}{L}\right)^2\left(1-\frac{2x}{L}\right)^2\right]\right\}dx$$

$$= \frac{2}{L\sqrt{Ma_\infty^2-1}}\int_0^L\left[2\alpha^2+8\left(\frac{t_m}{L}\right)^2\left(1-\frac{4x}{L}+\frac{4x^2}{L^2}\right)\right]dx$$

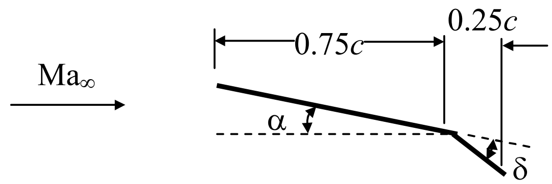
$$= \frac{4\alpha^2}{\sqrt{Ma_\infty^2-1}}+\frac{16}{L\sqrt{Ma_\infty^2-1}}\left(\frac{t_m}{L}\right)^2\left[\underbrace{L-2L+\frac{4}{3}L}_{\frac{1}{3}L}\right]$$

$$\boxed{\therefore\ C_D = \frac{4\alpha^2}{\sqrt{Ma_\infty^2-1}}+\frac{16}{3\sqrt{Ma_\infty^2-1}}\left(\frac{t_m}{L}\right)^2}$$

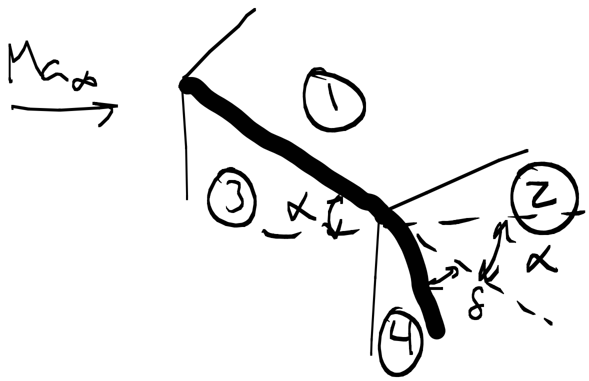
same as drag for a flat plate at angle of attack, α

thickness of airfoil increases the drag

Consider supersonic flow past a flat plate airfoil with a deflected flap as shown in the figure below. Plot the lift and drag coefficients a function of angle of attack, α, for various flap angles, δ. Note that these angles are assumed very small. Compare the lift and drag coefficients from your small angle approximation for $Ma_\infty = 2.0$, $\alpha = 3°$, and $\delta = 1°$ to the lift and drag coefficients on the airfoil using the oblique shock and expansion fan relations.



SOLUTION:

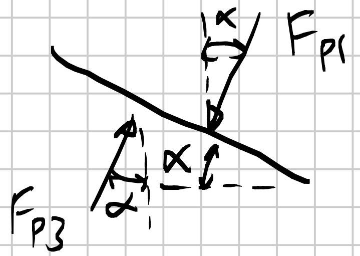


$$C_{p,u} = \frac{2}{\sqrt{Ma_\infty^2 - 1}} \left.\frac{dy}{dx}\right|_u \qquad C_{p,L} = \frac{-2}{\sqrt{Ma_\infty^2 - 1}} \left.\frac{dy}{dx}\right|_L$$

$$C_{p,1} = \frac{2}{\sqrt{Ma_\infty^2 - 1}} (-\alpha) \qquad C_{p,3} = \frac{-2}{\sqrt{Ma_\infty^2 - 1}} (-\alpha)$$

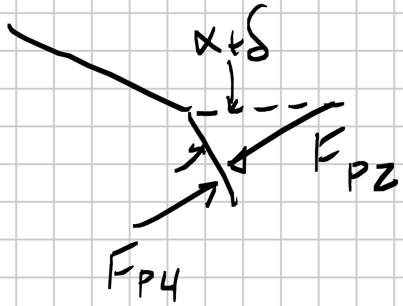
$$C_{p,2} = \frac{2}{\sqrt{Ma_\infty^2 - 1}} \left[-(\alpha + \delta)\right] \qquad C_{p,4} = \frac{-2}{\sqrt{Ma_\infty^2 - 1}} \left[-(\alpha + \delta)\right]$$

SOLUTION...



$$D_1 = -F_{p1}\sin\alpha \qquad L_1 = -F_{p1}\cos\alpha$$

$$D_3 = F_{p3}\sin\alpha \qquad L_3 = F_{p3}\cos\alpha$$



$$D_2 = -F_{p2}\sin(\alpha+\delta) \qquad L_2 = -F_{p2}\cos(\alpha+\delta)$$

$$D_4 = F_{p4}\sin(\alpha+\delta) \qquad L_4 = F_{p4}\cos(\alpha+\delta)$$

where
$$F_{p1} = \left[C_{p,1}\left(\tfrac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2\right) + p_\infty\right](0.75c)$$
$$F_{p2} = \left[C_{p,2}\left(\tfrac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2\right) + p_\infty\right](0.25c)$$
$$F_{p3} = \left[C_{p,3}\left(\tfrac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2\right) + p_\infty\right](0.75c)$$
$$F_{p4} = \left[C_{p,4}\left(\tfrac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2\right) + p_\infty\right](0.25c)$$

$$D = D_1 + D_2 + D_3 + D_4 = \left(-C_{p,1} + C_{p,3}\right)\sin\alpha\left(\tfrac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2\right)(0.75c)$$
$$+ \left(-C_{p,2} + C_{p,4}\right)\sin(\alpha+\delta)\left(\tfrac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2\right)(0.25c)$$

$$C_D = \frac{D}{\frac{1}{2}\gamma p_\infty \mathrm{Ma}_\infty^2 c} = \left(-C_{p,1} + C_{p,3}\right)(0.75\sin\alpha)$$
$$+ \left(-C_{p,2} + C_{p,4}\right)(0.25\sin(\alpha+\delta))$$

Solution...

$$C_D = \frac{2}{\sqrt{Ma_\infty^2-1}}\left[(\alpha+\alpha)(0.75\alpha) + (\alpha+\delta+\alpha+\delta)(0.25(\alpha+\delta))\right]$$

$$\boxed{\therefore C_D = \frac{4}{\sqrt{Ma_\infty^2-1}}\left[0.75\alpha^2 + 0.25(\alpha+\delta)^2\right]}$$

Following a similar approach for the lift coefficient:

$$C_L = (-C_{p,1}+C_{p,3})(0.75\cos\alpha) + (-C_{p,2}+C_{p,4})(0.25\cos(\alpha+\delta))$$

$$= \frac{2}{\sqrt{Ma_\infty^2-1}}\left[2\alpha(0.75) + 2(\alpha+\delta)(0.25)\right]$$

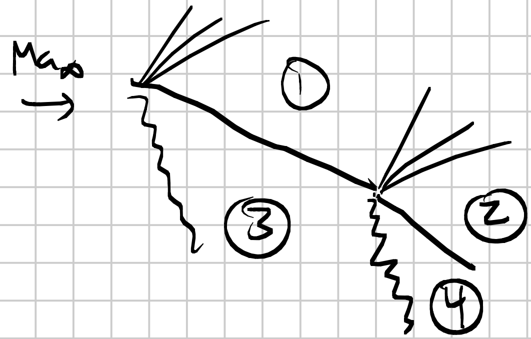
$$\boxed{\therefore C_L = \frac{4}{\sqrt{Ma_\infty^2-1}}\left[\alpha + 0.25\delta\right]}$$

For $Ma_\infty = 2.0$, $\alpha = 3^\circ$, and $\delta = 1^\circ$

$$\boxed{\begin{aligned} C_L &= 1.31\times10^{-1} \\ C_D &= 7.56\times10^{-3} \end{aligned}}$$

using small perturbation approximations

SOLUTION...



Region 1:

$$Ma_\infty = 2.0 \Rightarrow \nu_\infty = 26.38^\circ$$

$$\nu_1 = \nu_\infty + 3^\circ = 29.38^\circ$$

$$Ma_1 = 2.111 \xRightarrow{\text{isentropic relns}} \underline{\frac{p_1}{p_\infty} = 0.842}$$

Region 2:

$$Ma_1 = 2.111$$

$$\nu_2 = \nu_1 + 1^\circ = 30.38^\circ$$

$$Ma_2 = 2.149 \xRightarrow{\text{isentropic relns}} \frac{p_2}{p_1} = 0.943$$

$$\underline{\frac{p_2}{p_\infty}} = \left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_\infty}\right) = \underline{0.794}$$

Region 3:

$$\left.\begin{aligned} Ma_\infty &= 2.0 \\ \delta &= 3^\circ \end{aligned}\right\} \xRightarrow{\text{OSR}} \begin{aligned} Ma_3 &= 1.892 \\ \underline{p_3/p_\infty} &\underline{= 1.181} \end{aligned}$$

Region 4:

$$\left.\begin{aligned} Ma_3 &= 1.892 \\ \delta &= 1^\circ \end{aligned}\right\} \xRightarrow{\text{OSR}} \begin{aligned} Ma_4 &= 1.857 \\ \frac{p_4}{p_3} &= 1.056 \end{aligned}$$

$$\underline{p_4/p_\infty} = \left(\frac{p_4}{p_3}\right)\left(\frac{p_3}{p_\infty}\right) = \underline{1.247}$$

SOLUTION...

$$C_D = \frac{\left(-\frac{p_1}{p_\infty} + \frac{p_3}{p_\infty}\right)(0.75 \sin 3^\circ) + \left(-\frac{p_2}{p_\infty} + \frac{p_4}{p_\infty}\right)(0.25 \sin 4^\circ)}{\frac{1}{2}\gamma \mathrm{Ma}_\infty^2}$$

$$\boxed{\therefore C_D = 7.58 \times 10^{-3}}$$

$$C_L = \frac{\left(-\frac{p_1}{p_\infty} + \frac{p_3}{p_\infty}\right)(0.75 \cos 3^\circ) + \left(-\frac{p_2}{p_\infty} + \frac{p_4}{p_\infty}\right)(0.25 \cos 4^\circ)}{\frac{1}{2}\gamma \mathrm{Ma}_\infty^2}$$

$$\boxed{\therefore C_L = 1.31 \times 10^{-1}}$$

The small perturbation approximation gives nearly the same values as the oblique shock/expansion fan approach.

22. Unsteady, 1D Compressible Flow

Applications that might be approximated as being, 1D and unsteady:

a. accelerating piston in a cylinder
b. projectile moving through a cylinder
c. shock tube
d. start-up and shut-down transients in a wind tunnel

Governing equations for 1D, unsteady flow (ignoring viscous forces, body forces, and area changes):

continuity: $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0 \;\Rightarrow\; \frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0 \tag{329}$$

momentum: $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{330}$$

Assume that the flow is **isentropic**:

$$p=p(\rho) \quad\Rightarrow\quad \frac{\partial p}{\partial x}=\underbrace{\frac{dp}{d\rho}}_{=c^2}\frac{\partial \rho}{\partial x}=c^2\frac{\partial \rho}{\partial x} \tag{331}$$

Note that in the previous expression, the sound speed, c, has been used because the flow is isentropic.

To make our analysis here look similar in form to the analysis we used while investigating steady, 2D flows, let's re-write the velocity in the following manner:

$$u=\frac{\partial \phi}{\partial x}=\phi_x$$

where the subscript "x" signifies a partial differentiation with respect to x. Using this notation and substituting Eq. (331) into Eq. (330) and simplifying, Eqs. (329) and (330) become:

$$\frac{\partial \rho}{\partial t}+\phi_x\frac{\partial \rho}{\partial x}+\rho\phi_{xx}=0 \tag{332}$$

$$\phi_{xt}+\phi_x\phi_{xx}+\frac{c^2}{\rho}\frac{\partial \rho}{\partial x}=0 \tag{333}$$

Multiply Eq. (333) with dx and integrate with respect to x (note that Eq. (333) is a function of both x and t):

$$\phi_t+\tfrac{1}{2}\phi_x^2+\int_{\rho_0}^{\rho}\frac{c^2}{\rho}\,d\rho=f(t) \tag{334}$$

where $f(t)$ is an unknown function of time. Taking the partial derivative of Eq. (334) with respect to t gives:

$$\phi_{tt}+\phi_x\phi_{xt}+\frac{c^2}{\rho}\frac{\partial \rho}{\partial t}=f'(t) \tag{335}$$

We can substitute Eqs. (333) and (335) into Eq. (332) to give an expression that does not include the density:

$$f'(t)-\phi_{tt}-\phi_x\phi_{xt}-\phi_x\phi_{xt}-\phi_x^2\phi_{xx}+c^2\phi_{xx}=0$$

$$\left(c^2-\phi_x^2\right)\phi_{xx}-2\phi_x\phi_{xt}-\phi_{tt}=-f'(t) \tag{336}$$

The unknown function of time may be eliminated by defining:

$$\tilde{\phi} \equiv \phi + \int f(t)\,dt \tag{337}$$

$$\tilde{\phi}_x = \phi_x = u \qquad \Rightarrow \qquad \tilde{\phi}_{xx} = \phi_{xx}$$

$$\tilde{\phi}_t = \phi_t + f(t) \qquad \Rightarrow \qquad \tilde{\phi}_{tt} = \phi_{tt} + f'(t)$$

$$\tilde{\phi}_{xt} = \phi_{xt}$$

so that Eq. (336) becomes:

$$\boxed{\left(c^2 - u^2\right)\tilde{\phi}_{xx} - 2u\tilde{\phi}_{xt} - \tilde{\phi}_{tt} = 0} \tag{338}$$

Note also that Eqs. (334) and (335) become:

$$\tilde{\phi}_t + \tfrac{1}{2}\tilde{\phi}_x^2 + \int_{\rho_0}^{\rho} \frac{c^2}{\rho}\,d\rho = 0 \tag{339}$$

$$\tilde{\phi}_{tt} + \tilde{\phi}_x\tilde{\phi}_{xt} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial t} = 0 \tag{340}$$

Now let's examine the nature of Eq. (338) more closely. This 2nd order PDE in two-independent variables (x and t) will always be hyperbolic since:

$$B^2 - 4AC = 4u^2 - 4\left(c^2 - u^2\right)(-1) = 4c^2 > 0$$

where $A = c^2 - u^2$; $B = -2u$; $C = -1$.

The slope of the characteristic curves for Eq. (338) can be found using an approach similar to that used when investigating 2D, steady supersonic flows. We can write the following system of equations:

$$\begin{bmatrix} c^2 - u^2 & -2u & -1 \\ dx & dt & 0 \\ 0 & dx & dt \end{bmatrix} \begin{Bmatrix} \tilde{\phi}_{xx} \\ \tilde{\phi}_{xt} \\ \tilde{\phi}_{tt} \end{Bmatrix} = \begin{Bmatrix} 0 \\ d\tilde{\phi}_x \\ d\tilde{\phi}_t \end{Bmatrix} \quad \text{where} \quad \begin{matrix} d\tilde{\phi}_x = du \\ d\tilde{\phi}_t = d\phi + f' \end{matrix} \tag{341}$$

Across a characteristic curve, the derivatives of $\tilde{\phi}_x$ and $\tilde{\phi}_t$ are indeterminate. In order for this to occur, the determinant of the matrix on the LHS of the previous equation must be zero:

$$\begin{vmatrix} c^2 - u^2 & -2u & -1 \\ dx & dt & 0 \\ 0 & dx & dt \end{vmatrix} = \left(c^2 - u^2\right)dt^2 - dx^2 + 2u\,dx\,dt = 0$$

$$\left(\frac{dx}{dt}\right)^2 - 2u\left(\frac{dx}{dt}\right) - \left(c^2 - u^2\right) = 0$$

$$\boxed{\frac{dx}{dt} = u \pm c} \quad \text{and} \quad \frac{dt}{dx} = \frac{1}{u \pm c} \tag{342}$$

Slope of the characteristic curves in the *x-t* plane. The "+" sign represent a right-running characteristic and the "-" represents a left-running characteristic.

Since disturbances propagate along characteristic curves, we see that **disturbances will propagate at the speed of sound relative to the local flow velocity.**

In order to ensure that the velocity of the flow will always remain finite, we must also have finite velocity gradients. The velocity gradient can be found from the system of equations given in Eq. (341) using Kramer's rule:

$$\tilde{\phi}_{xx} = \frac{\partial u}{\partial x} = \frac{\begin{vmatrix} 0 & -2u & -1 \\ d\tilde{\phi}_x & dt & 0 \\ d\tilde{\phi}_t & dx & dt \end{vmatrix}}{\begin{vmatrix} c^2-u^2 & -2u & -1 \\ dx & dt & 0 \\ 0 & dx & dt \end{vmatrix}} = \frac{-d\tilde{\phi}_x dx + d\tilde{\phi}_t dt + 2ud\tilde{\phi}_x dt}{\left(c^2-u^2\right)dt^2 - dx^2 + 2udxdt} \tag{343}$$

Since along a characteristic curve the denominator of the previous expression is zero, the only way for the velocity gradient to remain finite is for the numerator to also be zero. Hence, we have the following relation:

$$-d\tilde{\phi}_x dx + d\tilde{\phi}_t dt + 2ud\tilde{\phi}_x dt = 0$$

$$du\left(\frac{dx}{dt} - 2u\right) = d\tilde{\phi}_t \tag{344}$$

The RHS of the previous equation is found with the aid of Eq. (339):

$$d\tilde{\phi}_t = -\tilde{\phi}_x d\tilde{\phi}_x - c^2\frac{d\rho}{\rho} = -udu - c^2\frac{d\rho}{\rho}$$

Substituting back into Eq. (344) and utilizing Eq. (342):

$$du\left(u \pm c - 2u\right) = -udu - c^2\frac{d\rho}{\rho}$$

$$\boxed{\mp\frac{du}{c} = \frac{d\rho}{\rho} = \frac{1}{c^2}\frac{dp}{\rho}} \quad \text{(Note: } \frac{d\rho}{\rho} = \frac{1}{\rho}\frac{dp}{c^2} \text{ since } c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s\text{.)} \tag{345}$$

Conditions along a characteristic curve. The "-" corresponds to a right-running characteristic while the "+" corresponds to a left-running characteristic.

Notes:

1. We could have also solved Eq. (341) for finite $\tilde{\phi}_{xt}$ or $\tilde{\phi}_{tt}$ to arrive at Eq. (345).

2. The numerical algorithms given during our previous notes on the method of characteristics for 2D, steady flow may also be applied here.

3. For an ideal gas, Eq. (345) can be written as:

$$\mp\frac{du}{c} = \frac{d\rho}{\rho} = \frac{1}{\gamma-1}\frac{dT}{T} = \frac{2}{\gamma-1}\frac{dc}{c}$$

$$\boxed{\pm du = \frac{2}{1-\gamma}dc} \tag{346}$$

Integrating along the characteristic curve gives:

$$\boxed{\pm\left(u_2-u_1\right) = \frac{2}{1-\gamma}\left(c_2-c_1\right)} \text{ or } \boxed{\Delta u = \begin{cases} \frac{2}{1-\gamma}\Delta c & \text{along a R-running characteristic} \\ \frac{2}{1-\gamma}\left(-\Delta c\right) & \text{along a L-running characteristic} \end{cases}} \tag{347}$$

Conditions along a characteristic curve in a perfect gas. The "+" corresponds to a right-running characteristic while the "-" corresponds to a left-running characteristic.

4. We can also determine how properties change <u>across</u> a characteristic curve by considering that by moving along a right-running characteristic we cross left-running characteristics and vice-versa.

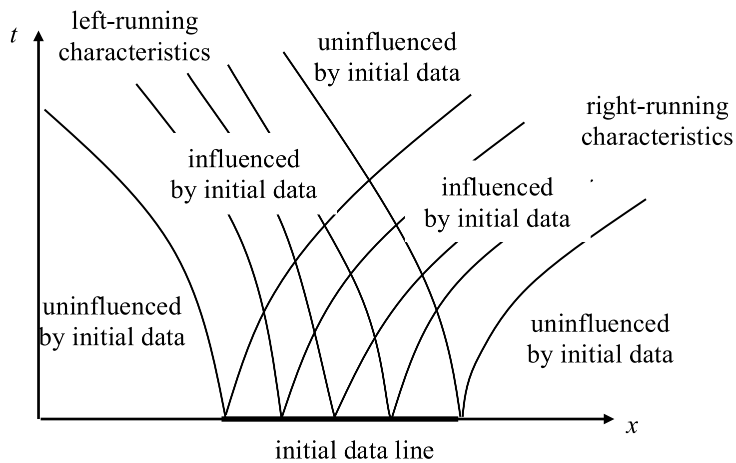
$$\boxed{\pm\frac{du}{c} = \frac{d\rho}{\rho} = \frac{1}{c^2}\frac{dp}{\rho}} \quad \text{(Note: compression} \Rightarrow dp > 0\text{, expansion} \Rightarrow dp < 0.) \tag{348}$$

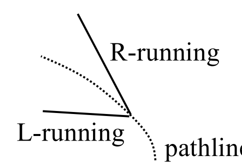
$$\boxed{\pm du = \frac{2}{\gamma - 1}dc} \tag{349}$$

$$\boxed{\pm(u_2 - u_1) = \frac{2}{\gamma - 1}(c_2 - c_1)} \tag{350}$$

Conditions <u>across</u> a characteristic curve in a perfect gas. The "+" corresponds to a right-running characteristic while the "-" corresponds to a left-running characteristic.

5. A sketch of characteristic curves on the *t*-*x* plane looks as follows:





6. For an isentropic flow of a perfect gas we have:

$$\frac{c_2}{c_1} = \left(\frac{T_2}{T_1}\right)^{1/2} = \left(\frac{p_2}{p_1}\right)^{\gamma-1/2\gamma} \tag{351}$$

so that Eq. (350) can be written as:

$$\pm\left(\frac{u_2}{c_1} - \frac{u_1}{c_1}\right) = \frac{2}{\gamma - 1}\left(\frac{c_2}{c_1} - 1\right)$$

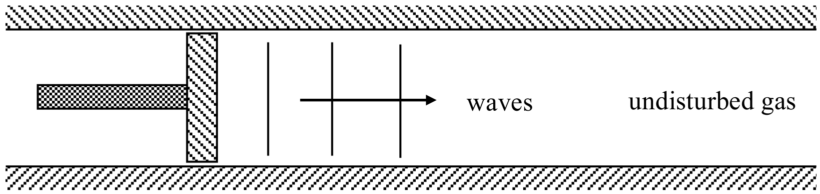
$$\boxed{\pm\left(\frac{u_2}{c_1} - \frac{u_1}{c_1}\right) = \frac{2}{\gamma - 1}\left[\left(\frac{p_2}{p_1}\right)^{\gamma-1/2\gamma} - 1\right]} \tag{352}$$

Conditions <u>across</u> a characteristic curve in a perfect gas. The "+" corresponds to a right-running characteristic while the "-" corresponds to a left-running characteristic.

7. The previous relations do <u>not</u> apply to a shock wave. Shock waves are non-isentropic processes and, hence, the isentropic assumption used in deriving the equations is not valid. In addition, shock waves travel at speeds <u>greater</u> than the sonic speed.

Simple Waves

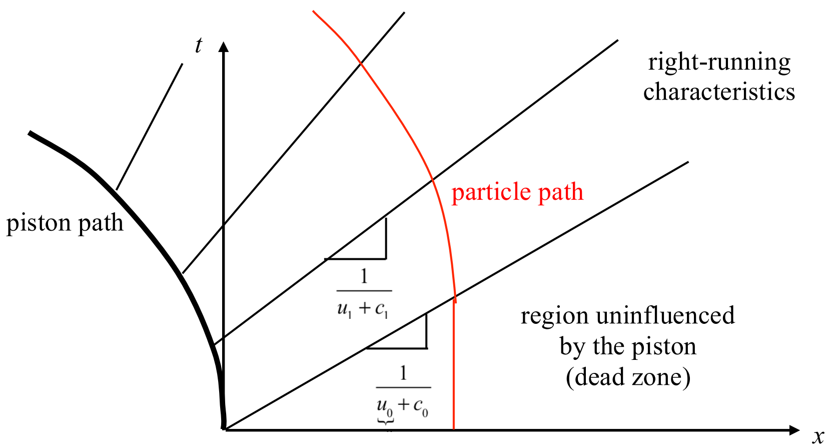
Flows involving simple waves contain either left or right-traveling waves but not both. A simple wave flow can be produced using an accelerating piston as shown in the figure below:



Let's consider two cases of simple waves in a perfect gas using this piston geometry. First we'll consider a piston moving to the left and then we'll consider a piston moving to the right.

Piston Moving to the Left

The movement of the piston can be shown on the *t*-*x* diagram. Note that we'll assume that the piston velocity increases with time.



At t=0, a sound wave leaves the piston and travels into the undisturbed fluid. This wave will travel at the sound speed in the undisturbed fluid, c_0, and thus is a straight line on the *t*-*x* plot. Across this right-running characteristic, we have from Eq. (349):

$$+du = \frac{2}{\gamma-1}dc \quad \left(\text{or } \Delta u = \frac{2}{1-\gamma}(-\Delta c) \text{ along a left running characteristic}\right) \tag{353}$$

Since $du < 0$ (the piston moves to the left so the fluid velocity should also move to the left), then $dc < 0$. Thus, the next pressure pulse will travel at a slightly slower speed, which corresponds to a larger slope on the *t*-*x* plot. Each characteristic curve will, in fact, be a straight line since u and c are constant in the regions between the pressure waves (refer to Eq. (342)). Thus, the characteristic lines diverge and the influence of the piston motion is "stretched" out as the waves propagate downstream.

Notes:
1. The motion of an individual fluid particle, known as a **pathline**, can be determined using:

$$\left.\frac{dx}{dt}\right|_{\text{particle}} = u\big|_{\text{particle}} \tag{354}$$

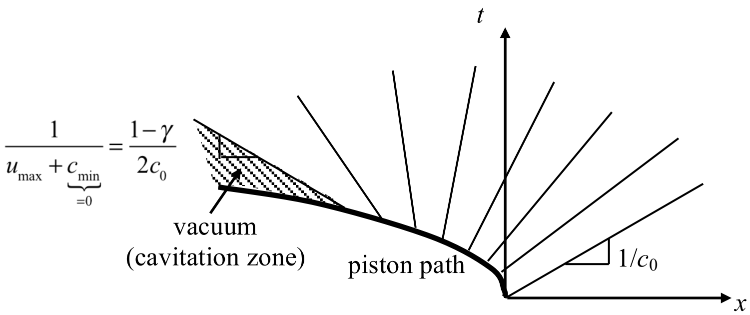
2. Now let's consider what happens if we accelerate the piston to the left from rest ($u_0 = 0$) to a very large speed. From Eq. (350) (we're crossing right-running characteristics) we have:

$$+\left(u-\underbrace{u_0}_{=0}\right)=\frac{2}{\gamma-1}\left(c-c_0\right) \quad \Rightarrow \quad \frac{c}{c_0}=\frac{\gamma-1}{2}\frac{u}{c_0}+1 \quad \Rightarrow \quad \frac{u_{\max}}{c_0}=\frac{-2}{1-\gamma}=\frac{2}{\gamma-1}$$

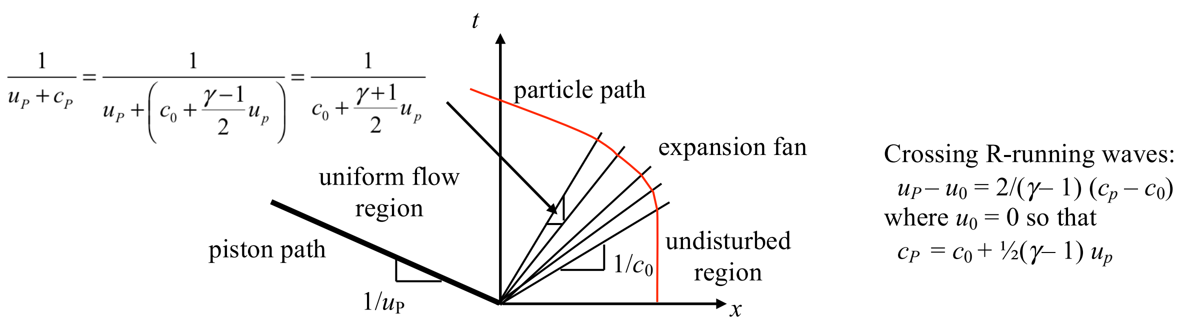
Since the piston is moving to the left, $u < 0$. Thus, the largest speed we can have that results in a non-negative speed of sound ($c \geq 0$) is:

$$\frac{u_{\max}}{c_0}=\frac{2}{\gamma-1} \tag{355}$$

This is known as the **escape speed**. If the piston continues to accelerate, a vacuum (called the **cavitation zone**) will form on the face of the piston as shown in the diagram below.

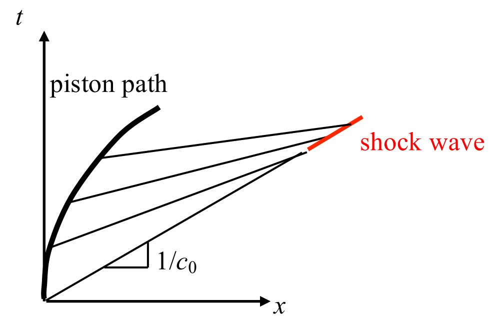


4. An impulsive withdrawal to the left results in an expansion fan as shown below:



Piston Moving to the Right

Now let's consider the case where the piston moves to the right to produce compression waves (refer to the figure shown below).



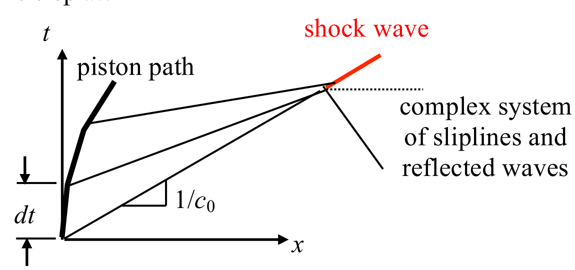
Across each compression wave (right-running characteristics as shown in the diagram above) we have from Eq. (349):

$$du = \frac{2}{\gamma - 1} dc$$

where $du > 0$. Hence, $dc > 0$ and the characteristic curves become steeper (refer to Eq. (342)). Eventually these compression waves will intersect and the isentropic assumption breaks down since the velocity gradients across the wave are no longer infinitesimal and viscous (irreversible) losses become significant. The point of the first intersection is defined as the start of a shock wave (refer to the previous figure).

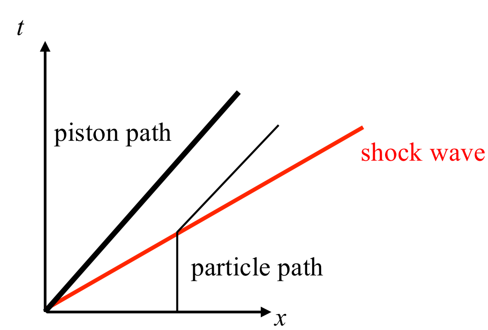
Notes:

1. Consider the diagram shown below where the piston is accelerated in very small, discrete increments of *du* each at every time step *dt*.



This section remains incomplete.

2. An impulsive acceleration to the right will immediately form a shock wave as shown in the following figure.



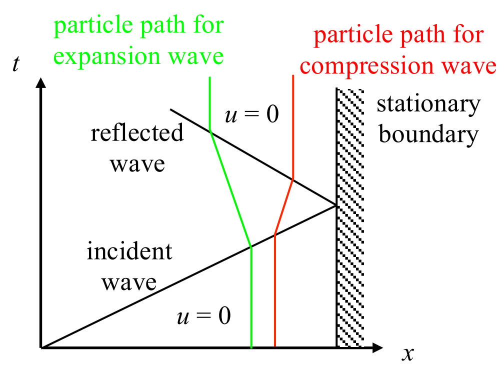
The strength of the shock wave will be such that the fluid velocity behind the shock will equal the piston velocity.

Interactions with Boundaries

Stationary Boundaries

Near a stationary wall the fluid velocity must equal zero. This implies that a wave must reflect in a similar sense as shown in the diagram below. For example, consider a right-running compression wave ($dp > 0$ and, according to Eq. (348), $du > 0$) impinging against a stationary wall. In order for the velocity to remain zero at the wall we must have for the reflected wave (a left-running characteristic), $du < 0$. Hence, according to Eq. (348) we see that $dp > 0$ and thus we have another compression wave.

Note that a fluid particle follows a compression wave and moves away from an expansion wave.



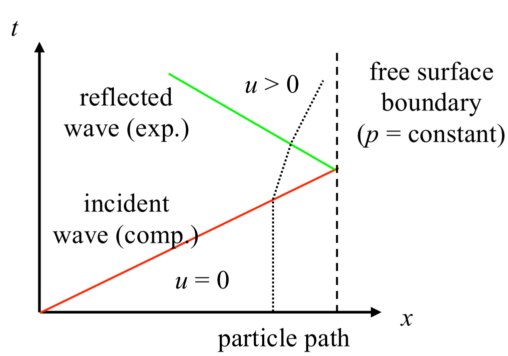
Across:
R-running: $du/c = 1/c^2\,(dp/\rho)$
L-running: $-du/c = 1/c^2\,(dp/\rho)$

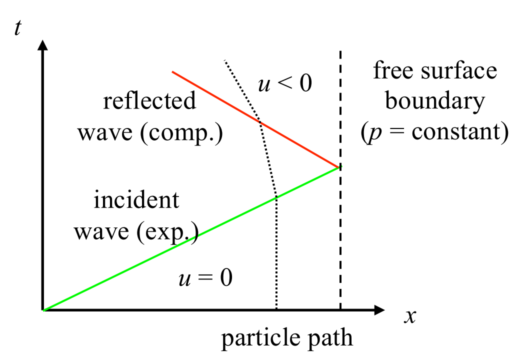
Free Surface Boundaries
This section remains incomplete.

Reflections from an open end are more complicated since we must consider four different cases: the flow may be either inflow or outflow and it may be either subsonic or supersonic.

Subsonic Outflow/Inflow
For low speed subsonic outflow or inflow, it is reasonable to assume that the pressure at the end of the duct is equal to the ambient pressure. As a result, waves will reflect in an unlike sense.

For example, consider a right-running compression wave ($dp > 0$, $du > 0$) impinging on an open end. In order for the pressure to remain constant at the open end (and equal to the ambient pressure), we must have $dp < 0$ which implies that the reflected wave is an expansion wave. Across the left-running reflected wave we observe from Eq. (348) that $du > 0$. A similar approach may be taken to determine the conditions for an incident expansion wave.



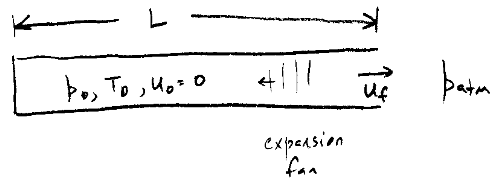


For high speed subsonic outflow or inflow, the unsteady effects at the open end must be included making the analysis much more difficult.

Supersonic Outflow/Inflow
Since for supersonic flow the flow velocity is larger than the propagation velocity of the reflected waves, the reflected waves are unable to propagate from the open end of the duct. Hence there are no reflected waves.

A diaphragm at the end of a 4 m long pipe containing air at a pressure of 200 kPa and a temperature of 30 °C suddenly ruptures causing an expansion wave to propagate down the pipe. Find the velocity at which the air is discharged from the pipe if the ambient air pressure is 103 kPa. Also find the velocity of the front and the back of the wave and hence find the time taken for the front of the wave to reach the end of the pipe.

SOLUTION:

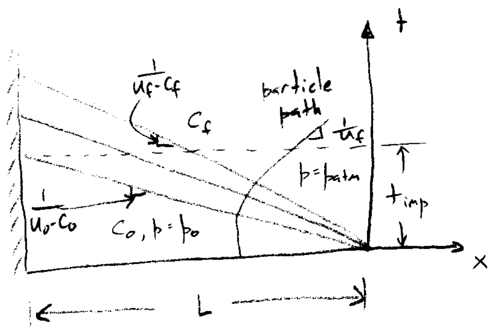


$L = 4\text{ m}$

$p_0 = 200\text{ kPa}$

$T_0 = 30°C = 303\text{ K}$

$p_{atm} = 103\text{ kPa}$



$c_0 = \sqrt{\gamma R T_0}$

$= 348.9 \text{ m/s}$

$\Rightarrow t_{imp} = \frac{L}{c_0} = \frac{4\text{ m}}{348.9\text{ m/s}}$

$\boxed{\therefore t_{imp} = 0.0115\text{ s}}$

- Across a left-running characteristic:

$$-du = \frac{2}{\gamma - 1} dc$$

$$\Rightarrow u_f - \cancel{u_0}^{\,0} = \frac{2}{1-\gamma}(c_f - c_0)$$

$$\Rightarrow \frac{u_f}{c_0} = \frac{2}{1-\gamma}\left(\frac{c_f}{c_0} - 1\right) \quad \text{but} \quad \frac{c_f}{c_0} = \frac{\sqrt{\gamma R T_f}}{\sqrt{\gamma R T_0}} = \left(\frac{T_f}{T_0}\right)^{1/2} = \left(\frac{p_f}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}$$

$$\Rightarrow \frac{u_f}{c_0} = \frac{2}{1-\gamma}\left[\left(\frac{p_f}{p_0}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right]$$

$\boxed{\therefore u_f = 157.8\text{ m/s}}$ using $c_0 = 348.9\text{ m/s}$, $\gamma = 1.4$, $p_f = 103\text{ kPa}$, $p_0 = 200\text{ kPa}$

Solution...

- The velocity of the back of the expansion fan is c_f w/r/t the local fluid velocity:

$$\frac{c_f}{c_o} = \left(\frac{p_f}{p_o}\right)^{\frac{\gamma-1}{2\gamma}}$$

$$\therefore \quad c_f = 317.3 \text{ m/s} \qquad \text{using} \quad \begin{aligned} c_o &= 348.9 \text{ m/s} \\ p_f &= 103 \text{ kPa} \\ p_o &= 200 \text{ kPa} \\ \gamma &= 1.4 \end{aligned}$$

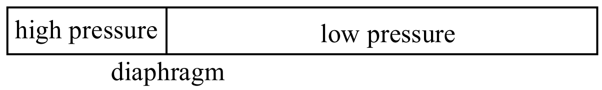
- The speed of the wave w/r/t the ground is

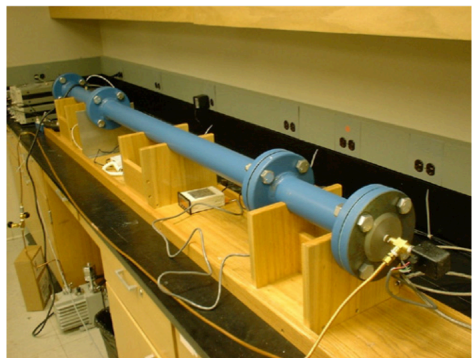
$$u_f - c_f = 157.8 \text{ m/s} - 317.3 \text{ m/s}$$

$$\boxed{\therefore \quad u_f - c_f = -159.5 \text{ m/s} \quad (\text{wave moves to left})}$$

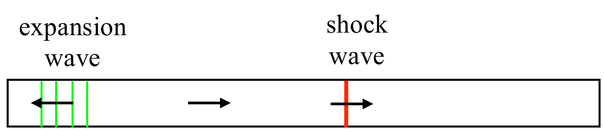
23. Description of a Shock Tube

A shock tube is a device consisting, in its simplest form, of a long tube of constant area divided into two sections by a diaphragm. One of the sections contains a high pressure gas (aka driver gas) while the other contains a low pressure gas (aka driven gas). A sketch of the device is shown below.





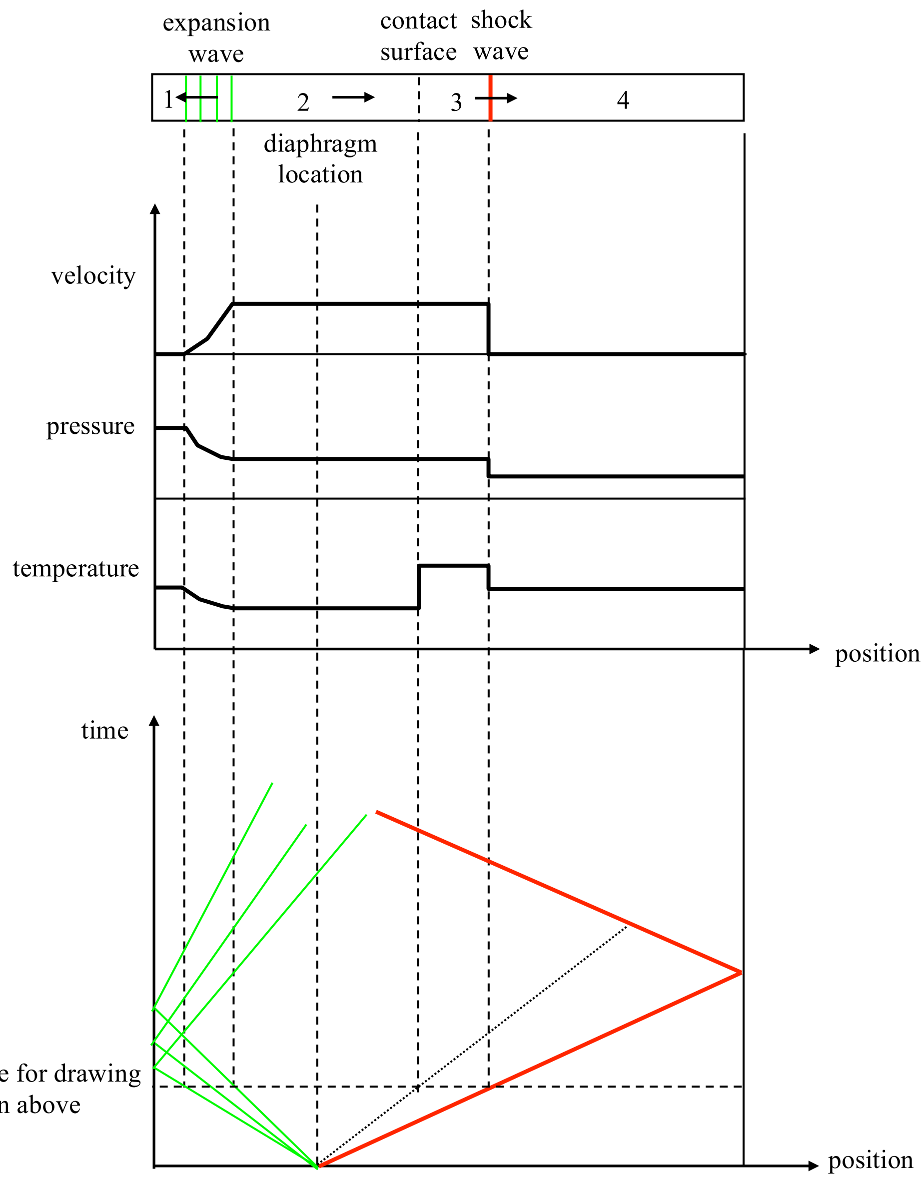
When the diaphragm between the two sections is broken, either by mechanical means or by increasing the pressure on the high pressure side and using a "scored" diaphragm designed to burst at a specified presssure, a shock wave propagates into the low pressure section and an expansion wave propagates into the high pressure section. Between the shock wave and the expansion wave is a region of uniform velocity.



Shock tubes are used:
- as an inexpensive, but short duration (usually on the order of milliseconds), wind tunnel,
- to study of transient aerodynamic effects,
- to study dynamic and thermal response,
- to study relaxation effects and reaction rates, and
- to generate high enthalpies for studying dissociation and ionization.

Analysis of the Flow in a Shock Tube

The velocity and pressure behind the shock wave must be equal to the velocity and pressure behind the expansion wave as shown in the figure below. The temperatures (and densities and entropies) are not necessarily equal however. If the temperature in the tube is initially uniform, then behind the expansion fan the temperature will be lower than the initial temperature while behind the shock wave the temperature will be higher than the initial temperature. The interface separating the two regions is called the **contact surface**. The contact surface is the interface dividing the gases that were originally separated by the diaphragm. Over time, diffusion will cause this interface to spread out.



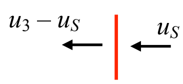
We know the following about the flow:

1. The gas properties are assumed identical on either side of the diaphragm, i.e., $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$ and $R_1 = R_2 = R_3 = R_4$.
2. The velocities in regions 1 and 4 are zero, i.e., $u_1 = u_4 = 0$.
3. The pressures and temperatures in regions 1 and 4 are known, i.e., p_1, p_4, T_1, and T_4 are known.
4. The pressure and velocity across the contact surface are equal, i.e., $p_2 = p_3$ and $u_2 = u_3$.

The remainder of the flow field is most easily analyzed using an iterative procedure as described below.

1. Assume a value for $p_2 = p_3$.

2. The value of u_2 can be determined from the pressure ratio, p_2/p_1, and a relationship developed previously in our notes concerning 1D, unsteady, and isentropic compressible flow:
$$-\frac{u_2}{c_1} = \frac{2}{\gamma-1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] \text{ (crossing L-running waves)} \tag{356}$$
where $c_1 = (\gamma R T_1)^{1/2}$.

3. Use the normal shock relations to determine Ma_S and u_3:
$$\frac{p_3}{p_4} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_S^2 - \frac{\gamma-1}{\gamma+1} \quad \text{(determine } \mathrm{Ma}_S\text{)} \tag{357}$$
$$\frac{-u_S}{u_3 - u_S} = \frac{(\gamma+1)\mathrm{Ma}_S^2}{(\gamma-1)\mathrm{Ma}_S^2 + 2} \tag{358}$$


velocities are given w/r/t the ground

where $u_S = \mathrm{Ma}_S(\gamma R T_4)^{1/2}$.

4. Is $u_3 = u_2$? If not, then go back to step 1. If so, then we've correctly determine the pressure and velocity in regions 2 and 3.

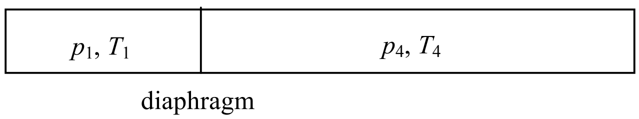
5. Now that the pressure and velocity in regions 1-4 are known, we can also determine the temperature in region 2 since we know the speed of sound there:
$$-\frac{u_2}{c_1} = \frac{2}{1-\gamma}\left[\frac{c_2}{c_1} - 1\right]$$
$$T_2 = \frac{c_2^2}{\gamma R}$$
or
$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$
We can also determine the temperature in region 3 using the normal shock relations:
$$\frac{T_3}{T_4} = \left[2 + (\gamma-1)\mathrm{Ma}_S^2\right]\left[\frac{2\gamma \mathrm{Ma}_S^2 - (\gamma-1)}{(\gamma+1)^2 \mathrm{Ma}_S^2}\right]$$

6. The speed of the contact surface can be determined by noting that it has the same velocity as the gas in regions 2 and 3, i.e., $u_{CS} = u_2 = u_3$.

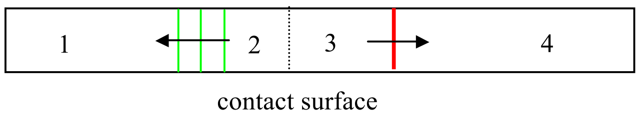
A shock tube containing air has a high pressure section at 300 kPa (abs) and a low pressure section at 30 kPa (abs). The temperature of the air is uniform at 15 °C. The diaphragm separating the two sections is suddenly ruptured. Find:

a. the velocity of the air between the shock wave and the expansion wave relative to the ground,
b. the speed of the shock wave relative to the ground, and
c. the speed of the front and back of the expansion fan relative to the ground.
d. Sketch the process on a *t*-*x* diagram.

SOLUTION:



$p_1 = 300$ kPa (abs)
$p_4 = 30$ kPa (abs)
$T_1 = T_4 = 15$ °C $= 288$ K
$u_1 = u_4 = 0$



Determine the speed of sound in regions 1 and 4.

$$c_1 = \sqrt{\gamma R T_1} \Rightarrow c_1 = 340.2 \text{ m/s (using } \gamma = 1.4 \text{ and } R = 287 \text{ J/(kg·K))} \quad (1)$$

$$c_4 = \sqrt{\gamma R T_4} \Rightarrow c_4 = 340.2 \text{ m/s} \quad (2)$$

Use the following algorithm to determine the properties in the various regions.

1. Assume a value for $p_2 = p_3$.

2. The value of u_2 can be determined from the pressure ratio, p_2/p_1, and a relationship developed previously in our notes concerning 1D, unsteady, and isentropic compressible flow:

$$-\frac{u_2}{c_1} = \frac{2}{\gamma - 1}\left[\left(\frac{p_2}{p_1}\right)^{\gamma-1/2\gamma} - 1\right] \quad (1.3)$$

where $c_1 = (\gamma R T_1)^{1/2}$.

3. Use the normal shock relations to determine Ma_S and u_3:

$$\frac{p_3}{p_4} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_S^2 - \frac{\gamma-1}{\gamma+1} \quad \text{(determine Ma}_S\text{)} \quad (1.4)$$

$$\frac{-u_S}{u_3 - u_S} = \frac{(\gamma+1)\mathrm{Ma}_S^2}{(\gamma-1)\mathrm{Ma}_S^2 + 2}$$

where $u_S = \mathrm{Ma}_S(\gamma R T_4)^{1/2}$.

(1.5)

4. Is $u_3 = u_2$? If not, then go back to step 1. If so, then we've correctly determine the pressure and velocity in regions 2 and 3.

5. Now that the pressure and velocity in regions 1-4 are known, we can also determine the temperature in region 2 since we know the speed of sound there:

$$-\frac{u_2}{c_1} = \frac{2}{1-\gamma}\left[\frac{c_2}{c_1} - 1\right]$$

$$T_2 = \frac{c_2^2}{\gamma R}$$

or

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\gamma-1/\gamma}$$

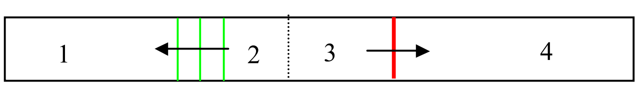
We can also determine the temperature in region 3 using the normal shock relations:

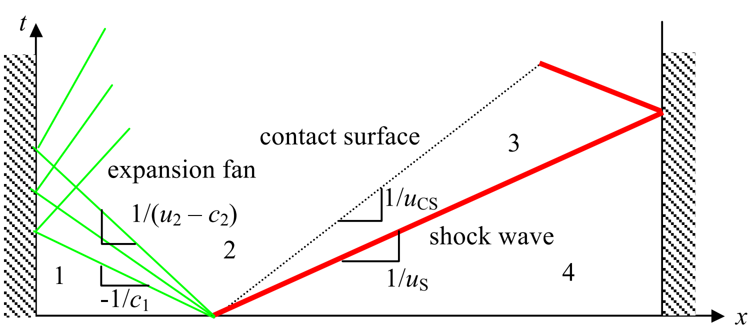
$$\frac{T_3}{T_4} = \left[2+(\gamma-1)\mathrm{Ma}_S^2\right]\left[\frac{2\gamma \mathrm{Ma}_S^2-(\gamma-1)}{(\gamma+1)^2\mathrm{Ma}_S^2}\right]$$

6. The speed of the contact surface can be determined by noting that it has the same velocity as the gas in regions 2 and 3, i.e., $u_{CS} = u_2 = u_3$.

Using the algorithm described above:

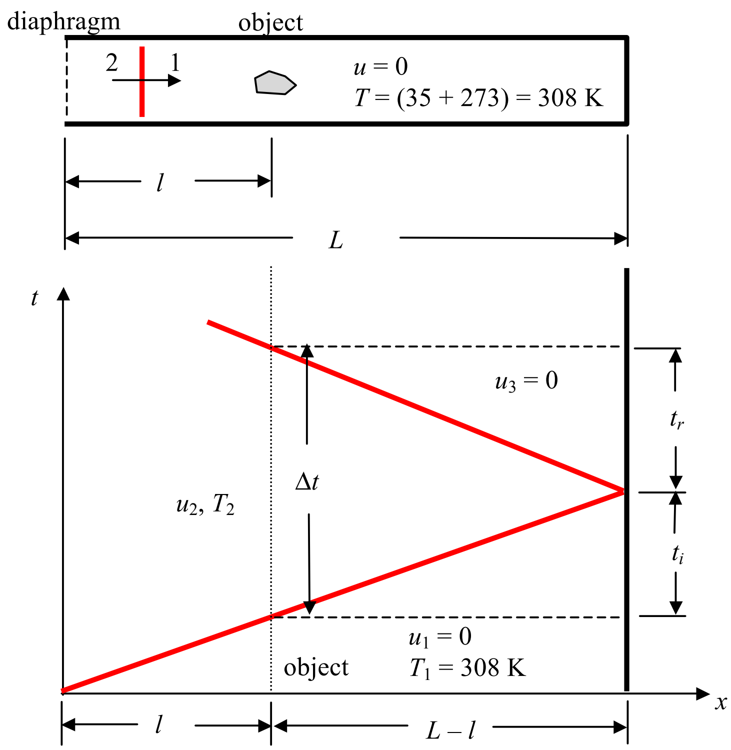
$\boxed{u_2 = u_3 = u_{CS}}$	=	$\boxed{279.3 \text{ m/s}}$	(answer to part (a))
p_2/p_1	=	0.2848	
$p_2 = p_3$	=	85.4 kPa	
Ma_S	=	1.607	
$\boxed{u_S}$	=	$\boxed{546.7 \text{ m/s}}$	(answer to part (b))
c_2	=	284.3 m/s	
$\boxed{u_2 - c_2}$	=	$\boxed{-5.0 \text{ m/s}}$	(answer to part (c) – back part of fan)
T_2	=	201.2 K	
T_3	=	401.1 K	
c_3	=	401.5 m/s	
$u_1 = u_4$	=	0	
c_1	=	340.2 m/s	
$\boxed{u_1 - c_1}$	=	$\boxed{-340.2 \text{ m/s}}$	(answer to part (c) – front part of fan)
c_4	=	340.2 m/s	





A shock tube is to be used to subject an object to momentary conditions of high pressure and temperature. To provide an adequate measuring time the tube is to be made long enough so that a period of 100 ms is provided between the time of passage over the body of the initial shock and the time of passage of the shock reflected from the closed end of the tube. The initial pressure ratio across the diaphragm is such as to yield an initial shock with a pressure ratio of 10 to 1, with the object located 3 m from the diaphragm. The initial temperature of the air in the shock tube is 35 °C. Determine a suitable length for the low-pressure end of the tube. Assume the contact surface travels much more slowly than the shock waves.

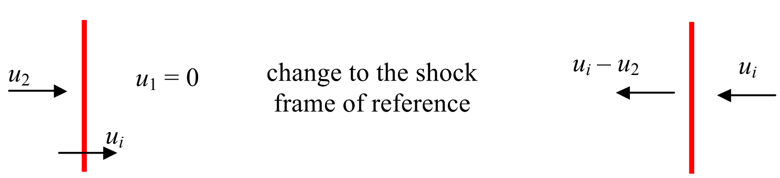
SOLUTION:



Use the given pressure ratio to determine the Mach number of the incident shock wave.

$$\frac{p_2}{p_1} = 10 = \frac{2\gamma}{\gamma+1}\text{Ma}_i^2 - \frac{\gamma-1}{\gamma+1} \Rightarrow \text{Ma}_i = 2.952 \tag{1}$$

Use this Mach number and the temperature to determine the speed of the incident shock wave relative to the ground.

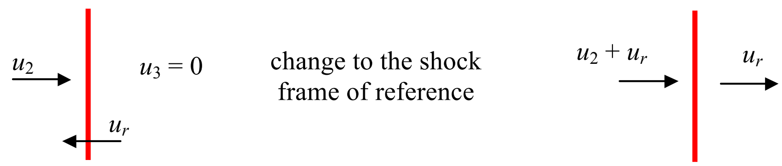


$$u_i = \text{Ma}_i\sqrt{\gamma R T_1} \Rightarrow u_i = 1038 \text{ m/s} \tag{2}$$

$$\frac{T_2}{T_1} = \left[2 + (\gamma-1)\text{Ma}_i^2\right]\left[\frac{2\gamma\text{Ma}_i^2 - (\gamma-1)}{(\gamma+1)^2\text{Ma}_i^2}\right] \Rightarrow T_2 = 807.9 \text{ K} \tag{3}$$

$$\frac{u_i}{u_i - u_2} = \frac{(\gamma+1)\mathrm{Ma}_i^2}{(\gamma-1)\mathrm{Ma}_i^2 + 2} \Rightarrow u_2 = 766.1\ \text{m/s} \quad \text{(wind velocity w/r/t ground following the shock)} \tag{4}$$

The air downstream of the reflected shock (w/r/t the ground) must be zero due to the presence of the stationary wall.



The velocity ratio across the shock may be related to the Mach number upstream of the shock (in the shock frame of reference):

$$\frac{u_2 + u_r}{u_r} = \frac{(\gamma+1)\mathrm{Ma}_r^2}{(\gamma-1)\mathrm{Ma}_r^2 + 2} \tag{5}$$

In addition, the Mach number of the flow into the shock may also be determined using the upstream flow conditions:

$$\mathrm{Ma}_r = \frac{u_2 + u_r}{\sqrt{\gamma R T_2}} \tag{6}$$

Solve Eqs. (5) and (6) iteratively to find $u_r = 425.6$ m/s and $\mathrm{Ma}_r = 2.092$.

The time for the incident shock to travel from the object ot the wall is:

$$t_i = \frac{L - l}{u_i} \tag{7}$$

and the time for the reflected shock to travel from the wall to the object is:

$$t_r = \frac{L - l}{u_r} \tag{8}$$

The goal is to have the total time, Δt, equal 100 ms:

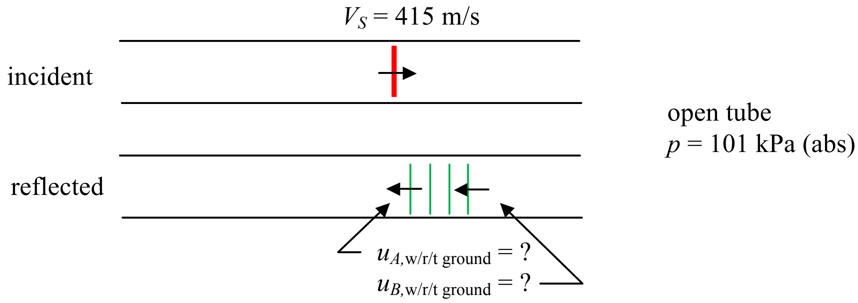
$$\Delta t = t_i + t_r = \frac{L - l}{u_i} + \frac{L - l}{u_r} \tag{9}$$

$$u_i u_r \Delta t = u_r (L - l) + u_i (L - l) = (u_r + u_i) L - (u_r + u_i) l \tag{10}$$

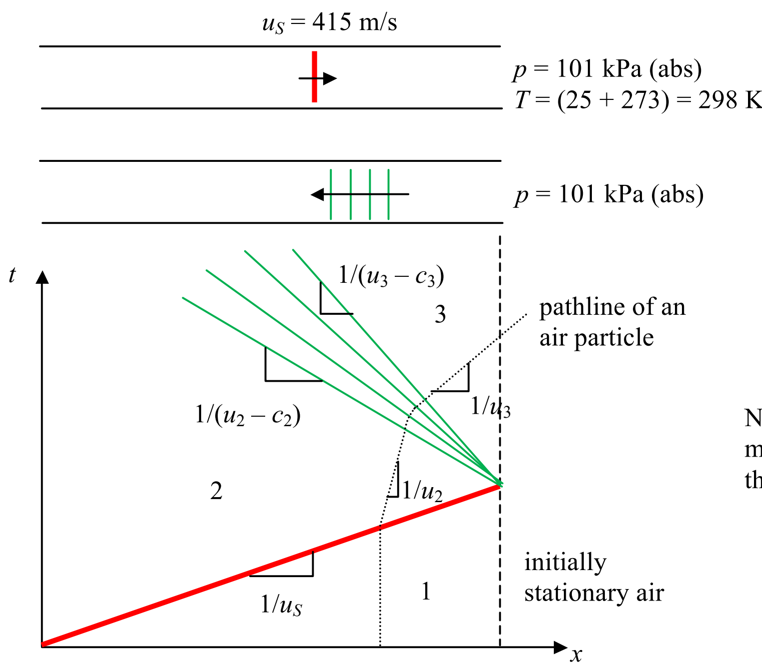
$$L = \frac{u_i u_r}{u_r + u_i} \Delta t + l \tag{11}$$

Substituting the given numbers, $\boxed{L = 33.2\ \text{m}}$.

A normal shock moves down an open-ended tube with a velocity of 415 m/s (with respect to the stationary air upstream of the shock). The ambient air pressure and temperature are 101 kPa (abs) and 25 °C upstream of the shock wave. Determine the velocity, with respect to the ground, of the **first and last** expansion waves that move down the tube after reflection of the shock from the open end.



SOLUTION:



Note: The speeds, u, are measured with respect to the ground.

The first and last reflected expansion waves are left running characteristics. Thus, the speed of these waves with respect to the ground will be:

$$u_{A,\text{w/r/t ground}} = u_2 - c_2 \tag{1}$$

$$u_{B,\text{w/r/t ground}} = u_3 - c_3 \tag{2}$$

The Mach number of the shock wave is:

$$\text{Ma}_S = \frac{u_S}{c_1} = \frac{u_S}{\sqrt{\gamma R T_1}} \Rightarrow \text{Ma}_S = 1.199 \tag{3}$$

where u_S = 415 m/s, γ = 1.4, R = 287 J/(kg.K), and T_1 = 298 K.

Using the normal shock relations, the Mach number and the velocity, temperature, and pressure ratios across the shock are:

$$\mathrm{Ma}_2^2 = \frac{(\gamma-1)\mathrm{Ma}_1^2 + 2}{2\gamma\mathrm{Ma}_1^2 - (\gamma-1)} \Rightarrow \mathrm{Ma}_2 = 0.8426 \text{ (relative to the shock)} \tag{4}$$

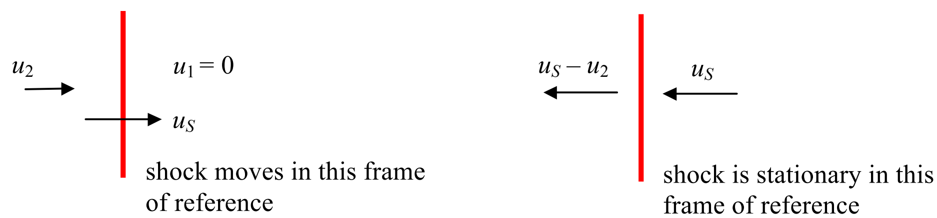
$$\frac{T_2}{T_1} = \left[2+(\gamma-1)\mathrm{Ma}_1^2\right]\frac{2\gamma\mathrm{Ma}_1^2-(\gamma-1)}{\left[(\gamma+1)\mathrm{Ma}_1\right]^2} \Rightarrow T_2/T_1 = 1.1276 \tag{5}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2 - \frac{\gamma-1}{\gamma+1} \Rightarrow p_2/p_1 = 1.5114 \tag{6}$$

$$\frac{p_2}{p_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_2/p_{02} = 0.6283 \tag{7}$$

$$\frac{T_2}{T_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{-1} \Rightarrow T_2/T_{02} = 0.8757 \tag{8}$$

$$\frac{u_S-u_2}{u_S} = \frac{2+(\gamma-1)\mathrm{Ma}_1^2}{(\gamma+1)\mathrm{Ma}_1^2} \Rightarrow (u_S-u_2)/u_S = 0.7460 \Rightarrow u_2 = 105.4 \text{ m/s} \tag{9}$$



The air velocity and speed of sound downstream of the shock wave are:

$$c_2 = \sqrt{\gamma R T_2} \Rightarrow c_2 = 367.4 \text{ m/s} \quad (T_2 = 336.0 \text{ K}) \tag{10}$$

$$p_2 = 153.1 \text{ kPa (abs)} \tag{11}$$

The shock wave reflects as an expansion fan from the free surface boundary in order to maintain a constant pressure boundary condition. The speed of the first expansion wave front relative to the ground is:

$$u_A = u_2 - c_2 \Rightarrow \boxed{u_A = -262.0 \text{ m/s (to the left)}} \tag{12}$$

The speed of sound in region 3 may be found from the pressure ratio and noting that the process in going from 2 to 3 is isentropic:

$$\frac{c_3}{c_2} = \left(\frac{p_3}{p_2}\right)^{\frac{\gamma-1}{2\gamma}} \Rightarrow c_3/c_2 = 0.9427 \Rightarrow c_3 = 346.4 \text{ m/s} \tag{13}$$

The velocity in region 3 may be found by noting that we're crossing left-running characteristic curves in going from region 2 to region 3:

$$u_3 - u_2 = -\frac{2}{\gamma-1}(c_3-c_2) = \frac{2}{1-\gamma}(c_3-c_2) \Rightarrow u_3 = 210.7 \text{ m/s} \tag{14}$$

Thus, the speed of the last expansion wave front relative to the ground is:

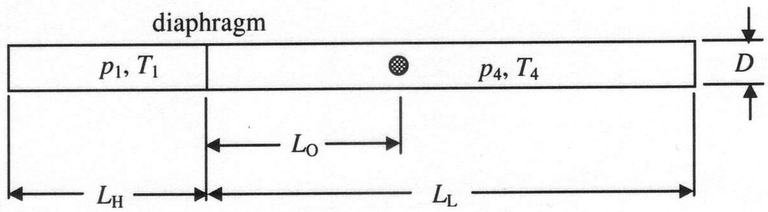
$$u_B = u_3 - c_3 \Rightarrow \boxed{u_B = -135.7 \text{ m/s (to the left)}} \tag{15}$$

Alternately, the velocity in region 3 may be found directly in terms of the pressure ratio using:

$$-\left(\frac{u_3}{c_2}-\frac{u_2}{c_2}\right) = \frac{2}{\gamma-1}\left[\left(\frac{p_3}{p_2}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] \Rightarrow u_3 = u_2 + \frac{2c_2}{\gamma-1}\left[1-\left(\frac{p_3}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\right] \Rightarrow u_3 = 210.7 \text{ m/s} \tag{16}$$

A shock tube is 10 m long with a 30 cm diameter. The high pressure section is 4 m long and contains air at 200 kPa; the low pressure section is 6 m long and contains air at 5 kPa. A test object is placed in the low pressure section 3 m from the diaphragm. Both sections initially contain air at 25 °C. The diaphragm is suddenly ruptured which causes a shock to move into the low pressure section. Determine the following:

a. the shock velocity,
b. the contact surface velocity,
c. the Mach number of the air behind the shock,
d. the time between the passage of the normal shock and the contact surface over the test object, and
e. the time between the passage of the normal shock and the reflected shock over the test object.
f. Provide an *x-t* diagram showing the location of the initial shock, reflected shock, and contact surface as a function of time.

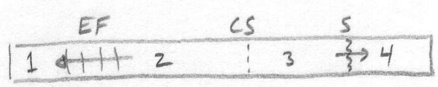


$D = 0.30$ m
$L_H = 4$ m
$L_L = 6$ m
$L_O = 3$ m
$p_1 = 200$ kPa
$p_4 = 5$ kPa
$T_1 = T_4 = 25$ °C

Assume air: $\gamma = 1.4$
$R = 287$ J/kg·K

SOLUTION:

- First solve for the pressure and velocity in regions 2 and 3 as shown:



- Use an iterative procedure:

1) Assume a value for $p_2 = p_3$

2) Determine u_2 using:
$$-\frac{u_2}{c_1} = \frac{2}{\gamma-1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right]$$
where $c_1 = \sqrt{\gamma R T_1}$

3) Determine u_s using:
$$\frac{p_3}{p_4} = \frac{2\gamma}{\gamma+1} Ma_s^2 - \frac{\gamma-1}{\gamma+1} \quad \text{(determine } Ma_s\text{)}$$
$$\frac{-u_s}{u_3 - u_s} = \frac{(\gamma+1) Ma_s^2}{(\gamma-1) Ma_s^2 + 2} \quad \text{(determine } u_3\text{)}$$
where $u_s = Ma_s \sqrt{\gamma R T_4}$

4) Is $u_2 = u_3$?

No.

- Use a spreadsheet to solve.

Solution...

- The temperature in region 2 can be found using the isentropic relations:

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

- The temperature in region 3 can be found using the shock relations:

$$\frac{T_3}{T_4} = \left[2+(\gamma-1)Ma_s^2\right]\left[\frac{2\gamma Ma_s^2-(\gamma-1)}{(\gamma+1)^2 Ma_s^2}\right]$$

$\gamma =$ 1.4
$R =$ 287 J/(kg.K)
$p_1 =$ 200 kPa
$p_4 =$ 5 kPa
$T_1 =$ 298 K
$T_4 =$ 298 K

$c_1 = c_4 =$ 346.0 m/s

$\Delta(p_2/p_1) =$ 0.00001

p_2/p_1	u_2	p_3/p_4	Ma_S	u_S	u_3	u_2-u_3
0.11930	453.2	4.772	2.057	711.9	453.1	0.06
0.11931	453.2	4.772	2.058	712.0	453.2	0.02
0.11932	453.2	4.773	2.058	712.0	453.2	-0.03
0.11933	453.1	4.773	2.058	712.0	453.2	-0.07
0.11934	453.1	4.774	2.058	712.1	453.3	-0.12
0.11935	453.1	4.774	2.058	712.1	453.3	-0.16
0.11936	453.1	4.774	2.058	712.1	453.3	-0.21
0.11937	453.1	4.775	2.058	712.1	453.3	-0.25
0.11938	453.1	4.775	2.058	712.2	453.4	-0.30
0.11939	453.1	4.776	2.058	712.2	453.4	-0.34

$u_2 = u_3 =$ 453.2 m/s
$p_2/p_1 =$ 0.1193
$p_2 = p_3 =$ 23.9 kPa
$Ma_S =$ 2.058
$u_S =$ 712.1 m/s
$c_2 =$ 255.4 m/s
$T_2 =$ 162.3 K
$T_3 =$ 517.1 K
$c_3 =$ 455.8 m/s

Solution...

- Thus, $\boxed{\begin{array}{l} U_S = 712.1 \text{ m/s} \\ U_{CS} = U_2 = U_3 = 453.2 \text{ m/s} \end{array}}$

- Mach # of air behind the shock

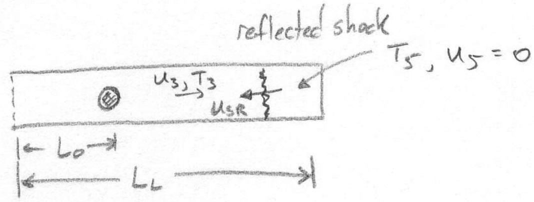
$$Ma_3 = \frac{U_3}{c_3} = \frac{453.2 \text{ m/s}}{455.8 \text{ m/s}} = \boxed{0.99 = Ma_3}$$

- $$\left.\begin{array}{l} t_S = L_0/U_S \\ t_{CS} = L_0/U_{CS} \end{array}\right\} \Rightarrow \Delta t_{S/CS} = t_{CS} - t_S = L_0/U_{CS} - L_0/U_S$$

$$= L_0\left(\frac{1}{U_{CS}} - \frac{1}{U_S}\right)$$

$$\boxed{\therefore \Delta t_{S/CS} = 0.0024 \text{ sec}} \quad (2.4 \text{ ms})$$

- $t_{S_{L_0}} = L_0/U_S$

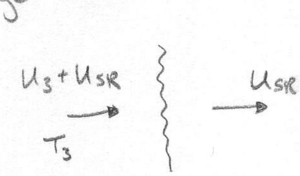


$t_{S_{L_L}} = L_L/U_S$

$$t_{S_{2L_L - L_0}} = t_{S_{L_L}} + \frac{(L_L - L_0)}{U_{SR}}$$

reflected shock velocity

- change frame of reference:



$$Ma_{SR} = \frac{U_3 + U_{SR}}{c_3} \quad \text{where } c_3 = \sqrt{\gamma R T_3}$$

$$\boxed{\begin{array}{l} \therefore \Delta t_{S_{L_0}/S_{2L_L-L_0}} = t_{S_{2L_L-L_0}} - t_{S_{L_0}} \\ = 12.8 \text{ ms} \end{array}}$$

where $t_{S_{L_0}} = 4.2 \text{ ms}$

$t_{S_{L_L}} = 8.4 \text{ ms}$

$t_{S_{2L_L-L_0}} = 17.0 \text{ ms}$

- Solve for U_{SR} using shock relations:

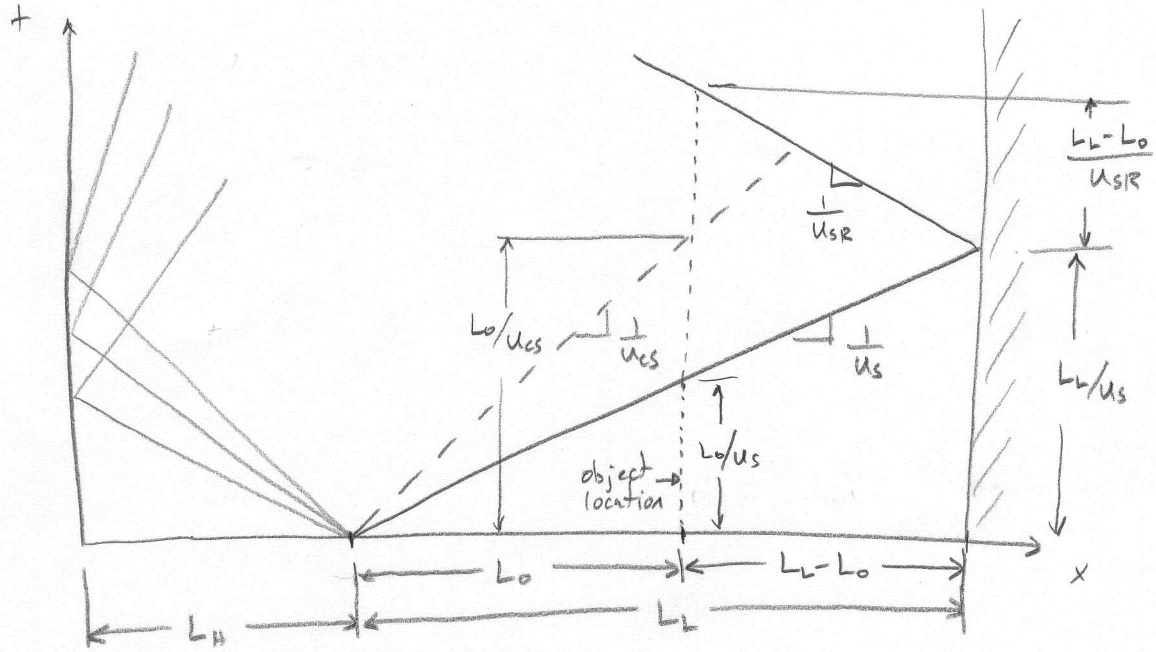
$$\frac{U_{SR}}{U_3 + U_{SR}} = \frac{2}{(\gamma+1) Ma_{SR}^2} + \frac{\gamma - 1}{\gamma + 1}$$

$$\Rightarrow \underline{U_{SR} = 349.5 \text{ m/s}}$$

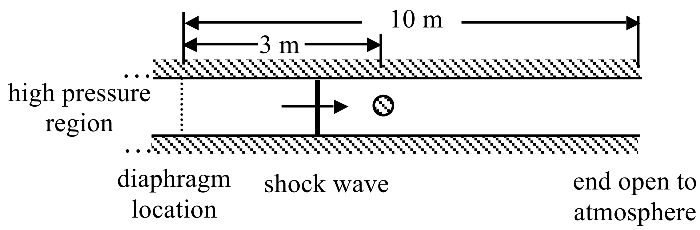
using $U_3 = 453.2 \text{ m/s}$

$c_3 = 455.8 \text{ m/s}$

Solution...


t
x
$\frac{L_L - L_0}{u_{SR}}$
$\frac{1}{u_{SR}}$
L_0/u_{CS}
$\frac{1}{u_{CS}}$
$\frac{1}{u_S}$
L_L/u_S
L_0/u_S
object location
L_0
$L_L - L_0$
L_H
L_L

High speed air flows past an object is to be studied using the shock tube shown below.



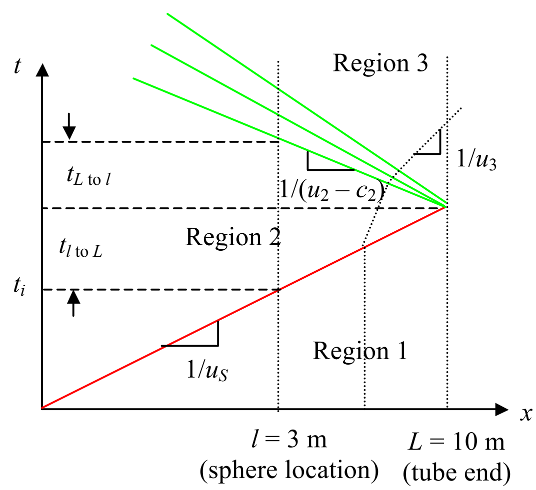
p_{atm} = 100 kPa (abs)
T = 293 K

The pressure rise across the shock wave is 4.5:1. The right hand side of the tube is open to the atmosphere.

a. Sketch the process on an *x*-*t* diagram.
b. How long will it take the incident shock wave to reach the object?
c. How much time passes between when the incident shock wave reaches the object to when the first reflected wave reaches the object?
d. What will be the air velocity out of the right side of the tube following the reflected waves?

You may ignore the contact surface for this problem.

SOLUTION:



$u_1 = 0$
p_1 = 100 kPa (abs)
T_1 = 293 K

u_2 = ?
p_2 = 450 kPa (abs) (pressure ratio of 4.5 to 1)
T_2 = ?

u_3 = ?
p_3 = 100 kPa (abs) (pressure at free surface is constant)
T_3 = ?

Note that a shock wave reflects from the free surface outlet boundary as an expansion fan in order to maintain a constant pressure.

Across the normal shock wave:

$$\frac{p_2}{p_1} = 4.5 \;\Rightarrow\; \text{Ma}_S = 2.0 \;\text{(using the normal shock relations)} \tag{1}$$

$$u_S = \text{Ma}_S\sqrt{\gamma R T_1} \;\Rightarrow\; \underline{u_S = 686.2 \text{ m/s}} \tag{2}$$

The time it takes for the incident shock to reach the object is:

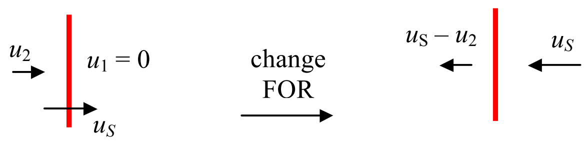
$$t_i = \frac{l}{u_S} \;\Rightarrow\; \boxed{t_i = 4.4 \text{ ms}} \tag{3}$$

The conditions in region 2 may be found using the normal shock relations.

$$\text{Ma}_S = 2.0 \;\Rightarrow\; \frac{T_2}{T_1} = 1.6875 \;\Rightarrow\; \underline{T_2 = 494.4 \text{ K}} \tag{4}$$

$$c_2 = \sqrt{\gamma R T_2} \;\Rightarrow\; \underline{c_2 = 445.7 \text{ m/s}} \tag{5}$$

The gas velocity behind the shock may be found by applying the normal shock relations in the shock wave's frame of reference (a steady frame of reference).



$$\text{Ma}_S = 2.0 \;\Rightarrow\; \frac{V_1}{V_2} = \frac{u_S}{u_S - u_2} = 2.6667 \;\Rightarrow\; u_2 = \left(\frac{2.6667 - 1}{2.6667}\right) u_S \tag{6}$$

$$\underline{u_2 = 428.9 \text{ m/s}} \tag{7}$$

Note that:

$$\text{Ma}_2 = u_2/c_2 \;\Rightarrow\; \text{Ma}_2 = 0.9623 \tag{8}$$

The time for the shock to travel from the object to the tube end is:

$$t_{l \text{ to } L} = \frac{L - l}{u_S} \;\Rightarrow\; t_{l \text{ to } L} = 10.2 \text{ ms} \tag{9}$$

The time for the reflected expansion wave to travel from the tube end to the object is:

$$t_{L \text{ to } l} = \frac{l - L}{u_2 - c_2} \;\Rightarrow\; 416.1 \text{ ms} \tag{10}$$

Thus, the total time between the passing of the shock wave and the first expansion wave is:

$$\Delta t = t_{l \text{ to } L} + t_{L \text{ to } l} \;\Rightarrow\; \boxed{\Delta t = 426.3 \text{ ms}} \tag{11}$$

The velocity in region 3 may be found by noting that we're crossing left-running characteristic curves in going from region 2 to region 3:

$$-\left(\frac{u_3}{c_2} - \frac{u_2}{c_2}\right) = \frac{2}{\gamma - 1}\left[\left(\frac{p_3}{p_2}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] \;\Rightarrow\; u_3 = u_2 + \frac{2c_2}{\gamma - 1}\left[1 - \left(\frac{p_3}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\right] \tag{12}$$

$$\boxed{u_3 = 859.8 \text{ m/s}} \tag{13}$$

Note that:

$$\frac{c_3}{c_2} = \left(\frac{p_3}{p_2}\right)^{\frac{\gamma-1}{2\gamma}} \;\Rightarrow\; c_3 = 359.5 \text{ m/s} \tag{14}$$

$$\text{Ma}_3 = \frac{u_3}{c_3} \;\Rightarrow\; \text{Ma}_3 = 2.392 \tag{15}$$