OBSERVERS FOR LINEAR SYSTEMS WITH UNKNOWN INPUTS

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ABSTRACT

In practice, it is often the case that systems can be modeled as having unknown inputs. For such systems, it may be necessary to estimate the states and inputs in order to achieve certain control objectives. In this thesis, we study linear systems with unknown inputs, and design observers to estimate the desired quantities. In particular, we consider the use of delayed observers, which enlarges the class of systems for which state and input estimation is possible.

We start by designing reduced-order observers that reconstruct the entire state. Our design procedure is quite general in that it encompasses the design of full-order observers via appropriate choices of design matrices. We provide necessary and sufficient conditions for the existence of such observers, as well as a characterization of the minimum delay required to reconstruct the entire state. Once the state observer is constructed, we show that it is straightforward to obtain an estimate of the unknown inputs.

We then present an investigation of partial state and input observers for linear systems with unknown inputs. Our approach characterizes the set of all linear functions of the states and inputs that can be reconstructed through a linear observer with a given delay, and directly produces the corresponding observer parameters. In comparison to previous work on partial observers, the main contributions of our work are (i) an algebraic procedure to design delayed linear observers with dimension no greater than that of the system, and (ii) a unified treatment of both state and input observers.

Finally, we consider a stochastic setup and present a method for constructing linear minimum-variance unbiased state estimators for discrete-time linear stochastic systems with unknown inputs. Our design provides a characterization of estimators with delay, which eases the established necessary conditions for existence of unbiased estimators with zero-delay. A consequence of using delayed estimators is that the noise affecting the system becomes correlated with the estimation error. We handle this correlation by increasing the dimension of the estimator appropriately.

To my family.

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CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

When designing control systems, there is frequently some degree of uncertainty surrounding the plant. For example, some of the plant parameters may not be exactly known [1, 2], or the plant may be subject to unmeasurable disturbances [3]. Similarly, in decentralized control systems, it may not be possible to have knowledge of the control signals generated by different controllers [4]. The system may also be affected by faults, the magnitude and characteristics of which cannot be predicted *a priori* [5].

These uncertainties can often be incorporated into the system model by treating them as unknown inputs. Traditional techniques for control system design must then be generalized to handle these uncertainties. For example, researchers have studied ways to reconstruct the unknown inputs by using only the output of the system and possibly the initial system state [6–11]. These investigations have revealed that it will generally be necessary to use delayed (or differentiated) outputs in order to reconstruct the inputs. A system is said to be *invertible* if it is possible to reconstruct the inputs in the above manner. Researchers have also extended the standard Luenberger state observers [12–14] to treat the problem of state estimation in linear systems with unknown inputs [15–25]. It has been shown that system invertibility is a necessary condition for the existence of unknown input observers [26], and thus delayed observers may be required in order to estimate the entire system state.

In this thesis, we study state and input observers for linear systems with unknown inputs. We start by examining the problem of observing the entire state and input in such systems, and develop a design procedure to obtain the observer parameters. We then investigate partial observers, which reconstruct a maximal subset of the states and inputs with a given delay. Finally, we study linear stochastic systems with unknown inputs, and develop an optimal state estimator that minimizes the mean square estimation error. Specifically, the main contributions of this thesis are:

1. The development of a unified design procedure for both reduced and full-order state

observers that incorporate delays.

2. The characterization of all possible linear functions of the state vector and input that can be observed with a given delay.

3. A design procedure for optimal state estimators with delays for linear systems operating in the presence of noise.

The fact that we consider delayed observers makes our investigation more general than many of the other works currently present in the literature.

1.2 Preliminaries and Notation

In this thesis, we will be considering discrete-time linear systems of the form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \ , \end{aligned} \tag{1.1}$$

with state vector $x \in \mathbb{R}^n$, unknown input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, and system matrices (A, B, C, D) of appropriate dimensions. Note that we omit known inputs in the above equations for clarity of development. We also assume without loss of generality that the matrix $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ is full column rank. This assumption can always be enforced by an appropriate transformation and renaming of the unknown inputs. Finally, while we consider discrete-time systems in our development, our approach applies equally well to continuous-time systems by replacing advances with differentiators (e.g., $y_{k+\alpha}$ should be replaced by the α'th derivative of $y(t)$).

The response of system (1.1) over $\alpha + 1$ time units is given by

$$\underbrace{\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+\alpha} \end{bmatrix}}_{Y_{k:k+\alpha}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha} \end{bmatrix}}_{\Theta_\alpha} x_k + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-1}B & CA^{\alpha-2}B & \cdots & D \end{bmatrix}}_{M_\alpha} \underbrace{\begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+\alpha} \end{bmatrix}}_{U_{k:k+\alpha}} \ . \tag{1.2}$$

The matrices Θ_α and M_α in the above equation can be expressed in a variety of ways.

We will be using the following identities in our derivations:

$$\Theta_\alpha = \begin{bmatrix} C \\ \Theta_{\alpha-1}A \end{bmatrix} = \begin{bmatrix} \Theta_{\alpha-1} \\ CA^\alpha \end{bmatrix} \ , \tag{1.3}$$

$$M_\alpha = \begin{bmatrix} D & 0 \\ \Theta_{\alpha-1}B & M_{\alpha-1} \end{bmatrix} = \begin{bmatrix} M_{\alpha-1} & 0 \\ C\zeta_{\alpha-1} & D \end{bmatrix} \ , \tag{1.4}$$

where

$$\zeta_{\alpha-1} \equiv \begin{bmatrix} A^{\alpha-1}B & A^{\alpha-2}B & \cdots & B \end{bmatrix} \ .$$

1.3 Previous Results

We now summarize some important results on system inversion and state observation. These results will be useful to our development later on.

1.3.1 Inversion

Consider the system response given by (1.2), and suppose that the value of the state x_k is known at time-step k. If we wish to determine u_k from the output, we require that the first m columns of M_α be linearly independent of each other and of the remaining αm columns of M_α. Otherwise, there exists a nonzero sequence of inputs $U_{k:k+\alpha}$ such that $M_\alpha U_{k:k+\alpha} = 0$, and this is indistinguishable from the input sequence $U_{k:k+\alpha} = 0$. Using (1.4), we note that the last αm columns of M_α have the same rank as $M_{\alpha-1}$. This reasoning was used by Sain and Massey to produce the following theorem for system invertibility [7].

Theorem 1.1 *For any nonnegative integer α,*

$$\mathit{rank}[M_\alpha] - \mathit{rank}[M_{\alpha-1}] \leq m$$

with equality if and only if the system in (1.1) *is invertible with delay α.*

If the condition in the above theorem holds, then there exists a matrix S such that

$$u_k = SY_{k:k+\alpha} - S\Theta_\alpha x_k \ ,$$

and this can be substituted into (1.1) to obtain x_{k+1}. Note that the expression on the left side of the inequality in the above theorem is a monotonically nondecreasing function of α, which indicates that more information can be obtained about the input by allowing a larger delay. An upper bound on the delay was obtained by Willsky, who studied system invertibility in [27].

Theorem 1.2 *Let q be the dimension of the nullspace of D. The system in* (1.1) *is invertible if and only if*

$$\mathit{rank}\,[M_{n-q+1}] - \mathit{rank}\,[M_{n-q}] = m \ ;$$

i.e., if (1.1) *is invertible, its inherent delay α cannot exceed $n-q+1$.*

1.3.2 State observation

As mentioned earlier in this chapter, the problem of state estimation for the system in (1.1) has also received considerable attention in the literature. The vast majority of these investigations focus on zero-delay observers (i.e., observers that only make use of the current measurement y_k to estimate x_k). The following theorem provides the well-established conditions for the existence of a zero-delay state observer for (1.1) [19, 25, 28].

Theorem 1.3 *The system* (1.1) *has an asymptotically stable state observer if and only if*

1. $\mathit{rank}\begin{bmatrix} D & 0 \\ CB & D \end{bmatrix} = m + \mathit{rank}[D] \ ,$

2. $\mathit{rank}\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \ \ \forall z \in \mathbb{C}, \ \ |z| \geq 1 \ .$

The values of z for which the second condition fails are called the *transmission zeros* of the system [29]. Thus, the second condition in the above theorem means that all transmission zeros of the system must be stable in order to estimate the entire system state. Comparing condition 1 to Theorem 1.1, we notice that the system must be invertible with a delay of one time-step in order to construct a zero-delay observer. One would then expect that this necessary condition could be met more easily through the use of a delayed observer. This fact was verified in [26] by defining a new output equation for the system (1.1), with Θ_α and M_α taking the place of the C and D matrices, respectively. These matrices were then substituted into the necessary conditions for zero-delay observers, and reduced to produce the following result.

Theorem 1.4 *The system* (1.1) *has an asymptotically stable state observer with delay α if and only if*

1. $rank[M_{\alpha+1}] - rank[M_\alpha] = m$,

2. $rank \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \ \forall z \in \mathbb{C}, \ |z| \geq 1$.

Note that the second condition in the above theorem is unchanged from the zero-delay case, which indicates that delays do not affect the transmission zeros of the system.

We are now ready to pursue our investigation of observers for linear systems with unknown inputs. The remainder of this thesis is organized as follows. In Chapter 2, we develop a design procedure for both full and reduced-order delayed observers to estimate the entire system state and input. In Chapter 3, we characterize the set of all linear functions of the state and input that can be observed with a given delay. In Chapter 4, we consider the case where the system is operating in the presence of noise, and develop a state estimator that is optimal in the sense of minimizing the mean square estimation error. We summarize and conclude the thesis in Chapter 5.

CHAPTER 2

FULL STATE AND INPUT OBSERVERS

2.1 Introduction

In this chapter, we study the problem of observing the entire system state and input in linear systems with unknown inputs. State observers for such systems have received considerable attention over the past few decades [16, 19, 22, 23, 25], and various methods of realizing both full and reduced-order zero-delay state observers have been presented. As discussed in Chapter 1, it may be necessary to utilize delayed observers in order to estimate the system state. While [26] established necessary and sufficient conditions for the existence of state observers with delays, no design procedure was provided. In [30], the authors handled delayed observers by constructing a higher-dimensional system that incorporated the delayed states into the new state vector. An observer was then constructed for this augmented system, and geometric conditions were given for the existence of such observers. In this chapter, we provide a unified design procedure for both reduced and full-order observers with delays, and present conditions for the existence of such observers. In contrast to the work in [30], the dimension of our observer is no greater than the dimension of the original system, and we present algebraic existence conditions. Our approach generalizes recently published work on full-order zero-delay state observers [25], and allows us to treat the full-order observer as a special case of a reduced-order observer where the dynamic portion reconstructs the entire state vector. Once we have constructed a state observer, we show that it is straightforward to obtain an estimate of the inputs.

2.2 State Observer

We first consider the problem of constructing an observer to estimate the system state. We start by determining the set of states which can be directly obtained from the output of the system over $\alpha + 1$ time-steps. The following theorem provides an answer to this problem.

Theorem 2.1 *For system* (1.1) *with response over* $\alpha+1$ *time-steps given by* (1.2)*, let*

$$t = rank \begin{bmatrix} \Theta_\alpha & M_\alpha \end{bmatrix} - rank \begin{bmatrix} M_\alpha \end{bmatrix} \ .$$

Then it is possible to perform a similarity transform on the system $\mathcal{S}$ *to obtain a new system* $\bar{\mathcal{S}}$ *such that exactly* t *of the states in* $\bar{\mathcal{S}}$ *are directly obtainable from the output of the system.*

Proof: Assume rank $\begin{bmatrix} \Theta_\alpha & M_\alpha \end{bmatrix}$ − rank $\begin{bmatrix} M_\alpha \end{bmatrix} = t$. This implies that there are t linearly independent vectors in the matrix Θ_α that cannot be written as a linear combination of vectors in M_α. Thus, there exists a matrix $\mathcal{P}$ of dimension $t \times (\alpha+1)p$ such that $\mathcal{P}\Theta_\alpha$ has full row-rank, and $\mathcal{P}M_\alpha = \mathbf{0}$. Define the similarity transformation matrix

$$\mathcal{T} \equiv \begin{bmatrix} \mathcal{P}\Theta_\alpha \\ \mathcal{H} \end{bmatrix} \ , \tag{2.1}$$

where the matrix $\mathcal{H}$ is chosen so that $\mathcal{T}$ has full rank. Note that $\mathcal{P}$ and $\mathcal{H}$ can be chosen so that $\mathcal{T}$ is orthogonal, if so desired. Consider the system $\bar{\mathcal{S}}$ with state-vector $\bar{x}_k = \begin{bmatrix} \bar{x}_{1,k} \\ \bar{x}_{2,k} \end{bmatrix} = \mathcal{T}x_k$. The system matrices in $\bar{\mathcal{S}}$ are given by

$$\begin{aligned} \bar{A} &\equiv \mathcal{T}A\mathcal{T}^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \ , \\ \bar{B} &\equiv \mathcal{T}B = \begin{bmatrix} \mathcal{P}\Theta_\alpha B \\ \mathcal{H}B \end{bmatrix} \ , \\ \bar{C} &\equiv C\mathcal{T}^{-1}, \quad \bar{D} \equiv D \ . \end{aligned} \tag{2.2}$$

Now it is readily seen from (1.2) that

$$\begin{aligned} \mathcal{P}Y_{k:k+\alpha} &= \mathcal{P}\Theta_\alpha\mathcal{T}^{-1}\bar{x}_k \\ &= \begin{bmatrix} I_t & 0 \end{bmatrix} \bar{x}_k \ , \end{aligned}$$

and thus the first t states of $\bar{x}_k$ are immediately obtained. ■

Remark 2.1 *The problem of determining a particular set of states from the (delayed) output was studied in [31], and the special case of perfect observability (i.e.,* $t = n$*) was studied in [4, 32]. The result in Theorem 2.1 appears to be new in that it deals with reconstructing a maximal subset of states from the output.*

To estimate the remaining $(n-t)$ states of $\bar{x}_k$ (i.e., $\bar{x}_{2,k}$), we construct a reduced-order observer of the form

$$\begin{aligned} z_{k+1} &= Ez_k + FY_{k:k+\alpha} \\ \psi_k &= z_k + GY_{k:k+\alpha} \ , \end{aligned} \tag{2.3}$$

where matrices E, F, and G are chosen such that $\psi_k \to \bar{x}_{2,k}$ as $k \to \infty$. Using (1.2), the observer error is given by

$$\begin{aligned} e_{k+1} &\equiv \psi_{k+1} - \bar{x}_{2,k+1} \\ &= Ez_k + FY_{k:k+\alpha} + GY_{k+1:k+\alpha+1} - \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \bar{x}_k - \mathcal{H}Bu_k \\ &= Ee_k + (F - EG)\,\Theta_\alpha x_k + G\Theta_\alpha A x_k + E\mathcal{H}x_k - \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \mathcal{T}x_k \\ &\quad + (F - EG)\,M_\alpha U_{k:k+\alpha} + G\Theta_\alpha Bu_k + GM_\alpha U_{k+1:k+\alpha+1} - \mathcal{H}Bu_k \ . \end{aligned}$$

Using the identities (1.3) and (1.4), the expression for the error can be written as

$$\begin{aligned} e_{k+1} = Ee_k &+ \begin{bmatrix} F - EG & 0 \end{bmatrix} \Theta_{\alpha+1}x_k + \begin{bmatrix} 0 & G \end{bmatrix} \Theta_{\alpha+1}x_k + E\mathcal{H}x_k - \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \mathcal{T}x_k \\ &+ \begin{bmatrix} F - EG & 0 \end{bmatrix} M_{\alpha+1}U_{k:k+\alpha+1} + \begin{bmatrix} 0 & G \end{bmatrix} M_{\alpha+1}U_{k:k+\alpha+1} - \mathcal{H}Bu_k \ . \end{aligned}$$

Partition the matrices F and G as

$$\begin{aligned} F &= \begin{bmatrix} F_0 & F_1 & \cdots & F_\alpha \end{bmatrix} \ , \\ G &= \begin{bmatrix} G_0 & G_1 & \cdots & G_\alpha \end{bmatrix} \end{aligned}$$

where each F_i and G_i are of dimension $(n-t) \times p$, and define

$$K \equiv \begin{bmatrix} F_0 - EG_0 & F_1 - EG_1 + G_0 & \cdots & F_\alpha - EG_\alpha + G_{\alpha-1} & G_\alpha \end{bmatrix} \ . \tag{2.4}$$

Note that since we are free to choose F and G, matrix K can be chosen to have any value we require. The error can then be expressed as

$$e_{k+1} = Ee_k + \left(E\mathcal{H} - \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \mathcal{T} + K\Theta_{\alpha+1} \right) x_k + KM_{\alpha+1}U_{k:k+\alpha+1} - \mathcal{H}Bu_k \ .$$

In order to force the error to go to zero, regardless of the values of x_k and the inputs, the following two conditions must hold:

1. E must be a stable matrix,

2. The matrix K must satisfy

$$KM_{\alpha+1} = \begin{bmatrix} \mathcal{H}B & 0 & \cdots & 0 \end{bmatrix}, \tag{2.5}$$

$$E\mathcal{H} = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \mathcal{T} - K\Theta_{\alpha+1} \ . \tag{2.6}$$

The solvability of condition (2.5) is given by the following theorem.

Theorem 2.2 *There exists a matrix K such that*

$$KM_{\alpha+1} = \begin{bmatrix} \mathcal{H}B & 0 & \cdots & 0 \end{bmatrix}$$

if and only if

$$rank\,[M_{\alpha+1}] - rank\,[M_\alpha] = m \ . \tag{2.7}$$

Proof: There exists a K satisfying (2.5) if and only if the matrix

$$R \equiv \begin{bmatrix} \mathcal{H}B & 0 & \cdots & 0 \end{bmatrix}$$

is in the space spanned by the rows of $M_{\alpha+1}$. This is equivalent to the condition

$$\text{rank} \begin{bmatrix} M_{\alpha+1} \\ R \end{bmatrix} = \text{rank}\,[M_{\alpha+1}] \ .$$

Using (1.4), we get

$$\begin{aligned}
\text{rank} \begin{bmatrix} M_{\alpha+1} \\ R \end{bmatrix} &= \text{rank} \begin{bmatrix} D & 0 \\ \Theta_\alpha B & M_\alpha \\ \mathcal{H}B & 0 \end{bmatrix} \\
&= \text{rank} \left(\begin{bmatrix} I & 0 & 0 \\ 0 & \mathcal{P} & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} D & 0 \\ \Theta_\alpha B & M_\alpha \\ \mathcal{H}B & 0 \end{bmatrix} \right) \\
&= \text{rank} \left(\begin{bmatrix} I & 0 & 0 \\ 0 & \mathcal{T} & 0 \\ 0 & \Theta_\alpha & I \end{bmatrix} \begin{bmatrix} D & 0 \\ B & 0 \\ 0 & M_\alpha \end{bmatrix} \right) \ .
\end{aligned}$$

By our assumption that the matrix $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ has full column rank, we get

$$\operatorname{rank}\begin{bmatrix} M_{\alpha+1} \\ R \end{bmatrix} = m + \operatorname{rank}\left[M_\alpha\right] \quad ,$$

thereby completing the proof. ■

Note that (2.7) is the condition for inversion of the inputs with known initial state, as given in [7]. If we set $\alpha = 0$, condition (2.7) becomes

$$\operatorname{rank}\begin{bmatrix} D & 0 \\ CB & D \end{bmatrix} = m + \operatorname{rank}\left[D\right] \quad ,$$

which is the well-known necessary condition for unknown-input observers with zero delay [19]. This is a fairly strict condition, and demonstrates the utility of a delayed observer. When designing such an observer, one can start with $\alpha = 0$ and increase α until a value is found that satisfies (2.7). Using Theorem 1.2, we see that an upper bound on the observer delay is $n-q$ time-steps, where q is the nullity of D. If (2.7) is not satisfied even for $\alpha = n-q$, it is not possible to estimate all the states in the system.

Remark 2.2 *As mentioned in Chapter 1, Condition* (2.7) *was obtained in [26] through a different method. The approach in that paper was to define a new output equation for the system, with* Θ_α *and* M_α *taking the place of the* C *and* D *matrices, respectively. These matrices were then substituted into the necessary conditions for zero-delay observers, and reduced to produce Equation* (2.7)*. While this approach is quite intuitive, it may result in unnecessarily large and redundant matrices when designing the observer parameters. Furthermore, the upper bound on the observer delay provided in [26] is* $\alpha = n - 1$*, which can be tightened by applying the result of Theorem 1.2.*

We now turn our attention to condition (2.6). Right-multiplying by $\mathcal{T}^{-1}$, we get the equivalent condition

$$\begin{bmatrix} 0 & E \end{bmatrix} = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} - K\Theta_{\alpha+1}\mathcal{T}^{-1} \quad . \tag{2.8}$$

From the above equation it is apparent that there is an additional constraint on K; namely, K times the first t columns of $\Theta_{\alpha+1}\mathcal{T}^{-1}$ must produce A_{21}. To satisfy this constraint, we define

$$\mathcal{T}_y \equiv \begin{bmatrix} \mathcal{P} \\ \Phi \end{bmatrix}, \quad \mathcal{J} \equiv \begin{bmatrix} \mathcal{T}_y & 0 \\ 0 & I_p \end{bmatrix} \quad , \tag{2.9}$$

where the matrix Φ is chosen so that $\mathcal{T}_y$ is square and invertible. Using (1.3) and (1.4), we get

$$\widehat{M} \equiv \mathcal{J} M_{\alpha+1} = \begin{bmatrix} 0 & 0 \\ \Phi M_\alpha & 0 \\ C\zeta_\alpha & D \end{bmatrix}, \quad \widehat{\Theta} \equiv \mathcal{J}\Theta_{\alpha+1}\mathcal{T}^{-1} = \begin{bmatrix} I_t & 0 \\ L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}, \tag{2.10}$$

where

$$\begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} = \begin{bmatrix} \Phi\Theta_\alpha \mathcal{T}^{-1} \\ CA^{\alpha+1}\mathcal{T}^{-1} \end{bmatrix} .$$

Since $\mathcal{J}$ is invertible, we can define a matrix $\bar{K}$ such that $K = \bar{K}\mathcal{J}$. Partitioning $\bar{K}$ as

$$\bar{K} = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix} ,$$

where $\bar{K}_1$ has t columns, Equations (2.5) and (2.8) become

$$\begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \Phi M_\alpha & 0 \\ C\zeta_\alpha & D \end{bmatrix} = \begin{bmatrix} \mathcal{H}B & 0 & \cdots & 0 \end{bmatrix} , \tag{2.11}$$

$$\begin{bmatrix} 0 & E \end{bmatrix} = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix} \begin{bmatrix} I_t & 0 \\ L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} . \tag{2.12}$$

It is clear from the above equations that $\bar{K}_1$ must be chosen such that

$$\bar{K}_1 = A_{21} - \bar{K}_2 \begin{bmatrix} L_1 \\ L_3 \end{bmatrix} ,$$

and so the problem is reduced to finding the matrix $\bar{K}_2$ satisfying Equations (2.11) and (2.12).

Recall that the first m columns of $M_{\alpha+1}$ must be linearly independent of each other and of the remaining $(\alpha+1)m$ columns (by Theorem 2.2), and so the rank of

$$\begin{bmatrix} \Phi M_\alpha & 0 \\ C\zeta_\alpha & D \end{bmatrix} \tag{2.13}$$

is $m + \mathrm{rank}[M_\alpha]$. Let $\mathcal{N}$ be a matrix whose rows form a basis for the left nullspace of the last $(\alpha+1)m$ columns of (2.13). In particular, we can assume without loss of generality that

$\mathcal{N}$ satisfies

$$\mathcal{N}\begin{bmatrix} \Phi M_\alpha & 0 \\ C\zeta_\alpha & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_m & 0 \end{bmatrix} \quad . \tag{2.14}$$

From (2.11), we see that $\bar{K}_2$ must be of the form

$$\bar{K}_2 = \widehat{K}\mathcal{N}$$

for some $\widehat{K} = \begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix}$, where $\widehat{K}_2$ has m columns. Equation (2.11) then becomes

$$\begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I_m & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{H}B & 0 \end{bmatrix} \quad , \tag{2.15}$$

from which it is obvious that $\widehat{K}_2 = \mathcal{H}B$ and $\widehat{K}_1$ is a free matrix.

Returning to Equation (2.12), we have

$$\begin{aligned} E &= A_{22} - \bar{K}_2 \begin{bmatrix} L_2 \\ L_4 \end{bmatrix} \\ &= A_{22} - \begin{bmatrix} \widehat{K}_1 & \mathcal{H}B \end{bmatrix} \mathcal{N} \begin{bmatrix} L_2 \\ L_4 \end{bmatrix} \quad . \end{aligned}$$

Defining

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \equiv \mathcal{N} \begin{bmatrix} L_2 \\ L_4 \end{bmatrix} \quad , \tag{2.16}$$

where ν_2 has m rows, we come to the final equation

$$E = (A_{22} - \mathcal{H}B\nu_2) - \widehat{K}_1\nu_1 \quad . \tag{2.17}$$

Recall that we require E to be a stable matrix, and this is only possible if the pair

$$(A_{22} - \mathcal{H}B\nu_2, \nu_1)$$

is detectable. This detectability condition can be stated in terms of the original system matrices as follows.

Theorem 2.3 *The rank condition*

$$\mathit{rank} \begin{bmatrix} zI - A_{22} + \mathcal{H}B\nu_2 \\ \nu_1 \end{bmatrix} = n - t, \;\; \forall z \in \mathbb{C}, \;\; |z| \geq 1$$

is satisfied if and only if

$$rank \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \;\; \forall z \in \mathbb{C}, \;\; |z| \geq 1 \;\; .$$

To prove the theorem, we make use of the following lemma, which is obtained by a simple modification of a theorem from [26].

Lemma 2.1 *The rank condition*

$$rank \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \;\; \forall z \in \mathbb{C}, \;\; |z| \geq 1$$

is satisfied if and only if

$$rank \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix} = n + m + rank[M_\alpha], \;\; \forall z \in \mathbb{C}, \;\; |z| \geq 1 \;\; .$$

We are now in place to prove Theorem 2.3.

Proof: We start by noting from (2.14) that

$$\begin{bmatrix} I_t & 0 \\ 0 & \mathcal{N} \end{bmatrix} \mathcal{J} \begin{bmatrix} D & 0 \\ \Theta_\alpha B & M_\alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_m & 0 \end{bmatrix} \;\; .$$

Let $\bar{\mathcal{N}}$ be a matrix whose rows form a basis for the left nullspace of M_α. We can then write

$$\begin{bmatrix} I_t & 0 \\ 0 & \mathcal{N} \end{bmatrix} \mathcal{J} = \mathcal{W} \begin{bmatrix} I_p & 0 \\ 0 & \bar{\mathcal{N}} \end{bmatrix} \tag{2.18}$$

for some invertible matrix $\mathcal{W}$. Through a series of nonsingular transformations, we obtain

$$\text{rank} \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix} = \text{rank} \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix}$$

$$
= \operatorname{rank} \begin{bmatrix} zI - A & -B & 0 & 0 \\ C & D & 0 & 0 \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B & 0 & 0 \\ zC & 0 & D & 0 \\ z\Theta_{\alpha-1}A & 0 & \Theta_{\alpha-1}B & M_{\alpha-1} \end{bmatrix}
$$

$$
= \operatorname{rank} \begin{bmatrix} zI - A & -B & 0 & 0 \\ C & D & 0 & 0 \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B & 0 & 0 \\ 0 & -zD & D & 0 \\ z\Theta_{\alpha-1}A & 0 & \Theta_{\alpha-1}B & M_{\alpha-1} \end{bmatrix}
$$

$$
= \operatorname{rank} \begin{bmatrix} zI - A & -B & 0 & 0 \\ C & D & 0 & 0 \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B & 0 & 0 \\ 0 & 0 & D & 0 \\ z\Theta_{\alpha-1}A & z\Theta_{\alpha-1}B & \Theta_{\alpha-1}B & M_{\alpha-1} \end{bmatrix} .
$$

Continuing in the above manner, we get

$$
\operatorname{rank} \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix} = \operatorname{rank} \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B & 0 \\ 0 & 0 & M_\alpha \end{bmatrix} ,
$$

and the rank of the top left submatrix in the above expression is given by

$$
\operatorname{rank} \begin{bmatrix} zI - A & -B \\ C & D \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} zI - \mathcal{T}A\mathcal{T}^{-1} & -\mathcal{T}B \\ C\mathcal{T}^{-1} & D \\ \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1} & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix}
$$

$$
= \operatorname{rank} \begin{bmatrix} zI - A_{11} & -A_{12} & -\mathcal{P}\Theta_\alpha B \\ -A_{21} & zI - A_{22} & -\mathcal{H}B \\ C\mathcal{T}^{-1}(:,1) & C\mathcal{T}^{-1}(:,2) & D \\ \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1}(:,1) & \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1}(:,2) & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix} ,
$$

where $\mathcal{T}^{-1}(:,1)$ represents the first t columns of $\mathcal{T}^{-1}$, and $\mathcal{T}^{-1}(:,2)$ represents the last $n-t$ columns. By the definition of $\mathcal{P}$, there exists a matrix $\mathcal{V}$ such that $\mathcal{P} = \mathcal{V}\bar{\mathcal{N}}$. Using the fact that $\mathcal{V}\bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1} = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}$, we get

$$\text{rank}\begin{bmatrix} zI-A & -B \\ C & D \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix} = \text{rank}\begin{bmatrix} zI_t & 0 & 0 \\ -A_{21} & zI-A_{22} & -\mathcal{H}B \\ C\mathcal{T}^{-1}(:,1) & C\mathcal{T}^{-1}(:,2) & D \\ \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1}(:,1) & \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1}(:,2) & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix} .$$

Using (2.10), (2.16), and (2.18), we left-multiply the last two block rows in the above matrix by $\mathcal{W}$ to obtain

$$\begin{aligned}\text{rank}\begin{bmatrix} zI-A & -B \\ C & D \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix} &= \text{rank}\begin{bmatrix} zI_t & 0 & 0 \\ -A_{21} & zI-A_{22} & -\mathcal{H}B \\ I_t & 0 & 0 \\ * & \nu_1 & 0 \\ * & \nu_2 & I_m \end{bmatrix} \\ &= t + \text{rank}\begin{bmatrix} zI-A_{22} & -\mathcal{H}B \\ \nu_1 & 0 \\ \nu_2 & I_m \end{bmatrix} \\ &= t + m + \text{rank}\begin{bmatrix} zI-A_{22}+\mathcal{H}B\nu_2 \\ \nu_1 \end{bmatrix} ,\end{aligned}$$

where $*$ represents unimportant matrices. This gives

$$\text{rank}\begin{bmatrix} zI-A & -B & 0 \\ C & D & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix} = t + m + \text{rank}[M_\alpha] + \text{rank}\begin{bmatrix} zI-A_{22}+\mathcal{H}B\nu_2 \\ \nu_1 \end{bmatrix} .$$

Using Lemma 2.1, we get the desired result. ■

We can now state the following theorem, whose proof is immediately given by the discussion so far.

Theorem 2.4 *The system $\mathcal{S}$ in* (1.1) *has an observer with delay α if and only if*

1. *rank* $[M_{\alpha+1}]$ − *rank* $[M_\alpha] = m$,
2. *rank* $\begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix} = n+m, \ \forall z \in \mathbb{C}, \ |z| \geq 1$.

Recall that the first condition in Theorem 2.4 means that the system is invertible with delay $\alpha + 1$. In fact, it has been shown in [11] that condition 2 is sufficient for the existence of a stable inverse for system $\mathcal{S}$. This fact leads to the following theorem.

Theorem 2.5 *The system $\mathcal{S}$ in* (1.1) *has an observer (possibly with delay) if and only if*

$$rank \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \ \ \forall z \in \mathbb{C}, \ \ |z| \geq 1 \ .$$

The result in the above theorem has also been noted in [33], which studied the problem of reconstructing the unknown inputs. The difference between Theorem 2.4 and Theorem 2.5 is that the latter does not provide a characterization of the delay in the observer. Note that the conditions in Theorem 2.4 (and the equivalent condition in Theorem 2.3) are a generalization of those given in [19, 25, 34] for the existence of zero-delay observers, and verify the conditions in [26].

Once the matrix $\widehat{K}_1$ is chosen to make E stable, we can obtain the matrix K in (2.5) and (2.6) by calculating

$$\begin{aligned} \bar{K}_2 &= \begin{bmatrix} \widehat{K}_1 & \mathcal{H}B \end{bmatrix} \mathcal{N} \\ \bar{K}_1 &= A_{21} - \bar{K}_2 \begin{bmatrix} L_1 \\ L_3 \end{bmatrix} \\ K &= \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix} \begin{bmatrix} \mathcal{T}_y & 0 \\ 0 & I_p \end{bmatrix} \ . \end{aligned} \tag{2.19}$$

We can then use (2.4) to map this K matrix to the observer gains F and G. Note that this mapping is not unique. In particular, one can choose $G_0 = G_1 = \cdots = G_{\alpha-1} = 0$, thereby getting

$$K = \begin{bmatrix} F_0 & F_1 & \cdots & F_\alpha - EG_\alpha & G_\alpha \end{bmatrix} \ .$$

This choice corresponds to using only the most delayed measurement in the output of the observer. Similarly, one can choose $F_1 = F_2 = \cdots = F_\alpha = 0$, which corresponds to using only the earliest measurement in the dynamic portion of the observer. Other combinations are also possible. Note that this freedom does not exist when designing a zero-delay observer. The final observer is given by Equation (2.3), and the estimate of the original system states is obtained as

$$\hat{x}_k = \mathcal{T}^{-1} \begin{bmatrix} \mathcal{P}Y_{k:k+\alpha} \\ \psi_k \end{bmatrix} \ . \tag{2.20}$$

Remark 2.3 *It is of interest to note that while we have pursued the development of a reduced-order observer, the above approach and conditions immediately apply to full-order observers as well. This is because a full-order observer can be viewed as a special case of a reduced-order observer, where the dynamic portion reconstructs the entire state. This can be accomplished by setting $\mathcal{P}$ to be an empty matrix (i.e., by choosing $t = 0$, $\mathcal{T} = \mathcal{H} = I_n$, $\mathcal{T}_y = I_{(\alpha+1)p}$).*

2.3 Input Observer

We now turn our attention to reconstructing the unknown inputs. Recall from the previous section that the system had to be invertible in order for a state observer to exist. This means that once we have constructed a state observer, it is straightforward to obtain an estimate of the inputs. The procedure is a generalization of a technique presented in [26] to the case where $D \neq 0$. We start by rewriting (1.1) as

$$\begin{bmatrix} x_{k+1} - Ax_k \\ y_k - Cx_k \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} u_k \quad . \tag{2.21}$$

Since $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ is assumed to be full column rank, there exists a matrix S such that

$$S \begin{bmatrix} B \\ D \end{bmatrix} = I_m \quad . \tag{2.22}$$

Left-multiplying (2.21) by S and replacing x_k with the estimated value $\hat{x}_k$ from (2.20), we get the estimate of the input to be

$$\hat{u}_k = S \begin{bmatrix} \hat{x}_{k+1} - A\hat{x}_k \\ y_k - C\hat{x}_k \end{bmatrix} \quad . \tag{2.23}$$

Since $\hat{x}_k \to x_k$ as $k \to \infty$, the above estimate will asymptotically approach the true value of the input. Since the estimate of $\hat{x}_{k+1}$ requires access to the output $y_{k+\alpha+1}$ (from (2.3)), the above estimate of the input will actually be delayed by $\alpha + 1$ time-steps. This agrees with (2.7), which indicates that the minimum delay for inversion of the inputs is $\alpha + 1$ time-steps.

2.4 Design Procedure

We now summarize the design steps that can be used in designing a delayed observer for the system given in (1.1).

1. Find the smallest α such that $\text{rank}[M_{\alpha+1}] - \text{rank}\,[M_\alpha] = m$. If the condition is not satisfied for $\alpha = n - \text{nullity}\,[D]$, it is not possible to reconstruct the entire system state.

2. Find the matrix $\mathcal{P}$ such that $\mathcal{P}\Theta_\alpha$ is full row rank and $\mathcal{P}M_\alpha$ is zero. Use Theorem 2.1 for a reduced-order observer, or set $\mathcal{P}$ to be the empty matrix for a full-order observer. Choose $\mathcal{H}$ and form the matrix $\mathcal{T}$ in (2.1) to obtain the transformed system given by (2.2). Also choose Φ and form the matrix $\mathcal{T}_y$ given in (2.9).

3. Find the matrix $\mathcal{N}$ satisfying (2.14).

4. Form the matrices $\widehat{\Theta}$ and $\left[\begin{smallmatrix}\nu_1\\ \nu_2\end{smallmatrix}\right]$ from (2.10) and (2.16).

5. If the detectability condition in Theorem 2.3 is satisfied, choose the matrix $\widehat{K}_1$ such that the eigenvalues of $E = (A_{22} - \mathcal{H}B\nu_2) - \widehat{K}_1\nu_1$ are stable.

6. Calculate K by using Equation (2.19).

7. Use (2.4) to map this K matrix to F and G.

8. Find the matrix S satisfying (2.22).

9. The final observer is given by Equation (2.3). An estimate of the original system states is obtained from (2.20), and an estimate of the unknown inputs is given by (2.23).

2.5 Example

Consider the system given by the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is found that condition (2.7) holds for $\alpha = 1$, so our observer must have a minimum delay of one time-step. Using Theorem 2.1, we find $t = 2$ and choose

$$\mathcal{P} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} ,$$
$$\mathcal{H} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} ,$$
$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Performing the similarity transformation, we get

$$\left[\begin{array}{c|c} A_{21} & A_{22} \end{array} \right] = \left[\begin{array}{cc|c} 1 & -1 & 1 \end{array} \right] .$$

The matrices $\widehat{M}$ and $\widehat{\Theta}$ from (2.10) are found to be

$$\widehat{M} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} , \quad \widehat{\Theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} .$$

In this example, the last $(\alpha + 1)m = 4$ columns of $\bar{M}$ have a rank of two, and thus the matrix $\mathcal{N}$ in (2.14) will only have two rows:

$$\mathcal{N} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

Equation (2.15) becomes

$$\begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix} \begin{bmatrix} I_2 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{H}B & 0 \end{bmatrix} ,$$

and since $\widehat{K}_2$ has $m = 2$ columns, $\widehat{K}_1$ is the empty matrix. This implies that we will have no freedom in choosing the eigenvalues of our observer.

Next, we use Equation (2.16) to obtain

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

Again, since ν_2 has $m = 2$ rows, ν_1 is the empty matrix. We now check the detectability of our system by computing

$$E = A_{22} - \mathcal{H}B\nu_2 = 0 \ ,$$

which implies that we are able to design a stable observer. Using (2.19), we get

$$\begin{aligned} \bar{K}_2 &= \begin{bmatrix} 1 & 0 & 2 & -2 \end{bmatrix} \ , \\ \bar{K}_1 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \ , \\ K &= \begin{bmatrix} 0 & 1 & -1 & 1 & 2 & -2 \end{bmatrix} \ . \end{aligned}$$

Finally, we obtain the F and G matrices by choosing $G_0 = 0$. Since $E = 0$, we have

$$\begin{aligned} F &= \begin{bmatrix} F_0 & F_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix} \ , \\ G &= \begin{bmatrix} G_0 & G_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & -2 \end{bmatrix} \ . \end{aligned}$$

The final observer is given by

$$\begin{aligned} z_{k+1} &= FY_{k:k+1} \ , \\ \psi_k &= z_k + GY_{k:k+1} \ , \end{aligned}$$

and an estimate of the original system states can be obtained via (2.20).

To test this observer, the system is simulated with an initial non-zero state, and driven by a sinusoidal input. The observer is initialized with an initial state of zero, and as seen from the plots in Figure 2.1, catches up with the system state to produce a perfect estimate that is delayed by one time-step. Note that the observer starts operation at $k = 1$, to account for the one-step delay.

To estimate the inputs, we find the matrix S satisfying (2.22) to be

$$S = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ .$$

Using (2.23), we get the results shown in Figure 2.2. Note that the estimate is delayed by $\alpha + 1 = 2$ time-steps, to account for the fact that we need y_{k+2} in order to reconstruct u_k.

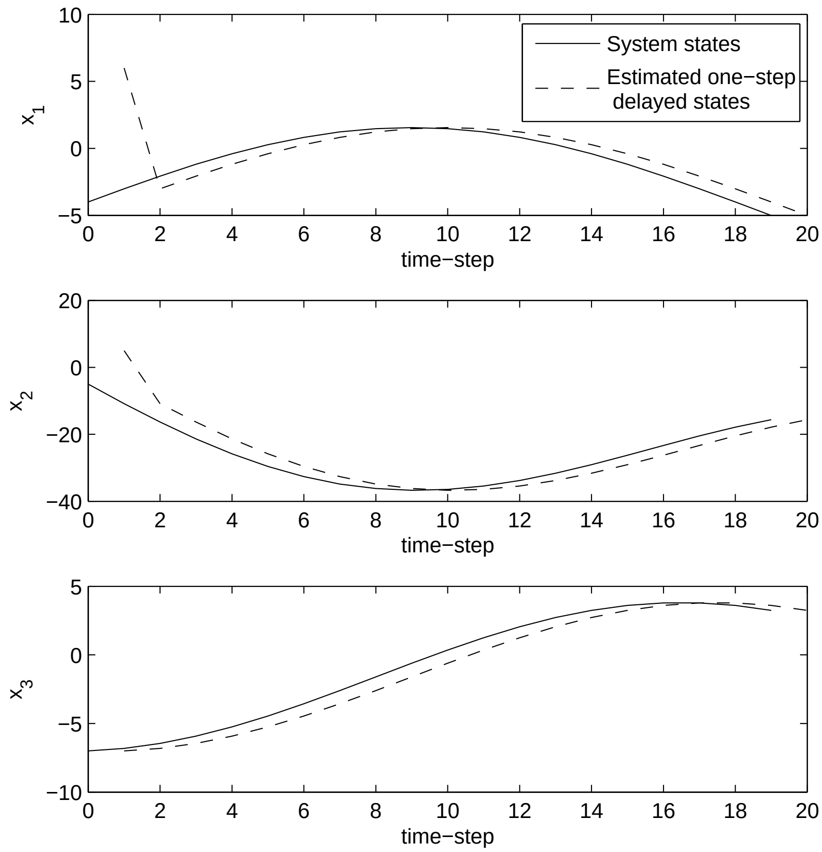

Figure 2.1: Simulation of state observer.

2.6 Summary

We have provided a characterization of unknown input observers with delays, and have developed a streamlined design procedure to obtain the observer parameters. Our approach is quite general in that it treats both reduced and full-order observers by selecting the design matrices appropriately. Once the state observer is constructed, it is straightforward to obtain an estimate of the unknown inputs.

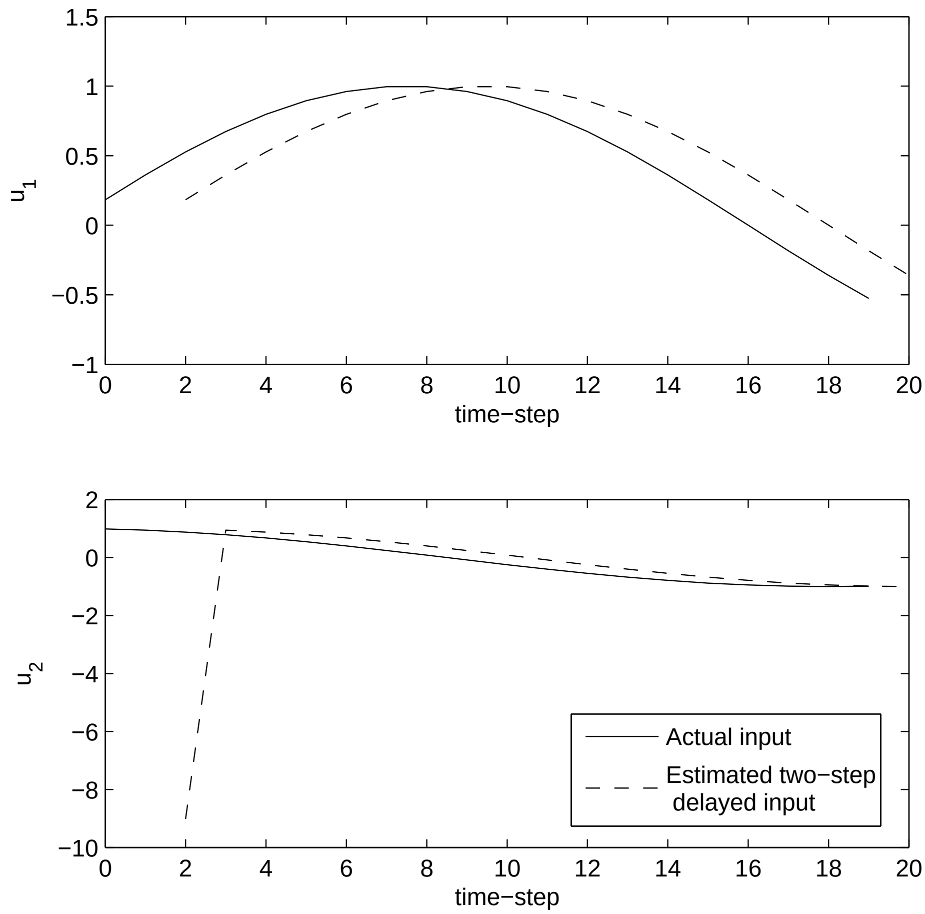

Figure 2.2: Simulation of input observer.

CHAPTER 3

PARTIAL STATE AND INPUT OBSERVERS

3.1 Introduction

We have already seen from the discussions in Chapters 1 and 2 that some strict conditions must be satisfied in order to estimate the entire system state. In particular, the requirement that the system be invertible (which might require the use of delayed measurements) means it will not be possible to build a full state observer for a large class of systems. Even if the system is invertible, the delay required to do so may be intolerably high. In this chapter, we present a method to determine the set of all linear functions of the state and input that can be reconstructed through a linear observer with a given delay.

The problem of reconstructing a particular function of the inputs and states was studied in [30] through a geometric approach. Delayed observers were handled in that paper by constructing a higher dimensional system which incorporated the delayed states and inputs into the new state vector. However, this approach might cause the dimension of the observer to be much larger than the dimension of the system. In [34] and [24], the authors examined partial state observers for continuous-time systems, but did not make use of delays (differentiators in continuous-time), and did not characterize the set of observable inputs. The problem of partial state observation was also studied in [35] for the specific case where the measurements are free of unknown inputs. That paper also did not allow the use of delays, and these two facts simplified the analysis considerably. The problem of partial input reconstruction was studied in [36] under the assumption that the initial system state is known. Consequently, the input observers constructed in that paper may be unstable. In comparison to the works considered above, the main contributions of this chapter are (i) an algebraic procedure to design delayed linear observers with dimension no greater than that of the system, and (ii) the characterization of all possible linear functions of the state and input that can be reconstructed with a given delay. Our approach directly produces the observer parameters in addition to the set of observable states and inputs. Our design does not assume knowledge of the initial system state, but can easily incorporate that information to increase the number of functionals that are reproduced.

3.2 Partial State Observer

We start by constructing a partial state observer for (1.1). This state observer will then be used in the next section to construct an input observer.

We wish to determine the set of linear functions of the state vector x_k that can be reproduced through a linear observer of the form

$$\begin{aligned} z_{k+1} &= Ez_k + FY_{k:k+\alpha} \\ \psi_k &= z_k + GY_{k:k+\alpha} \ , \end{aligned} \tag{3.1}$$

where the nonnegative integer α is the observer delay, and the matrices E, F and G are chosen such that $\psi_k \to Tx_k$ as $k \to \infty$ for some matrix T. The delay α is assumed to be a design parameter. The objective will be to find the matrix T of largest rank for which a stable observer can be constructed. We will show later in this section that the rows of T will then form a basis for all possible linear functions of the state. Using (1.2), the observer error is given by

$$\begin{aligned} e_{k+1} &\equiv \psi_{k+1} - Tx_{k+1} \\ &= Ez_k + FY_{k:k+\alpha} + GY_{k+1:k+1+\alpha} - TAx_k - TBu_k \\ &= Ee_k + (F - EG)\,\Theta_\alpha x_k + G\Theta_\alpha Ax_k + (ET - TA)\,x_k \\ &\quad + (F - EG)\,M_\alpha U_{k:k+\alpha} + G\Theta_\alpha Bu_k \\ &\quad + GM_\alpha U_{k+1:k+\alpha+1} - TBu_k \ . \end{aligned}$$

As in Chapter 2, we partition the matrices F and G as

$$\begin{aligned} F &= \begin{bmatrix} F_0 & F_1 & \cdots & F_\alpha \end{bmatrix} \ , \\ G &= \begin{bmatrix} G_0 & G_1 & \cdots & G_\alpha \end{bmatrix} \ , \end{aligned}$$

where F_i and G_i have p columns, and define

$$K \equiv \begin{bmatrix} F_0 - EG_0 & F_1 - EG_1 + G_0 & \cdots & F_\alpha - EG_\alpha + G_{\alpha-1} & G_\alpha \end{bmatrix} \ . \tag{3.2}$$

Using identities (1.3) and (1.4), the error can then be expressed as

$$e_{k+1} = Ee_k + (ET - TA + K\Theta_{\alpha+1})\,x_k + \left(KM_{\alpha+1} - \begin{bmatrix} TB & 0 & \cdots & 0 \end{bmatrix}\right) U_{k:k+\alpha+1} \ . \tag{3.3}$$

In order to force the error to go to zero regardless of the values of x_k and the inputs, the following conditions must hold:

1. E must be a stable matrix,
2. The matrix K must satisfy

$$KM_{\alpha+1} = \begin{bmatrix} TB & 0 & \cdots & 0 \end{bmatrix} \ , \tag{3.4}$$

$$ET - TA + K\Theta_{\alpha+1} = 0 \ . \tag{3.5}$$

We start with condition (3.4). Using (1.4), we can write this condition as

$$K \begin{bmatrix} D & 0 \\ \Theta_\alpha B & M_\alpha \end{bmatrix} = \begin{bmatrix} TB & 0 \end{bmatrix} \ . \tag{3.6}$$

Let $\mathcal{N}$ be a basis for the left nullspace of M_α (i.e., $\mathcal{N}M_\alpha = 0$). We can see from the above expression that the matrix K must be of the form

$$K = \bar{K} \begin{bmatrix} I_p & 0 \\ 0 & \mathcal{N} \end{bmatrix} \ , \tag{3.7}$$

for some $\bar{K}$. Equation (3.6) then becomes

$$\bar{K} \begin{bmatrix} D \\ \mathcal{N}\Theta_\alpha B \end{bmatrix} = TB \ , \tag{3.8}$$

where the rank of $\left[\begin{smallmatrix} D \\ \mathcal{N}\Theta_\alpha B \end{smallmatrix} \right]$ is given by

$$r = \text{rank}[M_{\alpha+1}] - \text{rank}[M_\alpha] \ .$$

We note that r is a monotonically nondecreasing function of α [7], and this shows that a larger value of α may allow greater freedom in choosing the matrix T. Returning to (3.8), there exists a pair of nonsingular matrices U and V such that

$$U \begin{bmatrix} D \\ \mathcal{N}\Theta_\alpha B \end{bmatrix} V = \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix} \ . \tag{3.9}$$

Define

$$\bar{K} = \widehat{K}U \tag{3.10}$$

for some $\widehat{K} \equiv \begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix}$, where $\widehat{K}_2$ has r columns. Right-multiplying Equation (3.8) by V, we get

$$\begin{aligned} \begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix} &= TBV \\ &\equiv T \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} \end{bmatrix} \ , \end{aligned} \tag{3.11}$$

where $\Gamma_{1,1}$ is the first r columns of BV, and $\Gamma_{1,2}$ is the last $m-r$ columns. From the above expression, we see that T must satisfy $T\Gamma_{1,2} = 0$. Let $\mathcal{N}_1$ be a basis for the left nullspace of $\Gamma_{1,2}$. This implies that T must be of the form

$$T = T_1 \mathcal{N}_1 \tag{3.12}$$

for some T_1, and thus we have to maximize the rank of T_1 in order to maximize the rank of T. If the rank of $\mathcal{N}_1$ is zero, it is not possible to reconstruct any of the states. Equation (3.11) also shows that $\widehat{K}_2 = T\Gamma_{1,1} = T_1\mathcal{N}_1\Gamma_{1,1}$, and $\widehat{K}_1$ is a free matrix.

Returning to condition (3.5), we use (3.7), (3.10), and (3.12) to obtain

$$ET_1\mathcal{N}_1 - T_1\mathcal{N}_1 A + \begin{bmatrix} \widehat{K}_1 & T_1\mathcal{N}_1\Gamma_{1,1} \end{bmatrix} U \begin{bmatrix} I_p & 0 \\ 0 & \mathcal{N} \end{bmatrix} \Theta_{\alpha+1} = 0 \ . \tag{3.13}$$

Defining

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \equiv U \begin{bmatrix} I_p & 0 \\ 0 & \mathcal{N} \end{bmatrix} \Theta_{\alpha+1} \ ,$$

where ν_2 has r rows, Equation (3.13) becomes

$$ET_1\mathcal{N}_1 - \begin{bmatrix} \widehat{K}_1 & T_1 \end{bmatrix} \begin{bmatrix} -\nu_1 \\ \mathcal{N}_1 \left(A - \Gamma_{1,1}\nu_2\right) \end{bmatrix} = 0 \ . \tag{3.14}$$

Since $\mathcal{N}_1$ has full row rank, there exists a nonsingular matrix V_1 such that

$$\mathcal{N}_1 V_1 = \begin{bmatrix} I_{r_1} & 0 \end{bmatrix} \ , \tag{3.15}$$

where r_1 is the rank of $\mathcal{N}_1$. Right-multiplying (3.14) by V_1, we get

$$\begin{bmatrix} ET_1 & 0 \end{bmatrix} - \begin{bmatrix} \widehat{K}_1 & T_1 \end{bmatrix} \begin{bmatrix} \Gamma_{2,1} & \Gamma_{2,2} \end{bmatrix} = 0 \ , \tag{3.16}$$

where

$$\begin{bmatrix}\Gamma_{2,1} & \Gamma_{2,2}\end{bmatrix} \equiv \begin{bmatrix} -\nu_1 \\ \mathcal{N}_1\left(A - \Gamma_{1,1}\nu_2\right)\end{bmatrix} V_1$$

and $\Gamma_{2,1}$ has r_1 columns. Let $\mathcal{N}_2 \equiv \begin{bmatrix}\mathcal{N}_{2,1} & \mathcal{N}_{2,2}\end{bmatrix}$ denote a basis for the left nullspace of $\Gamma_{2,2}$, where $\mathcal{N}_{2,2}$ has r_1 columns. From (3.16), we see that

$$\begin{bmatrix}\widehat{K}_1 & T_1\end{bmatrix} = T_2\mathcal{N}_2 \ , \tag{3.17}$$

for some matrix T_2, and thus we have

$$T_1 = T_2\mathcal{N}_{2,2} \ . \tag{3.18}$$

Substituting the above expressions into (3.16), we get

$$ET_2\mathcal{N}_{2,2} - T_2\mathcal{N}_2\Gamma_{2,1} = 0 \ .$$

We can now use the following algorithm to find values for T_2, T_1, and T. Note that this algorithm starts with $i = 2$, in order to maintain consistency with the notation used so far.

1. The equation at the ith iteration is

$$ET_i\mathcal{N}_{i,2} - T_i\mathcal{N}_i\Gamma_{i,1} = 0 \ . \tag{3.19}$$

2. Terminal conditions:

   (a) If the rank of $\mathcal{N}_{i,2}$ is zero, it is not possible to reconstruct any linear functions of the states. Stop iterating.

   (b) If $\mathcal{N}_{i,2}$ is full column rank, stop iterating.

3. If none of the terminal conditions are satisfied, let r_i denote the rank of $\mathcal{N}_{i,2}$. Find the nonsingular matrices U_i and V_i such that

$$U_i\mathcal{N}_{i,2}V_i = \begin{bmatrix} 0 & 0 \\ I_{r_i} & 0\end{bmatrix} \ .$$

4. Set $T_i = \bar{T}_iU_i$, and right-multiply (3.19) by V_i to obtain

$$E\bar{T}_i\begin{bmatrix} 0 & 0 \\ I_{r_i} & 0\end{bmatrix} - \bar{T}_iU_i\mathcal{N}_i\Gamma_{i,1}V_i = 0 \ .$$

5. Define
$$\begin{bmatrix}\Gamma_{i+1,1} & \Gamma_{i+1,2}\end{bmatrix} \equiv U_i \mathcal{N}_i \Gamma_{i,1} V_i \ ,$$
where $\Gamma_{i+1,1}$ has r_i columns.

6. Let $\mathcal{N}_{i+1}$ be a basis for the left nullspace of $\Gamma_{i+1,2}$, and let $\mathcal{N}_{i+1,2}$ be the last r_i columns of $\mathcal{N}_{i+1}$.

7. Set $\bar{T}_i = T_{i+1}\mathcal{N}_{i+1}$ and proceed to iteration $i+1$.

At this point, either the rank of $\mathcal{N}_{i,2}$ is zero, or $\mathcal{N}_{i,2}$ is full column rank. Note that one of these two conditions is guaranteed to occur after a sufficient number of iterations, since the rank of $\mathcal{N}_i$ will either decrease at each iteration or stay the same. The latter will occur only if $\mathcal{N}_i$ is a square nonsingular matrix, in which case $\mathcal{N}_{i,2}$ will have full column rank. To see why no state observer exists if the rank of $\mathcal{N}_{i,2}$ is $r_i = 0$, assume that the above iteration is performed one more time. In step 6 of the procedure, we see that $\mathcal{N}_{i+1,2}$ will be the empty matrix. Since this has rank zero, we can take it to have full column rank. The following analysis will then reveal that the matrix T will also have zero rank, implying that no state functionals can be obtained.

If $\mathcal{N}_{i,2}$ is full column rank, there exists a matrix U_i such that

$$U_i \mathcal{N}_{i,2} = \begin{bmatrix} 0 \\ I_{r_i} \end{bmatrix} \ . \tag{3.20}$$

Define $T_i = \bar{T}_i U_i$, and partition the matrix $\bar{T}_i$ as

$$\bar{T}_i = \begin{bmatrix} L & H \end{bmatrix} \ , \tag{3.21}$$

where H has r_i columns. The matrix T_1 from (3.18) is then given by

$$\begin{aligned} T_1 &= T_2 \mathcal{N}_{2,2} \\ &= \bar{T}_2 U_2 \mathcal{N}_{2,2} \\ &= T_3 \mathcal{N}_3 U_2 \mathcal{N}_{2,2} \\ &\ \ \vdots \\ &= \bar{T}_i U_i \mathcal{N}_i U_{i-1} \mathcal{N}_{i-1} \cdots U_3 \mathcal{N}_3 U_2 \mathcal{N}_{2,2} \ . \end{aligned} \tag{3.22}$$

Recall that we have to maximize the rank of T_1 in order to maximize the rank of T. Since the matrices V_i are nonsingular, we can repeatedly multiply the above expression on the right by $V_2, V_3, \ldots, V_{i-1}$ to get

$$\begin{aligned}
\mathrm{rank}[T_1] &= \mathrm{rank}[T_1 V_2] \\
&= \mathrm{rank}\left[\bar{T}_i U_i \mathcal{N}_i U_{i-1} \mathcal{N}_{i-1} \cdots U_3 \mathcal{N}_{3,2} V_3\right] \\
&= \mathrm{rank}\left[\bar{T}_i U_i \mathcal{N}_i U_{i-1} \mathcal{N}_{i-1} \cdots U_4 \mathcal{N}_{4,2} V_4\right] \\
&\vdots \\
&= \mathrm{rank}\left[\bar{T}_i U_i \mathcal{N}_{i,2}\right] \\
&= \mathrm{rank}\left[\begin{bmatrix} L & H \end{bmatrix} \begin{bmatrix} 0 \\ I_{r_i} \end{bmatrix}\right] \\
&= \mathrm{rank}[H] \ .
\end{aligned}$$

Since $\mathcal{N}_1$ is full row rank in (3.12), the rank of T will be the same as the rank of H, which is upper bounded by r_i. This shows that if $r_i = 0$, no state functionals can be produced.

We now continue with the construction of the observer. Substituting (3.21) into (3.19), we get

$$EH - \begin{bmatrix} L & H \end{bmatrix} U_i \mathcal{N}_i \Gamma_{i,1} = 0 \ . \tag{3.23}$$

Denoting

$$U_i \mathcal{N}_i \Gamma_{i,1} = \begin{bmatrix} \mathcal{C} \\ \mathcal{A} \end{bmatrix} \tag{3.24}$$

where $\mathcal{A}$ has r_i rows, Equation (3.23) becomes

$$EH - H\mathcal{A} - L\mathcal{C} = 0 \ . \tag{3.25}$$

The above equation is in the form of the Sylvester observer equation which is frequently encountered in functional observer design [37, 38].

To solve this equation, we first note that there is a nonsingular matrix P which transforms the pair $(\mathcal{A}, \mathcal{C})$ into the form

$$\begin{aligned}
\bar{\mathcal{A}} \equiv P\mathcal{A}P^{-1} &= \left[\begin{array}{cc|c} \mathcal{A}_o & 0 & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{\bar{o},s} & 0 \\ \hline \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{\bar{o},\bar{s}} \end{array}\right] \ , \\
\bar{\mathcal{C}} \equiv \mathcal{C}P^{-1} &= \left[\begin{array}{cc|c} \mathcal{C}_o & 0 & 0 \end{array}\right] \ ,
\end{aligned} \tag{3.26}$$

where the pair $(\mathcal{A}_o, \mathcal{C}_o)$ is observable, all modes of $\mathcal{A}_{\bar{o},s}$ are unobservable and stable, and all modes of $\mathcal{A}_{\bar{o},\bar{s}}$ are unobservable and unstable. The dimensions of $\mathcal{A}_o$, $\mathcal{A}_{\bar{o},s}$, and $\mathcal{A}_{\bar{o},\bar{s}}$ are taken to be $n_o \times n_o$, $n_{\bar{o},s} \times n_{\bar{o},s}$, and $n_{\bar{o},\bar{s}} \times n_{\bar{o},\bar{s}}$, respectively. Note that the above form is simply a slightly modified version of the Kalman observable canonical form [29]. Right-multiplying (3.25) by P^{-1} and setting $H = \bar{H}P$, we get

$$E\bar{H} - \bar{H}\bar{\mathcal{A}} - L\bar{\mathcal{C}} = 0 \ . \tag{3.27}$$

Let $L = \bar{H}L_1 + L_2$ for some L_1 and L_2. Partitioning $\bar{H}$ and L_1 as

$$\bar{H} = \begin{bmatrix} \bar{H}_1 & \bar{H}_2 \end{bmatrix}, \quad L_1 = \left[\begin{array}{c} L_{11} \\ L_{12} \\ \hline L_{13} \end{array}\right] \ ,$$

Equation (3.27) becomes

$$\begin{bmatrix} E\bar{H}_1 & E\bar{H}_2 \end{bmatrix} - \begin{bmatrix} \bar{H}_1 & \bar{H}_2 \end{bmatrix} \left[\begin{array}{cc|c} \mathcal{A}_o + L_{11}\mathcal{C}_o & 0 & 0 \\ \mathcal{A}_{21} + L_{12}\mathcal{C}_o & \mathcal{A}_{\bar{o},s} & 0 \\ \hline \mathcal{A}_{31} + L_{13}\mathcal{C}_o & \mathcal{A}_{32} & \mathcal{A}_{\bar{o},\bar{s}} \end{array}\right] - \left[\begin{array}{cc|c} L_2\mathcal{C}_o & 0 & 0 \end{array}\right] = 0 \ . \tag{3.28}$$

Since $(\mathcal{A}_o, \mathcal{C}_o)$ is observable, the matrix L_{11} can be chosen to place the eigenvalues of $\mathcal{A}_o + L_{11}\mathcal{C}_o$ at arbitrary (stable) locations. Denoting

$$\begin{aligned} \mathcal{A}_s &= \begin{bmatrix} \mathcal{A}_o + L_{11}\mathcal{C}_o & 0 \\ \mathcal{A}_{21} + L_{12}\mathcal{C}_o & \mathcal{A}_{\bar{o},s} \end{bmatrix} \ , \\ \Lambda &= \begin{bmatrix} \mathcal{A}_{31} + L_{13}\mathcal{C}_o & \mathcal{A}_{32} \end{bmatrix} \ , \end{aligned} \tag{3.29}$$

where the dimension of $\mathcal{A}_s$ is $n_s \times n_s$, we get

$$\begin{bmatrix} E\bar{H}_1 & E\bar{H}_2 \end{bmatrix} - \begin{bmatrix} \bar{H}_1\mathcal{A}_s + \bar{H}_2\Lambda & \bar{H}_2\mathcal{A}_{\bar{o},\bar{s}} \end{bmatrix} - \left[\begin{array}{cc|c} L_2\mathcal{C}_o & 0 & 0 \end{array}\right] = 0 \ . \tag{3.30}$$

To analyze the above equation, we will make use of the following lemma [39].

Lemma 3.1 *Suppose A and B are square matrices. The matrix equation*

$$AX - XB = 0$$

has the unique solution $X = 0$ if and only if A and B have no common eigenvalues.

Examining (3.30), we see that

$$E\bar{H}_2 - \bar{H}_2\mathcal{A}_{\bar{o},\bar{s}} = 0 \ .$$

For E to be stable, it must not share any eigenvalues with the matrix $\mathcal{A}_{\bar{o},\bar{s}}$. By Lemma 3.1, we see that $\bar{H}_2$ must be the zero matrix. Substituting this fact into (3.30), we get

$$E\bar{H}_1 - \bar{H}_1\mathcal{A}_s - \begin{bmatrix} L_2\mathcal{C}_o & 0 \end{bmatrix} = 0 \ .$$

To maximize the rank of $\bar{H}$, we choose $\bar{H}_1$ to be I_{n_s} and $L_2 = 0$, which gives us $E = \mathcal{A}_s$. From (3.21), we have

$$\begin{aligned} \bar{T}_i &= \begin{bmatrix} \bar{H}L_1 & \bar{H}P \end{bmatrix} \\ &= \begin{bmatrix} I_{n_s} & 0 \end{bmatrix} \begin{bmatrix} L_1 & P \end{bmatrix} \ , \end{aligned}$$

and we can use (3.22), (3.17) and (3.12) to obtain

$$T = \bar{T}_i U_i \mathcal{N}_i U_{i-1} \mathcal{N}_{i-1} \cdots U_3 \mathcal{N}_3 U_2 \mathcal{N}_{2,2} \mathcal{N}_1 \ , \tag{3.31}$$

$$\widehat{K}_1 = \bar{T}_i U_i \mathcal{N}_i U_{i-1} \mathcal{N}_{i-1} \cdots U_3 \mathcal{N}_3 U_2 \mathcal{N}_{2,1} \ . \tag{3.32}$$

From (3.7) and (3.10), we have

$$K = \begin{bmatrix} \widehat{K}_1 & T\Gamma_{1,1} \end{bmatrix} U \begin{bmatrix} I_p & 0 \\ 0 & \mathcal{N} \end{bmatrix} \ ,$$

and we can use (3.2) to map this K matrix to F and G. As discussed in Chapter 2, this mapping is not unique. The final state observer is given by (3.1).

Remark 3.1 *Many of the early papers on input reconstruction (invertibility) assumed the initial state of the system to be known [7, 8, 10, 36]. This assumption implies that the initial state observer error in* (3.3) *will be zero, and consequently, it may not be necessary for the matrix E to be stable. In this case, the matrix H in* (3.25) *can simply be taken to be the identity matrix, and this might increase the number of state and input functionals that can be observed.*

We now revisit our claim that the rows of T will form a basis for all possible linear functionals of the state vector. We first note that our algorithm produces a matrix T of maximum rank satisfying Equations (3.4)-(3.5), with corresponding matrices E and K.

Suppose that there exists another set of matrices $\tilde{T}$, $\tilde{E}$, and $\tilde{K}$ satisfying the same conditions, but such that the row space of $\tilde{T}$ is not completely contained within the row space of T. However, this would imply that the matrices

$$\mathcal{T} \equiv \begin{bmatrix} T \\ \tilde{T} \end{bmatrix}, \quad \mathcal{E} \equiv \begin{bmatrix} E & 0 \\ 0 & \tilde{E} \end{bmatrix}, \quad \mathcal{K} \equiv \begin{bmatrix} K \\ \tilde{K} \end{bmatrix}$$

also satisfy Equations (3.4)-(3.5), and the rank of $\mathcal{T}$ will be greater than that of T. This is not possible, since the rank of T is maximal. Thus, the row space of T contains all possible linear functionals of the state that can be obtained through a linear observer.

3.3 Partial Input Observer

We now turn our attention to determining the largest subset of the unknown inputs that can be reconstructed through a linear observer. Specifically, we seek an observer of the form

$$\eta_k = R\psi_k + SY_{k:k+\alpha} \ , \tag{3.33}$$

where $\eta_k \to T_u u_k$ as $k \to \infty$ for some matrix T_u, and ψ_k is the output of the state observer in (3.1). We use the output of the state observer in the above equation because it represents the largest amount of information that is available about the state of the system. The rank of T_u should be maximized in order to obtain the largest amount of information on the input u_k. As in the previous section, the rows of T_u will form a basis for all possible linear functions of the input.

To find the observer parameters and the matrix T_u, we left-multiply (1.2) by S to obtain

$$SY_{k:k+\alpha} = S\Theta_\alpha x_k + S \begin{bmatrix} D & 0 \\ \Theta_{\alpha-1}B & M_{\alpha-1} \end{bmatrix} U_{k:k+\alpha} \ . \tag{3.34}$$

In order to obtain $T_u u_k$ from the above equation, the matrix S must satisfy

$$S \begin{bmatrix} D & 0 \\ \Theta_{\alpha-1}B & M_{\alpha-1} \end{bmatrix} = \begin{bmatrix} T_u & 0 \end{bmatrix} \ .$$

Following the same procedure as in the partial state observer design, we note that S must be of the form

$$S = \bar{S} \begin{bmatrix} I_p & 0 \\ 0 & \bar{\mathcal{N}} \end{bmatrix} \tag{3.35}$$

for some $\bar{S}$, where $\bar{\mathcal{N}}$ is a basis for the left nullspace of $M_{\alpha-1}$. In the equation

$$\bar{S}\begin{bmatrix} D \\ \bar{\mathcal{N}}\Theta_{\alpha-1}B \end{bmatrix} = T_u \quad , \tag{3.36}$$

let $\bar{r}$ be the rank of $\left[\begin{smallmatrix} D \\ \bar{\mathcal{N}}\Theta_{\alpha-1}B \end{smallmatrix}\right]$. Then there exists a pair of nonsingular matrices $\bar{U}$ and $\bar{V}$ such that

$$\bar{U}\begin{bmatrix} D \\ \bar{\mathcal{N}}\Theta_{\alpha-1}B \end{bmatrix}\bar{V} = \begin{bmatrix} 0 & 0 \\ I_{\bar{r}} & 0 \end{bmatrix} \quad . \tag{3.37}$$

Define

$$\bar{S} = \widehat{S}\bar{U} \tag{3.38}$$

for some $\widehat{S} = \begin{bmatrix} \widehat{S}_1 & \widehat{S}_2 \end{bmatrix}$, where $\widehat{S}_2$ has $\bar{r}$ columns. Right-multiplying Equation (3.36) by $\bar{V}$, we get

$$\begin{aligned} \begin{bmatrix} \widehat{S}_1 & \widehat{S}_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ I_{\bar{r}} & 0 \end{bmatrix} &= T_u\bar{V} \\ &\equiv T_u\begin{bmatrix} \bar{V}_1 & \bar{V}_2 \end{bmatrix} \quad , \end{aligned} \tag{3.39}$$

where $\bar{V}_1$ is the first $\bar{r}$ columns of $\bar{V}$, and $\bar{V}_2$ is the last $m-\bar{r}$ columns. From the above expression, we see that T_u must satisfy $T_u\bar{V}_2 = 0$. Let $\mathcal{N}_v$ be a basis for the left nullspace of $\bar{V}_2$. This implies that T_u must be of the form

$$T_u = \bar{T}_u\mathcal{N}_v \tag{3.40}$$

for some $\bar{T}_u$, and thus the problem is reduced to maximizing the rank of $\bar{T}_u$. If the rank of $\mathcal{N}_v$ is zero, it is not possible to reconstruct any of the inputs. Equation (3.39) also shows that $\widehat{S}_2 = T_u\bar{V}_1 = \bar{T}_u\mathcal{N}_v\bar{V}_1$, and $\widehat{S}_1$ is a free matrix.

Defining

$$\begin{aligned} \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} &\equiv \bar{U}\begin{bmatrix} I_p & 0 \\ 0 & \bar{\mathcal{N}} \end{bmatrix} Y_{k:k+\alpha} \quad , \\ \begin{bmatrix} \hat{\nu}_1 \\ \hat{\nu}_2 \end{bmatrix} &\equiv \bar{U}\begin{bmatrix} I_p & 0 \\ 0 & \bar{\mathcal{N}} \end{bmatrix} \Theta_\alpha \quad , \end{aligned}$$

where $\hat{Y}_2$ and $\hat{\nu}_2$ have $\bar{r}$ rows, Equation (3.34) becomes

$$\begin{bmatrix} \widehat{S}_1 & \bar{T}_u \mathcal{N}_v \bar{V}_1 \end{bmatrix} \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \begin{bmatrix} \widehat{S}_1 & \bar{T}_u \mathcal{N}_v \bar{V}_1 \end{bmatrix} \begin{bmatrix} \hat{\nu}_1 \\ \hat{\nu}_2 \end{bmatrix} x_k + T_u u_k \ . \tag{3.41}$$

The above equation shows that we need access to the state functional

$$\begin{bmatrix} \widehat{S}_1 & \bar{T}_u \end{bmatrix} \begin{bmatrix} \hat{\nu}_1 \\ \mathcal{N}_v \bar{V}_1 \hat{\nu}_2 \end{bmatrix} x_k \tag{3.42}$$

in order to reconstruct $T_u u_k$. From the previous section, we recall that the output of the observer in (3.1) will be $T x_k$, and that the rows of T form a basis for all possible linear functions of the state. Thus we require that the functional in (3.42) be a linear combination of the rows in T. Mathematically, this means that there should exist a matrix Q such that

$$\begin{bmatrix} \widehat{S}_1 & \bar{T}_u \end{bmatrix} \begin{bmatrix} \hat{\nu}_1 \\ \mathcal{N}_v \bar{V}_1 \hat{\nu}_2 \end{bmatrix} = QT \ .$$

Rearranging, we have

$$\begin{bmatrix} \widehat{S}_1 & \bar{T}_u & Q \end{bmatrix} \underbrace{\begin{bmatrix} \hat{\nu}_1 \\ \mathcal{N}_v \bar{V}_1 \hat{\nu}_2 \\ -T \end{bmatrix}}_{\Pi} = 0 \ , \tag{3.43}$$

which implies that

$$\begin{bmatrix} \widehat{S}_1 & \bar{T}_u & Q \end{bmatrix} = T_\Pi \mathcal{N}_\Pi \ ,$$

where $\mathcal{N}_\Pi$ is a basis for the left nullspace of the matrix Π defined in (3.43) and T_Π is chosen to maximize the rank of $\bar{T}_u$. In particular, we can assume without loss of generality that $\mathcal{N}_\Pi$ is of the form

$$\mathcal{N}_\Pi = \begin{bmatrix} \widehat{S}_{11} & 0 & Q_1 \\ \widehat{S}_{12} & \bar{T}_{u,1} & 0 \\ \widehat{S}_{13} & \bar{T}_{u,2} & Q_2 \end{bmatrix} \ , \tag{3.44}$$

where the matrices $\left[\begin{smallmatrix} \bar{T}_{u,1} \\ \bar{T}_{u,2} \end{smallmatrix}\right]$ and Q_2 are full row rank. This form can always be obtained by left-multiplying $\mathcal{N}_\Pi$ by an appropriate nonsingular matrix. From the above expressions, we see that T_Π should be of the form $T_\Pi = \begin{bmatrix} 0 & I \end{bmatrix}$ in order to maximize the rank of $\bar{T}_u$. This results in

$$\bar{T}_u = \begin{bmatrix} \bar{T}_{u,1} \\ \bar{T}_{u,2} \end{bmatrix} \ ,$$

and allows us to use (3.40) to obtain

$$T_u = \begin{bmatrix} \bar{T}_{u,1} \\ \bar{T}_{u,2} \end{bmatrix} \mathcal{N}_v \ . \tag{3.45}$$

Substituting this into (3.41), we get

$$\begin{bmatrix} \widehat{S}_{12} & \bar{T}_{u,1}\mathcal{N}_v\bar{V}_1 \\ \widehat{S}_{13} & \bar{T}_{u,2}\mathcal{N}_v\bar{V}_1 \end{bmatrix} \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ Q_2 \end{bmatrix} Tx_k + \begin{bmatrix} \bar{T}_{u,1} \\ \bar{T}_{u,2} \end{bmatrix} \mathcal{N}_v u_k \ . \tag{3.46}$$

The above expression shows that the functional $\bar{T}_{u,1}\mathcal{N}_v u_k$ can be obtained directly from the output without relying on the value of the state, and the functional $\bar{T}_{u,2}\mathcal{N}_v u_k$ depends on the state functional $Q_2 T x_k$. As an aside, note that (3.44) also characterizes the set of state functionals which are directly available from the output. Specifically, the state functional $Q_1 T x_k$ can be obtained by setting $\widehat{S}_1 = \widehat{S}_{11}$ in (3.41).

Comparing (3.46) to (3.33), we see that R should be chosen as

$$R = -\begin{bmatrix} 0 \\ Q_2 \end{bmatrix} \ , \tag{3.47}$$

and S can obtained from (3.35) and (3.38) as

$$S = \begin{bmatrix} \widehat{S}_{12} & \bar{T}_{u,1}\mathcal{N}_v\bar{V}_1 \\ \widehat{S}_{13} & \bar{T}_{u,2}\mathcal{N}_v\bar{V}_1 \end{bmatrix} \bar{U} \begin{bmatrix} I_p & 0 \\ 0 & \bar{\mathcal{N}} \end{bmatrix} \ . \tag{3.48}$$

The combined state and input observer can be obtained from (3.1) and (3.33) as

$$\begin{aligned} z_{k+1} &= E z_k + F Y_{k:k+\alpha} \ , \\ \psi_k &= z_k + G Y_{k:k+\alpha} \ , \\ \eta_k &= R\psi_k + S Y_{k:k+\alpha} \ , \end{aligned} \tag{3.49}$$

where $\psi_k \to Tx_k$ and $\eta_k \to T_u u_k$ as $k \to \infty$.

3.4 Example

Consider a linearized model of the dynamics of a VTOL aircraft from [2]:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0.4422 & 0 & 0 & 0 \\ 3.5446 & 0 & 0 & 0 \\ -5.52 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} a_1(t) \\ s_1(t) \\ s_2(t) \\ p_1(t) \end{bmatrix}}_{u(t)}$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}}_{C} x(t) + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{D} u(t) \ , \tag{3.50}$$

where $a_1(t)$ represents an actuator fault, $s_1(t)$ and $s_2(t)$ represent sensor faults, and $p_1(t)$ represents a disturbance due to parameter uncertainties. We will assume that no prior information is available about the characteristics of these signals, and they will be modeled as unknown inputs. For this example, we will assume that the first derivative of the outputs is available. We wish to determine the largest subset of the states that can be obtained through an observer of the form

$$\begin{aligned} \dot{z}(t) &= Ez(t) + F \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \\ \psi(t) &= z(t) + G \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \quad . \end{aligned}$$

This is equivalent to $\alpha = 1$ for the discrete time observer in (3.1). From the discussion in Section 3.2, we see that we will require the following matrices to derive the observer parameters:

$$\begin{aligned} M_1 &= \begin{bmatrix} D & 0 \\ CB & D \end{bmatrix}, \quad \Theta_1 = \begin{bmatrix} C \\ CA \end{bmatrix} \quad , \\ M_2 &= \begin{bmatrix} D & 0 \\ \Theta_1 B & M_1 \end{bmatrix}, \quad \Theta_2 = \begin{bmatrix} C \\ \Theta_1 A \end{bmatrix} \quad . \end{aligned}$$

A basis for the left nullspace of M_1 is given by

$$\mathcal{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \ ,$$

and the matrix $\left[\begin{smallmatrix} D \\ \mathcal{N}\Theta_1 B \end{smallmatrix}\right]$ has a rank of 4. The matrices U and V from (3.9) are found to be

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.2614 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.4672 & 1 \end{bmatrix}, \quad V = I_4 \ .$$

From (3.11), we get

$$\Gamma_{1,1} = \begin{bmatrix} 0.4422 & 0 & 0 & 0 \\ 3.5446 & 0 & 0 & 0 \\ -5.52 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ ,$$

and $\Gamma_{1,2}$ is the empty matrix of dimension 4×0. A basis for the left nullspace of $\Gamma_{1,2}$ is taken to be $\mathcal{N}_1 = I_4$. This means that matrix $V_1 = I_4$ in (3.15), and

$$\Gamma_{2,1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0.3416 & -1.2272 & -0.1483 & -0.3696 \\ -0.3567 & 0.7064 & -0.4723 & -4.2660 \\ 0.0151 & 0.5208 & 0.6206 & 4.6356 \end{bmatrix} \ .$$

Once again, $\Gamma_{2,2}$ is the empty matrix of dimension 6×0. A basis for the left nullspace of $\Gamma_{2,2}$ is taken to be $\mathcal{N}_2 = I_6$, which means that the matrix $\mathcal{N}_{2,2}$ in (3.18) is $\mathcal{N}_{2,2} = \left[\begin{smallmatrix} 0 \\ I_4 \end{smallmatrix}\right]$. Since this is full column rank, one of the terminal conditions of the iteration is satisfied. The matrix

U_2 in (3.20) can be taken to be the identity matrix. From (3.24), we get

$$\left[\begin{array}{c} \mathcal{C} \\ \hline \mathcal{A} \end{array}\right] = U_2 \mathcal{N}_2 \Gamma_{2,1}$$
$$= \left[\begin{array}{rrrr} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0.3416 & -1.2272 & -0.1483 & -0.3696 \\ -0.3567 & 0.7064 & -0.4723 & -4.2660 \\ 0.0151 & 0.5208 & 0.6206 & 4.6356 \end{array}\right].$$

The matrix P in (3.26) is given by

$$P = \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0.9566 \\ 0 & 0 & 1 & 52.0961 \end{array}\right],$$

which gives

$$\bar{\mathcal{A}} \equiv P\mathcal{A}P^{-1} = \left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.3422 & 1.2046 & -1.0833 & 0 \\ \hline 0.4300 & 27.8401 & 0 & 4.0193 \end{array}\right],$$
$$\bar{\mathcal{C}} \equiv \mathcal{C}P^{-1} = \left[\begin{array}{rrr|r} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right].$$

This system has one unobservable and unstable mode at 4.0193 and one unobservable and stable mode at -1.0833. We choose to place the eigenvalues of the observable subsystem at $\{-10, -11\}$, and this requires that

$$L_{11} = \left[\begin{array}{rr} 10 & 0 \\ 0 & 11 \end{array}\right].$$

The matrices L_{12} and L_{13} will be taken to be the zero matrices. Following the discussion in Section 3.2, we choose $\bar{H} = \begin{bmatrix} I_3 & 0 \end{bmatrix}$, and this gives

$$E = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -11 & 0 \\ -0.3422 & 1.2046 & -1.0833 \end{bmatrix} .$$

From (3.31) and (3.32) we get

$$T = \bar{H} \begin{bmatrix} L_1 & P \end{bmatrix} U_2 \mathcal{N}_{2,2} \mathcal{N}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0.9566 \end{bmatrix} ,$$

$$\widehat{K}_1 = \bar{H} \begin{bmatrix} L_1 & P \end{bmatrix} U_2 \mathcal{N}_{2,1} = \begin{bmatrix} 10 & 0 \\ 0 & 11 \\ 0 & 0 \end{bmatrix} ,$$

which produces

$$K = \begin{bmatrix} 10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 11 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8.2097 & 0 & 0 & 0.9566 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

We will map this to the F and G matrices by using (3.2) and choosing $F_1 = 0$. This gives

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8.5511 & 0 & 0 & 0.1684 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -8.2097 & 0 & 0 & 0.9566 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Note that the entries corresponding to $\dot{y}(t)$ are zero in the above matrices, and thus the partial state observer is given by

$$\dot{z}(t) = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -11 & 0 \\ -0.3422 & 1.2046 & -1.0833 \end{bmatrix} z(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8.5511 & 0 & 0 & 0.1684 \end{bmatrix} y(t)$$

$$\psi(t) = z(t) + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -8.2097 & 0 & 0 & 0.9566 \end{bmatrix} y(t) .$$

We now turn our attention to determining the set of input functionals that can be observed. The matrix $\bar{\mathcal{N}}$ in (3.35) is given by

$$\bar{\mathcal{N}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

and the matrices $\bar{U}$ and $\bar{V}$ in (3.37) are found to be the same as U and V from the design of the state observer. The basis for the left nullspace of the empty matrix $\bar{V}_2$ in (3.39) is taken to be $\mathcal{N}_v = I_4$. Using (3.43), we get

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -0.0828 & 0.0613 & 0.0425 & -1.0301 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -0.0151 & -0.5208 & 0.3794 & -4.6356 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -0.9566 \end{bmatrix} ,$$

with corresponding left nullspace given by

$$\mathcal{N}_{\Pi} = \left[\begin{array}{rr|rrrr|rrr} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.297 & 0.752 & -3.770 & -0.971 & 0 & 1 & 0 & 0 & 0 \\ 0.076 & 0.944 & 0.916 & -0.983 & -1 & 0 & 0 & 0 & 0 \\ -0.072 & 0.054 & -0.876 & 0.984 & 0 & 0 & 0 & 0 & 1 \end{array}\right] ,$$

where the partitions correspond to $\widehat{S}_1$, $\bar{T}_u$ and Q from (3.43). From (3.45), we obtain

$$T_u = \begin{bmatrix} -3.770 & -0.971 & 0 & 1 \\ 0.916 & -0.983 & -1 & 0 \\ -0.876 & 0.984 & 0 & 0 \end{bmatrix} , \qquad (3.51)$$

and we use (3.47) and (3.48) to obtain

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.297 & 0 & -0.971 & 0.752 & -4.059 & 0 & 0 & 1 \\ 0.076 & -1 & -0.983 & 0.944 & 2.072 & 0 & 0 & 0 \\ -0.072 & 0 & 0.984 & 0.054 & -1.982 & 0 & 0 & 0 \end{bmatrix}.$$

To test these observers, the system in (3.50) is initialized with a nonzero initial state and driven by a set of unknown inputs. The initial state of the partial state observer is set to be zero, and the results of the simulation are shown in Figure 3.1. The output of the partial input observer and the corresponding input functionals are shown in Figure 3.2. As indicated by the above $\mathcal{N}_\Pi$ matrix, the first two input functionals are immediately obtained from the output, and the convergence of the third input functional depends on the convergence rate of the third state functional. Since the first two state and input functionals are directly available from the output, the actual and estimated signals for these functionals coincide in Figures 3.1 and 3.2. It is of interest to note that the second and third input functionals in (3.51) reproduce functions of the fault inputs $a_1(t)$, $s_1(t)$ and $s_2(t)$, and do not contain the disturbance input $p_1(t)$. These input functionals can therefore be used for fault detection and diagnosis.

3.5 Summary

We have provided a characterization of partial state observers for linear systems with unknown inputs. Our approach involves an iterative procedure which decouples the estimator error from the values of the state and unknown inputs. The overall procedure produces the observer parameters in addition to a characterization of all possible linear functions of the state. We have also used the partial state observer to determine all possible input functionals which can be observed from the output, and have constructed a partial input observer which produces these functionals. Both the state and input observers allow the use of delays, which enables a larger number of state and input functionals to be reproduced. The resulting observers are more general than those currently present in the literature, and can be used in applications such as fault diagnosis and control system design for uncertain systems.

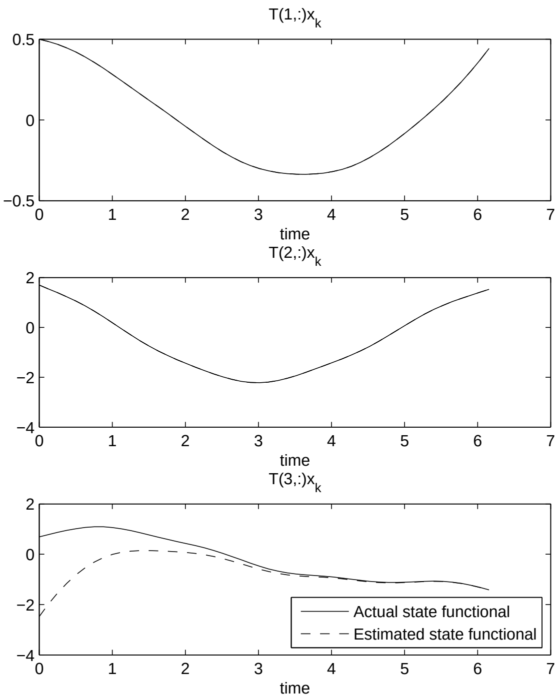

Figure 3.1: Simulation of partial state observer.

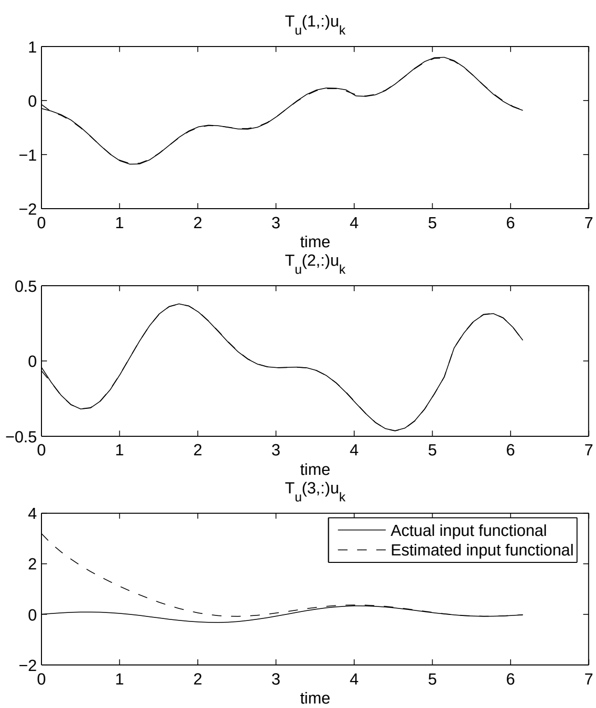

Figure 3.2: Simulation of partial input observer.

CHAPTER 4

OPTIMAL STATE ESTIMATORS

4.1 Introduction

In the preceding chapters, we have focused on constructing observers for deterministic systems. We now extend our investigation to the case where the system is affected by noise. The problem of state estimation in the presence of both unknown inputs and noise has been studied in [30, 40–43], among others. These papers have demonstrated that the necessary conditions for the existence of an unbiased zero-delay estimator are the same as those for zero-delay observers. Furthermore, it was shown in [43] that forcing the estimator to be unbiased might cause the system noise to become correlated with the estimation error (i.e., the noise behaves as if it is colored). In [30], Saberi et al. showed through a geometric approach that allowing delays in the estimator can relax the necessary conditions for optimal estimation. Delayed estimators were also studied in [44], but the design procedure outlined in that paper did not fully utilize the freedom in the estimator gain matrix, and ignored the correlation between the noise and error. Here, we study the general case of delayed estimators for linear systems with unknown inputs, and develop a methodology to construct optimal linear estimators. More specifically, our goal is to estimate the entire system state through linear recursive estimators that minimize the mean square estimation error. In addition, we require that the estimator be unbiased (i.e., the expected value of the estimation error must be zero). Our approach is an extension of the results in Chapter 2. The fact that we incorporate delays in our design procedure allows us to construct optimal estimators for a much larger class of systems than that considered by [40–43]. Our approach is more direct than the method in [30], which first transforms the estimator into a dual system and then uses techniques from H_2-optimal control in order to obtain the gain matrix. Furthermore, our estimators will generally be of smaller dimension than the estimators considered in [30], and our approach allows us to obtain a tighter bound on the maximum estimator delay. Our design procedure also makes full use of the freedom in the gain matrix, which produces better results than the method in [44]. In addition, our method avoids the problem of colored noise described in [43] by increasing the dimension of the estimator appropriately.

4.2 Preliminaries

Consider a discrete-time stochastic linear system $\mathcal{S}$ of the form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k + Du_k + v_k \ , \end{aligned} \tag{4.1}$$

with state vector $x \in \mathbb{R}^n$, unknown input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, and system matrices (A, B, C, D) of appropriate dimensions. The noise processes w_k and v_k are assumed to be uncorrelated, white, and zero mean, with covariance matrices Q_k and R_k, respectively. As in the previous chapters, we omit known inputs in the above equations for clarity of development, and assume without loss of generality that the matrix $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ is full column rank.

The response of system (4.1) over $\alpha + 1$ time units is given by

$$\begin{aligned} \underbrace{\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+\alpha} \end{bmatrix}}_{Y_{k:k+\alpha}} &= \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha} \end{bmatrix}}_{\Theta_\alpha} x_k + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-1}B & CA^{\alpha-2}B & \cdots & D \end{bmatrix}}_{M_\alpha} \underbrace{\begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+\alpha} \end{bmatrix}}_{U_{k:k+\alpha}} \\ &+ \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ C & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-1} & CA^{\alpha-2} & \cdots & C \end{bmatrix}}_{M_{w,\alpha}} \underbrace{\begin{bmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_{k+\alpha-1} \end{bmatrix}}_{W_{k:k+\alpha-1}} + \underbrace{\begin{bmatrix} v_k \\ v_{k+1} \\ \vdots \\ v_{k+\alpha} \end{bmatrix}}_{V_{k:k+\alpha}} . \end{aligned} \tag{4.2}$$

In the rest of the chapter, we will be using $E\{\cdot\}$ to denote the expected value of a stochastic parameter. The notation I_r represents the $r \times r$ identity matrix, and $(\cdot)^T$ indicates matrix transpose. We are now in place to proceed with the construction of an estimator for the states in $\mathcal{S}$.

4.3 Unbiased Estimation

We start by considering a linear recursive estimator of the same dimension as the state vector of $\mathcal{S}$. We will demonstrate that it will generally be necessary to use delayed measurements in order to construct an unbiased estimator, and that this will cause the system noise to behave as if it were colored.

Consider an estimator of the form

$$z_{k+1} = Az_k + K_k \left(Y_{k:k+\alpha} - \Theta_\alpha z_k\right) \ , \tag{4.3}$$

where the nonnegative integer α is the estimator delay, and the matrix K_k is chosen to (i) make the estimator unbiased and (ii) minimize the mean square error between z_{k+1} and x_{k+1}. Note that for $\alpha = 0$, the above estimator is in the form of the optimal estimator for systems with no unknown inputs [45]. Using (4.2), we obtain the estimation error as

$$\begin{aligned} e_{k+1} &\equiv z_{k+1} - x_{k+1} \\ &= \left(A - K_k\Theta_\alpha\right) z_k + K_k Y_{k:k+\alpha} - Ax_k - Bu_k - w_k \\ &= \left(A - K_k\Theta_\alpha\right) e_k + K_k V_{k:k+\alpha} + K_k M_\alpha U_{k:k+\alpha} - Bu_k \\ &\quad + K_k M_{w,\alpha} W_{k:k+\alpha-1} - w_k \ . \end{aligned} \tag{4.4}$$

In order for the estimator to be unbiased (i.e., $E\{e_k\} = 0$ for all k, regardless of the values of the unknown inputs), we require that

$$K_k M_\alpha = \begin{bmatrix} B & 0 & \cdots & 0 \end{bmatrix} \ . \tag{4.5}$$

The solvability of the above condition is given by the following theorem.

Theorem 4.1 *There exists a matrix K_k such that*

$$K_k M_\alpha = \begin{bmatrix} B & 0 & \cdots & 0 \end{bmatrix}$$

if and only if

$$rank\left[M_\alpha\right] - rank\left[M_{\alpha-1}\right] = m \ . \tag{4.6}$$

The proof of the above theorem is similar to the proof of Theorem 2.2 from Chapter 2, and so we omit it here. The result in the above theorem was also obtained in [44] for the specific case of $D = 0$. Once again, (4.6) is the condition for inversion of the inputs with known initial state [7], and demonstrates the utility of a delayed observer. An upper bound on α is provided by the following theorem.

Theorem 4.2 *Let q be the dimension of the nullspace of D. Then the delay of the unbiased estimator* (4.3) *will not exceed $\alpha = n - q + 1$ time-steps. If* (4.6) *is not satisfied for*

$\alpha = n - q + 1$, *then unbiased estimation of all the states is not possible with an estimator of the form given in* (4.3).

The proof of the above theorem is readily obtained by making use of the result in Theorem 1.2.

It is apparent from (4.4) that for $\alpha > 0$, the error will generally be a function of multiple time-samples of the noise processes w_k and v_k. In other words, the noise behaves as if it were colored. Darouach et al. [43] studied this situation for $\alpha \in \{0, 1\}$, and proposed certain strict conditions for the existence of unbiased optimal estimators with dimension n. Consequently, the estimators proposed in that paper can only be applied to a restricted class of systems. In the study of Kalman filters for systems with no unknown inputs, it has been shown that colored noise can be handled by increasing the dimension of the estimator [45, 46]. In the next section, we will apply this technique to construct an optimal estimator for the system in (4.1). The fact that our estimator uses delays allows it to be applied to a much larger class of systems than the estimators presented in [40–43].

4.4 Optimal Estimator

In this section, we will consider estimators for the case $\alpha > 0$. If Theorem 4.1 is satisfied for $\alpha = 0$, the noise in (4.4) will not be colored, and an optimal estimator of dimension n can be constructed. The resulting estimator is called an *optimal predictor* in [43], and can be found by using the method in that paper. To construct an optimal estimator for $\alpha > 0$, we will rewrite system $\mathcal{S}$ to obtain a new system $\bar{\mathcal{S}}$ given by

$$\begin{aligned} \bar{x}_{k+1} &= \bar{A}\bar{x}_k + \bar{B}u_k + \bar{B}_n n_k \\ y_k &= \bar{C}\bar{x}_k + Du_k \ , \end{aligned} \tag{4.7}$$

where

$$\bar{x}_k = \begin{bmatrix} x_k \\ W_{k:k+\alpha-2} \\ \hline V_{k:k+\alpha-1} \end{bmatrix} \ , \quad n_k = \begin{bmatrix} w_{k+\alpha-1} \\ v_{k+\alpha} \end{bmatrix} \ ,$$

$$\bar{A} = \left[\begin{array}{ccccc|ccccc} A & I_n & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I_p \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right], \quad \bar{B} = \left[\begin{array}{c} B \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{array}\right], \quad \bar{B}_n = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ I_n & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & I_p \end{array}\right],$$

$$\bar{C} = \left[\begin{array}{ccccc|ccccc} C & 0 & \cdots & 0 & 0 & I_p & 0 & \cdots & 0 & 0 \end{array}\right] .$$

Note that the state vector in this new system has dimension $\bar{n} = \alpha(n+p)$ for $\alpha > 0$. We will construct a linear recursive estimator for this augmented system, and demonstrate that the problem with colored noise from the last section no longer occurs. We will then extract an estimate of the original state vector from the estimate of the new state vector.

From (4.2), the output of this augmented system over $\alpha + 1$ time-steps is given by

$$Y_{k:k+\alpha} = \underbrace{\left[\begin{array}{cccc|cccc} C & 0 & \cdots & 0 & I_p & 0 & \cdots & 0 \\ CA & C & \cdots & 0 & 0 & I_p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ CA^{\alpha-1} & CA^{\alpha-2} & \cdots & C & 0 & 0 & \cdots & I_p \\ CA^{\alpha} & CA^{\alpha-1} & \cdots & CA & 0 & 0 & \cdots & 0 \end{array}\right]}_{\bar{\Theta}_\alpha} \bar{x}_k + M_\alpha U_{k:k+\alpha} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ C & I_p \end{bmatrix}}_{M_{n,\alpha}} n_k \ . \tag{4.8}$$

We notice that a portion of this equation is independent of the noise vector n_k, and so we can obtain some linear functions of the states directly from the output. The following theorem characterizes the set of states for which this is possible.

Theorem 4.3 *For system* (4.7) *with response over* $\alpha + 1$ *time-steps given by* (4.8)*, let* t *be the dimension of the left nullspace of* $M_{\alpha-1}$*. Then it is possible to perform a similarity transformation on the system* $\bar{\mathcal{S}}$ *to obtain a new system* $\widehat{\mathcal{S}}$ *such that exactly* t *of the states in* $\widehat{\mathcal{S}}$ *are directly obtainable from the output of the system.*

Proof: Let $\bar{\mathcal{P}}$ be a basis for the left nullspace of $M_{\alpha-1}$ (i.e., $\bar{\mathcal{P}} M_{\alpha-1} = 0$). Note that the dimension of $\bar{\mathcal{P}}$ is equal to

$$t = \alpha p - \mathrm{rank}[M_{\alpha-1}] \ . \tag{4.9}$$

Define the matrix $\mathcal{P} = \begin{bmatrix} \bar{\mathcal{P}} & 0 \end{bmatrix}$, where the zero matrix has p columns. Using (1.4), we see that $\mathcal{P}M_\alpha = 0$, and from (4.8), it is apparent that $\mathcal{P}\bar{\Theta}_\alpha$ is full row rank. Define the similarity transformation matrix

$$\mathcal{T} \equiv \begin{bmatrix} \mathcal{P}\bar{\Theta}_\alpha \\ \mathcal{H} \end{bmatrix} \ ,$$

where the matrix $\mathcal{H}$ is chosen so that $\mathcal{T}$ is invertible. In particular, $\mathcal{P}$ and $\mathcal{H}$ can be chosen so that $\mathcal{T}$ is orthogonal. Consider the system $\widehat{\mathcal{S}}$ with state-vector $\hat{x}_k = \begin{bmatrix} \hat{x}_{1,k} \\ \hat{x}_{2,k} \end{bmatrix} = \mathcal{T}\bar{x}_k$. The system matrices in $\widehat{\mathcal{S}}$ are given by

$$\begin{aligned}
\widehat{A} &\equiv \mathcal{T}\bar{A}\mathcal{T}^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \ , \\
\widehat{B} &\equiv \mathcal{T}\bar{B} = \begin{bmatrix} \mathcal{P}\bar{\Theta}_\alpha \bar{B} \\ \mathcal{H}\bar{B} \end{bmatrix} \ , \\
\widehat{B}_n &\equiv \mathcal{T}\bar{B}_n = \begin{bmatrix} \mathcal{P}\bar{\Theta}_\alpha \bar{B}_n \\ \mathcal{H}\bar{B}_n \end{bmatrix} \ , \\
\widehat{C} &\equiv \bar{C}\mathcal{T}^T, \ \ \widehat{D} \equiv D \ .
\end{aligned}$$

Now it is readily seen from (4.8) that

$$\begin{aligned}
\mathcal{P}Y_{k:k+\alpha} &= \mathcal{P}\bar{\Theta}_\alpha \mathcal{T}^T \hat{x}_k \\
&= \begin{bmatrix} I_t & 0 \end{bmatrix} \hat{x}_k \ ,
\end{aligned}$$

and thus the first t states of $\hat{x}_k$ are immediately obtained. ■

The remaining $(n-t)$ states of $\hat{x}_k$ evolve according to the equation

$$\begin{aligned}
\hat{x}_{2,k+1} &= A_{21}\hat{x}_{1,k} + A_{22}\hat{x}_{2,k} + \mathcal{H}\bar{B}u_k + \mathcal{H}\bar{B}_n n_k \\
&= A_{22}\hat{x}_{2,k} + A_{21}\mathcal{P}Y_{k:k+\alpha} + \mathcal{H}\bar{B}u_k + \mathcal{H}\bar{B}_n n_k \ .
\end{aligned} \tag{4.10}$$

Let Φ be a matrix such that $\begin{bmatrix} \bar{\mathcal{P}} \\ \Phi \end{bmatrix}$ is square and invertible. Define

$$\mathcal{G} \equiv \begin{bmatrix} \Phi & 0 \\ 0 & I_p \end{bmatrix}, \;\; \mathcal{J} \equiv \begin{bmatrix} \mathcal{P} \\ \mathcal{G} \end{bmatrix} \quad . \tag{4.11}$$

Using (1.3), (1.4) and (4.8), we note that

$$\begin{aligned} \mathcal{J} M_\alpha &= \begin{bmatrix} 0 \\ \mathcal{G} M_\alpha \end{bmatrix} \quad , \\ \mathcal{J} M_{n,\alpha} &= \begin{bmatrix} 0 \\ \mathcal{G} M_{n,\alpha} \end{bmatrix} = M_{n,\alpha} \quad , \\ \mathcal{J} \bar{\Theta}_\alpha \mathcal{T}^T &= \begin{bmatrix} I_t & 0 \\ L_1 & L_2 \end{bmatrix} \quad , \end{aligned}$$

where

$$\begin{bmatrix} L_1 & L_2 \end{bmatrix} = \mathcal{G} \bar{\Theta}_\alpha \mathcal{T}^T \quad . \tag{4.12}$$

Left-multiplying (4.8) by $\mathcal{J}$, we get

$$\begin{bmatrix} \mathcal{P} \\ \mathcal{G} \end{bmatrix} Y_{k:k+\alpha} = \begin{bmatrix} I_t & 0 \\ L_1 & L_2 \end{bmatrix} \hat{x}_k + \begin{bmatrix} 0 \\ \mathcal{G} M_\alpha \end{bmatrix} U_{k:k+\alpha} + \begin{bmatrix} 0 \\ \mathcal{G} M_{n,\alpha} \end{bmatrix} n_k \quad .$$

We see that the first t rows of the above equation do not contain any information about $\hat{x}_{2,k}$. The remaining rows can be written as

$$(\mathcal{G} - L_1 \mathcal{P}) \, Y_{k:k+\alpha} = L_2 \hat{x}_{2,k} + \mathcal{G} M_\alpha U_{k:k+\alpha} + \mathcal{G} M_{n,\alpha} n_k \quad . \tag{4.13}$$

To estimate $\hat{x}_{2,k}$, we use (4.10) and (4.13) to construct an estimator of the form

$$z_{k+1} = A_{22} z_k + A_{21} \mathcal{P} Y_{k:k+\alpha} + K_k \left((\mathcal{G} - L_1 \mathcal{P}) \, Y_{k:k+\alpha} - L_2 z_k \right) \quad , \tag{4.14}$$

where K_k is chosen to (i) make the estimator unbiased, and (ii) minimize the mean square error between z_{k+1} and $\hat{x}_{2,k+1}$. Using (4.10) and (4.13), we find the error between the two quantities to be

$$\begin{aligned} e_{k+1} &\equiv z_{k+1} - \hat{x}_{2,k+1} \\ &= (A_{22} - K_k L_2)\, e_k + \left(K_k \mathcal{G} M_{n,\alpha} - \mathcal{H}\bar{B}_n\right) n_k \\ &\quad + \left(K_k \mathcal{G} M_\alpha - \begin{bmatrix} \mathcal{H}\bar{B} & 0 & \cdots & 0 \end{bmatrix}\right) U_{k:k+\alpha} \ . \end{aligned} \tag{4.15}$$

Note that the noise in (4.15) is no longer colored from the perspective of the error, allowing us to construct an optimal estimator. For an unbiased estimator (i.e., $E\{e_k\} = 0$), we require that

$$K_k \mathcal{G} M_\alpha = \begin{bmatrix} \mathcal{H}\bar{B} & 0 & \cdots & 0 \end{bmatrix} \ . \tag{4.16}$$

The solvability of the above condition is given by the following theorem.

Theorem 4.4 *There exists a matrix K_k such that*

$$K_k \mathcal{G} M_\alpha = \begin{bmatrix} \mathcal{H}\bar{B} & 0 & \cdots & 0 \end{bmatrix}$$

if and only if

$$\mathit{rank}\,[M_\alpha] - \mathit{rank}\,[M_{\alpha-1}] = m \ .$$

Proof: Arguing as in the proof of Theorem 2.2, there exists a K_k satisfying (4.16) if and only if the matrix

$$S \equiv \begin{bmatrix} \mathcal{H}\bar{B} & 0 & \cdots & 0 \end{bmatrix}$$

is in the space spanned by the rows of $\mathcal{G} M_\alpha$. This is equivalent to the condition

$$\text{rank} \begin{bmatrix} \mathcal{G} M_\alpha \\ S \end{bmatrix} = \text{rank}\,[\mathcal{G} M_\alpha] \ . \tag{4.17}$$

Since $\mathcal{P} M_\alpha = 0$, we use (1.4) to get

$$\text{rank} \begin{bmatrix} \mathcal{G} M_\alpha \\ S \end{bmatrix} = \text{rank} \begin{bmatrix} \mathcal{P} M_\alpha \\ \mathcal{G} M_\alpha \\ S \end{bmatrix} = \text{rank} \begin{bmatrix} M_\alpha \\ S \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} D & 0 \\ \bar{\Theta}_{\alpha-1}\bar{B} & M_{\alpha-1} \\ \mathcal{H}\bar{B} & 0 \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} D & 0 \\ \bar{\Theta}_{\alpha-1}\bar{B} & M_{\alpha-1} \\ \bar{\mathcal{P}}\bar{\Theta}_{\alpha-1}\bar{B} & 0 \\ \mathcal{H}\bar{B} & 0 \end{bmatrix} .$$

Using the fact that $\bar{\mathcal{P}}\bar{\Theta}_{\alpha-1} = \mathcal{P}\bar{\Theta}_{\alpha}$ and our assumption that the matrix $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ has full column rank, we obtain

$$\operatorname{rank}\begin{bmatrix} \mathcal{G}M_\alpha \\ S \end{bmatrix} = \operatorname{rank}\begin{bmatrix} D & 0 \\ \bar{\Theta}_{\alpha-1}\bar{B} & M_{\alpha-1} \\ \bar{B} & 0 \end{bmatrix}$$
$$= m + \operatorname{rank}\left[M_{\alpha-1}\right] .$$

Finally, we note that $\operatorname{rank}[\mathcal{G}M_\alpha] = \operatorname{rank}[M_\alpha]$ in (4.17), and this concludes the proof. ■

The condition in the above theorem is the same as the one in Theorem 4.1. This means that the upper bound on the delay provided by Theorem 4.2 also applies to the reduced order estimator in (4.14).

To solve (4.16), we note that if the condition in Theorem 2.2 is satisfied, then the rank of $\mathcal{G}M_\alpha$ will also be $m + \operatorname{rank}[M_{\alpha-1}]$. Let $\mathcal{N}$ be a matrix whose rows form a basis for the left nullspace of the last αm columns of $\mathcal{G}M_\alpha$. In particular, we can assume without loss of generality that $\mathcal{N}$ satisfies

$$\mathcal{N}\mathcal{G}M_\alpha = \begin{bmatrix} 0 & 0 \\ I_m & 0 \end{bmatrix} . \tag{4.18}$$

Note that $\mathcal{G}M_\alpha$ has $(\alpha+1)p - t$ rows, and the last αm columns of the matrix have the same rank as $M_{\alpha-1}$. Using the expression for t given in (4.9), the number of rows in $\mathcal{N}$ is given by

$$\dim(\mathcal{N}) = (\alpha+1)p - t - \operatorname{rank}[M_{\alpha-1}]$$
$$= p .$$

From (4.16), we see that K_k must be of the form

$$K_k = \widehat{K}_k\mathcal{N} \tag{4.19}$$

for some $\widehat{K}_k = \begin{bmatrix} \widehat{K}_{1,k} & \widehat{K}_{2,k} \end{bmatrix}$, where $\widehat{K}_{1,k}$ has $p - m$ columns and $\widehat{K}_{2,k}$ has m columns. Equation (4.16) then becomes

$$\begin{bmatrix} \widehat{K}_{1,k} & \widehat{K}_{2,k} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I_m & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{H}\bar{B} & 0 \end{bmatrix} \ , \tag{4.20}$$

from which it is obvious that $\widehat{K}_{2,k} = \mathcal{H}\bar{B}$ and $\widehat{K}_{1,k}$ is a free matrix.

Returning to Equation (4.15), define

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \equiv \mathcal{N} L_2 \ , \tag{4.21}$$

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \equiv \mathcal{N}\mathcal{G} M_{n,\alpha} \ , \tag{4.22}$$

where ν_2 and ρ_2 have m rows. The expression for the estimation error from (4.15) can now be written as

$$e_{k+1} = \left(\left(A_{22} - \mathcal{H}\bar{B}\nu_2 \right) - \widehat{K}_{1,k}\nu_1 \right) e_k + \left(\mathcal{H} \left(\bar{B}\rho_2 - \bar{B}_n \right) + \widehat{K}_{1,k}\rho_1 \right) n_k \ . \tag{4.23}$$

Denoting

$$\begin{aligned} \mathcal{A} &= A_{22} - \mathcal{H}\bar{B}\nu_2 \\ &= \mathcal{H} \left(\bar{A}\mathcal{H}^T - \bar{B}\nu_2 \right) \ , \\ \mathcal{B} &= \mathcal{H} \left(\bar{B}\rho_2 - \bar{B}_n \right) \ , \\ \Pi_k &= \begin{bmatrix} Q_{k+\alpha-1} & 0 \\ 0 & R_{k+\alpha} \end{bmatrix} \ , \end{aligned} \tag{4.24}$$

the error covariance matrix is given by

$$\begin{aligned} \Sigma_{k+1} &\equiv E\{e_{k+1}e_{k+1}^T\} \\ &= \mathcal{A}\Sigma_k\mathcal{A}^T + \mathcal{B}\Pi_k\mathcal{B}^T - \widehat{K}_{1,k} \left(\mathcal{A}\Sigma_k\nu_1^T - \mathcal{B}\Pi_k\rho_1^T \right)^T \\ &\quad - \left(\mathcal{A}\Sigma_k\nu_1^T - \mathcal{B}\Pi_k\rho_1^T \right) \widehat{K}_{1,k}^T + \widehat{K}_{1,k} \left(\nu_1\Sigma_k\nu_1^T + \rho_1\Pi_k\rho_1^T \right) \widehat{K}_{1,k}^T \ . \end{aligned} \tag{4.25}$$

The matrix $\widehat{K}_{1,k}$ must be chosen to minimize the mean square error (or equivalently, the trace of the error covariance matrix). Recall from the definition of $\widehat{K}_k$ in (4.19) that $\widehat{K}_{1,k}$ will have $p - m$ columns. This means that there will be no freedom to minimize the mean square error if the number of outputs is equal to the number of unknown inputs. Taking the gradient of (4.25) with respect to $\widehat{K}_{1,k}$ and setting it equal to zero, we get the optimal gain to be

$$\widehat{K}_{1,k} = \left(\mathcal{A}\Sigma_k\nu_1^T - \mathcal{B}\Pi_k\rho_1^T\right)\left(\nu_1\Sigma_k\nu_1^T + \rho_1\Pi_k\rho_1^T\right)^{-1} \ . \tag{4.26}$$

Substituting this expression into (4.25), we get the optimal covariance update equation to be

$$\Sigma_{k+1} = \mathcal{A}\Sigma_k\mathcal{A}^T + \mathcal{B}\Pi_k\mathcal{B}^T - \left(\mathcal{A}\Sigma_k\nu_1^T - \mathcal{B}\Pi_k\rho_1^T\right)\left(\nu_1\Sigma_k\nu_1^T + \rho_1\Pi_k\rho_1^T\right)^{-1}\left(\mathcal{A}\Sigma_k\nu_1^T - \mathcal{B}\Pi_k\rho_1^T\right)^T \ . \tag{4.27}$$

The optimal gain and update equations in (4.26) and (4.27) require the calculation of a matrix inverse. If the inverse fails to exist, we can replace it with a pseudo-inverse [45]. The following theorem provides a sufficient condition for the existence of the inverse.

Theorem 4.5 *The matrix* $\left(\nu_1\Sigma_k\nu_1^T + \rho_1\Pi_k\rho_1^T\right)$ *is invertible if the matrix*

$$CQ_{k+\alpha-1}C^T + R_{k+\alpha}$$

is positive definite.

Proof: Let $\mathcal{N}_1 = \begin{bmatrix}\mathcal{N}_{11} & \mathcal{N}_{12}\end{bmatrix}$ represent the first $p-m$ rows of $\mathcal{N}$, where $\mathcal{N}_{12}$ has p columns. Using (4.22), (4.24) and the fact that $\mathcal{G}M_{n,\alpha} = \left[\begin{smallmatrix}0 & 0\\ C & I_p\end{smallmatrix}\right]$ we get

$$\rho_1\Pi_k\rho_1^T = \mathcal{N}_{12}\left(CQ_{k+\alpha-1}C^T + R_{k+\alpha}\right)\mathcal{N}_{12}^T \ .$$

It remains to show that the matrix $\mathcal{N}_{12}$ is full row rank. If $\mathcal{N}_{12}$ is not full row rank, there exists a nonzero row vector $\mathbf{d}$ such that

$$\mathbf{d}\begin{bmatrix}\mathcal{N}_{11} & \mathcal{N}_{12}\end{bmatrix} = \begin{bmatrix}\mathbf{f} & 0\end{bmatrix},$$

for some $\mathbf{f}$. From (4.18), (4.11), and (1.4) we have

$$\mathcal{N}_{11}\Phi M_{\alpha-1} + \mathcal{N}_{12}C\zeta_{\alpha-1} = 0 \ .$$

Multiplying on the left by $\mathbf{d}$, we get

$$\mathbf{f}\Phi M_{\alpha-1} = 0 \ .$$

Since the matrix $\left[\begin{smallmatrix}\bar{\mathcal{P}}\\ \Phi\end{smallmatrix}\right]$ is nonsingular and $\bar{\mathcal{P}}$ is a basis for the left nullspace of $M_{\alpha-1}$, the matrix $\Phi M_{\alpha-1}$ is full row rank. This implies that $\mathbf{f} = 0$, which means that $\mathbf{d}$ must also be the zero vector (since $\mathcal{N}_1$ is full row rank). Therefore, $\mathcal{N}_{12}$ is full row rank, which means that $\rho_1 \Pi_k \rho_1^T$ is positive definite if the condition given in the theorem is true. Since $\nu_1 \Sigma_k \nu_1^T$ is positive semidefinite, the theorem is proved. ■

We can now obtain an estimate of the original system states as follows. Using (4.19), we get the estimator gain in (4.14) to be

$$K_k = \begin{bmatrix} \widehat{K}_{1,k} & \mathcal{H}\bar{B} \end{bmatrix} \mathcal{N} \ . \tag{4.28}$$

The estimator is initialized with initial state $z_0 = \mathcal{H}E\{\bar{x}_0\}$, which will ensure that e_0 will have an expected value of zero. The estimate of the original state vector in (4.1) is given by

$$\begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{T}^T \begin{bmatrix} \mathcal{P}Y_{k:k+\alpha} \\ z_k \end{bmatrix} \ , \tag{4.29}$$

and the estimation error in the transformed coordinates is given by

$$\hat{e}_{k+1} = \begin{bmatrix} 0 \\ e_{k+1} \end{bmatrix} \ ,$$

where e_{k+1} is defined in (4.15). The error for the augmented system in (4.7) can then be obtained as

$$\bar{e}_{k+1} = \mathcal{T}^T \hat{e}_{k+1} = \mathcal{H}^T e_{k+1} \ , \tag{4.30}$$

and the error covariance matrix of the original state vector is given by

$$\begin{aligned} \Sigma_{x,k+1} &= \begin{bmatrix} I_n & 0 \end{bmatrix} E\{\bar{e}_{k+1}\bar{e}_{k+1}^T\} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{H}^T \Sigma_{k+1} \mathcal{H} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \ , \end{aligned} \tag{4.31}$$

where Σ_{k+1} is given by (4.27). The trace of the above covariance matrix will be the mean square estimation error for the state vector in the original system. The initial error covariance matrix for the update equation (4.27) can also be obtained from (4.30) as

$$\Sigma_0 = \mathcal{H}\bar{\Sigma}_0\mathcal{H}^T \ ,$$

where $\bar{\Sigma}_0$ is the initial error covariance matrix for the augmented system in (4.7).

Remark: At this point, it is worth noting that the estimator considered in this section may not necessarily be of minimum dimension. As discussed earlier, there might be certain conditions under which one could construct an optimal estimator of smaller dimension (e.g., see [43]). For the commonly considered case of $\alpha = 1$, the dynamic portion of our estimator will have dimension $n + p - t$. If $p - t$ is much smaller than n, the extra complexity will not have a large effect. In particular, if $D = 0$, then by Theorem 4.3 we have $t = p$, and the dimension of the estimator will not increase. Interestingly, this corresponds to the case of Markovian output noise described in [45].

4.5 Design Procedure

We now summarize the steps that can be used in designing a delayed estimator for the system given in (4.1).

1. Find the smallest α such that $\text{rank}[M_\alpha] - \text{rank}\,[M_{\alpha-1}] = m$. If the condition is not satisfied for $\alpha = n - \text{nullity}\,[D] + 1$, it is not possible to obtain an unbiased estimate of the entire system state.

2. Construct the augmented system given in (4.7).

3. Choose $\mathcal{P} = \begin{bmatrix} \bar{\mathcal{P}} & 0 \end{bmatrix}$ and $\mathcal{H}$ so that $\mathcal{T} = \left[\begin{smallmatrix} \mathcal{P}\bar{\Theta}_\alpha \\ \mathcal{H} \end{smallmatrix} \right]$ is orthogonal, and $\bar{\mathcal{P}}$ is a basis for the left nullspace of $M_{\alpha-1}$. Also choose Φ and form the matrix $\mathcal{G}$ given in (4.11).

4. Find the matrix $\mathcal{N}$ satisfying

$$\mathcal{N}\mathcal{G}M_\alpha = \begin{bmatrix} 0 & 0 \\ I_m & 0 \end{bmatrix} .$$

5. Form the matrices

$$\begin{bmatrix} L_1 & L_2 \end{bmatrix} = \mathcal{G}\bar{\Theta}_\alpha \mathcal{T}^T \ ,$$
$$\left[\begin{array}{c|c} \nu_1 & \rho_1 \\ \nu_2 & \rho_2 \end{array} \right] = \mathcal{N} \left[\begin{array}{c|c} L_2 & \mathcal{G}M_{n,\alpha} \end{array} \right] .$$

6. At each time-step k, calculate $\widehat{K}_{1,k}$ using Equation (4.26), and update the error covariance matrix using Equation (4.27).

7. Use (4.28) to obtain the estimator gain K_k.

8. The final estimator is given by Equation (4.14). The estimate of the original state vector is given by (4.29), with error covariance matrix given by (4.31).

4.6 Examples

4.6.1 Example 1

Consider the system given by the matrices

$$A = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$Q_k = 0.01 I_2, \quad R_k = 0.04 I_2 \ .$$

It is found that condition (4.6) holds for $\alpha = 2$, so our estimator must have a minimum delay of two time-steps. The state and noise vectors in the augmented system will be

$$\bar{x}_k = \begin{bmatrix} x_k \\ w_k \\ v_k \\ v_{k+1} \end{bmatrix}, \quad n_k = \begin{bmatrix} w_{k+1} \\ v_{k+2} \end{bmatrix} \ .$$

Using Theorem 1, we find $t = 2$ and choose

$$\bar{\mathcal{P}} = \begin{bmatrix} 0.64 & -0.34 & 0.34 & 0 \\ -0.36 & -0.34 & 0.34 & 0 \end{bmatrix},$$
$$\Phi = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$\mathcal{H} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.13 & 0 & 0.85 & -0.13 & 0.15 & -0.15 & 0 \\ 0.4 & -0.64 & 0 & -0.03 & 0.64 & 0.03 & -0.03 & 0 \\ -0.4 & -0.13 & 0 & 0.15 & 0.13 & 0.85 & 0.15 & 0 \\ 0.4 & 0.13 & 0 & -0.15 & -0.13 & 0.15 & 0.85 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

In this example, the last $\alpha m = 4$ columns of $\mathcal{G}M_\alpha$ have a rank of two, and thus, the matrix $\mathcal{N}$ in (4.18) will only have two rows:

$$\mathcal{N} = \begin{bmatrix} -0.9 & 1 & -1 & 0 \\ 0.5 & 0 & 0 & 0 \end{bmatrix} .$$

Thus, Equation (4.20) becomes

$$\begin{bmatrix} \hat{K}_{1,k} & \hat{K}_{2,k} \end{bmatrix} \begin{bmatrix} I_2 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{H}\bar{B} & 0 \end{bmatrix} ,$$

and since $\widehat{K}_{2,k}$ has $m = 2$ columns, $\widehat{K}_{1,k}$ is the empty matrix. This implies that we will have no freedom to minimize the trace of the covariance matrix.

From (4.12), we have

$$L_1 = \begin{bmatrix} 0.4367 & -0.7633 \\ 0.7429 & -0.4571 \\ 0.0826 & 0.0426 \\ 0.5286 & 0.1886 \end{bmatrix} ,$$

$$L_2 = \begin{bmatrix} 0 & 1.4538 & -0.3759 & 0.5462 & 1.4538 & 0 \\ 1 & 1.0545 & -0.7558 & -0.0545 & 0.0545 & 1 \\ 0 & 0.1754 & -0.0316 & 0.0246 & -0.0246 & 0 \\ 0.1 & 1.0699 & -0.2493 & 0.1301 & -0.1301 & 0 \end{bmatrix} ,$$

and we use Equations (4.21) and (4.22) to obtain

$$\begin{aligned} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} &= \mathcal{N} L_2 \\ &= \begin{bmatrix} 1 & -0.43 & -0.39 & -0.57 & -1.23 & 1 \\ 0 & 0.73 & -0.19 & 0.27 & 0.73 & 0 \end{bmatrix} , \\ \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} &= \mathcal{N} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \end{aligned}$$

Again, since ν_2 and ρ_2 have $m = 2$ rows, ν_1 and ρ_1 are empty matrices. Using (4.28), we get

$$K_k = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.1108 & 0.4436 & -0.4436 & 0 \\ -0.4888 & 0.4221 & -0.4221 & 0 \\ 0.1108 & -0.4436 & 0.4436 & 0 \\ -0.1108 & 0.4436 & -0.4436 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

The final estimator is given by (4.14) and an estimate of the original system states can be obtained via Equation (4.29). To test this estimator, the system is initialized with a random state with zero mean and covariance matrix I_2. A set of sinusoidal signals is used as the unknown inputs to the system. The estimator is initialized with an initial state of zero, and the resulting estimates are shown in Figure 4.1. The error covariance matrix of the system state is obtained from (4.31), and converges to

$$\Sigma_x = \begin{bmatrix} 0.1156 & -0.0320 \\ -0.0320 & 0.0400 \end{bmatrix} ,$$

which has a trace of 0.1556. The convergence of the trace (mean square error) is shown in Figure 4.2.

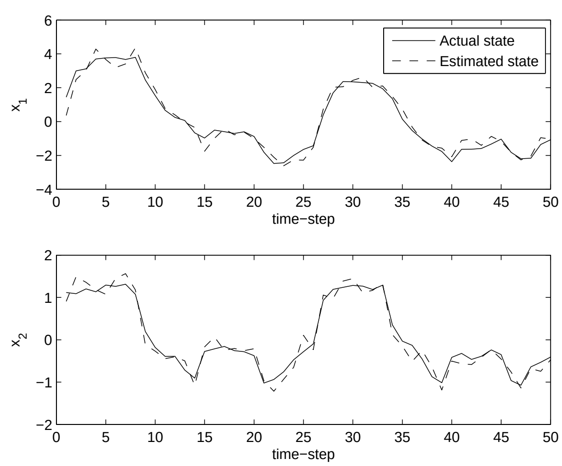

Figure 4.1: Simulation of system and estimator.

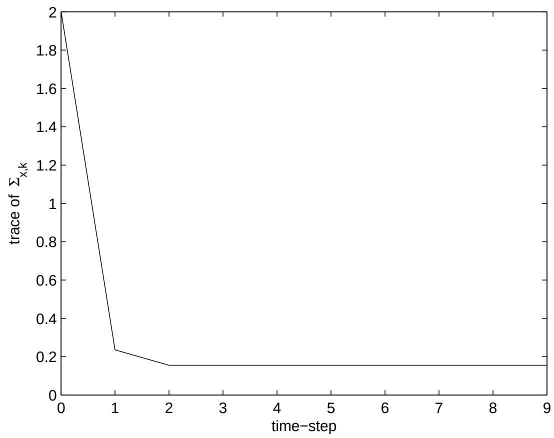

Figure 4.2: Mean square estimation error for the system state.

4.6.2 Example 2

Consider the following example from [44].

$$A = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$Q_k = 0.01 I_4, \;\; R_k = 0.01 I_2 \;\; .$$

Once again, we find that (4.6) is satisfied for $\alpha = 2$. Following the procedure outlined in this paper, we find that the error covariance matrix converges to

$$\Sigma_x = \begin{bmatrix} 0.0100 & 0.0029 & 0.0100 & 0.0029 \\ 0.0029 & 0.1359 & -0.0164 & 0.1038 \\ 0.0100 & -0.0164 & 0.0324 & 0.0061 \\ 0.0029 & 0.1038 & 0.0061 & 0.1041 \end{bmatrix},$$

which has a trace of 0.2824. In contrast, the trace of the error covariance matrix for the estimator constructed in [44] converges to 0.3778, which is approximately 34% worse than the performance achieved by our estimator. This discrepancy arises from the fact that the procedure in [44] ignores the correlation between the error and the noise, which leads to an incorrect covariance update equation.

4.7 Summary

We have provided a characterization of linear minimum-variance unbiased estimators for linear systems with unknown inputs, and have provided a design procedure to obtain the estimator parameters. We have shown that it will generally be necessary to use delayed measurements in order to obtain an unbiased estimate, and that these delays cause the system noise to become colored. We increase the dimension of our estimator in order to handle this colored noise.

As pointed out earlier, the estimators proposed in this chapter may not be of minimum dimension. An investigation of minimum dimension optimal estimators may provide an interesting direction for future research.

CHAPTER 5

SUMMARY AND CONCLUSIONS

In this thesis, we have considered the problem of constructing state and input observers for linear systems with unknown inputs. We started by studying the problem of observing the entire state and input, and developed a streamlined design procedure to obtain the observer parameters. Our approach is quite general in that it treats both reduced and full-order observers by selecting the design matrices appropriately. We then provided a characterization of partial state observers for linear systems with unknown inputs. Our approach involves an iterative procedure which decouples the estimator error from the values of the state and unknown inputs. The overall procedure produces a characterization of all possible linear functionals of the state that can be observed, along with the observer parameters. We have also used the partial state observer to determine all possible input functionals which can be observed from the output, and have constructed a partial input observer which produces these functionals. Finally, we studied linear minimum-variance unbiased state estimators for linear systems with unknown inputs. We have shown that it will generally be necessary to use delays in order to construct an unbiased estimator, which causes the system noise to behave as if it were colored. To handle this, we increased the dimension of the estimator appropriately. The observers described in this thesis can be used in applications such as fault detection and diagnosis, and control system design for uncertain systems.

There are some interesting directions for future research. One logical extension of our work is an investigation of unknown input observers for hybrid and switched linear systems. A zero-delay observer has recently been developed for switched systems [47], and we expect to develop delayed observers for such systems by using the ideas from our thesis. Another topic meriting further investigation is that of minimum dimension optimal estimators. As pointed out earlier, the estimators proposed in Chapter 4 may be larger than necessary. Therefore, an explicit characterization of the smallest dimension required for optimal estimation will further increase the attractiveness of our work.

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