Designing Stable Inverters and State Observers for Switched Linear Systems with Unknown Inputs

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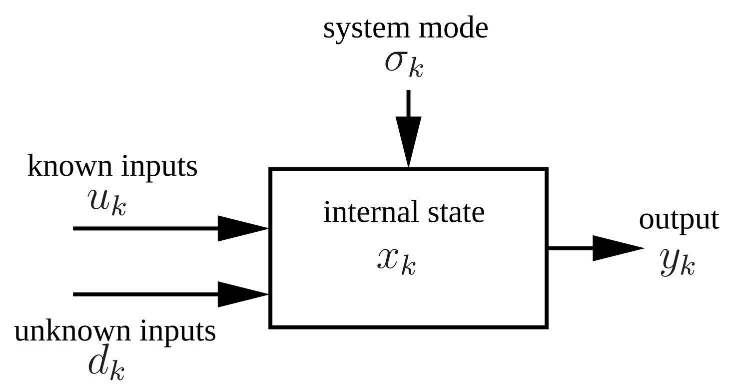
ABSTRACT

We present a method for estimating the inputs and states in discrete-time switched linear systems with unknown inputs. We first investigate the problem of system invertibility, which reconstructs the unknown inputs based on knowledge of the output of the system, the switching sequence, and the initial system state. We then relax the assumption on the knowledge of the initial system state, and construct observers that asymptotically estimate the state. Our design, which considers a general class of switched linear observers that switch modes based on the known (but arbitrary) switching sequence, shows that system invertibility is necessary in order to construct state observers. Furthermore, some portion of the observer gain must be used to recover the unknown inputs, and the remaining freedom must be used to ensure stability. The state of the observer is then used to asymptotically estimate the unknown inputs (i.e., it forms the dynamic portion of a stable inverter for the given switched system).

1. Overview

Many physical systems can be modeled as having

- System mode: $\sigma_k \in \Omega = \{1, 2, \ldots, N\}$
  - Specifies the dynamics of the system from N possibilities at time-step k
- Known inputs for each mode: $u_k \in \mathbb{R}^{q_{\sigma_k}}$
  - Control signals, etc.
- Unknown inputs for each mode: $d_k \in \mathbb{R}^{m_{\sigma_k}}$
  - Disturbances, faults, modeling uncertainties, etc.
- Internal state: $x_k \in \mathbb{R}^n$
  - Evolution: $x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k + E_{\sigma_k} d_k$
- Outputs: $y_k \in \mathbb{R}^p$
  - At time-step k: $y_k = C_{\sigma_k} x_k + D_{\sigma_k} u_k + F_{\sigma_k} d_k$



Known inputs can be easily handled in our analysis, so we drop them in rest of discussion

2. Objectives

Using the output of the system and the switching sequence:

- Estimate the unknown inputs d_k at each time-step
- Estimate the state of the system x_k at each time-step

3. System Invertibility

System is *invertible with delay* α if, at each time-step $k+\alpha$ $(k = 0, 1, \ldots)$, it is possible to uniquely recover the components of the unknown input d_k from the outputs of the system $y_0, y_1, \ldots, y_k, y_{k+1}, \ldots, y_{k+\alpha}$, the switching sequence $\sigma_0, \sigma_1, \ldots, \sigma_k, \sigma_{k+1}, \ldots, \sigma_{k+\alpha}$, and knowledge of the value of the state x_0.

3.1 Conditions for Invertibility

System output over $\alpha + 1$ time-steps is given by

$$\mathbf{y}_{k:k+\alpha} = \Theta_{\sigma_{k:k+\alpha}} x_k + \mathbf{M}_{\sigma_{k:k+\alpha}} \mathbf{d}_{k:k+\alpha} \tag{1}$$

where

$$\mathbf{y}_{k:k+\alpha} \triangleq \begin{bmatrix} y_k' & y_{k+1}' & \cdots & y_{k+\alpha}' \end{bmatrix}', \qquad \mathbf{d}_{k:k+\alpha} \triangleq \begin{bmatrix} d_k' & d_{k+1}' & \cdots & d_{k+\alpha}' \end{bmatrix}',$$

and $M_{\sigma_{k:k+\alpha}}$ and $\Theta_{\sigma_{k:k+\alpha}}$ are defined recursively as

$$\Theta_{\sigma_{k:k}} = C_{\sigma_k}, \quad M_{\sigma_{k:k}} = F_{\sigma_k},$$

$$\Theta_{\sigma_{k:k+\alpha}} = \begin{bmatrix} \Theta_{\sigma_{k:k+\alpha-1}} \\ C_{\sigma_{k+\alpha}} \prod_{j=0}^{\alpha-1} A_{\sigma_{k+j}} \end{bmatrix},$$

$$M_{\sigma_{k:k+\alpha}} = \begin{bmatrix} F_{\sigma_k} & 0 \\ \Theta_{\sigma_{k+1:k+\alpha}} E_{\sigma_k} & M_{\sigma_{k+1:k+\alpha}} \end{bmatrix}$$

Theorem 1:
System is invertible with delay α if and only if

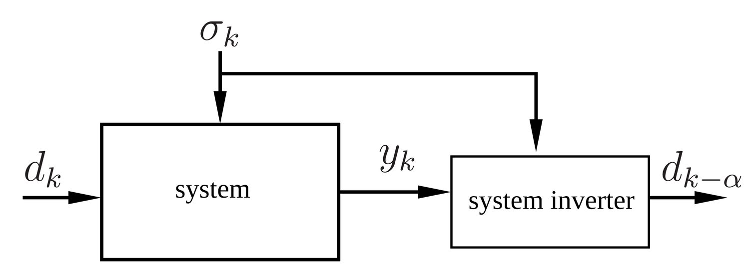
$$\text{rank}\left[M_{\sigma_{k:k+\alpha}}\right] - \text{rank}\left[M_{\sigma_{k+1:k+\alpha}}\right] = m_{\sigma_k}$$

for all switching paths $\sigma_{k:k+\alpha} \triangleq \{\sigma_k, \sigma_{k+1}, \ldots, \sigma_{k+\alpha}\}$, where $m_{\sigma_k} \triangleq \text{rank}\begin{bmatrix} E_{\sigma_k} \\ F_{\sigma_k} \end{bmatrix}$.

3.2 Construction of System Inverter

If condition in theorem is satisfied:

- There exists a matrix $\mathbf{R}_{\sigma_{k:k+\alpha}}$ for each switching sequence $\sigma_{k:k+\alpha}$ such that
$$\mathbf{R}_{\sigma_{k:k+\alpha}} \mathbf{M}_{\sigma_{k:k+\alpha}} = \begin{bmatrix} I_{m_{\sigma_k}} & 0 & \cdots & 0 \end{bmatrix}$$
- Multiply equation (1) by $\mathbf{R}_{\sigma_{k:k+\alpha}}$ to get
$$d_k = -\mathbf{R}_{\sigma_{k:k+\alpha}} \Theta_{\sigma_{k:k+\alpha}} x_k + \mathbf{R}_{\sigma_{k:k+\alpha}} \mathbf{y}_{k:k+\alpha}$$
- Let $\hat{d}_k$ and $\hat{x}_k$ be estimate of unknown input and system state, respectively:
$$\hat{d}_k = -\mathbf{R}_{\sigma_{k:k+\alpha}} \Theta_{\sigma_{k:k+\alpha}} \hat{x}_k + \mathbf{R}_{\sigma_{k:k+\alpha}} \mathbf{y}_{k:k+\alpha} \tag{2}$$
$$\hat{x}_{k+1} = A_{\sigma_k} \hat{x}_k + E_{\sigma_k} \hat{d}_k \tag{3}$$
- If we set $\hat{x}_0 = x_0$, then $\hat{x}_k = x_k$ and $\hat{d}_k = d_k$ for all k
- Equations (2) and (3) form the system inverter (with delay α)



4. State Estimation

- System inverter requires knowledge of initial state x_0
- If state is unknown, estimate it by constructing observer of the form
$$\hat{x}_{k+1} = J_{\sigma_{k:k+\alpha}} \hat{x}_k + K_{\sigma_{k:k+\alpha}} \mathbf{y}_{k:k+\alpha} \tag{4}$$
- Choose $J_{\sigma_{k:k+\alpha}}$ and $K_{\sigma_{k:k+\alpha}}$ so that $\hat{x}_k \to x_k$ as $k \to \infty$

4.1 Estimation Error Dynamics

- Examine estimation error
$$\begin{aligned} e_{k+1} &\triangleq \hat{x}_{k+1} - x_{k+1} \\ &= J_{\sigma_{k:k+\alpha}} e_k + \left(J_{\sigma_{k:k+\alpha}} - A_{\sigma_k} + K_{\sigma_{k:k+\alpha}} \Theta_{\sigma_{k:k+\alpha}}\right) x_k \\ &\quad + \left(K_{\sigma_{k:k+\alpha}} M_{\sigma_{k:k+\alpha}} - \begin{bmatrix} E_{\sigma_k} & 0 & \cdots & 0 \end{bmatrix}\right) \mathbf{d}_{k:k+\alpha} \end{aligned}$$
- Decouple states and unknown inputs from estimation error:
$$J_{\sigma_{k:k+\alpha}} = A_{\sigma_k} - K_{\sigma_{k:k+\alpha}} \Theta_{\sigma_{k:k+\alpha}} \tag{5}$$
$$K_{\sigma_{k:k+\alpha}} M_{\sigma_{k:k+\alpha}} = \begin{bmatrix} E_{\sigma_k} & 0 & \cdots & 0 \end{bmatrix} \tag{6}$$
- After decoupling, switched system $e_{k+1} = J_{\sigma_{k:k+\alpha}} e_k$ must be globally uniformly asymptotically stable (GUAS) to force $e_k \to 0$

4.2 Choosing Observer Gains

Theorem 2:
There exists $K_{\sigma_{k:k+\alpha}}$ satisfying equation (6) if and only if

$$\text{rank}\left[M_{\sigma_{k:k+\alpha}}\right] - \text{rank}\left[M_{\sigma_{k+1:k+\alpha}}\right] = m_{\sigma_k}$$

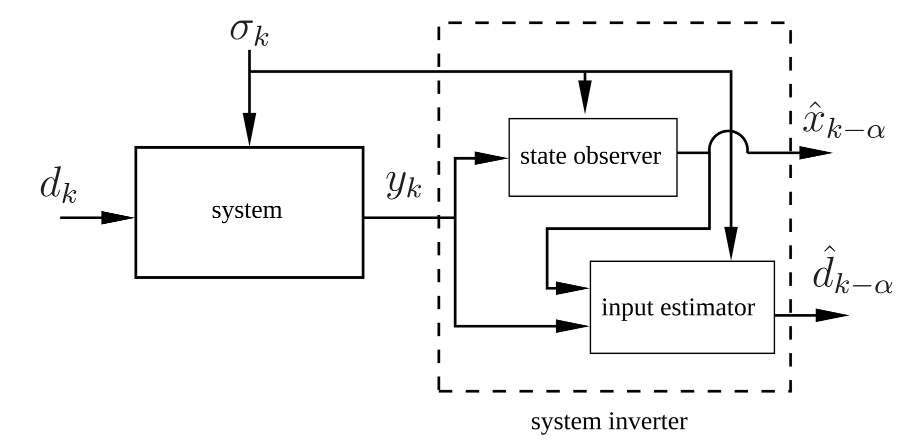
for all switching paths $\sigma_{k:k+\alpha} \triangleq \{\sigma_k, \sigma_{k+1}, \ldots, \sigma_{k+\alpha}\}$, where $m_{\sigma_k} \triangleq \text{rank}\begin{bmatrix} E_{\sigma_k} \\ F_{\sigma_k} \end{bmatrix}$.

- Comparing to Theorem 1, we see system must be invertible with delay α in order to construct state observer
- If condition is satisfied, solution to (6) can be parameterized as
$$K_{\sigma_{k:k+\alpha}} = \begin{bmatrix} L_{\sigma_{k:k+\alpha}} & E_{\sigma_k} \end{bmatrix} \mathcal{N}_{\sigma_{k:k+\alpha}} \tag{7}$$
for some constant matrices $\mathcal{N}_{\sigma_{k:k+\alpha}}$, and free matrices $L_{\sigma_{k:k+\alpha}}$
- Substitute into state decoupling equation (5) to get
$$\begin{aligned} e_{k+1} &= J_{\sigma_{k:k+\alpha}} e_k \\ &= \left(\mathcal{A}_{\sigma_{k:k+\alpha}} - L_{\sigma_{k:k+\alpha}} \mathcal{C}_{\sigma_{k:k+\alpha}}\right) e_k \end{aligned}$$
for some constant matrices $\mathcal{A}_{\sigma_{k:k+\alpha}}$ and $\mathcal{C}_{\sigma_{k:k+\alpha}}$
- Choose $L_{\sigma_{k:k+\alpha}}$ to obtain stability of e_k under arbitrary switching
  - e.g., Lyapunov methods, LMIs, dwell time, etc.
- Final observer given by (4), with $J_{\sigma_{k:k+\alpha}}$ from (5) and $K_{\sigma_{k:k+\alpha}}$ from (7)

5. Stable Inversion

To obtain a stable inverter:

- Replace dynamic portion of system inverter (equation (3)) with state observer (equation (4))
- Since output of observer $\hat{x}_k \to x_k$ as $k \to \infty$, we have $\hat{d}_k \to d_k$ as $k \to \infty$ in equation (2)
- Resulting inverter asymptotically estimates unknown inputs without knowledge of initial state x_0
- Dynamics of inverter are same as dynamics of observer (i.e., given by matrices $J_{\sigma_{k:k+\alpha}}$)
- Inverter is stable, since matrices $J_{\sigma_{k:k+\alpha}}$ were chosen to make estimation error GUAS



Note: Estimates of state and input are delayed by α time-steps

6. Summary of Contributions

- Studied estimation of unknown inputs and states in switched linear systems
- Presented test for system invertibility, along with a construction for system inverter
- When initial state is unknown, constructed a state observer to asymptotically estimate states
  - Showed that system must be invertible in order to construct state observer
- Obtained stable inverter by using state observer as dynamic portion of inverter

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