Distributed Consensus and Linear Functional Calculation in Networks: An Observability Perspective

Shreyas Sundaram and Christoforos N. Hadjicostis

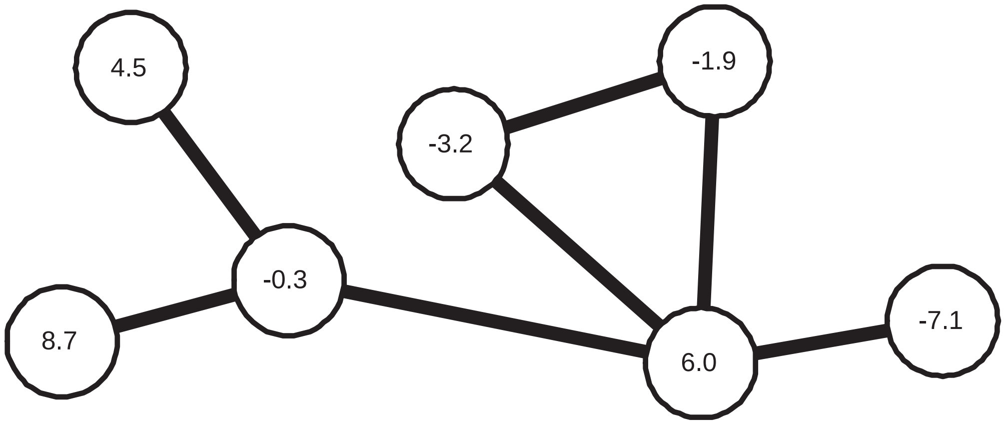
Coordinated Science Laboratory

and

Department of Electrical and Computer Engineering

University of Illinois at Urbana-Champaign

Overview

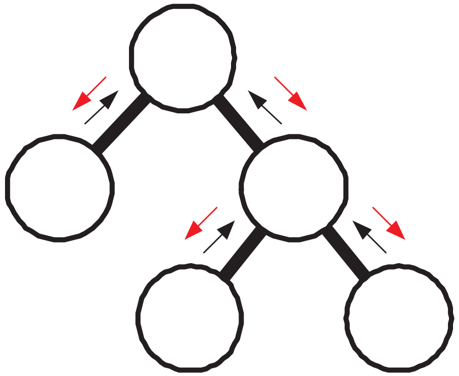


- Consider a network (directed or undirected) with nodes $\mathcal{X} = \{x_1, x_2, \ldots, x_N\}$
- Each node i has some initial value $x_i[0]$
  - e.g., temperature measurement, position, vote, fault status, etc.
- Each node can only receive information from its neighbors
- Objective: A subset of the nodes must calculate some function of the initial values
  - e.g., average, max, min, mode, etc.
  - **Consensus:** All nodes calculate the same function

Background on Consensus Protocols

Consensus problems studied for several decades (e.g., [Lynch, *Distributed Algorithms*])

- Simple solution: Flooding
  - Each node repeatedly sends all received values to neighbors
  - All nodes will know all initial values after several time-steps
- Another solution: “Convergecast”
  - Nodes organize themselves into a tree
  - Leaves propagate their values towards root
  - Root calculates function and broadcasts result back to leaves



- Many other protocols that deal with node/link faults, etc.

Linear Iterations for Consensus

- A different approach: Linear iterations
  - At each time-step, each node updates its value to be a linear combination of itself and its neighbors:

$$\underbrace{\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ \vdots \\ x_N[k+1] \end{bmatrix}}_{x[k+1]} = \underbrace{\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1N} \\ w_{21} & w_{22} & \cdots & w_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N1} & w_{N2} & \cdots & w_{NN} \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} x_1[k] \\ x_2[k] \\ \vdots \\ x_N[k] \end{bmatrix}}_{x[k]}$$

    - $w_{ij} = 0$ if x_j is not a neighbor of x_i
  - Choose weight matrix W so that $\lim_{k\to\infty} x[k] = \mathbf{1}\mathbf{c}'x[0]$
    - $\mathbf{1}$ is column vector with all 1's, and $\mathbf{c}'$ is some row vector
    - Special case: $\mathbf{c}' = \frac{1}{N}\mathbf{1}'$ (distributed averaging)
- Substantial analysis of convergence conditions available in the literature (e.g., see survey paper by Olfati-Saber, Proc. IEEE, Jan. 2007)

Convergence in Time-Invariant Graphs

- Linear iteration: $x[k+1] = Wx[k]$
- Necessary and sufficient conditions for $x[k] \rightarrow \mathbf{1}\mathbf{c}'x[0]$ [Xiao & Boyd, 2004]:
  - $W\mathbf{1} = \mathbf{1}$
  - $\mathbf{c}'W = \mathbf{c}'$
  - All other eigenvalues of W must have magnitude strictly less than 1
- Rate of convergence given by second largest eigenvalue of W
  - For faster convergence, minimize second largest eigenvalue by choosing weights appropriately [Xiao & Boyd, 2004]
- However, almost all existing methods only consider **asymptotic** convergence

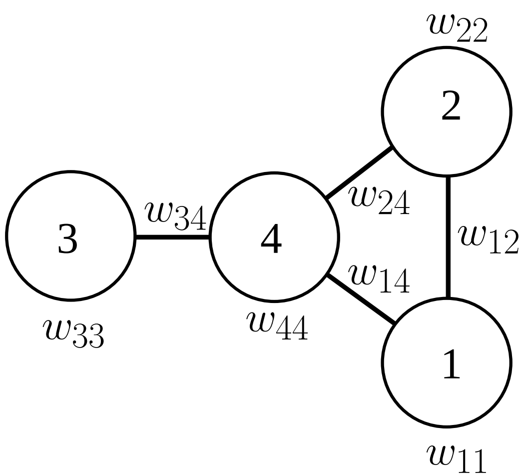
Contribution: We analyze linear iteration schemes using **observability theory**

- Allows nodes to reach consensus in **finite-time**

Modeling the Values Seen by Each Node

- Each node receives the values of its neighbors at each time-step
- Let $y_i[k] = E_i x[k]$ denote values received by node i at time-step k
  - Rows of E_i index portions of state-vector $x[k]$ that are available to node i
- e.g., for following graph,

$$x[k+1] = \begin{bmatrix} w_{11} & w_{12} & 0 & w_{14} \\ w_{12} & w_{22} & 0 & w_{24} \\ 0 & 0 & w_{33} & w_{34} \\ w_{14} & w_{24} & w_{34} & w_{44} \end{bmatrix} x[k], \quad y_2[k] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x[k]$$



The Observability Matrix

- Since $x[k] = W^k x[0]$, we have $y_i[k] = E_i x[k] = E_i W^k x[0]$
- Set of values seen by node i over $L+1$ time-steps is

$$\begin{bmatrix} y_i[0] \\ y_i[1] \\ y_i[2] \\ \vdots \\ y_i[L] \end{bmatrix} = \underbrace{\begin{bmatrix} E_i \\ E_i W \\ E_i W^2 \\ \vdots \\ E_i W^L \end{bmatrix}}_{\mathcal{O}_{i,L}} x[0]$$

- $\mathcal{O}_{i,L}$ is the **observability matrix** for the pair (W, E_i)
- Row-space of $\mathcal{O}_{i,L}$ characterizes all linear functionals that can be calculated by node i after $L+1$ time-steps

Calculating the Consensus Value

- Assume W is designed so that $\mathbf{c}'$ is in the row-space of the observability matrix for every node
- For each node i, find smallest L_i such that rank $\begin{bmatrix} \mathcal{O}_{i,L_i} \\ \mathbf{c}' \end{bmatrix} = \text{rank}\,[\mathcal{O}_{i,L_i}]$
- Find a vector Γ_i' for each i satisfying $\Gamma_i' \mathcal{O}_{i,L_i} = \mathbf{c}'$

Protocol:

- Nodes run linear iteration for $\max_i L_i + 1$ time-steps
- Each node i calculates $\mathbf{c}' x[0]$ after $L_i + 1$ time-steps as

$$\Gamma_i' \begin{bmatrix} y_i[0] \\ y_i[2] \\ \vdots \\ y_i[L_i] \end{bmatrix} = \Gamma_i' \mathcal{O}_{i,L_i} x[0] = \mathbf{c}' x[0]$$

The Observability Index

From observability theory:

- There exists a positive integer ν_i such that

  $$\text{rank}(\mathcal{O}_{i,0}) < \text{rank}(\mathcal{O}_{i,1}) < \cdots < \text{rank}(\mathcal{O}_{i,\nu_i-1}) = \text{rank}(\mathcal{O}_{i,\nu_i}) = \text{rank}(\mathcal{O}_{i,\nu_i+1}) = \cdots$$

- Rank of $\mathcal{O}_{i,L}$ increases until $L = \nu_i - 1$, and then stops increasing
- ν_i is known as the **observability index**

Implication for linear iteration model:

- Values seen by node i after time-step $\nu_i - 1$ are linear combinations of $y_i[0], y_i[1], \ldots, y_i[\nu_i - 1]$
- If node i can calculate $\mathbf{c}'x[0]$, it will require at most ν_i time-steps
- We show ν_i is upper-bounded by

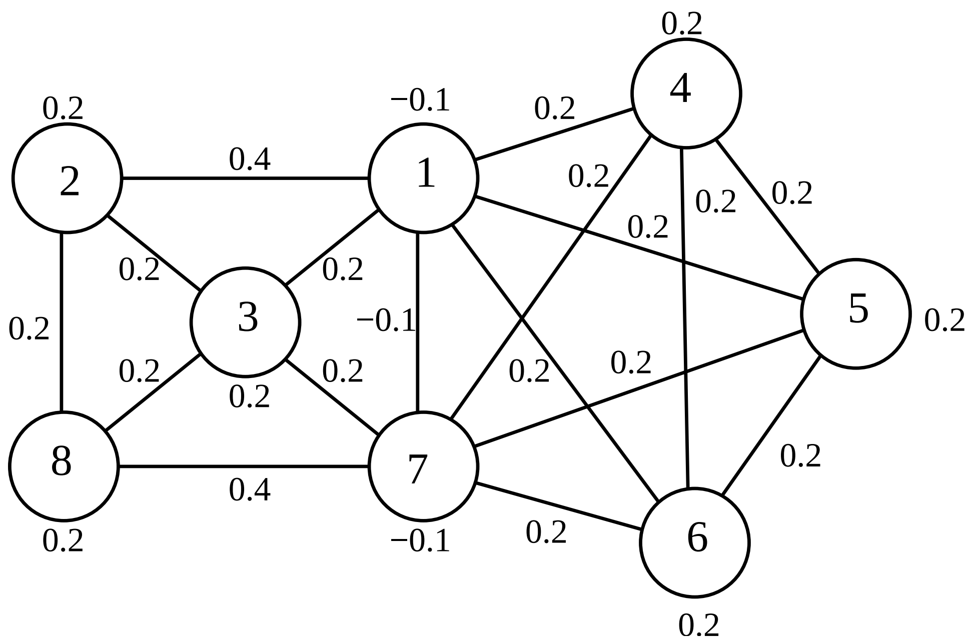
$$\nu_i \leq N - \text{degree}(i)$$

Row-Space of the Observability Matrix

- How do we choose the weight matrix so that $\mathbf{c}'$ is in the row-space of $\mathcal{O}_{i,\nu_i-1}$?
- For the linear iteration model, we show:
  - If μ is a simple eigenvalue of W, with right eigenvector $\mathbf{d}$ that has all entries nonzero, and left-eigenvector $\mathbf{c}'$, then $\mathbf{c}'$ is in the row-space of $\mathcal{O}_{i,\nu_i-1}$ for all i

- Recall: need $W\mathbf{1} = \mathbf{1}$ for asymptotic consensus
  - Any matrix that provides asymptotic consensus also allows finite-time consensus
- Do not need to restrict attention to matrices that provide asymptotic consensus
  - **Perron-Frobenius Theorem:**
    If W is nonnegative and the graph is strongly connected, then W has a simple eigenvalue μ with left and right-eigenvectors that have all positive entries.

Example: The Network and Weights



- Consider example from [Xiao & Boyd, 2004]
- Weights on edges and nodes chosen to maximize asymptotic rate of convergence
- W has simple eigenvalue 1, with $W\mathbf{1} = \mathbf{1}$, $\frac{1}{8}\mathbf{1}'W = \frac{1}{8}\mathbf{1}'$
  - Vector $\mathbf{c}' = \frac{1}{8}\mathbf{1}'$ will be in the row-space of observability matrix for every node

Example: Finding the Coefficient Vector

- Consider node 2 in the network

$$y_2[k] = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{E_2} x[k]$$

- Find smallest L_2 such that rank $\begin{bmatrix} \mathcal{O}_{2,L_2} \\ \frac{1}{8}\mathbf{1}' \end{bmatrix} = \text{rank}[\mathcal{O}_{2,L_2}]$
  - $L_2 = 0$: $\mathcal{O}_{2,0} = E_2$
  - $L_2 = 1$: $\mathcal{O}_{2,1} = \begin{bmatrix} E_2 \\ E_2 W \end{bmatrix}$ $\checkmark$
- Node 2 can calculate $\mathbf{c}'x[0]$ after two time-steps
- Calculate $\Gamma_2' =$

$$\begin{bmatrix} -0.2679 & -0.4464 & -0.3214 & -0.1964 & 0.6250 & -0.2098 & 2.6965 & -0.8795 \end{bmatrix}$$

Example: Running the Linear Iteration

- In this example, $L_i = 1$ for all nodes and consensus can be reached in 2 time-steps
- Suppose initial values of the nodes are

$$x[0] = \begin{bmatrix} -1.3256 & -13.6558 & 4.2533 & 5.8768 - 8.4647 & 14.9092 & 14.8916 & 2.6237 \end{bmatrix}'$$

  with mean 2.3885
- Run iteration $x[k+1] = Wx[k]$ for 2 time-steps:

$$\begin{bmatrix} x[0] & x[1] \end{bmatrix} = \begin{bmatrix} -1.3256 & -3.5040 \\ -13.6558 & -1.8860 \\ 4.2533 & 1.3574 \\ 5.8768 & 5.1774 \\ -8.4647 & 5.1774 \\ 14.9092 & 5.1774 \\ 14.8916 & 3.0078 \\ 2.6237 & 4.6009 \end{bmatrix}$$

Example: Calculating the Consensus Value

- Values seen by node 2:

$$y_2[0] = \begin{bmatrix} -1.3256 \\ -13.6558 \\ 4.2533 \\ 2.6237 \end{bmatrix}, \quad y_2[1] = \begin{bmatrix} -3.5040 \\ -1.8860 \\ 1.3574 \\ 4.6009 \end{bmatrix}$$

- Node 2 calculates $\mathbf{c}'x[0]$ as $\Gamma_2' \begin{bmatrix} y_2[0] \\ y_2[1] \end{bmatrix} = 2.3885$
- All other nodes calculate $\mathbf{c}'x[0]$ in the same manner
- Note: diameter and radius of network are both 2, so this scheme is time-optimal for consensus in this example

Finding the Coefficients: Decentralized Method

- Nodes require coefficient vectors Γ_i' to calculate $\mathbf{c}'x[0]$
- If W is not known to any node, can the nodes calculate their coefficients in a decentralized manner?
- Strategy: reconstruct observability matrix by running linear iterations with appropriate choices of initial conditions

Algorithm:

- Nodes run N different linear iterations, each for $N-1$ time-steps
  - For the j'th run, node j sets initial value to be 1 and others set initial value to be 0
  - Denote values seen by node i during time-step k of j'th run as $y_{i,j}[k]$

Finding the Coefficients: Decentralized Method

- Values seen by node i during j'th run:

$$\begin{bmatrix} y_{i,j}[0] \\ y_{i,j}[1] \\ \vdots \\ y_{i,j}[L_i] \end{bmatrix} = \mathcal{O}_{i,L_i} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

- After all runs are completed, node i has access to matrix

$$\begin{bmatrix} y_{i,1}[0] & y_{i,2}[0] & \cdots & y_{i,N}[0] \\ y_{i,1}[1] & y_{i,2}[1] & \cdots & y_{i,N}[1] \\ \vdots & \vdots & \ddots & \vdots \\ y_{i,1}[L_i] & y_{i,2}[L_i] & \cdots & y_{i,N}[L_i] \end{bmatrix} = \mathcal{O}_{i,L_i} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \mathcal{O}_{i,L_i}$$

- Node i can now calculate Γ_i' as before

Summary and Future Work

Summary:

- Used observability theory to study problem of distributed consensus and linear functional calculation
- Can obtain distributed consensus after running linear iteration for a finite number of time-steps
- Method does not depend on magnitudes of eigenvalues of weight matrix
- Nodes can learn their coefficients after running N linear iterations

Future Work:

- Choose weights to minimize number of time-steps for consensus
- Incorporate robustness to link/node faults
- Find more efficient methods to perform decentralized computation of coefficients