Spectral and Structural Properties of Random Interdependent Networks [*]

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Abstract

Random interdependent networks consist of a group of subnetworks where each edge between two different subnetworks is formed independently with probability p. In this paper, we investigate certain spectral and structural properties of such networks, with corresponding implications for certain variants of consensus and diffusion dynamics on those networks. We start by providing a characterization of the isoperimetric constant in terms of the inter-network edge formation probability p. We then analyze the algebraic connectivity of such networks, and provide an asymptotically tight rate of growth of this quantity for a certain range of inter-network edge formation probabilities. Next, we give bounds on the smallest eigenvalue of the grounded Laplacian matrix (obtained by removing certain rows and columns of the Laplacian matrix) of random interdependent networks for the case where the removed rows and columns correspond to one of the subnetworks. Finally, we study a property known as r-robustness, which is a strong indicator of the ability of a network to tolerate structural perturbations and dynamical attacks. Our results yield new insights into the structure and robustness properties of random interdependent networks.

Key words: Isoperimetric Constant; Robustness; Algebraic Connectivity; Grounded Laplacian Matrix; Random Interdependent Networks.

1 Introduction

The ever-increasing sophistication of today's large-scale engineered systems requires the development of new techniques to understand the implications of their interconnections. Various studies have shown that the topological structure of interconnections between the components of a decentralized system plays a fundamental role in the system's functioning. For instance, the convergence rate to the rendezvous point in a multi-agent setting with consensus dynamics is directly related to the algebraic connectivity of the interconnection network among the agents [27]. The interconnection structure in a decentralized system also crucially affects the robustness of the system against dynamical attacks

or random failures [16, 32, 36, 37]. While the classical literature on the structure of networks has traditionally focused on monolithic networks, there is an increasing realization that many large-scale networks are really "networks-of-networks," consisting of interdependencies between different subnetworks [13, 18, 25, 30]. Examples of interconnected networks include cyber and physical networks [31], policy influence and knowledge exchange networks in organizations [35] and different communities of individuals in social networks joined together by "weak ties" [15]. In this paper, we contribute to the understanding of interdependent networks by studying certain spectral and structural properties of such networks, namely *edge expansion*, *r-robustness*, *algebraic connectivity* and the *smallest eigenvalue of the grounded Laplacian matrix*. In addition to their topological implications, these properties also play a key role in certain variants of diffusion dynamics on networks, as we describe later.

We consider the class of *random interdependent networks*

consisting of k subnetworks, where each edge between nodes in different subnetworks is present independently with a certain probability p. Our model is fairly general in that we make no assumption on the topologies within the subnetworks, and captures Erdős-Rényi graphs and random k-partite graphs as special cases. The graph theoretic notion of *isoperimetric constant* (also called Cheeger constant), denoted by $i(G)$, is the key property that we use to derive our results. Our main result characterizes a threshold p_r for random k-partite networks to have $i(G) > r - 1$ where r is a positive integer. Furthermore, we prove that p_r is *also* the threshold for the minimum degree of the network to be r. This is potentially surprising given that $i(G) > r-1$ is a significantly stronger graph property than r-minimum-degree.

We then focus on the algebraic connectivity (defined as the second smallest eigenvalue of the Laplacian matrix) of random interdependent networks [6,10]. We show that when the inter-network edge formation probability p satisfies a certain condition, the algebraic connectivity of the network grows at a certain rate, *regardless* of the topologies within the subnetworks. Given the key role of algebraic connectivity in the speed of consensus dynamics on networks [27], our analysis demonstrates the importance of the edges that connect different communities in the network in terms of facilitating information spreading, in line with classical findings in the sociology literature [15]. Our result on algebraic connectivity of random interdependent networks is also applicable to the *stochastic block model* or *planted partition model* that has been widely studied in the machine learning literature [1,9,34], where it is assumed that the intra-network edges are also placed randomly with a certain probability. We also study the case where there are *stubborn* (or leader) nodes which do not change their states in the consensus dynamics. We characterize the rate of convergence induced by certain sets of stubborn nodes when the probability of edge formation between a normal node and a stubborn node is sufficiently high.

Finally, we analyze a metric known as r-robustness of networks. In recent years, the robustness of interdependent networks to intentional disruption or natural malfunctions has started to attract attention by a variety of researchers [12, 31, 36]. As we will describe later, r-robustness has strong connotations for the ability of networks to withstand structural and dynamical disruptions: it guarantees that the network will remain connected even if up to $r - 1$ nodes are removed from the neighborhood of *every* node in the network, and facilitates certain consensus dynamics that are resilient to adversarial nodes [8, 19, 33, 37, 38]. We identify a bound p_r for the probability of inter-network edge formation p such that for $p > p_r$, random interdependent networks with arbitrary intra-network topologies are guaranteed to be r-robust asymptotically almost surely.

2 Definitions

An undirected graph (network) is denoted by $G = (V, E)$ where V is the set of vertices (or nodes) and $E \subseteq V \times V$ is the set of edges. We denote the set $\mathcal{N}_i = \{v_j \in V \mid (v_i, v_j) \in E\}$ as the *neighbors* of node $v_i \in V$ in graph G. The *degree* of node v_i is $d_i = |\mathcal{N}_i|$, and d_{min} and d_{max} are the minimum and maximum degrees of the nodes in the graph, respectively. A graph $G' = (V', E')$ is called a *subgraph* of $G = (V, E)$, denoted as $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E \cap \{V' \times V'\}$. A network G has node connectivity r if removal of any $r - 1$ (or fewer) nodes does not disconnect G. For an integer $k \in \mathbb{Z}_{\geq 2}$, a graph G is k-partite if its vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$ such that there are no edges between nodes within any of those sets.

The *adjacency matrix* for the graph $G = (V, E)$ is a matrix $A \in \{0, 1\}^{n \times n}$ whose (i, j) entry is 1 if $(v_i, v_j) \in E$, and zero otherwise. The *Laplacian matrix* for the graph is given by $L = D - A$, where D is the degree matrix with $D = \text{diag}(d_1, d_2, \ldots, d_n)$. The eigenvalues of the Laplacian are real and nonnegative, and are denoted by $0 = \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G)$.

3 Background and Application

We now introduce the structural properties that we will study in this paper, along with their implications for diffusion and consensus dynamics.

3.1 Isoperimetric Constant

The *edge-boundary* of a set of nodes $S \subset V$ is given by $\partial S = \{(v_i, v_j) \in E \mid v_i \in S, v_j \in V \setminus S\}$. The *isoperimetric constant* of G is defined as [3]

$$i(G) \triangleq \min_{S \subset V, |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}. \tag{1}$$

By choosing S as the vertex with the smallest degree we obtain $i(G) \leq d_{min}$. Our results on the isoperimetric constant of random interdependent networks (given in Section 5) are at the heart of many of the subsequent results that we provide in this paper.

3.2 Algebraic Connectivity

The second smallest eigenvalue of the Laplacian matrix $\lambda_2(G)$, is called the *algebraic connectivity* of the graph and is related to the isoperimetric constant by [3]

$$\frac{i(G)^2}{2d_{max}} \leq \lambda_2(G) \leq 2i(G). \tag{2}$$

Here, we focus on the application of algebraic connectivity in certain types of consensus (or diffusion) dynamics. Consider a multi-agent setting with n agents and interaction topology modeled by the graph $G = (V, E)$, where each node of G corresponds to an agent. There is an edge between two nodes in the graph G if their corresponding agents communicate or exchange information. Each agent $v_i \in V$ has an initial state $x_i(0) \in \mathbb{R}$ (an opinion, decision, measurement, etc.) which evolves over time as a function of the states of v_i's neighbors:

$$\dot{x}_i(t) = \sum_{v_j \in \mathcal{N}_i} (x_j(t) - x_i(t)).$$

The system-wide dynamics can then be represented by

$$\dot{X}(t) = -LX(t), \tag{3}$$

where $X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ denotes the vector of states of all of the nodes and L denotes the Laplacian matrix. When G is a connected graph, the state of all of the agents converges to $\mathbf{1}^T X(0)/n$ (the average of the initial values) and the asymptotic convergence rate is given by $\lambda_2(G)$ [5, 27]. We provide a tight bound for $\lambda_2(G)$ in Section 6 when the underlying graph belongs to the class of random interdependent networks.

3.3 Smallest Eigenvalue of the Grounded Laplacian

Next, consider the consensus setting with a group of agents $S \subset V$ who keep their states constant, i.e., $\forall v_s \in S, \exists x_s \in \mathbb{R}$ such that $x_s(t) = x_s, \forall t \geq 0$. Depending on the context, these agents are called *stubborn agents*, *zealots*, or *leaders* [14, 22, 29]. Let X_F and X_S denote the states of the follower and stubborn agents, respectively. Then the equation (3) can be written as

$$\begin{bmatrix} \dot{X}_F(t) \\ \dot{X}_S(t) \end{bmatrix} = - \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} X_F(t) \\ X_S(t) \end{bmatrix}. \tag{4}$$

Matrices L_{21} and L_{22} are both zero by the definition of stubborn agents. Matrix L_{11} is called the *grounded Laplacian* of the system and is denoted by $L_{11} = L_g(S)$; we drop the argument [1] S whenever it is clear from the context. It can be shown that the state of each follower agent asymptotically converges to a convex combination of the values of the stubborn agents with convergence rate given by $\lambda(L_g)$, the *smallest eigenvalue of the grounded Laplacian* [4]. In Section 7, we obtain a bound for $\lambda(L_g)$ when G is a random interdependent network.

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[1] We refer to S as the set of grounded nodes.

3.4 The Notion of r-Robustness

In this paper, we use the following metric known as r-*robustness* to study robustness of networks against structural and dynamical disruptions.

Definition 1 ([19]) *Let $r \in \mathbb{N}$. A subset S of nodes in the graph $G = (V, E)$ is said to be r-reachable if there exists a node $v_i \in S$ such that $|\mathcal{N}_i \setminus S| \geq r$. A graph $G = (V, E)$ is said to be r-robust if for every pair of nonempty, disjoint subsets $V_1, V_2 \subseteq V$, either V_1 or V_2 is r-reachable.*

Simply put, an r-reachable set contains a node that has r neighbors outside that set, and an r-robust graph has the property that no matter how one chooses two disjoint nonempty sets, at least one of those sets is r-reachable. The property of r-robustness is significantly stronger than the property of r-connectivity (an r-connected network is a network with node connectivity equal to r) or r-minimum degree; an r-robust graph will remain connected even after up to $r - 1$ nodes are removed from the neighborhood of *every* remaining node, while an r-connected graph will only guarantee connectedness after the removal of $r - 1$ nodes *in total* [19, 37]. Indeed, the gap between the robustness and node connectivity (and minimum degree) parameters can be arbitrarily large, as illustrated by the graph G in Fig. 1a. While the minimum degree and node connectivity of the graph G is $n/4$, it is only 1-robust (consider subsets $V_1 \cup V_2$ and $V_3 \cup V_4$).

The following result shows that the isoperimetric constant $i(G)$ provides a lower bound on the robustness parameter.

Lemma 1 *Let r be a positive integer. If $i(G) > r - 1$, then the graph is at least r-robust.*

PROOF. If $i(G) > r - 1$, then every set of nodes $S \subset V$ of size up to $\frac{n}{2}$ has at least $(r-1)|S|+1$ edges leaving that
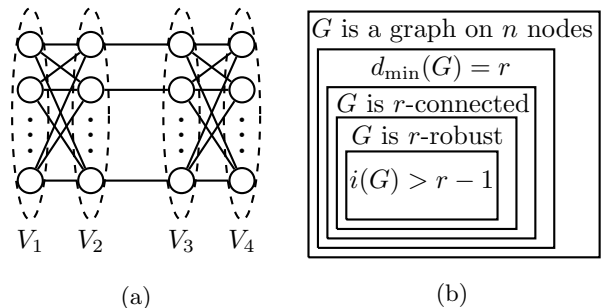

Fig. 1. (a) Graph $G = (V, E)$ with $V = V_1 \cup V_2 \cup V_3 \cup V_4$ and $|V_i| = \frac{n}{4}$, $1 \leq i \leq 4$. All of the nodes in V_1 (V_3) are connected to all of the nodes in V_2 (V_4). There is a one to one connection between nodes in V_2 and nodes in V_3. (b) Relationships between different topological metrics of robustness.

set (by the definition of $i(G)$). By the pigeonhole principle [17], at least one node in S has at least r neighbors outside S. Now for any two disjoint non-empty sets S_1 and S_2, at least one of these sets has size at most $\frac{n}{2}$, and thus is r-reachable. Therefore, the graph is r-robust. □

Note that together with (2), the above lemma implies that any graph is at least $\lceil \frac{\lambda_2(G)}{2} \rceil$-robust. As an example of Lemma 1, consider the graph G in Fig. 1a which has isoperimetric constant of at most 0.5 (since the edge boundary of $V_1 \cup V_2$ has size $n/4$), but is 1-robust. The relationships between these different graph-theoretic measures of robustness are summarized in Fig. 1b.

In order to see the application of r-robustness in consensus dynamics, consider the case where agents synchronously update their states according to the following filtering dynamics [19, 37]. Given $F \in \mathbb{N}$, at each time step, each node receives the values of its neighbors, disregards the largest F and the smallest F values (that are larger or smaller than its own value), and updates its state as

$$x_i[k+1] = w_{ii}[k]x_i[k] + \sum_{j \in \mathcal{R}_i[k]} w_{ij}[k]x_j[k], \quad (5)$$

where $\mathcal{R}_i[k]$ represents the set of nodes whose values were adopted by node i at time step k, and $w_{ii}[k]$ and $w_{ij}[k]$ are bounded away from zero and satisfy $\sum_{j \in \mathcal{R}_i[k]} w_{ij}[k] = 1, \forall i \in V, \ k \in \mathbb{Z}_{\geq 0}$. Then, consensus of the agents is guaranteed if and only if the underlying graph G is at least $(F+1)$-robust [19]. Furthermore, if there are up to F adversarial (arbitrarily behaving) nodes in the neighborhood of every normal node, then under these dynamics, all normal nodes will converge to consensus in the convex hull of the initial values of the normal nodes as long as the graph is $(2F+1)$-robust [19, 37].

4 Random Interdependent Networks

In this paper, we investigate the properties that we discussed in the last section for the class of *random interdependent networks*, which we define below.

Definition 2 *An interdependent network G is denoted by a tuple $G = (G_1, G_2, \ldots, G_k, G_p)$ where $G_l = (V_l, E_l)$ for $l = 1, 2, \ldots, k$ are called the subnetworks of the network G, and $G_p = (V_1 \cup V_2 \cup \ldots \cup V_k, E_p)$ is a k-partite network with $E_p \subseteq \cup_{l \neq t} V_l \times V_t$ specifying the interconnection (or inter-network) topology.*

For the rest of this paper, we assume that $|V_1| = |V_2| = \cdots = |V_k| = n$ and that the number of subnetworks k is at least 2.

Definition 3 *Define the probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, where the sample space Ω_n consists of all possible interdependent networks $(G_1, G_2, \ldots, G_k, G_p)$ and the index $n \in \mathbb{N}$ denotes the number of nodes in each subnetwork. The σ-algebra $\mathcal{F}_n$ is the power set of Ω_n and the probability measure $\mathbb{P}_n$ associates a probability $\mathbb{P}(G_1, G_2, \ldots, G_k, G_p)$ to each network $G = (G_1, G_2, \ldots, G_k, G_p)$. A random interdependent network is a network $G = (G_1, G_2, \ldots, G_k, G_p)$ drawn from Ω_n according to the given probability distribution.*

Note that deterministic structures for the subnetworks or interconnections can be obtained as a special case of the above definition where $\mathbb{P}(G_1, G_2, \ldots, G_k, G_p)$ is 0 for interdependent networks not containing those specific structures; for instance, a *random k-partite graph* is obtained by allocating a probability of 0 to interdependent networks where any of the G_l for $1 \leq l \leq k$ is nonempty. Through an abuse of notation, we will refer to random k-partite graphs simply by G_p in this paper. Similarly, Erdős-Rényi random graphs on kn nodes are obtained as a special case of the above definition by choosing the edges in $G_1, G_2, \ldots, G_k$ and G_p independently with a common probability p.

In this paper, we focus on the case where the k-partite network G_p is independent of G_l for $1 \leq l \leq k$. Specifically, we assume that each possible edge of G_p is present independently with probability p (possibly a function of n). We will characterize certain properties of such networks as $n \to \infty$, captured by the following definition.

Definition 4 *For a random interdependent network, we say that a property $\mathcal{P}$ holds asymptotically almost surely (a.a.s.) if the probability measure of the set of graphs with property $\mathcal{P}$ (over the probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$) tends to 1 as $n \to \infty$.*

Application of Consensus Dynamics on Random Interdependent Networks

Consider a society with multiple communities, where individuals have inter-community and intra-community links. Modeling the interaction among individuals by an interdependent network (where each subnetwork represents a community), the consensus (or opinion) dynamics described in the previous section and the set of properties that we will study later in this paper can be interpreted as follows.

- The case where individuals update their opinions by aggregating the opinions of all of their neighbors corresponds to the standard consensus dynamics given by equation (3). Then, the algebraic connectivity of the interdependent network determines how fast information spreads throughout the network.
- Equation (4) models the situation where one of the communities acts as a *leader community* (i.e., its mem-

bers keep their opinions fixed). In this case, the smallest eigenvalue of the grounded Laplacian matrix determines the speed at which the follower nodes converge to the steady state.

- Finally, the filtering dynamics in equation (5) generalize DeGroot opinion dynamics [7] by allowing each node to discard the F highest and F lowest opinions of their neighbors before averaging the rest [37]. The notion of r-robustness (for $r \in \mathbb{N}$) enables us to understand the ability of the network to facilitate consensus under such dynamics even when some individuals behave in an adversarial or erratic manner.

The following theorem describes the implications of our results for consensus dynamics that operate over random interdependent networks.

Theorem 2 *Let $G = (G_1, G_2, \ldots, G_k, G_p)$ be a random interdependent network with probability of inter-network edge formation p that satisfies the condition $\limsup_{n \to \infty} \frac{\ln n}{np} < 1$. Consider a multi-agent setting with consensus dynamics and interaction topology between the agents modeled by G.*

(1) If all nodes run the consensus dynamics (3), then the rate of convergence is given by $\Theta(np)$ a.a.s.

(2) Suppose that all of the nodes in $G_1 = (V_1, E_1)$ are leaders, i.e., $\forall v_i \in V_1, \exists x_i \in \mathbb{R}$ such that $x_i(t) = x_i, \ \forall t \geq 0$. Assuming that the rest of the nodes are followers, they converge to the steady state with convergence rate $\Theta(np)$ a.a.s.

(3) There exists $F \in O(np)$ (a.a.s) such that if the nodes update their states according to the filtering dynamics given in (5), they will converge to consensus, even if there are up to F adversarial nodes in the neighborhood of every normal node.

The proof of the above theorem follows from Theorems 6, 7 and 11 (proved in the rest of the paper) and the discussion in Section 3. Note that these results are independent of the topologies inside the subnetworks and thus illustrates the crucial role of inter-network edges on consensus dynamics that operate over random interdependent networks, in line with classical findings from the sociology literature [15].

5 Isoperimetric Constant of Random Interdependent Networks

This section provides three important results about the isoperimetric constant of random interdependent networks (which ties together the remainder of the results in the paper). We start with the following lemma, giving a threshold for the isoperimetric constant of a random k-partite network to be greater than $r - 1$. Our focus on k-partite networks in the next two results serves two purposes. First, by leveraging the knowledge that there are no edges between nodes within any subnetwork, we

are able to provide a *sharp* threshold for the isoperimetric constant of such networks (described later). Second, these results will allow us to subsequently provide a bound for the isoperimetric constant of general random interdependent networks (with arbitrary intra-network topologies) by leveraging the fact that adding edges to a graph does not decrease the isoperimetric constant.

Lemma 3 *Consider a random k-partite network G_p with inter-network edge formation probability $p = p(n)$. Let $x = x(n)$ be some function satisfying $x \to \infty$ as $n \to \infty$. Then for any positive integer r and $k \geq 2$,*

(1) If $p(n) = \frac{\ln n + (r-1) \ln \ln n + x(n)}{(k-1)n}$, then $i(G_p) > r - 1$ (and thus the minimum degree is at least r) a.a.s.

(2) If $p(n) = \frac{\ln n + (r-1) \ln \ln n - x(n)}{(k-1)n}$, then the minimum degree is at most $r - 1$ (and thus $i(G_p) \leq r - 1$) a.a.s.

The proof of the above lemma is provided in Appendix A. The above result shows that the function $t(n) = \frac{\ln n + (r-1) \ln \ln n}{(k-1)n}$ forms a (sharp) threshold for the property $i(G_p) > r - 1$, and also for the graph to have minimum degree r (a significantly weaker property). [2]

Lemma 3 provides the condition under which the isoperimetric constant is higher or lower than $r - 1$ (a constant value). Next, we will investigate a coarser rate of growth for the inter-network edge formation p, and show that for such probability functions, the isoperimetric constant scales as $\Theta(np)$. This will play a role in Sections 6 and 8, where we investigate algebraic connectivity and robustness of random interdependent networks.

Lemma 4 *Consider a random k-partite network G_p and assume that the inter-network edge formation probability p satisfies $\limsup_{n \to \infty} \frac{\ln n}{(k-1)np} < 1$. Fix any $\epsilon \in (0, \frac{1}{2}]$. Then there exists a constant α (that depends on p) such that the minimum degree d_{min}, maximum degree d_{max} and isoperimetric constant $i(G_p)$ a.a.s. satisfy*

$$\alpha np \leq i(G_p) \leq d_{min} \leq d_{max}$$
$$\leq n(k-1)p \left(1 + \sqrt{3} \left(\frac{\ln n}{(k-1)np} \right)^{\frac{1}{2} - \epsilon} \right). \quad (6)$$

The proof of the above lemma is omitted due to the space constraints, and can be found in [24].

So far in this section, we have been focused on random k-partite graphs. We now use these results to provide a

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[2] Loosely speaking, if $p(n)$ is "bigger" than $t(n)$ (in the sense specified by the lemma), then the stated properties a.a.s. hold and if $p(n)$ is "less" than $t(n)$, the properties a.a.s. do not hold.

bound for the isoperimetric constant of random interdependent graphs (with arbitrary topologies within the subnetworks).

Lemma 5 *Let $G = (G_1, G_2, \ldots, G_k, G_p)$ be a random interdependent network and assume that the probability of inter-network edge formation p satisfies $\limsup_{n \to \infty} \frac{\ln n}{(k-1)np} < 1$. Then $i(G) = \Theta(np)$.*

The proof of Lemma 5 is given in Appendix B. In the following sections, we build on these results to study the spectral and structural properties of random interdependent networks (with corresponding implications for consensus and diffusion dynamics that operate over these networks), as illustrated in Fig. 2.
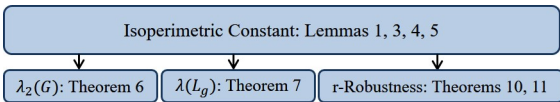

Fig. 2. We use the concept of isoperimetric constant to characterize other properties of random interdependent networks.

Remark 1 *The results that we developed in this section for isoperimetric constant of random interdependent and k-partite networks **only** depend on the inter-network edge formation probability p. This fact demonstrates the important role of inter-network edges on structural and spectral properties of such networks. Moreover, since our results in the rest of this paper are directly derived from Lemmas 3, 4 and 5, they are also **only** dependent on the inter-network edge formation probability p and are independent of intra-network topologies. A deeper investigation of the role of the subnetwork topologies would potentially lead to further refinements of our results, and is left as a venue for future work.*

Remark 2 *It is worth noting that Lemmas 4 and 5 generalize the results in [29] on the isoperimetric constant of Erdős-Rényi random networks. This is due to the fact that the class of random interdependent networks that we consider includes Erdős-Rényi random graphs as a special case where edges in $G_1, G_2, \cdots, G_k$ and G_p are formed independently with a fixed probability of p.*

6 Algebraic Connectivity of Random Interdependent Networks

The algebraic connectivity of interdependent networks has started to receive attention in recent years. The authors of [30] analyzed the algebraic connectivity of deterministic interconnected networks with one-to-one weighted symmetric inter-network connections. The recent paper [20] studied the algebraic connectivity of a mean field model of interdependent networks where each

subnetwork has an identical structure, and the interconnections are all-to-all with appropriately chosen weights. Spectral properties of random interdependent networks (under the moniker of *planted partition models*) have also been studied in research areas such as algorithms and machine learning [1, 9, 26]. Here, we leverage our results from the previous section to provide a bound on the algebraic connectivity for random interdependent networks.

Theorem 6 *Consider a random interdependent network $G = (G_1, G_2, \ldots, G_k, G_p)$ and assume that the probability of inter-network edge formation p satisfies $\limsup_{n \to \infty} \frac{\ln n}{(k-1)np} < 1$. Then $\lambda_2(G) = \Theta(np)$ a.a.s.*

PROOF. First note that by Lemma 5, there exists a constant $\gamma > 0$ such that $i(G) \leq \gamma np$ a.a.s. Hence from inequality (2), we have $\lambda_2(G) = O(np)$ a.a.s.

Next, we prove the lower bound on $\lambda_2(G)$. Consider the k-partite subgraph G_p of the network G. By Lemma 4 and the inequality (2), we know that $\lambda_2(G_p) \geq \bar{\alpha} np$ for some positive constant $\bar{\alpha}$ a.a.s. Since adding edges to a graph does not decrease the algebraic connectivity of that graph [2], we have $\lambda_2(G) \geq \lambda_2(G_p) \geq \bar{\alpha} np$ a.a.s. $\square$

In order to validate the bound in Theorem 6, we have drawn in Fig. 3 the ratio of the algebraic connectivity to np for a random interdependent network with two subnetworks and $p = 0.1$. The subnetworks have random structure where any two nodes (inside a subnetwork) are connected independently with a fixed probability of 0.02. As we can see, this ratio converges to a small range (between 0.8 and 1) as predicted by Theorem 6.
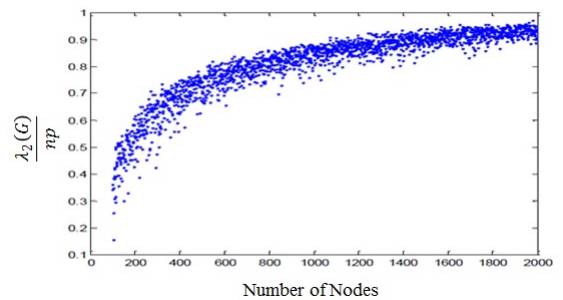

Fig. 3. Ratio of algebraic connectivity to np for a random interdependent network with two subnetworks.

Theorem 6 demonstrates the importance of inter-network edges on the algebraic connectivity of the overall network when $\limsup_{n \to \infty} \frac{\ln n}{(k-1)np} < 1$. This requirement on the growth rate of p cannot be reduced if one wishes to stay agnostic about the probability distributions over the topologies of the subnetworks. Indeed, by Lemma 3, if $\limsup_{n \to \infty} \frac{\ln n}{(k-1)np} > 1$, a random k-partite graph will have at least one isolated node a.a.s.

and thus has algebraic connectivity equal to zero a.a.s. In this case the quantity $\frac{\ln n}{(k-1)n}$ forms a *coarse threshold* for the algebraic connectivity being 0, or growing as $\Theta(np)$. On the other hand, if one had further information about the probability distributions over the subnetworks, one could potentially relax the condition on p required in the above results. For instance, as mentioned in Section 4, when each of the k subnetworks is an Erdős-Rényi graph formed with probability p, then the entire interdependent network is an Erdős-Rényi graph on kn nodes; in this case, the algebraic connectivity is $\Omega(np)$ a.a.s. as long as $\limsup_{n\to\infty} \frac{\ln n}{knp} < 1$ [29]. This constraint on p differs by a factor of $\frac{k}{k-1}$ from the expression in Theorem 6.

7 Smallest Eigenvalue of the Grounded Laplacian in Random Interdependent Networks

In the previous section, we investigated the rate of convergence in consensus dynamics on random interdependent networks through the notion of algebraic connectivity. The goal of this section is to examine the convergence rate of consensus dynamics when agents are partitioned as leaders and followers. This can been done by studying the smallest eigenvalue of the grounded Laplacian matrix (obtained by removing certain rows and columns from the Laplacian) as discussed in Section 3.3. Specifically, we consider the case where all of the rows and columns corresponding to the nodes in one of the subnetworks are removed as described in Section 3.3. This represents the situation where all of the nodes in the grounded subnetwork act as leaders in consensus dynamics, while the rest of the nodes are followers.

Theorem 7 *Consider a random interdependent network $G = (G_1, G_2, \ldots, G_k, G_p)$, and suppose that one of the subnetworks consists only of leader nodes and the rest of the nodes in the network are followers. Assume that the probability of edge formation between a follower and leader node is denoted by p and satisfies $\limsup_{n\to\infty} \frac{\ln n}{np} < 1$. Then the smallest eigenvalue of the grounded Laplacian satisfies $\lambda(L_g) = \Theta(np)$.*

In order to prove this theorem, we use a simplified version of Theorem 1 in [28] which is stated below.

Lemma 8 ([28]) *Consider a graph $G = (V, E)$ and suppose $S \subset V$ is a set of grounded nodes. For $v_i \in V \setminus S$, let β_i be the number of grounded nodes in node v_i's neighborhood. Then $\min_{i\in\mathcal{V}\setminus S} \beta_i \le \lambda(L_g) \le \max_{i\in\mathcal{V}\setminus S} \beta_i$.*

We will use Lemma 8 to prove Theorem 7. Without loss of generality, assume that V_1 consists entirely of leader nodes and all other nodes are followers. Note that Theorem 7 only depends on the probability of edge formation between a leader and a follower node and thus only
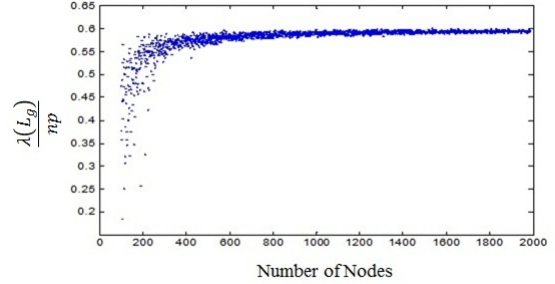

Fig. 4. Ratio of the smallest eigenvalue of the grounded Laplacian matrix to np in a random interdependent network with two subnetworks.

the interconnections between V_1 and each of the subnetworks V_i, $2 \le i \le k$, matter. Since the subgraphs induced by V_1 and V_i are bipartite networks, we use Lemma 4 to obtain a bound on the degree of the nodes in V_i. We then use Lemma 8 to relate the obtained bound to the smallest eigenvalue of the grounded Laplacian matrix. Below, we provide the formal proof.

PROOF. [Proof of Theorem 7] For any node $v_j \in V_i$ where $2 \le i \le k$, let β_j^i be the number of neighbors of node v_j in the set V_1 (assume that V_1 consists entirely of leader nodes). Consider the inter-network topology $H_i = (V_i \cup V_1, E_H^i)$ between nodes in V_i and V_1, i.e, $E_H^i = E_p \cap (V_i \times V_1)$. It is clear that H_i is a bipartite network and thus β_j^i is the degree of nodes $v_j \in V_i$ in the network H_i. Therefore, by Lemma 4 (with $k = 2$), for the specified range of p, there exist α and γ such that

$$\alpha np \le d_{min}(H_i) \le \min_{1\le j\le n} \beta_j^i$$
$$\max_{1\le j\le n} \beta_j^i \le d_{\max}(H_i) \le \gamma np,$$

a.a.s. The above inequalities hold for all $2 \le i \le k$ a.a.s. Since k is a constant value, we conclude that there must exist $\alpha', \gamma' > 0$ such that $\alpha' np \le \min_{i\in\mathcal{V}\setminus S} \beta_i$ and $\max_{i\in\mathcal{V}\setminus S} \beta_i \le \gamma' np$ a.a.s and thus by Lemma 8, $\lambda(L_g) = \Theta(np)$ a.a.s. $\square$

Fig. 4 demonstrates the ratio of the smallest eigenvalue of the grounded Laplacian matrix to np in a random interdependent network with two subnetworks and $p = 0.1$. The subnetworks have a random structure where any two nodes are connected independently with a fixed probability of 0.02. As we can see, the ratio asymptotically remains between 0.5 and 0.6, corroborating the bound in Theorem 7.

Remark 3 *The analysis of grounded Laplacian matrices for scenarios where the leaders are spread across multiple subnetworks is more challenging. The existing analytical bounds in the literature for general grounded Laplacians*

require more information about the network topology [29], and thus further research is required to obtain bounds for random interdependent networks with arbitrary subnetwork topologies and general leader sets.

8   Robustness of Random Interdependent Networks

In this section we focus on characterizing the conditions under which random interdependent networks are r-robust. We will first consider random k-partite networks, and show that they exhibit phase transitions at certain thresholds for the probability p; this will allow us to subsequently characterize the robustness of random interdependent networks with arbitrary subnetwork topology.

Lemma 9 *For any positive integers r and $k \geq 2$,*

$$t(n) = \frac{\ln n + (r-1)\ln\ln n}{(k-1)n}$$

is a threshold for r-robustness of random k-partite graphs.

PROOF. Consider a random k-partite graph G_p with edge formation probability $p(n) = \frac{\ln n + (r-1)\ln\ln n + x}{(k-1)n}$, where $r \in \mathbb{N}$ is a constant and $x = x(n)$ is some function satisfying $x \to \infty$ as $n \to \infty$. By Lemma 3, we know that $i(G_p) > r - 1$. Therefore, by Lemma 1, G_p is at least r-robust a.a.s.

Next consider $p(n) = \frac{\ln n + (r-1)\ln\ln n - x}{(k-1)n}$, where $x = x(n)$ satisfies $x \to \infty$ as $n \to \infty$. Lemma 3 indicates that the minimum degree of a random k-partite graph G_p is less than r a.a.s. Hence, G_p is not r-robust a.a.s (by the relationships shown in Fig. 1b). □

Together with Lemma 3, the above result indicates that the properties of r-robustness and r-minimum-degree (and correspondingly, r-connectivity) all share the same threshold function in random k-partite graphs, despite the fact that r-robustness is a significantly stronger property than the other two properties. Recent work has shown that these properties share the same thresholds in Erdős-Rényi random graphs [37] and random intersection graphs [38], and Lemma 9 adds random k-partite networks to this list.

Next we now consider general random interdependent networks with arbitrary topologies within the subnetworks. Note that any general random interdependent network can be obtained by first drawing a random k-partite graph, and then adding additional edges to fill out the subnetworks. Using the fact that r-robustness is a *monotonic graph property* (i.e., adding edges to an r-robust graph does not decrease the robustness parameter), we obtain the following result.

Theorem 10 *Consider a random interdependent network $G = (G_1, G_2, \ldots, G_k, G_p)$. Assume that the inter-network edge formation probability satisfies $p(n) \geq \frac{\ln n + (r-1)\ln\ln n + x}{(k-1)n}$, $r \in \mathbb{Z}_{\geq 1}$ and $x = x(n)$ is some function satisfying $x \to \infty$ as $n \to \infty$. Then G is r-robust a.a.s.*

We conclude this section by characterizing the robustness of random interdependent networks under a coarser rate of growth in p.

Theorem 11 *Consider a random interdependent network $G = (G_1, G_2, \ldots, G_k, G_p)$ with inter-network edge formation probability $p = p(n)$ that satisfies $\limsup_{n\to\infty} \frac{\ln n}{(k-1)np} < 1$. Then G is $\Omega(np)$-robust a.a.s.*

PROOF. Following the same methodology that we established in this paper, we first prove that if p satisfies the condition of the theorem, then the k-partite random graph G_p is $\Omega(np)$-robust. For $p = p(n)$ satisfying the given condition, we have $i(G_p) = \Theta(np)$ a.a.s. from Lemma 4. By Lemma 1, the robustness parameter of G_p is $\Omega(np)$ a.a.s. Next note that, since adding edges to a network does not decrease the robustness parameter, the interdependent network $G = (G_1, G_2, \ldots, G_k, G_p)$ is $\Omega(np)$-robust a.a.s. □

Remark 4 *Since the robustness parameter is always less than the minimum degree of the graph by Lemma 1, under the $\limsup_{n\to\infty} \frac{\ln n}{(k-1)np} < 1$ condition, G_p is $O(np)$-robust a.a.s. from Lemma 4. We also showed in the proof of Theorem 11 that under the given condition, G_p is $\Omega(np)$-robust. Therefore, if the inter-network edge formation probability satisfies the $\limsup_{n\to\infty} \frac{\ln n}{(k-1)np} < 1$, the k-partite subgraph G_p is $\Theta(np)$-robust.*

9   Summary

We studied certain spectral and structural properties of random interdependent networks. We started by analyzing the isoperimetric constant of random k-partite graphs, and showed that the properties $i(G) > r - 1$ and r-minimum-degree share the same threshold function. We then investigated three important characteristics of random interdependent networks, namely the algebraic connectivity, the smallest eigenvalue of the grounded Laplacian matrix and r-robustness, and provided tight asymptotic rates of growth for these quantities on random interdependent networks for certain ranges of inter-network edge formation probabilities (regardless of the subnetwork topologies). Our results lead to insights about consensus and diffusion dynamics that operate over random interdependent networks.

There are various interesting avenues for future research, including a deeper investigation of the role of the sub-

network topologies, and other probability distributions over the inter-network edges (outside of Bernoulli inter-connections).

A   Proof of Lemma 3

PROOF. First consider the case that the inter-network edge formation probability is $p = \frac{\ln n + (r-1)\ln\ln n + x}{(k-1)n}$ where $x \to \infty$ when $n \to \infty$. For technical reasons, we assume that $x(n) = o(\ln\ln n)$; this is not restrictive as the result will hold even if $x(n)$ grows faster than this bound due to the fact that $i(G_p) > r - 1$ and minimum degree being at least r are both monotonic properties [11].

We have to show that for any set of vertices of size m, $1 \le m \le nk/2$, there are at least $m(r-1)+1$ edges that leave the set a.a.s. Consider a set $S \subset V_1 \cup V_2 \cup \cdots \cup V_k$ with $|S| = m$. Assume that the set S contains s_l nodes from V_l for $1 \le l \le k$ (i.e., $|S \cap V_l| = s_l \ge 0$). Define E_S as the event that $m(r - 1)$ or fewer edges leave S. Note that $|\partial S|$ is a binomial random variable with parameters $\sum_{l=1}^{k} s_l \left(\sum_{t=1,t\neq l}^{k} (n - s_t) \right)$ and p. We have that

$$\sum_{l=1}^{k} s_l \left(\sum_{t=1,t\neq l}^{k} (n - s_t) \right) = \sum_{l=1}^{k} s_l(n(k-1) - m + s_l)$$

$$= n(k-1)m - m^2 + \sum_{l=1}^{k} s_l^2. \tag{A.1}$$

Then we have

$$\mathbf{Pr}(E_S) = \tag{A.2}$$
$$\sum_{i=0}^{m(r-1)} \binom{n(k-1)m - m^2 + \sum_{l=1}^{k} s_l^2}{i} \times$$
$$p^i (1-p)^{n(k-1)m - m^2 + \sum_{l=1}^{k} s_l^2 - i}.$$

For $s_l \in \mathbb{R}, 1 \le l \le k$, we have $\sum_{l=1}^{k} s_l^2 \ge \frac{\left(\sum_{l=1}^{k} s_l \right)^2}{k}$, which is a direct consequence of the Cauchy-Schwartz inequality. Thus $0 \le m^2 - \sum_{l=1}^{k} s_l^2 \le \frac{(k-1)m^2}{k}$, and applying this to the inequality (A.2), we get

$$\mathbf{Pr}(E_S) \tag{A.3}$$
$$\le \sum_{i=0}^{m(r-1)} \binom{n(k-1)m}{i} p^i (1-p)^{n(k-1)m - \frac{(k-1)m^2}{k} - i}.$$

Next note that $k \ge 2$ and for $1 \le i \le m(r-1)$, we have

$$\frac{\binom{n(k-1)m}{i} p^i (1-p)^{n(k-1)m - \frac{(k-1)m^2}{k} - i}}{\binom{n(k-1)m}{i-1} p^{i-1} (1-p)^{n(k-1)m - \frac{(k-1)m^2}{k} - (i-1)}}$$
$$= \frac{n(k-1)m - i + 1}{i} \times \frac{p}{1-p}$$
$$\ge \frac{n(k-1) - r + 1}{r - 1} \times \frac{p}{1-p},$$

which is lower bounded by some constant strictly larger than 1 for sufficiently large n. Thus there exists some constant $R > 0$ such that

$$\mathbf{Pr}(E_S) \tag{A.4}$$
$$\le \sum_{i=0}^{m(r-1)} \binom{n(k-1)m}{i} p^i (1-p)^{n(k-1)m - \frac{(k-1)m^2}{k} - i}$$
$$\le R \binom{n(k-1)m}{m(r-1)} p^{m(r-1)} \times$$
$$(1-p)^{n(k-1)m - \frac{(k-1)m^2}{k} - m(r-1)}.$$

Define P_m as the probability that there exists a set of nodes T such that $|T| = m$ and $|\partial T| \le m(r-1)$. Then using the inequality $\binom{n}{m} \le (\frac{ne}{m})^m$ yields

$$P_m \le \sum_{\substack{|S|=m, \\ S \subset \cup_{l=1}^{k} V_l}} \mathbf{Pr}(E_S)$$
$$\le R \binom{nk}{m} \binom{n(k-1)m}{m(r-1)} p^{m(r-1)} \times$$
$$(1-p)^{n(k-1)m - \frac{(k-1)m^2}{k} - m(r-1)}$$
$$\le R \left(\frac{nke}{m} \right)^m \left(\frac{n(k-1)mep}{m(r-1)} \right)^{m(r-1)} \times \tag{A.5}$$
$$(1-p)^{n(k-1)m - \frac{(k-1)m^2}{k} - m(r-1)}$$
$$= R \Bigg(\frac{ke^r}{(r-1)^{r-1}(1-p)^{r-1}} \times$$
$$\frac{n(1-p)^{n(k-1)}(n(k-1)p)^{r-1}}{m(1-p)^{\frac{(k-1)m}{k}}} \Bigg)^m$$
$$\le R \left(\frac{c_1 n(1-p)^{n(k-1)}(n(k-1)p)^{r-1}}{m(1-p)^{\frac{(k-1)m}{k}}} \right)^m,$$

where c_1 is a constant satisfying $\frac{ke^r}{(r-1)^{r-1}(1-p)^{r-1}} \le c_1 < \frac{2ke^r}{(r-1)^{r-1}}$ for sufficiently large n. Recalling the function $p(n) = \frac{\ln n + (r-1)\ln\ln n + x}{(k-1)n}$ and using the inequality $1 -$

$p \le e^{-p}$ yields

$$P_m \le R \left(\frac{c_1 n e^{-n(k-1)p} (n(k-1)p)^{r-1}}{m(1-p)^{\frac{(k-1)m}{k}}} \right)^m$$

$$= R \left(c_1 \left(\frac{\ln n + (r-1)\ln\ln n + x}{\ln n} \right)^{r-1} \times \right.$$

$$\left. \frac{e^{-x}}{m(1-p)^{\frac{(k-1)m}{k}}} \right)^m$$

$$\le R \left(\frac{c_2 e^{-x}}{m(1-p)^{\frac{(k-1)m}{k}}} \right)^m .$$

Due to the fact that $\frac{\ln n + (r-1)\ln\ln n + x}{\ln n} < 2$ for sufficiently large n, c_2 is a constant upper bound for $c_1 \left(\frac{\ln n + (r-1)\ln\ln n + x}{\ln n} \right)^{r-1}$ such that $0 < c_2 < c_1 2^{r-1}$. Next, we substitute the Taylor series expansion $\ln(1-p) = -\sum_{i=1}^{\infty} \frac{p^i}{i}$ for $p \in [0,1)$ in the above inequality to obtain

$$P_m \le R \left(\frac{c_2 e^{-x} e^{-\frac{(k-1)m}{k}\ln(1-p)}}{m} \right)^m$$

$$= R \left(\frac{c_2 e^{-x} e^{\frac{(k-1)mp}{k}} \exp\{\frac{(k-1)m}{k}p^2 \sum_{i=2}^{\infty}\frac{p^{i-2}}{i}\}}{m} \right)^m .$$

Since we have $\sum_{i=2}^{\infty}\frac{p^{i-2}}{i} < \sum_{i=2}^{\infty}p^{i-2} = \frac{1}{1-p}$ and $\frac{(k-1)m}{k}p^2 \to 0$ as $n \to \infty$, there exists a constant c_3 such that $0 < \frac{(k-1)m}{k}p^2 \sum_{i=2}^{\infty}\frac{p^{i-2}}{i} < c_3 < 1$ for sufficiently large n. Therefore,

$$P_m \le R \left(c_2 e^{c_3} \frac{e^{-x}e^{\frac{(k-1)mp}{k}}}{m} \right)^m = R \left(c_4 \frac{e^{-x}e^{\frac{(k-1)mp}{k}}}{m} \right)^m ,$$

where $0 < c_4 = c_2 e^{c_3} < \frac{k e^{r+1} 2^r}{(r-1)^{r-1}}$.

Consider the function $f(m) = \frac{e^{\frac{(k-1)mp}{k}}}{m}$. Since $\frac{df}{dm} = \frac{e^{\frac{(k-1)mp}{k}}(\frac{(k-1)mp}{k}-1)}{m^2}$, we have that $\frac{df}{dm} < 0$ for $m < \frac{k}{(k-1)p}$ and $\frac{df}{dm} > 0$ for $m > \frac{k}{(k-1)p}$. Therefore, $f(m) \le \max\{f(1), f(nk/2)\}$ for $m \in \{1, 2, \ldots, \lfloor nk/2 \rfloor\}$. We have

$$f(nk/2) = \frac{2e^{\frac{(k-1)nkp}{2k}}}{nk} = \frac{2}{k}e^{(-\ln n + (r-1)\ln\ln n + x)/2}.$$

Since $\ln\ln n = o(\ln n)$, we have that $f(nk/2) = o(1)$. Moreover, $1 < f(1) = e^{\frac{(k-1)p}{k}} < e$ and thus for sufficiently large n we must have $f(m) \le f(1) < e$. Therefore,

$$P_m \le R(c_4 e^{1-x})^m.$$

Let P_0 be the probability that there exists a set S with size $nk/2$ or less such that $|\partial S| \le |S|(r-1)$. Then by the union bound we have

$$P_0 \le \sum_{m=1}^{\lfloor nk/2 \rfloor} P_m \le \sum_{m=1}^{\infty} R(c_4 e^{1-x})^m = \frac{Rc_4 e^{1-x}}{1 - c_4 e^{1-x}} = o(1),$$

since $x \to \infty$ as $n \to \infty$. Thus $i(G_p) > r - 1$ a.a.s. This also implies that G_p has minimum degree at least r a.a.s. (by the relationships shown in Fig. 1b).

To complete the proof, we have to show that for $p = \frac{\ln n + (r-1)\ln\ln n - x}{(k-1)n}$ where $x \to \infty$ when $n \to \infty$, $i(G_p) \le r - 1$ a.a.s. In order to prove this, we show the stronger result that G_p has a node with degree less than or equal to $r - 1$. Again for technical reasons, we assume that $x = o(\ln\ln n)$.

Consider the vertex set $V_1 = \{v_1, \ldots, v_n\}$, and define the random variable $X = X_1 + X_2 + \cdots + X_n$ where $X_i = 1$ if the degree of node v_i is less than r and zero otherwise. The goal is to show that if $p(n) = \frac{\ln n + (r-1)\ln\ln n - x}{(k-1)n}$, then $\mathbf{Pr}(X = 0) \to 0$ asymptotically almost surely. This means that for the specified $p(n)$, there exists a node in V_1 with degree less than r with high probability. Thus by selecting that single node as the set S, we have $i(G_p) \le \frac{|\partial S|}{|S|} \le r - 1$ which is exactly what we want to show.

The random variable X is zero if and only if $X_i = 0$ for $1 \le i \le n$. Since X_i and X_j are identically distributed and independent when $i \ne j$, we have

$$\mathbf{Pr}(X = 0) = \mathbf{Pr}(X_1 = 0)^n = (1 - \mathbf{Pr}(X_1 = 1))^n$$
$$\le e^{-n\mathbf{Pr}(X_1=1)}, \quad \text{(A.6)}$$

where the last inequality is due to the fact that $1 - p \le e^{-p}$ for $p \ge 0$. Now, note that

$$n\mathbf{Pr}(X_1 = 1) = n \sum_{i=0}^{r-1} \binom{n(k-1)}{i} p^i (1-p)^{n(k-1)-i}$$

$$\ge n \binom{n(k-1)}{r-1} p^{r-1} (1-p)^{n(k-1)-r+1}$$

$$\ge n \binom{n(k-1)}{r-1} p^{r-1} (1-p)^{n(k-1)}, \quad \text{(A.7)}$$

where the last inequality is obtained by using the fact that $0 < (1-p)^{r-1} \le 1$ for $r \ge 1$. Using the fact that $\binom{n(k-1)}{r-1} = \Omega(n^{r-1})$ for constant r and k, and $(1-p)^{n(k-1)} = e^{n(k-1)\ln(1-p)} = \Omega(e^{-n(k-1)p})$ when $np^2 \to 0$ (which is satisfied for the function p that we

are considering above), the inequality (A.7) becomes

$$n\mathbf{Pr}(X_1 = 1) = \Omega\left(n^r p^{r-1} e^{-n(k-1)p}\right)$$
$$= \Omega\left(\frac{(\ln n + (r-1)\ln\ln n - x)^{r-1}}{(\ln n)^{r-1}}e^x\right)$$
$$= \Omega(e^x),$$

where in the above we used $p = \frac{\ln n + (r-1)\ln\ln n - x}{(k-1)n}$. Thus we must have that $\lim_{n\to\infty} n\mathbf{Pr}(X_1 = 1) = \infty$, which proves that $\mathbf{Pr}(X = 0) \to 0$ as $n \to \infty$ (from (A.6)). Therefore, there exists a node with degree less than r a.a.s. and consequently, $i(G_p) \leq r - 1$ a.a.s. $\square$

B   Proof of Lemma 5

PROOF. First we show that $i(G) \leq \gamma np$ for some $\gamma > 0$ a.a.s. Consider the set of nodes V_1 in the first subnetwork G_1. The number of edges between V_1 and all other V_j, $2 \leq j \leq k$ is a binomial random variable $B(n^2(k-1), p)$ and thus $\mathbb{E}[|\partial V_1|] = n^2(k-1)p$. Using the Chernoff bound [21] for the random variable $|\partial V_1|$, we have (for $0 < \delta < 1$)

$$\mathbf{Pr}(|\partial V_1| \geq (1+\delta)\mathbb{E}[|\partial V_1|]) \leq e^{\frac{-\mathbb{E}(|\partial V_1|)\delta^2}{3}}. \quad (B.1)$$

Choosing $\delta = \frac{\sqrt{3}}{\sqrt{\ln n}}$, the upper bound in the expression above becomes $\exp\left(-\frac{n^2(k-1)p}{\ln n}\right)$. Since $\ln n < n(k-1)p$ for n sufficiently large and for p satisfying the condition in the proposition, the right hand side of inequality (B.1) goes to zero as $n \to \infty$. Thus $|\partial V_1| \leq (1+o(1))\mathbb{E}[|\partial V_1|]$ a.a.s. Therefore

$$i(G) = \min_{\substack{|A| \leq \frac{nk}{2}, \\ A \subseteq V_1 \cup V_2 \cup \cdots \cup V_k}} \frac{|\partial A|}{|A|} \leq \frac{|\partial V_1|}{|V_1|}$$
$$\leq \frac{(1+o(1))n^2(k-1)p}{n}$$
$$\leq \gamma np,$$

a.a.s. for some $\gamma > 0$.

Next, we have to show that $i(G) \geq \alpha np$ for some $\alpha > 0$. Consider the k-partite subgraph of network G which is denoted by G_p. By Lemma 4, we know that there exists $\alpha > 0$ such that $i(G_p) \geq \alpha np$ a.a.s. Adding edges does not decrease the isoperimetric constant (by definition of the isoperimetric constant) and thus $i(G) \geq i(G_p) \geq \alpha np$ a.a.s. $\square$

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