# Multi-Layer Network Formation via a Colonel Blotto Game

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***Abstract*—We introduce a two-player network formation game based on the classical Colonel Blotto game. We consider a scenario where there is a common set of nodes and each player in the game designs a network layer by purchasing a set of edges between these nodes. We assume that players have a limited budget with which to bid on each edge and the utility of a given set of edges to a player is a function of the resulting network layer. We characterize the ranges of player budgets for which the game admits pure Nash equilibria for utility functions that depend on the component sizes and diameter of the formed networks.**

***Index Terms*—Multi-layer Network Formation, Game Theory, Nash Equilibrium, Colonel Blotto Game**

## I. Introduction

The characteristics of large-scale networks (such as the Internet, the power grid, social, economical and biological networks) have been actively studied over the past several decades. These networks demonstrate non-trivial structural properties (i.e., patterns of relationships and interactions between the various nodes) which play a fundamental role in their functioning. A classical approach to model the formation of such networks is to have each node connect to a random subset of the other nodes (chosen according to an appropriate probability distribution) [1], [2]. While this provides a tractable method for studying certain random phenomena (such as propagation of a disease in a society), it does not necessarily capture the driving principles beneath social, economic or engineered networks, where agents have discretion about connections that they form with other individuals. In other words, these kinds of networks are often formed strategically by one or more designers in order to maximize certain utility functions; the resulting network can be analyzed using game-theoretic notions of equilibria and efficiency [3]–[6].

The selection of an optimal configuration or design of a network occurs in many different application contexts including transportation, telecommunication and electric power systems. Many real-world networks inherently consist of *multiple* layers of relationships between the same set of nodes. Examples include friendship and professional relationships in social networks, policy influence and knowledge exchange in organizational networks [7], and coupled communication and energy infrastructure networks [8]. Despite numerous works on different aspects of multi-layer networks over the past few years [7], [9]–[11], the problem of *strategic multi-layer network formation* has been addressed only recently [12].

In this paper, we consider a scenario with multiple network designers, each building a different layer of a network over a common set of nodes. We model this as a multi-layer network formation game where the network designers are the players. We assume that the players have finite budgets and directly compete with each other to purchase edges from a common set. An example of this would be when multiple telecommunications companies allocate their budgets to improve their services between pairs of cities, with the company that spends the most on a given edge winning the communication link on that edge. The proposed network formation game in this paper is based on the classical *Colonel Blotto* game [13]–[15]. We consider utilities for players that depend on the component sizes and diameter of the formed networks, and characterize the range of player budgets for which the game admits pure Nash equilibria. The developed game also has applications to attack and defense problems in networks [5], [16].

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## II. Definitions

An undirected network (or graph) is denoted by $G = (N, E)$ where $N = \{1, 2, \ldots, n\}$ is the set of nodes (or vertices) and $E \subseteq \{(i, j) | i, j \in N, i \neq j\}$. The set of all possible graphs on $N$ is denoted by $G^N$. Two nodes are said to be directly connected (or neighbors) if there is an edge between them. A path from node $v_1$ to $v_k$ in graph $G$ is a sequence of distinct nodes $v_1 v_2 \cdots v_k$ where there is an edge between any consecutive nodes of the sequence. The length of a path is the number of edges in the sequence. The distance between nodes $i$ and $j$ in graph $G$ is the length of the shortest path between those two nodes, and is denoted by $d_G(i, j)$. The diameter of a graph is the largest distance between any two of its nodes, i.e., $\max_{i,j \in N, i \neq j} d_G(i, j)$. A cycle is a path of length two or more from a node to itself.

A tree is a connected acyclic graph. The complement of graph $G = (N, E)$ is denoted by $\bar{G} = (N, \bar{E})$, where $\bar{E} = E^c \setminus E$ and $E^c$ is the set of all possible edges between nodes in $N$. Two graphs on the same set of nodes are said to be disjoint if their edge sets are disjoint.

## III. Colonel Blotto Network Formation Game

We consider a setting for competitive network formation where two players are each given a budget and there is a common set of nodes. These players compete with each other to purchase edges between these nodes. As an example, consider two competing transportation companies. Each company has a fixed budget to spend on service between some or all pairs of cities. Under an idealized allocation rule, the company that spends more on a given edge wins that edge, and the utility to a company is a function of all of the edges that it wins. Similar examples can be formulated for telecommunication companies bidding on spectrum or communication links, and wireless networks where a transmitter and jammer are competing to send or disrupt information, respectively, with fixed power budgets [17].

Games of this form, where multiple players compete in a bidding war across multiple battlefields, are traditionally known as *Colonel Blotto* games; here we will extend such games to the network formation setting and characterize the types of equilibria that occur for different utility functions and budgets of the players. We start by reviewing the classical version of this game.

### A. The Colonel Blotto Game

The classical Colonel Blotto (CB) game is defined as follows [13], [14].

*Definition 1:* There are two players $P_1$ and $P_2$ with $S_1$ and $S_2$ units of resources, respectively. There are $n$ battlefields, and the players choose their actions simultaneously. The strategy space of player $P_l$, $l \in \{1, 2\}$ is given by

$$X^l = \{(x_1^l, x_2^l, \dots, x_n^l) \in \mathbb{R}^n_{\geq 0} | \sum_{i=1}^{n} x_i^l = S_l\}.$$

The amount of resources allocated by player $P_l$ to field $i$ is $x_i^l$. There is a zero sum game between the players in each field $i$ where the player with the higher amount of allocated resources to that field receives a payoff of $+1$ and the loser receives $-1$; both players receive 0 when they allocate equal resources to a field (although other tie-breaking rules can also be considered). The total payoff to each of the players is given by $u_1 = \sum_{i=1}^{n} sgn(x_i^1 - x_i^2)$ and $u_2 = -u_1$. □

Using connections to simultaneous first-price all-pay auctions, [15] showed that this game has no pure Nash equilibrium when $\frac{1}{n} < \frac{S_2}{S_1} \leq 1$, but does have an equilibrium in mixed strategies. Other variants of this game such as having more than two players, different types of functions for scoring over battles, and fields with different values have also been addressed in the literature [13], [14]. Here, we introduce the Colonel Blotto game into the competitive network formation setting, and characterize the set of equilibria that can occur.

### B. The Colonel Blotto Network Formation Game

*Definition 2:* The CB Network Formation Game consists of two players $P_1$ and $P_2$, and a set of nodes $N = \{1, 2, \dots, n\}$. These players play a CB game where the set of battlefields are the $\binom{n}{2}$ edges between these nodes. Player $P_l$, $l \in \{1, 2\}$, has a positive budget $S_l$, and strategy space

$$X^l = \{(x_{12}^l, x_{13}^l, \dots, x_{n-1,n}^l) \in \mathbb{R}^{\binom{n}{2}}_{\geq 0} | \sum_{i,j \in N, i \neq j} x_{ij}^l = S_l\}, \tag{1}$$

where entry $x_{ij}^l$ indicates the allocation by player $P_l$ to edge $(i, j)$. Each pair of strategies $(x^1, x^2) \in X^1 \times X^2$ induces graphs $G_1(x^1, x^2)$ and $G_2(x^1, x^2)$, where $G_l(x^1, x^2)$ is formed from edges in which player $l$ has the highest allocation for $l \in \{1, 2\}$. There is a utility function $u_l : G^N \times G^N \to \mathbb{R}$ for $l \in \{1, 2\}$ that determines the utility of each player based on the formed networks. The pair of strategies $(x^1, x^2)$ is said to be a pure Nash equilibrium if and only if $x^1 \in \operatorname{argmax}_{x \in X^1} u_1(G_1(x, x^2), G_2(x, x^2))$ and $x^2 \in \operatorname{argmax}_{x \in X^2} u_2(G_1(x^1, x), G_2(x^1, x))$. For simplicity, we denote $G_l(x^1, x^2)$ by just $G_l$ in the rest of the paper. □

*Remark 1:* The chosen strategy $x^l \in X^l$ of player $P_l$ in equation (1) can equivalently be viewed as a weighted graph[1] $F_l$, where each player has allocated a nonzero amount of investment on each of the links and zero investment on the links of its complement network. We refer to the strategies of the players as their *chosen network* or *investment vector*. □

Note that the above game is a version of the CB game where the battlefields exhibit *strategic complementarities* (i.e., the value of a given battlefield, or edge, depends on the set of all battlefields won by that player). While CB games with complementarities have been studied previously (e.g., [14]), the difference in our setting is that the complementarities arise from network characteristics (such as distance and connectedness), which lends additional structure to the problem.

In the rest of this section, we will assume without loss of generality that player $P_1$ has a budget $S_1 = 1$, and $S_2 \leq 1$. The following facts will be useful for characterizing the equilibria of the game for various utility functions.

*Lemma 1:* Consider the CB Network Formation Game with budgets $S_1 = 1$ and $S_2 \leq S_1$, and node set $N = \{1, 2, \dots, n\}$.

1) If $S_2 < \frac{2}{n(n-1)}$, there is an investment vector allowing player $P_1$ to win all edges, regardless of the strategy of player $P_2$.
2) If $\frac{2}{n(n-1)} \leq S_2 \leq \frac{2}{n}$, there is an investment vector for player $P_1$ such that player $P_2$ cannot win a component of size $n$.
3) If $\frac{2(m-1)}{n(n-1)} < S_2 \leq 1$ where $2 \leq m \leq n$, then for any given investment vector of player $P_1$, player $P_2$ can win a star network with at least $m$ nodes.

*Proof:* The first case is trivial as it suffices for player $P_1$ to allocate $\frac{2}{n(n-1)}$ on all of the edges.

For the second case, note that $P_2$ needs at least $n-1$ edges to win a component of size $n$. If player $P_1$ allocates $\frac{2}{n(n-1)}$ on each edge, player $P_2$ can never win a component of size $n$ since that would require a total investment larger than $\frac{2}{n}$.

Finally, in the last case, let $S_i^j$ denote a star network of size $m$ centered at node $i$ with $m-1$ peripheral nodes chosen from the set $N \setminus \{i\}$ (the index $j$ will be used to enumerate such star networks). It is clear that for any $i \in N$, there exists $\binom{n-1}{m-1}$ of these star networks. Denote the sum of the investments of player 1 on the edges of the star network $S_i^j$ by $b_i^j$. Then we have

$$\sum_{i=1}^{n} \sum_{j=1}^{\binom{n-1}{m-1}} b_i^j = 2\binom{n-2}{m-2}.$$

This is due to the fact that each edge $(u, v)$ is counted $2\binom{n-2}{m-2}$ times, where $\binom{n-2}{m-2}$ is the number of possible star networks on node $u$ that contains the edge $(u, v)$ (and similarly for node $v$). Therefore, there must exist a node $i$ with $b_i^j$ such that

$$b_i^j \leq \frac{2\binom{n-2}{m-2}}{n\binom{n-1}{m-1}} = \frac{2(m-1)}{n(n-1)}.$$

[1] A weighted graph associates a real number (which must be nonnegative in the CB game) to every edge in the graph.

By allocating his or her resources appropriately, player $P_2$ can win all of the edges in the star network $S_i^j$. ■

*Remark 2:* Note that when $S_2 < 1$, for any given investment vector of player $P_2$, player $P_1$ can win all edges of the network by simply matching $P_2$'s investment everywhere, and then spreading the excess budget evenly over all edges. When $S_2 = 1$, no player can win all edges because that would require a total investment larger than 1. However, for any given investment vector of player $P_2$, player $P_1$ can win $\binom{n}{2} - 1$ edges as follows: choose an edge where $P_2$ has a nonzero investment $r$, match $P_2$'s investment on all other edges and then distribute an additional $r$ evenly over all those edges. The same is true for $P_2$ by symmetry. These arguments are independent of the utility function and are standard in the study of CB games [13]. □

We now study the CB network formation game for two natural utility functions.

### C. Colonel Blotto Network Formation with Respect to Largest Component

In this section we will define the utility $u_l$ of each player to be an increasing function of the size of the largest component in their formed network, capturing the notion that players wish their network to provide paths between as many nodes as possible. For instance, having a larger component is advantageous for a telecommunications company that provides service among a group of cities. This has also been a measure of robustness in the network attack and defense literature, where the defender aims at maximizing the size of his component in the presence of adversaries [3], [5]. The following proposition characterizes Nash equilibria in terms of the range of resources available to player 2.

*Proposition 1:* Assume that $\frac{2(m-1)}{n(n-1)} < S_2 \leq 1$ where $2 \leq m \leq n$. Let $G_l \in G^N$ for $l \in \{1, 2\}$ denote the network formed by player $P_l$ as the outcome of the game. Define the utility functions of the players in the CB network formation game as

$$u_1(G_1, G_2) = g_1(C(G_1)), \;\; u_2(G_1, G_2) = g_2(C(G_2)),$$

where $g_1(\cdot)$ and $g_2(\cdot)$ are increasing functions and $C(G)$ denotes the size of the largest component in graph $G$. Then for any Nash equilibrium pair of actions $x^1 \in X^1$ and $x^2 \in X^2$, we have that $C(G_1) = n$ and $C(G_2) \geq m$.

*Proof:* By Remark 2, it is clear that the formed network by player 1 must be a connected network. Furthermore, when $\frac{2(m-1)}{n(n-1)} < S_2 \leq 1$, Lemma 1 indicates that for any given strategy chosen by player $P_1$, player $P_2$ can always choose his strategy such that his outcome network has a component of size at least $m$. ■

Note that there are many equilibria of the form described in the above proposition, as in any complete graph on $n$ nodes, there are $\lfloor \frac{n}{2} \rfloor$ disjoint spanning trees [18]. For any range of resources available for player $P_2$, any two of these trees form an equilibrium when each player allocates his budget entirely to the edges on his tree.

By Lemma 1, we know that for $S_2 \leq \frac{2}{n(n-1)}$, there exist strategies for player $P_1$ such that player $P_2$ cannot win a single edge. Thus in the case of $S_2 \leq \frac{2}{n(n-1)}$, there are instances of Nash equilibria in which player 2 forms the empty network.

### D. Colonel Blotto Network Formation with Respect to Diameter

Proposition 1 considered the case where each player only cares about the size of the largest component in his or her network; however, the *diameter* of the network is often also of interest in network formation and design [19], [20]. Here, we study the CB network formation game when the utility of each player is decreasing in the diameter of the network.

*Proposition 2:* Let $G_l \in G^N$ for $l \in \{1, 2\}$ denote the network formed by player $P_l$ as the outcome of the game under strategies $x^1 \in X^1$ and $x^2 \in X^2$. Define the utility functions of the players in the CB network formation game as

$$u_1(G_1, G_2) = h_1(D(G_1)), \;\; u_2(G_1, G_2) = h_2(D(G_2)),$$

where $h_1(\cdot)$ and $h_2(\cdot)$ are decreasing functions and $D(G)$ denotes the diameter of graph $G$. Then the pair of actions $x^1 \in X^1$ and $x^2 \in X^2$ are in Nash equilibrium if and only if one of the following two conditions holds.

1) $S_2 = S_1$ and both $G_1$ and $G_2$ have a diameter of 2.
2) $S_2 \leq \frac{2}{n}$ and $G_1 = G^c$ and $G_2 = G^e$.

*Proof:* We first argue that there is no (pure) Nash equilibrium when $\frac{2}{n} < S_2 < 1$. By Lemma 1, for any given strategy $x^1 \in X^1$, player $P_2$ can choose a strategy $x^2 \in X^2$ such that he obtains a star, which has diameter 2. However for any given strategy $x^2 \in X^2$, player $P_1$ can choose $x^1 \in X^1$ such that she wins all edges and obtains a diameter of 1 (as argued in Remark 2). Thus there is no pure Nash equilibrium for this range of $S_2$.

Suppose $S_2 = 1$. Note from Remark 2 that no player can win all edges, and thus no player can achieve a diameter less than 2. However, for any given investment by one player, the other has a strategy that wins all but one of the edges, thereby winning a network of diameter 2. Thus all Nash equilibria under $S_2 = S_1 = 1$ satisfy the property that both players win a network of diameter 2.

Now suppose that $S_2 \leq \frac{2}{n}$. In this case, by Lemma 1, for any given investment of player $P_2$, there always exists a strategy for player $P_1$ to win all edges and obtain a diameter of 1. Thus any Nash equilibrium must have player $P_1$ winning all edges and $P_2$ winning none. When player $P_1$ invests $\frac{2}{n(n-1)}$ on all edges, player $P_2$ can never obtain a connected network, and thus any investment vector $x^2 \in X^2$ with less than $\frac{2}{n(n-1)}$ on each edge yields a Nash equilibrium. ■

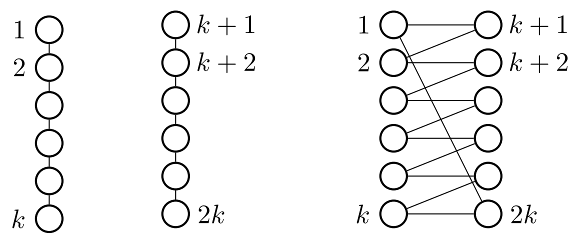
Based on Proposition 2 for $S_1 = S_2$, players are in Nash equilibrium if and only if their formed networks each have a diameter of 2. By definition, the networks $G_1$ and $G_2$ formed in the CB network formation game are edge-disjoint. It is easy to check that for $n = 5$, the cycle graph 1-2-3-4-5-1 and its complement both have diameter of 2. We now show that there exist such graphs for any $n \geq 5$. We will use the following notation.

*Definition 3:* Let $G_i = (N_i, E_i)$, $1 \leq i \leq T$, be a set of graphs. Then $G = (N, E) = [G_1, G_2, \ldots, G_T]$ is a graph such that $N = \cup_{i=1}^T N_i$ and $E = \cup_{i=1}^T E_i$. □

*Proposition 3:* Suppose that we have $|N| = 2k$ nodes for $k \geq 3$. Let graph $G_1$ consist of nodes $\{1, 2, \ldots, k\}$, $G_2$ consist of nodes $\{k+1, k+2, \ldots, 2k\}$, and $G_3$ consist of all nodes $\{1, 2, \ldots, 2k\}$. Define the edge sets of $G_1$, $G_2$ and $G_3$ as follows (depicted in Figure 1):

1) $G_1$ is the path $1 - 2 - 3 - \cdots - k$.
2) $G_2$ is the path $(k+1) - (k+2) - \cdots - (2k)$.
3) $G_3$ is the cycle $1 - (k+1) - 2 - (k+2) - \cdots - (2k-1) - k - (2k) - 1$.

Define graphs $F_1 = [\bar{G}_1, \bar{G}_2, G_3]$ and $F_2 = \bar{F}_1$. Then both $F_1$ and $F_2$ have a diameter of 2. When $|N| = 2k+1$ for some $k \geq 3$, it suffices to connect the node $(2k+1)$ to all nodes $\{k+1, k+2, \ldots, 2k\}$ in graph $F_1$.




(a) $G_1$ (b) $G_2$ (c) $G_3$

Fig. 1

*Proof:* The proof of this proposition is straightforward but tedious, and thus we provide only a sketch of the proof in the interest of space. For $k = 3$ the proof can be verified by examining the graph directly, and thus suppose $k \geq 4$. In graph $F_1$, consider node 1. This node is directly connected to nodes in the set $\{3, 4, \ldots, k, k+1, 2k\}$. It has a path of length 2 to nodes in the set $\{2, k+3, k+4, \ldots, 2k-1\}$ via node $k+1$. Finally, it has a path of length 2 to $k+2$ via node $2k$. Next consider node 2. This node is directly connected to nodes in the set $\{4, 5, \ldots, k, k+1, k+2\}$. It has a path of length 2 to node 3 via node $k+2$. Finally, it has a path of length 2 to nodes in the set $\{k+3, \ldots, 2k\}$ via $k+1$. This reasoning can be repeated for all nodes in the network to show that $F_1$ has a diameter of 2, and similarly that $F_2$ has a diameter of 2. ■

*Remark 3:* Note that the graphs provided above are not unique in having the property that both the graph and its complement have a diameter of 2. For example, consider the Erdos-Renyi graph $G(n, p)$ on $n$ nodes, where each of the possible $\binom{n}{2}$ edges is independently present with probability $p \in [0, 1]$. When $p$ is constant in $(0, 1)$, it is well known that asymptotically (in $n$) almost surely the graph $G(n, p)$ will have diameter 2 [21]. Noting that the complement of $G(n, p)$ is an Erdos-Renyi graph $G(n, 1-p)$, we see that an Erdos-Renyi graph and its complement will both have diameter 2 with probability tending to 1 as $n \to \infty$. □

## IV. Conclusions

In this paper, we introduced a competitive two-player network formation game based on the classical Colonel Blotto game, and characterized the range of player budgets for which this game admits pure Nash equilibria under two different utility functions. There are interesting avenues for further research on this problem, including other classes of utility functions in the CB game that we defined here. While we have only considered pure Nash equilibria in this paper, characterizing mixed strategies in the CB setting would provide further insights into the formation process. Also, a mechanism to incorporate stochasticity and partial information into the network formation process would be of value in modeling the formation of realistic networks.

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## References

[1] P. Erdos and A. Renyi, "On random graphs," *Publicationes of Mathematicae Debrecen*, vol. 6, pp. 290–297, 1959.
[2] D. Watts and S. Strogatz, "Collective dynamics of small-world networks," *Nature*, vol. 393, no. 4, pp. 440–442, 1998.
[3] L. Blume, D. Easley, J. Kleinberg, R. Kleinberg, and E. Tardos, "Network formation in the presence of contagious risk," in *Proceedings of the 12th ACM Conference on Electronic Commerce*, 2011, pp. 1–10.
[4] M. Brautbar and M. Kearns, "A clustering coefficient network formation game," in *Proceedings of the 4th International Conference on Algorithmic Game Theory*, 2011, pp. 224–235.
[5] S. Goyal and A. Vigier, "Robust networks," *Working Paper*, 2011.
[6] M. Kearns, S. Judd, and Y. Vorobeychik, "Behavioral experiments on a network formation game," in *Proceedings of the 13th ACM Conference on Electronic Commerce*, 2012, pp. 690–704.
[7] K. Weiss, M. Hamann, M. Kinney, and H. Marsh, "Knowledge exchange and policy influence in a marine resource governance network," *Global Environmental Change*, vol. 22, no. 1, pp. 178 – 188, 2012.
[8] C. M. Schneider, N. A. M. Araujo, S. Havlin, and H. J. Herrmann, "Toward designing robust coupled networks," *arXiv:1106.3234 [cond-mat.stat-mech]*, 2011.
[9] M. Beccuti, G. Franceschinis, M. Kaâniche, and K. Kanoun, "Critical information infrastructure security," 2009, ch. Multi-level Dependability Modeling of Interdependencies between the Electricity and Information Infrastructures, pp. 48–59.
[10] J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, "Robustness of a network formed by $n$ interdependent networks with a one-to-one correspondence of dependent nodes," *Phys. Rev. E*, vol. 85, p. 066134, 2012.
[11] J. Gao, S. V. Buldyrev, H. E. Stanley, and S. Havlin, "Networks formed from interdependent networks," *Nature Physics*, vol. 8, pp. 40–48, 2012.
[12] E. Moradi-Shahrivar and S. Sundaram, "Strategic multi-layer network formation," in *Proceedings of the IEEE Conference on Decision and Control*, 2013, pp. 582–587.
[13] R. Golman and S. E. Page, "General Blotto game: Games of strategic allocative mismatch," *Public Choice*, vol. 138, no. 3-4, pp. 279–299, 2009.
[14] M. Shubik and R. J. Weber, "Systems defense games: Colonel Blotto, command and control," *Naval Research Logistics Quarterly*, vol. 28, no. 2, pp. 281–287, 1981.
[15] B. Roberson, "The Colonel Blotto game," *Economic Theory*, vol. 29, no. 1, pp. 1–24, 2006.
[16] A. Gueye, J. C. Walrand, and V. Anantharam, "Design of network topology in an adversarial environment," in *Proceedings of the First International Conference on Decision and Game Theory for Security*, 2010, pp. 1–20.
[17] Y. Wu, B. Wang, and K. Liu, "Optimal power allocation strategy against jamming attacks using the Colonel Blotto game," in *IEEE Global Telecommunications Conference*, 2009, pp. 1–5.
[18] E. M. Palmer, "On the spanning tree packing number of a graph: a survey," *Discrete Math.*, vol. 230, no. 1-3, pp. 13–21, 2001.
[19] M. O. Jackson, *Social and Economic Networks*. Princeton University Press, 2008.
[20] E. Tardos and T. Wexler, "Network formation games and the potential function method," in *Algorithmic Game Theory*, 2007, pp. 487–516.
[21] R. Diestel, *Graph Theory*. Springer-Verlag: Heidelberg, 2005.