Coherence and Convergence Rate in Networked Dynamical Systems

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***Abstract*— We study two metrics in stochastic consensus dynamics with leaders or stubborn agents: network coherence (defined in terms of the system $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms), and convergence rate. We allow each agent to maintain an individual level of stubbornness in deviating from its initial values. We give bounds on the convergence rate and present sufficient conditions under which the bounds become tight. Moreover we study the effect of the level of stubbornness of the agents on network coherence and convergence rate. We then characterize these two metrics in random regular graphs and Erdos-Renyi random graphs. From a leader selection point of view, we show that maximizing $\mathcal{H}_\infty$ coherence is equivalent to maximizing convergence rate. Moreover we study conditions under which the optimal leader for maximizing $\mathcal{H}_2$ coherence differs from the optimal leader for maximizing convergence rate, and conversely, provide sufficient conditions on the network for a single leader to maximize both metrics simultaneously.**

I. Introduction

There are two approaches that are commonly adopted to optimize the performance of networked dynamical systems: (i) engineering the interconnections (i.e., topology) of the network [1], [2] and (ii) changing the roles or dynamics of certain members of the network. The deviations in dynamics in the latter approach may be due to certain nodes acting as *leaders*, or due to nodes maintaining a certain level of "stubbornness" in changing their states [3], [4]. Several metrics have been considered in the literature for selecting leaders in a given network [5]–[10]. Here, we focus on two particular metrics: *convergence rate* and *network coherence*. The former considers how quickly the chosen set of leaders (or stubborn nodes) can cause the remaining nodes to converge to their steady state values, while the latter considers the deviations of the non-leader nodes from their steady state values when their state updates are affected by noise.

In the context of convergence rate, the papers [10]–[12] consider the interplay between the stubborn agents and the topology of the network, while [13] studies leader selection in multi-agent systems for maximizing the convergence rate. In the context of network coherence, various papers have investigated the problem of choosing leaders to maximize coherence (minimize system $\mathcal{H}_2$ norm) using different algorithms [9], [14], [15] and centrality metrics [6], [16]. There has also been an investigation of bounds on coherence and how it scales with network topology [7], [17], [18].

Our work in this paper contributes to the above literature in the following ways. (1) We introduce a new notion of network coherence, termed $\mathcal{H}_\infty$ coherence, based on the system $\mathcal{H}_\infty$ norm. (2) We give bounds on the smallest eigenvalue of the grounded Laplacian matrix which characterizes the convergence rate and $\mathcal{H}_\infty$ coherence in the network in the presence of both fully and partially stubborn agents. (3) We discuss the effect of the number of stubborn agents and the level of stubbornness on the coherence and convergence rate of the network. (4) We use the above results to give a tight characterization of coherence in random regular graphs and Erdos-Renyi random graphs. (5) We analyze the relationship between the two coherence metrics and convergence rate in networks; in particular we show that maximizing $\mathcal{H}_\infty$ coherence in a leader selection problem is equivalent to maximizing convergence rate. Moreover, we study conditions under which the optimal leader for maximizing $\mathcal{H}_2$ coherence differs from the optimal leader for maximizing convergence rate (or $\mathcal{H}_\infty$ coherence), and conversely, provide sufficient conditions on the network for a single leader to maximize both metrics simultaneously.

II. Notation

We denote an undirected graph by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ is the set of nodes (or vertices) and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. The neighbors of node $v_i \in \mathcal{V}$ are given by the set $\mathcal{N}_i = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$. The degree of node v_i is denoted by $d_i \triangleq |\mathcal{N}_i|$. For a given set of nodes $X \subset \mathcal{V}$, the *edge-boundary* (or just boundary) of the set is given by $\partial X \triangleq \{(v_i, v_j) \in \mathcal{E} \mid v_i \in X, v_j \in \mathcal{V} \setminus X\}$. The adjacency matrix of the graph is given by an $n \times n$ binary matrix A, where element $A_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and zero otherwise. The Laplacian matrix of the graph is given by $L \triangleq D - A$, where $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$. The eigenvalues of the Laplacian are real and nonnegative, and are denoted by $0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_n(L)$. For a given subset $\mathcal{S} \subset \mathcal{V}$ of nodes (which we term *grounded nodes*), the *grounded Laplacian* induced by $\mathcal{S}$ is denoted by $L_g(\mathcal{S})$ or simply L_g, and is obtained by removing the rows and columns of L corresponding to the nodes in $\mathcal{S}$. When there is a single grounded node (i.e., $\mathcal{S} = \{v_i\}$ for some $v_i \in \mathcal{V}$), we denote the grounded Laplacian by L_{gi}. The smallest eigenvalue of any square matrix M is denoted by $\lambda(M)$. For matrix A with complex entries, A^* is the conjugate transpose of A.

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III. Network Dynamics

Consider a connected network consisting of n agents $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. Agent $v_j \in \mathcal{V}$ has a scalar real-valued state $\psi_j(t)$, which evolves over time as a function of the states of v_j's neighbors in the network. Each agent $v_j \in \mathcal{V}$ has a certain level of *stubbornness* in deviating from its initial state, captured by a scalar $k_j \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. We partition the set of agents into three sets: the *followers* $\mathcal{F} = \{v_j \in \mathcal{V} \mid k_j = 0\}$, *partially stubborn agents* $\mathcal{P} = \{v_j \in \mathcal{V} \mid 0 < k_j < \infty\}$, and the *fully stubborn agents* (or *leaders*) $\mathcal{S} = \{v_j \in \mathcal{V} \mid k_j = \infty\}$. We assume without loss of generality that the fully stubborn agents are placed last in the ordering of the agents. The state dynamics for the followers and partially stubborn agents are of the form

$$\dot{\psi}_j(t) = \sum_{v_i \in \mathcal{N}_j} (\psi_i(t) - \psi_j(t)) - k_j(\psi_j(t) - \psi_j(0)) + w_j(t), \quad v_j \in \mathcal{F} \cup \mathcal{P}, \quad (1)$$

where $w_j(t)$ is a zero mean white noise process. The state dynamics of the fully stubborn agents are given by

$$\dot{\psi}_j(t) = 0, \;\; v_j \in \mathcal{S}. \quad (2)$$

Aggregating the states of all follower and partially stubborn agents into a vector $\psi_F(t) \in \mathbb{R}^{n-|\mathcal{S}|}$, and the states of all fully stubborn agents into a vector $\psi_{\mathcal{S}}(t)$ (note that $\psi_{\mathcal{S}}(t) = \psi_{\mathcal{S}}(0)$ for all $t \geq 0$), equation (1) yields the dynamics

$$\dot{\psi}_F(t) = -\bar{L}_g \psi_F(t) + K\psi_F(0) + J\psi_{\mathcal{S}}(0) + w(t), \quad (3)$$

where $\bar{L}_g = L_g + K$, L_g is the grounded Laplacian induced by the fully stubborn agents, $K = \mathrm{diag}(k_1, k_2, \ldots, k_{n-|\mathcal{S}|})$ indicates the level of stubbornness of the agents, and J is a submatrix of the graph Laplacian capturing the influence of the fully stubborn agents on the other agents. The vector $w(t)$ is a vector representing a zero-mean white noise process with covariance I. When all agents are followers, the above dynamics degenerate to the classical consensus dynamics [1]. If the underlying network is connected and there is at least one fully or partially stubborn agent, the matrix $\bar{L}_g$ is diagonally dominant with at least one strictly diagonally dominant row. Thus from [19], we conclude that it is a positive definite matrix and $\bar{L}_g^{-1}$ is a nonnegative matrix. In this case, the dynamics given by (3) are asymptotically stable, and the rate of convergence is determined by the smallest eigenvalue of $\bar{L}_g$. Moreover, from the nonnegativity of $\bar{L}_g^{-1}$ we conclude (using the Perron-Frobenius theorem) that the eigenvector corresponding to the largest eigenvalue of $\bar{L}_g^{-1}$ can be taken to be nonnegative. We will use this property in the next section.

Let $\bar{\psi}_F(t)$ denote the expected value of $\psi_F(t)$, and define the error term $e(t) = \psi_F(t) - \bar{\psi}_F(t)$. The *network disorder* due to noise is measured via the $\mathcal{H}_2$ norm of the error dynamics of (3) as follows [7]

$$\mathrm{trace}(\lim_{t\to\infty} \mathbb{E}(e(t)e(t)^T)) = \frac{1}{2}\mathrm{trace}(\bar{L}_g^{-1}). \quad (4)$$

A network with small value of disorder in (4) is called a *coherent network* [17]; since the above metric is based on the $\mathcal{H}_2$ norm of a system with transfer function $G(s) = (sI + \bar{L}_g)^{-1}$, we refer to it as $\mathcal{H}_2$ coherence. For the case where there are no stubborn agents, the steady state variance of the states in the consensus network is given by $\mathrm{trace}(L^{\dagger}) = \sum_{i\neq 1} \frac{1}{\lambda_i(L)}$ [17]. In this paper, in addition to the above quantity, we study another metric for network coherence in terms of the $\mathcal{H}_\infty$ norm of the system, defined as $\mathcal{H}_\infty \triangleq \sup_{\omega\in\mathbb{R}} ||G(j\omega)||$, where $||G(j\omega)|| = \lambda_{max}^{\frac{1}{2}}(G^*(j\omega)G(j\omega))$ is the maximum singular value of the transfer function matrix [20]. The following proposition characterizes the $\mathcal{H}_\infty$ norm in terms of $\lambda(\bar{L}_g)$.

Proposition 1: The system $\mathcal{H}_\infty$ norm of the error dynamics of (3) is $\frac{1}{\lambda(\bar{L}_g)}$.

Proof: For system (3) we have $G(j\omega) = (j\omega I + \bar{L}_g)^{-1}$. Thus we have

$$\begin{aligned} G(j\omega)^*G(j\omega) &= (-j\omega I + \bar{L}_g)^{-1}(j\omega I + \bar{L}_g)^{-1} \\ &= \underbrace{(\omega^2 I + \bar{L}_g^2)^{-1}}_{\mathcal{A}^{-1}}. \end{aligned} \quad (5)$$

We know that $\mathcal{A} > 0$ (positive definite) which yields $\mathcal{A}^{-1} > 0$. Thus finding $\sup_\omega \lambda_{max}(\mathcal{A}^{-1})$ is equivalent to finding $\inf_\omega \lambda_{min}(\mathcal{A})$. Since $\lambda_{min}(\mathcal{A}) = \omega^2 + \lambda_{min}(\bar{L}_g^2)$, we have $\inf_\omega \lambda_{min}(\mathcal{A}) = \lambda_{min}(\bar{L}_g^2)$, proving the proposition. ■

Analogously to the notion of an $\mathcal{H}_2$ coherent network, a network with a small $\mathcal{H}_\infty$ norm will be referred to as an $\mathcal{H}_\infty$ coherent network.

Remark 1: According to the above proposition, maximizing $\lambda(\bar{L}_g)$ in a leader selection problem results in maximizing convergence rate and maximizing $\mathcal{H}_\infty$ coherence of the network simultaneously. However the leaders maximizing the $\mathcal{H}_\infty$ coherence may not necessarily be optimal for $\mathcal{H}_2$ coherence, as will be discussed later. □

In this paper, we will thus focus on the following two metrics: (1) the smallest eigenvalue $\lambda(\bar{L}_g)$ of the grounded Laplacian, characterizing the convergence rate and $\mathcal{H}_\infty$ coherence in dynamics with partially or fully stubborn agents, and (2) the trace of the inverse of the grounded Laplacian, $\mathrm{trace}(\bar{L}_g^{-1})$, characterizing the $\mathcal{H}_2$ coherence of the network.

IV. Bounds on the smallest eigenvalue of $\bar{L}_g$

In this section we provide bounds on the smallest eigenvalue of $\bar{L}_g$; our analysis builds upon and generalizes the proofs and results in [12] to the case where some of the nodes are partially stubborn, and demonstrates the role of the stubbornness parameters k_i in the convergence rate and $\mathcal{H}_\infty$ coherence.

Theorem 1: Consider a connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with a set of grounded nodes $\mathcal{S} \subset \mathcal{V}$. Let the grounded nodes be placed last in the ordering of nodes and let $k_1, k_2, \ldots, k_{n-|\mathcal{S}|}$ be nonnegative real numbers, with $K = \mathrm{diag}(k_1, k_2, \ldots, k_{n-|\mathcal{S}|})$. Let L_g be the grounded Laplacian matrix for $\mathcal{G}$, and define $\bar{L}_g = L_g + K$. Let β_i be the number of grounded nodes in node v_i's neighborhood, and let $\mathbf{x}$ be the nonnegative eigenvector corresponding to $\lambda(\bar{L}_g)$, normalized such that the largest component is $x_{max} = 1$.

Then

$$\max\left\{\min_{i\in\mathcal{V}\setminus\mathcal{S}}\{\beta_i+k_i\},\left(\frac{|\partial\mathcal{S}|+\mathcal{K}}{n-|\mathcal{S}|}\right)x_{min}\right\}\leq\lambda(\bar{L}_g)\leq$$
$$\min_{\emptyset\neq X\subseteq\mathcal{V}\setminus\mathcal{S}}\frac{|\partial X|+\sum_{v_i\in X}k_i}{|X|}\leq\frac{|\partial\mathcal{S}|+\mathcal{K}}{n-|\mathcal{S}|}\leq\max_{i\in\mathcal{V}\setminus\mathcal{S}}\{\beta_i+k_i\}, \tag{6}$$

where x_{min} is the smallest eigenvector component in $\mathbf{x}$ and $\mathcal{K}=\sum_{i=1}^{n-|\mathcal{S}|}k_i$. □

Proof: The proof of the lower bound $\frac{|\partial\mathcal{S}|+\mathcal{K}}{n-|\mathcal{S}|}x_{min}$ and the tightest and the second tightest upper bounds are the same as the proof of Theorem 1 in [12] and are omitted here. The extreme upper bound is due to the fact that $\sum_{i=1}^{n-|\mathcal{S}|}(\beta_i+k_i)=|\partial\mathcal{S}|+\mathcal{K}$ which gives $\frac{|\partial\mathcal{S}|+\mathcal{K}}{n-|\mathcal{S}|}\leq\max\{\beta_i+k_i\}$. For the lower bound $\min_{i\in\mathcal{V}\setminus\mathcal{S}}\{\beta_i+k_i\}$, we left-multiply the eigenvector equation $\bar{L}_g\mathbf{x}=\lambda(\bar{L}_g)\mathbf{x}$ by the vector consisting of all 1's to get

$$\lambda(\bar{L}_g)\sum_{i=1}^{n-|\mathcal{S}|}x_i=\sum_{i=1}^{n-|\mathcal{S}|}(\beta_i+k_i)x_i\geq\min_i\{\beta_i+k_i\}\sum_{i=1}^{n-|\mathcal{S}|}x_i,$$

which gives $\lambda(\bar{L}_g)\geq\min_i\{\beta_i+k_i\}$ as required. ■

In the following lemma we provide a sufficient condition under which the smallest component of the eigenvector corresponding to $\lambda(\bar{L}_g)$ goes to 1 and consequently the bound (6) becomes tight. This result is a straightforward extension of the bound given in [12], and thus the proof is omitted.

Lemma 1: Let $\mathbf{x}$ be the eigenvector corresponding to the smallest eigenvalue of $\bar{L}_g$. Then the smallest eigenvector component of $\mathbf{x}$ satisfies

$$x_{min}\geq1-\frac{2\sqrt{|\mathcal{S}||\partial\mathcal{S}|+\mathcal{K}(\mathcal{K}+2|\mathcal{S}|)}}{\lambda_2(\hat{L})}, \tag{7}$$

where $\hat{L}_{(n-|\mathcal{S}|)\times(n-|\mathcal{S}|)}$ is the Laplacian matrix formed by removing the fully stubborn agents and their incident edges. □

Thus, in networks where the number of fully stubborn agents (and their incident edges) and the aggregate level of stubbornness grow slowly compared to the algebraic connectivity of the network, the bounds on the smallest eigenvalue in (6) become tight.

Another possibility for the tightness of the bound (6) is the case where the level of stubbornness of the agents is larger than a certain value k. In this case according to the lower bound $\min_{i\in\mathcal{V}\setminus\mathcal{S}}\{\beta_i+k_i\}$ we have $\lambda(\bar{L}_g)\geq k$, regardless of the network connectivity and the location of the fully stubborn agents. In this case the lower bound containing x_{min} may be loose as discussed in the following example.

Example 1: Consider a line graph of length n with one fully stubborn agent at one end and the rest of the nodes having a stubbornness value of k. By increasing the length of the line, x_{min} and its corresponding lower bound go to zero while from the other lower bound we have $\lambda(\bar{L}_g)\geq k$. □

The following definition leads to another condition under which $\lambda(\bar{L}_g)$ is bounded away from zero.

Definition 1 ([21]): A subset of vertices $\mathcal{X}\subset\mathcal{V}$ is an f-dominating set if any vertex $v_i\in\mathcal{V}\setminus\mathcal{X}$ is connected to at least f vertices in $\mathcal{X}$. □

Based on the above definition, if the set of fully stubborn agents is an f-dominating set, then in (6) we have $\min_{i\in\mathcal{V}\setminus\mathcal{S}}\{\beta_i+k_i\}\geq f$ which gives $\lambda(\bar{L}_g)\geq f$, regardless of the stubbornness of the agents and the connectivity of the network.

V. The Effect of Increasing Stubbornness

We now turn our attention to analyzing the impact of increasing stubbornness on the convergence rate and the network coherence. It was shown in [9] that adding additional fully stubborn agents to a network with a single fully stubborn agent will increase the network $\mathcal{H}_2$ coherence (4). The following simple result extends this to both fully and partially stubborn agents.

Proposition 2: Let L_{gi} be the grounded Laplacian matrix for a network with just one fully stubborn agent v_i. Then

$$\mathrm{trace}(\bar{L}_g^{-1})\leq\mathrm{trace}(L_{gi}^{-1}),\quad\lambda(L_{gi})\leq\lambda(\bar{L}_g), \tag{8}$$

where $\bar{L}_g$ corresponds to a network consisting of any combination of fully or partially stubborn agents containing v_i as a fully stubborn agent.

Proof: Let $\mathcal{S}$ be the set of fully stubborn agents, containing v_i. Let L_g be the grounded Laplacian induced by $\mathcal{S}$, and let L_{gi} be the grounded Laplacian induced by v_i. By the interlacing theorem [19], we have $\lambda_j(L_{gi})\leq\lambda_j(L_g)$ for $j=1,2,\ldots,n-|\mathcal{S}|$. Thus $\mathrm{trace}(L_{gi}^{-1})\geq\mathrm{trace}(L_g^{-1})$. Now consider the case where some agents are partially stubborn, in addition to the fully stubborn agents $\mathcal{S}$. We have $\bar{L}_g=L_g+K$, where K is the diagonal matrix containing the stubbornness parameters. According to Weyl's inequality $\lambda_j(\bar{L}_g)\geq\lambda_j(L_g)$ for $j=1,2,\ldots,n-|\mathcal{S}|$, and thus we have $\mathrm{trace}(\bar{L}_g^{-1})\leq\mathrm{trace}(L_g^{-1})\leq\mathrm{trace}(L_{gi}^{-1})$, and $\lambda_1(L_{gi})\leq\lambda_1(\bar{L}_g)$ as required. ■

Remark 2: Proposition 2 indicates that the worst case scenario for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ coherence and convergence rate among all combinations of fully or partially stubborn agents is the case where there is one fully stubborn agent. □

VI. Coherence in Random Graphs

In this section we characterize network coherence in Erdos-Renyi random graphs and random regular graphs. To avoid the issue of choosing the stubbornness parameters in random graphs, here we will focus on cases where there are no partially stubborn agents. However, note from Proposition 2 that both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ coherences increase as agents become more stubborn, and thus the values of network disorder for non-stubborn agents discussed here are upper bounds for the case where partially stubborn agents are present in the network.

A. Erdos-Renyi Random Graphs

Definition 2: An Erdos-Renyi (ER) random graph $\mathcal{G}(n,p)$ is a graph on n nodes, where each edge between two distinct nodes is present independently with probability p (which could be a function of n). We say that a graph property holds *asymptotically almost surely* if the probability of drawing a graph with that property goes to 1 as $n \to \infty$. Let Ω_n be the set of all undirected graphs on n nodes. For a given graph function $f : \Omega_n \to \mathbb{R}_{\geq 0}$ and another function $g : \mathbb{N} \to \mathbb{R}_{\geq 0}$, we say $f(\mathcal{G}(n,p)) \leq (1 + o(1))g(n)$ asymptotically almost surely if there exists some function $h(n) \in o(1)$ such that $f(\mathcal{G}(n,p)) \leq (1+h(n))g(n)$ with probability tending to 1 as $n \to \infty$. Lower bounds of the above form have an essentially identical definition. □

Before discussing network coherence in ER random graphs we recall the following theorem for the smallest eigenvalue of the grounded Laplacian (with no partially stubborn agents) in such graphs.

Theorem 2 ([12]): Consider the Erdos-Renyi random graph $\mathcal{G}(n,p)$, where the edge probability p satisfies $\limsup_{n\to\infty} \frac{\ln n}{np} < 1$. Let $\mathcal{S}$ be a set of grounded nodes chosen uniformly at random with $|\mathcal{S}| = o(\sqrt{np})$. Then the smallest eigenvalue $\lambda(L_g)$ of the grounded Laplacian satisfies $(1-o(1))|\mathcal{S}|p \leq \lambda(L_g) \leq (1+o(1))|\mathcal{S}|p$ asymptotically almost surely. □

The above theorem covers a broad range of edge-probability functions, and includes constant p as a special case. Based on the above theorem, we obtain the following result for the $\mathcal{H}_\infty$ coherence in random graphs.

Theorem 3: Consider a random graph $\mathcal{G}(n,p)$ with $\limsup_{n\to\infty} \frac{\ln n}{np} < 1$. Let $\mathcal{S} \subset \mathcal{V}$ be a set of grounded nodes chosen uniformly at random with $|\mathcal{S}| = o(\sqrt{np})$. Then for $\mathcal{H}_\infty$ coherence we have

$$(1 - o(1))\frac{1}{|\mathcal{S}|p} \leq \frac{1}{\lambda(L_g)} \leq (1 + o(1))\frac{1}{|\mathcal{S}|p}, \qquad (9)$$

asymptotically almost surely. □

Using Theorem 2, we obtain the following result for the $\mathcal{H}_2$ coherence in ER random graphs with constant p.

Theorem 4: Consider a random graph $\mathcal{G}(n,p)$ with constant p. Let $\mathcal{S} \subset \mathcal{V}$ be a set of grounded nodes chosen uniformly at random with $|\mathcal{S}| = o(\sqrt{n})$. Then for $\mathcal{H}_2$ coherence we have

$$(1 - o(1))\frac{|\mathcal{S}|+1}{2|\mathcal{S}|p} \leq \frac{1}{2}\,\mathrm{trace}(L_g^{-1}) \leq (1 + o(1))\frac{|\mathcal{S}|+1}{2|\mathcal{S}|p}, \qquad (10)$$

asymptotically almost surely. □

Proof: For each node $v_i \in \mathcal{V} \setminus \mathcal{S}$, let β_i denote the number of grounded nodes that are in the neighborhood of v_i, i.e., $\beta_i \triangleq |\mathcal{S} \cup \mathcal{N}_i|$. We can then write the grounded Laplacian matrix L_g as $L_g = \bar{L} + E$, where $E = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_{n-|\mathcal{S}|})$ and $\bar{L}$ is the Laplacian matrix for the graph induced by the nodes $\mathcal{V} \setminus \mathcal{S}$ (recall that we are assuming that there are no partially stubborn nodes). Using Weyl's inequality for $i = 1, 2, ..., n - |\mathcal{S}|$, we have $\lambda_i(\bar{L}) \leq \lambda_i(L_g) \leq \lambda_i(\bar{L}) + |\mathcal{S}|$. Thus we have,

$$\frac{1}{2}\left(\sum_{i=2}^{n-|\mathcal{S}|}\left(\frac{1}{\lambda_i(\bar{L}) + |\mathcal{S}|}\right) + \frac{1}{\lambda_1(L_g)}\right) \leq \frac{1}{2}\,\mathrm{trace}(L_g^{-1})$$
$$\leq \frac{1}{2}\left(\sum_{i=2}^{n-|\mathcal{S}|}\frac{1}{\lambda_i(\bar{L})} + \frac{1}{\lambda_1(L_g)}\right). \qquad (11)$$

Noting that $\bar{L}$ is the Laplacian matrix for an Erdos-Renyi random graph on $n - |\mathcal{S}|$ nodes with constant p, for $n - |\mathcal{S}| = \Omega(n)$ we have $(1 - o(1))(n - |\mathcal{S}|)p \leq \lambda_2(\bar{L}) \leq (1 + o(1))(n - |\mathcal{S}|)p$ and $(1 - o(1))(n - |\mathcal{S}|)p \leq \lambda_{n-|\mathcal{S}|}(\bar{L}) \leq (1+o(1))(n-|\mathcal{S}|)p$ asymptotically almost surely [22]. Thus according to (11), Theorem 3 and considering the fact that $|\mathcal{S}| = o(\sqrt{n})$, the result is obtained. ■

Remark 3: With a single fully stubborn agent and constant p, both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ coherence metrics are within $(1\pm o(1))\frac{1}{p}$ asymptotically almost surely. □

Theorems 2 and 3 apply to any edge probability p satisfying $p(n) \geq \frac{c \ln n}{n}$, for constant $c > 1$. For any p in this range, the second smallest eigenvalue of the graph Laplacian satisfies $\lambda_2(L) = \Theta(np)$ asymptotically almost surely [12]. Thus, for this more general class of edge probabilities, we have the following looser bound on $\mathcal{H}_2$ coherence.

Corollary 1: Consider a random graph $\mathcal{G}(n,p)$ where the edge probability satisfies $p \geq \frac{c \ln n}{n}$ for any $c > 1$ and a set of grounded nodes $\mathcal{S} \subset \mathcal{V}$ chosen uniformly at random such that $|\mathcal{S}| = o(\sqrt{np})$. Then for $\mathcal{H}_2$ coherence we have

$$\frac{1}{2}\,\mathrm{trace}(L_g^{-1}) = \Theta(\frac{1}{p}), \qquad (12)$$

asymptotically almost surely.

Proof: By Theorem 2, we have $(1 - o(1))|\mathcal{S}|p \leq \lambda(L_g) \leq (1+o(1))|\mathcal{S}|p$ for the regime of p mentioned in the corollary. According to the Cauchy interlacing theorem and [12] we have $\beta np \geq 2d_{max} \geq \lambda_n(L) \geq \lambda_i(L_g) \geq \lambda_i(L) \geq \lambda_2(L) \geq \alpha np$ for some $\alpha, \beta > 0$ and $i = 2, ..., n - |\mathcal{S}|$. Adding all the eigenvalues into $\mathrm{trace}(L_g^{-1})$ gives the result. ■

B. Random Regular Graphs

Definition 3: Let $\Omega_{n,d}$ be the set of all undirected graphs on n nodes where every node has degree d (note that this assumes that nd is even). A *random d-regular graph*, denoted $\mathcal{G}_{n,d}$ is a graph drawn uniformly at random from $\Omega_{n,d}$. □

In these graphs, the algebraic connectivity satisfies

$$\lambda_2(L) \geq d - 2\sqrt{d-1} - \epsilon, \qquad (13)$$

asymptotically almost surely, for any $\epsilon > 0$ [23]. Thus for any constant integer $d > 2$, the algebraic connectivity $\lambda_2(L)$ will be bounded away from zero asymptotically almost surely. In the following proposition, we show that for any combination of the stubborn agents in a random regular graph, the network disorder will grow at most linearly with the network size.[1]

[1]There are some graphs in which the network disorder grows faster than the network size, e.g., d-dimensional grids for $d = 1, 2$ in which the network disorder is $O(n^2)$ and $O(n \log(n))$, respectively [17].

Proposition 3: Let $\mathcal{G}_{n,d}$ be a random d-regular graph on n nodes, with a set of fully stubborn agents $\mathcal{S}$. Then for sufficiently large (constant) d both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are $O(n)$ asymptotically almost surely.

Proof: For the case of a single fully stubborn agent v_i, it was shown in [12] that $\lambda(L_{gi}) = \Theta(\frac{1}{n})$ asymptotically almost surely for a random d-regular graph with sufficiently large d. This implies that the $\mathcal{H}_\infty$ norm in this case is $\Theta(n)$. Moreover, for $j = 2, 3, \ldots, n-1$, we have $\lambda_j(L_{gi}) \geq \lambda_2(L_{gi}) \geq \lambda_2(L) \geq \alpha d$ asymptotically almost surely for some $\alpha > 0$ and sufficiently large d; the second last inequality is due to the interlacing theorem, and the last inequality is due to (13). Thus for the $\mathcal{H}_2$ norm we have $\frac{1}{2}\,\mathrm{trace}(L_{gi}^{-1}) = \frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{\lambda_j(L_{gi})} = \frac{1}{2\lambda(L_{gi})} + \frac{1}{2}\sum_{j=2}^{n-1}\frac{1}{\lambda_j(L_{gi})} = \Theta(n) + O(\frac{n}{d}) = \Theta(n)$. For more than one fully stubborn agent, we use Proposition 2 to obtain the desired result. ■

VII. THE RELATIONSHIP BETWEEN TWO LEADER SELECTION PROBLEMS

While we have thus far taken the perspective of agents being *stubborn* in deviating from their initial values, an alternate perspective is to view fully stubborn agents as *leaders* that are trying to steer the network to a desired state [9], [13]. In particular, one may wish to choose the leader agents in order to maximize convergence rate [13], [10], or to maximize network coherence [9], [17], [6]. According to Remark 1, maximizing convergence rate via leader selection always results in maximizing $\mathcal{H}_\infty$ coherence as well. However, this is not the case for $\mathcal{H}_2$ coherence. In this section, we focus on the two objectives: convergence rate (or $\mathcal{H}_\infty$ coherence) and $\mathcal{H}_2$ coherence. We give an example of a graph where the best single leaders to maximize each of the objectives are far away from each other in the network, and then provide a sufficient condition for a single leader to maximize both objectives simultaneously.

A. An Example With Different Optimal Leaders for Each Objective

As shown in [6], the optimal single leader for maximizing $\mathcal{H}_2$ coherence is the node with maximal *information centrality* defined as $IC(\mathcal{G}) = \max_{i\in\mathcal{V}}[\frac{1}{n}\sum_j \gamma_{ij}]^{-1}$, where γ_{ij} is the sum of the lengths of *all* paths between nodes v_i and v_j in the network. In tree graphs, the information central vertex and the closeness central vertex (a vertex whose summation of distances to the rest of the vertices is minimum) in the network are the same.

For the metric of convergence rate, we define a *grounding centrality* for each node v_i, which is equal to the smallest eigenvalue of the grounded Laplacian induced by that node. Thus, the single best leader in terms of maximizing convergence rate and $\mathcal{H}_\infty$ coherence is the one with largest grounding centrality [11]. In the following example, we show that the grounding central vertex and the information central vertex can be far from each other in a graph.

Example 2: A *broom tree*, $B_{n,\Delta}$, is a star S_Δ with Δ leaf vertices and a path of length $n - \Delta - 1$ attached to the center of the star, as illustrated in Fig. 1 [24]. Consider the

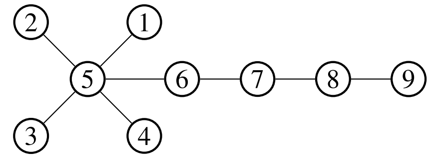

Fig. 1: Broom tree with $\Delta = 4$, $n = 9$.

broom tree $B_{2\Delta+1,\Delta}$. By numbering the vertices as shown in Fig. 1, for $\Delta = 500$, we find (numerically) that the grounding central vertex is vertex 614. The information central vertex is located at the middle of the star (vertex 501). The deviation of the grounding central vertex from the information central vertex increases as Δ increases. □

We now discuss conditions under which a single leader can optimize both objectives simultaneously.

B. A Sufficient Condition for a Single Leader to Optimize Both Objectives

Definition 4: The *resistance distance* r_{ij} between two vertices v_i and v_j in a graph is the equivalent resistance between these two points when we treat each edge of the graph as a 1Ω resistor. The *effective resistance* of vertex v_i is $R_i = \sum_{j\neq i} r_{ij}$. □

It can be shown that the resistance distance between v_i and v_j is the j-th diagonal element of L_{gi}^{-1}, where v_i is a single grounded vertex [25]. Thus, the effective resistance of vertex v_i is

$$R_i = \mathrm{trace}(L_{gi}^{-1}). \tag{14}$$

Moreover, the resistance distance between vertices v_i and v_j is given by [25]

$$r_{ij} = (e_i - e_j)^T L_{gk}^{-1}(e_i - e_j), \tag{15}$$

where $k \notin \{i, j\}$ is the index of an arbitrary vertex which becomes grounded and e_i is a vector of zeros except for a 1 in the element corresponding to the i-th vertex. Thus from (14) and (15), the effective resistance of v_i is

$$\begin{aligned} R_i = \mathrm{trace}(L_{gi}^{-1}) &= r_{ik} + \sum_{j\neq i}(e_i - e_j)^T L_{gk}^{-1}(e_i - e_j) \\ &= \mathrm{trace}(L_{gk}^{-1}) + n r_{ik} - 2S_i^k, \end{aligned} \tag{16}$$

where S_i^k is the summation of the elements of the i-th row (or column) in L_{gk}^{-1}. From (16) we have $\mathrm{trace}(L_{gk}^{-1}) - \mathrm{trace}(L_{gi}^{-1}) = 2S_i^k - n r_{ik}$. Thus for v_k to be a better leader than v_i for $\mathcal{H}_2$ coherence, it is sufficient to have

$$2\bar{S} - n r_{ik} \leq 0, \tag{17}$$

where $\bar{S} = \max_j S_j^k$ is the maximum row sum in L_{gk}^{-1}. On the other hand, from [11] we know that $\bar{S}x_{min} \leq \lambda_{max}(L_{gk}^{-1}) \leq \bar{S}$. Combining this with (17) yields $\lambda_{max}(L_{gk}^{-1}) \leq \frac{n r_{ik} x_{min}}{2}$ as a sufficient condition for v_k to be a better leadership candidate than v_i for the objective of $\mathcal{H}_2$ coherence. This sufficient condition can be more conveniently framed as $\lambda(L_{gk}) \geq \frac{2}{n r_{ik} x_{min}}$. From [25] we know that $r_{ik} \geq \max\{\frac{1}{d_i}, \frac{1}{d_k}\}$ where d_i and d_k are the degrees of vertices of v_i and v_k respectively. Thus a sufficient condition for the above inequality to hold is $\lambda(L_{gk}) \geq \frac{2\min\{d_i,d_k\}}{nx_{min}}$. A sufficient condition for this, based on (6) (with $\mathcal{S} = \{v_k\}$ and $\mathcal{K} = 0$), is

$$\frac{d_k x_{min}}{n-1} \geq \frac{2\min\{d_i, d_k\}}{nx_{min}}. \quad (18)$$

On the other hand, for v_k to be a better leader compared to v_i for maximizing convergence rate, according to (6) it is sufficient to have $\frac{d_k x_{min}}{n-1} \geq \frac{d_i}{n-1}$ which gives $d_k \geq \frac{d_i}{x_{min}}$, where x_{min} is again the smallest eigenvector component of $\mathbf{x}$, the eigenvector corresponding to $\lambda(L_{gk})$. Combining this with (18), a sufficient condition for v_k to be a better leader than v_i for both objectives simultaneously is

$$d_k \geq \max\left\{\frac{d_i}{x_{min}}, \frac{2d_i(n-1)}{nx_{min}^2}\right\}. \quad (19)$$

Since $\frac{2(n-1)}{nx_{min}} \geq 1$ for $n \geq 2$, we obtain the following result.

Proposition 4: Consider a connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. Node $v_k \in \mathcal{V}$ will simultaneously be the best single leader to maximize both $\mathcal{H}_2$ coherence and convergence rate (and thus $\mathcal{H}_\infty$ coherence) if $d_k \geq \frac{2d_i(n-1)}{nx_{min}^2}$ for all $v_i \in \mathcal{V} \setminus \{v_k\}$, where x_{min} is the smallest component of the eigenvector corresponding to the smallest eigenvalue of the grounded Laplacian L_{gk}.

Example 3: Consider an ER random graph $\mathcal{G}(n, p)$ with $p = \frac{c \ln n}{n}$ for some $c > 1$. The degree of each vertex in the graph is $d_i = \Theta(np)$ and the algebraic connectivity is $\lambda_2(L) = \Theta(np)$ asymptotically almost surely [12]. Suppose we wish to connect a single leader node v_{n+1} to this network in such a way that it is the single best leader for maximizing both convergence rate and coherence. Choose any $(2 + \epsilon)d_{max}$ nodes in the network for v_{n+1} to connect to for any $\epsilon > 0$, where d_{max} is the maximum degree of any node in the network. Let $L_{g,n+1}$ be the grounded Laplacian induced by v_{n+1}. From (7) (with no partially stubborn agents), the eigenvector for $\lambda(L_{g,n+1})$ has smallest component $x_{min} \geq 1 - \frac{2\sqrt{d_{n+1}}}{\lambda_2(L)}$ which goes to 1 asymptotically almost surely (since $d_{n+1} = (2+\epsilon)d_{max} = \Theta(np)$ and $\lambda_2(L) = \Theta(np)$). Thus the condition $d_{n+1} \geq \frac{2d_i(n-1)}{nx_{min}^2}$ will be satisfied for this node asymptotically almost surely. □

VIII. Summary and Conclusions

We considered convergence rate and network coherence in continuous-time consensus dynamics with partially and fully stubborn agents. We provided bounds on these quantities in terms of the stubbornness parameters and number of edges leaving the stubborn agents. Our bounds allowed us to provide tight characterizations of the coherence in Erdos-Renyi and random regular graphs. We analyzed conditions for a single leader to simultaneously optimize both coherence and convergence rate. Analyzing network coherence in other well known networks such as scale-free networks and random geometric networks is an avenue for future research.

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