

Resource Sharing Games with Failures and Heterogeneous Risk Attitudes

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Abstract—We study a setting where a set of players simultaneously invest in a shared resource. The resource has a probability of failure and a return on investment, both of which are functions of the total investment by all players. We use a simple reference dependent preference model to capture players with heterogeneous risk attitudes (risk seeking, risk neutral and risk averse). We show the existence and uniqueness of a pure strategy Nash equilibrium in this setting and examine the effect of different risk attitudes on players’ strategies in the presence of uncertainty. In particular, we show that at the equilibrium, risk averse players are pushed out of the resource by risk seeking players. We compare the failure probabilities in the decentralized (game-theoretic) and centralized settings, and show that our proposed game belongs to the class of best response potential games, for which there are simple dynamics that allow all players to converge to the equilibrium.

I. INTRODUCTION

Resource sharing games are strategic games in which players compete for a set of common resources. Players choose to invest (wealth, traffic, etc.) in a selected set of resources, and receive utility from each resource as a function of their investment and other players’ investments. A classic instance of such games is a *congestion game*, where the utility of a resource decreases with the total investment in the resource, i.e., as the resource becomes more “congested” [1]. Different formulations of congestion games have been applied to model several real-world applications such as load balancing [2], communication networks [3] and transportation networks [4]. One can also consider scenarios in which greater investment in a shared resource results in greater overall utility. Online collaborative environments [5] are an example of this.

In many scenarios, it is possible that resources might fail, resulting in a loss of investment for the players. In such situations, it is important to understand the strategic investment decisions made by players, and their implications for the potential failure of the resource. In this paper, we consider games with a single resource with a possibility of failure. The resource has a predefined rate of return and a failure probability associated with it, both of which are functions of total investment by the players. We consider settings in which the rate of return decreases (as in congestion games) and increases (as in online collaborative environments) with total investment. In addition, we model the failure probability as an increasing function of total

investment. This is reflective of several real-world settings where greater load on a resource makes it more susceptible to failure [6].

Probabilistic failure of resources adds uncertainty to the utility of the players. Studies from behavioral economics show that individuals typically exhibit different risk attitudes while making decisions under uncertainty. In this paper, we consider a reference dependent model of decision making, popularly known as “loss aversion” motivated by prospect theory [7]. Loss or gain of utility is measured with respect to a reference point. For a risk averse player, her decrease in utility under loss of investment is greater than the face-value of the investment; for risk seeking players, the opposite is true. In our formulation, the risk attitude of a player is modeled by a single parameter, which we vary across players to capture heterogeneity in their preferences (e.g., risk averse, risk neutral and risk seeking behavior).

We show existence and uniqueness of a pure Nash equilibrium (PNE) of our game for some natural classes of rate of return and failure probability functions. We allow players to have zero investment as a feasible strategy, so that they can effectively drop out of the game if necessary. Our results show that risk seeking players invest more than risk averse players, which drives the failure probability higher, and sufficiently risk averse players drop out of the game at a PNE. Only a set of most risk seeking players have a nonzero investment at the PNE, and we derive conditions to identify this set of players. The exact number of players who have dropped out depends on the risk attitudes of all the players. We compare the worst case failure probability at a PNE with the social welfare maximizing solution, and provide examples of decentralized dynamics that lead to convergence to the PNE of the game.

A. Related Work

Penn et al., [6] studied a class of congestion games, sometimes referred to as *resource selection games*, with resource failures. In their proposed model, the strategy set of a player is any subset of a finite set of available resources, and therefore discrete. The player receives a fixed return as long as at least one of her chosen resources does not fail. The failure probability of a resource is modeled as an increasing function of the number players sharing the resource. Ashlagi et al., [8] examine a version of this game where the number of players in the game is not known. Games with agent failures have also been proposed in [9]. The existence of a PNE and the impact of agent failures on the *Price of Anarchy*

This research was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

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(PoA) were studied for the settings mentioned above under the assumption that players are risk neutral.

Nikolova and Steir-Moses [10] introduced uncertainty to congestion games on networks by adding zero-mean noise to the congestion cost functions of the edges of the graph. Their model marked a departure from risk neutral agents to ones who have mean-variance risk preferences [11]. In related work [12], the authors studied a traffic assignment model with uncertain path costs and heterogeneous risk averse players. The risk preference of a player is captured by the degree to which she minimizes the maximum possible deviation from the mean congestion cost. Hoy [13] examined the existence of PNEs in atomic splittable congestion games with uncertainty, capturing risk preferences with utility functions that exhibited a diminishing marginal return property. Recently Piliouras et al., [14] studied the effects of several different risk preferences of players on the price of anarchy in stochastic congestion games.

Our setting is different from the above works in several ways. We model players with heterogeneous risk appetites in the presence of uncertainty due to failures. Our formulation subsumes expected utility maximization (risk neutrality) as a special case. We consider a simple reference dependent model of risk preferences motivated by prospect theory [7]. The risk preference of a player is captured by a single parameter, which allows us to model heterogeneous preferences while keeping the analysis tractable. To gain insights into strategic behavior in the heterogeneous risk averse setting, we focus on the single resource case in this paper and leave the multiple resource case for future work. Our risk preference model is motivated by the study of loss aversion behavior of players in settings such as mechanism design and rent seeking contests [15], [16], and we extend the application of such models to the resource sharing games setting.

II. PROBLEM FORMULATION

Consider a single resource, and let $\mathcal{N} = \{1, 2, \dots, |\mathcal{N}|\}$ be the set of players. Each player $i \in \mathcal{N}$ has an associated risk preference $k_i \in \mathbb{R}_+$, termed her *loss aversion index* [17], and a strategy space S_i . Without loss of generality, let players be ordered such that $k_1 \leq k_2 \leq \dots \leq k_n$. We refer to each element $x_i \in S_i$ as a strategy or investment of player i in the resource. We assume that S_i is a non-empty convex and compact subset of the nonnegative real numbers \mathbb{R}_+ for every player. In particular, we only consider strategy sets that are closed intervals of real numbers. Let $S = \prod_{i \in \mathcal{N}} S_i$ be the product of strategy spaces. Following standard notation, we denote $S_{-i} = \prod_{j \in \mathcal{N}, j \neq i} S_j$ as the joint strategy space of all players except i . The utility of player i is a function $u_i : S_i \times S_{-i} \rightarrow \mathbb{R}$, which we will specify below.

The resource is characterized by two functions; a rate of return $r : \mathbb{R}_+ \rightarrow \mathbb{R}$ and a probability of failure $p : \mathbb{R}_+ \rightarrow [0, 1]$. We assume that the rate of return and failure probability functions depend only on the total investment $x_T = \sum_{i \in \mathcal{N}} x_i$. Further, we let $r(x_T)$ and $p(x_T)$ be twice continuously differentiable. For simplicity of notation, let $y_i = \sum_{j=1, j \neq i}^n x_j$, $x_j \in S_j$, be the total investment of all

players other than i . The utility of player i is defined as

$$u_i(x_i, y_i) = \begin{cases} -k_i x_i, & \text{with probability } p(x_i + y_i) \\ x_i r(x_i + y_i) & \text{otherwise.} \end{cases} \quad (1)$$

When the resource fails, the players lose their investment entirely and receive no return. Instead, they receive a negative utility corresponding to the loss weighted by their loss aversion index. When the resource does not fail, a gain of $x_i r(x_i + y_i)$ is returned to the player.

The utility function in equation (1) belongs to the class of *reference dependent* preference models for decision making under uncertainty. This was popularized by Kahneman and Tversky in their seminal paper [7]. In their exact formulation, the utility for an outcome z is of the form

$$u_i(z) = \begin{cases} (z - z_0)^\alpha, & \text{when } z \geq z_0 \\ -k(z_0 - z)^\beta & \text{otherwise,} \end{cases} \quad (2)$$

where z_0 is the reference point with respect to which losses and gains are defined. Two additional parameters, α and β , are used to shape the utility function to have some desired properties, and k is the parameter that controls the risk aversion behavior. To simultaneously capture the heterogeneity in preferences of the players while keeping the analysis tractable, we choose $\alpha = \beta = 1$ and vary k among the players. We refer to k as the “loss aversion index”, in keeping with the terminology in [17]. An individual with $k > 1$ weights her losses more than gains of equal magnitude. In the behavioral economics literature, such an individual is typically referred to as “loss averse”. In this paper, we use the term “risk averse” synonymously with this notion of loss aversion. In contrast, risk seeking players have loss aversion indices in the interval $[0, 1)$. Risk neutral players have $k = 1$, which is the typical expectation maximization scenario. Thus, a larger value of k implies a higher preference towards loss aversion. We choose the reference point z_0 to be zero so that players compare their loss and gain to the case where they do not participate in the game.

The players are expected utility maximizers with respect to the utility function given by equation (1) and thus maximize

$$\begin{aligned} \mathbb{E}(u_i) &= (1 - p(x_T))x_i r(x_T) + p(x_T)(-k_i x_i) \\ &= x_i [r(x_T)(1 - p(x_T)) - k_i p(x_T)] \\ &\triangleq x_i f_i(x_T). \end{aligned} \quad (3)$$

Here $f_i(x_T)$ is the “effective rate of return” of player i . The expected utility formulation resembles the player costs (negative of utility) in atomic splittable congestion games [18], [3]. A common assumption in those formulations is that the cost functions are non-decreasing and convex. This implies that atomic splittable congestion games are instances of concave games [19], for which existence and uniqueness of PNE has been shown. However, monotonicity and convexity do not hold in equation (3) for a large class of functions $r(x_T)$ and $p(x_T)$ (as we discuss later). In addition, different players experience different effective congestion depending on their loss aversion indices, unlike the homogeneous congestion costs studied in [18], [3]. Nonetheless, we obtain results

showing existence and uniqueness of PNE for a fairly broad class of functions.

We consider $p(x_T)$ to be any strictly increasing function such that $p(\bar{x}) = 1$ for some $0 < \bar{x} < \infty$. For convenience, we assume that $\bar{x} = 1$. Any investment by a player with value greater than or equal to 1 will result in certain failure of the resource, and a loss for her. We allow zero as a feasible investment strategy so that a player can choose not to invest anything at all, effectively opting out of the game. Therefore, we restrict ourselves to the strategy set $S_i = [0, 1]$ for each player (including 1 to make S_i a compact subset of \mathbb{R}_+). We denote this resource sharing game as $\Gamma(\mathcal{N}, \{S_i\}_{i \in \mathcal{N}}, \{u_i\}_{i \in \mathcal{N}})$; note that the loss aversion indices for the users are contained within the utility functions in equation (1).

In the rest of this section, we characterize some properties of this game for general rate of return and increasing failure probability functions. After that, we will focus on specific classes of functions to obtain stronger results pertaining to the equilibrium of the game.

A. Social Welfare

We start by considering the optimal (centralized) allocation, defined with respect to the typical social welfare function, $\Psi(x_1, x_2, \dots, x_{|\mathcal{N}|}) = \sum_{i \in \mathcal{N}} \mathbb{E}(u_i)$, [20].

Proposition 1: In the social welfare maximizing solution, only the most risk seeking player has a nonzero investment.

Proof: The social welfare Ψ has the form

$$\begin{aligned} \Psi &= \sum_{i=1}^n x_i [r(x_T)(1 - p(x_T)) - k_i p(x_T)] \\ &= x_T r(x_T)(1 - p(x_T)) - p(x_T) \sum_{i=1}^n k_i x_i. \end{aligned}$$

For any x_T , the allocation $x_1 = x_T$ and $x_i = 0, \forall i > 1$, maximizes Ψ . In the event of a tie for k_1 , any allocation to all the players with the same k_1 results in the same optimal total investment and social welfare. ■

B. Characteristics of the PNE

We now characterize some useful properties of pure Nash equilibria for resource sharing games with failures and heterogeneous risk preferences. The following proposition shows that in any PNE of the game (if one exists), the only players with a nonzero investment are a set of most risk seeking players (regardless of the form of $r(\cdot)$ and $p(\cdot)$).

Proposition 2: At a PNE, if $x_i = 0$, then all players $j, j > i$, have $x_j = 0$.

Proof: It is easy to see that $f_i(x_T)$ and $\mathbb{E}(u_i)$ in equation (3) are strictly decreasing in k_i for fixed x_T . Note that a player will invest zero if and only if $f_i(x_T) \leq 0$. Thus $\forall x \in \mathbb{R}_+$, if $f_i(x) \leq 0$, then $f_j(x) \leq 0, \forall j > i$, and a player with negative effective rate of return prefers to drop out with zero investment. ■

Let $d \in \mathbb{N}$ be the largest index among all of the players with a strictly positive investment at a given PNE. We refer to the set of players $\{1, 2, \dots, d\}$ as the support of the PNE. When a player has a positive investment, it is strictly less

than 1. This investment satisfies the first order condition of each player's utility, given by

$$\frac{d\mathbb{E}(u_i)}{dx_i} = x_i f'_i(x_T) + f_i(x_T) = 0. \quad (4)$$

We know that $f_i(x_T) > 0$ for $i \in \{1, 2, \dots, d\}$, because otherwise player i will receive a non positive payoff and would drop out. Thus, $f'_i(x_T) < 0$ at a PNE for every player i in the support. We define $h_i(x_T)$ as the negative of the normalized investment of player i at a PNE, i.e.,

$$h_i(x_T) := -\frac{x_i}{x_T} = \frac{f_i(x_T)}{x_T f'_i(x_T)}. \quad (5)$$

It follows that

$$\sum_{i=1}^d h_i(x_T) = -1. \quad (6)$$

Proposition 3: The investment of a player i at a PNE is decreasing in her loss aversion index when $p(x_T)$ is non-decreasing in x_T .

Proof: Let x_T be the total investment of all the players in the PNE. Consider two players with loss aversion indices k_1 and k_2 , such that $k_2 > k_1$. Let their investments be x_1 and x_2 . If $x_1 = 0$, i.e., $f_1(x_T) \leq 0$, then $f_2(x_T) < 0$, and therefore $x_2 = 0$. So it suffices to concentrate on the case where $x_1 > 0$. From equation (4),

$$x_1 f'_1(x_T) + f_1(x_T) = 0.$$

Recall that $f'_1(x_T) < 0$. It follows that

$$\begin{aligned} f'_2(x_T) &= r'(x_T)(1 - p(x_T)) - r(x_T)p'(x_T) - k_2 p'(x_T) \\ &< f'_1(x_T) < 0, \end{aligned}$$

since $k_2 > k_1$ and $p'(x_T) > 0$.

Suppose $x_2 > x_1$. We then show a contradiction to equation (4). Formally,

$$\begin{aligned} x_2 f'_2(x_T) + f_2(x_T) &< x_2 f'_2(x_T) + f_1(x_T) \\ &< x_1 f'_1(x_T) + f_1(x_T) = 0. \end{aligned}$$

Therefore, $x_2 \leq x_1$. ■

Since x_T is the same for all players, it follows that $h_i(x_T)$ is strictly increasing in k_i .

The above results hold for arbitrary functions $r(x_T)$ and (in the case of Proposition 3) non-decreasing $p(x_T)$. In the rest of the paper, we will restrict our attention to certain classes of $r(x_T)$ and $p(x_T)$, and prove the existence and uniqueness of PNE for such functions.

III. EXISTENCE AND UNIQUENESS OF PNE

In this section, we focus on resource sharing games Γ satisfying the following assumption.

Assumption 1: Let $\Gamma(\mathcal{N}, \{S_i\}_{i \in \mathcal{N}}, \{u_i\}_{i \in \mathcal{N}})$ be a class of resource sharing games with the following properties.

- 1) The failure probability $p(x_T)$ is convex, strictly increasing and continuously differentiable with $p(1) = 1$.
- 2) The rate of return $r(x_T)$ is concave, strictly monotonic and continuously differentiable.
- 3) Player i 's strategy set is $S_i = [0, 1], \forall i \in \mathcal{N}$.

□

We will consider both the case where the rate of return is strictly decreasing in total investment, and the case where it is strictly increasing. When $r(x_T)$ is strictly decreasing, we further assume that $r(x_T) > 0$ whenever $0 \leq x_T \leq 1$.¹ When $r(x_T)$ is strictly increasing, we put no such restriction.

Let \bar{S}_{-i} denote the space of total investment by the players other than player i . Formally, $\bar{S}_{-i} = \{x \in \mathbb{R}_+ | \exists x_j \in S_j, j \in \mathcal{N}, j \neq i, \text{ such that } x = \sum_{j=1, j \neq i}^{|\mathcal{N}|} x_j\}$. Let $b_i(y_i)$ be a best response of player i to total investment y_i of other players, i.e., $b_i(y_i) \in \operatorname{argmax}_{x \in S_i} \mathbb{E}(u_i)$. We drop the subscript i in the following analysis for notational convenience as the results hold for every player i . We show that $b(y)$ is unique and continuous for $y \in \bar{S}$, where $\bar{S} = [0, |\mathcal{N}| - 1]$. We then prove the existence of a PNE using Brouwer's fixed point theorem. We will use the following identities derived from the expected utility in equation (3):

$$\begin{aligned} \frac{d\mathbb{E}(u)}{dx} &= [xr'(x+y) + r(x+y)](1-p(x+y)) \\ &\quad - xr(x+y)p'(x+y) \\ &\quad - k(p(x+y) + xp'(x+y)), \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{d^2\mathbb{E}(u)}{dx^2} &= [xr''(x+y) + 2r'(x+y)](1-p(x+y)) \\ &\quad - 2[xr'(x+y) + r(x+y)]p'(x+y) \\ &\quad - xr(x+y)p''(x+y) \\ &\quad - k(2p'(x+y) + xp''(x+y)). \end{aligned} \quad (8)$$

Lemma 1: For each player in resource sharing games satisfying Assumption 1, there exists a $\bar{y} \in [0, 1)$ such that 0 is a best response if and only if $y \geq \bar{y}$. Furthermore, there exists an open interval $\mathcal{I} \subset [0, \bar{y})$ such that if $y < \bar{y}$, then all best responses are positive, and each best response $b(y)$ satisfies $b(y) + y \in \mathcal{I}$. Furthermore, $f(x) > 0$ and $f'(x) < 0, \forall x \in \mathcal{I}$.

Proof: We consider decreasing and increasing rate of return functions separately.

Case 1: $r(x_T)$ is strictly decreasing.

In this case, it is easy to see that the effective rate of return f is strictly decreasing.

If $f(0) \leq 0$, then the only best response of the player is to drop out of the game, so in this case $\bar{y} = 0$.

Now consider effective rate of return functions for which $f(0) > 0$. Since $f(1) < 0$, it follows that there exists a unique $\bar{y} \in (0, 1)$ at which $f(\bar{y}) = 0$. Thus $b(y) = 0$ for $y \geq \bar{y}$, as any positive investment would result in negative utility for the player. For any $y < \bar{y}$, $f(x+y) > 0$ when $0 < x < \bar{y} - y$. Since there exists x for which $\mathbb{E}(u) = xf(x+y) > 0$, $b(y) > 0$ in the range $y \in [0, \bar{y})$. Thus, $b(y) > 0$ if and only if $y \in [0, \bar{y})$. In addition, $x_T = (b(y) + y) \in (0, \bar{y})$ since expected utility must be positive at the best response. We denote the interval $(0, \bar{y})$ as \mathcal{I} .

Case 2: $r(x_T)$ is strictly increasing.

¹If this is not the case, we can restrict the strategy set of the players to the interval $[0, \bar{x}]$ instead, where $r(\bar{x}) = 0$. When $x_T > \bar{x}$, no player would be willing to invest as $r(x_T)$ and consequently $f_i(x_T)$ will be negative. For convenience we assume $\bar{x} \geq 1$, so that r is positive over $[0, 1]$.

Here, the effective rate of return $f(x_T)$ is concave in x_T when $r(x_T) \geq 0$ as demonstrated below:

$$\begin{aligned} \frac{df(x_T)}{dx} &= r'(x_T)(1-p(x_T)) - (r(x_T) + k)p'(x_T), \quad (9) \\ \frac{d^2f(x_T)}{dx^2} &= r''(x_T)(1-p(x_T)) - 2r'(x_T)p'(x_T) \\ &\quad - (r(x_T) + k)p''(x_T) < 0. \end{aligned}$$

If $f(x_T) < 0, \forall x_T \in [0, 1]$, then the only best response for the player is to invest 0 and drop out of the game. As argued in the previous case, here $\bar{y} = 0$.

Thus suppose $f(x_T) > 0$ for some $x_T \in [0, 1]$, and define $\hat{x}_T := \inf\{x \in [0, 1] | f(x) > 0\}$. Then $r(x_T) \geq 0, \forall x_T \in [\hat{x}_T, 1]$, and it follows that $f(x_T)$ is concave when $x_T \in [\hat{x}_T, 1]$. Furthermore if $\hat{x}_T > 0$, then $f'(\hat{x}_T) > 0$. Now there are two possible cases arising from whether $f'(\hat{x}_T)$ is positive or negative, and we analyze them separately.

- Suppose $f'(\hat{x}_T) > 0$. Since $f'(1) < 0$ from equation (9), and f is concave in $(\hat{x}, 1]$, there must be a unique maximum of f , i.e., $\exists \hat{z} \in (\hat{x}_T, 1]$ such that $f'(\hat{z}) = 0$. From the concavity of f , $f'(x_T) < 0$ for $x_T > \hat{z}$. In addition, since $f(1) < 0, \exists \bar{y} \in (\hat{z}, 1]$ where $f(\bar{y}) = 0$.
- Otherwise suppose $\hat{x}_T = 0$ and $f'(\hat{x}_T) < 0$. This is identical to the previous case with decreasing rate of return. We define $\hat{z} = 0$ and it is easy to see that $\exists \bar{y} \in [0, 1]$ with $f(\bar{y}) = 0$.

When the total investment y of all other players satisfies $y \geq \bar{y}$, then $b(y) = 0$. Otherwise $b(y) > 0$ using a similar argument as in the previous case. Here we must have $f(b(y) + y) > 0$ and $f'(b(y) + y) < 0$, as argued after equation (4). We define \mathcal{I} as the region of interest in which $f'(x_T) < 0$ and $f(x_T) > 0$, i.e., $\mathcal{I} = (\hat{z}, \bar{y}) \subset [0, 1]$. Thus, whenever $b(y) > 0, y + b(y) \in \mathcal{I}$. ■

Lemma 2: The best response mapping $b(y)$ is unique for $y \in \bar{S}$.

Proof: We use the results of Lemma 1 to show that the expected utility has a single maximum in the interval \mathcal{I} .

Case 1: $r(x_T)$ is strictly decreasing.

When the best response $b(y)$ is strictly positive, it is a solution to $\frac{d\mathbb{E}(u)}{dx} = 0$. It is easy to see that for r and p satisfying the conditions in Assumption 1, the last two terms in equation (7) are strictly negative. Thus, any solution must have $[xr'(x+y) + r(x+y)] > 0$, for which $\frac{d^2\mathbb{E}(u)}{dx^2} < 0$ as is evident from equation (8). Thus, in the region where $[xr'(x+y) + r(x+y)] > 0, \mathbb{E}(u)$ is concave.

We know that r is concave and decreasing. Thus $[xr'(x+y) + r(x+y)]$ is strictly decreasing. At $x = 0$ and $y < \bar{y}$, we have $[xr'(x+y) + r(x+y)] > 0$. Let $[cr'(c+y) + r(c+y)] = 0$ for $c \in \mathbb{R}_+$. Thus, the only region in which $[xr'(x+y) + r(x+y)]$ is positive is $[0, c]$. As a result, we can have exactly one maximum of $\mathbb{E}(u)$ as a root of $\frac{d\mathbb{E}(u)}{dx}$. Hence $b(y)$ is unique for $y \in [0, \bar{y})$. Furthermore, by Lemma 1, $b(y) = 0, \forall y \geq \bar{y}$.

Case 2: $r(x_T)$ is strictly increasing.

Recall from Lemma 1 that $\forall x \in \mathbb{R}_+$ such that $x + y \in \mathcal{I}$, we have $f(x+y) > 0$ and $f'(x+y) < 0$. In this region, the

expected utility is concave, i.e.,

$$\frac{d^2\mathbb{E}(u)}{dx^2} = xf''(x+y) + 2f'(x+y) < 0.$$

Since the best response satisfies $b(y) + y \in \mathcal{I}$, $b(y)$ must be a maximum of $\mathbb{E}(u)$, and therefore it is unique.

This concludes the proofs for both cases. \blacksquare

To summarize, for both increasing and decreasing rate of return functions, there exists $0 \leq \bar{y} < 1$ such that $b(y) = 0$ if and only if $y \geq \bar{y}$. Otherwise when $x > 0$ is a best response to y , it satisfies the first order condition in equation (4). Define $g(x+y) = -\frac{f(x+y)}{f'(x+y)}$, which satisfies $g(b(y)+y) = b(y)$ when $b(y) > 0$. We first prove a useful property of $g(x+y)$.

Proposition 4: Let $r(x_T)$ and $p(x_T)$ satisfy the properties in Assumption 1, and let \mathcal{I} be the interval defined in Lemma 1. Then $g(x+y)$ is a strictly decreasing function of x and y when $x+y \in \mathcal{I}$.

The proof of the above proposition is given in the Appendix. We are now ready to show the monotonicity of the best response function.

Lemma 3: The best response mapping $b(y)$ is non-increasing in $y \in \bar{S}$.

Proof: Let $x_1 = b(y_1)$ and $x_2 = b(y_2)$, where $y_1, y_2 \in \bar{S}$ with $y_2 > y_1$. Let $y_2 < \bar{y}$ (recall that $f(\bar{y}) = 0$), otherwise the result follows trivially. According to the definition of $g(x+y)$, $x_1 = g(x_1+y_1)$ and $x_2 = g(x_2+y_2)$. Recall from the proof of Lemma 1 that $b(y)+y \in \mathcal{I}$. Thus, if $x_2 > x_1$, it follows from Proposition 4 that $x_2 = g(x_2+y_2) < g(x_1+y_1) = x_1$, which is a contradiction. \blacksquare

Lemma 4: The best response mapping $b(y)$ is continuous for $y \in \bar{S}$.

Proof: When $y \geq \bar{y}$, $b(y) = 0$, and is therefore continuous for all $y > \bar{y}$, and continuous from the right at \bar{y} . When $\bar{y}-\epsilon < y < \bar{y}$, any $x > \epsilon$ would result in $f(x+y) \leq 0$, while otherwise $xf(x+y) > 0$. Hence, it must be true that $0 < b(y) < \epsilon$. Thus $b(y)$ is continuous at \bar{y} .

Now let $y_1 \in [0, \bar{y})$, and let $x_1 = b(y_1)$. Note that $x_1+y_1 < \bar{y}$ by Lemma 1. Choose $\epsilon > 0$ small enough so that $x_1+y_1+\epsilon < \bar{y}$, and define $y_2 = y_1+\epsilon$, with corresponding positive best response $x_2 = b(y_2)$. From Lemma 3, we know that $0 < x_2 \leq x_1$. If $x_1 < \epsilon$ then $x_1-x_2 < \epsilon$, and the best response is right-continuous. Otherwise, recall from equation (4) that

$$x_i f'(x_i+y_i) + f(x_i+y_i) = 0,$$

for $i = \{1, 2\}$. Let $x = x_1 - \epsilon$ so that $x+y_2 = x_1+y_1$. It follows that

$$\begin{aligned} x f'(x+y_2) + f(x+y_2) &= x f'(x_1+y_1) + f(x_1+y_1) \\ &= -\epsilon f'(x_1+y_1) > 0, \end{aligned}$$

i.e., $\frac{d\mathbb{E}(u)}{dx} > 0$ when $y = y_2$ and $x = x_1 - \epsilon$. We will show that $\frac{d\mathbb{E}(u)}{dx}$ becomes zero in the interval $(x_1 - \epsilon, x_1)$, so that $x_2 \geq x = x_1 - \epsilon$.

When the rate of return is decreasing, we see from equation (8) that $\mathbb{E}(u)$ is concave whenever $\frac{d\mathbb{E}(u)}{dx}$ is positive, and thus for $y = y_2$, $\frac{d\mathbb{E}(u)}{dx}$ strictly decreases in the interval

$(x_1 - \epsilon, x_1)$ until it becomes negative. Therefore, the best response $x_2 \in (x_1 - \epsilon, x_1)$.

When the rate of return is increasing, $x+y_2 = x_1+y_1 \in \mathcal{I}$, where $\mathcal{I} = (\hat{z}, \bar{y})$ as defined in Lemma 1. Since x_2 is a best response, the resulting total investment $x_2+y_2 \in \mathcal{I}$ as well. If $x_1 - \epsilon + y_2 \in \mathcal{I}$, then from Lemma 2, $\mathbb{E}(u)$ is concave in x in $(x_1 - \epsilon, x_1)$ when $y = y_2$. Therefore $\frac{d\mathbb{E}(u)}{dx}$ is strictly decreasing in x and we have $x_1 - \epsilon \leq x_2 \leq x_1$. Otherwise, if $x_1 - \epsilon + y_2 < \hat{z}$, then $x_2 > x_1 - \epsilon$, as $x_2 + y_2 > \hat{z}$.

An identical argument holds when y_1 is decreased by a small amount. Thus, whenever y_2 is in a small neighborhood of y_1 , x_2 is in a small neighborhood of x_1 . This concludes the proof. \blacksquare

Theorem 1: Resource sharing games Γ satisfying Assumption 1 admit a PNE.

Proof: The strategy set of each player i , S_i , is a convex and compact subset of the Euclidean space, and so is the joint strategy space, $S \subset \mathbb{R}^{|\mathcal{N}|}$. Define a mapping, $T : S \rightarrow S$, such that $T(x_1, \dots, x_{|\mathcal{N}|}) = (b_1(y_1), \dots, b_{|\mathcal{N}|}(y_{|\mathcal{N}|}))$, where $y_i = x_T - x_i$. From Lemma 2, T is well defined and unique over the entire domain, and from Lemma 4, it is continuous. An application of Brouwer's fixed point theorem [21] yields a strategy profile $s = \{x_i^*\}_{i \in \mathcal{N}} \in S$ which is invariant under the best response mapping, and therefore it is a PNE of the game. \blacksquare

Note that since $p(x_T^*) = 1$ for $x_T^* \geq 1$, we must have $x_T^* < 1$ at a PNE. Otherwise any player with a positive investment could opt out and the strategy profile would not remain invariant under the best response mapping. To show uniqueness of PNE, we first prove the following Lemma.

Lemma 5: The negative of the normalized investment at a PNE with total investment x_T , $h_i(x_T) = \frac{f_i(x_T)}{x_T f'_i(x_T)}$, is a strictly increasing function of x_T for every player i in the support of the PNE.

Proof: Note that $h_i(x_T) = -\frac{g_i(x_T)}{x_T}$, where $g_i(x_T) = x_i$ at a PNE with total investment x_T , and $g_i(x_T) > 0$ for player i in the support. Therefore,

$$\frac{dh_i(x_T)}{dx_T} = -\left[\frac{g'_i(x_T)}{x_T} - \frac{g_i(x_T)}{x_T^2} \right].$$

From Proposition 4, $g'_i(x_T) < 0$, since the total investment x_T at the PNE must be in \mathcal{I} whenever player i is in the support. Therefore, $\frac{dh_i(x_T)}{dx_T} > 0$. \blacksquare

Theorem 2: The PNE for Γ under Assumption 1 is unique.

Proof: First, we show that the PNE is unique up to the number of players. Suppose otherwise, and let x_{T1} and x_{T2} be two different PNEs with respective supports $Supp_1(\Gamma) = \{1, 2, \dots, d_1\}$ and $Supp_2(\Gamma) = \{1, 2, \dots, d_2\}$, and let $d_2 > d_1$ without loss of generality. Since $d_2 \notin Supp_1(\Gamma)$, $f_{d_2}(x_{T1}) < 0$, while $f_{d_2}(x_{T2}) > 0$. Therefore $x_{T1} > x_{T2}$. We provide a contradiction to this.

For both x_{T1} and x_{T2} , equation (6) holds true. Thus,

$$\begin{aligned} & \sum_{j=1}^{d_1} -h_j(x_{T2}) + \sum_{j=d_1+1}^{d_2} -h_j(x_{T2}) = 1 \\ \implies & \sum_{j=1}^{d_1} -h_j(x_{T2}) < 1 = \sum_{j=1}^{d_1} -h_j(x_{T1}). \end{aligned}$$

From Lemma 5, we know that if $x_{T1} > x_{T2}$, we would have

$$-h_j(x_{T2}) > -h_j(x_{T1}), \forall j \in \{1, 2, \dots, d_1\},$$

which is a contradiction.

Now, for a given number of players, x_T at a PNE is unique, which follows from strict monotonicity of $h_j(x_T)$. Then the first order conditions give the unique individual investments of the players. ■

Corollary 1: Let Γ_1, Γ_2 be two resource sharing games with supports at the PNE denoted by $Supp(\Gamma_1) = \{1, 2, \dots, d_1\}$ and $Supp(\Gamma_2) = \{1, 2, \dots, d_2\}$. Let $k_{i,j}, i \in \{1, 2\}, j \in Supp(\Gamma_i)$ be the loss aversion index of player j in Γ_i . Further assume that $d_2 > d_1$. If $k_{1,i} = k_{2,i}, \forall i \in \{1, 2, \dots, d_1\}$, then $x_{T1}^* < x_{T2}^*$. □

IV. INEFFICIENCY OF PNE

To characterize the effect of selfish (game-theoretic) optimization by each player, we compare $p(\hat{x}_T)$, the failure probability of the resource at the socially optimal solution \hat{x}_T , to the failure probability at a PNE. From Proposition 1, we know that \hat{x}_T is the investment of the most risky player k_1 that maximizes her utility when playing in isolation.

Define Γ_n^k to be the family of resource sharing games satisfying Assumption 1, with n players and $k_1 = k$. Two games in this family differ only in the loss aversion indices of the players other than player 1. We define the inefficiency of the family of games Γ_n^k as

$$\eta = \sup \frac{p(x_T^*)}{p(\hat{x}_T)}, \quad (10)$$

where x_T^* is the total investment at a PNE for a game in Γ_n^k and the supremum is taken over the PNEs of the games in the family Γ_n^k . Note that \hat{x}_T is the same for all games in the family.

The following theorem shows that for the family of games Γ_n^k , the highest failure probability occurs when all players have loss aversion index k .

Theorem 3: For a given family $\Gamma_n^{k_1}$ of resource sharing games, the highest x_T^* occurs when $k_i = k_1, \forall i \in \{2, 3, \dots, n\}$.

Proof: Consider an instance of $\Gamma_1 \in \Gamma_n^{k_1}$, and let $\mathbf{k}^1 = \{k_1^1, k_2^1, \dots, k_n^1\}$ be the vector of loss aversion indices of the players, with $k_1^1 = k_1$, and $k_n^1 > k_1$. Let the PNE of Γ_1 have support $\{1, 2, \dots, d\}$ with $d > 1$ and total investment x_{T1} . Suppose that $k_d^1 > k_1^1$ and denote by r the smallest index in $\{1, 2, \dots, d-1\}$ such that $k_r^1 < k_{r+1}^1 = k_{r+2}^1 = \dots = k_d^1$.

Now, construct a new vector of loss aversion indices, \mathbf{k}^2 such that

$$k_j^2 = \begin{cases} k_j^1 & \text{for } j \in \{1, 2, \dots, r\} \cup \{d+1, d+2, \dots, n\}, \\ k_r^1 & \text{otherwise.} \end{cases}$$

The new vector \mathbf{k}^2 has the loss aversion indices unchanged for players not in the support, and for all players with indices less than or equal to r . All players previously in the support with index bigger than r now have their loss aversion index equal to k_r^1 . Let x_{T2} be the total investment at the PNE for the new game Γ_2 with loss aversion indices \mathbf{k}^2 . We now show that $x_{T2} > x_{T1}$.

If the support of the new game is smaller, then all players with index k_r^1 drop out in Γ_2 , while r was still part of the support in Γ_1 . Thus, following the argument in Theorem 2, we have $x_{T2} > x_{T1}$.

On the other hand, suppose the support of Γ_2 is $d_2 \geq d$. Denote by h_j^i the negative of the normalized investment by the player with index j in $\Gamma_i, i \in \{1, 2\}$. Assume to the contrary that $x_{T1} > x_{T2}$. From Lemma 5 we have

$$h_j^1(x_{T1}) > h_j^2(x_{T2}), \forall j \in \{1, 2, \dots, r\},$$

as those players' loss aversion indices remain unchanged.

Now for all players $j \in \{r+1, \dots, d\}$, we have $h_j^1(x_{T1}) > h_j^2(x_{T1}) > h_j^2(x_{T2})$. The first inequality follows from $h_j^i(x_T)$ being strictly increasing in k_j^i , which is a consequence of Proposition 3. The second inequality follows from Lemma 5 and $x_{T1} > x_{T2}$.

With similar arguments as in Theorem 2, for both x_1 and x_2 , equation (6) holds true. Thus,

$$\begin{aligned} & \sum_{j=1}^d -h_j^2(x_{T2}) + \sum_{j=d+1}^{d_2} -h_j^2(x_{T2}) = 1 \\ \implies & \sum_{j=1}^d -h_j^2(x_{T2}) \leq \sum_{j=1}^d -h_j^1(x_{T1}). \end{aligned}$$

But this is a contradiction, from the above arguments. So again it must be that $x_{T2} > x_{T1}$.

If we continue this procedure of obtaining a new vector of loss aversion profiles from an arbitrary starting vector, we will reach a game which has a support of players having loss aversion indices k_1 and with total investment at least as large as in the original support. From Corollary 1, when all players have the same k , more players at a PNE results in a greater x_T . Since the above analysis holds for any initial vector \mathbf{k}^1 with $k_1^1 = k_1$, the total investment for the game with all loss aversion indices equal to k_1 is greater than the total investment for any other vector of coefficients. ■

When all players have the same loss aversion index k , then all of them participate in the PNE and have equal investment. If the individual investment of the players is x , then the total investment is $x_T = nx$, where $n = |\mathcal{N}|$. The investment of each player satisfies the first order condition $f(nx) + xf'(nx) = 0$. By summing over all players, we obtain x_T as the solution to the following equation.

$$nf(x_T) + x_T f'(x_T) = 0,$$

where f is the effective rate of return for the common loss aversion index. We now discuss two specific rate of return and failure probability functions, for which we characterize the support at the PNE and obtain the exact inefficiency η when the number of players is large. The question of worst

case inefficiency in general resource sharing games under Assumption 1 remains open for future work.

A. Constant rate of return, linear $p(x_T)$

Consider the special case where $r(x_T) = b > 0$ and $p(x_T) = x_T$ on $[0, 1]$. The expected utility of player i is,

$$\mathbb{E}(u_i) = x_i[b(1 - x_T) - k_i x_T] = x_i[b - (b + k_i)x_T].$$

It is easy to see that $\mathbb{E}(u_i)$ is concave in x_i . Any player with a positive investment at the PNE must satisfy

$$x_i + x_T = \frac{b}{b + k_i}, \forall i \in \text{Supp}(\mathcal{N}).$$

This is a system of linear equations, $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A} = \mathbf{I} + \mathbb{1}\mathbb{1}^T$. Application of the Sherman–Morrison formula [22] gives the unique solution to the above system of equations as

$$x_i^* = \frac{b}{b + k_i} - \frac{1}{n + 1} \sum_{j=1}^n \frac{b}{b + k_j},$$

where n is the size of the support. Thus, for each player in the support,

$$\frac{1}{b + k_i} > \frac{1}{n + 1} \sum_{j=1}^n \frac{1}{b + k_j}.$$

If all players have identical loss aversion index k , the total investment at the PNE is

$$x_T^* = \frac{|\mathcal{N}|}{|\mathcal{N}| + 1} \frac{b}{b + k}.$$

Note that the social welfare maximizing investment is $\hat{x}_T = \frac{1}{2} \frac{b}{b + k_1}$, which is x_T^* for $|\mathcal{N}| = 1$. The inefficiency of the PNE is

$$\eta = \sup \frac{x_T^*}{\hat{x}_T} = \frac{2|\mathcal{N}|}{|\mathcal{N}| + 1}.$$

For large n , η approaches 2 from below.

B. Proportional rate of return, linear $p(x_T)$

Suppose $r(x_T) = ax_T$, with $a > 0$. The expected utility of player i is

$$\mathbb{E}(u_i) = x_i x_T [a(1 - x_T) - k_i].$$

For player i to opt out (i.e., $f_i(x_T) \leq 0$), we need

$$x_T \geq 1 - \frac{k_i}{a}.$$

Straightforward calculations yield that the worst case failure probability at a PNE is at most 1.5 times that of social optimum for large number of players. In particular, for $|\mathcal{N}|$ player games,

$$\sup \frac{x_T^*}{\hat{x}_T} = \frac{3(|\mathcal{N}| + 1)}{2(|\mathcal{N}| + 2)}.$$

V. CONVERGENCE TO NASH EQUILIBRIUM

Having established the existence of PNE for the classes of functions in Assumptions 1, we now show that the players can dynamically reach the PNE by observing each other's actions and optimally updating their investments. We will first start by showing that under Assumptions 1, Γ belongs to a class of games known as *Weak Strategic Substitutes*

(WSTS) game with aggregation, first defined by Dubey, et al. [23]. The definition is reproduced here for clarity.

Definition 1: A strategic game $\Gamma(\mathcal{N}, \{S_i\}_{i \in \mathcal{N}}, \{u_i\}_{i \in \mathcal{N}})$ is a WSTS game with aggregation if the player utilities are defined as, $u_i : S_i \times \bar{S}_{-i} \rightarrow \mathbb{R}$ and there exists a function $b_i : \bar{S}_{-i} \rightarrow \mathbb{R}$ for every $i \in \mathcal{N}$ such that:

- 1) $b_i(x) \in \operatorname{argmax}_{t \in S_i} u_i(t, x), \forall x \in \bar{S}_{-i}$,
- 2) b_i is continuous on \bar{S}_{-i} and
- 3) for $x, y \in \bar{S}_{-i}$ and $x > y$, $b_i(x) \leq b_i(y)$.

□

As argued in Remark 2 in [23], a WSTS game with unique best response is an instance of *best response potential games* [24].

For the resource sharing games Γ considered in this paper, the expected utility of a player only depends on the sum of other players' investments. When Γ satisfies Assumption 1, then Lemma 2, Lemma 4, and Lemma 3 verify the three properties in Definition 1, respectively.

In addition, since the best response function of each player is unique (by Lemma 2), it follows that $\Gamma(\mathcal{N}, \{S_i\}_{i \in \mathcal{N}}, \{u_i\}_{i \in \mathcal{N}})$ belongs to the class of best-response potential games. The following two simple dynamics, i.e., sequential best response dynamics [24] and simultaneous better reply dynamics [23], converge to the PNE of Γ . We briefly review the two dynamics here for clarity, and note that more details can be found in [24], [23].

A. Sequential best response dynamics

Let $\{s(t)\}_{t=1}^\infty = \{s_1(t), \dots, s_{|\mathcal{N}|}(t)\}_{t=1}^\infty$ be a sequence of strategy profiles such that player i updates her strategy when $t \bmod |\mathcal{N}|$ is equal to i . The strategy update is defined as:

$$s_i(t) = b_i \left(\sum_{j=1, j \neq i}^{|\mathcal{N}|} s_j(t-1) \right).$$

This sequential update procedure is shown to converge to the unique PNE [24].

B. Simultaneous better reply dynamics

Let $\{s(t)\}_{t=1}^\infty = \{s_1(t), \dots, s_{|\mathcal{N}|}(t)\}_{t=1}^\infty$ be a sequence of strategy profiles. Every player i maintains a conjecture of investment by every other player $\sigma_j^i(t)$, with an arbitrary starting value $\sigma_j^i(1)$. Let $\sigma_{-i}(t)$ denote the total investment of all other players according to the conjecture maintained by player i in time t . At every time period t , player i computes the unique best response $s_i(t)$ to her present conjecture $\sigma_{-i}(t)$ of strategies of other players. Conjectures are defined recursively as follows:

$$\sigma_j^i(t) = \lambda_t s_j(t-1) + (1 - \lambda_t) \sigma_j^i(t-1), \forall j \in \mathcal{N} \setminus \{i\},$$

where $\{\lambda_t\}_{t=1}^\infty$ is a sequence of positive numbers such that $\sum_{t=1}^\infty \lambda_t = \infty$ and $\sum_{t=1}^\infty \lambda_t^2 < \infty$. By Remark 2 of [23], the sequence $\{s_i(t)\}_{i \in \mathcal{N}}$ converges to the PNE of the game as $t \rightarrow \infty$.

VI. CONCLUSION

In this paper, we studied a general class of resource sharing games with probabilistic loss of investment for the players due to resource failure. We introduced a framework to model heterogeneity of risk preferences of players, and showed the existence of a pure strategy Nash equilibrium for certain natural instances of the game. We gave examples of inefficiency and convergence dynamics for this class of games. The generalization of this model to the case of multiple resources remains an important avenue for future work. Our current model also assumes that the loss aversion indices of all the players is common knowledge. This assumption can be relaxed to a Bayesian setting, where players have distributional information about the loss aversion indices of other players. The resulting Bayesian Nash equilibrium would be interesting to characterize. Finally, one could consider the problem of designing incentives, such as a taxing policy, such that failure probability at an equilibrium can be maintained within a desired limit. This mechanism design aspect of the problem can be applied to several real world scenarios such as transportation networks, cloud computing, etc.

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APPENDIX

Proof of Proposition 4: Recall that $g(x+y) = -\frac{f(x+y)}{f'(x+y)}$. Then,

$$\frac{dg(x+y)}{dx} = -\frac{(f'(x+y))^2 - f(x+y)f''(x+y)}{(f'(x+y))^2}. \quad (11)$$

Case 1: $r(x_T)$ is strictly decreasing.

For clarity of presentation, denote $f(x+y)$, $r(x+y)$ and $p(x+y)$ as f , r , p , respectively. We have

$$\begin{aligned} f &= r(1-p) - kp, \\ f' &= r'(1-p) - rp' - kp', \\ f'' &= r''(1-p) - 2r'p' - rp'' - kp'', \\ f'^2 &= (r'(1-p))^2 + (p'(r+k))^2 \\ &\quad - 2r'p'(r+k)(1-p), \\ f f'' &= (r(1-p) - kp)(r''(1-p) - rp'' - kp'') \\ &\quad + 2r'p'kp - 2r'p'r(1-p). \end{aligned}$$

Note that $f > 0$, as $x+y \in \mathcal{I}$. Let $\gamma = r''(1-p) - rp'' - kp'' < 0$. Then,

$$\begin{aligned} f f'' &= f\gamma + 2r'p'kp - 2r'p'r(1-p), \\ f'^2 - f f'' &= (r'(1-p))^2 + (p'(r+k))^2 \\ &\quad - 2r'p'(r+k)(1-p) \\ &\quad - f\gamma - 2r'p'kp + 2r'p'r(1-p) \\ &= (r'(1-p))^2 + (p'(r+k))^2 - f\gamma \\ &\quad - 2r'p'kp - 2r'p'k(1-p) > 0. \end{aligned}$$

Therefore, $\frac{dg(x+y)}{dx} < 0$. By symmetry we also have $\frac{dg(x+y)}{dy} < 0$.

Case 2: $r(x_T)$ is strictly increasing.

When $x+y \in \mathcal{I}$, $f(x+y) > 0$. Recall from the proof of Lemma 1 that $f''(x_T) < 0$ for all $x_T \in \mathcal{I}$. Thus, it follows directly from equation (11) that $\frac{dg(x+y)}{dx} < 0$. ■