Spectral Properties of the Grounded Laplacian Matrix with Applications to Consensus in the Presence of Stubborn Agents

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***Abstract*— We study linear consensus and opinion dynamics in networks that contain stubborn agents. Previous work has shown that the convergence rate of such dynamics is given by the smallest eigenvalue of the grounded Laplacian induced by the stubborn agents. Building on this, we define a notion of centrality for each node in the network based upon the smallest eigenvalue obtained by removing that node from the network. We show that this centrality can deviate from other well known centralities. We then characterize certain properties of the smallest eigenvalue and corresponding eigenvector of the grounded Laplacian in terms of the graph structure and the expected absorption time of a random walk on the graph.**

I. Introduction

Collective behavior in networks of agents has been studied in a variety of communities including sociology, physics, biology, economics, computer science and engineering [1], [2]. A topic that has received particular interest is that of opinion dynamics and consensus in networks, where the agents repeatedly update their opinions or states via interactions with their neighbors [3]–[5]. For certain classes of interaction dynamics, various conditions have been provided on the network topology that guarantee convergence to a common state [6]–[9].

Aside from providing conditions under which convergence occurs, the question of what value the agents converge to is also of importance. In particular, the ability of certain individual agents in the network to excessively influence the final value can be viewed as both a benefit and a drawback, depending on whether those agents are viewed as leaders or adversaries. The effect of individual agents' initial values on the final consensus value has been studied in [10] [11]. When a subset of agents is fully stubborn (i.e., they refuse to update their value), it has been shown that under a certain class of linear update rules, the values of all other agents asymptotically converge to a convex combination of the stubborn agent's values [12].

Given the ability of individuals to influence linear opinion dynamics by keeping their values constant, a natural metric to consider is the speed at which the population converges to the final value for a given set of stubborn agents or leaders. The convergence rate is dictated by spectral properties of certain matrices; in continuous-time dynamics, this is the *grounded Laplacian* matrix [13]. There are various recent works that investigate the *leader selection* problem, where the goal is to select a set of leaders (or stubborn agents) to maximize the convergence rate [12], [14]–[16]. Similarly, one can consider the problem of leader selection problem in networks in the presence of noise [17], where the main goal is to minimize the steady state error covariance of the followers [18].

In this paper, we continue the investigation of convergence rate in continuous-time linear consensus dynamics with stubborn agents. Specifically, given the key role of the grounded Laplacian in this setting, we provide a characterization of certain spectral properties of this matrix, adding to the literature on such matrices [12]–[14], [19]. We define a natural centrality metric, termed grounding centrality, based on the smallest eigenvalue of the grounded Laplacian induced by each node. This captures the importance of the node in the network via the rate of convergence it would induce in the consensus dynamics if chosen as a stubborn agent. We show that this centrality metric can deviate from other common centrality metrics such as closeness centrality, betweenness centrality and degree centrality. We then provide bounds on the smallest eigenvalue of the grounded Laplacian, along with certain properties of the associated eigenvector. Finally, we compare the smallest eigenvalue to the maximum expected absorption time for a random walk on the underlying graph. We provide bounds based on the deviation in components of the eigenvector for the smallest eigenvalue. Our results on the spectrum of the grounded Laplacian are of independent interest, with applications to various settings [13], [19].

The structure of the paper is as follows. In Section III we introduce the mathematical formulation of the consensus problem in the presence of stubborn agents, and show the role of the grounded Laplacian matrix. We introduce the notion of grounding centrality in Section IV. Section V introduces some spectral properties of the grounded Laplacian matrix for general graphs as well as some unique properties for special graphs such as trees. In Section VI we make a connection to absorbing random walks on Markov chains and provide bounds on the smallest eigenvalue of the grounded Laplacian in terms of the spread of the components in the associated eigenvector. Section VII concludes the paper.

II. Definitions and Notation

We use $\mathcal{G}(\mathcal{V},\mathcal{E})$ to denote a weighted and undirected graph where $\mathcal{V}$ is the set of vertices (or nodes) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. The *neighbors* of vertex $v_i \in \mathcal{V}$ in graph $\mathcal{G}$ are given by the set $\mathcal{N}_i = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$. The weight associated with an edge connecting vertex $v_i \in \mathcal{V}$ to vertex $v_j \in \mathcal{V}$ is denoted by w_{ij}. We take the weights to be nonnegative and symmetric (i.e., $w_{ij} = w_{ji}$).

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We define $d_i = \sum_{v_j \in \mathcal{N}_i} w_{ij}$; in unweighted graphs where each $w_{ij} \in \{0,1\}$, d_i simply denotes the degree of vertex v_i. We will be considering a nonempty subset of vertices $\mathcal{S} \subset \mathcal{V}$ to be *stubborn*, and assume without loss of generality that the stubborn agents are placed last in an ordering of the agents. We say a vertex $v_i \in \mathcal{V} \setminus \mathcal{S}$ is an α-vertex if $\mathcal{N}_i \cap \mathcal{S} \neq \emptyset$, and say v_i is a β-vertex otherwise. Vertices in $\mathcal{V} \setminus \mathcal{S}$ will also be called *followers*.

III. Basic Models

A. Consensus Model

Consider a network of agents described by the connected and undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, representing the structure of the system, and a set of differential equations describing the interactions between each pair. In the study of consensus and continuous-time opinion dynamics [7], each agent $v_i \in \mathcal{V}$ starts with an initial scalar state (or opinion) $y_i(t)$, which it continuously updates as a function of the states of its neighbors. A commonly studied version of these dynamics involves a linear update rule of the form

$$\dot{y}_i(t) = \sum_{v_j \in \mathcal{N}_i} w_{ij}(y_j(t) - y_i(t)), \tag{1}$$

where w_{ij} is the weight assigned by node v_i to the state of its neighbor v_j. Aggregating the state of all of the nodes into the vector $Y(t) = \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \end{bmatrix}^T$, equation (1) can be written as

$$\dot{Y} = -LY, \tag{2}$$

where L is the *weighted graph Laplacian* defined as $L = D - A$ with degree matrix $D = diag(d_1, d_2, \ldots, d_n)$ and weighted adjacency matrix A containing the weight w_{ij} in entry (i,j).

For an undirected graph $\mathcal{G}$ with symmetric weights (i.e., $w_{ij} = w_{ji}$ for all $i \neq j$), L is a symmetric matrix with real eigenvalues that can be ordered sequentially as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2\Delta$ where $\Delta = \max_i \sum_{v_j \in \mathcal{N}_i} w_{ij}$. When the graph $\mathcal{G}$ is connected, the second smallest eigenvalue λ_2 is positive and all nodes asymptotically reach consensus on the value $y_{eq} = \frac{c^T Y(0)}{c^T \mathbf{1}}$ where c is the left eigenvector of L corresponding to λ_1 [7].

B. Consensus In The Presence of Stubborn Agents

Assume that there is a subset $S \subset \mathcal{V}$ of agents whose opinions are kept constant throughout time, i.e., $\forall v_s \in S$, $\exists y_s \in \mathbb{R}$ such that $y_s(t) = y_s \ \forall t \in \mathbb{R}_{\geq 0}$. Such agents are known as *stubborn agents* or *leaders* (depending on the context) [12], [14]. In this case the dynamics (2) can be written in the matrix form

$$\begin{bmatrix} \dot{Y}_F(t) \\ \dot{Y}_S(t) \end{bmatrix} = -\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Y_F(t) \\ Y_S(t) \end{bmatrix}, \tag{3}$$

where Y_F and Y_S are the states of the followers and stubborn agents, respectively. Since the stubborn agents keep their values constant, the matrices L_{21} and L_{22} are zero. It can be shown that the state of each follower asymptotically converges to a convex combination of the values of the stubborn agents, and that the rate of convergence is asymptotically given by the smallest eigenvalue of L_{11} [12]. The matrix L_{11} is called the *grounded Laplacian* formed by removing the rows and columns corresponding to the stubborn agents from L. In the rest of the paper, the grounded Laplacian formed by removing the rows and columns of L corresponding to the vertices in set $S \subset \mathcal{V}$ will be denoted by L_{gS}, and L_{gs} if $S = \{v_s\}$. When the set S is fixed and clear from the context, we will simply use the notation L_g to denote the grounded Laplacian. For any given set S, we denote the smallest eigenvalue of the grounded Laplacian matrix by $\lambda(L_{gS})$ or simply λ.

Remark 1: When the graph $\mathcal{G}$ is connected, the grounded Laplacian matrix is a positive definite matrix and its inverse is a nonnegative matrix (i.e., a matrix whose elements are nonnegative) [20]. From the Perron-Frobenius (P-F) theorem, the eigenvector associated with the smallest eigenvalue of the grounded Laplacian can be chosen to be nonnegative (elementwise). Furthermore, when the stubborn agent is not a cut vertex, the eigenvector associated with the smallest eigenvalue can be chosen to have all elements positive. □

There have been various recent investigations of graph properties that impact the convergence rate for a given set of stubborn agents, leading to the development of algorithms to find approximately optimal sets of stubborn agents to maximize the convergence rate [12], [14], [16]. Motivated by the fact that the smallest eigenvalue of the grounded Laplacian dominates the convergence rate, in the next section we define a notion of *grounding centrality* for each node in the network, corresponding to how quickly that node would lead the rest of the network to steady state if chosen to be stubborn. We show via an example that grounding centrality can deviate from other common centrality metrics in networks, and then in subsequent sections, we provide bounds on the grounding centrality based on graph theoretic properties.

IV. Grounding Centrality

There are various metrics for evaluating the importance of individual nodes in a network. Common examples include eccentricity (the largest distance from the given node to any other node), closeness centrality (the sum of the distances from the given node to all other nodes in the graph), degree centrality (the degree of the given node) and betweenness centrality (the number of shortest paths between all nodes that pass through the given node) [21] [22]. In addition to the above centrality metrics (which are purely based on position in the network), one can also derive centrality metrics that pertain to certain classes of dynamics occurring on the network. For example, [23] assigned a centrality score to each node based on its component in a left-eigenvector of the system matrix. Similarly, [24] studied discrete time consensus dynamics and proposed centrality metrics to capture the influence of forceful agents. The discussion on convergence rate induced by each node in the last section also lends itself to a natural dynamical centrality metric, defined as follows.

Definition 1: Consider a weighted graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, with weight w_{ij} for edge $(i,j) \in \mathcal{E}$. The *grounding centrality* of each vertex $v_s \in \mathcal{V}$, denoted by $I(s)$, is $I(s) = \lambda(L_{gs})$. The set of *grounding central* vertices in the graph $\mathcal{G}$ is given by $IC(\mathcal{G}) = \text{argmax}_{v_s \in \mathcal{V}} \lambda(L_{gs})$. □

According to the above definition, a grounding central vertex $v_s \in IC(\mathcal{G})$ is a vertex that maximizes the asymptotic convergence rate if chosen as a stubborn agent (or leader), over all possible choices of single stubborn agents.

It was shown in [12] and [16] that the convergence time in a network in the presence of stubborn agents (or leaders) is upper bounded by an increasing function of the distance from the stubborn agents to the rest of the network. In the case of a single stubborn agent, the notion of distance from that agent to the rest of the network is similar to that of closeness centrality and eccentricity. While this is a natural approximation for the grounding centrality (and indeed plays a role in the upper bounds provided in those papers), there are graphs where the grounding centrality can deviate from other well known centralities, as shown below.

Example 1: A *broom tree*, $B_{n,\Delta}$, is a star S_Δ with Δ leaf vertices and a path of length $n-\Delta-1$ attached to the center of the star, as illustrated in Fig. 1 [25].

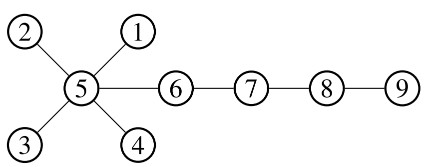

Fig. 1: Broom tree with $\Delta = 4$, $n = 9$.

Define the closeness central vertex as a vertex whose summation of distances to the rest of the vertices is minimum, the degree central vertex as a vertex with maximum degree in the graph and the center of the graph as a vertex with smallest eccentricity [22]. Consider the broom tree $B_{2\Delta+1,\Delta}$. By numbering the vertices as shown in Fig. 1, for $\Delta = 500$, we find (numerically) that the grounding central vertex is vertex 614, and the center of the graph is 750. The closeness and degree and betweenness central vertices are located at the middle of the star (vertex 501). The deviation of the grounding central vertex from the other centralities and the center of this graph increases as Δ increases. □

As discussed in the previous section, the problem of characterizing the grounding centrality of vertices using graph-theoretic properties is an ongoing area of research [12]–[14], [16]. In the next section, we develop some bounds for $\lambda(L_g)$ by studying certain spectral properties of the grounded Laplacian.

V. Spectral properties of the grounded Laplacian matrix L_g

Theorem 1: Consider a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with a single stubborn agent $v_s \in \mathcal{V}$. Define $\bar{w} = \max_{v_i \in \mathcal{N}_s} w_{si}$ to be the largest edge weight between any α-vertex and the stubborn agent. Then the smallest eigenvalue of the weighted grounded Laplacian matrix L_{gs} satisfies $0 \leq \lambda \leq \bar{w}$. □

Proof: The lower bound is simply obtained from the Cauchy interlacing theorem, which states that

$$0 = \lambda_1(L) \leq \lambda(L_{gs}) \leq \lambda_2(L),$$

where $\lambda_2(L)$ is the second smallest eigenvalue of the graph Laplacian matrix and is known as the algebraic connectivity of graph $\mathcal{G}$ [26]. For the upper bound, let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} \end{bmatrix}^T$ denote the eigenvector associated with the smallest eigenvalue. The eigenvector equation for the i-th vertex is given by

$$d_i x_i - \sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} w_{ij} x_j = \lambda x_i. \tag{4}$$

Adding all of these eigenvector equations (or equivalently, multiplying both sides of the eigenvector equation corresponding to $\lambda(L_{gs})$ by a vector consisting of all 1's), we have

$$\mathbf{1}^T L_g \mathbf{x} = \mathbf{1}^T \lambda \mathbf{x}. \tag{5}$$

The left hand side of (5) equals $\sum_{v_j \in \mathcal{N}_s} w_{sj} x_j$ and the right hand side equals $\lambda \sum_{v_j \in \mathcal{V} \setminus \{v_s\}} x_j$. Thus (5) gives

$$\sum_{v_j \in \mathcal{N}_s} w_{sj} x_j = \lambda \sum_{v_j \in \mathcal{V} \setminus \{v_s\}} x_j. \tag{6}$$

Using the facts that $\bar{w} = \max_{v_j \in \mathcal{N}_s} w_{sj}$, $\mathcal{N}_s \subseteq \mathcal{V} \setminus \{v_s\}$ and all eigenvector components are nonnegative, equation (6) yields $\lambda \leq \bar{w} \frac{\sum_{v_j \in \mathcal{N}_s} x_j}{\sum_{v_j \in \mathcal{V} \setminus \{v_s\}} x_j}$, from which the result follows. ■

The upper bound given above for $\lambda(L_g)$ is tighter than the upper bound obtained from the Cauchy interlacing theorem. This difference becomes more apparent when we consider the fact that in certain graphs (e.g., unweighted Erdos-Renyi random graphs), the algebraic connectivity grows unboundedly [27] while $\lambda(L_g)$ remains bounded. The above theorem also provides the following characterization of the smallest eigenvalue of the grounded Laplacian in the case of multiple stubborn agents.

Corollary 1: Given a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with a set $S \subset \mathcal{V}$ of stubborn agents, for each α-vertex v_j, define

$$w_{Sj} = \sum_{v_s \in \mathcal{N}_j \cap S} w_{sj}.$$

The smallest eigenvalue of the grounded Laplacian L_{gS} satisfies $0 \leq \lambda \leq \bar{w}$ where $\bar{w} = \max_{v_j \in \mathcal{V} \setminus S} w_{Sj}$.

Proof: When $|S| = 1$, the result follows from Theorem 1. If $|S| > 1$, we construct a new graph $\bar{\mathcal{G}}$ with vertex set $(\mathcal{V} \setminus S) \cup \{v_s\}$, with a single stubborn agent v_s connected to all α-vertices in $\mathcal{V} \setminus S$. For each α-vertex v_j, define the weight on the edge (v_s, v_j) as $w_{sj} = w_{Sj}$ defined in the corollary. Since the rows and columns corresponding to the stubborn agents are removed to obtain the grounded Laplacian matrix, the grounded Laplacian for graph $\mathcal{G}$ is the same as the grounded Laplacian for graph $\bar{\mathcal{G}}$. This converts the network with multiple stubborn agents into a network with a single stubborn agent and the result follows from Theorem 1. ■

For the case of graphs with homogeneous weights w, Theorem 1 shows that $0 \leq \lambda \leq w$, and in particular, $\lambda \leq 1$ for unweighted graphs. The following result shows that the upper bound is reached if and only if the leader agents are directly connected to every other agent.

Proposition 1: Given a connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with homogeneous positive weights w and a stubborn agent $v_s \in \mathcal{V}$, we have $\lambda(L_{gs}) = w$ if and only if $\mathcal{N}_s = \mathcal{V} \setminus \{v_s\}$.

Proof: We start by rearranging (6) to obtain

$$\lambda \sum_{v_j \in \mathcal{V} \setminus \{\mathcal{N}_s \cup \{v_s\}\}} x_j = (w - \lambda) \sum_{v_j \in \mathcal{N}_s} x_j. \quad (7)$$

The *if* part is trivially true according to (7). In order to show the *only-if* part, we prove by contradiction. Suppose $\lambda = w$ and there exists a β-vertex in $\mathcal{G}$ (i.e., a vertex that is not a neighbor of v_s). According to (7) its eigenvector component should be zero and according to (4) the eigenvector components of its neighbors are zero. If L_{gs} is irreducible, i.e., the stubborn agent v_s is not a cut vertex, since $\mathcal{G}$ is connected, the eigenvector will be $\mathbf{x} = \mathbf{0}$. If L_{gs} is reducible, i.e., the stubborn agent v_s is a cut vertex, by removing v_s the graph $\mathcal{G}$ is partitioned into separate components. If there exists a β-vertex in the component of $\mathcal{G}$ whose eigenvector elements are nonzero, since this component is connected, by the same argument all of the eigenvector elements in this component will be zero. As the eigenvector elements of other components are zero we have $\mathbf{x} = \mathbf{0}$. ■

Throughout the rest of the paper we focus on the case of a single stubborn agent $v_s \in \mathcal{V}$ and homogenous weights $w = 1$, and denote the resulting grounded Laplacian by L_{gs} (or L_g). Note that changing the weight w only scales the degrees of the vertices and the eigenvalues of L_{gs}, but does not affect the eigenvector corresponding to $\lambda(L_{gs})$. We also assume that the stubborn agent does not connect to all of the other agents, as this case is trivially covered by the above result.

The following result will be helpful for providing a tighter bound on the smallest eigenvalue of the grounded Laplacian for certain choices of stubborn agents.

Proposition 2: Let $\mathbf{x}$ be the eigenvector corresponding to the smallest eigenvalue of L_{gs}. For each vertex $v_i \in \mathcal{V} \setminus \{v_s\}$, define $a_{v_i} = \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{|\mathcal{N}_i \setminus \{v_s\}|}$ if $|\mathcal{N}_i \setminus \{v_s\}| > 0$, and $a_{v_i} = 0$ otherwise. Then for each β-vertex v_i we have $x_i > a_{v_i}$ and for each α-vertex v_i we have $x_i \leq a_{v_i}$.

Proof: Rearranging the eigenvector equation (4), we have $x_i = \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i - \lambda}$. From Theorem 1 we know that $0 < \lambda \leq 1$. Thus we have

$$\begin{aligned} \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i} < x_i &= \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i - \lambda} \\ &\leq \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i - 1}. \end{aligned} \quad (8)$$

Since we have $a_{v_i} = \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i}$, $\forall v_i \in \mathcal{V} \setminus \{\mathcal{N}_s \cup \{v_s\}\}$ and $a_{v_i} = \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i - 1}$, $\forall v_i \in \mathcal{N}_s$, according to (8) we have $x_i > a_{v_i}$, $\forall v_i \in \mathcal{V} \setminus \{\mathcal{N}_s \cup \{v_s\}\}$ and $x_i \leq a_{v_i}$, $\forall v_i \in \mathcal{N}_s$. In the special case of $\mathcal{N}_i = \{v_s\}$, v_i becomes an isolated vertex after removing v_s. The 1×1 block corresponding to vertex v_i in L_{gs} has eigenvalue 1, and thus is not the block of L_{gs} containing the smallest eigenvalue (by Proposition 1 and the assumption that v_s does not connect to all nodes). In this case $a_{v_i} = 0$ which gives $x_i = a_{v_i}$. ■

According to the above proposition, the eigenvector element of an α-vertex is less than the average value of its neighbors' eigenvector entries. Thus it does not have the maximum eigenvector component among its neighbors. Similarly the eigenvector component of a β-vertex is greater than the average value of its neighbors' components and it does not have the minimum eigenvector component among its neighbors.

Corollary 2: For any β-vertex v, there is a decreasing sequence of eigenvector components starting from v that ends at an α-vertex.

Proof: Since each β-vertex has a neighbor with smaller eigenvector component, starting from any β-vertex there is a path consisting of vertices that have decreasing eigenvector components. If this sequence does not finish at an α-vertex it finishes at a β-vertex. There exists another vertex in the neighborhood of that vertex with smaller eigenvector component. Thus the decreasing sequence must finish at one of the α-vertices. ■

This leads to the following corollary; a vertex is said to be in the i-th layer if its shortest path to the stubborn agent has length i.

Corollary 3: The minimum eigenvector component in layers i and j, where $i > j$, occurs in layer j.

Proof: Let v and $\bar{v}$ be the vertices with minimum eigenvector components among the vertices in layers i and j respectively. According to Corollary 2 there is a path starting from v and ending at an α-vertex, making a decreasing sequence of eigenvector components. Since $i > j$ this path contains a vertex v' in layer j. Thus according to Corollary 2 we have $x_v \geq x_{v'} \geq x_{\bar{v}}$ which proves the claim. As a result the global minimum eigenvector component belongs to one of the α-vertices. ■

The above results lead to the following tighter upper bounds for $\lambda(L_{gs})$, as compared to Theorem 1, for a special class of graphs which contains trees.

Proposition 3: If $\mathcal{G}$ is a connected graph and the stubborn agent v_s has $d_s = 1$, then $\lambda \leq \frac{1}{|\mathcal{V} \setminus \{v_s\}|}$.

Proof: Let $k = |\mathcal{V} \setminus \{v_s\}|$, and let v_k be the single α-vertex in $\mathcal{G}$. According to Corollary 3, its eigenvector component is smaller than the eigenvector components of all of the other $k-1$ vertices in $\mathcal{V} \setminus \{v_s\}$ and according to (7) we have

$$x_k = \frac{\lambda}{1-\lambda}(x_1 + x_2 + \ldots + x_{k-1}) \geq \frac{\lambda(k-1)}{1-\lambda} x_k,$$

which gives $\lambda \leq \frac{1}{k}$. ■

If v_s is a cut vertex in $\mathcal{G}(\mathcal{V}, \mathcal{E})$, removing it causes the graph $\mathcal{G}$ to be partitioned into multiple components. The grounded Laplacian L_{gs} is block diagonal, where each block is the grounded Laplacian matrix for one of the components.

In this case the eigenvalues of L_{gs} are the union of the eigenvalues of each block, and the smallest eigenvalue of each block can be bounded as above, leading to the following corollary.

Corollary 4: If the stubborn agent v_s is a cut vertex and each component formed by removing v_s has a single α-vertex, then $\lambda \leq \frac{1}{k}$ where k is the size of the largest component in the graph induced by $\mathcal{V} \setminus \{v_s\}$.

As mentioned above, the class of graphs considered in Corollary 4 contains trees, because each vertex in a tree is a cut vertex and all of its incident edges are cut edges.

VI. Bounds on Grounding Centrality via Absorption Time

The convergence properties of linear consensus dynamics with stubborn agents are closely related to certain properties of random walks on graphs, including mixing times, commute times, and absorption probabilities [11], [12], [14], [15]. In this section we discuss the relationship between grounding centrality and the expected absorption time of an absorbing random walk on the underlying graph. To this end, we first review some properties of the inverse of the grounded Laplacian matrix.

A. Properties of the Inverse of the Grounded Laplacian Matrix

As discussed in Remark 1, when the graph $\mathcal{G}$ is connected, for any $v_s \in \mathcal{V}$, the inverse of the grounded Laplacian matrix $L_{gs}^{-1}(\mathcal{G})$ exists and is a nonnegative matrix. In this case, an alternative definition for the grounding centrality of v_s from the one in Definition 1 is that it is the maximum eigenvalue of L_{gs}^{-1}, with $IC(\mathcal{G}) = \text{argmin}_{v_s \in \mathcal{V}} \lambda_{max}(L_{gs}^{-1})$. Since the eigenvector corresponding to the largest eigenvalue of $L_{gs}^{-1}(\mathcal{G})$ is the same as the eigenvector for the smallest eigenvalue of $L_{gs}(\mathcal{G})$, the properties discussed in Remark 1 continue to hold (i.e., this eigenvector can be chosen to be nonnegative, and strictly positive if v_s is not a cut vertex).

One of the consequences of the P-F theorem applied to L_{gs}^{-1} is that the largest eigenvalue satisfies

$$\lambda_{\max}(L_{gs}^{-1}) \leq \max_i \{[L_{gs}^{-1}]_i \mathbf{1}\}, \tag{9}$$

where $[L_{gs}^{-1}]_i$ is the i-th row of L_{gs}^{-1}. Let

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} \end{bmatrix}^T$$

be the eigenvector corresponding to $\lambda_{\max}(L_{gs}^{-1})$. The element x_i in this eigenvector is associated with the i-th vertex. As this eigenvector has nonnegative elements, we can normalize it such that $\max_i x_i = 1$. Let $\Delta = x_{max} - x_{min} = 1 - x_{min} \geq 0$ be the difference between the maximum and the minimum entries of eigenvector $\mathbf{x}$. Since all of the elements of L_{gs}^{-1} and $\mathbf{x}$ are nonnegative, and we have $\mathbf{x} \geq (1-\Delta)\mathbf{1} = x_{min}\mathbf{1}$ elementwise, we have

$$[L_{gs}^{-1}]_i[(1-\Delta)\mathbf{1}] \leq [L_{gs}^{-1}]_i \mathbf{x} = \lambda_{max}(L_{gs}^{-1}) x_i.$$

Since $0 \leq x_i \leq 1$ we have $[L_{gs}^{-1}]_i[(1-\Delta)\mathbf{1}] \leq \lambda_{max}(L_{gs}^{-1})$. Combined with (9), this gives

$$\max_i \{[L_{gs}^{-1}]_i[(1-\Delta)\mathbf{1}]\} \leq \lambda_{\max}(L_{gs}^{-1}) \leq \max_i \{[L_{gs}^{-1}]_i \mathbf{1}\}. \tag{10}$$

By minimizing over all choices of stubborn agent $v_s \in \mathcal{V}$ from (10) we have

$$\begin{aligned} \min_s \max_i \{[L_{gs}^{-1}]_i(1-\Delta)\mathbf{1}\} &\leq \min_s \lambda_{\max}(L_{gs}^{-1}) \\ &\leq \min_s \max_i \{[L_{gs}^{-1}]_i \mathbf{1}\}. \end{aligned} \tag{11}$$

As $\Delta \to 0$ the upper bound and the lower bound of (11) approach $\min_s \lambda_{\max}(L_{gs}^{-1})$.

Equations (10) and (11) provide bounds on the grounding centrality of each vertex in the graph and the grounding centrality of vertices in $IC(\mathcal{G})$, respectively. We now relate the bounds in (11) to an absorbing random walk on the underlying graph.

B. Relationship of Grounding Centrality to an Absorbing Random Walk on Graphs

We start with the following preliminary definitions about absorbing Markov chains.

Definition 2: A Markov chain is a sequence of random variables $X_1, X_2, X_3, ...$ with the property that given the present state, the future and past states are independent. Mathematically

$$\begin{aligned} Pr(X_{n+1} = x | X_1 = x_1, X_2 = x_2, ..., X_n = x_n) \\ = Pr(X_{n+1} = x | X_n = x_n). \end{aligned}$$

A state x_i of a Markov chain is called absorbing if it is impossible to leave it, i.e., $Pr(X_{n+1} = x_i | X_n = x_i) = 1$. A Markov chain is absorbing if it has at least one absorbing state and if from every state it is possible to go to an absorbing state. A state that is not absorbing is called a transient state [19]. □

If there are r absorbing states and t transient states, the transition matrix will have the canonical form

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}, \quad P^n = \begin{bmatrix} Q^n & \bar{R} \\ 0 & I \end{bmatrix}, \tag{12}$$

where $Q_{t\times t}$, $R_{t\times r}$ and $\bar{R}_{t\times r}$ are some nonzero matrices, $0_{r\times t}$ is a zero matrix and $I_{r\times r}$ is an identity matrix. The first t states are transient and the last r states are absorbing. The probability of going to state x_j from state x_i is given by entry p_{ij} of matrix P. Furthermore entry (i, j) of the matrix P^n is the probability of being in state x_j after n steps when the chain is started in state x_i.

The *fundamental matrix* for P is given by [19]

$$N = \sum_{j=0}^{\infty} Q^j = (I - Q)^{-1}. \tag{13}$$

The entry n_{ij} of N gives the expected number of time steps that the process is in the transient state x_j when it starts from the transient state x_i. Furthermore the i-th entry of $N\mathbf{1}$ is the expected number of steps before the chain is absorbed, given that the chain starts in the state x_i. In the context of a random walk on a given graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ containing one absorbing vertex v_s, the probability of going from transient vertex v_i to the transient vertex v_j is $P_{ij} = A_{ij}/d_i$ where A is the adjacency matrix and d_i is the degree of v_i. Thus the matrix Q in (12) becomes $Q = D_{gs}^{-1}A_{gs}$ where A_{gs} and D_{gs} are the grounded degree and grounded adjacency matrix, respectively (obtained by removing the rows and columns corresponding to the absorbing state v_s from those matrices).

To relate the absorbing walk to the grounded Laplacian, note that $L_{gs}^{-1} = (D_{gs} - A_{gs})^{-1} = (I - D_{gs}^{-1}A_{gs})^{-1}D_{gs}^{-1}$. Comparing to (13), we have

$$N_s = L_{gs}^{-1}D_{gs}. \tag{14}$$

where the index s denotes that vertex v_s is an absorbing state. This leads to the following result.

Proposition 4: Given graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ and a stubborn agent $v_s \in \mathcal{V}$, let d_{max} and d_{min} denote the maximum and minimum degrees of vertices in $\mathcal{V}\setminus\{v_s\}$, respectively. Then

$$\frac{1}{d_{max}}\max_i\{[N_s]_i[(1-\Delta)\mathbf{1}]\} \leq \lambda_{max}(L_{gs}^{-1}) \leq \frac{1}{d_{min}}\max_i\{[N_s]_i\mathbf{1}\}, \tag{15}$$

where $[N_s]_i\mathbf{1}$ is the expected absorption time of a random walk starting at $v_i \in \mathcal{V}\setminus\{v_s\}$ with absorbing vertex v_s.

Proof: Substituting (14) into (10) gives

$$\frac{1}{d_{max}}\max_i\{[N_s]_i[(1-\Delta)\mathbf{1}]\} \leq \max_i\{[N_sD_{gs}^{-1}]_i[(1-\Delta)\mathbf{1}]\} \leq \lambda_{\max}(L_{gs}^{-1}) \leq \max_i\{[N_sD_{gs}^{-1}]_i\mathbf{1}\} \leq \frac{1}{d_{min}}\max_i\{[N_s]_i\mathbf{1}\},$$

which proves the claim. ■

Remark 2: Taking the minimum over all possible choices of absorbing vertex in (15) gives

$$\frac{1}{d_{max}}\min_s\max_i\{[N_s]_i[(1-\Delta)\mathbf{1}]\} \leq \min_s\lambda_{max}(L_{gs}^{-1}) \leq \frac{1}{d_{min}}\min_s\max_i\{[N_s]_i\mathbf{1}\}.$$

The upper and lower bounds approach each other in graphs where d_{max} and d_{min} are equal and $\Delta \to 0$ (i.e., the smallest eigenvector component goes to 1). In this case, the grounding central vertex becomes a vertex that if is chosen as the absorbing vertex, the maximum expected absorption time in the random walk on $\mathcal{G}$ is minimized. □

VII. Conclusion

We analyzed spectral properties of the grounded Laplacian matrix in the context of linear consensus dynamics with stubborn agents. We defined a natural centrality metric based upon the smallest eigenvalue of the grounded Laplacian, and provided bounds on this centrality using graph-theoretic properties. An avenue for future research is to explore additional relationships between the network topology and grounding centrality.

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