State Estimation for Linear Systems with Unknown Inputs: Unknown Input Norm-Observers and BIBOBS Stability

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***Abstract*— We study state estimation of linear systems with unknown inputs. When the system is not strongly observable (strongly detectable), one cannot exactly (asymptotically) reconstruct the states without further information about the system or inputs; in this case, various formulations have been studied that require additional information about the nature of the unknown inputs (e.g., Kalman filtering and set-membership filtering). In this paper, we consider a formulation where the unknown inputs and initial condition of the system are bounded in magnitude. The objective is to construct an unknown input norm-observer which estimates an upper bound for the norm of the states. In order to characterize the existence of such an observer, we propose a notion of bounded-input-bounded-output-bounded-state (BIBOBS) stability; this concept supplements various system properties, including detectability, bounded-input-bounded-output (BIBO) stability, bounded-input-bounded-state (BIBS) stability, and input-output-to-state stability (IOSS). We provide checkable conditions on the system matrices under which a general class of linear systems is BIBOBS stable, and show that the set of modes of the system with magnitude 1 plays a key role.**

I. Introduction

Estimation and filtering have long been one of the central topics of control and signal processing. Specifically, the problem of estimating system states in the presence of unknown inputs has been studied extensively in the literature (e.g., see [1], [2], [3]). Such unknown inputs can be used to model uncertainty in the systems, including disturbances or faults [1], noise [2], and attacks [4]. A typical approach to decouple the influence of unknown inputs is to construct an *unknown input observer* [3].

For linear systems with no constraints on the inputs, it has been shown that there exists an asymptotic observer which recovers the states despite the unknown inputs if and only if the system is *strongly detectable* (or equivalently, has no unstable invariant zeros) [5], [6], [7]. If one wants to reconstruct the states in finite time, the system must be *strongly observable* (or have no nonzero invariant zeros). When the system is not strongly detectable, one can either assume further information about the initial condition or the unknown inputs in order to perform state estimation, or relax the estimation objective. A common approach, following the line of the Kalman filter, is optimal filtering theory which models the initial condition and inputs as random variables and stochastic processes, respectively, with certain statistics [8]. One can also take a deterministic approach to model the uncertainties, using the so-called set-membership framework where the initial condition and unknown inputs are assumed to be bounded in some set (e.g., ellipsoids or polytopes) and the goal is to find the set of possible states that are consistent with the observations [9], [10]. Furthermore, partial observers have also been considered [11].

In many applications, rather than aiming to reconstruct the states exactly, it suffices to provide an upper bound for the norm of the states. In [12], [13], the authors proposed the notion of a *norm-estimator* for nonlinear systems which is driven by *known* inputs and outputs of the system and returns an estimate for the norm of the states. They showed that the system admits a norm-estimator if and only if it satisfies a property termed uniform IOSS (UIOSS). Moreover, they showed that one can construct a norm-estimator based on a certain type of Lyapunov function.

In this paper, we study the problem of estimating the states of discrete-time linear systems with unknown inputs when the system is not strongly detectable. Specifically, we consider a setting similar to the set-membership filtering approach where the norm of the initial condition and unknown inputs are bounded by some known constants. We extend the concept of norm-estimation to the unknown input case by defining an *unknown input norm-observer*, and determine the conditions under which such an observer exists.

To solve this problem, we propose a notion of stability termed *bounded-input-bounded-output-bounded-state (BIBOBS) stability*, which is a fundamental property that is related to various existing system properties, including detectability, BIBO stability, BIBS stability, and IOSS (as we will discuss later in this paper). In addition to its implications for unknown input norm-observers, the concept of BIBOBS stability has applications to the *false data injection* problem studied in [14], [15]. In this setting, the system is a fault detection filter, the output is interpreted as the residue of the filter, the bound on the output represents the threshold above which an alarm is raised, and the states are interpreted as the estimation error; the goal of the attacker is to maximize the error while remaining undetected. If the attacker is constrained to apply bounded inputs (a scenario which is not considered in [14], [15]), BIBOBS stability is required for preventing worst-case attacks (i.e., those causing arbitrarily large error without triggering the alarm [16]).

To the best of our knowledge, a characterization of BIBOBS stability has not been provided in the literature. We

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provide this characterization for the class of diagonalizable linear systems, which leads to a construction of an unknown input norm-observer. We show that the set of marginally stable eigenvalues (i.e., those with magnitude 1) plays a key role: other than through unobservable strictly unstable eigenvalues (i.e., those with magnitude bigger than 1), the only way for bounded inputs to drive the state unbounded while keeping the output bounded is by manipulating marginally stable eigenvalues. As we show, care must be taken to identify the subset of marginally stable eigenvalues that can be manipulated in this way.

The rest of this paper is organized as follows. In Section II, we define the concept of an unknown input norm-observer for linear systems with unknown inputs. In Section III, we propose the notion of BIBOBS stability and provide necessary and sufficient conditions for a class of linear systems to be BIBOBS stable. We also illustrate the results via some examples. In Section IV, we discuss BIBOBS stability in the context of other existing stability notions, and conclude in Section V.

A. Notation and Terminology

We will be using the following notation. For a matrix M, let $\text{rank}(M)$ and M^T be its rank and transpose, respectively. Further define the null space and range space of a matrix M by $\mathcal{N}(M)$ and $\mathcal{R}(M)$, respectively. The Euclidean norm of a vector and the corresponding induced matrix norm are both denoted by $\|\cdot\|$. For a signal z, we will denote its supremum norm over time interval $[0,k]$ by $\|z\|_{[0,k]} = \max_{0\le j\le k}\|z[j]\|$ [17]. A function $\alpha:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0)=0$. If a $\mathcal{K}$-function is also unbounded, then it is said to be of class $\mathcal{K}_\infty$. A function $\beta:\mathbb{R}_{\ge 0}\times\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot,k)$ is of class $\mathcal{K}$ and $\beta(r,k)\to 0$ as $k\to\infty, \forall r\ge 0$ [13].

II. Unknown Input Norm-observers for Linear Systems with Unknown Inputs

We consider the discrete-time linear system

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned} \tag{1}$$

with state vector $x\in\mathbb{R}^n$, output $y\in\mathbb{R}^p$, unknown input $u\in\mathbb{R}^m$, and system matrices (A,B,C,D) of appropriate dimensions. The initial condition of the system is x_0. The unknown inputs u may represent disturbances, faults, attacks, or other uncontrolled uncertainties. For convenience and ease of exposition, we assume henceforth that the system is diagonalizable.

The system (1) is said to be *strongly observable* if for any initial condition x_0 and any sequence of unknown inputs $\{u[k]\}$, there exists some positive integer L such that x_0 can be recovered from $y[0:L]$, where $y[0:L]$ is the vector of outputs over $L+1$ time steps. It is well known that (e.g., see [18]) the system is strongly observable if and only if

$$\text{rank}(\begin{bmatrix}\mathcal{O}_n & \mathcal{J}_n\end{bmatrix}) = n + \text{rank}(\mathcal{J}_n) \tag{2}$$

where

$$\mathcal{O}_n = \begin{bmatrix} C^T & (CA)^T & \cdots & (CA^n)^T\end{bmatrix}^T$$

is the *observability matrix* and

$$\mathcal{J}_n = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^{n-2}B & CA^{n-3}B & \cdots & D \end{bmatrix}$$

is the *invertibility matrix* of system (1). Thus, if equality (2) holds, then we can recover $x[k]$ by using $y[k:k+n]$ (without any information about $u[k:k+n]$). A relaxation of strong observability is the concept of strong detectability. A system is strongly detectable if the state asymptotically decays to zero when the output is identically zero, *regardless* of the inputs to the system [5]. Thus, strong detectability characterizes the ability to reconstruct the states *asymptotically* regardless of the unknown inputs.

However, in many applications, strong observability or strong detectability of the system may not hold; for example, if the matrix B has full column rank and there are more unknown inputs than outputs (i.e., $m>p$ in system (1)), the system cannot be strongly observable or strongly detectable [18]. In this case, we need to relax the objective of exactly reconstructing the states.

In practice, the inputs (either known or unknown) are often bounded in magnitude; for example, when the unknown inputs represent attacks on the system, it may be reasonable to assume that the inputs are bounded due to physical constraints (such as actuator saturation) or resource limitations. Furthermore, one may also have prior information about certain properties of the initial condition of the system. Thus, in this section, we consider a setting similar to the set-membership approach where we assume that the norm of the initial condition $\|x_0\|$ and the inputs are upper bounded by some constants $x_0^{\max}>0$ and $u_{\max}>0$, respectively, i.e., $\|x_0\|\le x_0^{\max}$ and $\|u\|_{[0,\infty)}\le u_{\max}$. Instead of attempting to reconstruct the possible set of states consistent with the outputs as in the set-membership approach, our objective is to build an unknown input norm-observer for the states of the system by utilizing the information on the bounds of the initial condition and unknown inputs, defined as follows.

Definition 1 (Unknown Input Norm-observer): For system (1), we say that there exists an unknown input norm-observer $\hat{x}$ of the norm of the states $\|x\|$ if there exist functions $\gamma_1,\gamma_2,\gamma_3\in\mathcal{K}_\infty$ such that $\hat{x}[k]=\gamma_1(x_0^{\max})+\gamma_2(\|y\|_{[0,k]})+\gamma_3(u_{\max})$ and $\|x[k]\|\le\hat{x}[k], \forall k, x_0, u$. □

The above definition of an unknown input norm-observer is similar to the concept of norm-estimation studied in [13]; the difference is that we do not use information on the unknown inputs except an upper bound and we do not require the influence of the initial condition to asymptotically decay to zero. As we will see later in the next section, it turns out that these differences make the characterization of the unknown input norm-observer different from the concept of UIOSS proposed in [13].

III. BIBOBS STABILITY

In order to characterize system properties that allow norm-estimation, we start by introducing the following definition of bounded-input-bounded-output-bounded-state (BIBOBS) stability.

Definition 2 (BIBOBS Stability): The system (1) is said to be **BIBOBS stable** if there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\|x[k]\| \leq \alpha_1(\|x_0\|) + \alpha_2(\|y\|_{[0,k]}) + \alpha_3(\|u\|_{[0,k]}), \forall k, x_0, u.$$

□

In words, BIBOBS stability characterizes the ability of bounded disturbances to affect the state while remaining undetected (via the outputs); in the next section, we will discuss the relationship of BIBOBS stability to other established notions (such as BIBO stability, BIBS stability and IOSS). The following result relates BIBOBS stability to the norm-estimation objective described in the previous section.

Theorem 1: The system (1) is BIBOBS stable if and only if it admits an unknown input norm-observer. □

Proof: If the system is BIBOBS stable, in Definition 2, we can just replace $\|x_0\|$ and $\|u\|_{[0,k]}$, $\forall k$, by $x_0^{\max}$ and $u_{\max}$, respectively, and we get an unknown input norm-observer. When the system is not BIBOBS stable, as we will prove later in Lemma 2, for any $x_0^{\max}, u_{\max} > 0$, there exist some initial condition and input sequence such that the outputs are bounded but the states become unbounded, and thus, there does not exist any unknown input norm-observer. ■

In the rest of this section, we derive the conditions under which the system (1) is BIBOBS stable. As we will see, the constraint on the norm of the inputs limits their ability to drive the state of the system to be unbounded while remaining undetected via the outputs, and the eigenvalues with magnitude 1 play an essential role under these constraints. Note that a construction method for the unknown input norm-observer will follow as a byproduct of the proof. We will start with the following definition.

Definition 3: For an eigenvalue λ of the matrix A, we say that λ is **strictly unstable** if it has magnitude bigger than 1, and **marginally stable** if it has magnitude 1. □

To give a simple characterization for BIBOBS stability, we perform a similarity transformation on the system (1) via a matrix H to obtain

$$\begin{aligned} x_H[k+1] &= A_H x_H[k] + B_H u[k] \\ y[k] &= C_H x_H[k] + Du[k], \end{aligned} \quad (3)$$

where $x_H = Hx$, $A_H = HAH^{-1}$, $B_H = HB$, and $C_H = CH^{-1}$ have the form

$$x_H = \begin{bmatrix} x_{co}^T & x_{c\bar{o}}^T & x_u^T & \vdots & x_d^T \end{bmatrix}^T,$$

$$A_H = \left[\begin{array}{ccc:c} A_{co} & 0 & A_{13} & A_{14} \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_u & 0 \\ \hdashline 0 & 0 & 0 & A_d \end{array}\right], B_H = \left[\begin{array}{c} B_{co} \\ B_{c\bar{o}} \\ 0 \\ \hdashline 0 \end{array}\right],$$

$$C_H = \begin{bmatrix} C_{co} & 0 & C_u & \vdots & C_d \end{bmatrix}.$$

In the above representation, the subsystem (A_{co}, B_{co}, C_{co}) is both controllable and observable, the subsystem $(A_{c\bar{o}}, B_{c\bar{o}}, 0)$ is controllable but not observable, A_u is a diagonal matrix containing the stable and strictly unstable eigenvalues of the uncontrollable components of A, and A_d is a diagonal matrix containing the marginally stable eigenvalues of the uncontrollable components of A. If some of these eigenvalues do not exist, we consider their corresponding components to have size 0 (or not exist). Note that for any linear system (1) which is diagonalizable, one can always find a H matrix to obtain the above transformation. Specifically, one can first transform the system into its Kalman canonical form and then apply a further transformation to convert the uncontrollable subsystem into diagonal form. Further note that $\|H\|^{-1}\|x_H\| \leq \|x\| \leq \|H^{-1}\|\|x_H\|$.

Now we group all the states except x_d into a vector x_1, i.e., $x_H = \begin{bmatrix} x_1^T & x_d^T \end{bmatrix}^T$. Denote the components of A_H, B_H and C_H associated with x_1 by A_1, B_1 and C_1, respectively, i.e., $A_H = \begin{bmatrix} A_1 & A_2 \\ 0 & A_d \end{bmatrix}$ where $A_2 = \begin{bmatrix} A_{14}^T & A_{24}^T & 0 \end{bmatrix}^T$, $B_H = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, and $C_H = \begin{bmatrix} C_1 & C_d \end{bmatrix}$. The following theorem is our main result in this paper and characterizes BIBOBS stability for diagonalizable linear systems.

Theorem 2: The system (1) is BIBOBS stable if and only if in the form (3), the pair (A_1, C_1) is detectable. □

From Theorem 2, it is easy to see that BIBOBS stability is strictly weaker than detectability, as we do not require the pair (A_d, C_d) to be observable (recall that this subsystem is diagonal and consists entirely of marginally stable eigenvalues). Here we split the proof of the above theorem into the following two lemmas. The first lemma proves the sufficiency part of the theorem.

Lemma 1: The system (1) is BIBOBS stable if in the form (3), the pair (A_1, C_1) is detectable. □

Proof: If the pair (A_1, C_1) is detectable, then there exists some matrix L such that the matrix $A_1 + LC_1$ is stable. Let $y_1[k] = C_1 x_1[k]$. Using the same trick as in [13] for characterizing IOSS, we can construct the following observer

$$\hat{x}_1[k+1] = A_1\hat{x}_1[k] + B_1 u[k] + A_2 x_d[k] + L(C_1\hat{x}_1[k] - y_1[k])$$

with the property that if $\hat{x}_1[0] = x_1[0]$, then $\hat{x}_1[k] = x_1[k], \forall k, u$. Thus, we know that

$$\begin{aligned} x_1[k] &= (A_1 + LC_1)^k x_1[0] \\ &+ \sum_{i=0}^{k-1} (A_1 + LC_1)^{k-1-i} \left(B_1 u[i] + A_2 x_d[i] - Ly_1[i]\right). \end{aligned}$$

Let λ_1 be the eigenvalue of $A_1 + LC_1$ with largest magnitude. Since $A_1 + LC_1$ is stable, we know that $|\lambda_1| < 1$ and we can choose some constant δ such that $|\lambda_1| < \delta < 1$. Then there exists some constant $K > 0$ [13] such that $\forall k$,

$$\begin{aligned} &\|x_1[k]\| \leq K\delta^k \|x_1[0]\| \\ &+ \frac{K}{1-\delta}\left(\|B_1\|\|u\|_{[0,k]} + \|A_2\|\|x_d\|_{[0,k]} + \|L\|\|y_1\|_{[0,k]}\right). \end{aligned}$$

Note that $y_1[k] = y[k] - C_d x_d[k] - Du[k]$ and thus $\|y_1\|_{[0,k]} \leq \|y\|_{[0,k]} + \|C_d\|\|x_d\|_{[0,k]} + \|D\|\|u\|_{[0,k]}$. Since A_d is a diagonal matrix with marginally stable eigenvalues, we know that $\|x_d\|_{[0,k]} = \|x_d[0]\|, \forall k$. Thus, for all k,

$$\begin{aligned}\|x_H[k]\| &= \| \begin{bmatrix} x_1^T[k] & x_d^T[k] \end{bmatrix}^T \| \leq \|x_d[k]\| + \|x_1[k]\| \\ &\leq \|x_d[0]\| + K\delta^k \|x_1[0]\| + \\ &\frac{K}{1-\delta}\left(\|B_1\|\|u\|_{[0,k]} + \|A_2\|\|x_d[0]\|\right) + \\ &\frac{K\|L\|}{1-\delta}\left(\|y\|_{[0,k]} + \|C_d\|\|x_d[0]\| + \|D\|\|u\|_{[0,k]}\right) \\ &\leq [1 + K\delta^k + \frac{K}{1-\delta}(\|A_2\| + \|L\|\|C_d\|)]\|x_H[0]\| + \\ &\frac{K\|L\|}{1-\delta}\|y\|_{[0,k]} + \frac{K}{1-\delta}(\|B_1\| + \|L\|\|D\|)\|u\|_{[0,k]}.\end{aligned} \tag{4}$$

Thus, there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that $\forall k, x_H[0], u$,

$$\|x_H[k]\| \leq \alpha_1(\|x_H[0]\|) + \alpha_2(\|y\|_{[0,k]}) + \alpha_3(\|u\|_{[0,k]}).$$

Since $\|x\| \leq \|H^{-1}\|\|x_H\|$, we know that the original system (1) is BIBOBS stable, completing the proof. ■

Remark 1: If the system satisfies the condition for BIBOBS stability, a construction of the unknown input norm-observer follows by inequality (4). Specifically, in Definition 1 (the definition of unknown input norm observer), we can choose the functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ as follows:

$$\begin{aligned}\gamma_1(x_0^{\max}) &= [1 + K\delta^k + \\ &\frac{K}{1-\delta}(\|A_2\| + \|L\|\|C_d\|)]\|H^{-1}\|\|H\|x_0^{\max}, \\ \gamma_2(\|y\|_{[0,k]}) &= \frac{K\|L\|}{1-\delta}\|H^{-1}\|\|y\|_{[0,k]}, \\ \gamma_3(u_{\max}) &= \frac{K}{1-\delta}(\|B_1\| + \|L\|\|D\|)\|H^{-1}\|u_{\max}.\end{aligned}$$

□

Example 1: To illustrate the result in Lemma 1, consider the following system:

$$A = \left[\begin{array}{cc:c} 2 & & \\ & 1 & \\ \hdashline & & 1 \end{array}\right] = \left[\begin{array}{c:c} A_1 & A_2 \\ \hdashline & 1 \end{array}\right], B = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \\ \hdashline 0 & 0 \end{array}\right] = \left[\begin{array}{c} B_1 \\ \hdashline 0 \end{array}\right],$$

$$C = \left[\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{c:c} C_1 & 0 \end{array}\right], \ D = 0.$$

Since (A_1, C_1) is observable, by Lemma 1, we know that the system is BIBOBS stable. Then we can choose a matrix L such that the matrix $A_1 + LC_1$ is stable; for example, choose $L = \begin{bmatrix} -1.2 & 0.5 \\ 0 & -0.5 \end{bmatrix}$ such that $\text{eig}(A_1 + LC_1) = \{0.8, 0.5\}$. We have the bound $\|(A_1 + LC_1)^k\| \leq 5 \times 0.8^k$ (i.e., $K = 5$ and $\delta = 0.8$). Since $\frac{K\|L\|}{1-\delta} < 33$, $\frac{K\|B_1\|}{1-\delta} < 41$, $H = I$, and $\|A_2\| = \|C_d\| = \|D\| = 0$, by using the functions in Remark 1, we can get an unknown input norm-observer $\hat{x}$ satisfying $\|x[k]\| \leq \hat{x}[k], \forall k, x_0, u$, as follows:

$$\hat{x}[k] = (1 + 5 \times 0.8^k)x_0^{\max} + 33\|y\|_{[0,k]} + 41u_{\max}.$$

□

The next lemma proves the necessity part of Theorem 2. The idea behind the proof is that if the condition in Theorem 2 fails, for any constant upper bounds $x_0^{\max}, u_{\max} > 0$, one can choose an initial state and a sequence of inputs $\{u[k]\}$ such that $\|y[k]\| \leq y_{\max}$ for all k and any $y_{\max} > 0$, while $\|x[k]\| \to \infty$. In particular, if the matrix $A_{c\bar{o}}$ contains marginally stable eigenvalues, we will show that a bounded input can persistently excite the states corresponding to those eigenvalues while the output stays bounded.

Lemma 2: The system (1) is BIBOBS stable only if in the form (3), the pair (A_1, C_1) is detectable. □

Proof: Let the dimension of A_1 be n_1. When the pair (A_1, C_1) is not detectable, there exists some $|\lambda| \geq 1$ such that $\text{rank}(\begin{bmatrix} \lambda I - A_1 \\ C_1 \end{bmatrix}) < n_1$. The analysis for the case where $|\lambda| > 1$ is covered by the standard argument for undetectable systems: there exists some eigenvector w of A_1 such that $w \in \mathcal{N}(C_1)$. Choose w as the initial condition for the subsystem (A_1, C_1), zero initial condition for the subsystem (A_d, C_d), and let the inputs be identically zero. This results in the output being zero for all time, while $\|x[k]\| \to \infty$.

Now we only need to consider the case where $|\lambda| = 1$. In this case, since the subsystem (A_{co}, C_{co}) is observable, λ must be an eigenvalue of $A_{c\bar{o}}$. Choose the initial condition of the subsystem corresponding to the states $\begin{bmatrix} x_u^T & x_d^T \end{bmatrix}^T$ to be zero. Then from the form of the transformed system in (3), we can focus on the following subsystem:

$$\begin{aligned}\begin{bmatrix} x_{co}[k+1] \\ x_{c\bar{o}}[k+1] \end{bmatrix} &= \begin{bmatrix} A_{co} & 0 \\ A_{21} & A_{c\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co}[k] \\ x_{c\bar{o}}[k] \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} C_{co} & 0 \end{bmatrix} \begin{bmatrix} x_{co}[k] \\ x_{c\bar{o}}[k] \end{bmatrix} + Du[k].\end{aligned} \tag{5}$$

Denote the dimension of $\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix}$ by n_c, and denote the controllability matrix of subsystem (5) by $\mathcal{C}_{n_c-1}$. Let the eigenvector of $A_{c\bar{o}}$ associated with λ be v. Choose the initial condition of subsystem (5) to be $\begin{bmatrix} 0 \\ v \end{bmatrix}$ and for $i = 0, 1, 2, \ldots$, choose the input sequence over time interval $[in_c, (i+1)n_c - 1]$ to be that $\mathcal{C}_{n_c-1}u[in_c : (i+1)n_c - 1] = \begin{bmatrix} 0 \\ \lambda^{(i+1)n_c}v \end{bmatrix}$. Note that since the subsystem (5) is controllable, $\mathcal{C}_{n_c-1}$ has full rank and such an input sequence always exists. Further note that since $|\lambda| = 1$, the supremum norm of the input sequence is bounded. One can check that under this choice of initial condition and inputs, for any integer $i \geq 0$,

$$\begin{aligned}\begin{bmatrix} x_{co}[in_c] \\ x_{c\bar{o}}[in_c] \end{bmatrix} &= \begin{bmatrix} 0 \\ (i+1)\lambda^{in_c}v \end{bmatrix} \\ y[in_c : (i+1)n_c - 1] &= \mathcal{J}_{n_c-1}u[in_c : (i+1)n_c - 1].\end{aligned}$$

We can see that in this case, the states become unbounded while the inputs and outputs are bounded and thus, the system is not BIBOBS stable. Note that since we can always scale the initial condition and inputs, the above analysis holds for any upper bounds $x_0^{\max}$ and $u_{\max}$.

Combining the above analysis, we know that if in the form (3), the pair (A_1, C_1) is not detectable, the system is not BIBOBS stable, completing the proof. ■

Example 2: To illustrate the result in Lemma 2, consider the following system:

$$A=\left[\begin{array}{cc:c}2 & & \\ 1 & 1 & \\ \hdashline & & 1\end{array}\right]=\left[\begin{array}{c:c}A_1 & \\ \hdashline & 1\end{array}\right], B=\left[\begin{array}{cc}1 & 1\\ 1 & 0\\ \hdashline 0 & 0\end{array}\right]=\left[\begin{array}{c}B_1\\ \hdashline 0\end{array}\right],$$

$$C=\left[\begin{array}{cc:c}1 & 0 & 0\end{array}\right]=\left[\begin{array}{c:c}C_1 & 0\end{array}\right], D=0.$$

Note that the controllable but not observable subsystem $(1, \begin{bmatrix}1 & 0\end{bmatrix}, 0)$ is marginally stable. Thus, by Lemma 2, we know that the system is not BIBOBS stable. Specifically, choose $x[0]=\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$, and $u[k]=\begin{bmatrix}\frac{6}{11} & -\frac{4}{11}\end{bmatrix}^T$ when k is even and $u[k]=\begin{bmatrix}\frac{3}{11} & -\frac{7}{11}\end{bmatrix}^T$ when k is odd. Then we have $x[k]=\begin{bmatrix}0 & 1+\frac{k}{2} & 0\end{bmatrix}^T$ when k is even and $x[k]=\begin{bmatrix}\frac{2}{11} & \frac{17}{11}+\frac{k-1}{2} & 0\end{bmatrix}^T$ when k is odd, and $y[k]=0$ when k is even and $y[k]=\frac{2}{11}$ when k is odd; thus, the states become unbounded while the outputs are always bounded. □

Remark 2: From the proof of Theorem 2, we can see that the result in Theorem 2 also applies to the case where only the marginally stable eigenvalues of the uncontrollable components are diagonalizable (while the matrix A is not necessarily diagonalizable). □

IV. A Discussion of BIBOBS Stability

In this section, we discuss the relationships between BIBOBS stability and other classical system properties.

A. Related Linear System Properties

We first consider the concept of detectability, which is typically defined for systems without inputs and means that if the output is zero, the state decays to zero. Intuitively, if a system is detectable, then the states associated with unstable eigenvalues are observable and thus the output carries information about this set of states; since bounded inputs cannot drive the states associated with stable eigenvalues to be unbounded, the system must be BIBOBS stable. The relationship between BIBOBS stability and detectability is summarized as follows.

Proposition 1: For system (1), detectability implies BIBOBS stability, but not vice versa.

Proof: As demonstrated by Theorem 2, detectability is strictly stronger than BIBOBS stability (e.g., the system in Example 1 is BIBOBS stable but not detectable). Another way to show this is to use the concept of IOSS. In [13], the authors showed that if a linear system is detectable, then it is also IOSS. As we will discuss later in Table I, IOSS is strictly stronger than BIBOBS stability (i.e., for a system to be BIBOBS stable, the influence of the initial condition does not necessarily decay to 0). Thus, for linear systems, detectability implies BIBOBS stability. ■

Next we consider BIBO stability. For a linear system with the initial condition being 0, the system is said to be BIBO stable if bounded inputs always result in bounded outputs. Since the state does not play a role in the definition of BIBO stability, BIBO stability does not imply BIBOBS stability. In the converse direction, a BIBOBS stable system does not have to be BIBO stable since the output of a BIBOBS stable system can be unbounded with bounded input; a system is still BIBOBS stable as long as we can observe that the states become unbounded. To see the difference between BIBO stability and BIBOBS stability, consider the following example.

Example 3: Consider the following two scalar systems

$$x[k+1]=2x[k]+u[k], \qquad y[k]=0. \tag{6}$$

$$x[k+1]=2x[k]+u[k], \qquad y[k]=x[k]. \tag{7}$$

System (6) is BIBO stable (since the output is always zero) but not BIBOBS stable (since we can let the input always be zero and drive the state to be unbounded by any nonzero initial condition). System (7) is BIBOBS stable (since the output is the state) but not BIBO stable (since bounded input can result in unbounded output). □

Now we consider the notion of BIBS stability. A system is said to be BIBS stable if bounded input always results in bounded state. The relationship between BIBS stability and BIBOBS stability is as follows.

Proposition 2: For system (1), BIBS stability implies BIBOBS stability, but not vice versa.

Proof: One direction is easy to show: since for a BIBS stable system bounded input always results in bounded state and the state-output mapping is linear, the system must be BIBOBS stable. For the other direction, consider again the system (7) in Example 3; the system is BIBOBS stable but not BIBS stable. ■

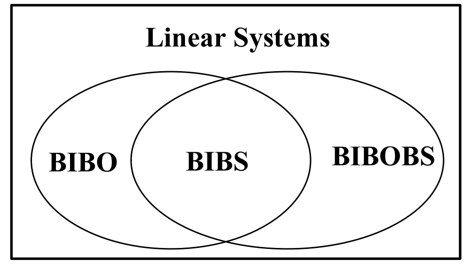

Fig. 1. Venn diagram for the classes of linear systems that are BIBO stable, BIBS stable and BIBOBS stable.

To summarize these relationships, we have the following result. See Figure 1 for the relationships between BIBO stability, BIBS stability and BIBOBS stability.

Proposition 3: The system (1) is BIBS stable if and only if it is both BIBO stable and BIBOBS stable.

Proof: Note that for linear systems, BIBS stability also implies BIBO stability. Thus, if a linear system is BIBS stable then it must be both BIBO stable and BIBOBS stable. If a system is both BIBO stable and BIBOBS stable, then in Definition 2, we can replace the function of the outputs by some $\mathcal{K}_\infty$ function of the inputs, which implies that the system is also BIBS stable. ■

TABLE I

COMPARISON OF DIFFERENT NOTIONS OF STABILITY.

BIBS Stability: $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\|x[k]\| \leq \alpha_1(\|x_0\|) + \alpha_2(\|u\|_{[0,k]}), \forall k, x_0, u$	ISS: $\exists \alpha \in \mathcal{K}_\infty, \beta \in \mathcal{KL}_\infty$ such that $\|x[k]\| \leq \beta(\|x_0\|, k) + \alpha(\|u\|_{[0,k]}), \forall k, x_0, u$
BIBO Stability: $\exists \alpha \in \mathcal{K}_\infty$ such that $\|y[k]\| \leq \alpha(\|u\|_{[0,k]}), \forall k, u$	IOS: $\exists \alpha \in \mathcal{K}_\infty, \beta \in \mathcal{KL}_\infty$ such that $\|y[k]\| \leq \beta(\|x_0\|, k) + \alpha(\|u\|_{[0,k]}), \forall k, x_0, u$
BIBOBS Stability: $\exists \alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that $\|x[k]\| \leq \alpha_1(\|x_0\|) + \alpha_2(\|y\|_{[0,k]}) + \alpha_3(\|u\|_{[0,k]}), \forall k, x_0, u$	IOSS: $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty, \beta \in \mathcal{KL}_\infty$ such that $\|x[k]\| \leq \beta(\|x_0\|, k) + \alpha_1(\|u\|_{[0,k]}) + \alpha_2(\|y\|_{[0,k]}), \forall k, x_0, u$

B. Interpretation in Nonlinear Setting

Although we focus on linear systems in this paper, the concept of BIBOBS stability can also be applied to nonlinear systems, and further insights can be obtained by comparing it with a set of properties that are normally defined for such systems. In Table I, we summarize the stability notions of interest; for a comprehensive discussion of these properties, we refer to [19]. Note that *ISS* represents input-to-state stability and *IOS* represents input-output stability. Further note that we also include ISS-type definitions for BIBO stability and BIBS stability by using the class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

From Table I, we can see that the concept of BIBOBS stability fits naturally in the landscape of stability theory. Specifically, we can categorize the notions in Table I by two criteria. The first criterion is whether the output is taken into account: the notions in the first two rows do not consider output. The second criterion is whether the definition requires the influence of the initial condition to decay asymptotically: the notions in the second column all have this requirement.

Among these notions, BIBOBS stability is very similar to IOSS with the only difference being in the term related to x_0. It is easy to see that IOSS is stronger than BIBOBS stability: for a system to be BIBOBS stable, the impact of initial conditions does not have to asymptotically decay over time. Along this line, it is interesting to compare another property which imposes a further constraint on the term related to x_0. In [17], the authors proposed a notion of $\mathcal{KL}$ norm-observability for nonlinear systems; a system is $\mathcal{KL}$ norm-observable if it is IOSS and in the definition of IOSS, the function β can be chosen to decay arbitrarily fast in the second argument. Since $\mathcal{KL}$ norm-observability and IOSS are equivalent to observability and detectability for linear systems, respectively, both of them are stronger than BIBOBS stability.

V. SUMMARY

We studied the problem of constructing an unknown input norm-observer for linear systems with unknown inputs, which can be regarded as a relaxed estimation objective for cases where perfect estimation cannot be achieved. In order to characterize conditions for the existence of such an observer, we proposed the notion of BIBOBS stability and provided system properties that prevent bounded inputs from driving the state to be unbounded while keeping the outputs bounded. We showed that under certain conditions, the inputs can be chosen so that the states corresponding to the eigenvalues with magnitude 1 are persistently excited while the states of the other systems are maintained in a bounded orbit; thus, care must be taken to avoid such situations. Following the discussion in Section IV-B, it will be of interest to extend the concept of BIBOBS stability to nonlinear systems and explore the conditions for this property to hold.

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