# Automatic Differentiation of Functional Programs and its use for Probabilistic Programming

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Part of this talk covers joint work with Barak A. Pearlmutter.

www.autodiff.org

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**Seventh European Workshop on AD**

INRIA SOPHIA ANTIPOLIS · University of Hertfordshire · Cranfield University Shrivenham · RWTH AACHEN UNIVERSITY

# Seventh European Workshop on Automatic Differentiation
# Monday 24th - Tuesday 25th November 2008
# Oxford-Man Institute of Quantitative Finance
# Oxford University
# Oxford, UK

This two day workshop is the seventh in a series providing a forum for the presentation of theoretical developments in, and applications of, Automatic Differentiation (AD) and adjoint methods. It will be informal, with no published proceedings, so allowing for the discussion of work in progress and presentations by those new to the subject. The workshop is kindly being hosted by Prof. Mike Giles of the Oxford-Man Institute of Quantitative Finance.

We welcome every researcher who wishes to present their work on any aspect of AD or its applications. However, at this workshop we would encourage contributions in the area of

- **Applications and Challenges in Economics and Finance**

Confirmed speakers include:
**Luca Capriotti** - CREDIT SUISSE
**Oliver Pironneau** - Université Paris VI
**Luca Guerrieri** - US Federal Reserve Bank
**Benjamin Skrainka** - University College London

If you are interested in giving a 25 minute talk on any aspect of AD, or any applications you think suitable for AD then please contact Dr Martin Bücker at buecker@sc.rwth-aachen.de. We particularly welcome contributions from research students and those new to AD.

**URGENT**
Accommodation for this workshop has been arranged at Linton Lodge, Oxford which must be booked in the next few days -- see details under Travel and Accomodation

Contact: webmaster@autodiff.org

Username:
Password:
login

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(define ($f$ $x$) $2x^3$)

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(define (f x) $2x^3$) $\rightsquigarrow$ (define (f′ x) $6x^2$)

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# Nesting

```
(sqrt (sqrt x))
```

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$$\max_x \min_y f(x, y)$$

# The Essence of Forward-Mode AD

$$f(c+\varepsilon) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}\varepsilon + \frac{f''(c)}{2!}\varepsilon^2 + \cdots + \frac{f^{(i)}(c)}{i!}\varepsilon^i + \cdots$$

Taylor, B. (1715). *Methodus Incrementorum Directa et Inversa*. London.

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- evaluate $f$

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$(\mathcal{D}\, f)$ is $\mathcal{O}(1)$ relative to $f$ (both space and time).

# Arithmetic on Complex Numbers

$a + b\mathrm{i}$

Hamilton, W. R. (1837). *Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time*. Transactions of the Royal Irish Academy, **17**(1):293–422.

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$\langle a, b \rangle$

$$\langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle = \langle (a_1 + a_2), (b_1 + b_2) \rangle$$
$$\langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle = \langle (a_1 \times a_2 - b_1 \times b_2), (a_1 \times b_2 + a_2 \times b_1) \rangle$$

Hamilton, W. R. (1837). *Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time*. Transactions of the Royal Irish Academy, **17**(1):293–422.

# Arithmetic on Dual Numbers

$x + x'\varepsilon$

Clifford, W. K. (1873). *Preliminary Sketch of Bi-quaternions*. Proceedings of the London Mathematical Society, **4**:381–95.

# Arithmetic on Dual Numbers

$x + x'\varepsilon$
$\varepsilon^2 = 0$, but $\varepsilon \neq 0$

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# Arithmetic on Dual Numbers

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$$(x_1 + x_1'\varepsilon) + (x_2 + x_2'\varepsilon) = (x_1 + x_2) + (x_1' + x_2')\varepsilon$$
$$(x_1 + x_1'\varepsilon) \times (x_2 + x_2'\varepsilon) = (x_1 \times x_2) + (x_1 \times x_2' + x_2 \times x_1')\varepsilon + (x_1' + x_2')\varepsilon^2$$

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$\langle x, x' \rangle$

Clifford, W. K. (1873). *Preliminary Sketch of Bi-quaternions*. Proceedings of the London Mathematical Society, **4**:381–95.

# Arithmetic on Dual Numbers

$x + x'\varepsilon$
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$$(x_1 + x_1'\varepsilon) \times (x_2 + x_2'\varepsilon) = (x_1 \times x_2) + (x_1 \times x_2' + x_2 \times x_1')\varepsilon$$

$\langle x, x' \rangle$

$$\langle x_1, x_1' \rangle + \langle x_2, x_2' \rangle = \langle (x_1 + x_2), (x_1' + x_2') \rangle$$
$$\langle x_1, x_1' \rangle \times \langle x_2, x_2' \rangle = \langle (x_1 \times x_2), (x_1 \times x_2' + x_2 \times x_1') \rangle$$

Clifford, W. K. (1873). *Preliminary Sketch of Bi-quaternions*. Proceedings of the London Mathematical Society, **4**:381–95.

# Dynamic Overloading: SCMUTILS

```
(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define +
 (let ((+ +))
  (lambda (x1 x2)
   (make-bundle (+ (primal x1) (primal x2))
                (+ (tangent x1) (tangent x2))))))

(define *
 (let ((+ +) (* *))
  (lambda (x1 x2)
   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
                   (* (tangent x1) (primal x2)))))))

(define ((D f) x) (tangent (f (make-bundle x 1))))
```

# Dynamic Overloading: SCMUTILS

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(define-structure bundle primal tangent)
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                (+ (tangent x1) (tangent x2))))))

(define *
 (let ((+ +) (* *))
  (lambda (x1 x2)
   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
                   (* (tangent x1) (primal x2)))))))

(define ((D f) x) (tangent (f (make-bundle x 1))))

(define (f x) (* 2 (* x (* x x))))
```

# Dynamic Overloading: SCMUTILS

```
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 (let ((+ +) (* *))
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(define ((D f) x) (tangent (f (make-bundle x 1))))

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(𝒟 f)
```

# Dynamic Overloading: SCMUTILS

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   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
                   (* (tangent x1) (primal x2)))))))

(define ((D f) x) (tangent (f (make-bundle x 1))))

(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

# Dynamic Overloading: SCMUTILS

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 (let ((+ +))
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   (make-bundle (+ (primal x1) (primal x2))
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(define *
 (let ((+ +) (* *))
  (lambda (x1 x2)
   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
                   (* (tangent x1) (primal x2)))))))

(define ((D f) x) (tangent (f (make-bundle x 1))))

(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

# Dynamic Overloading: SCMUTILS

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(define +
 (let ((+ +))
  (lambda (x1 x2)
   (make-bundle (+ (primal x1) (primal x2))
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   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
                   (* (tangent x1) (primal x2)))))))

(define ((D f) x) (tangent (f (make-bundle x 1))))

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(𝒟 f)
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(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient

# Dynamic Overloading: SCMUTILS

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 (let ((+ +))
  (lambda (x1 x2)
   (make-bundle (+ (primal x1) (primal x2))
                (+ (tangent x1) (tangent x2))))))

(define *
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   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
                   (* (tangent x1) (primal x2)))))))

(define ((D f) x) (tangent (f (make-bundle x 1))))

(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

# Dynamic Overloading: SCMUTILS

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(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
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 (let ((+ +))
  (lambda (x1 x2)
   (make-bundle (+ (primal x1) (primal x2))
                (+ (tangent x1) (tangent x2))))))

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 (let ((+ +) (* *))
  (lambda (x1 x2)
   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
                   (* (tangent x1) (primal x2)))))))

(define ((D f) x) (tangent (f (make-bundle x 1))))

(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

# Dynamic Overloading: SCMUTILS

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(define +
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  (lambda (x1 x2)
   (make-bundle (+ (primal x1) (primal x2))
                (+ (tangent x1) (tangent x2))))))

(define *
 (let ((+ +) (* *))
  (lambda (x1 x2)
   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
                   (* (tangent x1) (primal x2)))))))

(define ((D f) x) (tangent (f (make-bundle x 1))))

(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

# Dynamic Overloading: SCMUTILS

```
(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define ((D f) x)
 (fluid-let ((+ (lambda (x1 x2)
                  (make-bundle (+ (primal x1) (primal x2))
                               (+ (tangent x1) (tangent x2)))))
             (* (lambda (x1 x2)
                  (make-bundle (* (primal x1) (primal x2))
                               (+ (* (primal x1) (tangent x2))
                                  (* (tangent x1) (primal x2)))))))
  (tangent (f (make-bundle x 1)))))
```

```
(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

# Dynamic Overloading: SCMUTILS

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(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define ((D f) x)
 (fluid-let ((+ (lambda (x1 x2)
                  (make-bundle (+ (primal x1) (primal x2))
                               (+ (tangent x1) (tangent x2)))))
             (* (lambda (x1 x2)
                  (make-bundle (* (primal x1) (primal x2))
                               (+ (* (primal x1) (tangent x2))
                                  (* (tangent x1) (primal x2)))))))
  (tangent (f (make-bundle x 1)))))
```

```
(define (f x) (* 2 (* x (* x x))))

(𝒟 f)
(𝒟 (𝒟 f))
(𝒟 (lambda (x) ... (𝒟 (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)                                    AD_TOP = f
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = gf
AD_IVARS = x, gx
AD_DVARS = gf, gresult
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)                                   AD_TOP = f
double precision x, f                           AD_IVARS = x
f = 2.0d0*x*x*x                                 AD_DVARS = f
end

function gf(x, gx, gresult)                     AD_TOP = gf
double precision x, gx, gf, gresult             AD_IVARS = x, gx
gf = 2.0d0*x*x*x                                AD_DVARS = gf, gresult
gresult = 6.0d0*x*x*gx
end

function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
gresult = 6.0d0*x*x*gx
ggresult = 6.0d0*x*x*ggx+12.0d0*x*gx*gx
end
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)                                    AD_TOP = f
double precision x, f                            AD_IVARS = x
f = 2.0d0*x*x*x                                  AD_DVARS = f
end

function gf(x, gx, gresult)                      AD_TOP = gf
double precision x, gx, gf, gresult              AD_IVARS = x, gx
gf = 2.0d0*x*x*x                                 AD_DVARS = gf, gresult
gresult = 6.0d0*x*x*gx
end

function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
gresult = 6.0d0*x*x*gx
ggresult = 6.0d0*x*x*ggx+12.0d0*x*gx*gx
end
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)                                    AD_TOP = f
double precision x, f                            AD_IVARS = x
f = 2.0d0*x*x*x                                  AD_DVARS = f
end

function gf(x, gx, gresult)                      AD_TOP = gf
double precision x, gx, gf, gresult              AD_IVARS = x, gx
gf = 2.0d0*x*x*x                                 AD_DVARS = gf, gresult
gresult = 6.0d0*x*x*gx
end

function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
gresult = 6.0d0*x*x*gx
ggresult = 6.0d0*x*x*ggx+12.0d0*x*gx*gx
end
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)                                    AD_TOP = f
double precision x, f                            AD_IVARS = x
f = 2.0d0*x*x*x                                  AD_DVARS = f
end

function gf(x, gx, gresult)                      AD_TOP = gf
double precision x, gx, gf, gresult              AD_IVARS = x, gx
gf = 2.0d0*x*x*x                                 AD_DVARS = gf, gresult
gresult = 6.0d0*x*x*gx
end

function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
gresult = 6.0d0*x*x*gx
ggresult = 6.0d0*x*x*ggx+12.0d0*x*gx*gx
end
```

Fast but inconvenient

# Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = gf
AD_IVARS = x, gx
AD_DVARS = gf, gresult
AD_PREFIX = h
```

```
function hgf(x, hx, gx, hgx, gresult, hgresult, hresult)
double precision x, hx, gx, hgx, hgf, hresult, gresult, hgresult
hgf = 2.0d0*x*x*x
hresult = 6.0d0*x*x*hx
gresult = 6.0d0*x*x*gx
hgresult = 6.0d0*x*x*hgx+12.0d0*x*gx*hx
end
```

Fast but inconvenient

# Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...
```

# Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...
```

# Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

# Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow

# Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
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Slow

# Static Overloading: FADBAD++

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double f(double x) {return 2*x*x*x;}
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Slow and inconvenient

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Slow and inconvenient

# Our API for Functional Forward AD

Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^n}$

# Our API for Functional Forward AD

$$\texttt{bundle} \quad : \quad \mathbb{R}^n \times \overline{\mathbb{R}^h} \to (\mathbb{R}^n \triangleright \overline{\mathbb{R}^h})$$

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Differential geometry bundles points $\mathbb{R}^n$ in a manifold with tangent vectors $\overline{\mathbb{R}^h}$
`j*` maps a function to its *push forward*

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Generalize to arbitrary types

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Generalize to arbitrary types

What is the tangent of a discrete value or a function?

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Can abbreviate $\tau \rhd \overline{\tau'}$ as $\overrightarrow{\tau}$

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Convenient

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Convenient and fast

# A property

# A property

$x : \mathbb{R}^n$

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$x : \mathbb{R}^n$ $\overline{x'} : \mathbb{R}^n$

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$x : \mathbb{R}^n$ $\overline{x'} : \mathbb{R}^n$ $f : \mathbb{R}^n \to \mathbb{R}^m$

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$$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$$

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$x : \mathbb{R}^n$ $\quad \overline{x'} : \mathbb{R}^n$ $\quad f : \mathbb{R}^n \to \mathbb{R}^m$

$((\mathcal{J}\ f)\ x)[i,j] = \frac{\partial f(x)[i]}{\partial x[j]}$ $\quad ((\mathcal{J}\ f)\ x) : \mathbb{R}^{m \times n}$

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`(primal ((j* f) (bundle x` $\overline{x'}$ `)))` $= (f\ x)$

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`(primal ((j* f) (bundle x x'))) =` $(f\ x)$

`(tangent ((j* f) (bundle x x'))) =` $((\mathcal{J}\ f)\ x) \times \overline{x'}$

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`(tangent ((j* f) (bundle x x'))) =` $(((\mathcal{J}\ f)\ x)\ \overline{x'})$

`((j* f) x) = (bundle (f (primal x)) (((`$\mathcal{J}\ f$`) (primal x)) (tangent x)))`

# A property

$x : \mathbb{R}^n$ $\quad \overline{x'} : \mathbb{R}^n$ $\quad f : \mathbb{R}^n \to \mathbb{R}^m$

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rearrangement function: $(\forall i)(\exists j)(f\ x)[i] = x[j]$

# A property

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when $f$ is a rearrangement function

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Problem avoided if we take $\overline{\texttt{\#t}'} = \texttt{\#t}$

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# Modularity

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# *Breaking* Modularity

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$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overleftrightarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots \mathcal{H} \overleftrightarrow{f}\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ \overleftarrow{f} \triangleq \ldots \textsc{NewtonsMethod}\ \overleftarrow{f}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq (\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

$$\textsc{Deviation} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \triangleq \text{argmin}\ \overleftarrow{\textsc{Deviation}}$$

Fermi, E. (1946). *The Development of the first chain reaction pile*. Proceedings of the American Philosophy Society, **90**:20–4.

# *Breaking* Modularity

$$\nabla \overleftarrow{f}\ \mathbf{x} \triangleq \ldots \overleftarrow{f}\ \mathbf{x} \ldots$$

$$\mathcal{H} \overleftrightarrow{f}\ \mathbf{x} \triangleq \ldots \overleftrightarrow{f} \ldots \mathbf{x} \ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overleftrightarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots \mathcal{H} \overleftrightarrow{f}\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ \overleftarrow{f} \triangleq \ldots \textsc{NewtonsMethod}\ \overleftarrow{f}\ \overleftrightarrow{f}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

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# *Breaking* Modularity

$$\nabla \overleftarrow{f}\ \mathbf{x} \triangleq \ldots \overleftarrow{f}\ \mathbf{x} \ldots$$

$$\mathcal{H} \overrightarrow{\overleftarrow{f}}\ \mathbf{x} \triangleq \ldots \overrightarrow{\overleftarrow{f}} \ldots \mathbf{x} \ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ \overleftarrow{f}\ \overrightarrow{\overleftarrow{f}}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots \mathcal{H} \overrightarrow{\overleftarrow{f}}\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ \overleftarrow{f}\ \overrightarrow{\overleftarrow{f}} \triangleq \ldots \textsc{NewtonsMethod}\ \overleftarrow{f}\ \overrightarrow{\overleftarrow{f}}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{NeutronFlux} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{NeutronFlux}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

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# *Breaking* Modularity

$$\nabla \overleftarrow{f}\ \mathbf{x} \triangleq \ldots \overleftarrow{f}\ \mathbf{x} \ldots$$

$$\mathcal{H} \overleftrightarrow{f}\ \mathbf{x} \triangleq \ldots \overleftrightarrow{f} \ldots \mathbf{x} \ldots$$

$$\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots$$

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$$\text{argmin}\ \overleftarrow{f}\ \overleftrightarrow{f} \triangleq \ldots \textsc{NewtonsMethod}\ \overleftarrow{f}\ \overleftrightarrow{f}\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

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$$\textsc{Deviation} \overset{\textsc{Tapenade}}{\rightsquigarrow} \overleftarrow{\textsc{Deviation}}$$

$$\mathbf{r}^* \triangleq \text{argmin}\ \overleftarrow{\textsc{Deviation}}\ \overleftarrow{\overleftarrow{\textsc{Deviation}}}$$

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# *Breaking* Modularity

$$
\begin{array}{lcl}
\nabla \overleftarrow{f}\ \mathbf{x} & \triangleq & \ldots \overleftarrow{f}\ \mathbf{x} \ldots \\
\mathcal{H} \overleftrightarrow{f}\ \mathbf{x} & \triangleq & \ldots \overleftrightarrow{f} \ldots \mathbf{x} \ldots \\
\textsc{GradientDescent}\ \overleftarrow{f}\ \mathbf{x}_0 & \triangleq & \ldots \mathbf{x}_{i+1} := \ldots \nabla \overleftarrow{f}\ \mathbf{x}_i \ldots \\
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\text{argmin}\ \overleftarrow{f}\ \overleftrightarrow{f} & \triangleq & \ldots \textsc{NewtonsMethod}\ \overleftarrow{f}\ \overleftrightarrow{f}\ \mathbf{x}_0 \ldots \\
\textsc{NeutronFlux}\ \mathbf{r} & \triangleq & \boxed{\textit{classified}} \\
\textsc{NeutronFlux} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overleftarrow{\textsc{NeutronFlux}} \\
\overleftarrow{\textsc{NeutronFlux}} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overleftrightarrow{\textsc{NeutronFlux}} \\
\textsc{Deviation}\ \mathbf{r} & \triangleq & ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2 \\
\textsc{Deviation} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overleftarrow{\textsc{Deviation}} \\
\overleftarrow{\textsc{Deviation}} & \overset{\textsc{Tapenade}}{\rightsquigarrow} & \overleftrightarrow{\textsc{Deviation}} \\
\mathbf{r}^* & \triangleq & \text{argmin}\ \overleftarrow{\textsc{Deviation}}\ \overleftrightarrow{\textsc{Deviation}}
\end{array}
$$

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# *Restoring* Modularity

$$\nabla f\ \mathbf{x} \triangleq$$

$$\mathcal{H} f\ \mathbf{x} \triangleq$$

$$\textsc{GradientDescent}\ f\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla f\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ f\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla f\ \mathbf{x}_i \ldots \mathcal{H} f\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ f \triangleq \ldots \textsc{GradientDescent}\ f\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

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# *Restoring* Modularity

$$\nabla f\ \mathbf{x} \triangleq ((\overrightarrow{\mathcal{J}}\ f)\ \mathbf{x} \rhd \overline{\mathbf{e}_1'}), \ldots, ((\overrightarrow{\mathcal{J}}\ f)\ \mathbf{x} \rhd \overline{\mathbf{e}_n'})$$

$$\mathcal{H} f\ \mathbf{x} \triangleq$$

$$\textsc{GradientDescent}\ f\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla f\ \mathbf{x}_i \ldots$$

$$\textsc{NewtonsMethod}\ f\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla f\ \mathbf{x}_i \ldots \mathcal{H} f\ \mathbf{x}_i \ldots$$

$$\text{argmin}\ f \triangleq \ldots \textsc{GradientDescent}\ f\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

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# *Restoring* Modularity

$$\nabla f\ \mathbf{x} \triangleq \ldots (\overleftarrow{\mathcal{J}}\ f)\ \mathbf{x} \ldots$$

$$\mathcal{H} f\ \mathbf{x} \triangleq$$

$$\textsc{GradientDescent}\ f\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla f\ \mathbf{x}_i \ldots$$

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$$\text{argmin}\ f \triangleq \ldots \textsc{GradientDescent}\ f\ \mathbf{x}_0 \ldots$$

$$\textsc{NeutronFlux}\ \mathbf{r} \triangleq \boxed{\textit{classified}}$$

$$\textsc{Deviation}\ \mathbf{r} \triangleq ((\textsc{NeutronFlux}\ \mathbf{r}) - \textsc{NeutronFlux}_{\text{critical}})^2$$

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# *Restoring* Modularity

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$$\mathcal{H} f\ \mathbf{x} \triangleq \ldots (\overrightarrow{\mathcal{J}}\ (\overleftarrow{\mathcal{J}}\ f)) \ldots \mathbf{x} \ldots$$

$$\textsc{GradientDescent}\ f\ \mathbf{x}_0 \triangleq \ldots \mathbf{x}_{i+1} := \ldots \nabla f\ \mathbf{x}_i \ldots$$

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# Having your cake and eating it too

- Convenient
- Fast

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  - $\mathcal{D}$ formulated as a higher-order function in the language
- Fast

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- Convenient
  - $\mathcal{D}$ formulated as a higher-order function in the language
  - no arbitrary restrictions
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  - higher-order derivatives
    `(`$\mathcal{D}$ `(`$\mathcal{D}$ `f))`
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    `(`$\mathcal{D}$ `(`$\mathcal{D}$ `f))`
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    `(`$\mathcal{D}$ `(lambda (...) ... (`$\mathcal{D}$ `(lambda (...) ...)) ...))`
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  - $\mathcal{D}$ formulated as a higher-order function in the language
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  - higher-order derivatives
    `(`$\mathcal{D}$ `(`$\mathcal{D}$ `f))`
  - nesting
    `(`$\mathcal{D}$ `(lambda (...) ... (`$\mathcal{D}$ `(lambda (...) ...)) ...))`
- Fast
  - $\mathcal{D}$ implemented by reflective transformation of environments and code associated with closures
  - compile away reflection with partial evaluation implemented by flow analysis

# Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f)
 ...)
```

# Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f)
 ...)

(𝒟 (lambda (x) 2x^3))
```

# Monovariant Flow Analysis: 0-CFA

```
(define (D f:(λx 2x^3))
 ...)

(D (lambda (x) 2x^3))
```

# Monovariant Flow Analysis: 0-CFA

```
(define (D f:(λx 2x^3))
  ...:(λx 6x^2))

(D (lambda (x) 2x^3))
```

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3)$)
 ...:$(\lambda x\ 6x^2)$)

($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3)$)
 ...:$(\lambda x\ 6x^2)$)

($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$

($\mathcal{D}$ (lambda (x) $3x^4$))

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4)$)
 ...:$(\lambda x\ 6x^2)$)

($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$

($\mathcal{D}$ (lambda (x) $3x^4$))

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4)$)
  ...:$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$)

($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$

($\mathcal{D}$ (lambda (x) $3x^4$))

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4)$)
  ...:$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$)

($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$

($\mathcal{D}$ (lambda (x) $3x^4$)):$(\lambda x\ 12x^3)$

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4)$)
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($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$

($\mathcal{D}$ (lambda (x) $3x^4$)):$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ 2x^3) \cup (\lambda x\ 3x^4)$)
 ...:$(\lambda x\ 6x^2) \cup (\lambda x\ 12x^3)$)

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# Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f)
 ...)
```

# Monovariant Flow Analysis: 0-CFA

```
(define (D f)
 ...)

(D (D (lambda (x) e^2x)))
```

# Monovariant Flow Analysis: 0-CFA

```
(define (𝒟 f:(λx e^2x))
 ...)

(𝒟 (𝒟 (lambda (x) e^2x)))
```

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\, e^{2x})$)
  ...:$(\lambda x\, 2e^{2x})$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) $e^{2x}$)))

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\, e^{2x})$)
  ...:$(\lambda x\, 2e^{2x})$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) $e^{2x}$)):$(\lambda x\, 2e^{2x})$)

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ e^{2x}) \cup (\lambda x\ 2e^{2x})$)
  ...:$(\lambda x\ 2e^{2x})$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) $e^{2x}$)):$(\lambda x\ 2e^{2x})$)

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ e^{2x}) \cup (\lambda x\ 2e^{2x})$)
  ...:$(\lambda x\ 2e^{2x}) \cup (\lambda x\ 4e^{2x})$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) $e^{2x}$)):$(\lambda x\ 2e^{2x})$)

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ e^{2x}) \cup (\lambda x\ 2e^{2x})$)
  ...:$(\lambda x\ 2e^{2x}) \cup (\lambda x\ 4e^{2x})$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) $e^{2x}$)):$(\lambda x\ 2e^{2x}) \cup (\lambda x\ 4e^{2x})$)

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f: $(\lambda x\ e^{2x}) \cup (\lambda x\ 2e^{2x}) \cup (\lambda x\ 4e^{2x})$)
  ...: $(\lambda x\ 2e^{2x}) \cup (\lambda x\ 4e^{2x})$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) $e^{2x}$)): $(\lambda x\ 2e^{2x}) \cup (\lambda x\ 4e^{2x})$)

# Monovariant Flow Analysis: 0-CFA

(define ($\mathcal{D}$ f:$(\lambda x\ e^{2x}) \cup (\lambda x\ 2e^{2x}) \cup (\lambda x\ 4e^{2x}) \cup \ldots$)
 ...:$(\lambda x\ 2e^{2x}) \cup (\lambda x\ 4e^{2x}) \cup \ldots$)

($\mathcal{D}$ ($\mathcal{D}$ (lambda (x) $e^{2x}$)):$(\lambda x\ 2e^{2x}) \cup (\lambda x\ 4e^{2x}) \cup \ldots$)

# Polyvariant Flow Analysis: $k$-CFA

## with Bounded Context Sensitivity

```
(define (𝒟 f) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

# Polyvariant Flow Analysis: $k$-CFA

## with Bounded Context Sensitivity

```
(define (𝒟 f) ...)
```

```
(define (g ...) ... (𝒟 (lambda (x) 2x^3)) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

# Polyvariant Flow Analysis: $k$-CFA

## with Bounded Context Sensitivity

`(define (`$\mathcal{D}_{\texttt{g}}$ `f) ...)`

`(define (g ...) ... (`$\mathcal{D}$ `(lambda (x)` $2x^3$`)) ...)`

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

# Polyvariant Flow Analysis: $k$-CFA

## with Bounded Context Sensitivity

(define ($\mathcal{D}_{\mathsf{g}}$ f:($\lambda x\ 2x^3$)) ...)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)) ...)

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# Polyvariant Flow Analysis: $k$-CFA

## with Bounded Context Sensitivity

`(define (`$\mathcal{D}_{\texttt{g}}$ `f:`$(\lambda x\ 2x^3)$`) ...:`$(\lambda x\ 6x^2)$`)`

`(define (g ...) ...` $(\mathcal{D}$ `(lambda (x)` $2x^3$`)) ...)`

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# Polyvariant Flow Analysis: $k$-CFA

## with Bounded Context Sensitivity

`(define (`$\mathcal{D}_{\mathtt{g}}$ `f:`$(\lambda x\ 2x^3)$`) ...:`$(\lambda x\ 6x^2)$`)`

`(define (g ...) ... (`$\mathcal{D}$ `(lambda (x)` $2x^3$`)):`$(\lambda x\ 6x^2)$ `...)`

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# Polyvariant Flow Analysis: $k$-CFA

## with Bounded Context Sensitivity

(define ($\mathcal{D}_{\mathsf{g}}$ f:$(\lambda x\ 2x^3)$) ...:$(\lambda x\ 6x^2)$)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$ ...)

(define (h ...) ... ($\mathcal{D}$ (lambda (x) $3x^4$)) ...)

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

# Polyvariant Flow Analysis: $k$-CFA

## with Bounded Context Sensitivity

```
(define (D_g f:(λx 2x^3)) ...:(λx 6x^2))

(define (D_h f) ...)

(define (g ...) ... (D (lambda (x) 2x^3)):(λx 6x^2) ...)

(define (h ...) ... (D (lambda (x) 3x^4)) ...)
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

# Polyvariant Flow Analysis: $k$-CFA

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(define ($\mathcal{D}_g$ f:$(\lambda x\ 2x^3)$) ...:$(\lambda x\ 6x^2)$)

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# Polyvariant Flow Analysis: $k$-CFA

## with Bounded Context Sensitivity

(define ($\mathcal{D}_g$ f:$(\lambda x\ 2x^3)$) ...:$(\lambda x\ 6x^2)$)

(define ($\mathcal{D}_h$ f:$(\lambda x\ 3x^4)$) ...:$(\lambda x\ 12x^3)$)

(define (g ...) ... ($\mathcal{D}$ (lambda (x) $2x^3$)):$(\lambda x\ 6x^2)$ ...)

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`(define (g ...) ... (`$\mathcal{D}$ `(lambda (x)` $2x^3$`)):`$(\lambda x\ 6x^2)$ `...)`

`(define (h ...) ... (`$\mathcal{D}$ `(lambda (x)` $3x^4$`)):`$(\lambda x\ 12x^3)$ `...)`

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

# Polyvariant Flow Analysis: *k*-CFA

## with Bounded Context Sensitivity

```
(define ((compose n f) x)
 (if (zero? n) x ((compose (- n 1) f) (f x))))
```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

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Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

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```

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(define (𝒟compose:compose f:g′) ...)
```

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```

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

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```

⋮

Shivers, III, O. G. (1991). *Control-Flow Analysis of Higher-Order Languages* ***or*** *Taming Lambda*. Ph.D. thesis, CMU.

# Polyvariant Flow Analysis

## with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \rightarrow v$$

# Polyvariant Flow Analysis

## with Unbounded Context Sensitivity

$$\mathcal{E} : e \times \sigma \to v$$

$$v ::= \texttt{\#t} \mid \texttt{\#f} \mid \texttt{()} \mid \mathbb{R} \mid (v_1, v_2) \mid \langle \sigma, e \rangle$$

$$\sigma ::= \{x_1 \mapsto v_1, \ldots\}$$

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Memoize $\overline{\mathcal{E}}$ indexed (by suitable equivalence relations on) $e$ and $\overline{\sigma}$.

# Polyvariant Flow Analysis

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Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.

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Side benefit: union-free
No tags, tag checking, tag dispatching, indirect calls

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Not suitable for arbitrary (i.e., typical SCHEME, ML, HASKELL, etc.) programs.
*I*s suitable for FORTRAN-like programs.
*Necessary* for migrating reflective source-code transformation to compile time.
Side benefits: union-free, no cyclic abstract values
No tags, tag checking, tag dispatching, indirect calls
Allows complete unboxing: no allocation, reclamation, indirection

# Game Theory

| | | $B$ | | | | |
|---|---|---|---|---|---|---|
| | | $b_1$ | $\dots$ | $b_j$ | $\dots$ | $b_n$ |
| | $a_1$ | | | | | |
| | $\vdots$ | | $\ddots$ | $\vdots$ | | |
| $A$ | $a_i$ | | $\dots$ | $\text{PAYOFF}(a_i, b_j)$ | $\dots$ | |
| | $\vdots$ | | | $\vdots$ | $\ddots$ | |
| | $a_m$ | | | | | |

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

# Game Theory

| | | $B$ | | | | |
|---|---|---|---|---|---|---|
| | | $b_1$ | $\dots$ | $b_j$ | $\dots$ | $b_n$ |
| | $a_1$ | | | | | |
| | $\vdots$ | | $\ddots$ | $\vdots$ | | |
| $A$ | $a_i$ | | $\dots$ | $\textsc{Payoff}(a_i, b_j)$ | $\dots$ | |
| | $\vdots$ | | | $\vdots$ | $\ddots$ | |
| | $a_m$ | | | | | |

$$\max_{a \in A} \min_{b \in B} \textsc{Payoff}(a, b)$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

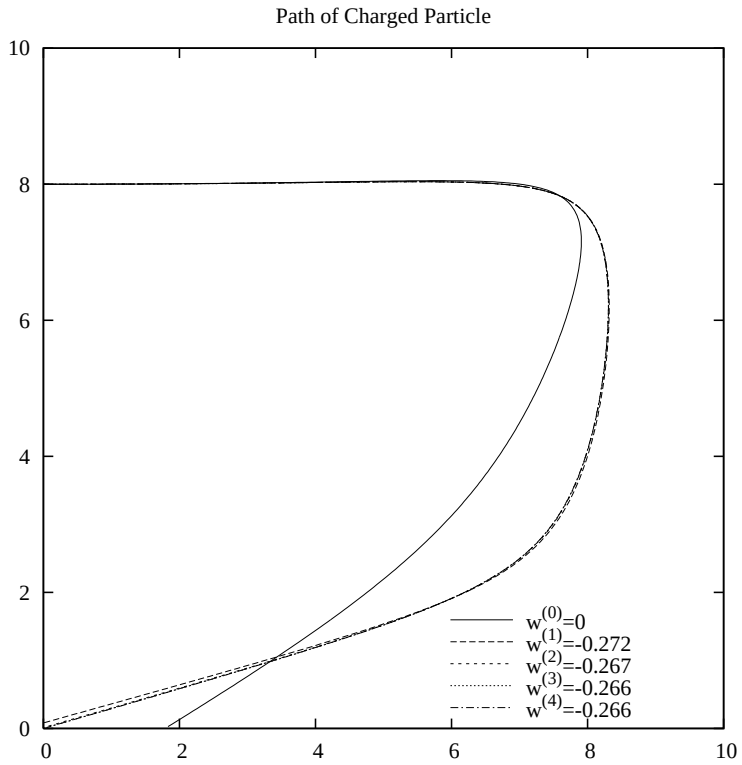
# Game Theory

| | | $\mathbb{R}^n$ | |
|---|---|---|---|
| | | $\mathbf{b}$ | |
| | $\vdots$ | $\vdots$ | $\vdots$ |
| $\mathbb{R}^m$ | $\mathbf{a}$ | $\text{PAYOFF}(\mathbf{a}, \mathbf{b})$ | |
| | $\vdots$ | $\vdots$ | $\vdots$ |

$$\max_{\mathbf{a} \in \mathbb{R}^m} \min_{\mathbf{b} \in \mathbb{R}^n} \text{PAYOFF}(\mathbf{a}, \mathbf{b})$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

# Cathode Ray Tubes




$$
\begin{aligned}
\text{potential: } p(\mathbf{x}; w) &= \|\mathbf{x} - (10, 10 - w)\|^{-1} + \|\mathbf{x} - (10, 0)\|^{-1} \\
\ddot{\mathbf{x}}(t) &= -\nabla_{\mathbf{x}} p(\mathbf{x})|_{\mathbf{x}=\mathbf{x}(t)} \\
\dot{\mathbf{x}}(t + \Delta t) &= \dot{\mathbf{x}}(t) + \Delta t\, \ddot{\mathbf{x}}(t) \\
\mathbf{x}(t + \Delta t) &= \mathbf{x}(t) + \Delta t\, \dot{\mathbf{x}}(t) \\
\text{When: } x_1(t + \Delta t) &\leq 0 \\
\text{let: } \Delta t_f &= -x_1(t)/\dot{x}_1(t) \\
t_f &= t + \Delta t_f \\
\mathbf{x}(t_f) &= \mathbf{x}(t) + \Delta t_f \dot{\mathbf{x}}(t) \\
\text{Error: } E(w) &= x_0(t_f)^2 \\
\text{Find: } & \operatorname*{argmin}_w E(w)
\end{aligned}
$$

Sprague, C. S. and George, R. H. (1939). *Cathode Ray Deflecting Electrode*. US Patent 2,161,437.

George, R. H. (1940). *Cathode Ray Tube*. US Patent 2,222,942.

# Performance Comparison

| | particle | | | | saddle | | | |
|---|---|---|---|---|---|---|---|---|
| | FF | FR | RF | RR | FF | FR | RF | RR |
| STALIN∇ | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| ADIFOR | 1.52 | ■ | ■ | ■ | 2.07 | ■ | ■ | ■ |
| TAPENADE | 3.40 | ■ | ■ | ■ | 2.56 | ■ | ■ | ■ |
| FADBAD++ | 65.69 | ■ | ■ | ■ | 22.44 | ■ | ■ | ■ |
| MLTON | 53.89 | 88.88 | 16.08 | 28.06 | 40.39 | 51.21 | 1.86 | 2.67 |
| OCAML | 160.50 | 340.35 | 147.91 | 263.66 | 107.71 | 156.33 | 6.75 | 13.51 |
| SML/NJ | 106.21 | 182.45 | 105.04 | 185.15 | 84.38 | 106.01 | 3.55 | 6.31 |
| GHC | 165.22 | ■ | ■ | ■ | 121.18 | ■ | ■ | ■ |
| BIGLOO | 505.90 | 761.40 | 104.81 | 228.56 | 423.69 | 440.25 | 15.77 | 24.59 |
| CHICKEN | 1120.37 | 2026.31 | 425.60 | 1872.85 | 889.58 | 1144.65 | 35.73 | 68.94 |
| GAMBIT | 444.13 | 752.63 | 138.34 | 256.30 | 362.65 | 420.48 | 14.08 | 23.87 |
| IKARUS | 192.07 | 312.28 | 61.79 | 114.87 | 158.88 | 205.97 | 6.75 | 11.40 |
| LARCENY | 726.59 | 1108.18 | 144.55 | 270.14 | 571.81 | 613.65 | 19.14 | 29.77 |
| MIT SCHEME | 1472.26 | 2500.00 | 309.66 | 591.36 | 1243.26 | 1428.57 | 51.36 | 79.10 |
| MZC | 2073.26 | 3434.64 | 340.30 | 655.83 | 2436.26 | 1996.40 | 72.45 | 150.02 |
| MZSCHEME | 2344.70 | 4076.16 | 409.95 | 843.68 | 2000.89 | 2332.43 | 80.78 | 134.00 |
| SCHEME->C | 391.42 | 605.26 | 109.77 | 198.43 | 324.95 | 328.84 | 12.74 | 18.28 |
| SCMUTILS | 3321.20 | ■ | ■ | ■ | 2800.71 | ■ | ■ | ■ |
| STALIN | 208.10 | 366.08 | 51.84 | 91.86 | 166.96 | 212.93 | 7.68 | 11.40 |

- ■ not implemented but could implement
- ■ not implemented in existing tool
- ■ can't implement

# Gradient-Based Optimization

```
(define (e i n)
 (if (zero? n)
     '()
     (cons (if (zero? i) 1.0 0.0)
           (e (- i 1) (- n 1)))))
```

# Gradient-Based Optimization

```
(define (e i n)
 (if (zero? n)
     '()
     (cons (if (zero? i) 1.0 0.0)
           (e (- i 1) (- n 1)))))

(define ((gradient f) x)
 (let ((n (length x)))
  (map (lambda (i) (tangent ((j* f) (bundle x (e i n)))))
       (iota n))))
```

# Gradient-Based Optimization

```
(define (e i n)
 (if (zero? n)
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     (cons (if (zero? i) 1.0 0.0)
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(define ((gradient f) x)
 (let ((n (length x)))
  (map (lambda (i) (tangent ((j* f) (bundle x (e i n)))))
       (iota n))))

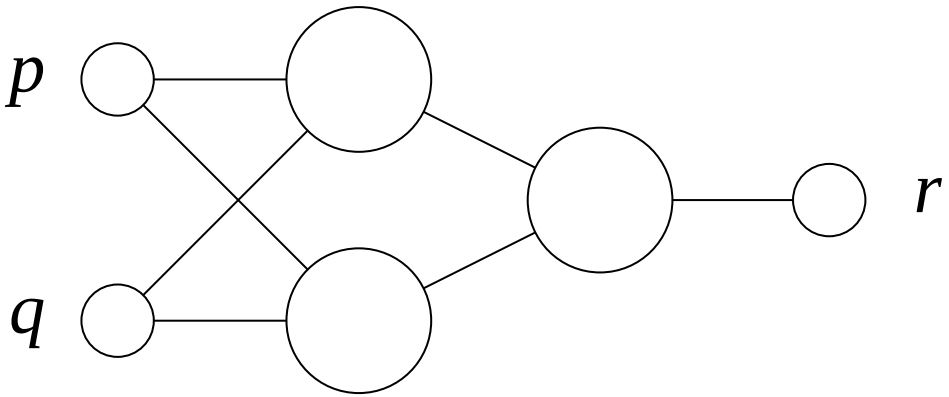
(define (gradient-ascent f x0 n eta)
 (if (zero? n)
     (list x0 (f x0) ((gradient f) x0))
     (gradient-ascent f
                      (zip (lambda (xi gi) (+ xi (* eta gi)))
                           x0
                           ((gradient f) x0))
                      (- n 1)
                      eta)))
```

# Gradient-Based Optimization

```
(define ((gradient f) x) (cdr ((cdr ((*j f) (*j x))) 1.0)))
```

```
(define (gradient-ascent f x0 n eta)
 (if (zero? n)
     (list x0 (f x0) ((gradient f) x0))
     (gradient-ascent f
                      (zip (lambda (xi gi) (+ xi (* eta gi)))
                           x0
                           ((gradient f) x0))
                      (- n 1)
                      eta)))
```

# Neural Networks




| $p$ | $q$ | $r$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

Rumelhart, D. E., Hinton, G. E., and Williams, R. J. (1986). *Learning representations by back-propagating errors*. Nature, **323**:533–6.

# Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))
```

# Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
 (map (sum-activities activities) ws-layer))
```

# Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
 (map (sum-activities activities) ws-layer))

(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))
```

# Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
 (map (sum-activities activities) ws-layer))

(define (sigmoid x) (/ 1 (+ (exp (- 0 x)) 1)))

(define ((forward-pass ws-layers) in)
 (if (null? ws-layers)
     in
     ((forward-pass (cdr ws-layers))
      (map sigmoid (sum-layer in (car ws-layers))))))
```

# Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

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 (map (sum-activities activities) ws-layer))

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 (if (null? ws-layers)
     in
     ((forward-pass (cdr ws-layers))
      (map sigmoid (sum-layer in (car ws-layers))))))

(define ((error-on-dataset dataset) ws-layers)
 ((fold + 0)
  (map (lambda ((list in target))
         (* 0.5 (magnitude-squared (v- ((forward-pass ws-layers) in) target))))
       dataset)))
```

# Neural Networks in VLAD

```
(define ((sum-activities activities) bias ws)
 ((fold + bias) (zip * ws activities)))

(define (sum-layer activities ws-layer)
 (map (sum-activities activities) ws-layer))

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(define ((error-on-dataset dataset) ws-layers)
 ((fold + 0)
  (map (lambda ((list in target))
         (* 0.5 (magnitude-squared (v- ((forward-pass ws-layers) in) target))))
       dataset)))

(gradient-descent (error-on-dataset '(((0 0) (0))
                                      ((0 1) (1))
                                      ((1 0) (1))
                                      ((1 1) (0))))
                  '(((0 -0.284227 1.16054) (0 0.617194 1.30467))
                    ((0 -0.084395 0.648461)))
                  1000.0
                  0.3)
```

# Performance Comparison

| | backprop | | |
|---|---|---|---|
| | Fs | Fv | R |
| STALIN∇ | 1.00 | ■ | 1.00 |
| ADIFOR | 11.84 | 2.68 | ■ |
| TAPENADE | 11.35 | 4.33 | 6.24 |
| ADIC | 16.33 | 3.93 | ■ |
| ADOL–C | 12.34 | 3.89 | 35.53 |
| CPPAD | 42.15 | ■ | 23.69 |
| FADBAD++ | 98.96 | 33.15 | 53.03 |
| MLTON | 73.94 | ■ | 37.94 |
| OCAML | 157.75 | ■ | 149.14 |
| SML/NJ | 142.71 | ■ | 94.97 |
| GHC | ■ | ■ | ■ |
| BIGLOO | 577.45 | ■ | 306.60 |
| CHICKEN | 1391.75 | ■ | 971.91 |
| GAMBIT | 545.20 | ■ | 341.73 |
| IKARUS | 216.42 | ■ | 147.49 |
| LARCENY | 955.98 | ■ | 486.64 |
| MIT SCHEME | 1900.04 | ■ | 1141.22 |
| MZC | 2439.93 | ■ | 1571.52 |
| MZSCHEME | 3477.86 | ■ | 1866.28 |
| SCHEME->C | 484.24 | ■ | 233.75 |
| SCMUTILS | 4544.48 | ■ | ■ |
| STALIN | 832.68 | ■ | 367.84 |

- ■ not implemented but could implement
- ■ not implemented in existing tool
- ■ can't implement

# Probabilistic Lambda Calculus

$P =$ **if** $x_0$ **then** $0$ **else if** $x_1$ **then** $1$ **else** $2$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

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$\Pr(x_0 \mapsto \textbf{true}) = p_0$ $\quad$ $\Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$

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$$\prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v | p_0, p_1) = p_0(1 - p_0)^3 p_1 (1 - p_1)^2$$

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# Probabilistic Prolog

```
p(0).
p(X):-q(X).
q(1).
q(2).
```

# Probabilistic Prolog

$\Pr(\texttt{p(0).}) = p_0$
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  ((lambda-expression? expression)
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    (lambda (tagged-distribution)
     (evaluate
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      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
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(gradient-ascent
 (lambda (p)
  (let ((tagged-distribution
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                   (list Pr(x0 ↦ true) = p0   Pr(x0 ↦ false) = 1 − p0
                         Pr(x1 ↦ true) = p1   Pr(x1 ↦ false) = 1 − p1
                         ...))))
    (map-reduce
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     (lambda (value)
      (likelihood value tagged-distribution))
     '(0 1 2 2))))
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```

# Probabilistic Prolog

```
(define (proof-distribution term clauses)
 (let ((offset ...))
  (map-reduce
   append
   '()
   (lambda (clause)
    (let ((clause (alpha-rename clause offset)))
     (let loop ((p (clause-p clause))
                (substitution (unify term (clause-term clause)))
                (terms (clause-terms clause)))
      (if (boolean? substitution)
          '()
          (if (null? terms)
              (list (make-double p substitution))
              (map-reduce
               append
               '()
               (lambda (double)
                (loop (* p (double-p double))
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               (proof-distribution
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              (list (make-double p substitution))
              (map-reduce
               append
               '()
               (lambda (double)
                (loop (* p (double-p double))
                      (append substitution (double-substitution double))
                      (rest terms)))
               (proof-distribution
                (apply-substitution substitution (first terms)) clauses)))))))
   clauses)))
```

# Probabilistic Prolog

```
(define (proof-distribution term clauses)
 (let ((offset ...))
  (map-reduce
   append
   '()
   (lambda (clause)
    (let ((clause (alpha-rename clause offset)))
     (let loop ((p (clause-p clause))
                (substitution (unify term (clause-term clause)))
                (terms (clause-terms clause)))
      (if (boolean? substitution)
          '()
          (if (null? terms)
              (list (make-double p substitution))
              (map-reduce
               append
               '()
               (lambda (double)
                (loop (* p (double-p double))
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               append
               '()
               (lambda (double)
                (loop (* p (double-p double))
                      (append substitution (double-substitution double))
                      (rest terms)))
               (proof-distribution
                (apply-substitution substitution (first terms)) clauses)))))))
   clauses)))
```

# Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
  (let ((clauses (list Pr(p(0).) = p_0
                       Pr(p(X):-q(X).) = 1 − p_0
                       Pr(q(1).) = p_1
                       Pr(q(2).) = 1 − p_1)))
   (map-reduce
    *
    1.0
    (lambda (query)
     (likelihood (proof-distribution query clauses)))
    '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
 0.1)
```

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    '(p(0) p(1) p(2) p(2)))))
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    '(p(0) p(1) p(2) p(2)))))
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    *
    1.0
    (lambda (query)
     (likelihood (proof-distribution query clauses)))
    '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
 0.1)
```

# Generated Code

```
static void f2679(double a_f2679_0,double a_f2679_1,double a_f2679_2,double a_f2679_3){
  int t272381=((a_f2679_2==0.)?0:1);
  double t272406;
  double t272405;
  double t272404;
  double t272403;
  double t272402;
  if((t272381==0)){
    double t272480=(1.-a_f2679_0);
    double t272572=(1.-a_f2679_1);
    double t273043=(a_f2679_0+0.);
    double t274185=(t272480*a_f2679_1);
    double t274426=(t274185+0.);
    double t275653=(t272480*t272572);
    double t275894=(t275653+0.);
    double t277121=(t272480*t272572);
    double t277362=(t277121+0.);
    double t277431=(t277362*1.);
    double t277436=(t275894*t277431);
    double t277441=(t274426*t277436);
    double t277446=(t273043*t277441);
    ...
    double t1777107=(t1774696+t1715394);
    double t1777194=(0.-t1745420);
    double t1778533=(t1777194+t1419700);
    t272406=a_f2679_0;
    t272405=a_f2679_1;
    t272404=t277446;
    t272403=t1778533;
    t272402=t1777107;}
  else {...}
  r_f2679_0=t272406;
  r_f2679_1=t272405;
  r_f2679_2=t272404;
  r_f2679_3=t272403;
  r_f2679_4=t272402;}
```

# Performance Comparison

| | probabilistic-lambda-calculus | | probabilistic-prolog | |
|---|---|---|---|---|
| | F | R | F | R |
| STALIN∇ | 1.00 | 1.00 | 1.00 | 1.00 |
| MLTON | 106.45 | 124.95 | 789.41 | 483.47 |
| OCAML | 215.73 | 538.68 | 1207.13 | 1534.61 |
| SML/NJ | 197.75 | 272.45 | 2448.02 | 1471.94 |
| GHC | ■ | ■ | ■ | ■ |
| BIGLOO | 832.92 | 1048.11 | 14422.16 | 8286.06 |
| CHICKEN | 2305.98 | 3283.00 | 66948.70 | 37792.84 |
| GAMBIT | 879.88 | 1153.86 | 24316.03 | 13649.81 |
| IKARUS | 437.46 | 531.10 | 8242.92 | 4845.86 |
| LARCENY | 1651.01 | 1673.22 | 25589.62 | 14833.53 |
| MIT SCHEME | 3491.10 | 4130.19 | 85819.57 | 48335.38 |
| MZC | 5289.17 | 5929.14 | 154206.95 | 83480.27 |
| MZSCHEME | 6235.78 | 7134.71 | 166129.12 | 91630.70 |
| SCHEME->C | 682.15 | 794.31 | 10530.66 | 5980.27 |
| SCMUTILS | 6456.99 | ■ | 80100.23 | ■ |
| STALIN | 1240.73 | 1137.41 | 22511.79 | 10986.43 |

- ■ not implemented but could implement, including FORTRAN, C, and C++
- ■ not implemented in existing tool
- ■ can't implement

*It is, of course, not excluded that the range of arguments or range of values of a function should consist wholly or partly of functions. The derivative, as this notion appears in the elementary differential calculus, is a familiar mathematical example of a function for which both ranges consist of functions.*

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Gottfried Leibniz

|

Jacob Bernoulli

|

Johann Bernoulli

|

Leonhard Euler

|

Joseph Louis Lagrange

|

Simeon Poisson

|

Michel Chasles

|

Hubert Anson Newton

|

Eliakim Hastings Moore

|

Oswald Veblen

|

Alonzo Church