Binomial Checkpointing for Arbitrary Programs with No User Annotation

Jeffrey Mark Siskind, qobi@purdue.edu
Barak Avrum Pearlmutter, barak@pearlmutter.net

PURDUE UNIVERSITY

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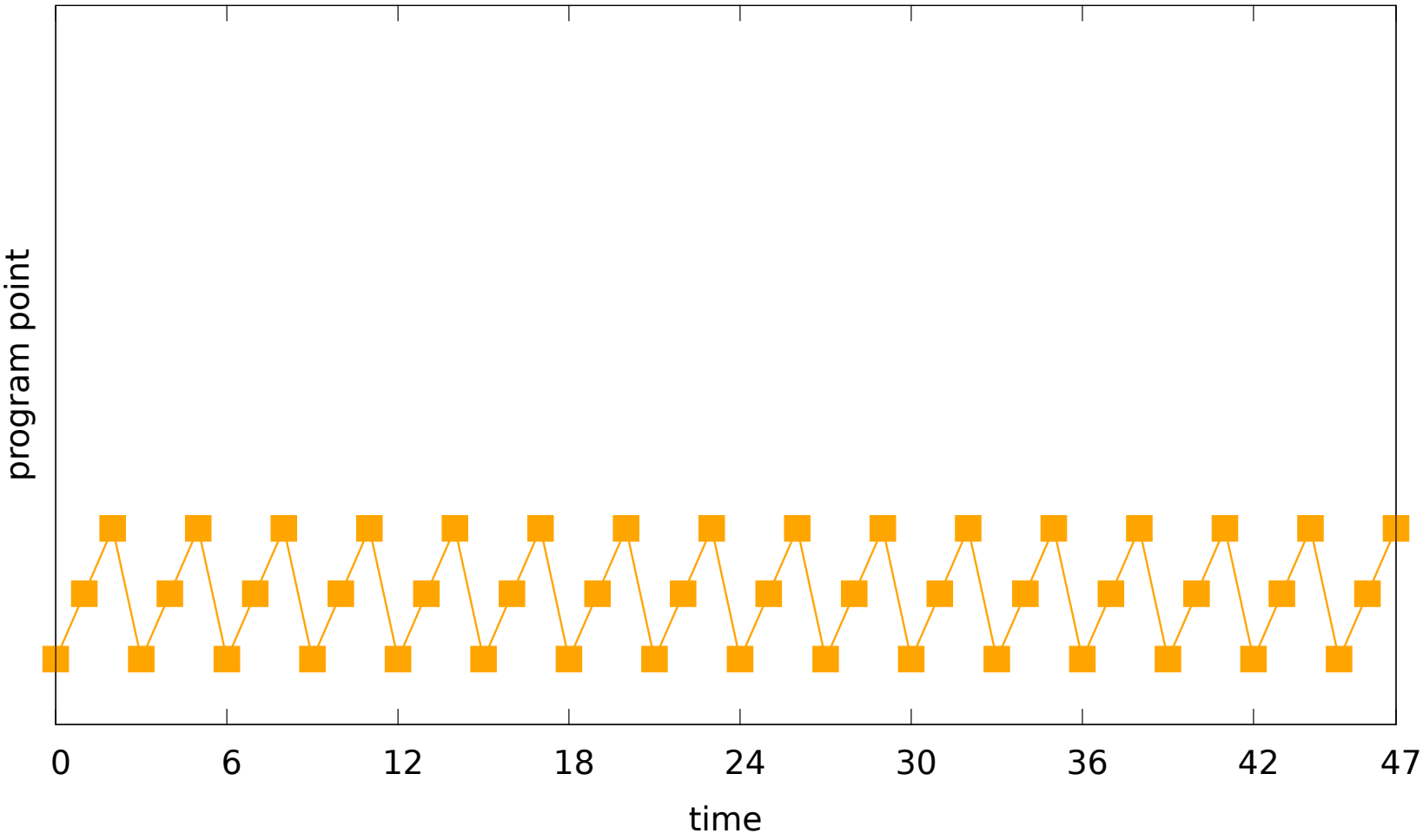
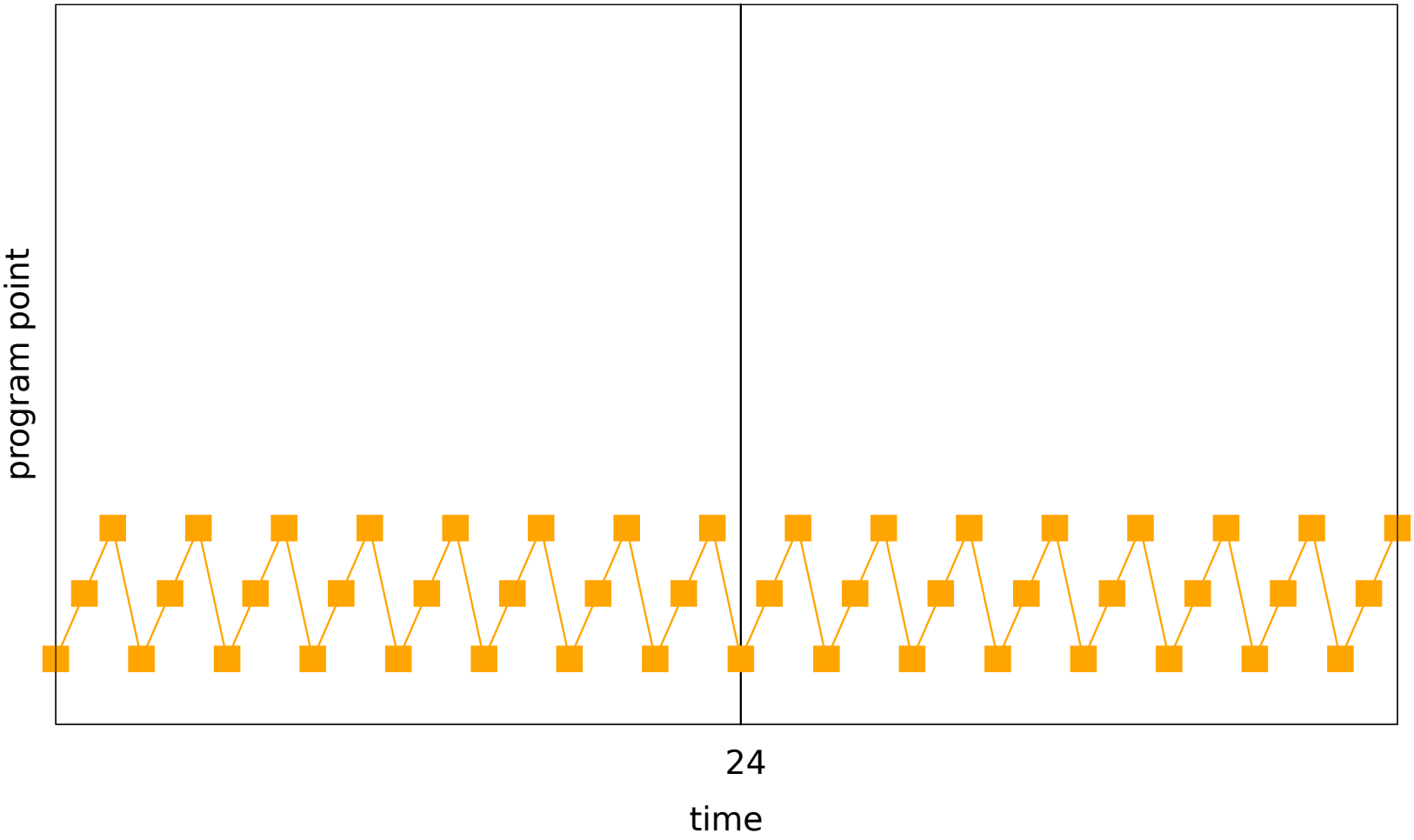
Execution Trace of Loop



Execution Trace of Loop

Easy to make regular and uniform checkpoints



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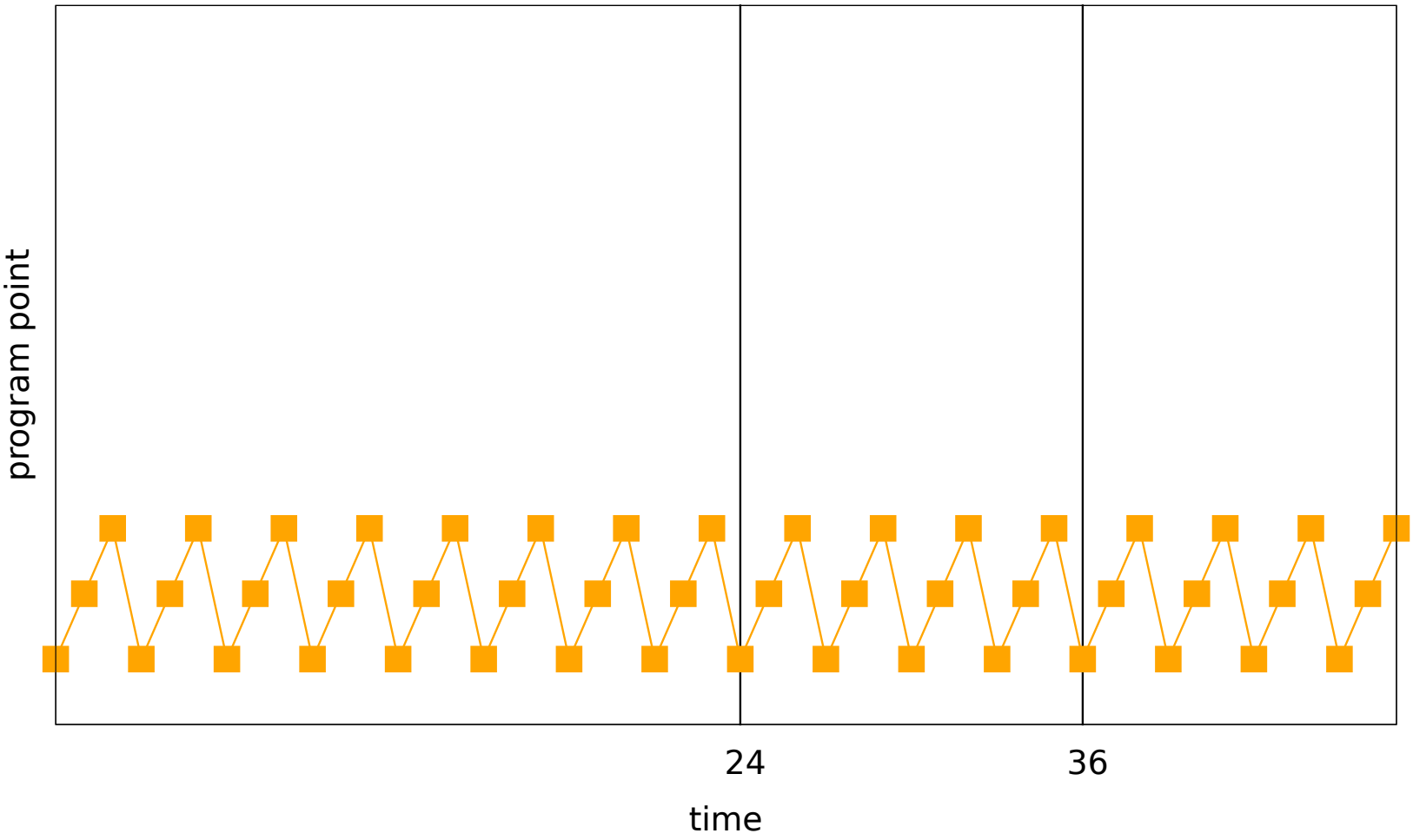

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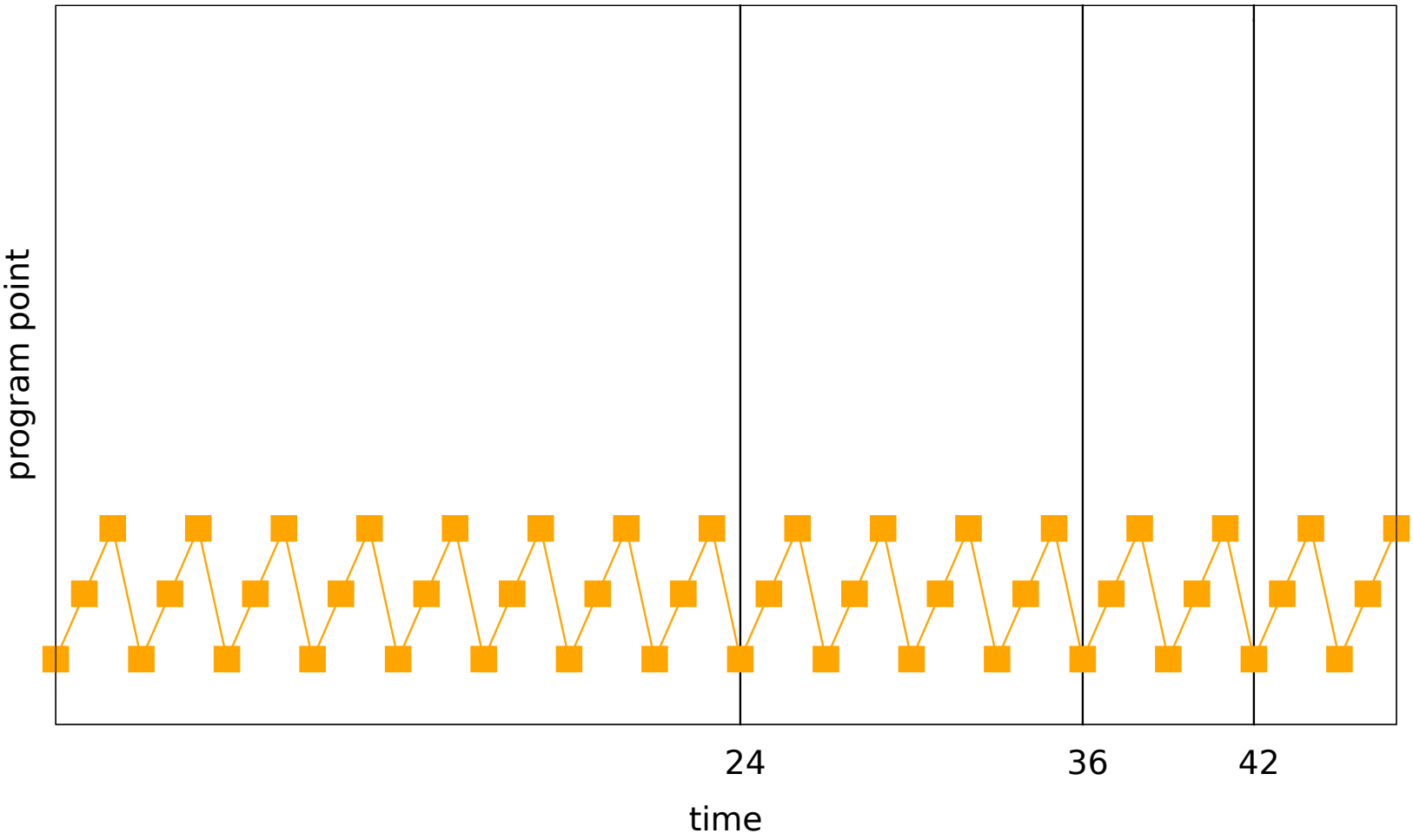

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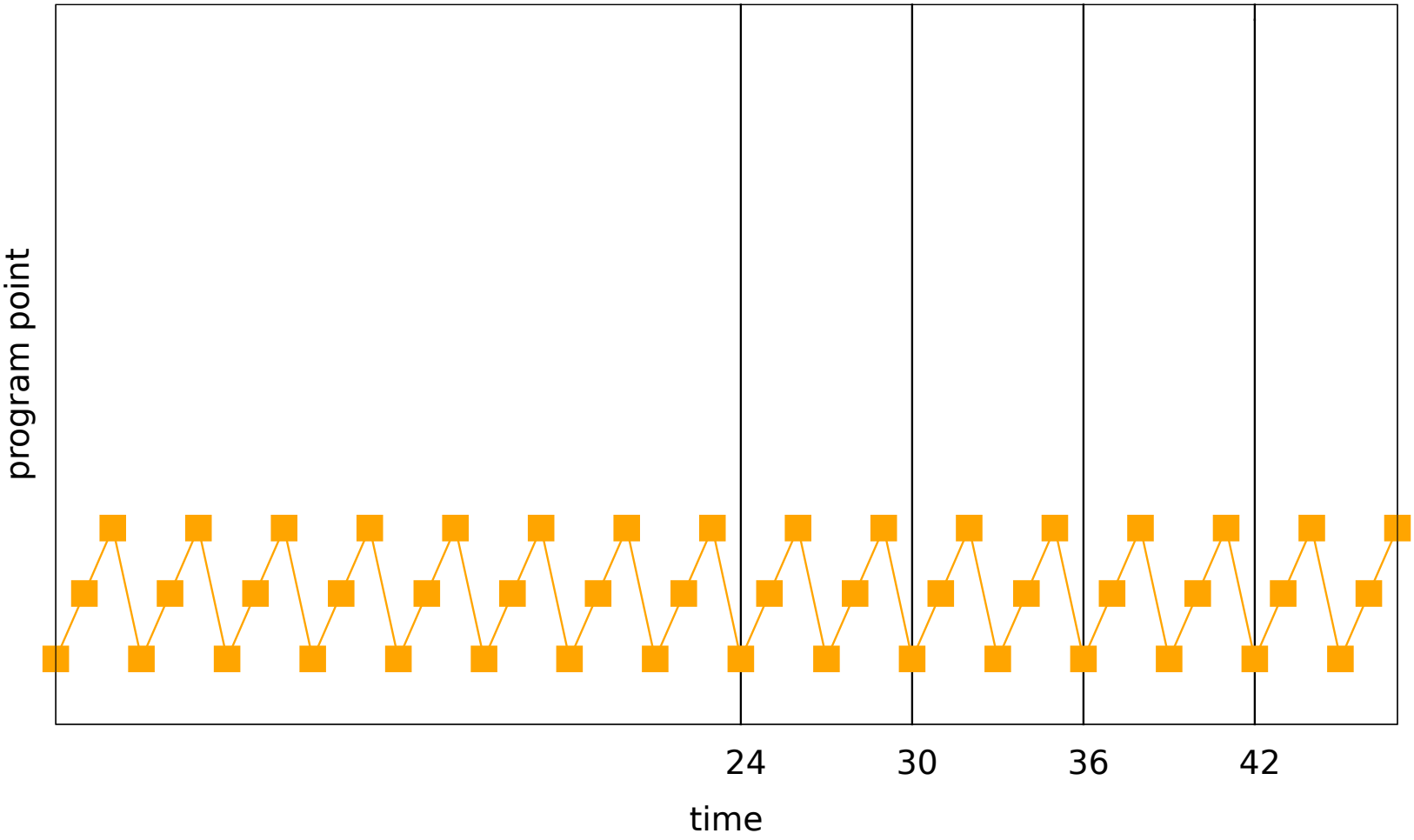

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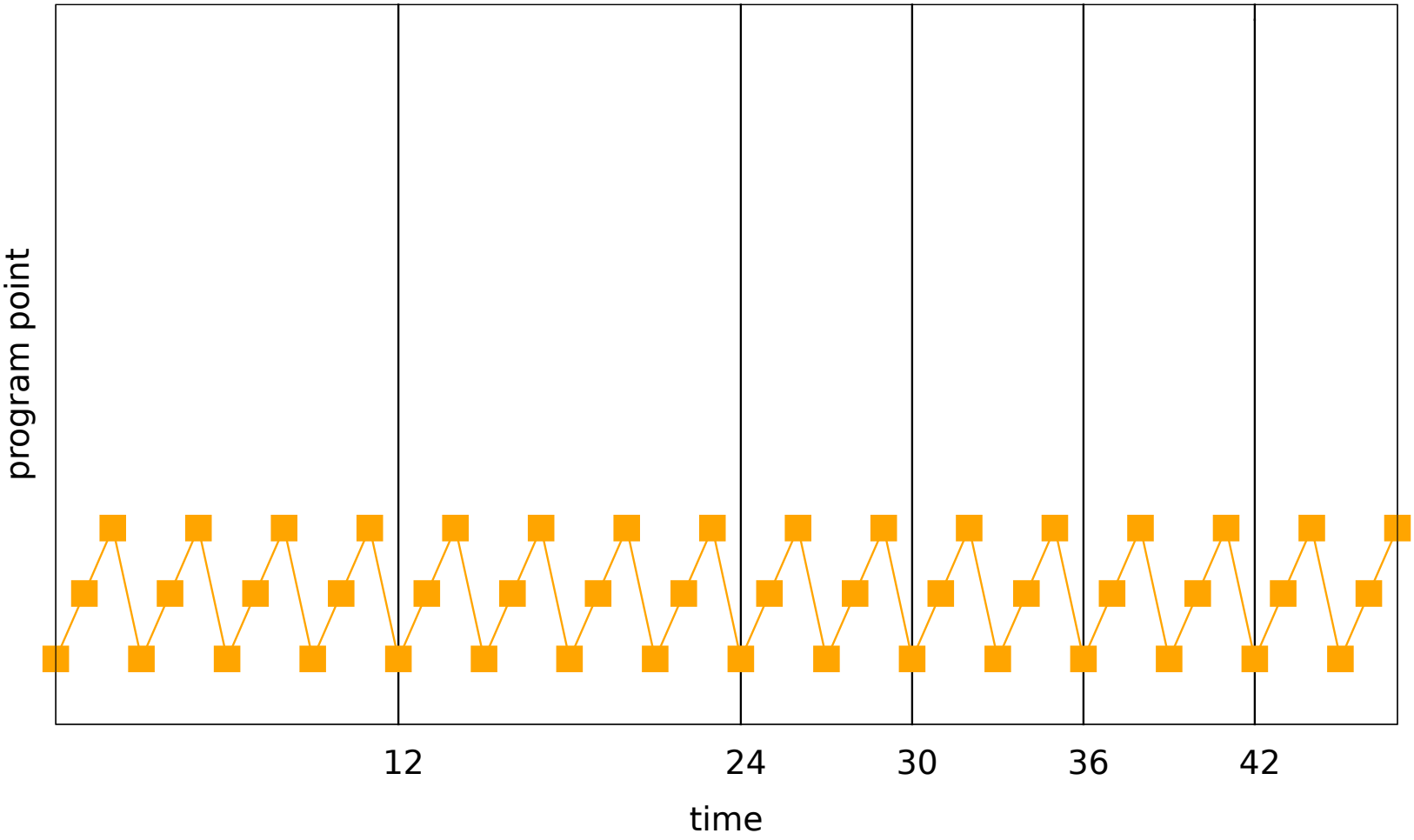

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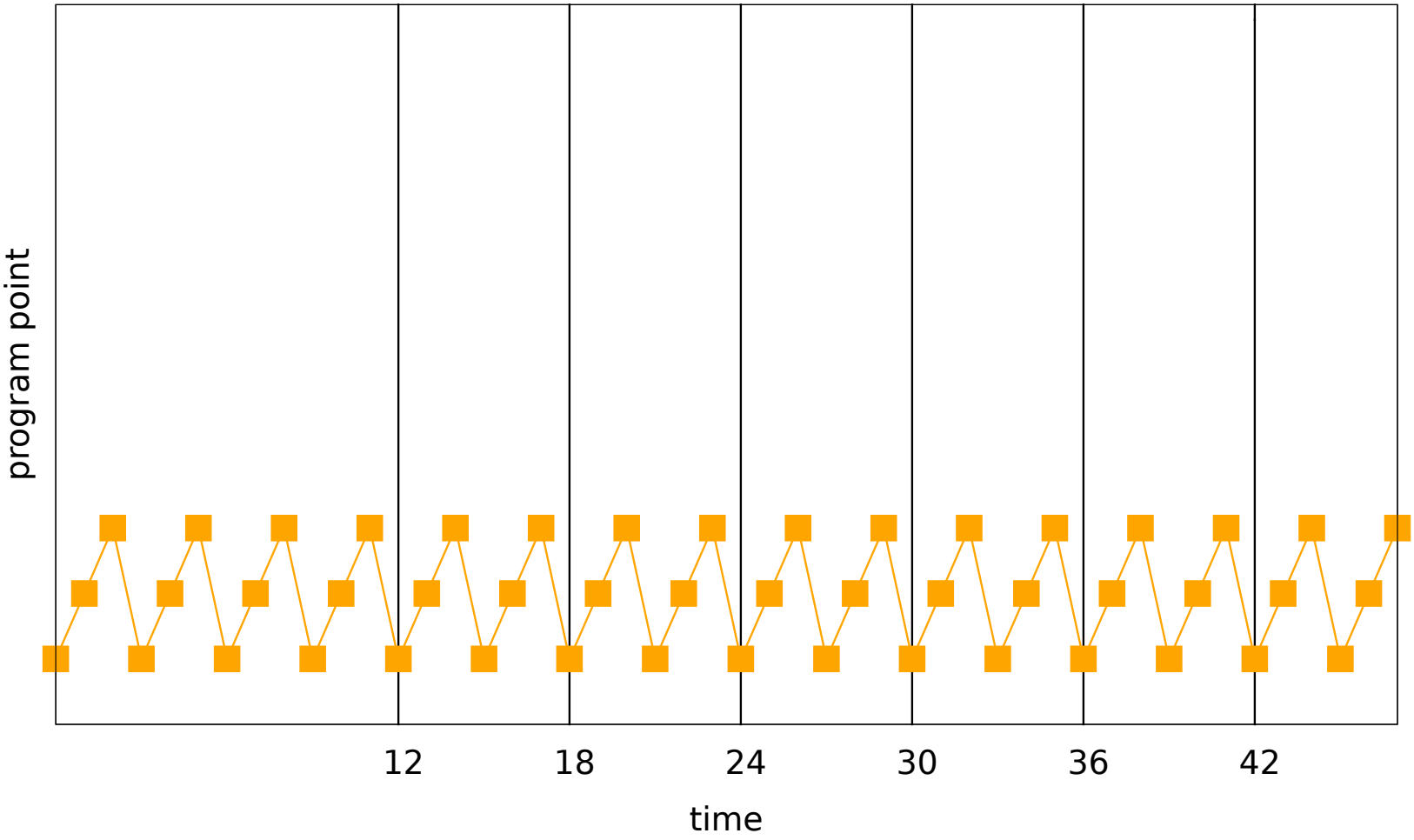

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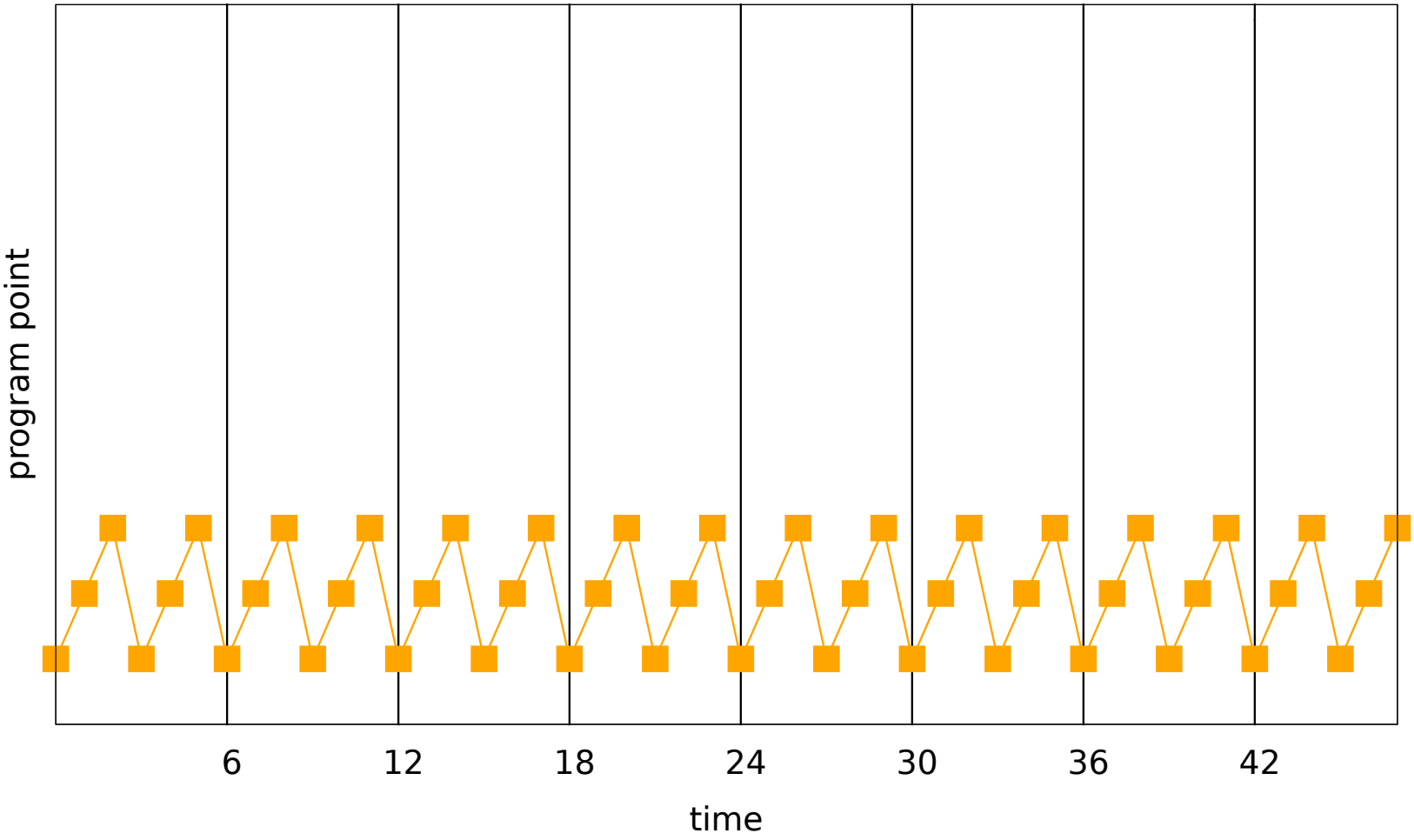
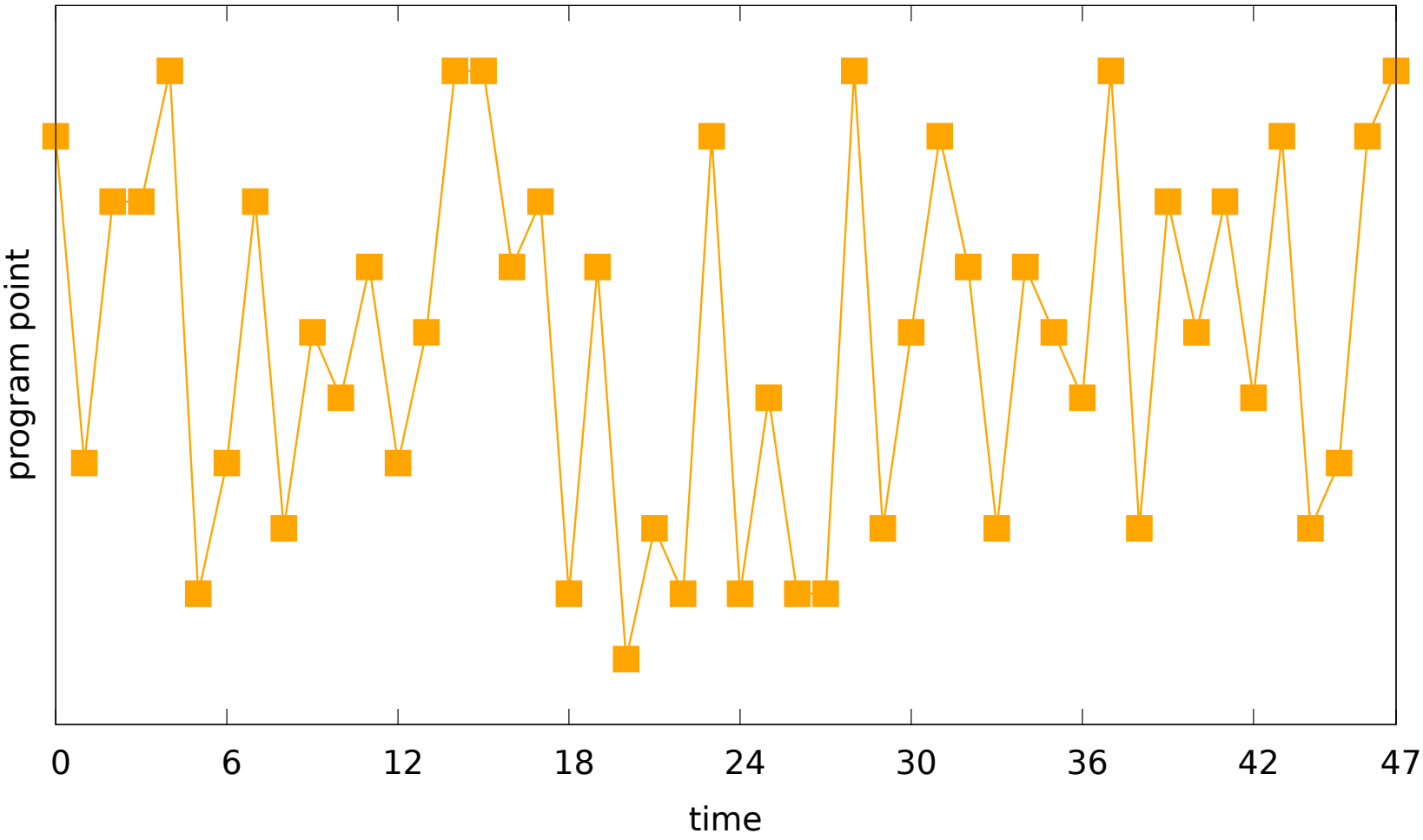


Execution Trace of Arbitrary Code



Execution Trace of Arbitrary Code

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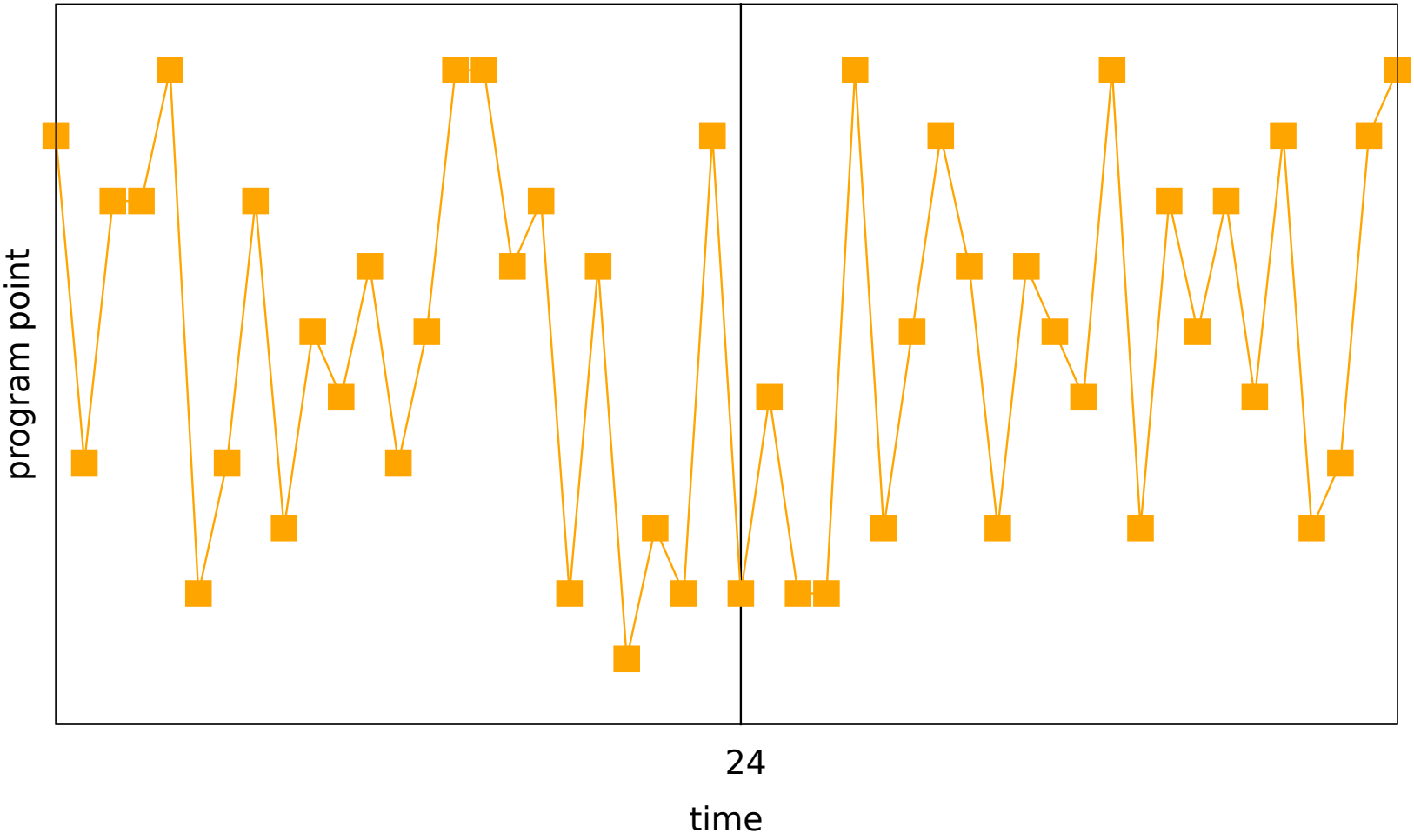

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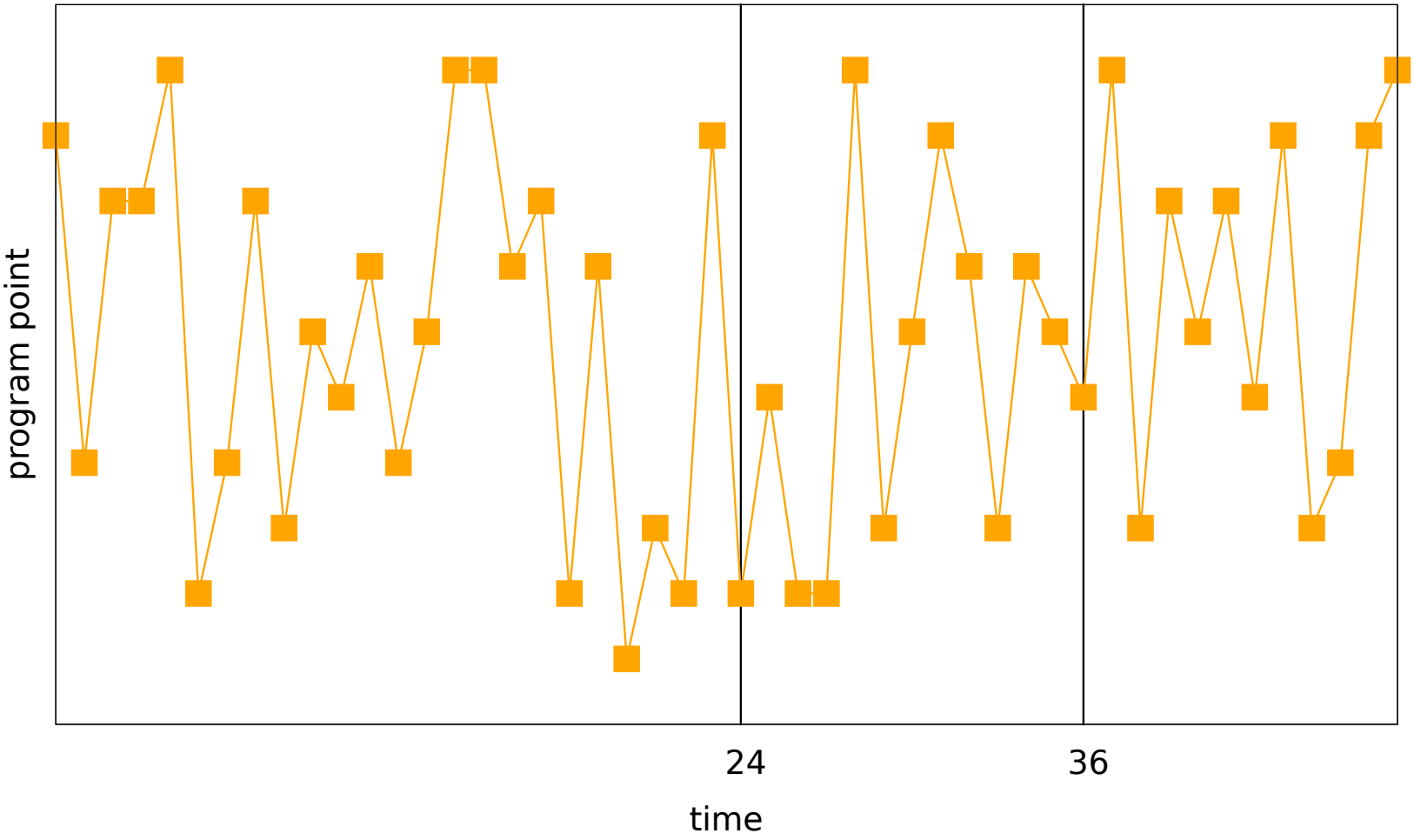

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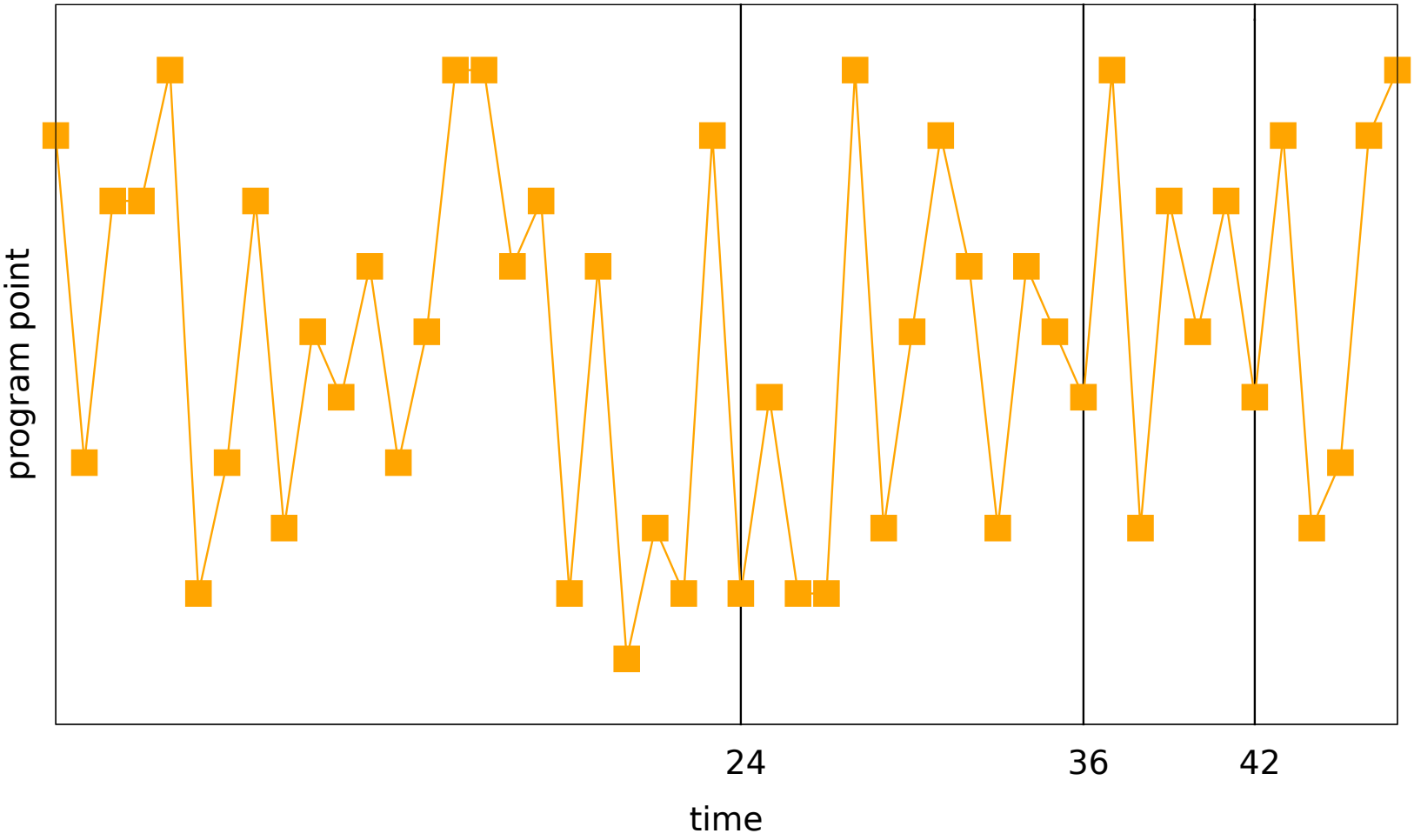

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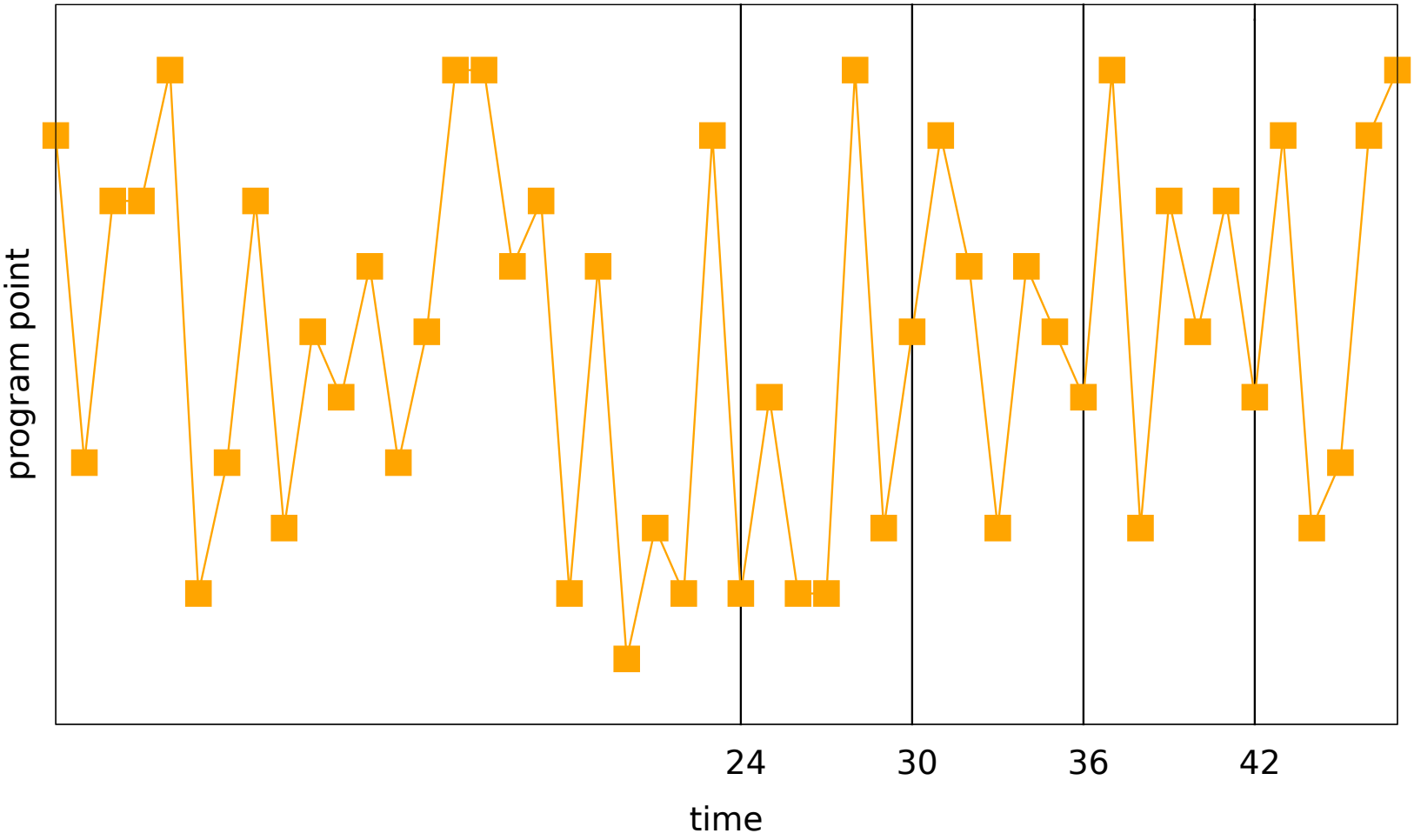

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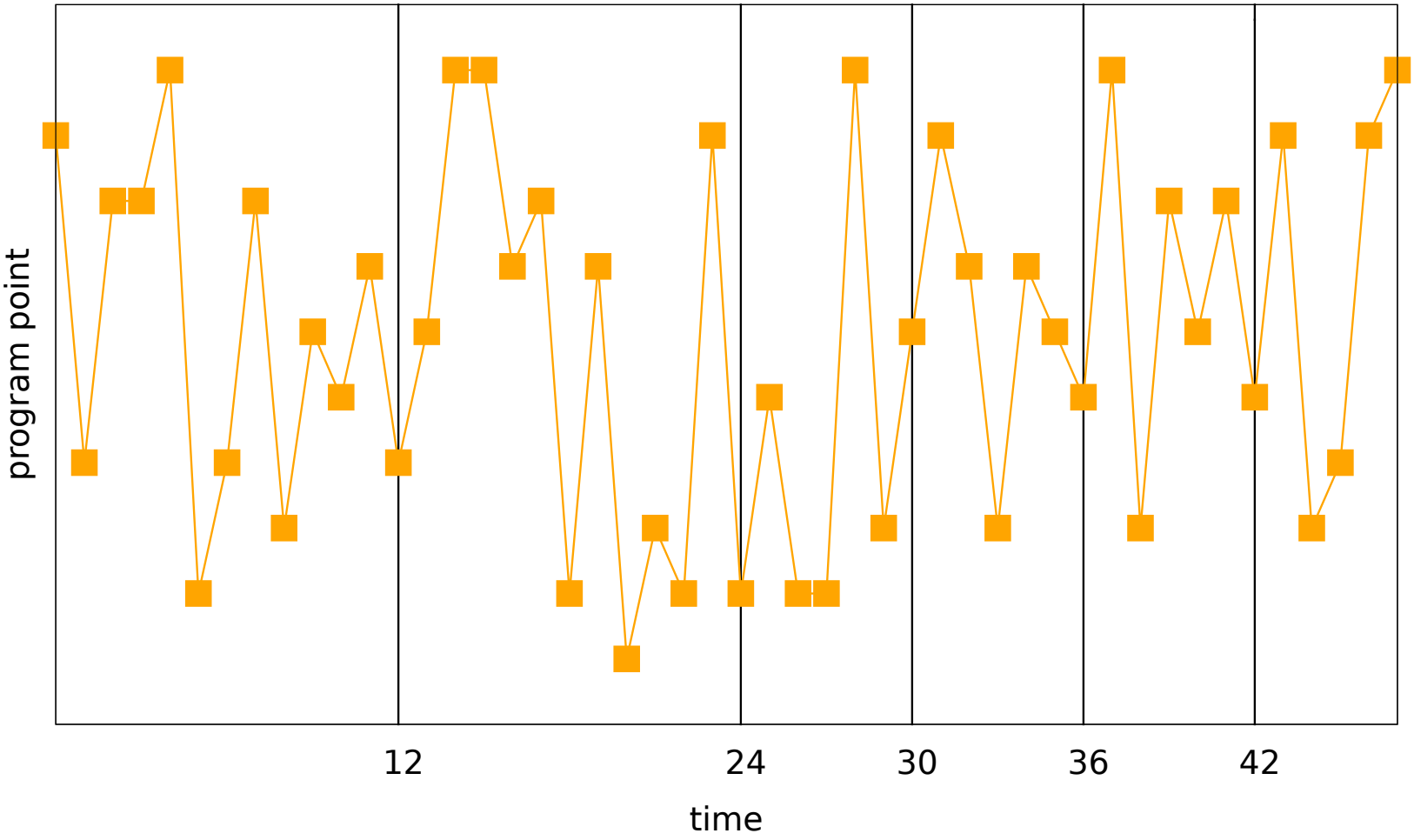

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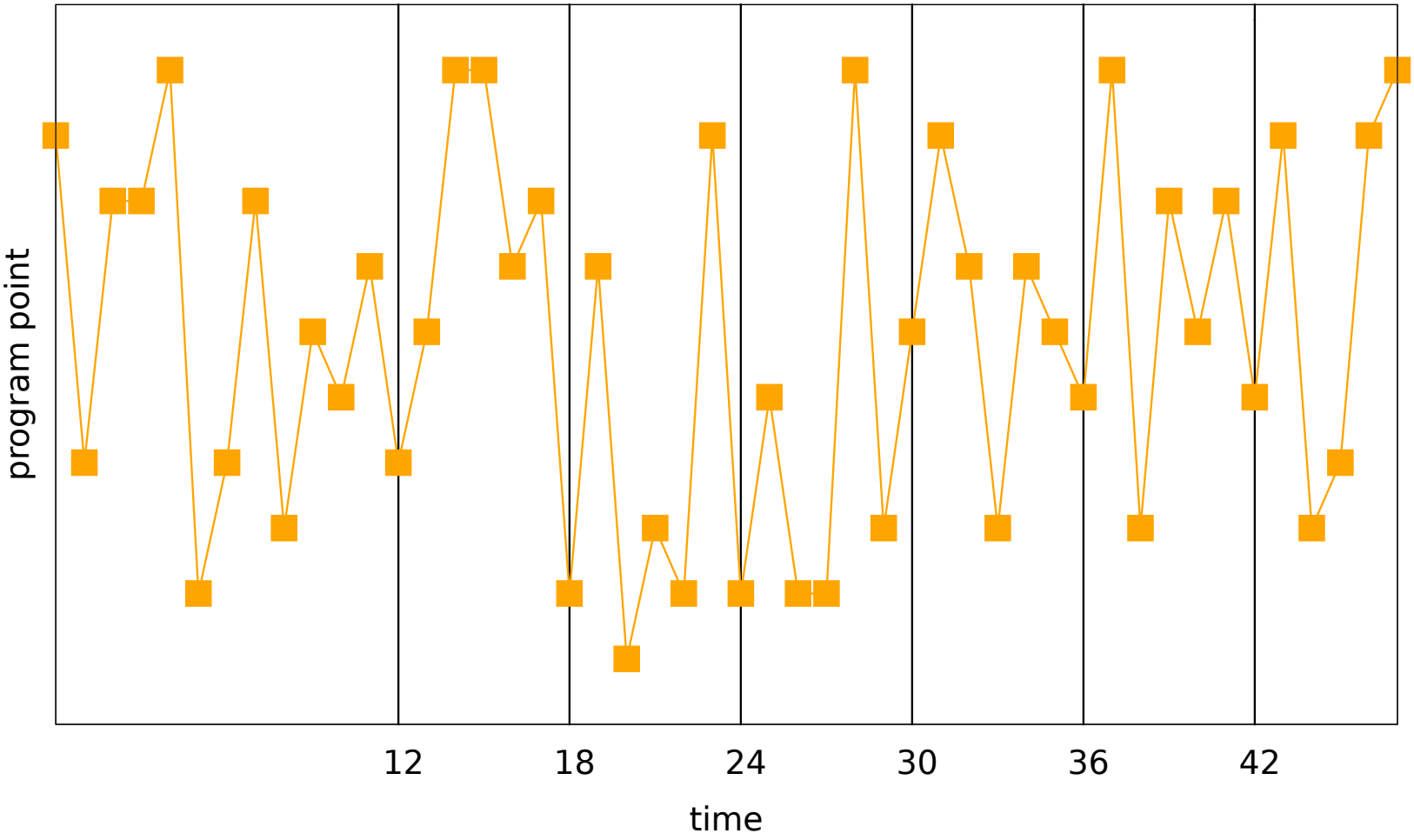

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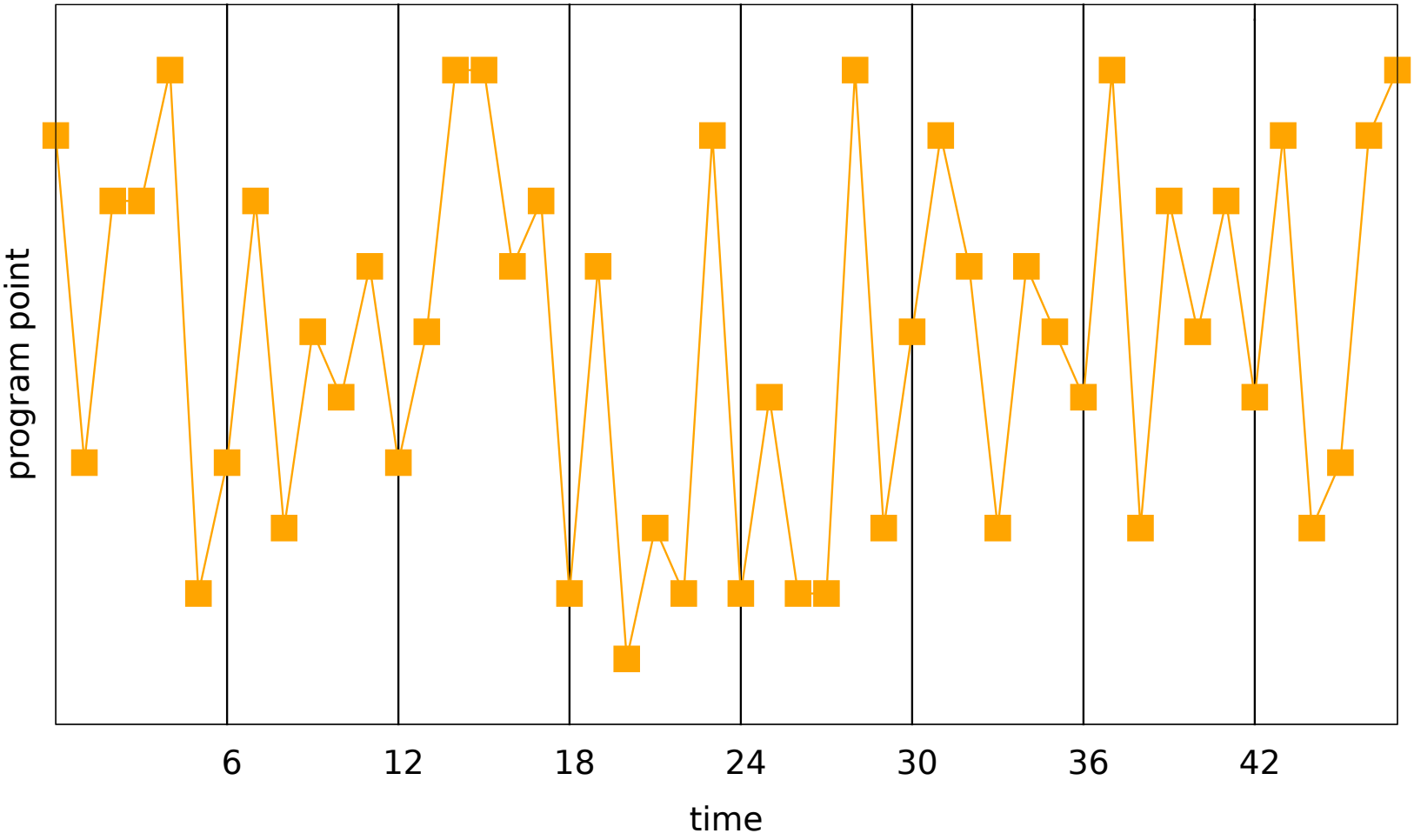

Key Idea

```
function main(w)
    local x = f(w)
    local y = h(g(x))
    local z = p(y)
    return z
end
```

$\rightsquigarrow$

```
function main(w)
    for i = 1, 5
    if i==1 then
        local x = f(w)
    elseif i==2 then
        local t = g(x)
    elseif i==3 then
        local y = h(t)
    elseif i==4 then
        local z = p(y)
    elseif i==5 then
        return z
    end
end
```

Algorithm for Bisection Checkpointing

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overset{\leftarrow}{\mathcal{J}}\ f\ x\ \grave{y}$ (0)

inductive case:

$h \circ g = f$ (1)

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PRIMOPS $f\ x \mapsto (y, n)$	Return $y = f(x)$ along with the number n of steps needed to compute y.
CHECKPOINT $f\ x\ n \mapsto u$	Run the first n steps of the computation of $f(x)$ and return a checkpoint u.
RESUME $u \mapsto y$	If $u = (\text{CHECKPOINT}\ f\ x\ n)$, return $y = f(x)$.

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$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u}$ (4)

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overset{\leftarrow}{\mathcal{J}}\ f\ x\ \grave{y}$ (0)

inductive case:

$h \circ g = f$ (1)

$u = g\ x$ (2)

$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y}$ (3)

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inductive case: $(y, 2n) = \text{PRIMOPS}\ f\ x$ (1)

$u = g\ x$ (2)

$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y}$ (3)

$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u}$ (4)

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inductive case: $(y, 2n) = \text{PRIMOPS}\ f\ x$ (1)

$u = \text{CHECKPOINT}\ f\ x\ n$ (2)

$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y}$ (3)

$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u}$ (4)

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

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$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y}$ (3)

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Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overset{\leftarrow}{\mathcal{J}}\ f\ x\ \grave{y}$ (0)

inductive case: $(y, 2n) = \text{PRIMOPS}\ f\ x$ (1)

$u = \text{CHECKPOINT}\ f\ x\ n$ (2)

$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ h\ u\ \grave{y}$ (3)

$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u}$ (4)

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

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inductive case:

$(y, 2n) = \text{PRIMOPS}\ f\ x$ (1)

$u = \text{CHECKPOINT}\ f\ x\ n$ (2)

$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ (\lambda u.\text{RESUME}\ u)\ u\ \grave{y}$ (3)

$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u}$ (4)

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

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$(y, 2n) = \text{PRIMOPS}\ f\ x$ (1)

$u = \text{CHECKPOINT}\ f\ x\ n$ (2)

$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ (\lambda u.\text{RESUME}\ u)\ u\ \grave{y}$ (3)

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$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\ (\lambda u.\text{RESUME}\ u)\ u\ \grave{y}$ (3)

$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{u}$ (4)

Bisection Checkpointing via General Checkpointing API

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\, f\ x\ \grave{y}$:

base case ($f\ x$ fast):

$$(y, \grave{x}) = \overset{\leftarrow}{\mathcal{J}}\, f\ x\ \grave{y} \tag{0}$$

inductive case:

$$(y, 2n) = \text{PRIMOPS}\ f\ x \tag{1}$$

$$u = \text{CHECKPOINT}\ f\ x\ n \tag{2}$$

$$(y, \grave{u}) = \overset{\checkmark}{\mathcal{J}}\, (\lambda u.\text{RESUME}\ u)\ u\ \grave{y} \tag{3}$$

$$(u, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\, (\lambda x.\text{CHECKPOINT}\ f\ x\ n)\ x\ \grave{u} \tag{4}$$

CPS Conversion

```
function f(x)
   return q(p(g(x), h(x)))
end
```

$\rightsquigarrow$

```
function f(c, x)
   return g(function(t1)
               return h(function(t2)
                           return p(function(t3)
                                       return q(c, t3)
                                    end, t1, t2)
                        end, x)
            end, x)
end
```

CPS Conversion is a Program Transformation

$$
\begin{aligned}
[x]_c &\rightsquigarrow c\ x \\
[\lambda x.e]_c &\rightsquigarrow c\ \lambda c\ x.[e]_c \\
[e_1\ e_2]_c &\rightsquigarrow [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
e_0 &\rightsquigarrow [e_0]_{\lambda x.x}
\end{aligned}
$$

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[e_1\ e_2]_c &\rightsquigarrow [e_1]\lambda x_1.[e_2]\lambda x_2.x_1\ c\ x_2 \\
e_0 &\rightsquigarrow [e_0]\lambda x.x
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CPS Conversion is a Program Transformation

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\ x \\
[\lambda x.e]_c & \rightsquigarrow & c\ \lambda c\ x.[e]_c \\
[e_1\ e_2]_c & \rightsquigarrow & [e_1]_{\lambda x_1.[e_2]_{\lambda x_2.x_1\ c\ x_2}} \\
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e_0 &\rightsquigarrow [e_0]_{\lambda x.x}
\end{aligned}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_c & \rightsquigarrow & c\ x \\
[\lambda x.e]_c & \rightsquigarrow & c\ (\lambda c\ x.[e]_c) \\
[e_1\ e_2]_c & \rightsquigarrow & [e_1](\lambda\ x_1.[e_2](\lambda\ x_2.x_1\ c\ x_2)\ )
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n} & \rightsquigarrow & c\ \ n\ \ \ \ x \\
[\lambda x.e]_{c,n} & \rightsquigarrow & c\ \ n\ \ \ \ \lambda c\ n\ \ x.[e]_{c,n} \\
[e_1\ e_2]_{c,n} & \rightsquigarrow & [e_1]_{(\lambda n_1\ \ x_1.[e_2]_{(\lambda n_2\ \ x_2.x_1\ c\ n_2\ \ x_2),n_1}\ ),\ n}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n} & \rightsquigarrow & c\ (n+1)\quad x \\
[\lambda x.e]_{c,n} & \rightsquigarrow & c\ (n+1)\quad \lambda c\ n\quad x.[e]_{c,n} \\
[e_1\ e_2]_{c,n} & \rightsquigarrow & [e_1]_{(\lambda n_1\quad x_1.[e_2]_{(\lambda n_2\quad x_2.x_1\ c\ n_2\quad x_2),n_1}\quad),(n+1)}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ x \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l}
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
\left[x\right]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ x \\
\left[\lambda x.e\right]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ \lambda c\ n\ l\ x.\left[e\right]_{c,n,l} \\
\left[e_1\ e_2\right]_{c,n,l} & \rightsquigarrow & \left[e_1\right]_{(\lambda n_1\ l_1\ x_1.\left[e_2\right]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
\left\langle e\right\rangle_{c,n,l} & \rightsquigarrow & \textbf{if } n = l \textbf{ then } [\![c, n, \lambda c\ n\ l.e]\!] \textbf{ else } e
\end{array}
$$

CPS Conversion to Support the General Checkpointing API

$$
\begin{array}{lcl}
[x]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ x \\
[\lambda x.e]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ \lambda c\ n\ l\ x.[e]_{c,n,l} \\
[e_1\ e_2]_{c,n,l} & \rightsquigarrow & [e_1]_{(\lambda n_1\ l_1\ x_1.[e_2]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
\langle e \rangle_{c,n,l} & \rightsquigarrow & \textbf{if } n = l \textbf{ then } [\![c, n, \lambda c\ n\ l.e]\!] \textbf{ else } e
\end{array}
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\left[\lambda x.e\right]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ \lambda c\ n\ l\ x.\left[e\right]_{c,n,l} \\
\left[e_1\ e_2\right]_{c,n,l} & \rightsquigarrow & \left[e_1\right]_{(\lambda n_1\ l_1\ x_1.\left[e_2\right]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
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CPS Conversion to Support the General Checkpointing API

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\left[x\right]_{c,n,l} & \rightsquigarrow & c\ (n+1)\ l\ x \\
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\left[e_1\ e_2\right]_{c,n,l} & \rightsquigarrow & \left[e_1\right]_{(\lambda n_1\ l_1\ x_1.\left[e_2\right]_{(\lambda n_2\ l_2\ x_2.x_1\ c\ n_2\ l_2\ x_2),n_1,l_1}),(n+1),l} \\
\left\langle e\right\rangle_{c,n,l} & \rightsquigarrow & \textbf{if } n = l \textbf{ then } [\![c, n, \lambda c\ n\ l.e]\!] \textbf{ else } e
\end{array}
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CPS Conversion to Support the General Checkpointing API

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\begin{array}{lcl}
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Space and Time Complexity

computation length	t
maximal live storage	w
space for checkpoints	$O(w \log t)$
space for tape	$O(w)$
time to (re)compute primal	$O(t \log t)$
time for reverse sweep	$O(t)$

FORTRAN Example

```
      subroutine f(x, y)
      n = 100003
      y = x
c$ad binomial-ckp n+1 30 1
      do i = 1, n
         m = l(x, i)
         do j = 1, m
            y = y*y
            y = sqrt(y)
         end do
      end do
      end
```

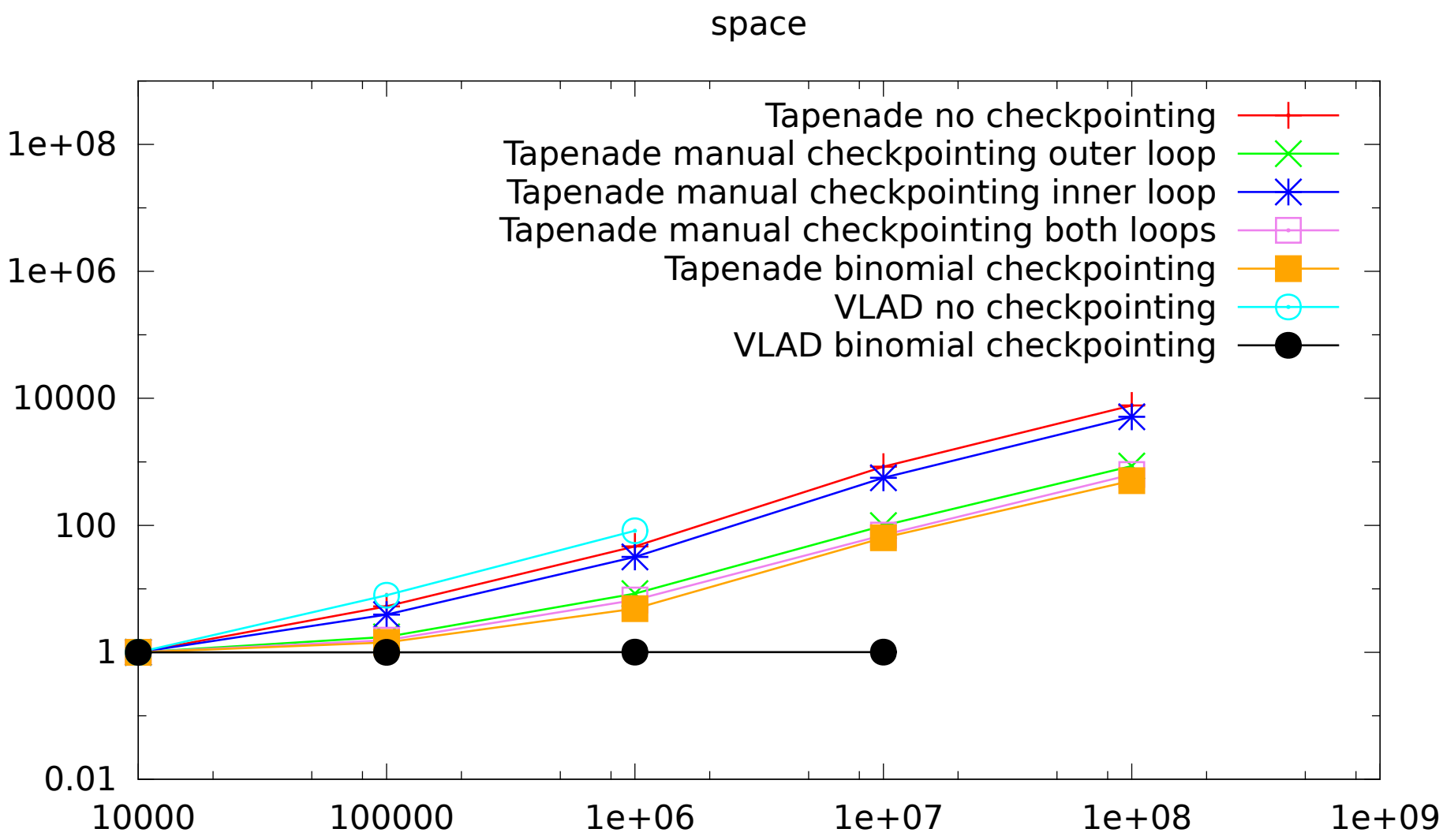
VLAD Example

```
(define (f x)
 (let ((n 100003))
  (let outer ((i 1) (y x))
   (if (> i n)
       y
       (outer (+ i 1)
              (let ((m (l x i)))
               (let inner ((j 1) (y y))
                (if (> j m)
                    y
                    (inner (+ j 1)
                           (sqrt (* y y)))))))))))
```

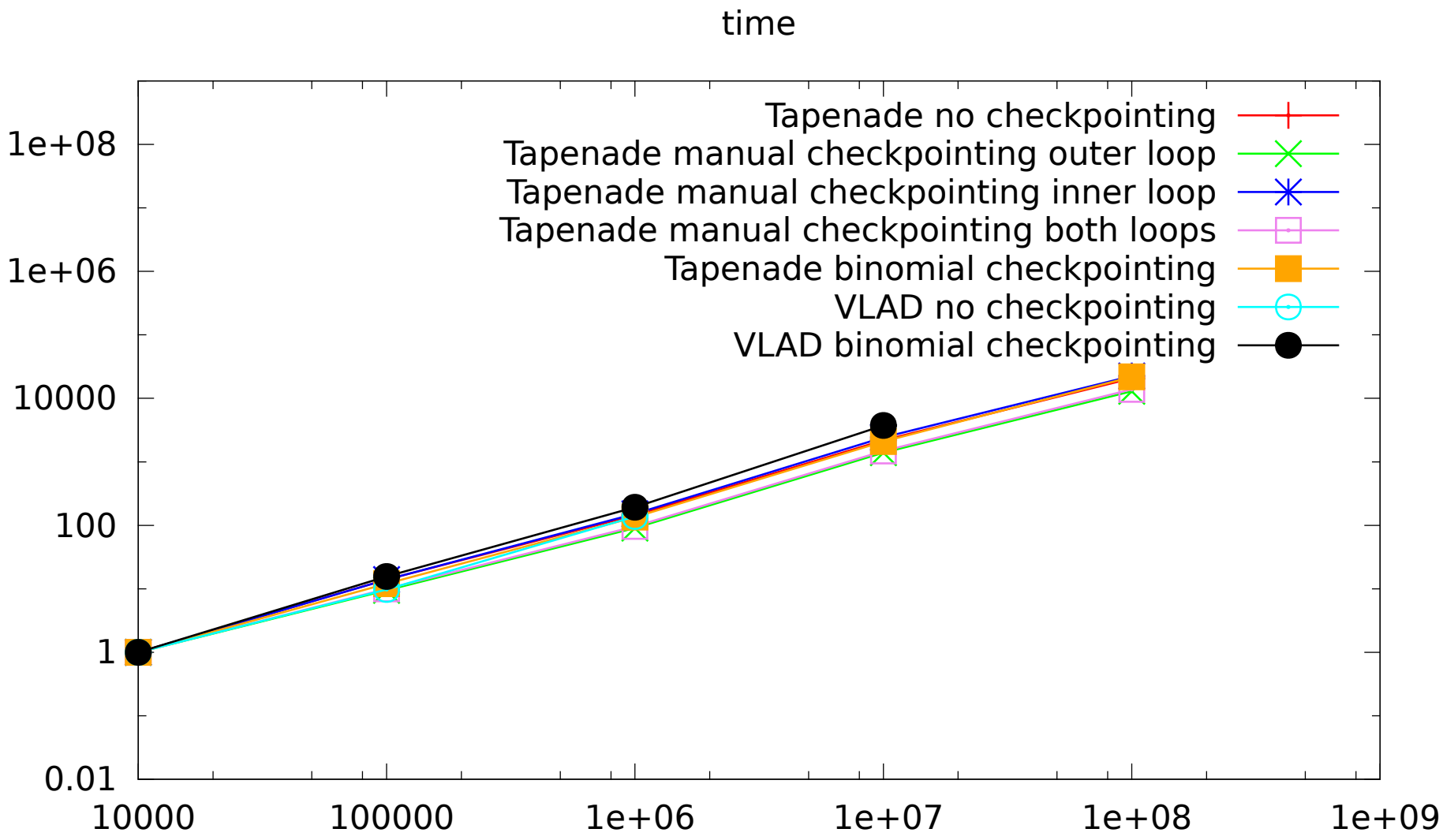
Space and Time Complexity of our Example

computation length	t	$= O(n)$
maximal live storage	w	$= O(1)$
space for checkpoints	$O(w \log t)$	$= O(\log n)$
space for tape	$O(w)$	$= O(1)$
time to (re)compute primal	$O(t \log t)$	$= O(n \log n)$
time for reverse sweep	$O(t)$	$= O(n)$

Space Usage of our Example



Time Usage of our Example



Take-Home Message

Checkpointing

- is traditionally formulated around loop iterations
- but can be extended to arbitrary code
- that doesn't have same-size iterations of a single loop
- using CPS to make arbitrary code look like it does

metaphor: a CPU is an instruction-execution loop