Scheme as a framework for Deep Learning

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Joint work with Barak Avrum Pearlmutter and Hamad Ahmed

Our Work

Our Work

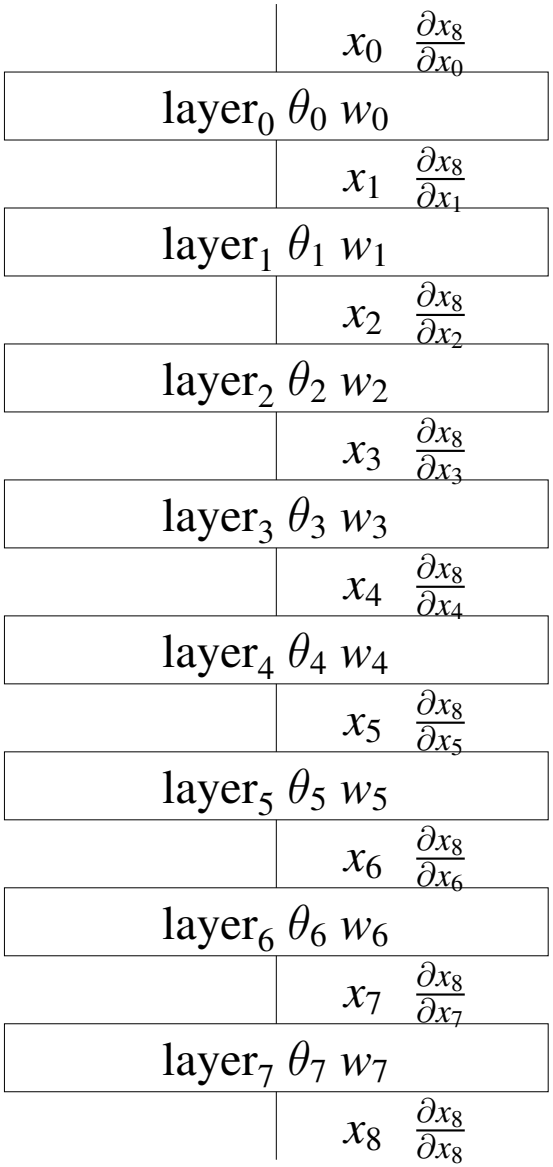
AD in functional programs.

Our Work

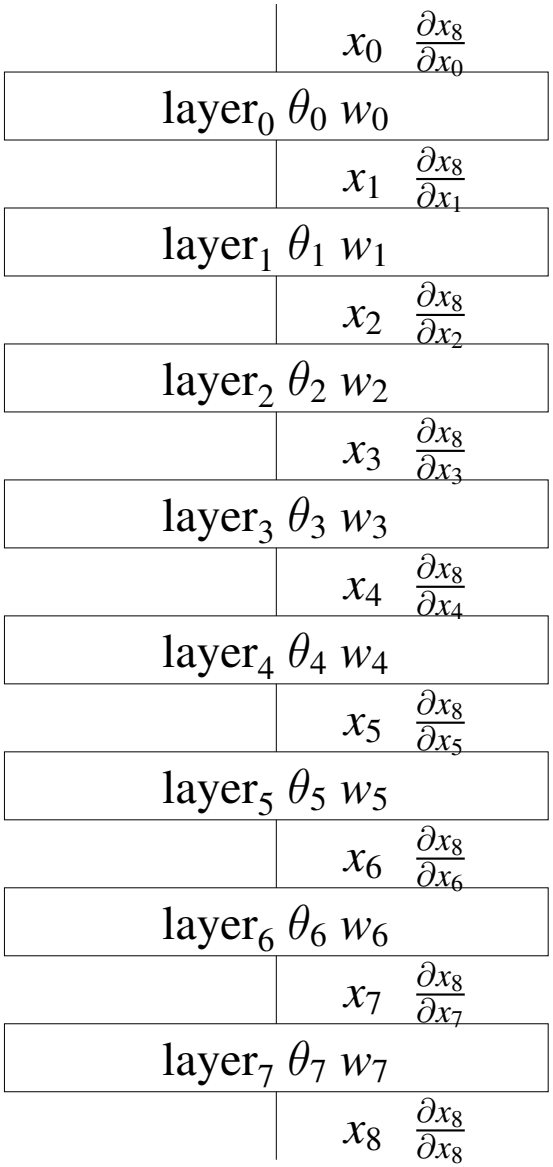
AD in functional programs.

AD is easier in functional programs.

A Neural Network



A Neural Network is a (Functional) Program



$$
\begin{aligned}
&\text{net}\ [\theta_0, \ldots, \theta_7]\ [w_0, \ldots, w_7]\ x_0 \triangleq \\
&\quad \textbf{let}\ \ x_1 = \text{layer}_0\ \theta_0\ w_0\ x_0 \\
&\qquad\quad x_2 = \text{layer}_1\ \theta_1\ w_1\ x_1 \\
&\qquad\quad x_3 = \text{layer}_2\ \theta_2\ w_2\ x_2 \\
&\qquad\quad x_4 = \text{layer}_3\ \theta_3\ w_3\ x_3 \\
&\qquad\quad x_5 = \text{layer}_4\ \theta_4\ w_4\ x_4 \\
&\qquad\quad x_6 = \text{layer}_5\ \theta_5\ w_5\ x_5 \\
&\qquad\quad x_7 = \text{layer}_6\ \theta_6\ w_6\ x_6 \\
&\qquad\quad x_8 = \text{layer}_7\ \theta_7\ w_7\ x_7 \\
&\quad \textbf{in}\ x_8
\end{aligned}
$$

A (Functional) Program

$$
\begin{array}{l}
f\ [w_0, w_1]\ [x_0, x_1] \stackrel{\triangle}{=} \\
\quad \textbf{let} \quad t_0 = w_0 \times x_0 \\
\qquad\quad t_1 = w_1 \times x_1 \\
\qquad\quad\ \ y = t_0 + t_1 \\
\quad \textbf{in}\ y
\end{array}
$$

A (Functional) Program is a (Neural) Network

$$
\begin{array}{l}
f\ [w_0, w_1]\ [x_0, x_1] \triangleq \\
\quad \textbf{let} \quad t_0 = w_0 \times x_0 \\
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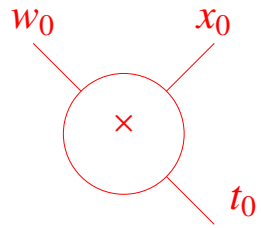
$$



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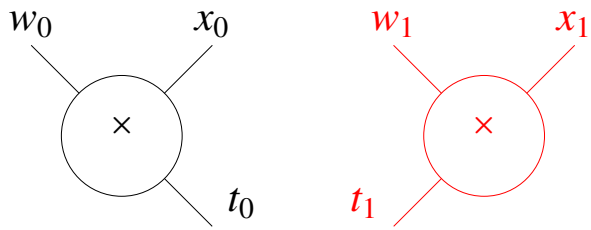

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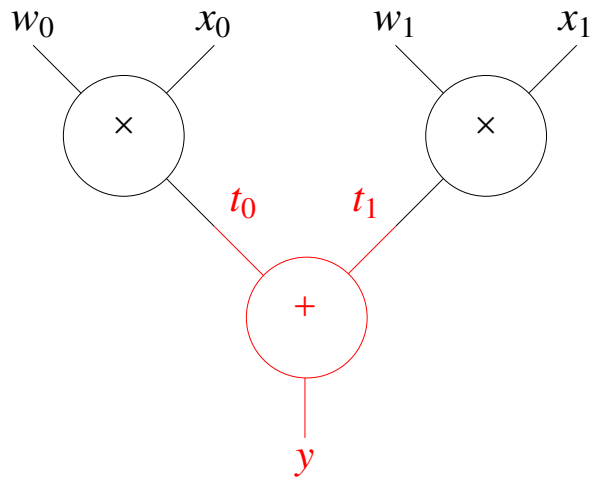
\end{array}
$$



A (Functional) Program is a (Neural) Network

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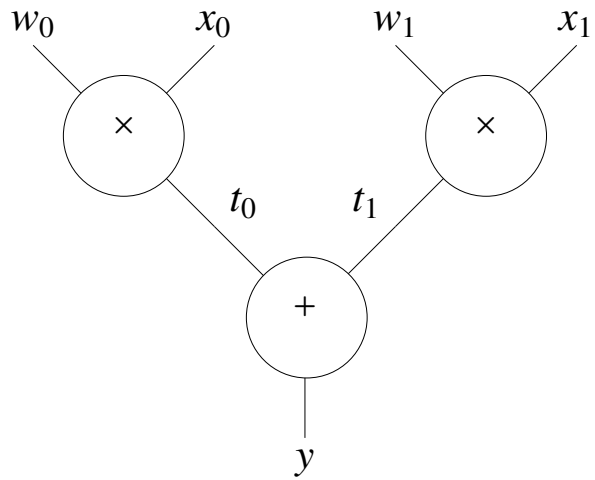
\end{array}
$$



Some Observations

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- Deep learning network ‘frameworks’ are domain specific (functional) programming languages.

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- Can perform backpropagation on (functional) programs
  by having an execution of the program generate a network.

Some Observations

- Deep learning network ‘frameworks’ are domain specific (functional) programming languages.
- A deep neural network is a long running (functional) program.
- Can perform backpropagation on (functional) programs
  by having an execution of the program generate a network.
  This is called reverse-mode automatic differentiation (AD).

Outline

1. Migrate reflective AD through partial evaluation

2. Implementing checkpointing reverse mode through CPS

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2. Implementing checkpointing reverse mode through CPS

Key Idea

Migrate reflective source-to-source transformation
from run time to compile time
with abstract interpretation

Traditional AD by Source-to-Source Transformation

Preprocessor at Compile Time

```
function g(x)
   return x+1
end


function f(x)
   return 2*g(x)
end



... derivative(f, 3) ...
```

Traditional AD by Source-to-Source Transformation

Preprocessor at Compile Time

```
function g(x)
   return x+1
end


function f(x)
   return 2*g(x)
end


local y, y_tangent = f_forward(3, 1)
... y_tangent ...
```

Traditional AD by Source-to-Source Transformation

Preprocessor at Compile Time

```
function g(x)
   return x+1
end


function f_forward(x, x_tangent)
   local y, y_tangent = g_forward(x, x_tangent)
   return return 2*y, 2*y_tangent
end

local y, y_tangent = f_forward(3, 1)
... y_tangent ...
```

Traditional AD by Source-to-Source Transformation

Preprocessor at Compile Time

```
function g_forward(x, x_tangent)
   local y, y_tangent = x, x_tangent
   return x+1, x_tangent
end

function f_forward(x, x_tangent)
   local y, y_tangent = g_forward(x, x_tangent)
   return return 2*y, 2*y_tangent
end

local y, y_tangent = f_forward(3, 1)
... y_tangent ...
```

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f)
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end")
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
                         return return 2*y, 2*y_tangent
                      end"
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
                         return return 2*y, 2*y_tangent
                      end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end")
```

--

Source-to-Source Transformation at Run Time

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function f(x)
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code(f) ==> "function f(x)
                return 2*g(x)
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transform("function f(x)
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           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
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                      end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end") ==> f_forward
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
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                      end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end") ==> f_forward

called_by(f)
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
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                      end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end") ==> f_forward

called_by(f) ==> {g}
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                        local y, y_tangent = g_forward(x, x_tangent)
                        return return 2*y, 2*y_tangent
                     end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end") ==> f_forward

called_by(f) ==> {g}

function derivative(f, x)
   for g in called_by(f) do compile(transform(code(g))) end
   local y, y_tangent = compile(transform(code(f)))(x, 1)
   return y_tangent
end
--
```

But How Can We Make This Efficient?

```
while not converged() do
    x = x-eta*derivative(f, x)
end
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... add(x, y) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add(x, y) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
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      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

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function scalar_add(x, y)
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function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add_1(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)
   if DOUBLE=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add_1(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)
   if false then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add_1(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

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function scalar_add(x, y)
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function vector_add(x, y)
   local n = x:size(1)
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      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)



      return scalar_add(x, y)

end

local x = DOUBLE, y = DOUBLE
... add_1(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
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   end
   return z
end

function add_1(DOUBLE, DOUBLE)



      return scalar_add(x, y)

end

local x = 3, y = 4
... scalar_add(x, y) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

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function scalar_add(x, y)
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   end
   return z
end

function add_1(DOUBLE, DOUBLE)



      return scalar_add(x, y)

end

local x = 3, y = 4
... x+y ...
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... add(x, y) ...
```

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      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
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end

function vector_add(x, y)
   local n = x:size(1)
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   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_2(ARRAY, ARRAY)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add_2(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_2(ARRAY, ARRAY)
   if ARRAY=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add_2(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
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function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_2(ARRAY, ARRAY)
   if true then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add_2(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_2(ARRAY, ARRAY)

      return vector_add(x, y)



end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add_2(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
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   local n = x:size(1)
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function add(x, y)
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      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... vector_add(x, y) ...
```

A Single Powerful Optimization

```
{x = e1, y = e2}.x
```

A Single Powerful Optimization

```
{x = e1, y = e2}.x ⇝ e1
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- can eliminate storage allocation

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A Single Powerful Optimization

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```

- can eliminate storage allocation
- can eliminate storage reclamation
- can eliminate storage writes
- can eliminate storage reads
- can eliminate dead code

The Kind of Code People Write in Dynamic Languages

```
function map(f, x)
   y = torch.Tensor(x:size(1))
   for i = 1, x:size(1) do
      y[i] = f(x[i])
   end
   return y
end

function reduce(g, i, x)
   y = i
   for i = 1, x:size(1) do
      y = g(y, x[i])
   end
   return y
end

reduce(function(x, y) return x+y end,
       0,
       map(function(x) return x*x end, torch.Tensor({u, v, w, x, y})))
```

--

The Kind of Code People Write in Dynamic Languages

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reduce(function(x, y) return x+y end,
       0,
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u*u + v*v + w*w + x*x + y*y

--
```

Key Idea

You need this anyway
to compile dynamic languages efficiently

Same mechanism can support AD

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative(g, x)
   local y, y_tangent = compile(transform(code(g)))(x, 1)


   return y_tangent
end
```

```
... derivative(f, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative_1(g, x)
   local y, y_tangent = compile(transform(code(g)))(x, 1)


   return y_tangent
end
```

```
... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative_1(FUNCTION_F, x)
   local y, y_tangent = compile(transform(code(FUNCTION_F)))(x, 1)

   return y_tangent
end

... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative_1(FUNCTION_F, x)
   local y, y_tangent = compile(transform("function f(x)
                                              return 2*x
                                           end"))(x, 1)

   return y_tangent
end
```

```
... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative_1(FUNCTION_F, x)
   local y, y_tangent = compile("function f_forward(x, x_tangent)
                                    local y, y_tangent = 2*x, 2*x_tangent
                                    return y, y_tangent
                                 end")(x, 1)
   return y_tangent
end
```

```
... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end

function f_forward(x, x_tangent)
   local y, y_tangent = 2*x, 2*x_tangent
   return y, y_tangent
end

function derivative_1(FUNCTION_F, x)
   local y, y_tangent = f_forward(x, 1)



   return y_tangent
end


... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

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function f_forward(x, x_tangent)
   local y, y_tangent = 2*x, 2*x_tangent
   return y, y_tangent
end

function derivative(g, x)
   local y, y_tangent = compile(transform(code(g)))(x, 1)



   return y_tangent
end

local y, y_tangent = f_forward(x, 1)
... y_tangent ...
```

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- separates AD from optimization

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A Single Powerful Optimization

- separates AD from optimization
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  (forward mode is 28 lines; reverse mode is 155 lines)
- tape is a data structure (in the language)
- many AD optimizations (like TBR) fall out
- makes it easier to get it right
- makes it easier to get it to nest

Essence of Forward Transform

$$
\begin{array}{rcl}
\overrightarrow{c} & \rightsquigarrow & \overrightarrow{\mathcal{J}}\ c \\
\overrightarrow{\lambda x.e} & \rightsquigarrow & \lambda\ \overrightarrow{x}\ .\ \overrightarrow{e} \\
\overrightarrow{e_1\ e_2} & \rightsquigarrow & \overrightarrow{e_1}\ \overrightarrow{e_2} \\
\overrightarrow{\textbf{letrec}\ x_1 = e_1; \ldots; x_n = e_n\ \textbf{in}\ e} & \rightsquigarrow & \textbf{letrec}\ \overrightarrow{x_1} = \overrightarrow{e_1}; \ldots; \overrightarrow{x_n} = \overrightarrow{e_n}\ \textbf{in}\ \overrightarrow{e} \\
\overrightarrow{e_1, e_2} & \rightsquigarrow & \overrightarrow{e_1}\ \overrightarrow{,}\ \overrightarrow{e_2}
\end{array}
$$

Essence of Reverse Transform

$$
\begin{array}{rcl}
\overleftarrow{x = c} & \rightsquigarrow & \overleftarrow{x} = \overleftarrow{\mathcal{J}}\, c \\
\overleftarrow{x_1 = x_2} & \rightsquigarrow & \overleftarrow{x_1} = \overleftarrow{x_2} \\
\overleftarrow{x = \lambda x.e} & \rightsquigarrow & \overleftarrow{x} = \overleftarrow{\lambda x.e} \\
\overleftarrow{x = x_1\ x_2} & \rightsquigarrow & \overleftarrow{x}, \overleftarrow{\bar{x}} = \overleftarrow{x_1}\ \overleftarrow{x_2} \\
\overleftarrow{x = x_1, x_2} & \rightsquigarrow & \overleftarrow{x} = \overleftarrow{x_1}\ \overleftarrow{,}\ \overleftarrow{x_2}
\end{array}
$$

$$
\begin{array}{rcl}
\overline{x_1 = x_2} & \rightsquigarrow & \overleftarrow{x_2} \mathrel{+}= \overleftarrow{x_1} \\
\overline{x = \lambda x.e} & \rightsquigarrow & \overleftarrow{\lambda x.e} \mathrel{+}= \overleftarrow{x} \\
\overline{x = x_1\ x_2} & \rightsquigarrow & \overleftarrow{x_1}\ \overleftarrow{,}\ \overleftarrow{x_2} \mathrel{+}= \overleftarrow{\bar{x}}\ \overleftarrow{x} \\
\overline{x = x_1, x_2} & \rightsquigarrow & \overleftarrow{x_1}\ \overleftarrow{,}\ \overleftarrow{x_2} \mathrel{+}= \overleftarrow{x}
\end{array}
$$

$$
\overleftarrow{\lambda x.\mathbf{let}\ b_1; \ldots; b_n \mathbf{in}\ y} \quad \rightsquigarrow \quad \lambda \overleftarrow{x}.\mathbf{let}\ \overleftarrow{b_1}; \ldots; \overleftarrow{b_n} \mathbf{in}\ \overleftarrow{y}, \lambda \overleftarrow{y}.\mathbf{let}\ \overline{b_n}; \ldots; \overline{b_1} \mathbf{in}\ \overleftarrow{x}
$$

Game Theory

		B				
		b_1	$\dots$	b_j	$\dots$	b_n
	a_1					
	$\vdots$		$\ddots$	$\vdots$		
A	a_i		$\dots$	$\textsc{Payoff}(a_i, b_j)$	$\dots$	
	$\vdots$			$\vdots$	$\ddots$	
	a_m					

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Game Theory

		B				
		b_1	$\dots$	b_j	$\dots$	b_n
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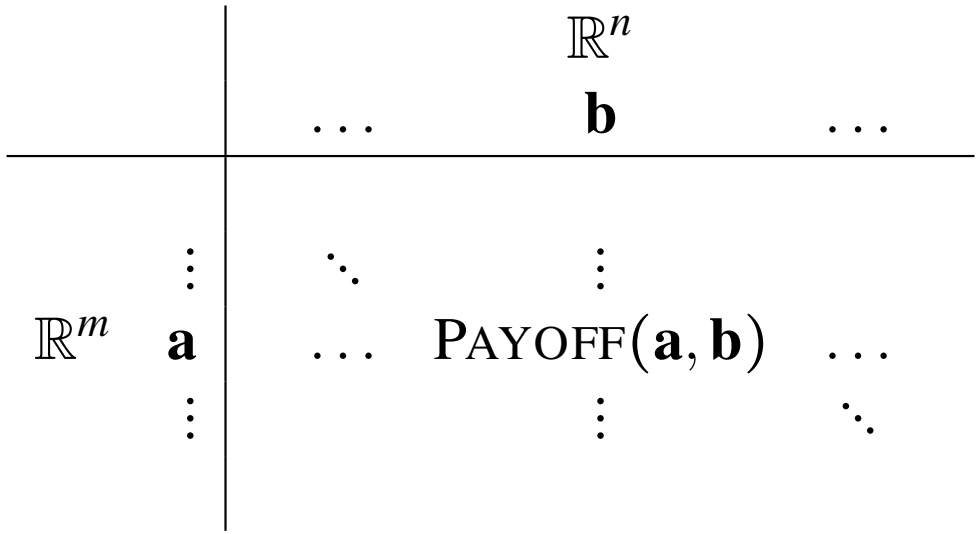
	$\vdots$		$\ddots$	$\vdots$		
A	a_i		$\dots$	$\textsc{Payoff}(a_i, b_j)$	$\dots$	
	$\vdots$			$\vdots$	$\ddots$	
	a_m					

$$\max_{a \in A} \min_{b \in B} \textsc{Payoff}(a, b)$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Game Theory



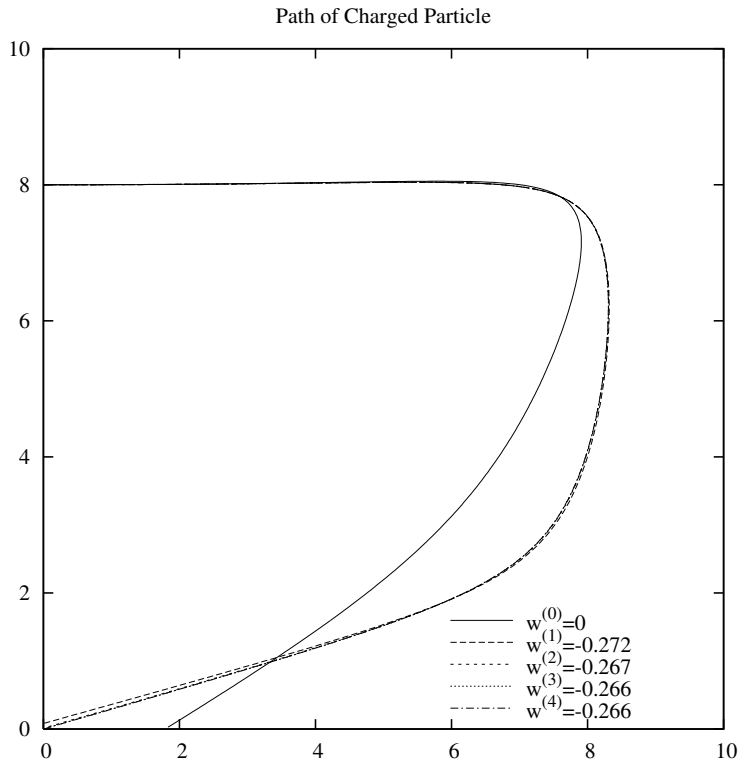
$$\max_{\mathbf{a} \in \mathbb{R}^m} \min_{\mathbf{b} \in \mathbb{R}^n} \textsc{Payoff}(\mathbf{a}, \mathbf{b})$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Code

```
(letrec ((loop
          (lambda (i r)
           (if (zero? i)
               r
               (loop (- i 1)
                     (let* ((start (list (real 1) (real 1)))
                            (f (lambda (x1 y1 x2 y2)
                                 (- (+ (sqr x1) (sqr y1))
                                    (+ (sqr x2) (sqr y2)))))
                            ((list x1* y1*)
                             (multivariate-argmin-F
                              (lambda ((list x1 y1))
                               (multivariate-max-F
                                (lambda ((list x2 y2)) (f x1 y1 x2 y2))
                                start))
                              start))
                            ((list x2* y2*)
                             (multivariate-argmax-F
                              (lambda ((list x2 y2)) (f x1* y1* x2 y2))
                              start)))
                       (list (list (write-real x1*) (write-real y1*))
                             (list (write-real x2*) (write-real y2*)))))))))
 (loop (real 1000) (list (list (real 0) (real 0)) (list (real 0) (real 0)))))
```

Cathode Ray Tubes



$$
\begin{aligned}
\text{potential: } p(\mathbf{x}; w) &= \|\mathbf{x} - (10, 10 - w)\|^{-1} + \|\mathbf{x} - (10, 0)\|^{-1} \\
\ddot{\mathbf{x}}(t) &= -\nabla_{\mathbf{x}} p(\mathbf{x})|_{\mathbf{x}=\mathbf{x}(t)} \\
\dot{\mathbf{x}}(t + \Delta t) &= \dot{\mathbf{x}}(t) + \Delta t \ddot{\mathbf{x}}(t) \\
\mathbf{x}(t + \Delta t) &= \mathbf{x}(t) + \Delta t \dot{\mathbf{x}}(t) \\
\text{When: } x_1(t + \Delta t) &\leq 0 \\
\text{let: } \Delta t_f &= -x_1(t)/\dot{x}_1(t) \\
t_f &= t + \Delta t_f \\
\mathbf{x}(t_f) &= \mathbf{x}(t) + \Delta t_f \dot{\mathbf{x}}(t) \\
\text{Error: } E(w) &= x_0(t_f)^2 \\
\text{Find:} & \quad \operatorname*{argmin}_w E(w)
\end{aligned}
$$

Sprague, C. S. and George, R. H. (1939). *Cathode Ray Deflecting Electrode*. US Patent 2,161,437.

George, R. H. (1940). *Cathode Ray Tube*. US Patent 2,222,942.

Code

```
(define (naive-euler w)
 (let* ((charges
         (list (list (real 10) (- (real 10) w)) (list (real 10) (real 0))))
        (x-initial (list (real 0) (real 8)))
        (xdot-initial (list (real 0.75) (real 0)))
        (delta-t (real 1e-1))
        (p (lambda (x)
            ((reduce + (real 0))
             ((map (lambda (c) (/ (real 1) (distance x c)))) charges)))))
  (letrec ((loop (lambda (x xdot)
                   (let* ((xddot (k*v (real -1) ((gradient-F p) x)))
                          (x-new (v+ x (k*v delta-t xdot))))
                    (if (positive? (list-ref x-new 1))
                        (loop x-new (v+ xdot (k*v delta-t xddot)))
                        (let* ((delta-t-f (/ (- (real 0) (list-ref x 1))
                                             (list-ref xdot 1)))
                               (x-t-f (v+ x (k*v delta-t-f xdot))))
                         (sqr (list-ref x-t-f 0))))))))
   (loop x-initial xdot-initial))))

(letrec ((loop
          (lambda (i r)
           (if (zero? i)
               r
               (loop (- i 1)
                     (let* ((w0 (real 0))
                            ((list w*)
                             (multivariate-argmin-F
                              (lambda ((list w)) (naive-euler w)) (list w0))))
                      (write-real w*)))))))
 (loop (real 1000) (real 0)))
```

Probabilistic Lambda Calculus

P = **if** x_0 **then** 0 **else if** x_1 **then** 1 **else** 2

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

P = **if** x_0 **then** 0 **else if** x_1 **then** 1 **else** 2

$\Pr(x_0 \mapsto \textbf{true}) = p_0$ $\quad$ $\Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$

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$$\Pr(\mathcal{E}(P) = 0 | p_0, p_1) = p_0$$
$$\Pr(\mathcal{E}(P) = 1 | p_0, p_1) = (1 - p_0)p_1$$
$$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$$

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$$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$$

$$\prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v | p_0, p_1) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

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$$\operatorname*{argmax}_{p_0, p_1} \prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v | p_0, p_1) = \left\langle \frac{1}{4}, \frac{1}{3} \right\rangle$$

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Probabilistic Prolog

```
p(0).
p(X):-q(X).
q(1).
q(2).
```

Probabilistic Prolog

$\Pr(\texttt{p(0).}) = p_0$
$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$
$\Pr(\texttt{q(1).}) = p_1$
$\Pr(\texttt{q(2).}) = 1 - p_1$

Probabilistic Prolog

$\Pr(\texttt{p(0).}) = p_0$
$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$
$\Pr(\texttt{q(1).}) = p_1$
$\Pr(\texttt{q(2).}) = 1 - p_1$

$\Pr(\texttt{?-p(0).}) = p_0$
$\Pr(\texttt{?-p(1).}) = (1 - p_0)p_1$
$\Pr(\texttt{?-p(2).}) = (1 - p_0)(1 - p_1)$

Probabilistic Prolog

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$\Pr(\texttt{?-p(2).}) = (1 - p_0)(1 - p_1)$

$$\prod_{q \in \{\texttt{p(0)},\texttt{p(1)},\texttt{p(2)},\texttt{p(2)}\}} \Pr(\texttt{?-}q\texttt{.}) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

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Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
          (map-tagged-distribution
           (lambda (value) (value tagged-distribution))
           (evaluate (application-callee expression) environment))))))
```

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     (evaluate
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   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
               (evaluate (application-argument expression)
                         environment)))
          (map-tagged-distribution
           (lambda (value) (value tagged-distribution))
           (evaluate (application-callee expression) environment))))))
```

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
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Probabilistic Lambda Calculus

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Probabilistic Lambda Calculus

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    (lambda (tagged-distribution)
     (evaluate
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Probabilistic Lambda Calculus

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  ((constant-expression? expression)
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```

Probabilistic Lambda Calculus

```
(gradient-ascent
 (lambda (p)
  (let ((tagged-distribution
         (evaluate if x0 then 0 else if x1 then 1 else 2
                   (list Pr(x0 ↦ true) = p0   Pr(x0 ↦ false) = 1 − p0
                         Pr(x1 ↦ true) = p1   Pr(x1 ↦ false) = 1 − p1
                         ...))))
    (map-reduce
     *
     1.0
     (lambda (value)
      (likelihood value tagged-distribution))
     '(0 1 2 2))))
 '(0.5 0.5)
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Probabilistic Lambda Calculus

```
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                          ...))))
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                         ...))))
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     1.0
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Probabilistic Lambda Calculus

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 0.1)
```

Probabilistic Lambda Calculus

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(gradient-ascent
 (lambda (p)
  (let ((tagged-distribution
         (evaluate if x0 then 0 else if x1 then 1 else 2
                   (list Pr(x0 ↦ true) = p0  Pr(x0 ↦ false) = 1 − p0
                         Pr(x1 ↦ true) = p1  Pr(x1 ↦ false) = 1 − p1
                         ...))))
    (map-reduce
     *
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Probabilistic Lambda Calculus

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Probabilistic Lambda Calculus

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```

Probabilistic Prolog

```
(define (proof-distribution term clauses)
 (let ((offset ...))
  (map-reduce
   append
   '()
   (lambda (clause)
    (let ((clause (alpha-rename clause offset)))
     (let loop ((p (clause-p clause))
                (substitution (unify term (clause-term clause)))
                (terms (clause-terms clause)))
      (if (boolean? substitution)
          '()
          (if (null? terms)
              (list (make-double p substitution))
              (map-reduce
               append
               '()
               (lambda (double)
                (loop (* p (double-p double))
                      (append substitution (double-substitution double))
                      (rest terms)))
               (proof-distribution
                (apply-substitution substitution (first terms)) clauses)))))))
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               (proof-distribution
                (apply-substitution substitution (first terms)) clauses)))))))
   clauses)))
```

Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
  (let ((clauses (list Pr(p(0).) = p0
                       Pr(p(X):-q(X).) = 1 − p0
                       Pr(q(1).) = p1
                       Pr(q(2).) = 1 − p1)))
   (map-reduce
    *
    1.0
    (lambda (query)
     (likelihood (proof-distribution query clauses)))
    '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
  (let ((clauses (list Pr(p(0).) = p0
                       Pr(p(X):-q(X).) = 1 − p0
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     (likelihood (proof-distribution query clauses)))
    '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
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    *
    1.0
    (lambda (query)
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    '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
  (let ((clauses (list Pr(p(0).) = p_0
                       Pr(p(X):-q(X).) = 1 - p_0
                       Pr(q(1).) = p_1
                       Pr(q(2).) = 1 - p_1)))
   (map-reduce
    *
    1.0
    (lambda (query)
     (likelihood (proof-distribution query clauses)))
    '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
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    *
    1.0
    (lambda (query)
     (likelihood (proof-distribution query clauses)))
    '(p(0) p(1) p(2) p(2)))))
'(0.5 0.5)
1000.0
0.1)
```

Generated Code

```
static void f2679(double a_f2679_0,double a_f2679_1,double a_f2679_2,double a_f2679_3){
  int t272381=((a_f2679_2==0.)?0:1);
  double t272406;
  double t272405;
  double t272404;
  double t272403;
  double t272402;
  if((t272381==0)){
    double t272480=(1.-a_f2679_0);
    double t272572=(1.-a_f2679_1);
    double t273043=(a_f2679_0+0.);
    double t274185=(t272480*a_f2679_1);
    double t274426=(t274185+0.);
    double t275653=(t272480*t272572);
    double t275894=(t275653+0.);
    double t277121=(t272480*t272572);
    double t277362=(t277121+0.);
    double t277431=(t277362*1.);
    double t277436=(t275894*t277431);
    double t277441=(t274426*t277436);
    double t277446=(t273043*t277441);
    ...
    double t1777107=(t1774696+t1715394);
    double t1777194=(0.-t1745420);
    double t1778533=(t1777194+t1419700);
    t272406=a_f2679_0;
    t272405=a_f2679_1;
    t272404=t277446;
    t272403=t1778533;
    t272402=t1777107;}
  else {...}
  r_f2679_0=t272406;
  r_f2679_1=t272405;
  r_f2679_2=t272404;
  r_f2679_3=t272403;
  r_f2679_4=t272402;}
```

Benchmarks

		backprop		
		Fs	Fv	R
VLAD	STALIN∇	1.00	■	1.00
FORTRAN	ADIFOR	15.51	3.35	■
	TAPENADE	14.97	5.97	6.86
C	ADIC	22.75	5.61	■
C++	ADOL-C	12.16	5.79	32.77
	CPPAD	54.74	■	29.24
	FADBAD++	132.31	46.01	60.71
ML	MLTON	95.20	■	39.90
	OCAML	202.01	■	156.93
	SML/NJ	181.93	■	102.89
HASKELL	GHC	■	■	■
SCHEME	BIGLOO	743.26	■	360.07
	CHICKEN	1626.73	■	1125.24
	GAMBIT	671.54	■	379.63
	IKARUS	279.59	■	165.16
	LARCENY	1203.34	■	511.54
	MIT SCHEME	2446.33	■	1113.09
	MZC	1318.60	■	754.47
	MZSCHEME	1364.14	■	772.10
	SCHEME->C	597.67	■	280.93
	SCMUTILS	5889.26	■	■
	STALIN	435.82	■	281.27

Damned Benchmarks

		particle				saddle			
		FF	FR	RF	RR	FF	FR	RF	RR
VLAD	STALIN∇	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
FORTRAN	ADIFOR	2.05	■	■	■	5.44	■	■	■
	TAPENADE	5.51	■	■	■	8.09	■	■	■
C	ADIC	■	■	■	■	■	■	■	■
C++	ADOL-C	■	■	■	■	■	■	■	■
	CPPAD	■	■	■	■	■	■	■	■
	FADBAD++	93.32	■	■	■	60.67	■	■	■
ML	MLTON	78.13	111.27	45.95	32.57	114.07	146.28	12.27	10.58
	OCAML	217.03	415.64	352.06	261.38	291.26	407.67	42.39	50.21
	SML/NJ	153.01	226.84	270.63	192.13	271.84	299.76	25.66	23.89
HASKELL	GHC	209.44	■	■	■	247.57	■	■	■
SCHEME	BIGLOO	627.78	855.70	275.63	187.39	1004.85	1076.73	105.24	89.23
	CHICKEN	1453.06	2501.07	821.37	1360.00	2276.69	2964.02	225.73	252.87
	GAMBIT	578.94	879.39	356.47	260.98	958.73	1112.70	89.99	89.23
	IKARUS	266.54	386.21	158.63	116.85	424.75	527.57	41.27	42.34
	LARCENY	964.18	1308.68	360.68	272.96	1565.53	1508.39	126.44	112.82
	MIT SCHEME	2025.23	3074.30	790.99	609.63	3501.21	3896.88	315.17	295.67
	MZC	1243.08	1944.00	740.31	557.45	2135.92	2434.05	194.49	187.53
	MZSCHEME	1309.82	1926.77	712.97	555.28	2371.35	2690.64	224.61	219.29
	SCHEME->C	582.20	743.00	270.83	208.38	910.19	913.66	82.93	69.87
	SCMUTILS	4462.83	■	■	■	7651.69	■	■	■
	STALIN	364.08	547.73	399.39	295.00	543.68	690.64	63.96	52.93

Statistics

		probabilistic-lambda-calculus		probabilistic-prolog	
		F	R	F	R
VLAD	STALIN∇	1.00	1.00	1.00	1.00
FORTRAN	ADIFOR	■	■	■	■
	TAPENADE	■	■	■	■
C	ADIC	■	■	■	■
C++	ADOL-C	■	■	■	■
	CPPAD	■	■	■	■
	FADBAD++	■	■	■	■
ML	MLTON	129.11	114.88	848.45	507.21
	OCAML	249.40	499.43	1260.83	1542.47
	SML/NJ	234.62	258.53	2505.59	1501.17
HASKELL	GHC	■	■	■	■
SCHEME	BIGLOO	983.12	1016.50	12832.92	7918.21
	CHICKEN	2324.54	3040.44	44891.04	24634.44
	GAMBIT	1033.46	1107.26	26077.48	14262.70
	IKARUS	497.48	517.89	8474.57	4845.10
	LARCENY	1658.27	1606.44	25411.62	14386.61
	MIT SCHEME	4130.88	3817.57	87772.39	49814.12
	MZC	2294.93	2346.13	57472.76	31784.38
	MZSCHEME	2721.35	2625.21	60269.37	33135.06
	SCHEME->C	811.37	803.22	10605.32	5935.56
	SCMUTILS	7699.14	■	83656.17	■
	STALIN	956.47	1994.44	15048.42	16939.28

Take-Home Message

Powerful and efficient AD can be attained by:

- integrating AD into compiler
- formulating AD as one of many compiler transformations
- using abstract interpretation to migrate AD transformation from run time to compile time

Outline

A (Brief) History of Backpropagation aka Reverse-Mode AD

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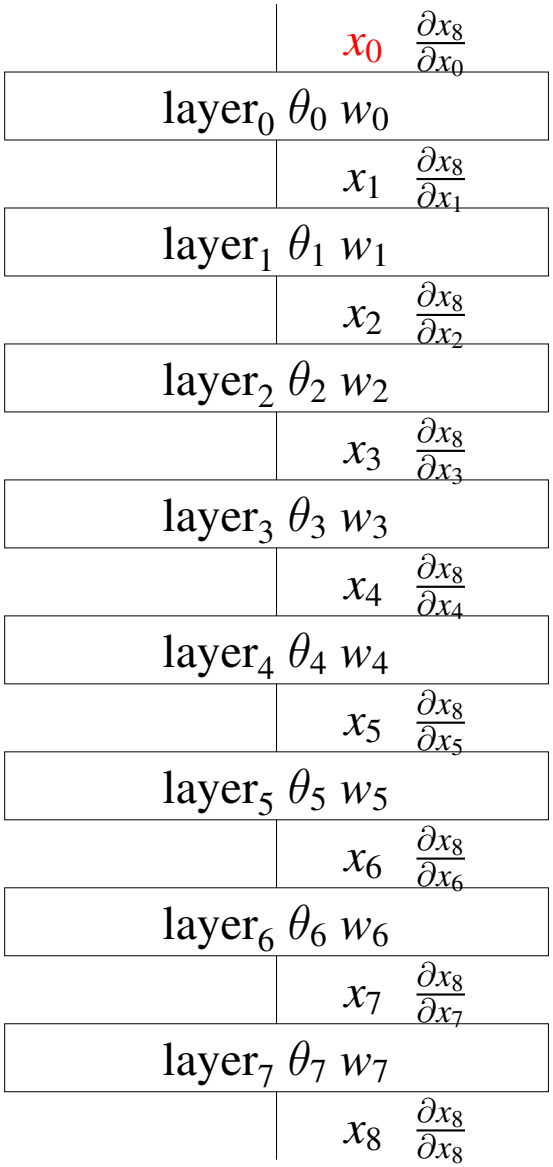

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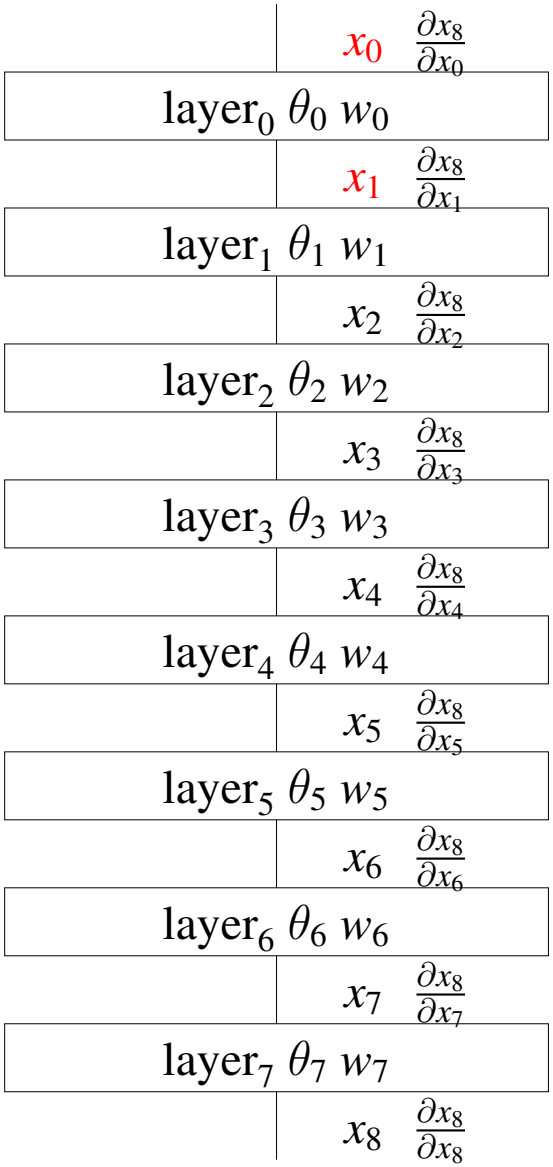

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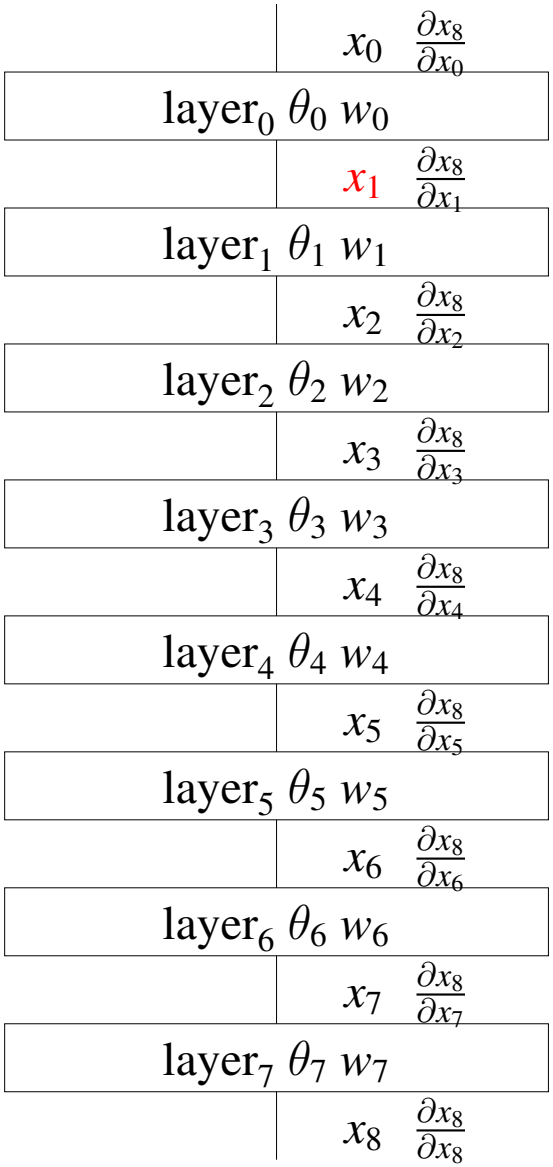

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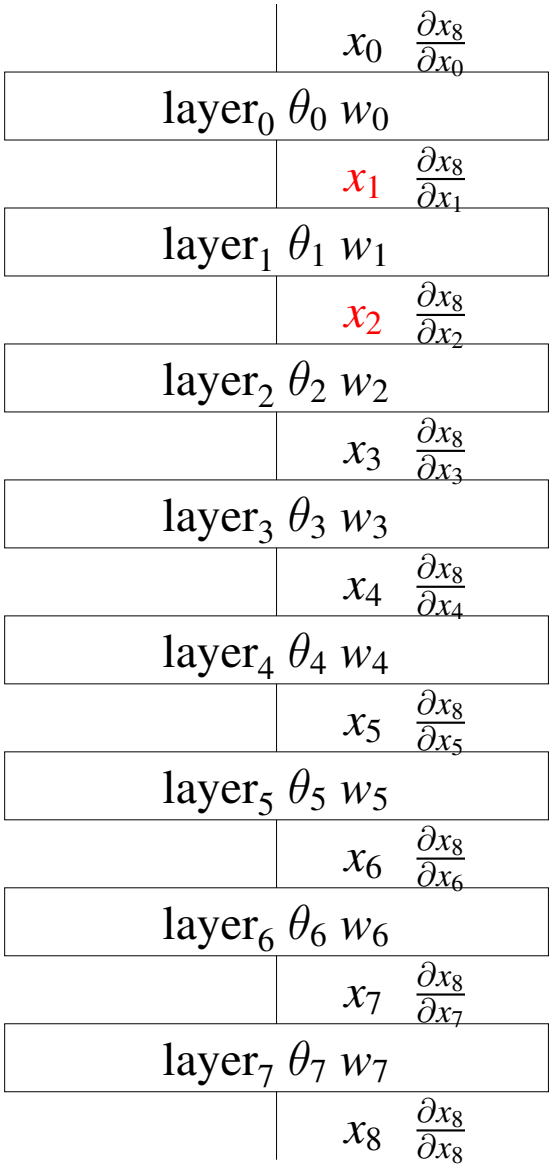

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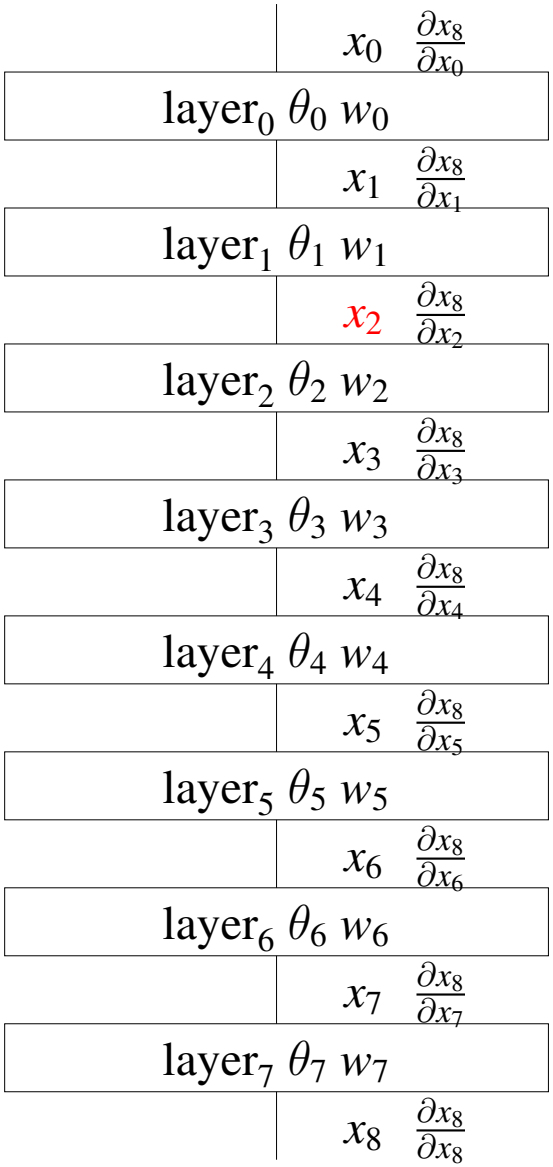

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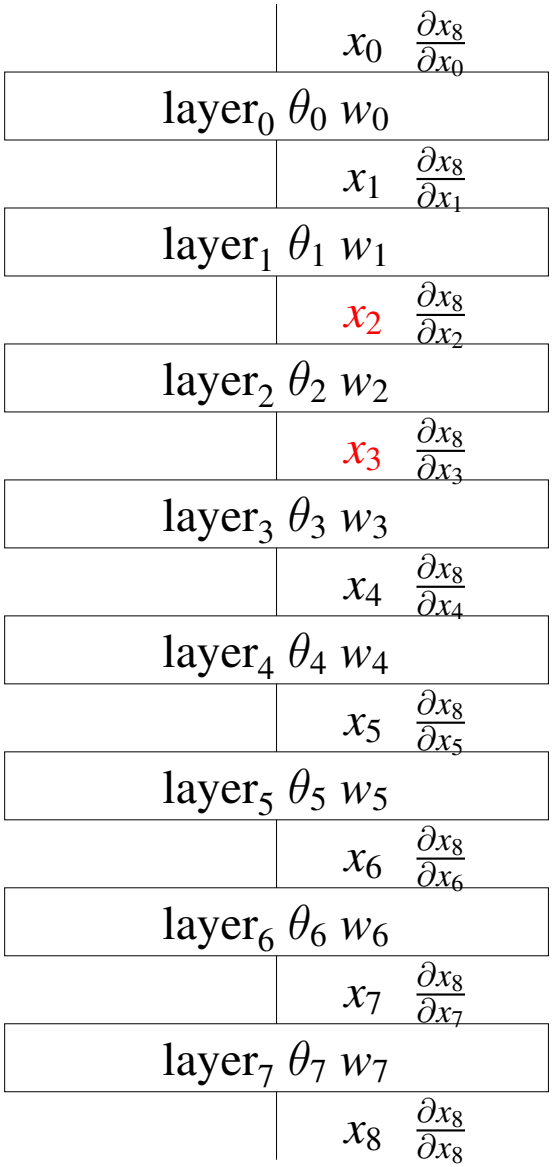

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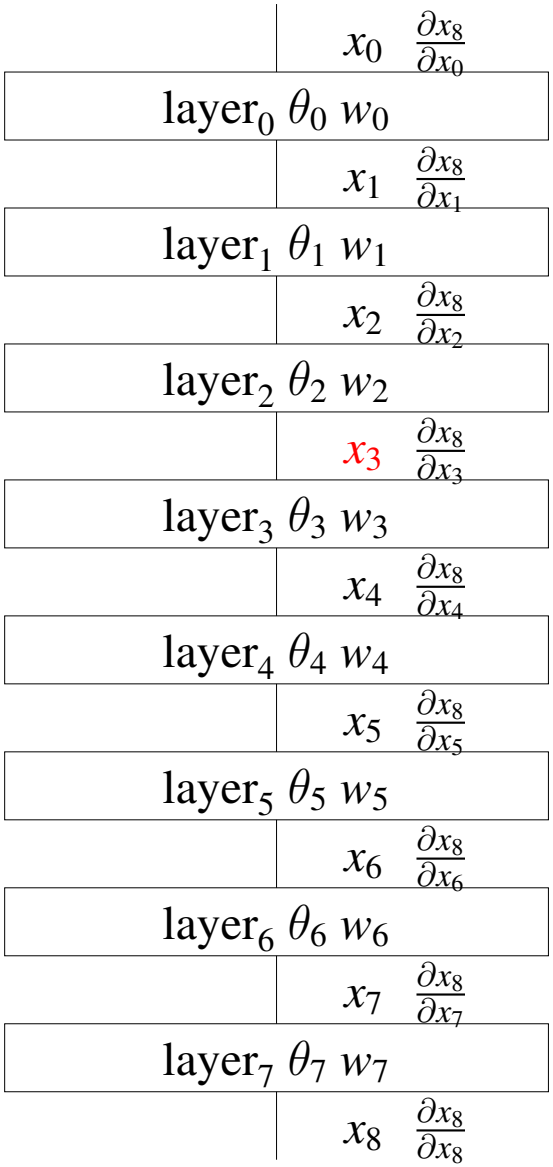

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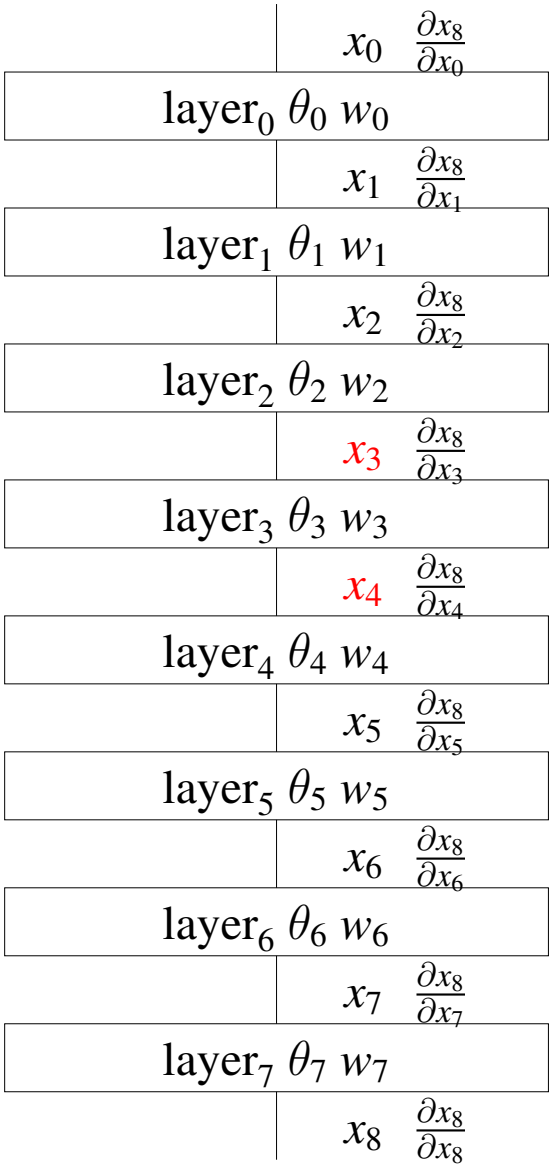

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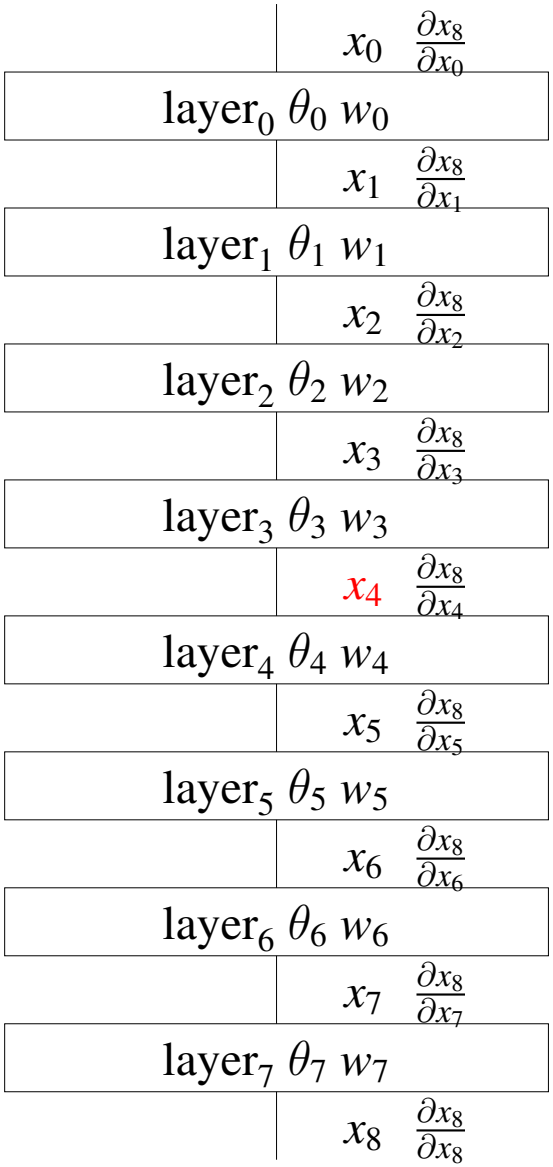

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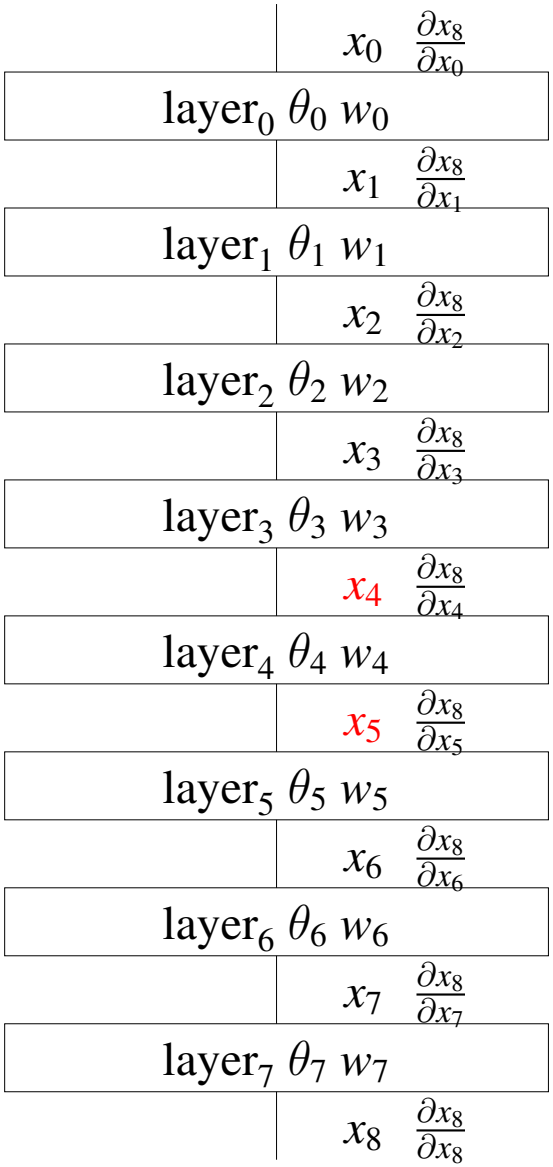

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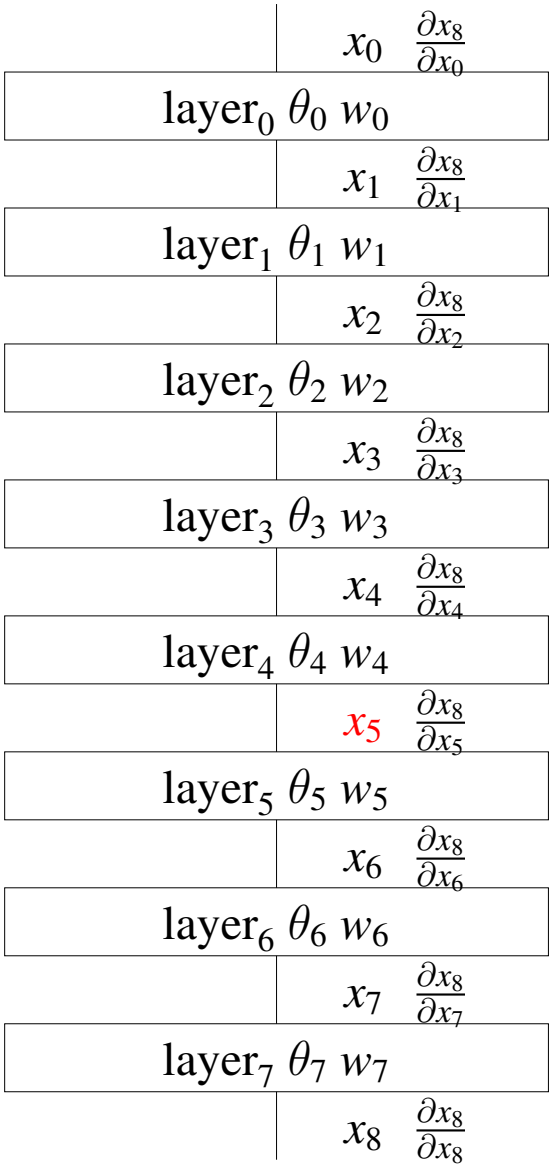

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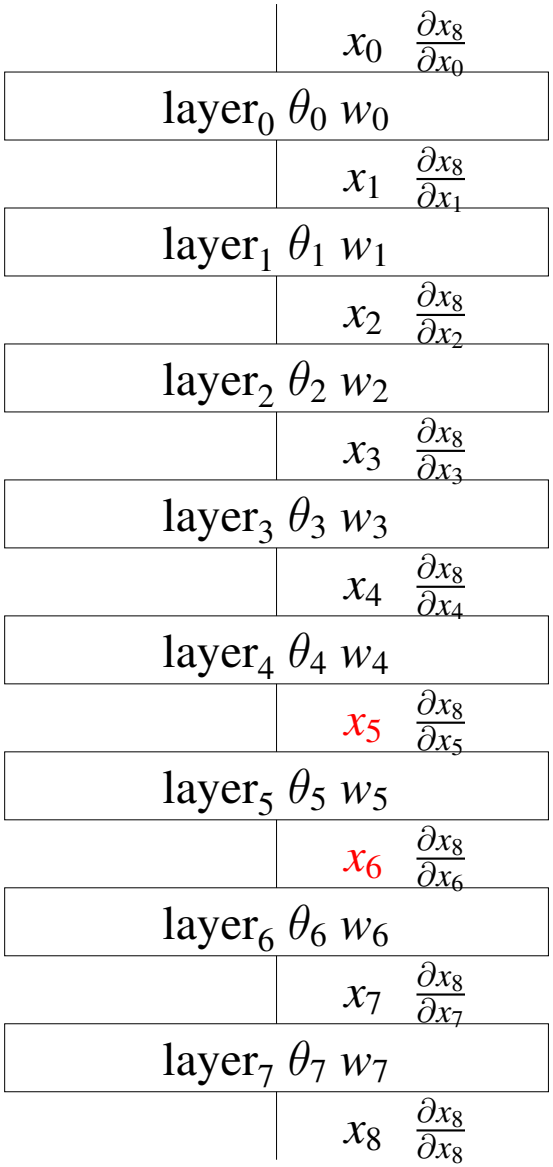

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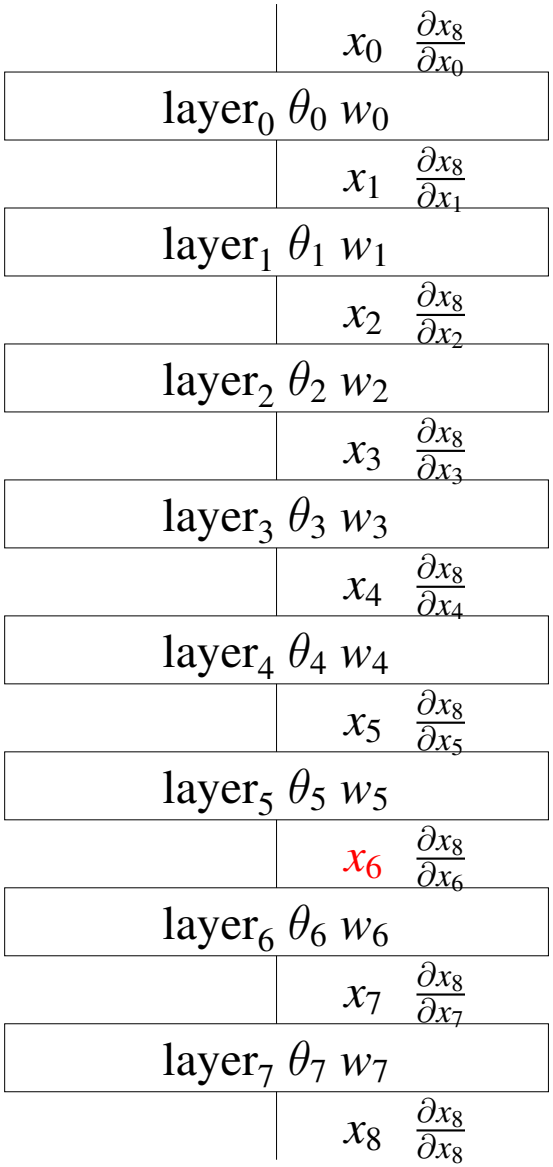

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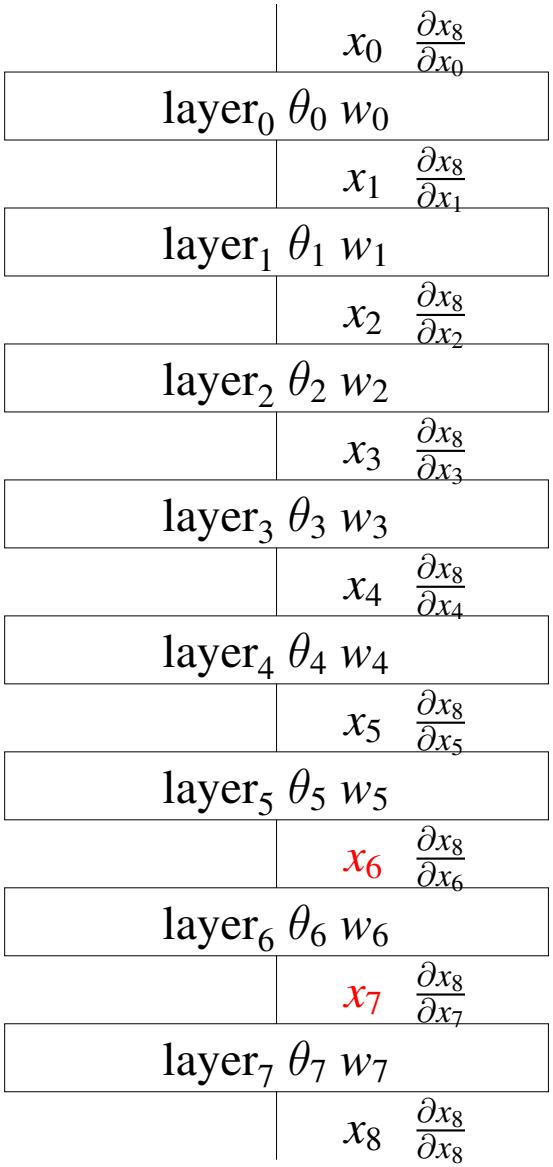

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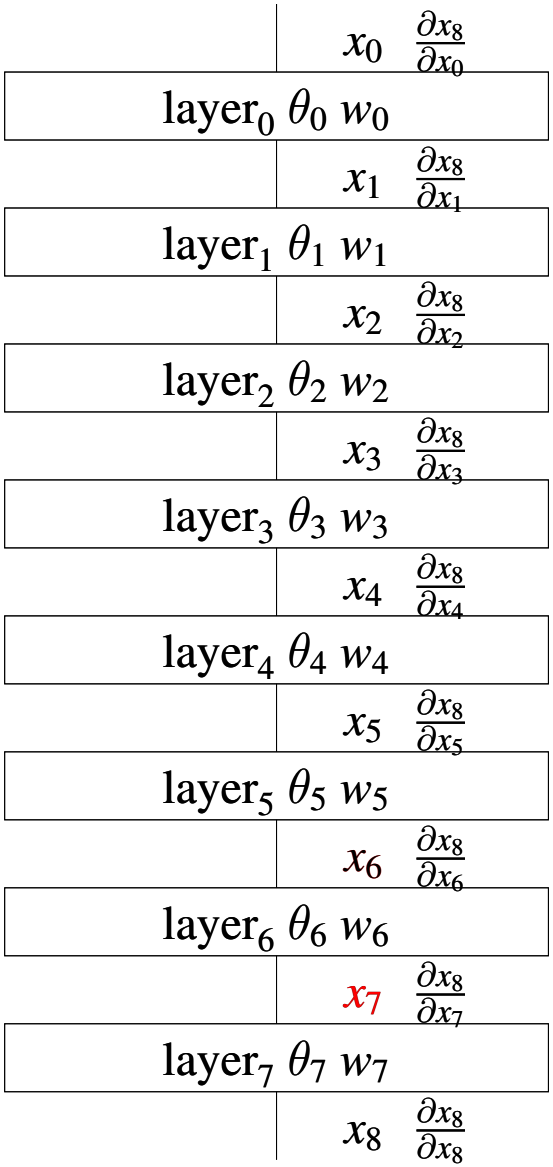

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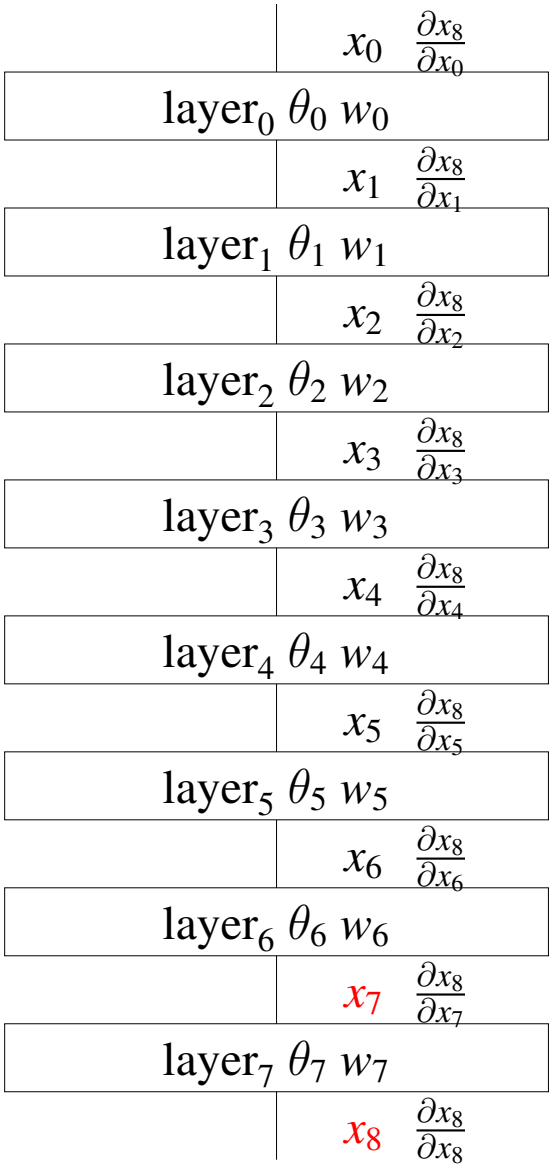

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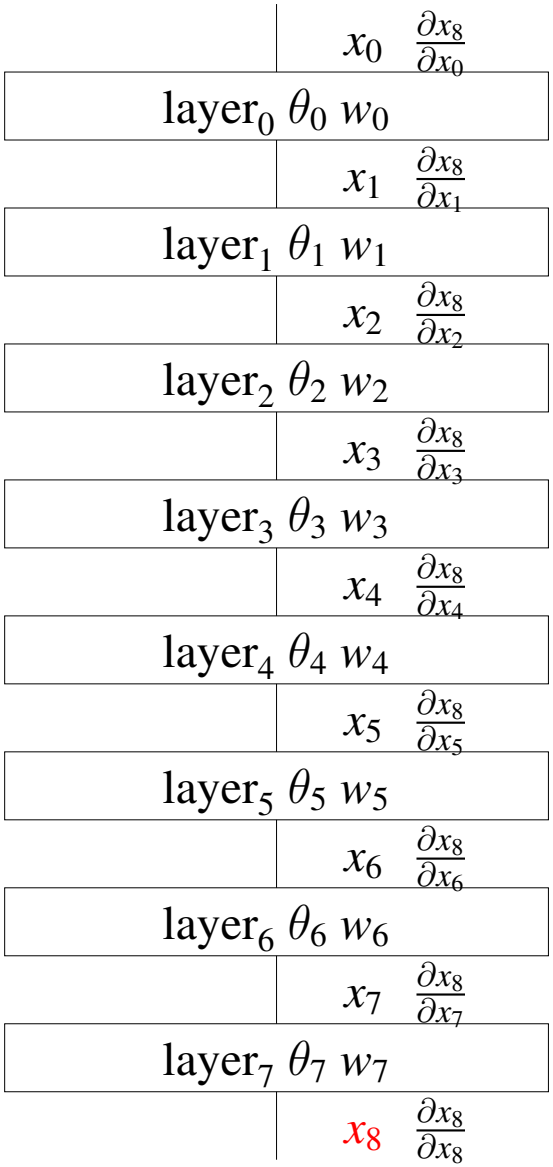

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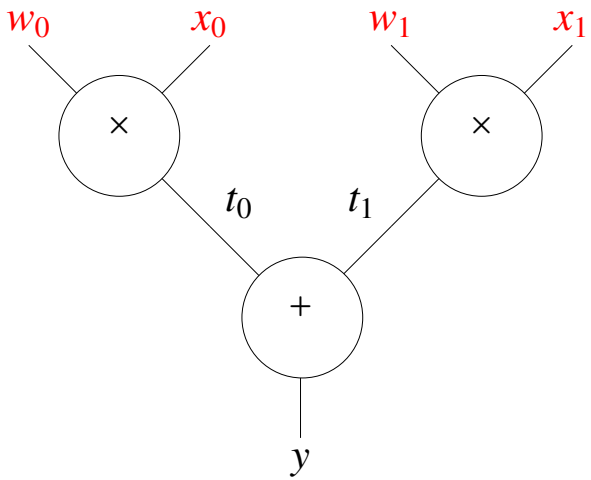

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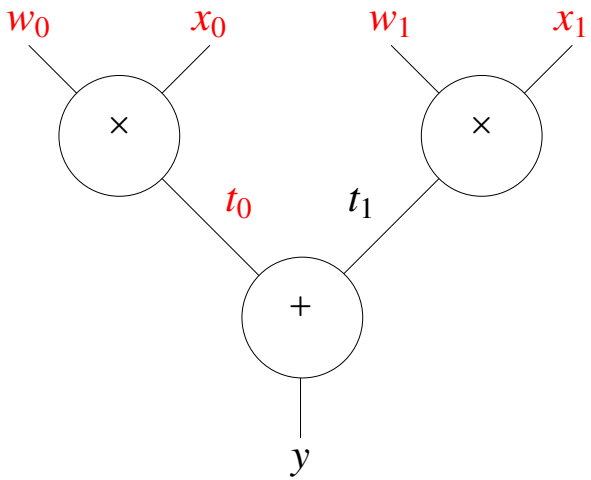

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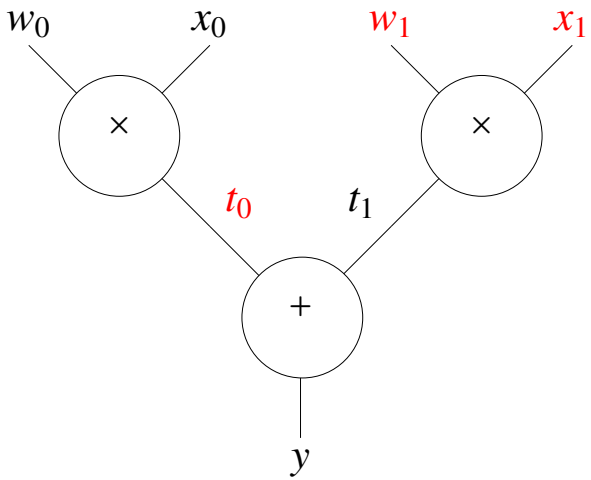

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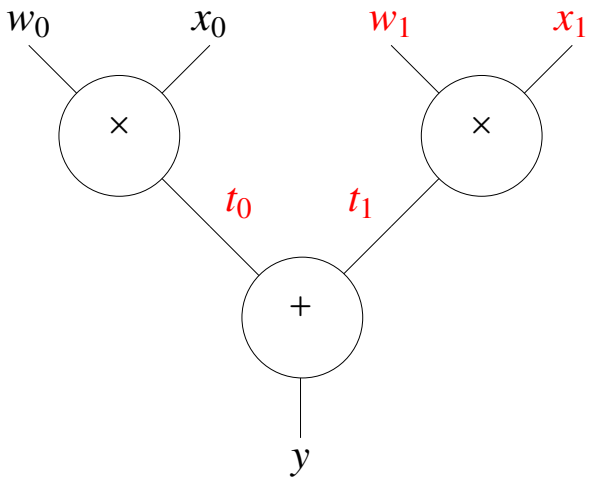

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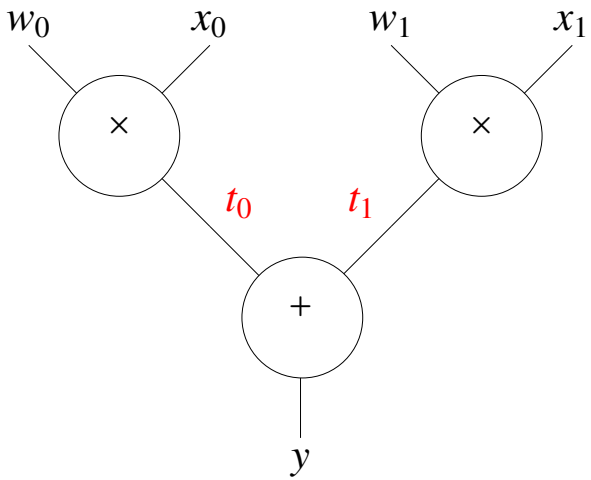

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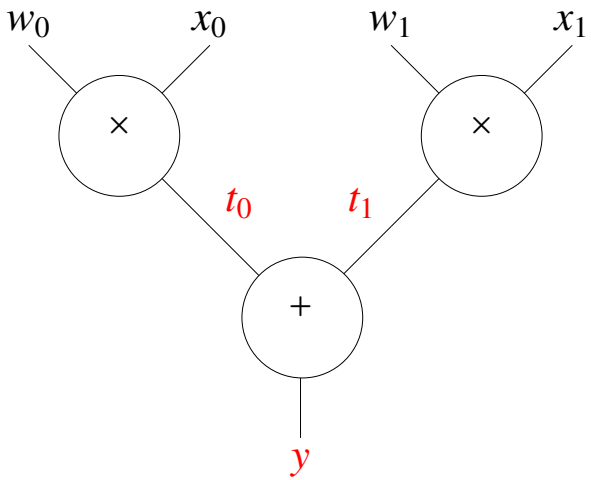

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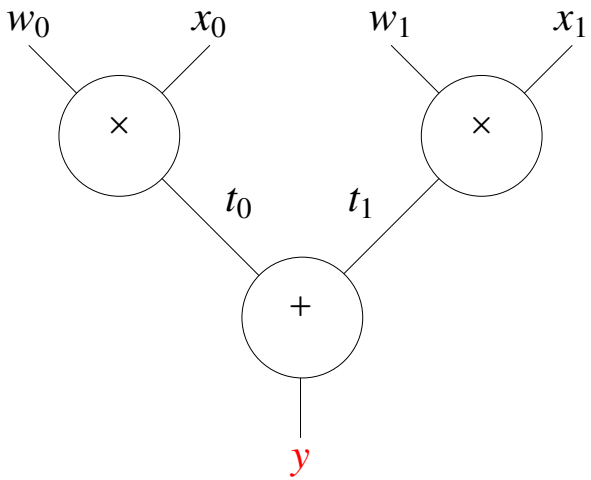

Some Observations

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- Only need to store live variables.

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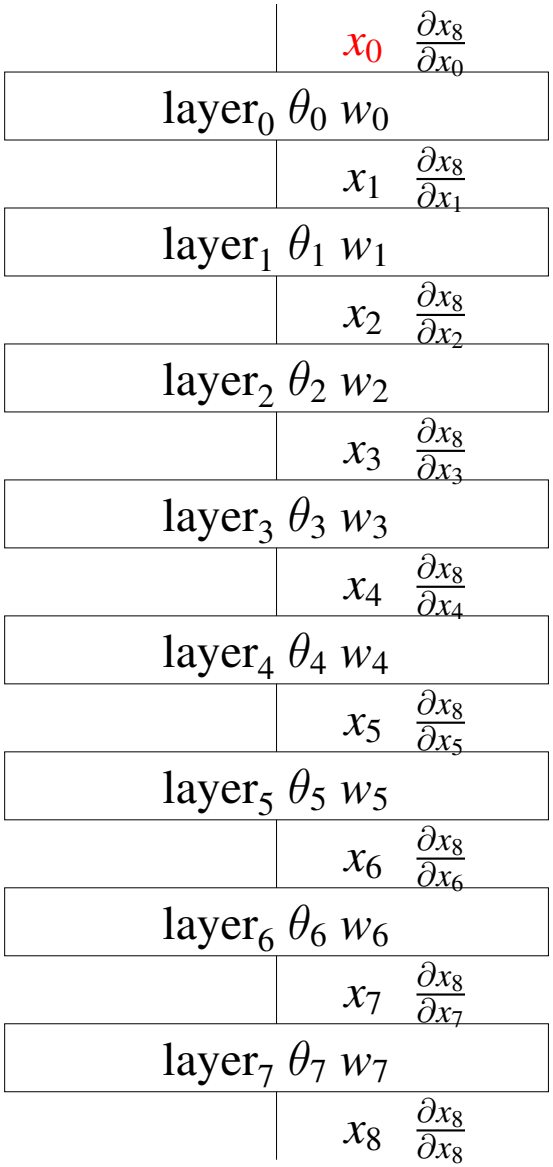
- Most deep learning frameworks store all intermediate variables to allow subsequent backpropagation.

Evaluating a Neural Network to Allow Subsequent Backpropagation



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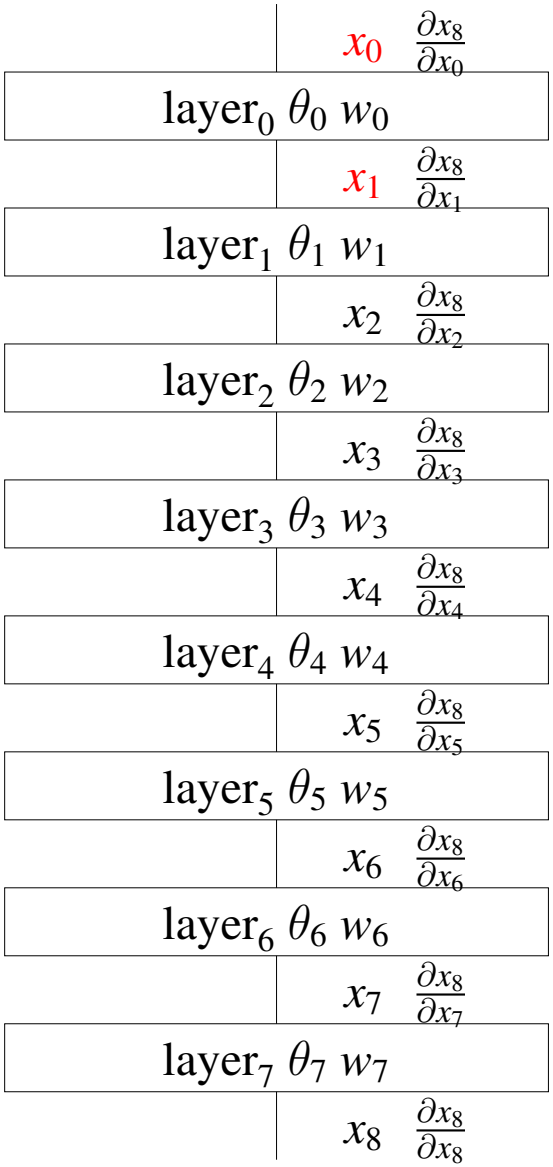

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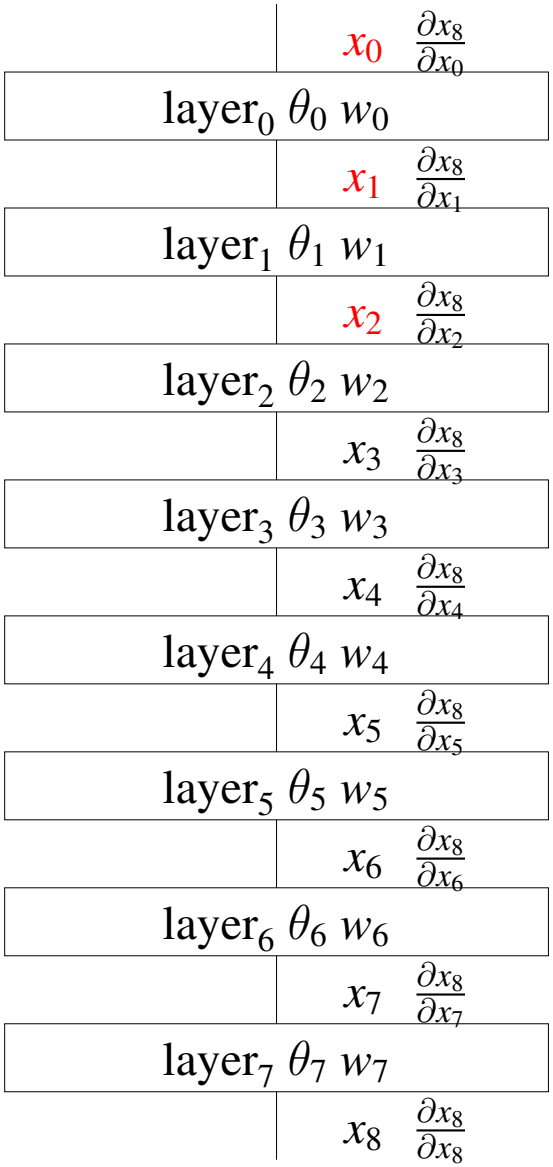

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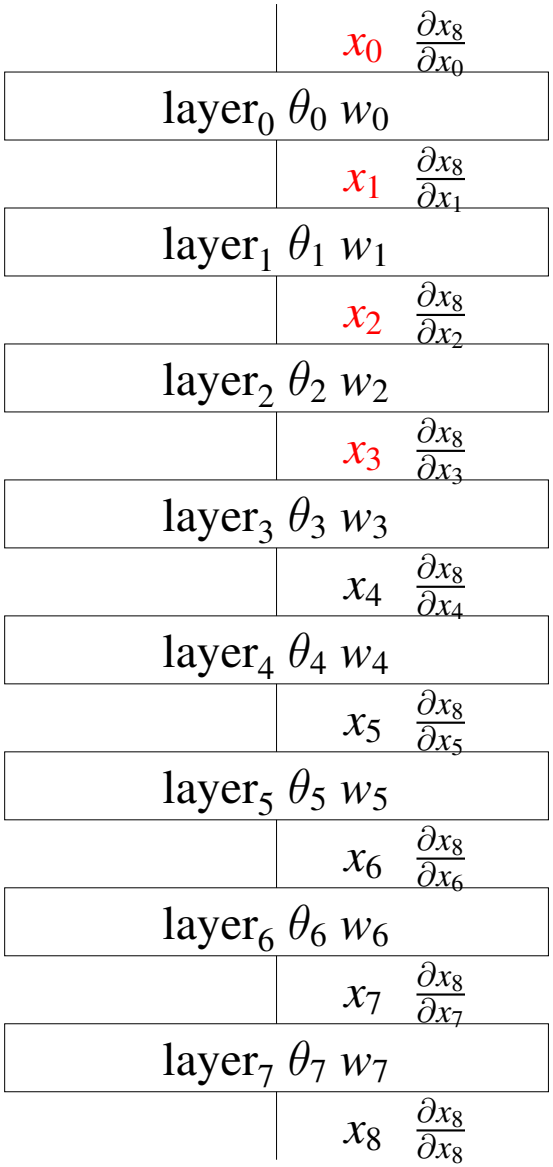

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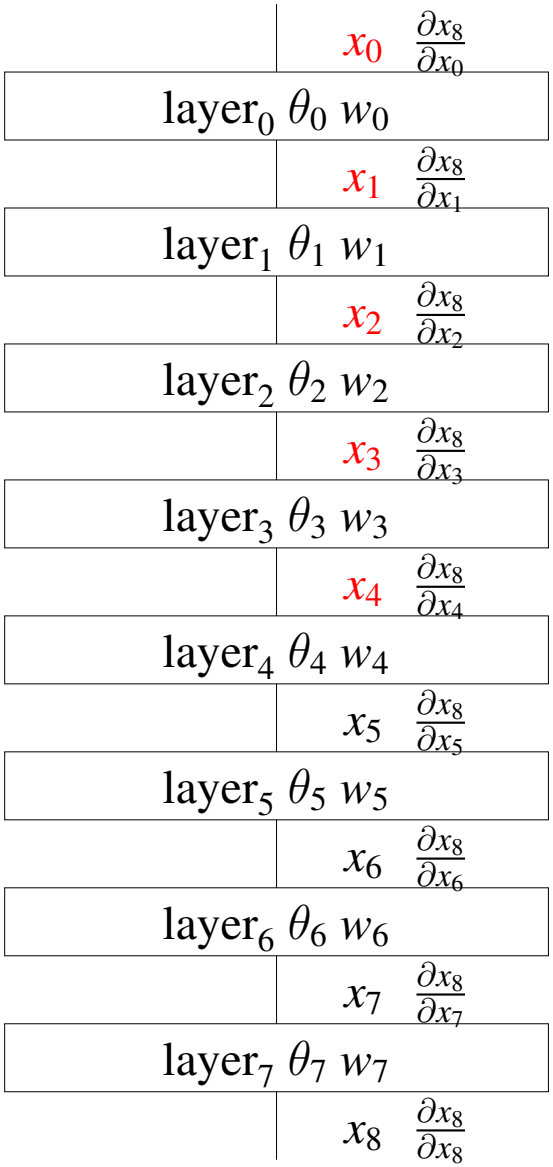

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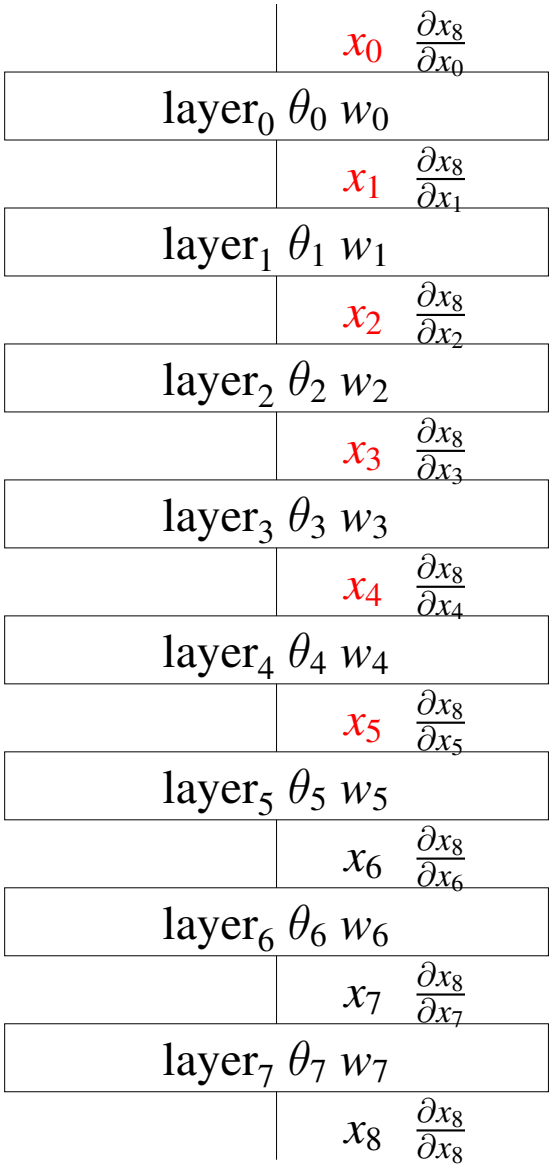

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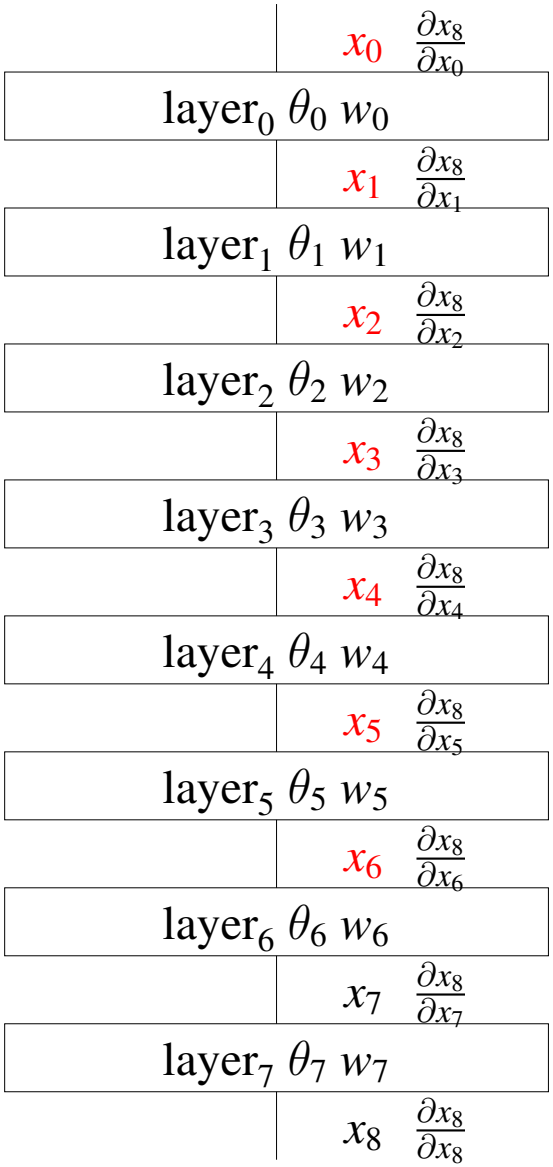

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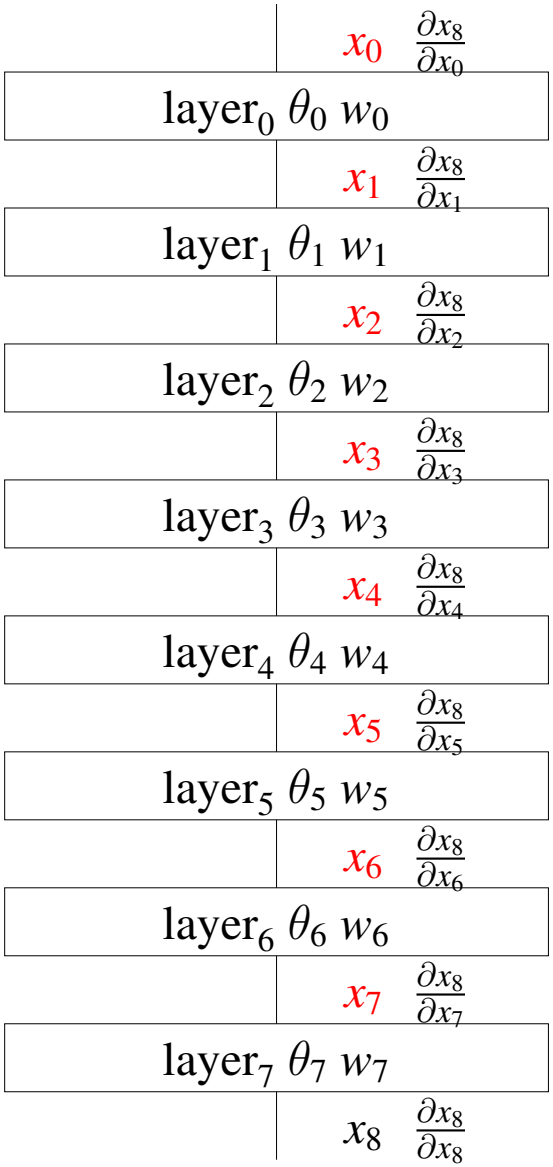

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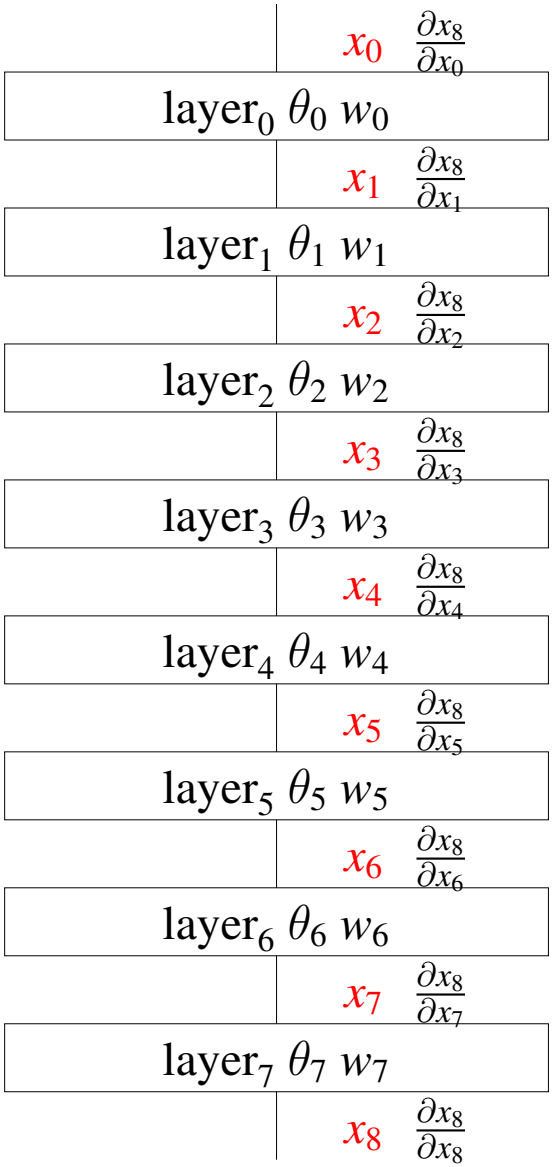

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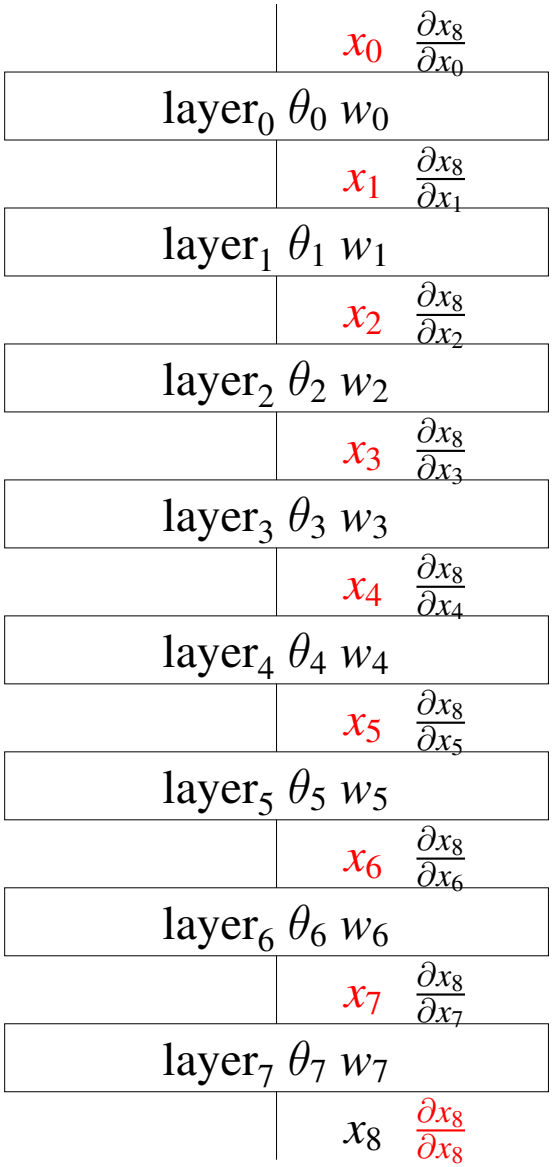

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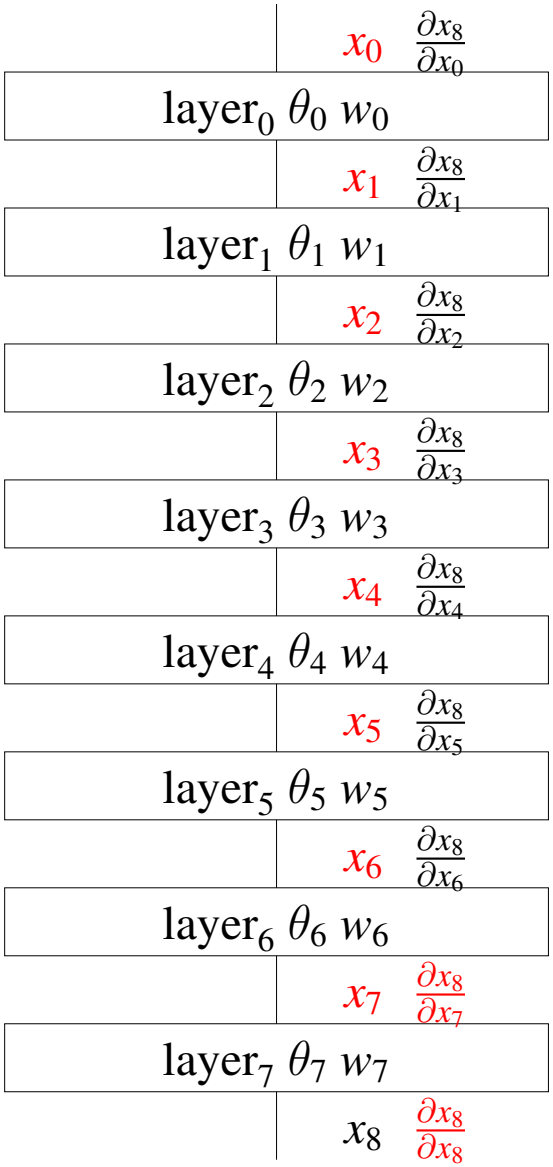

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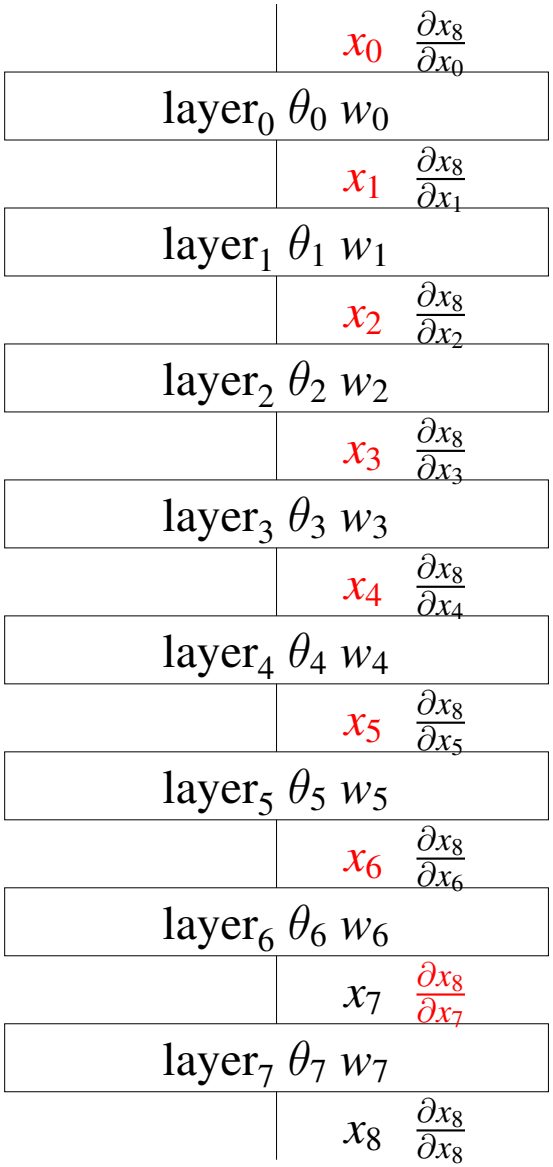

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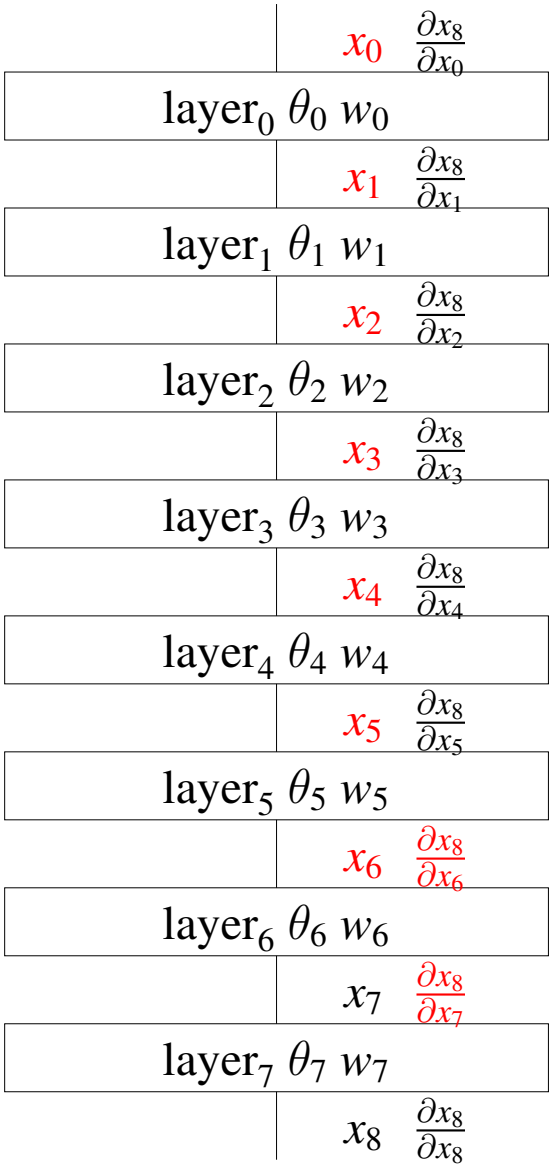

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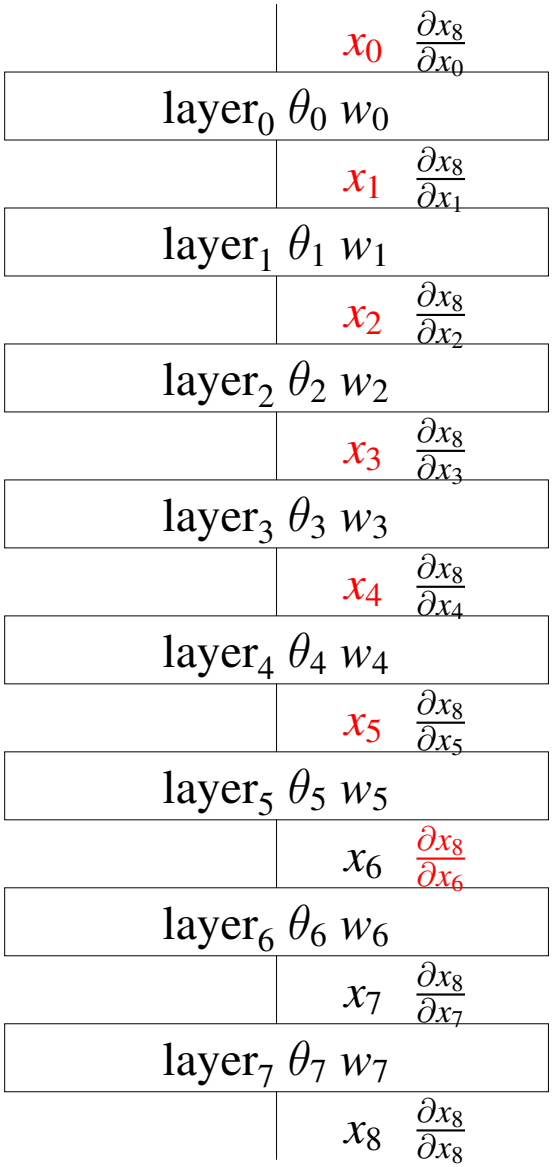

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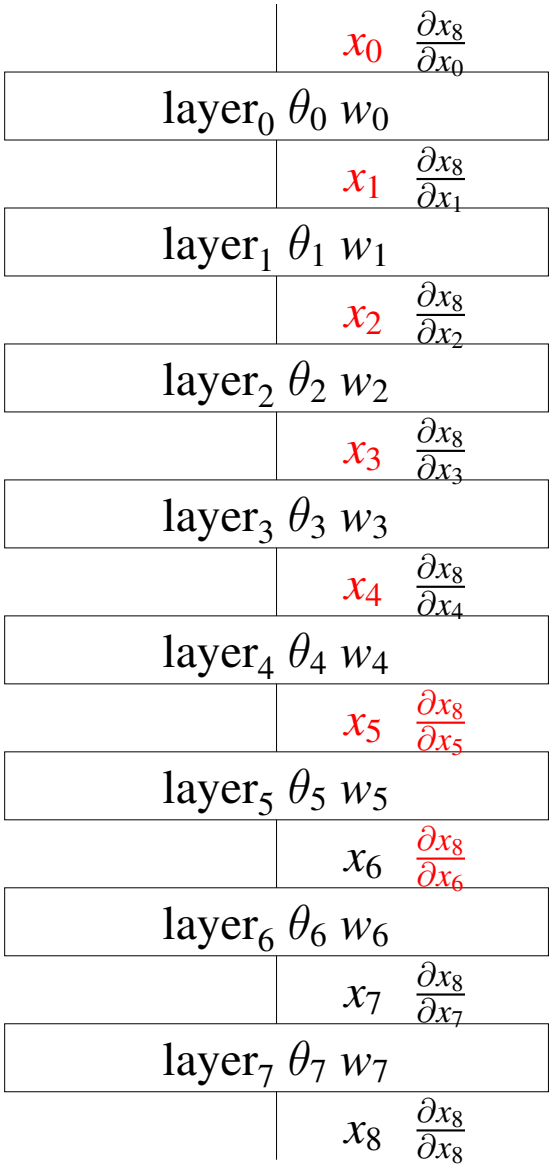

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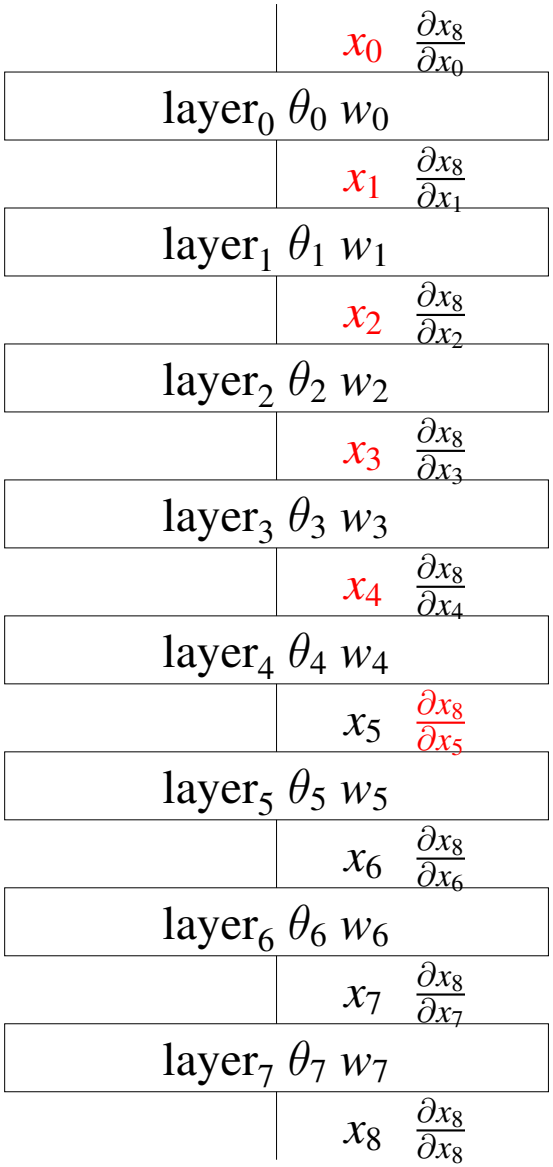

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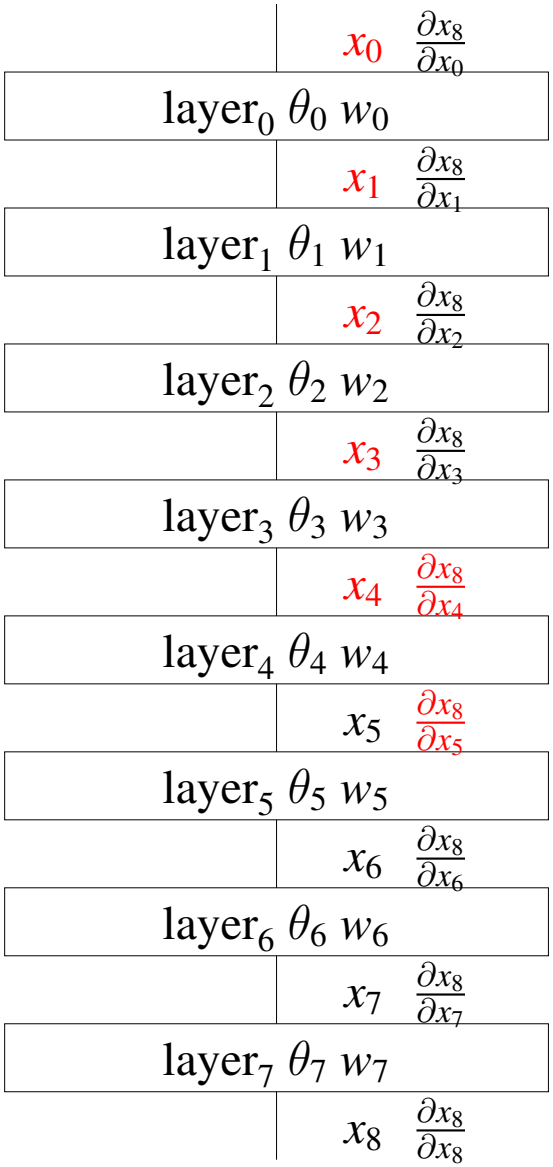

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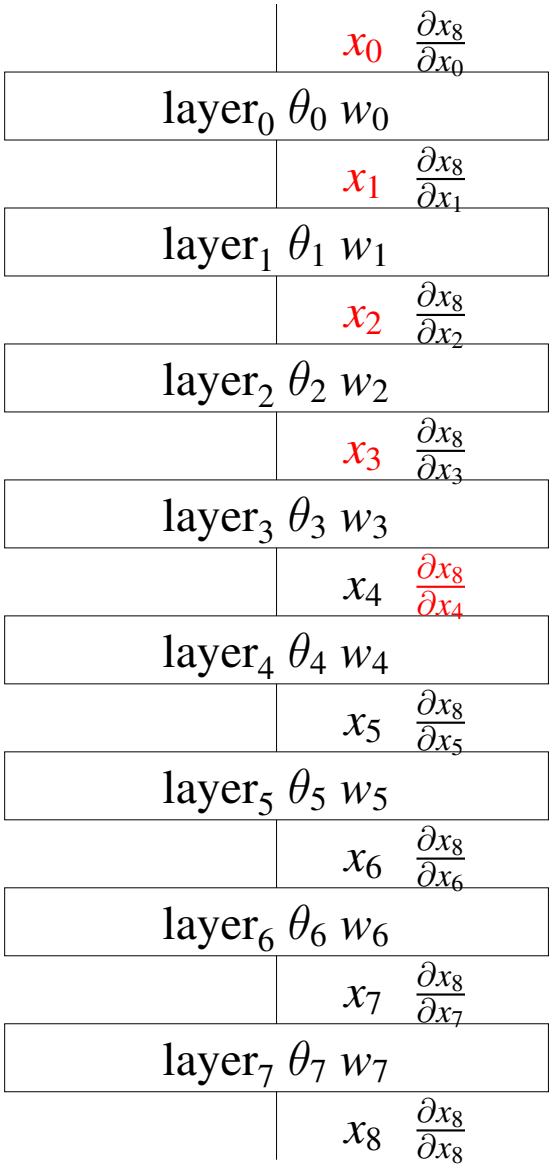

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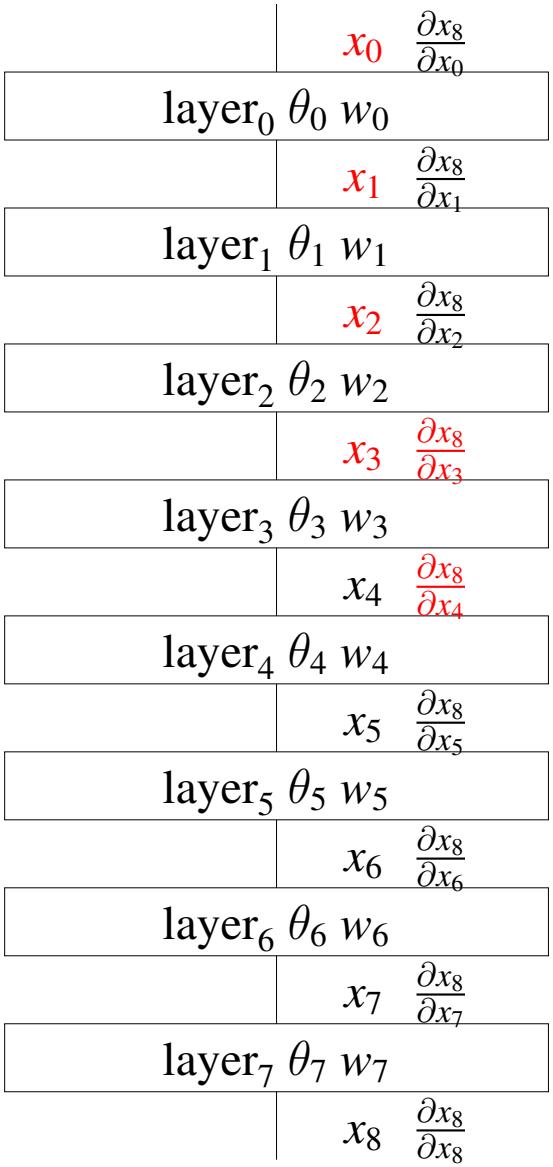

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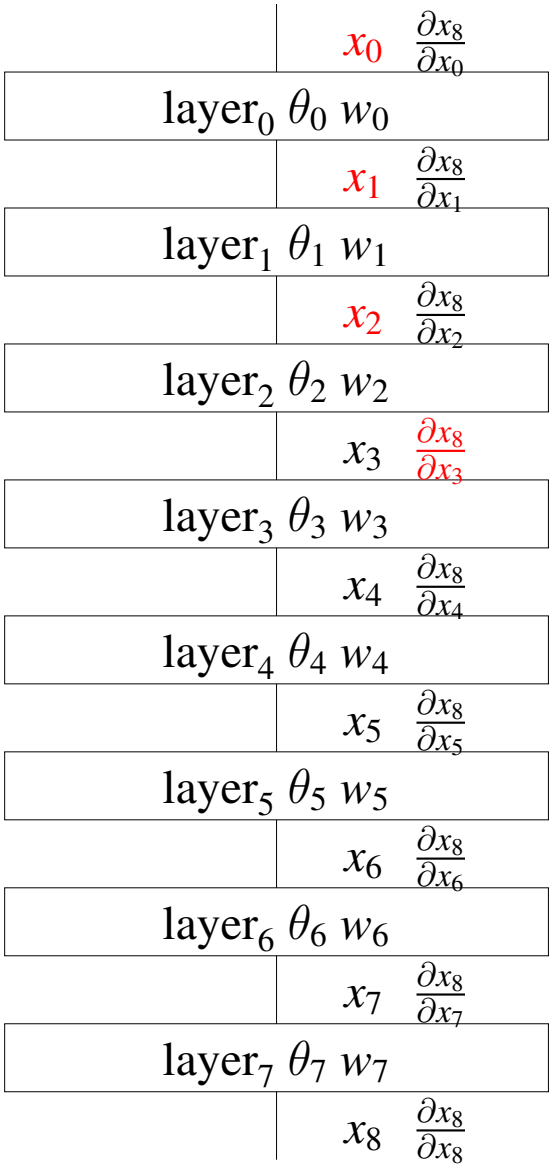

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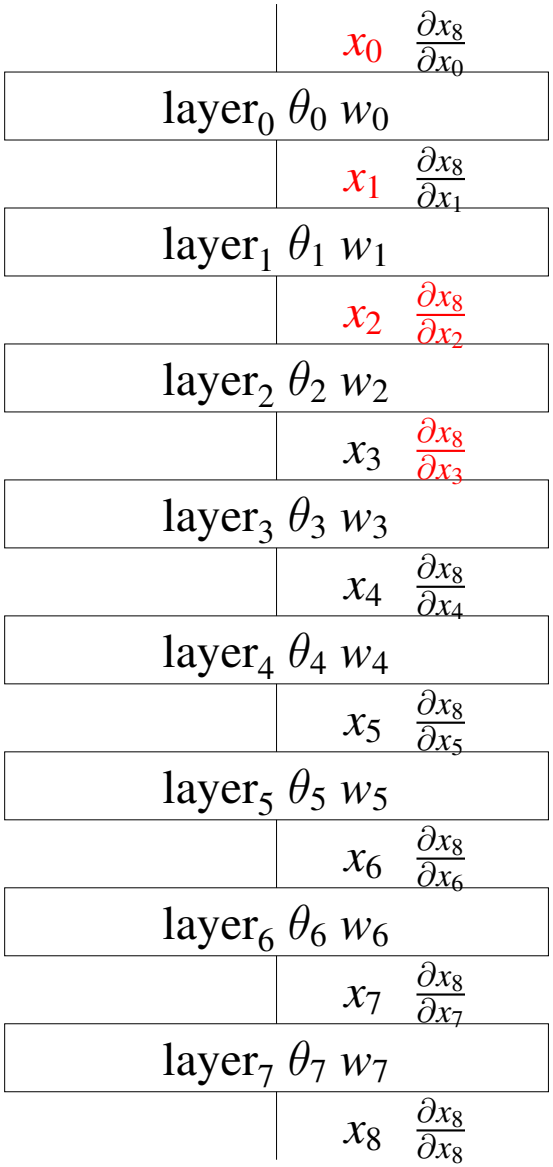

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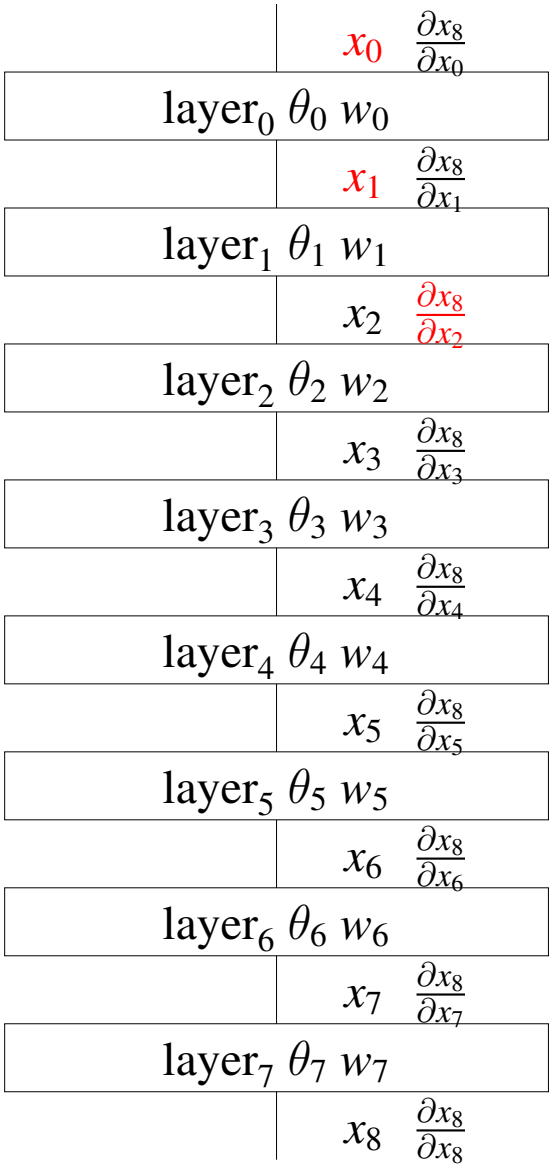

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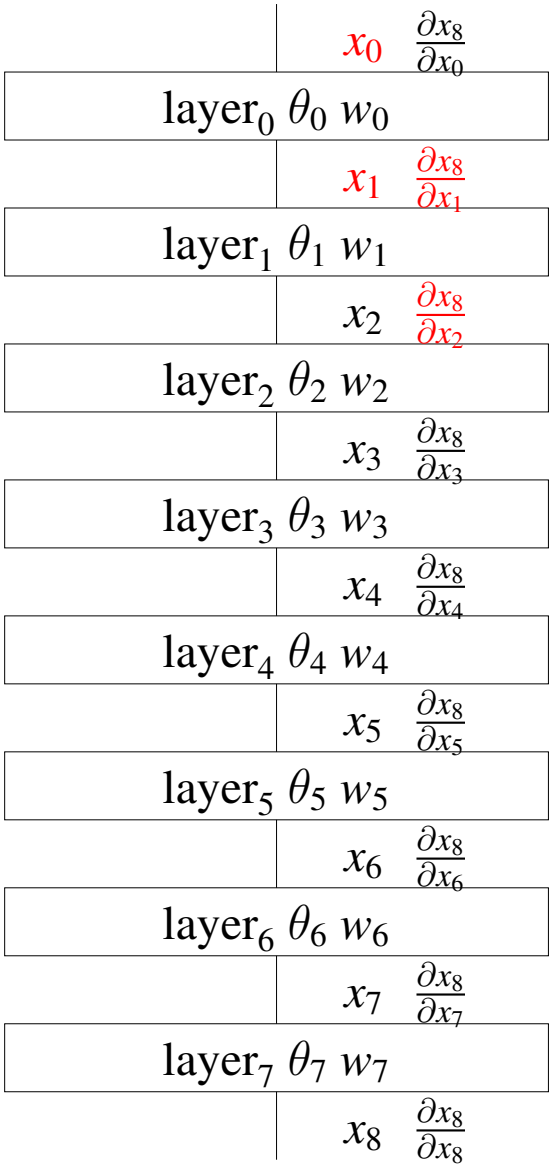

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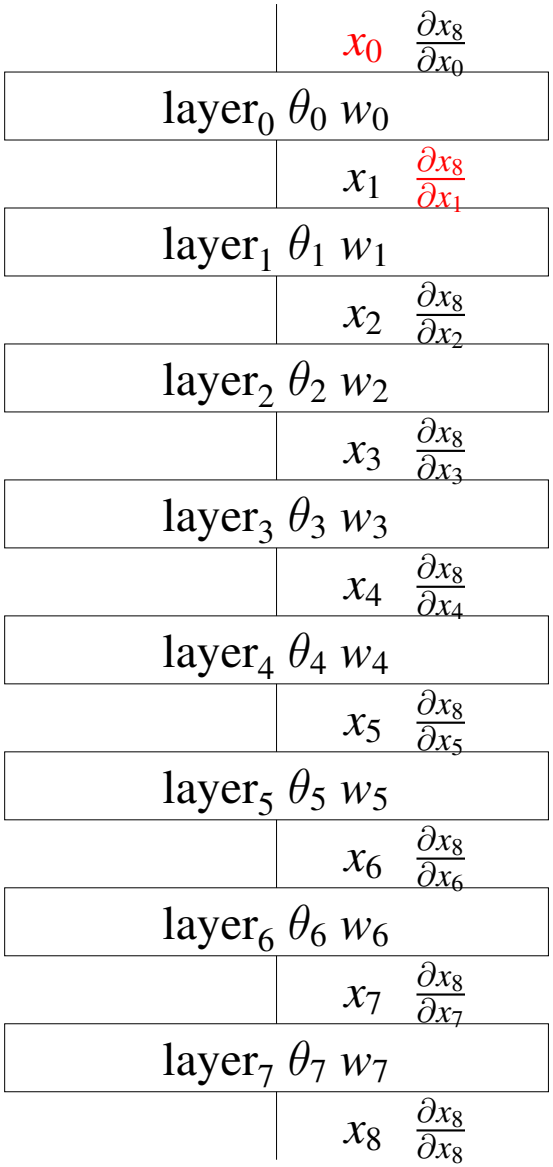

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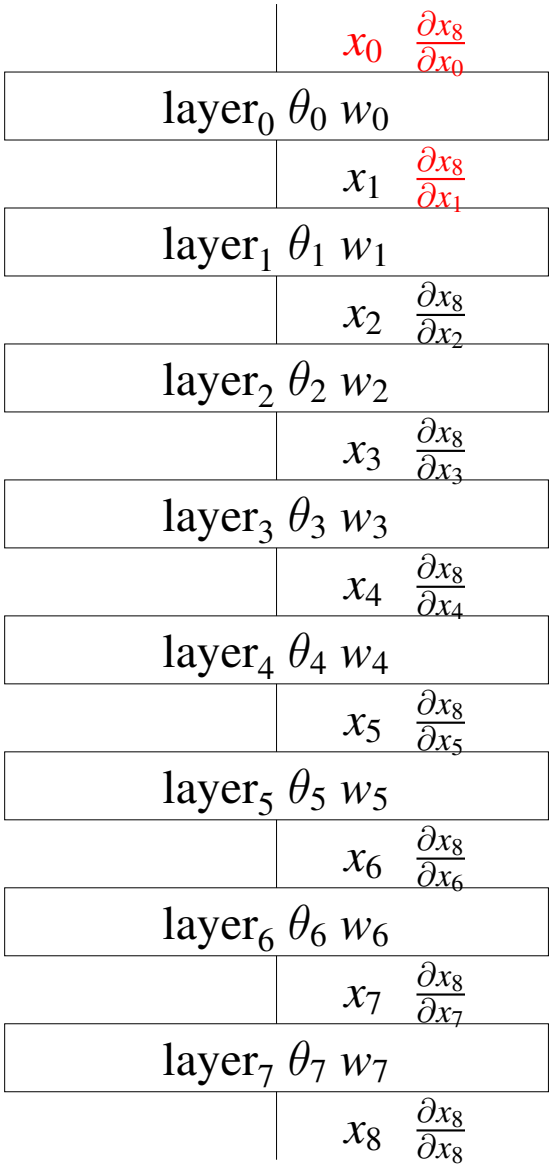

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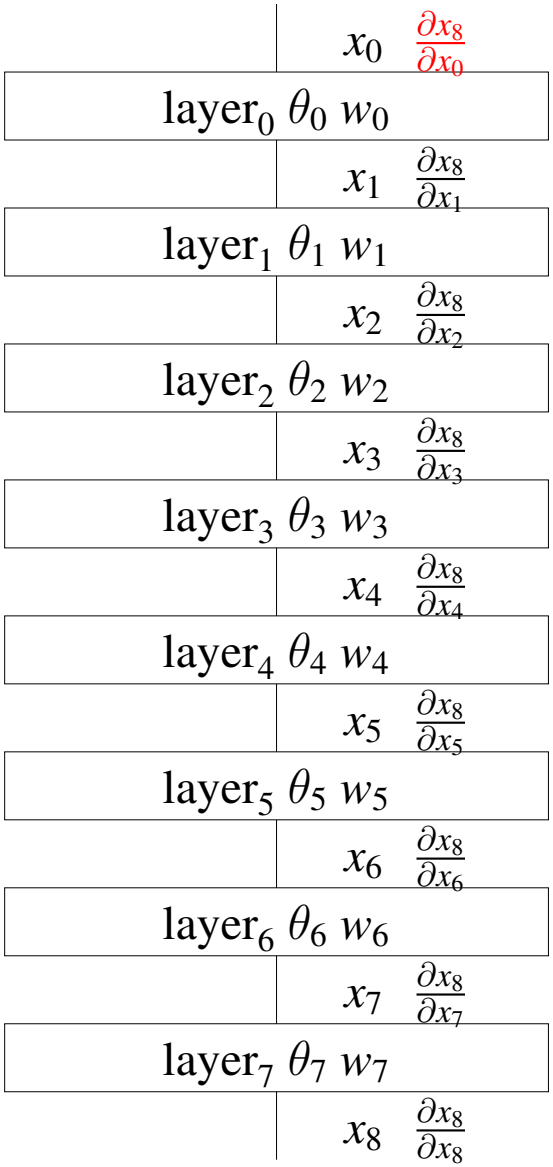

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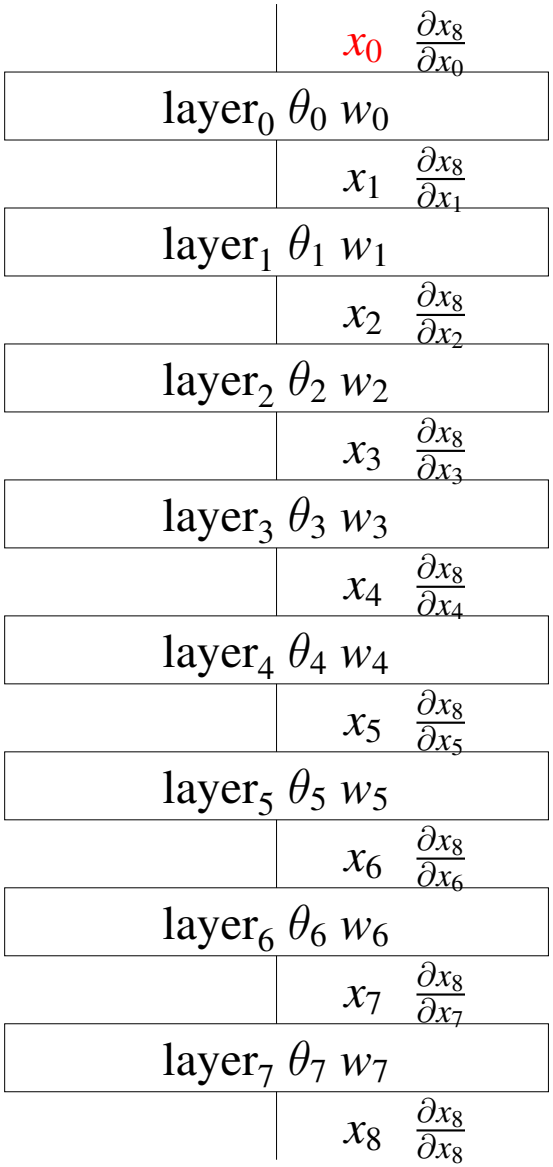
Backpropagation in a Neural Network with Checkpointing



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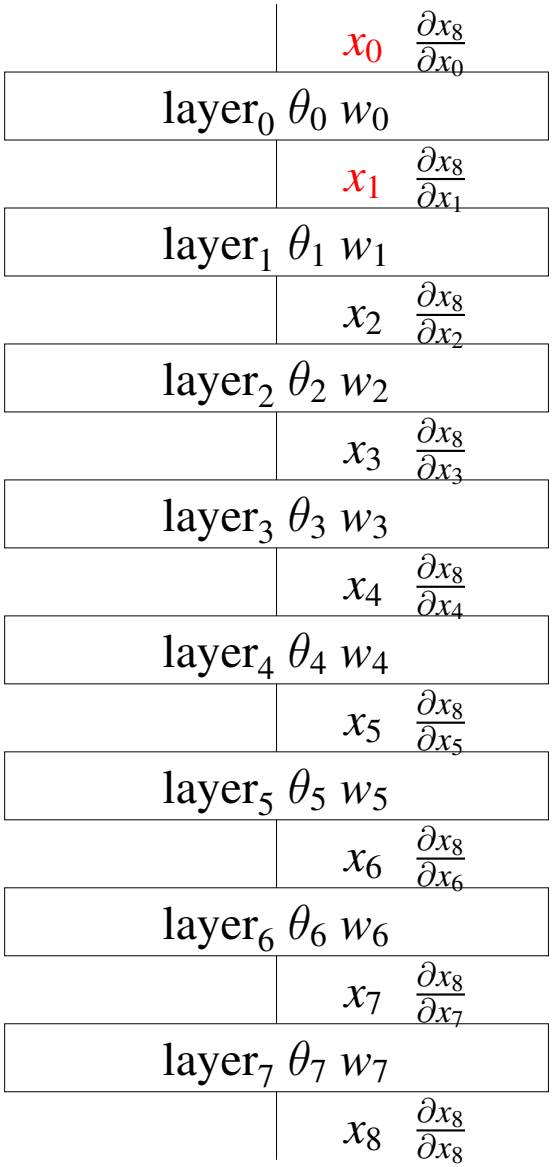

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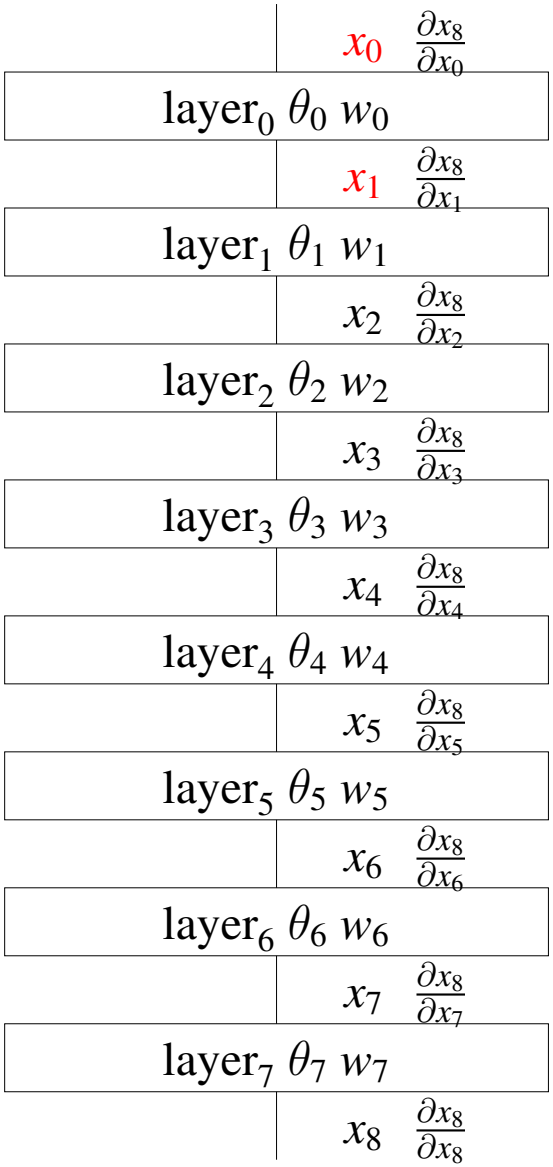

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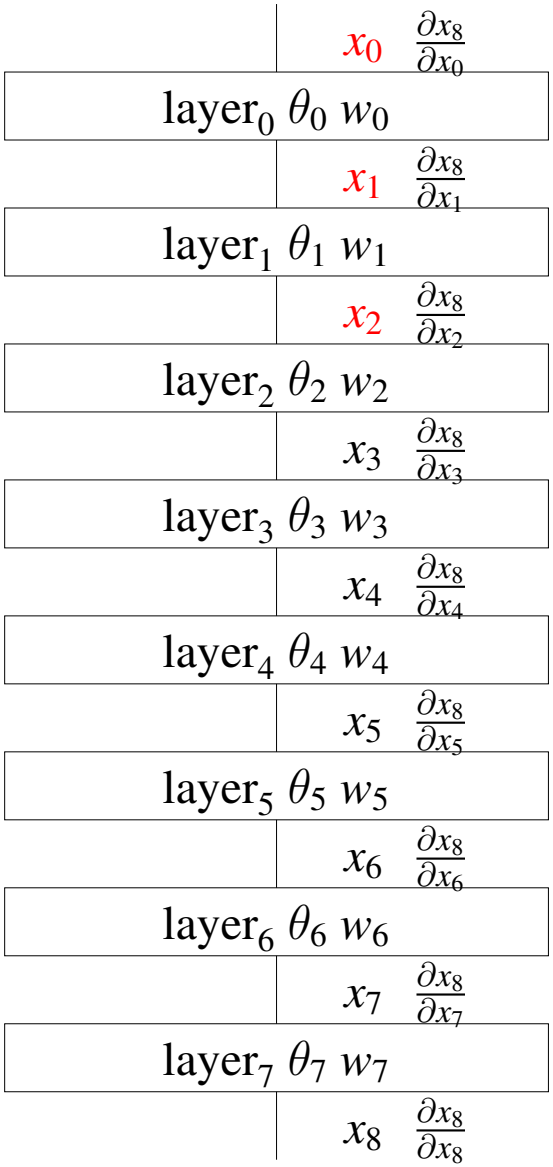

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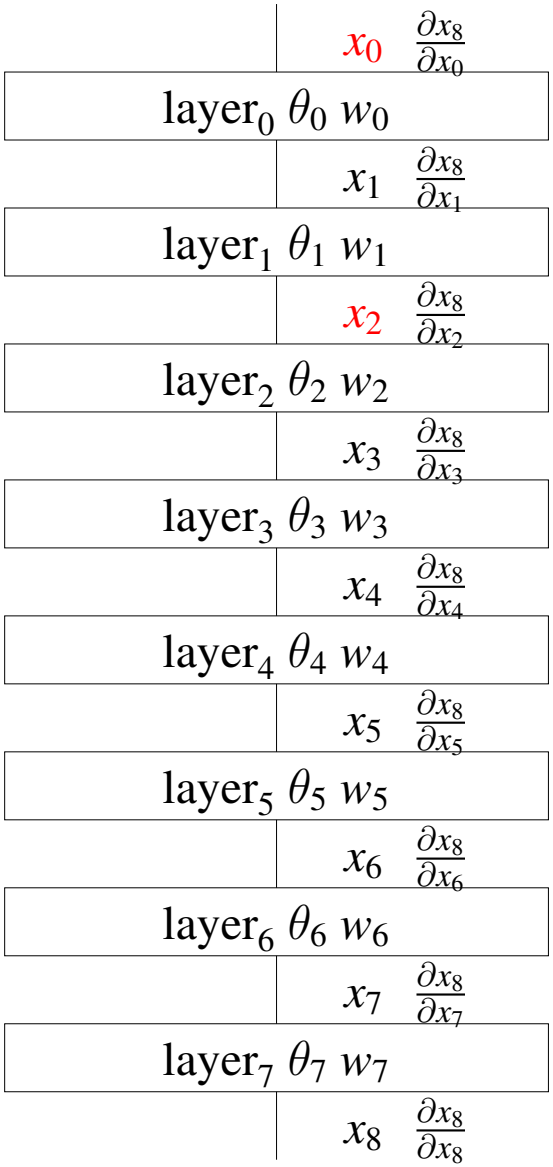

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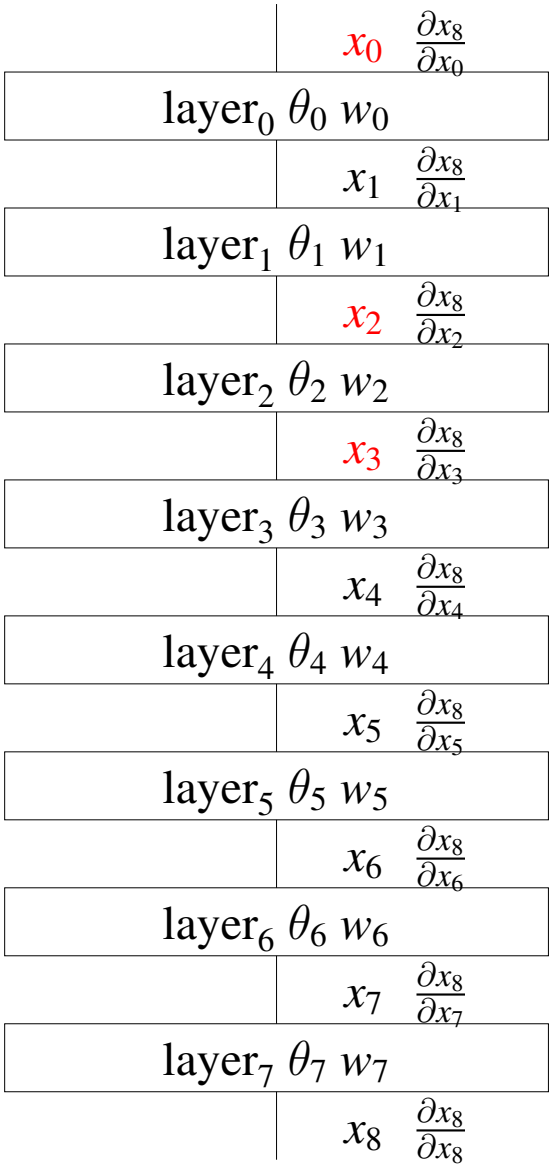

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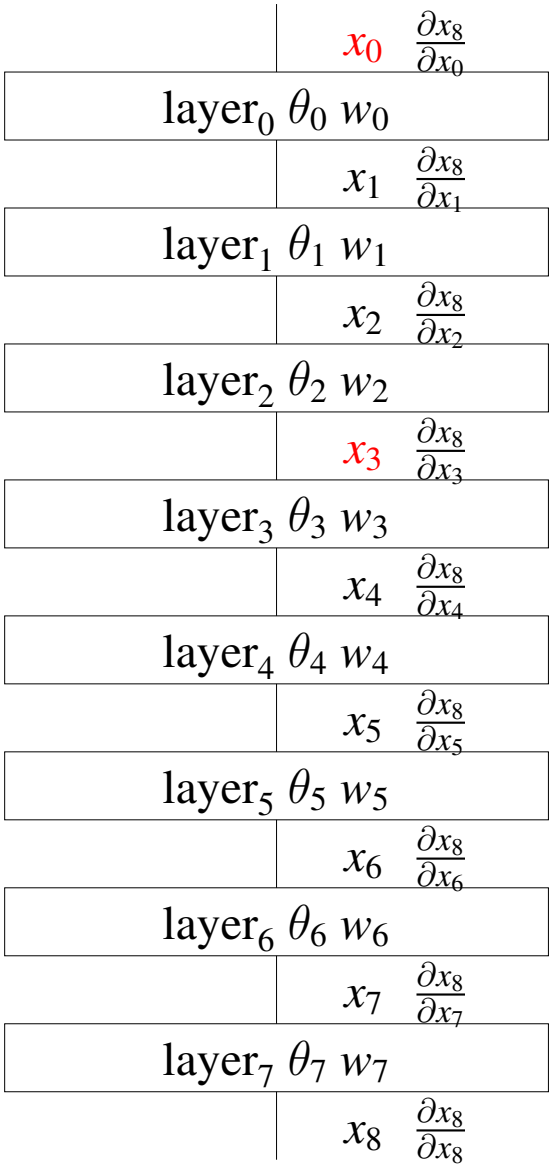

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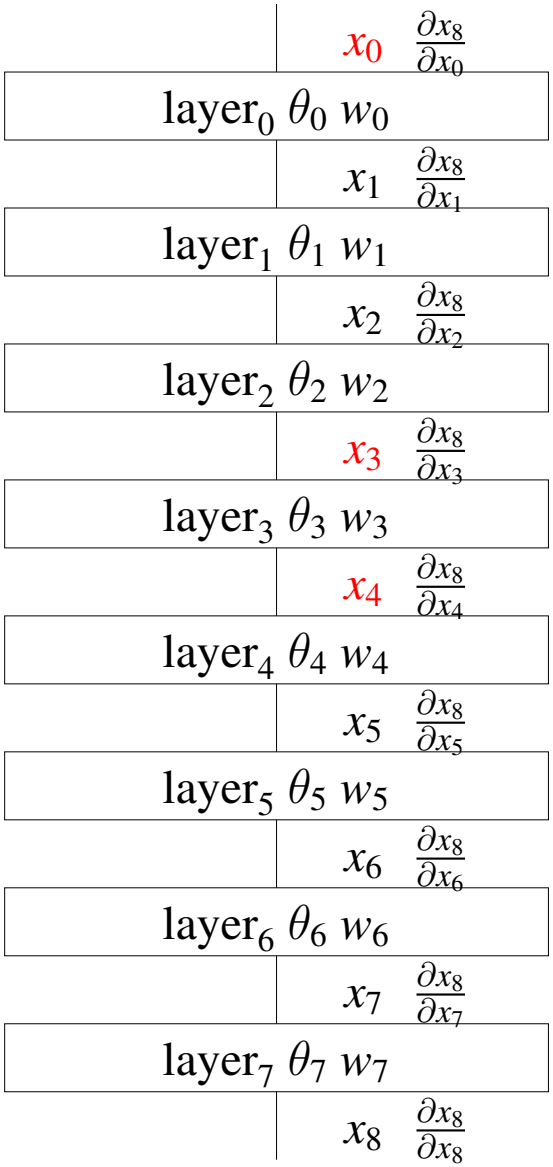

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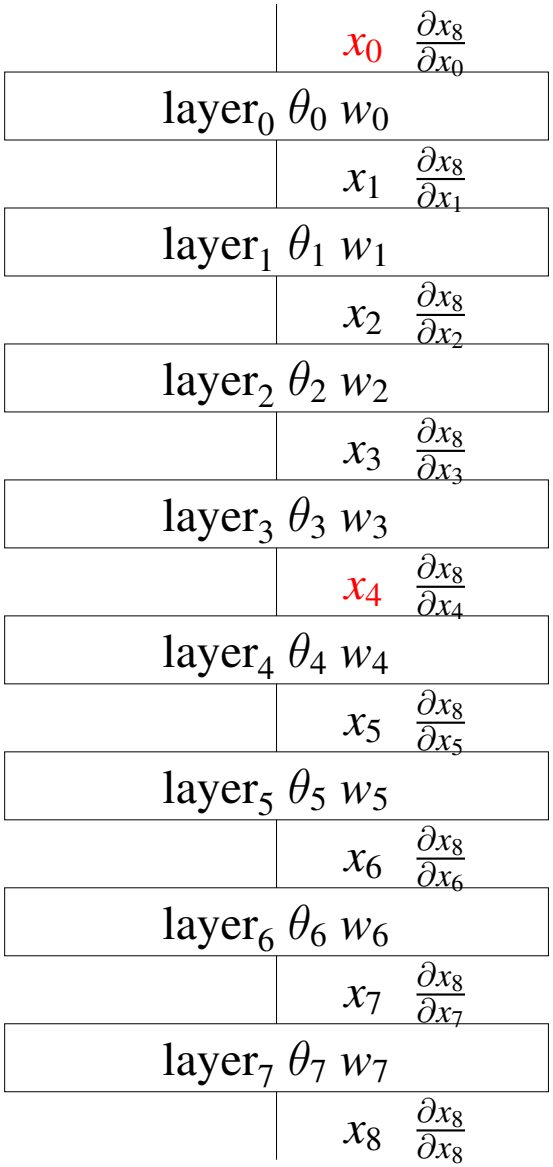

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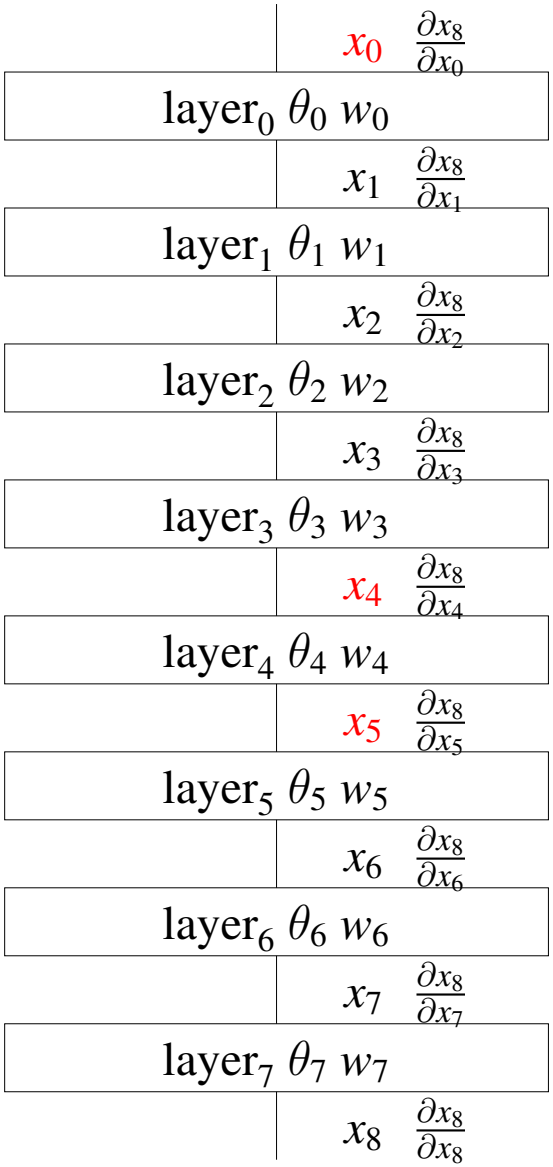

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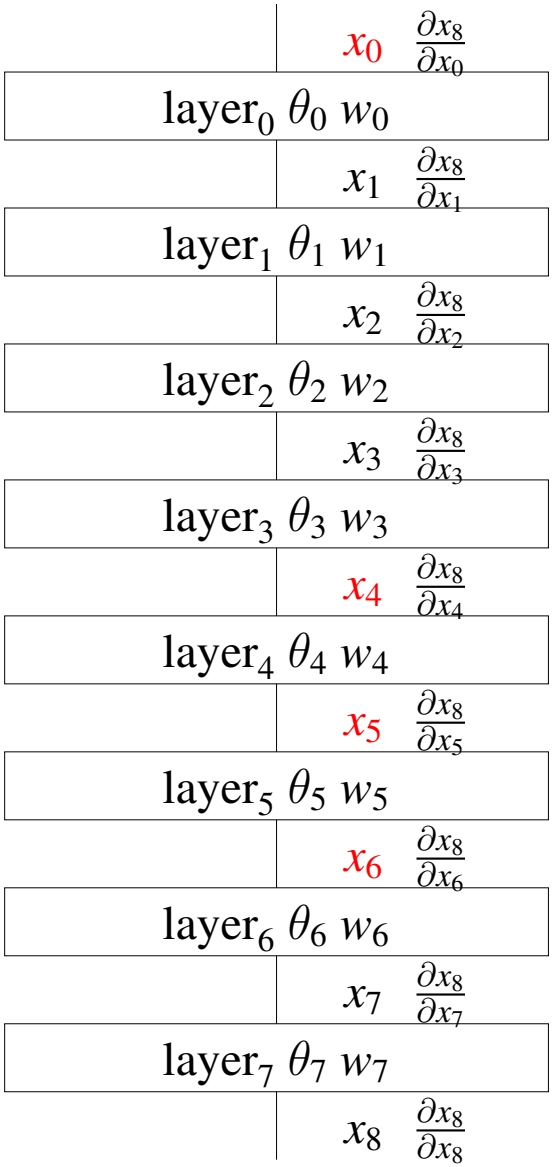

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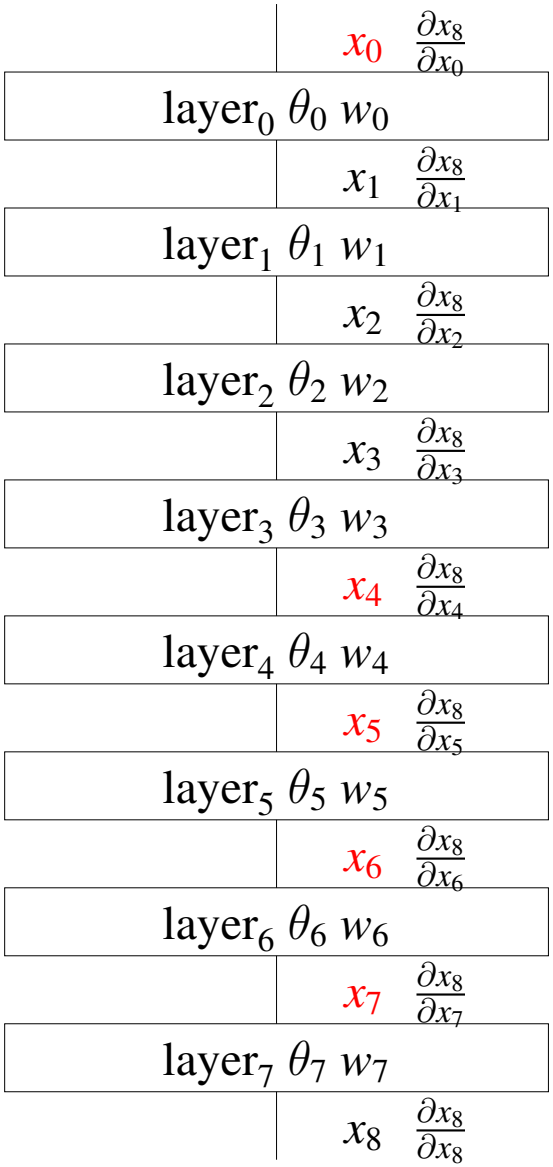

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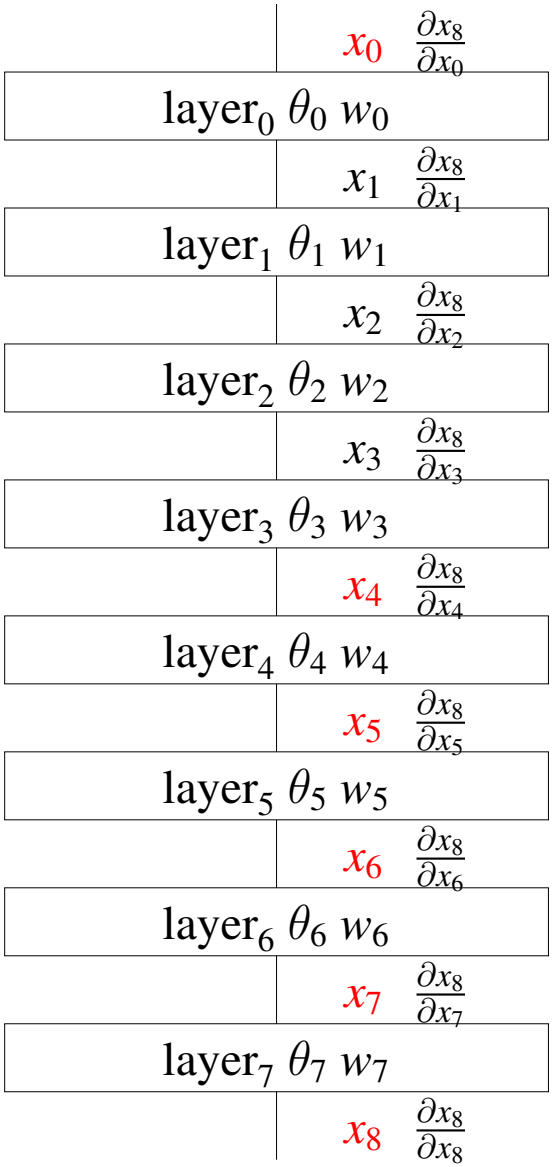

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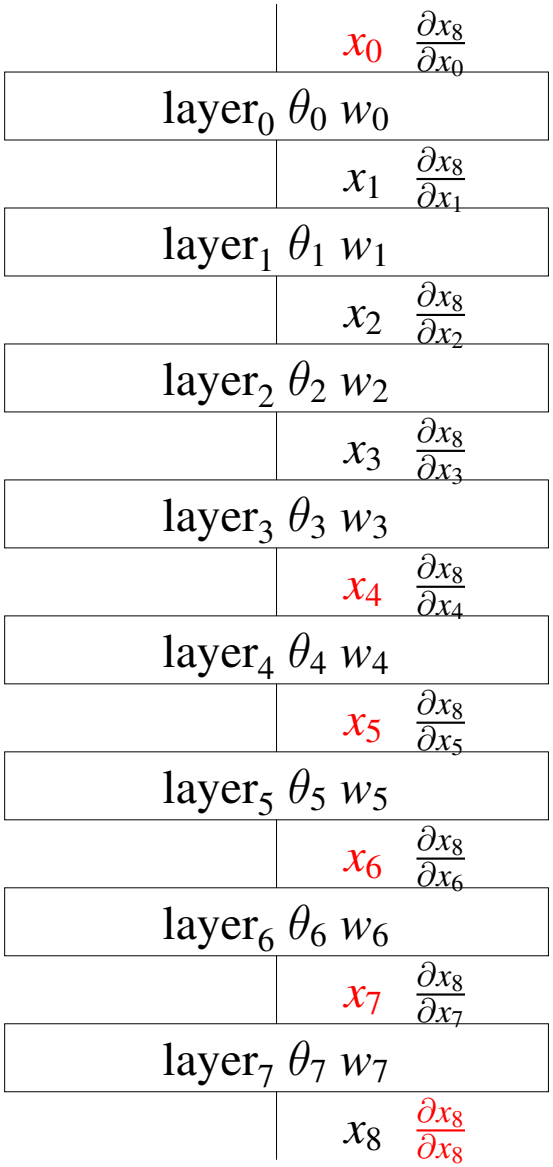

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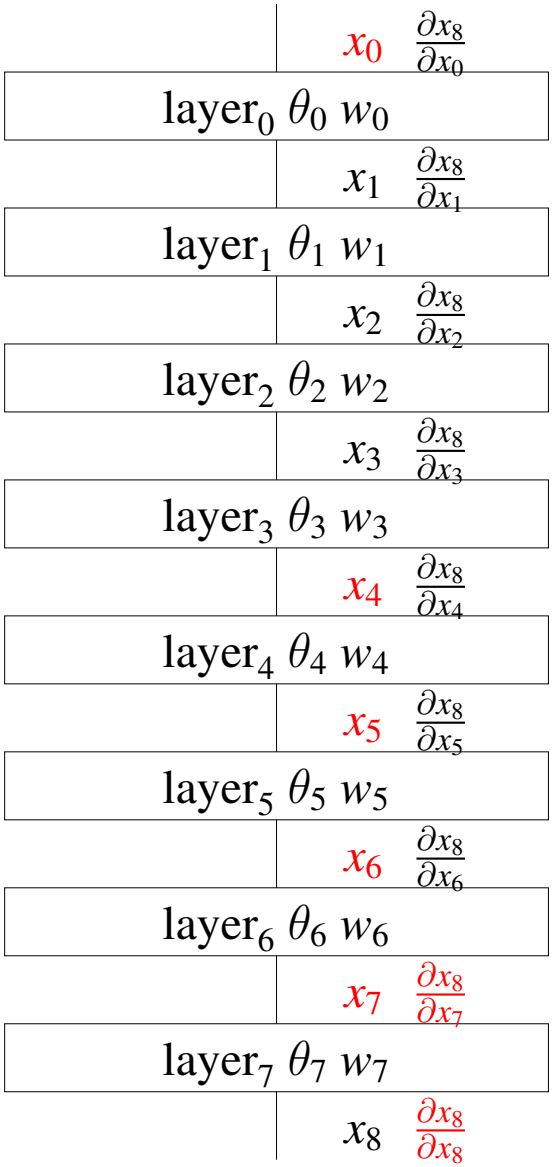

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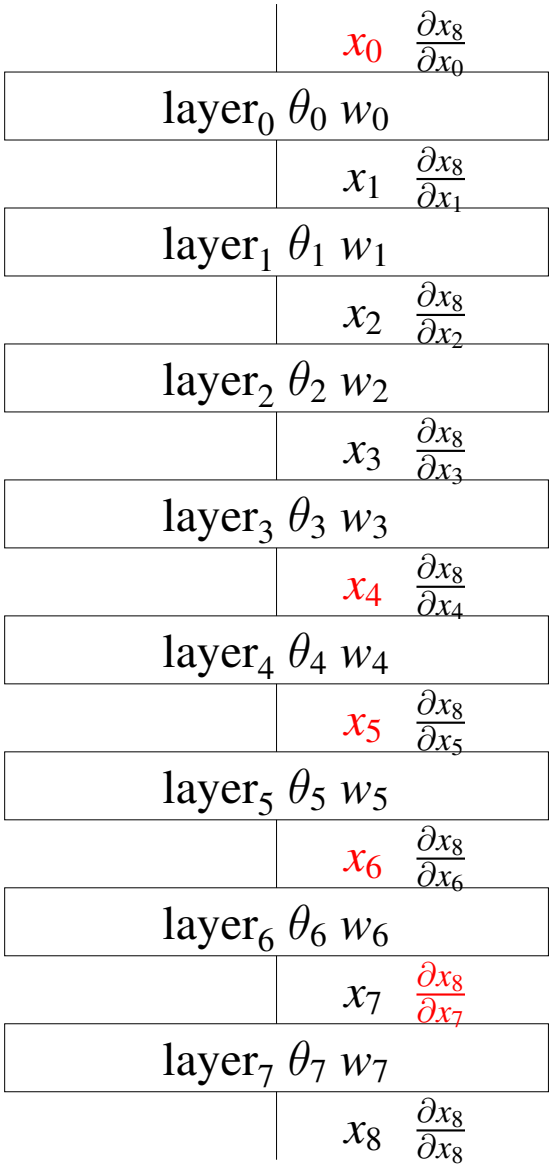

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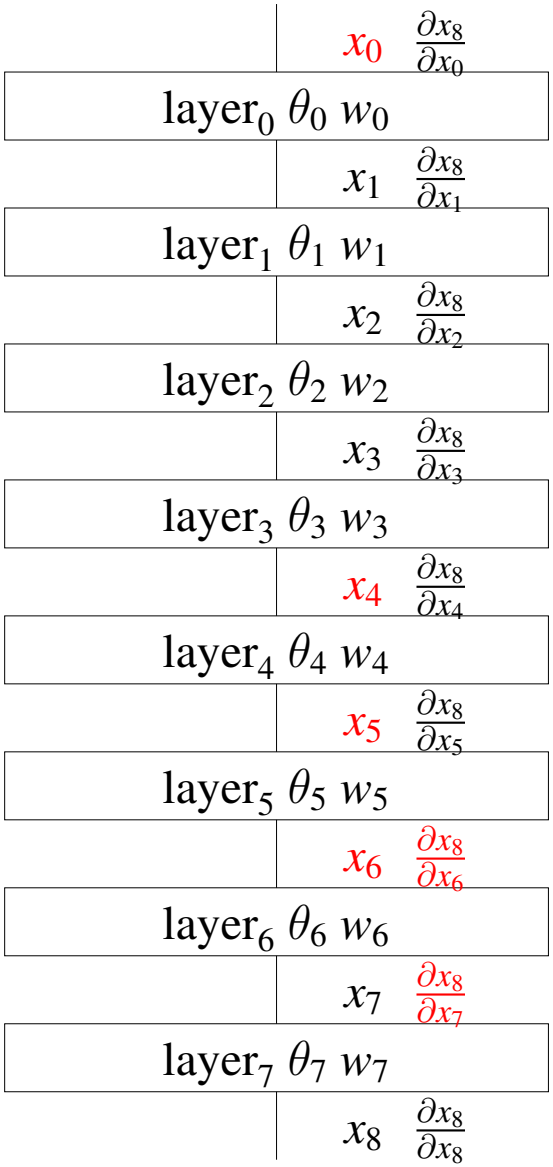

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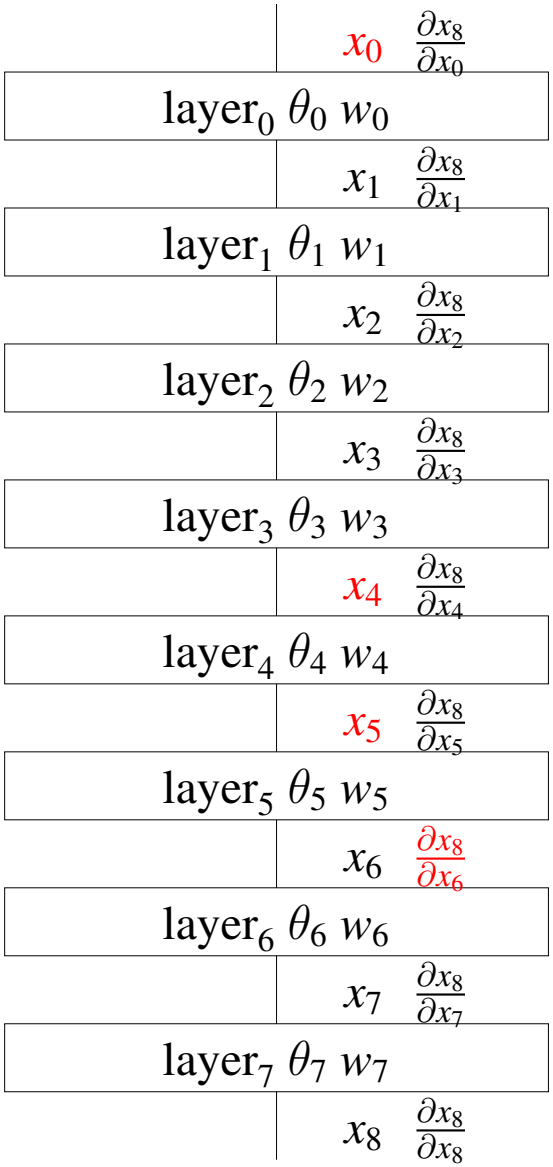

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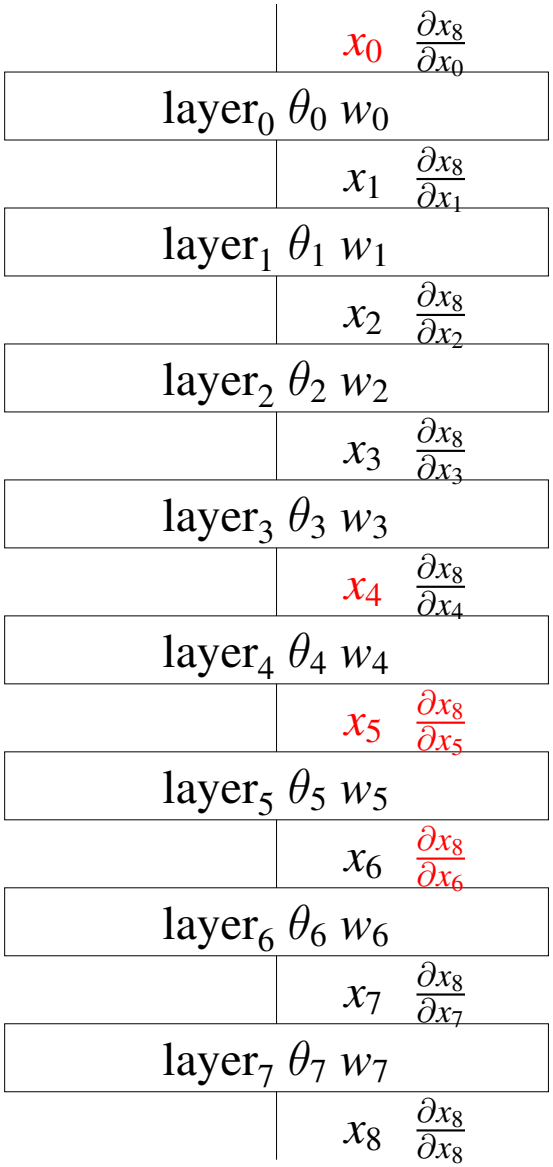

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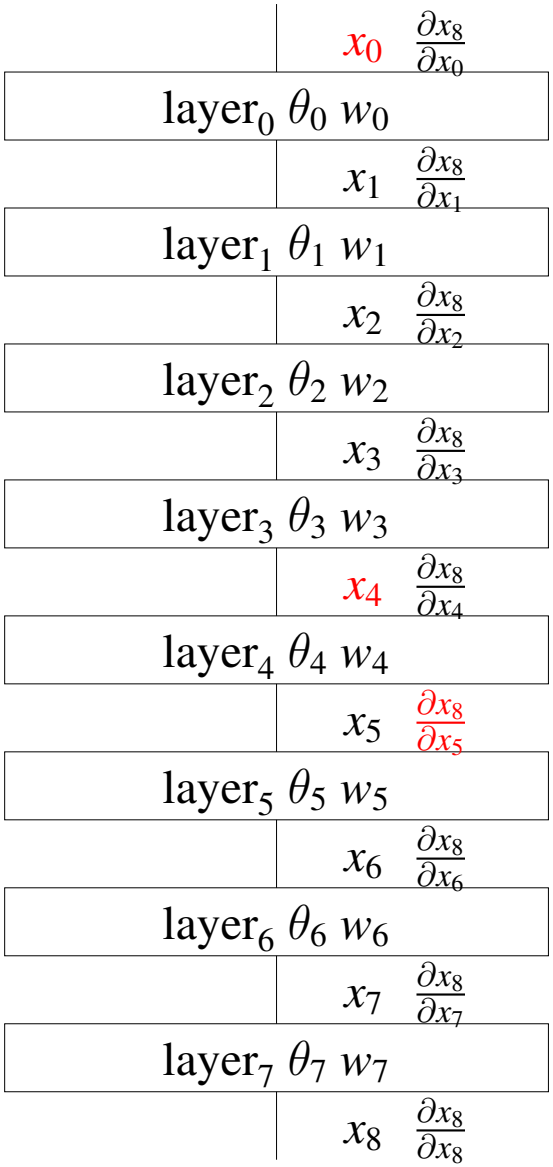

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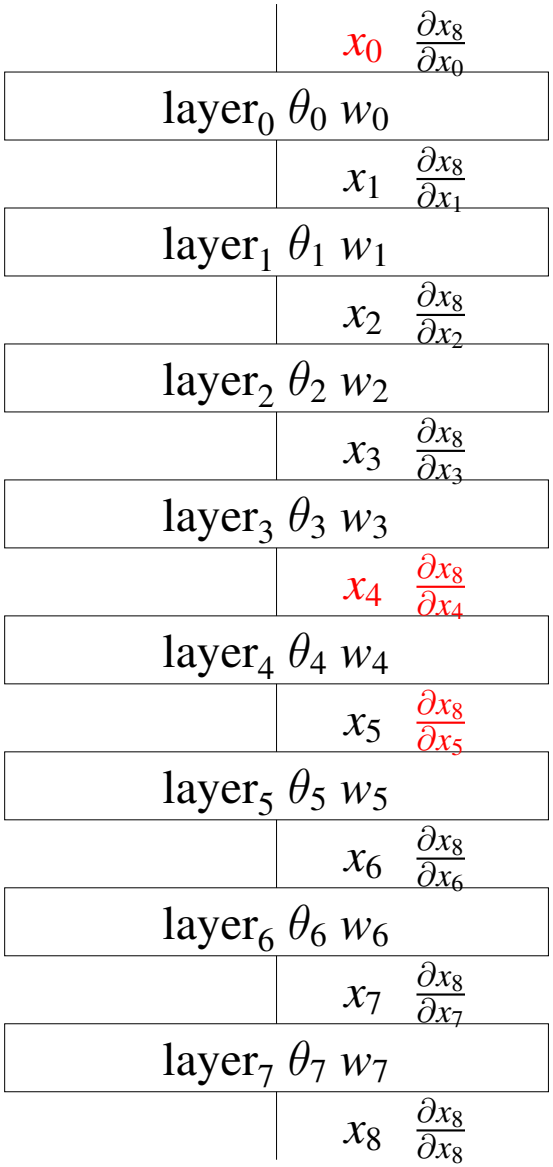

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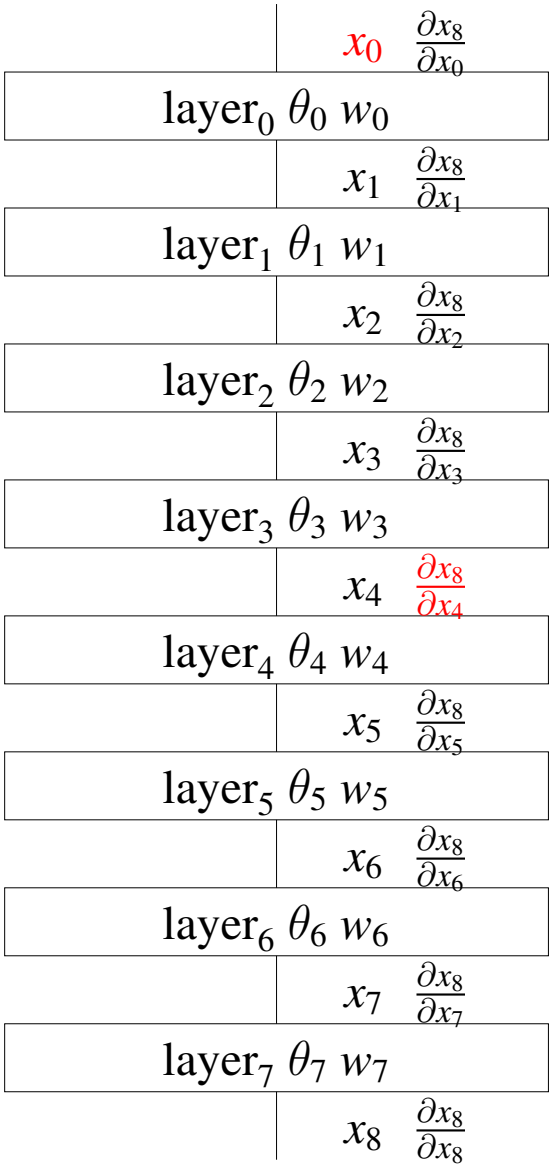

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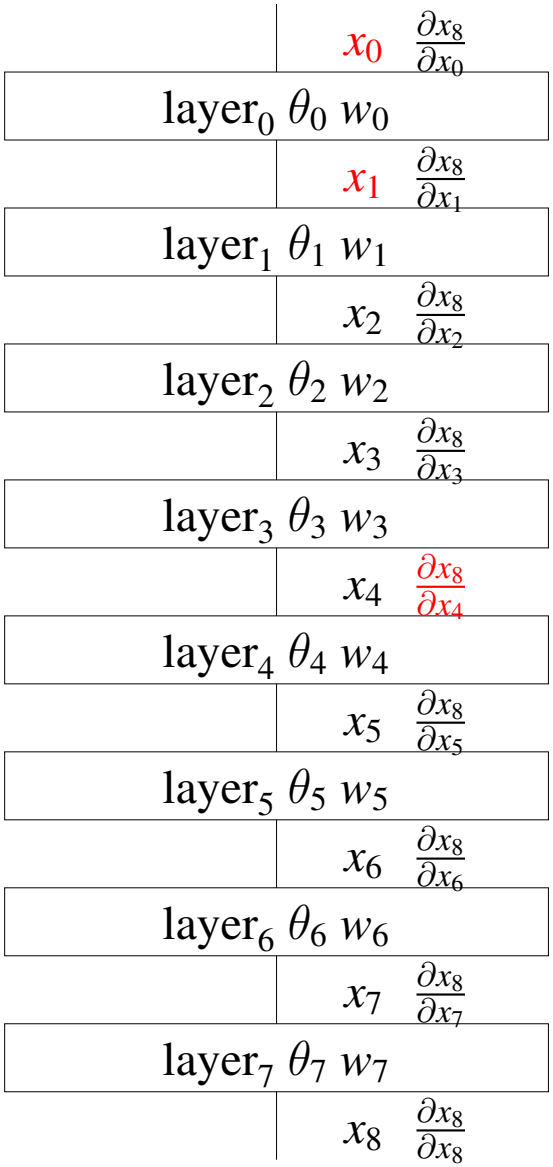

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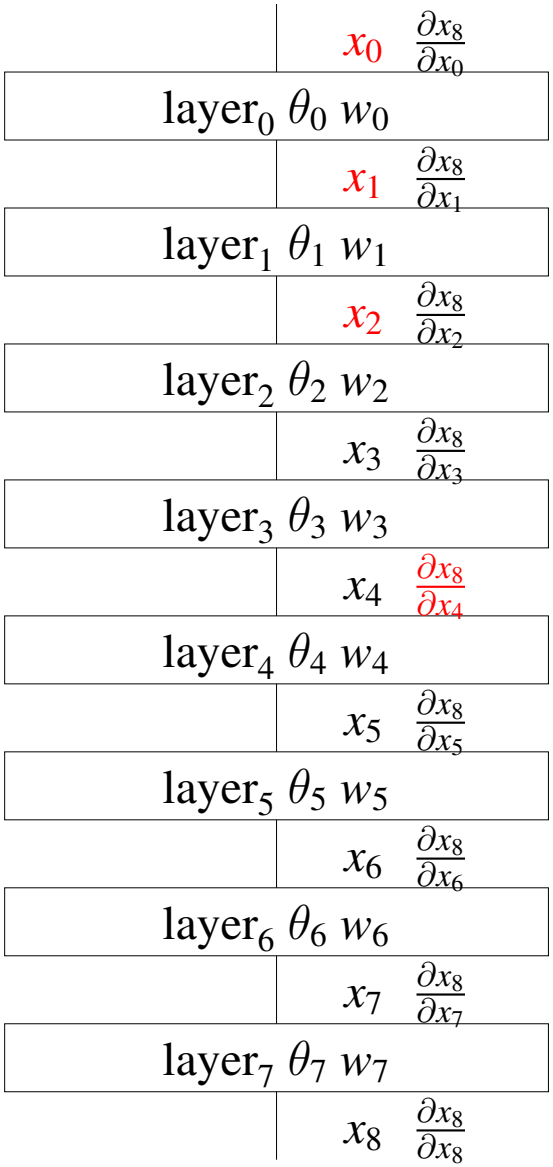

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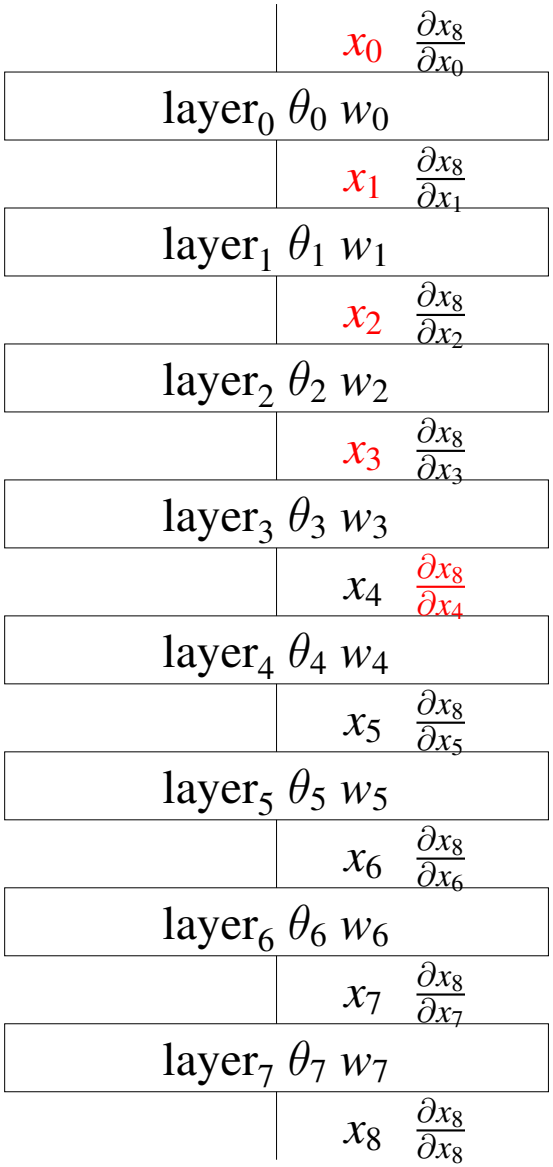

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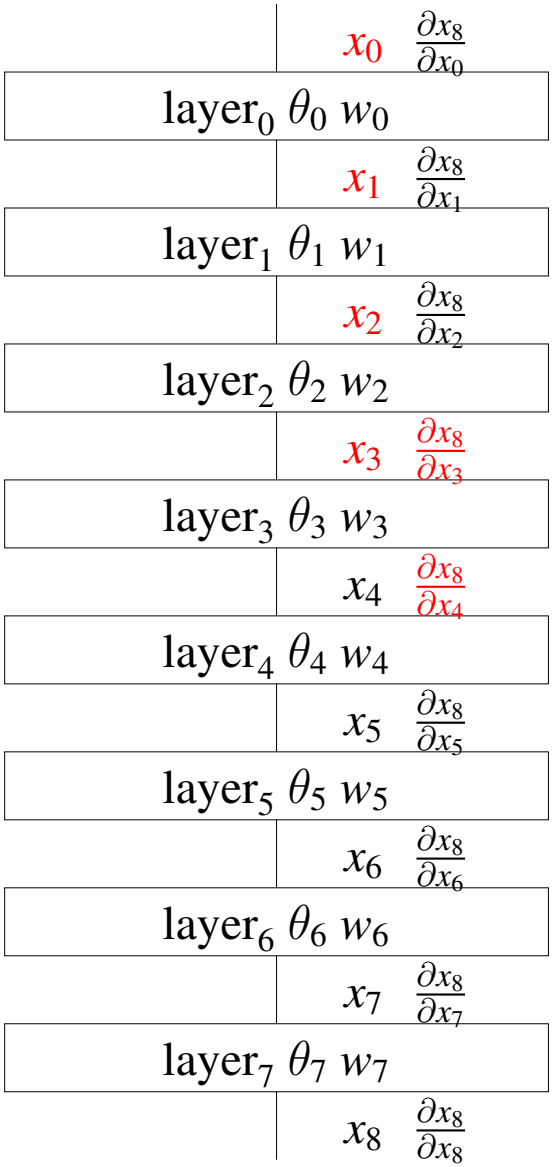

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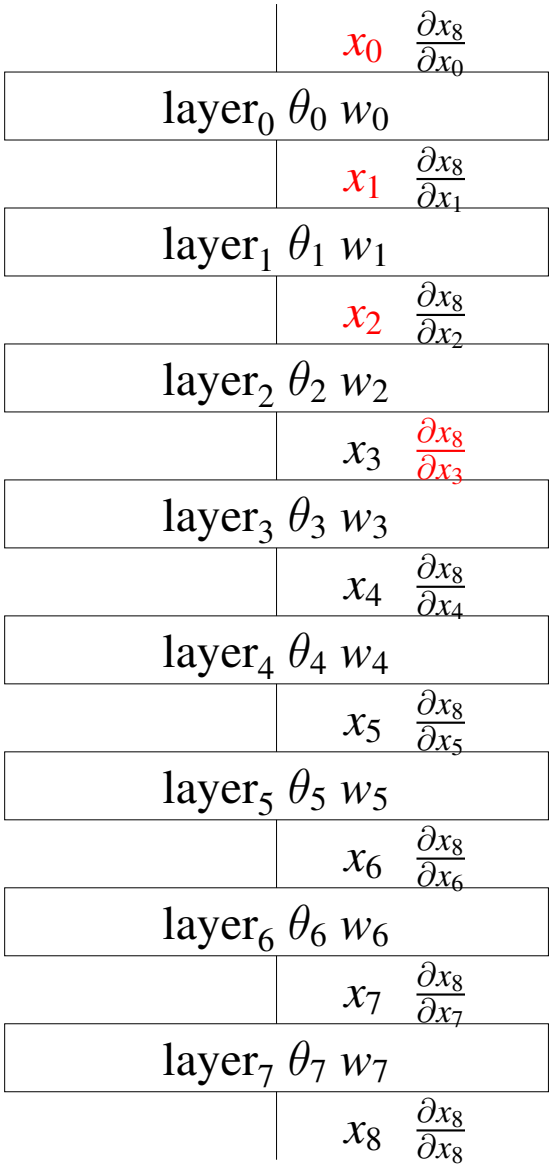

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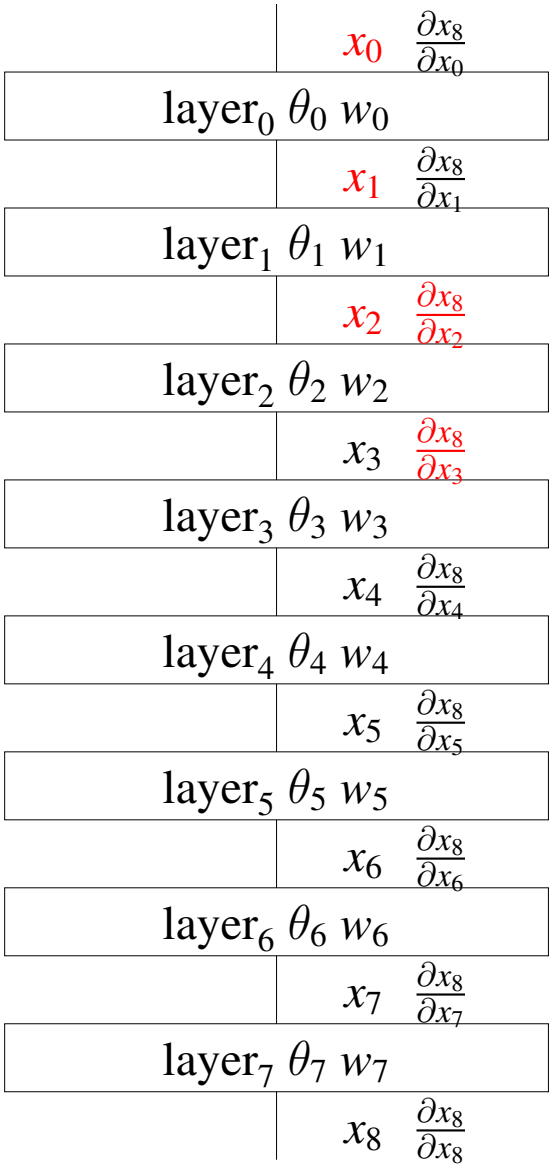

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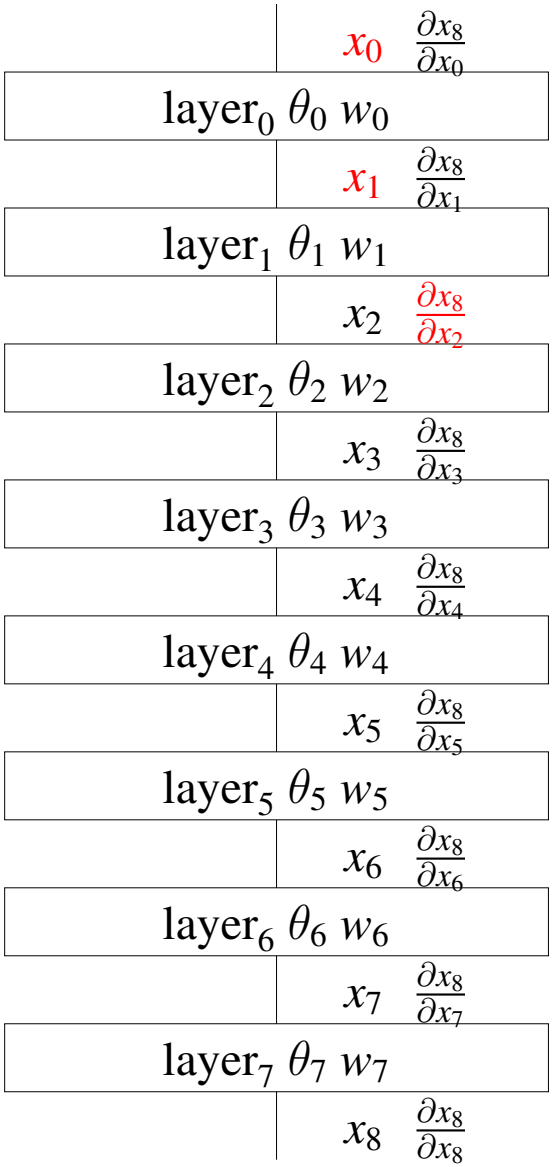

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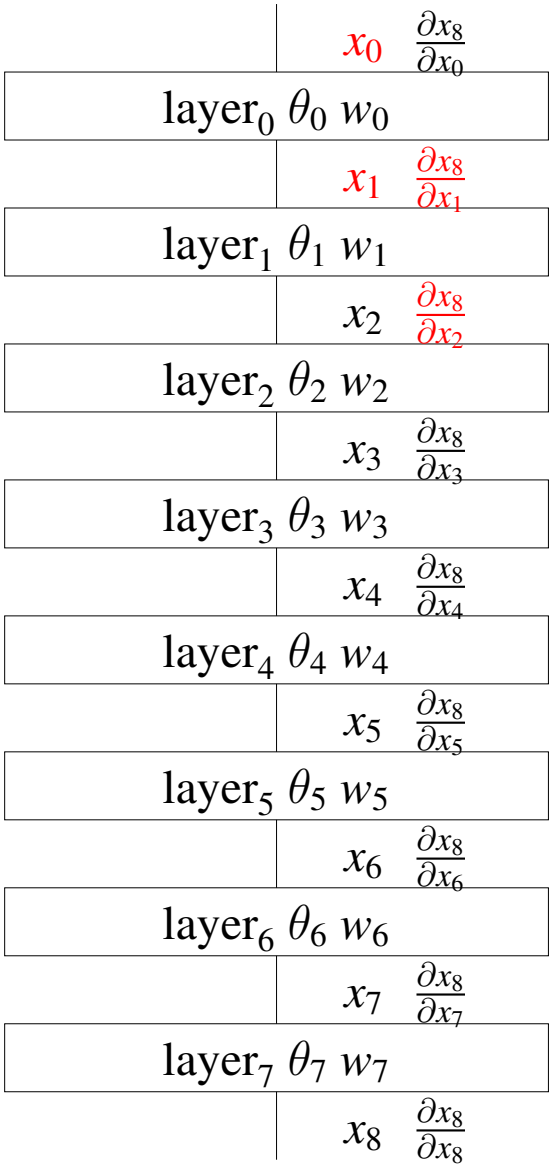

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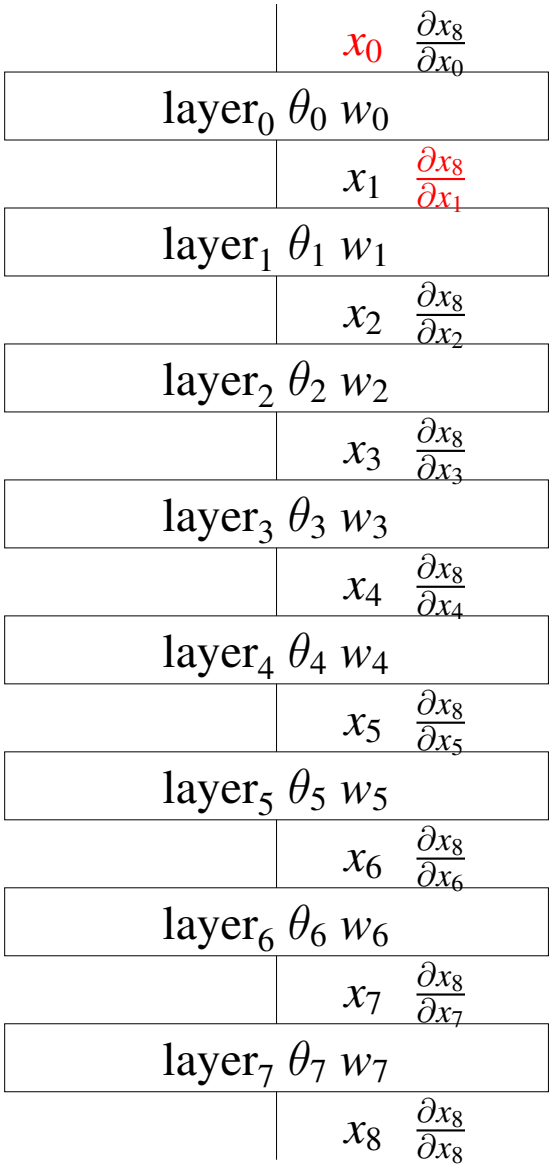

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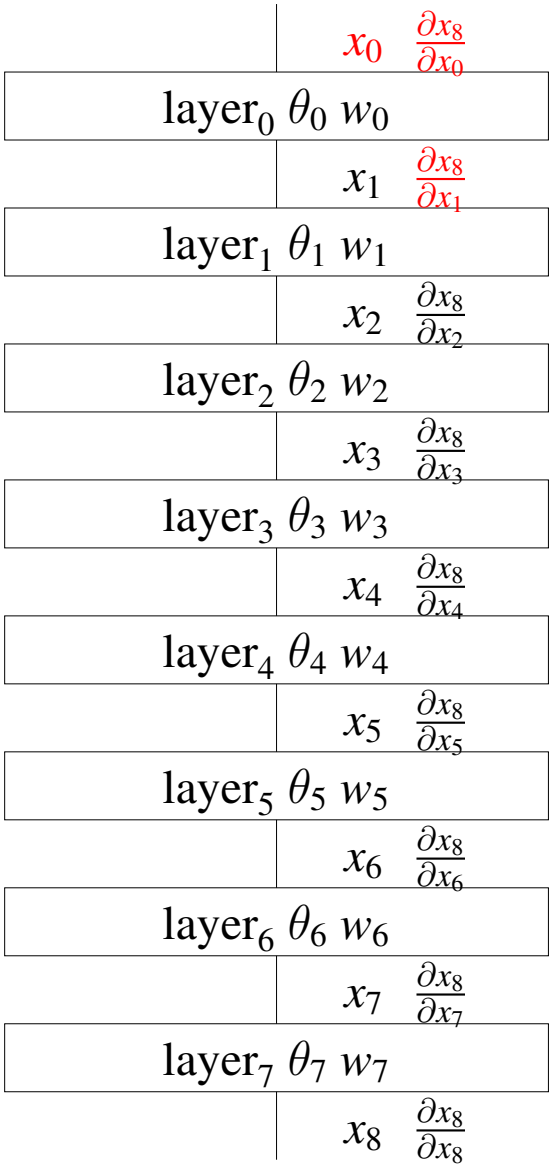

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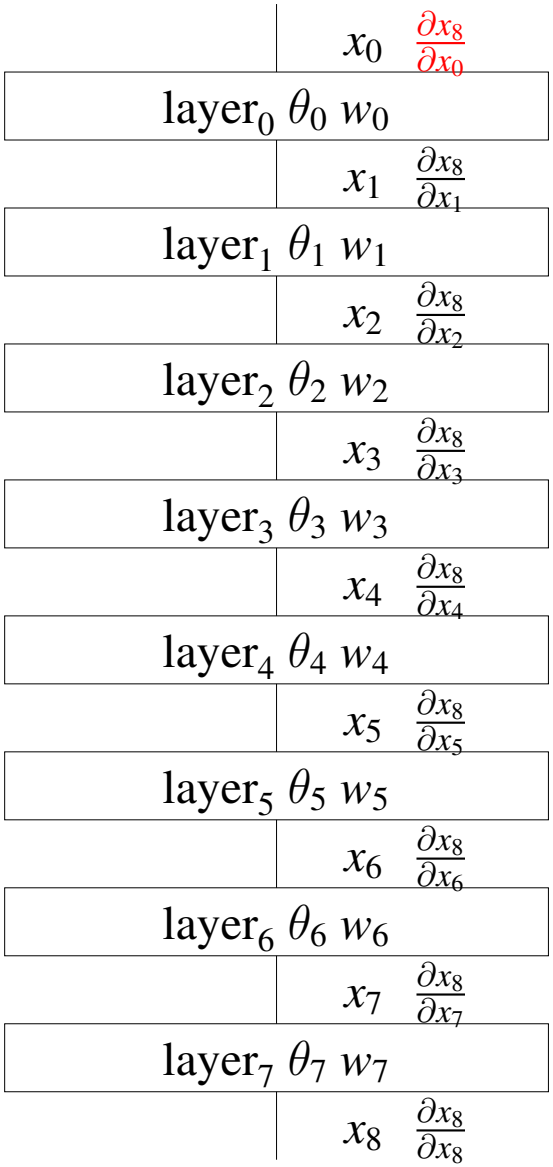

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- Can perform divide-and-conquer.

Divide-and-Conquer Checkpointing

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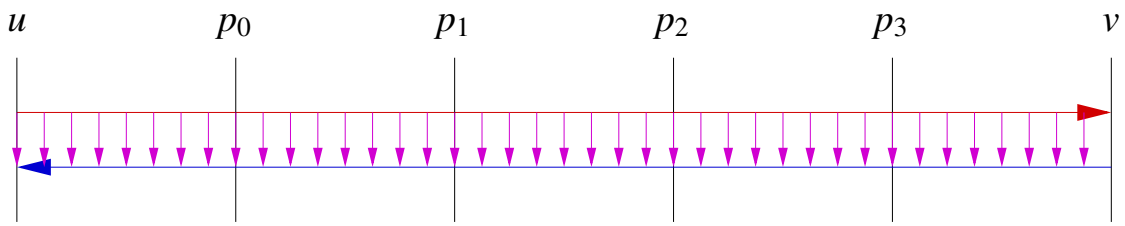

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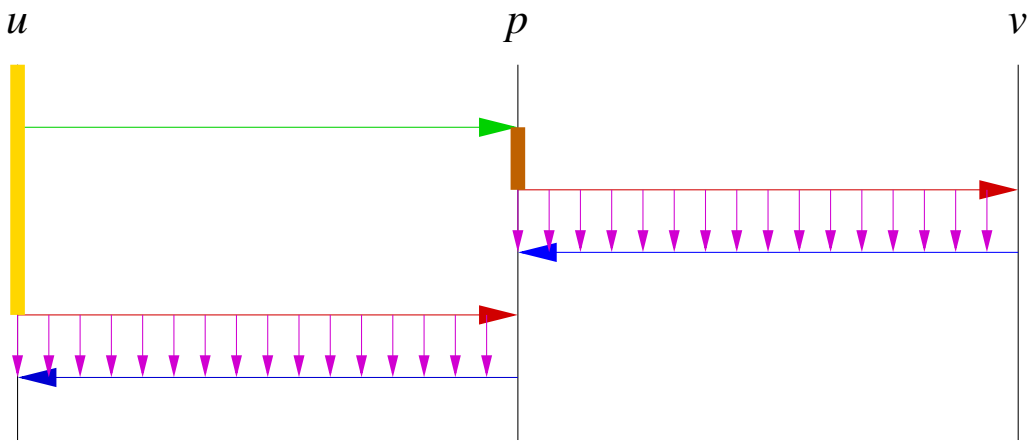

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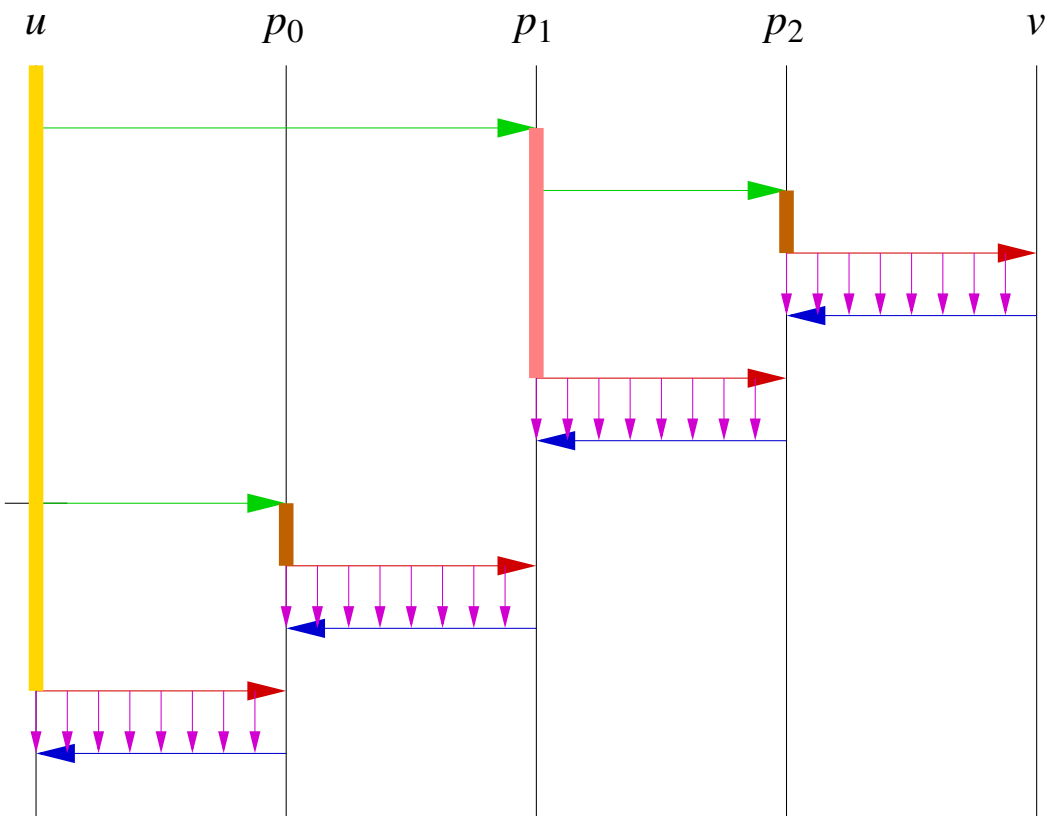

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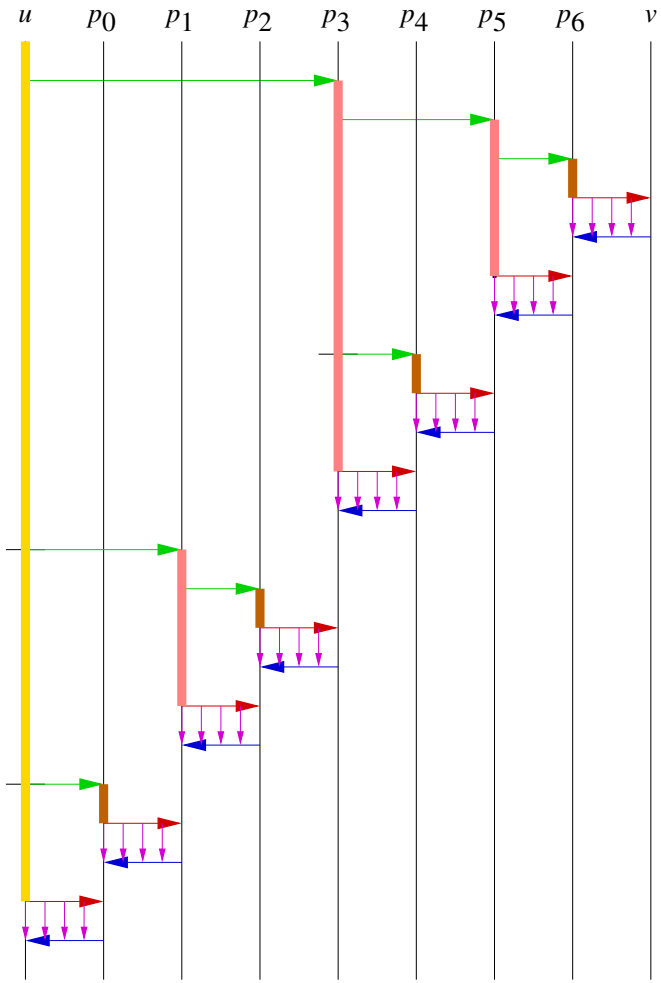

Complexity of Divide-and-Conquer Checkpointing

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A (Brief) History of Divide-and-Conquer Checkpointing

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Implemented for DO Loops

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```
      do 10 i=1, n
         ...
10    continue
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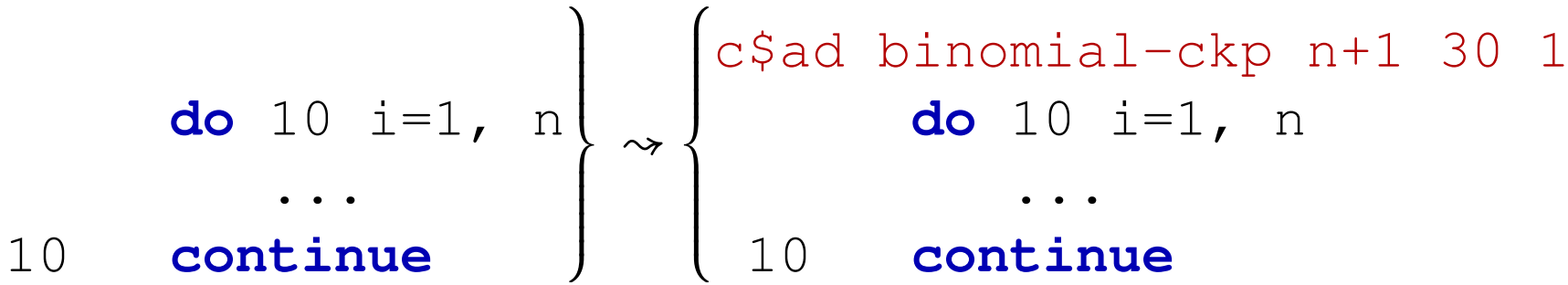

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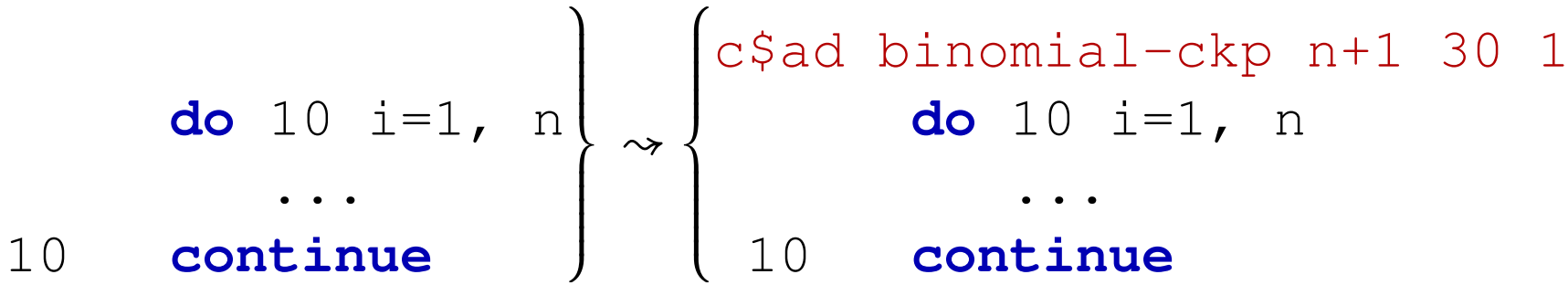

https://www-sop.inria.fr/tropics/tapenade/faq.html

Assuming that the final number of iterations N is known, and assuming that each iteration has the same runtime cost,

Desiderata

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A (deep) neural network has no loops (except inside primitives).

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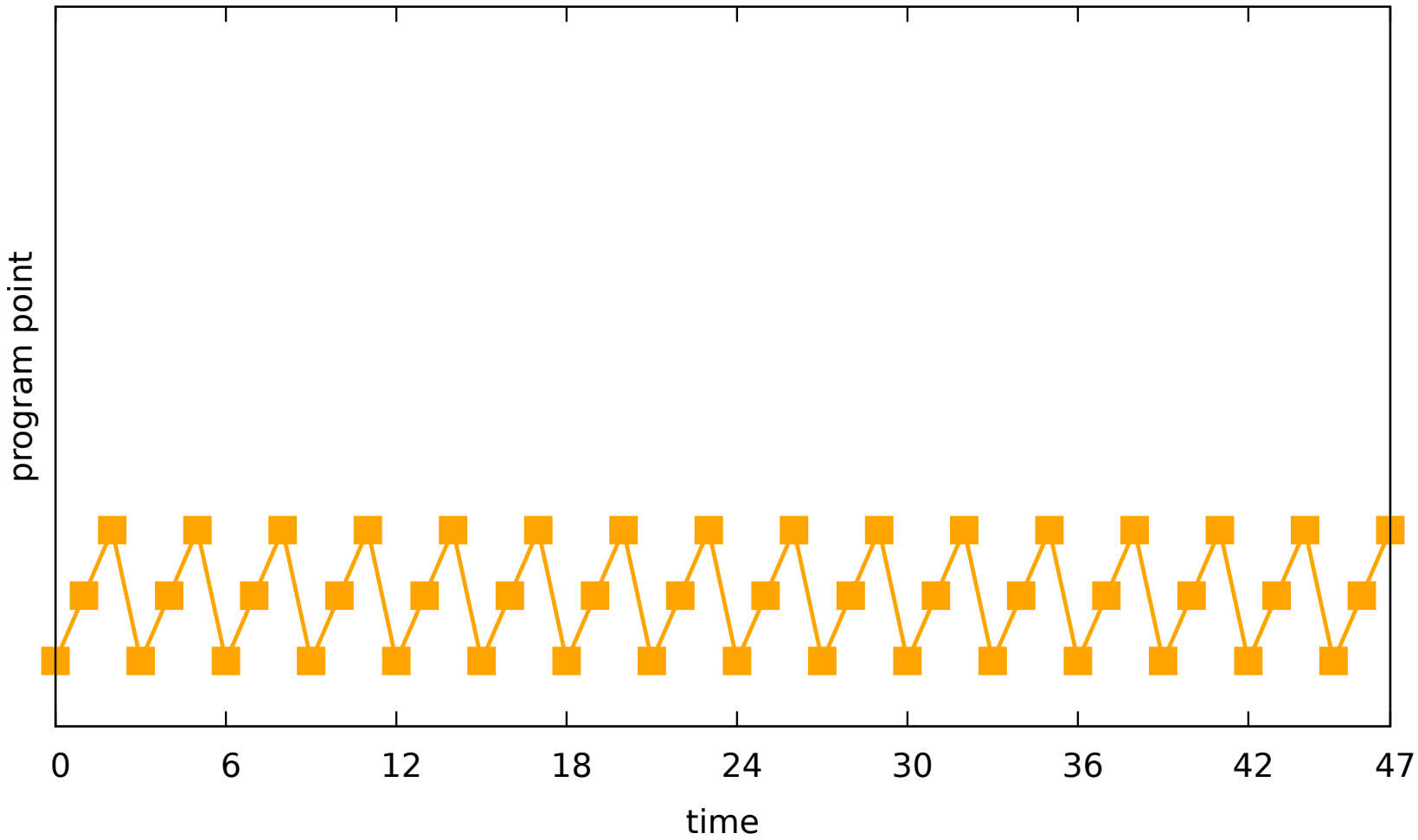
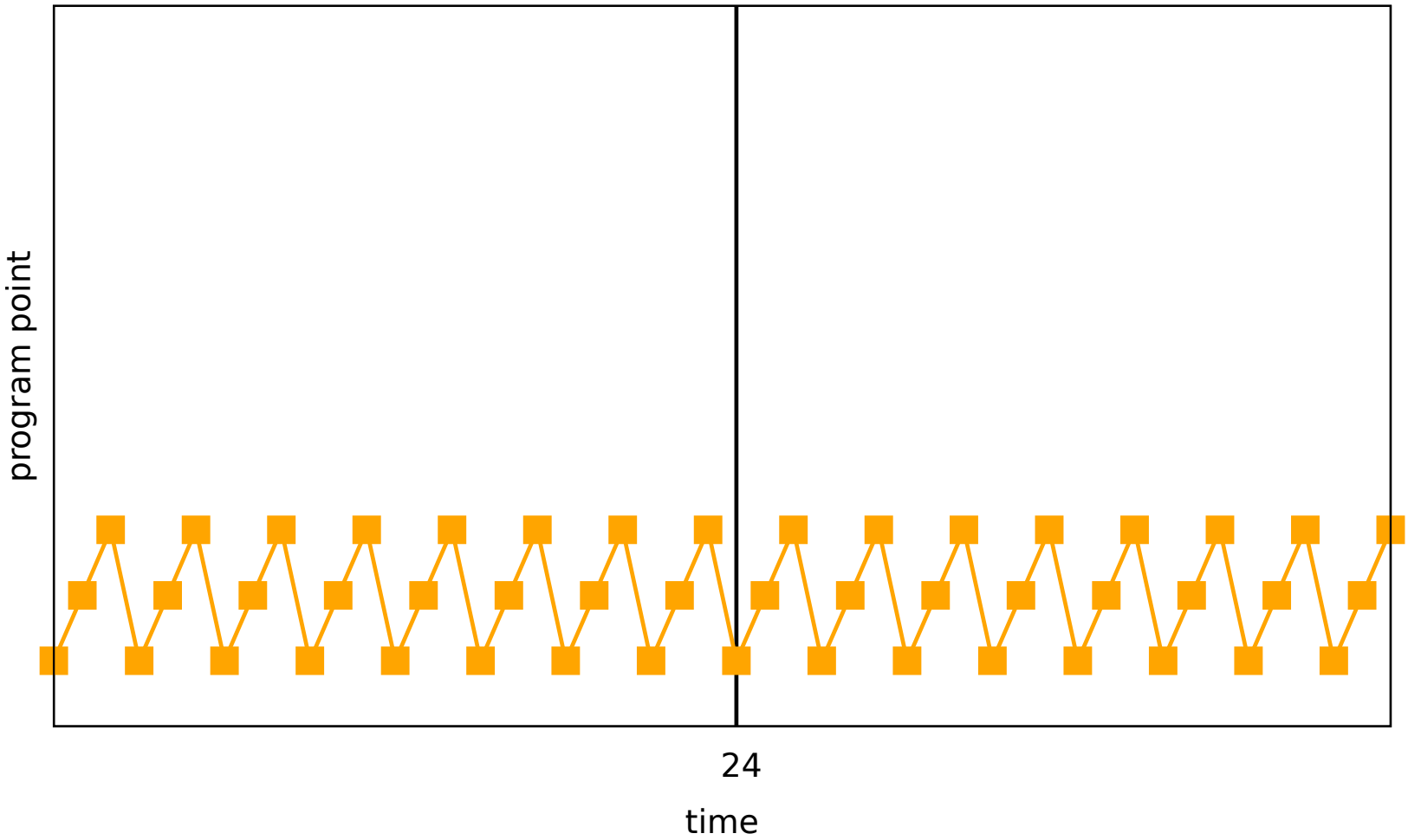
Want to implement for arbitrary code (not just a single DO loop).

Execution Trace of Loop



Execution Trace of Loop

Easy to make regular and uniform checkpoints



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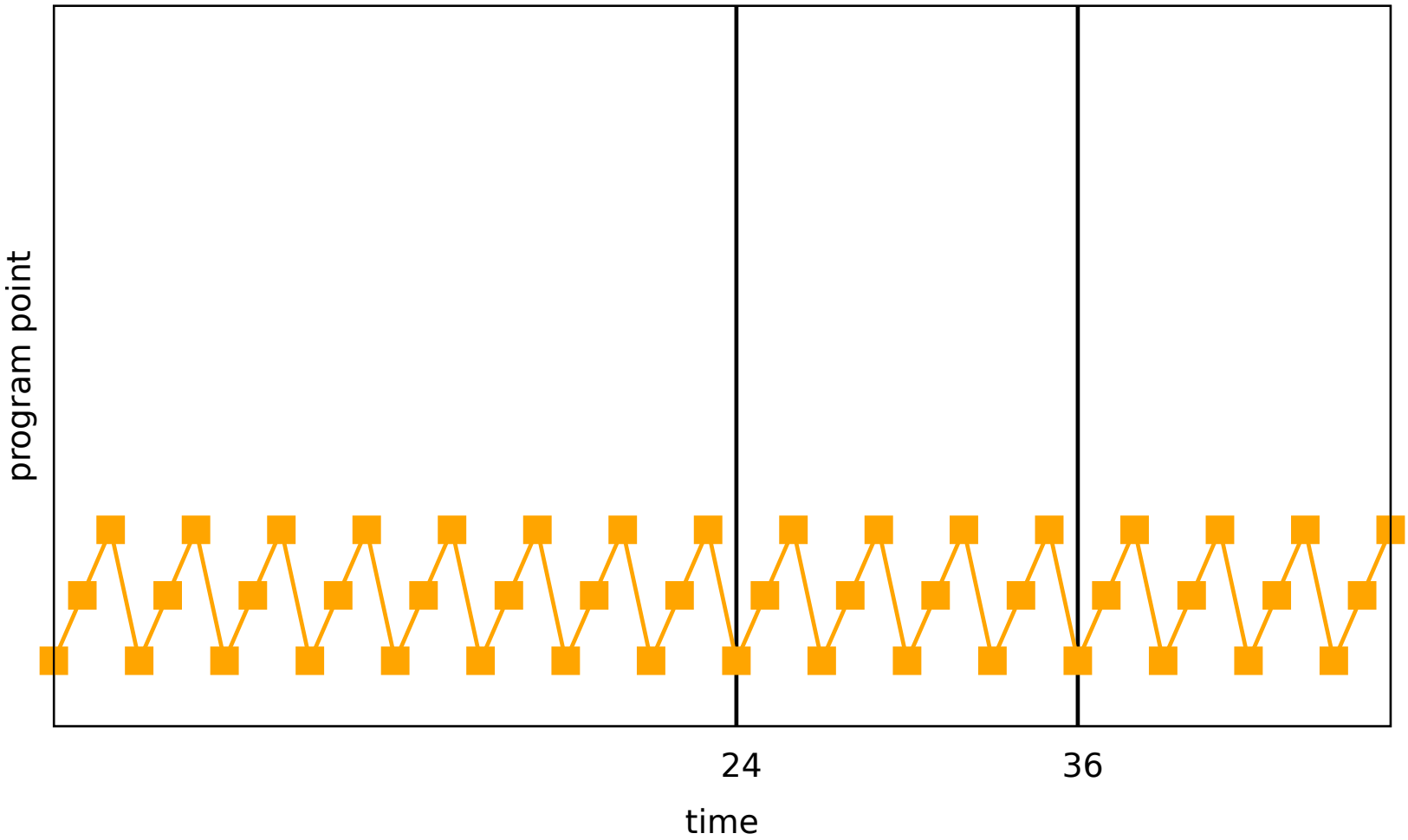

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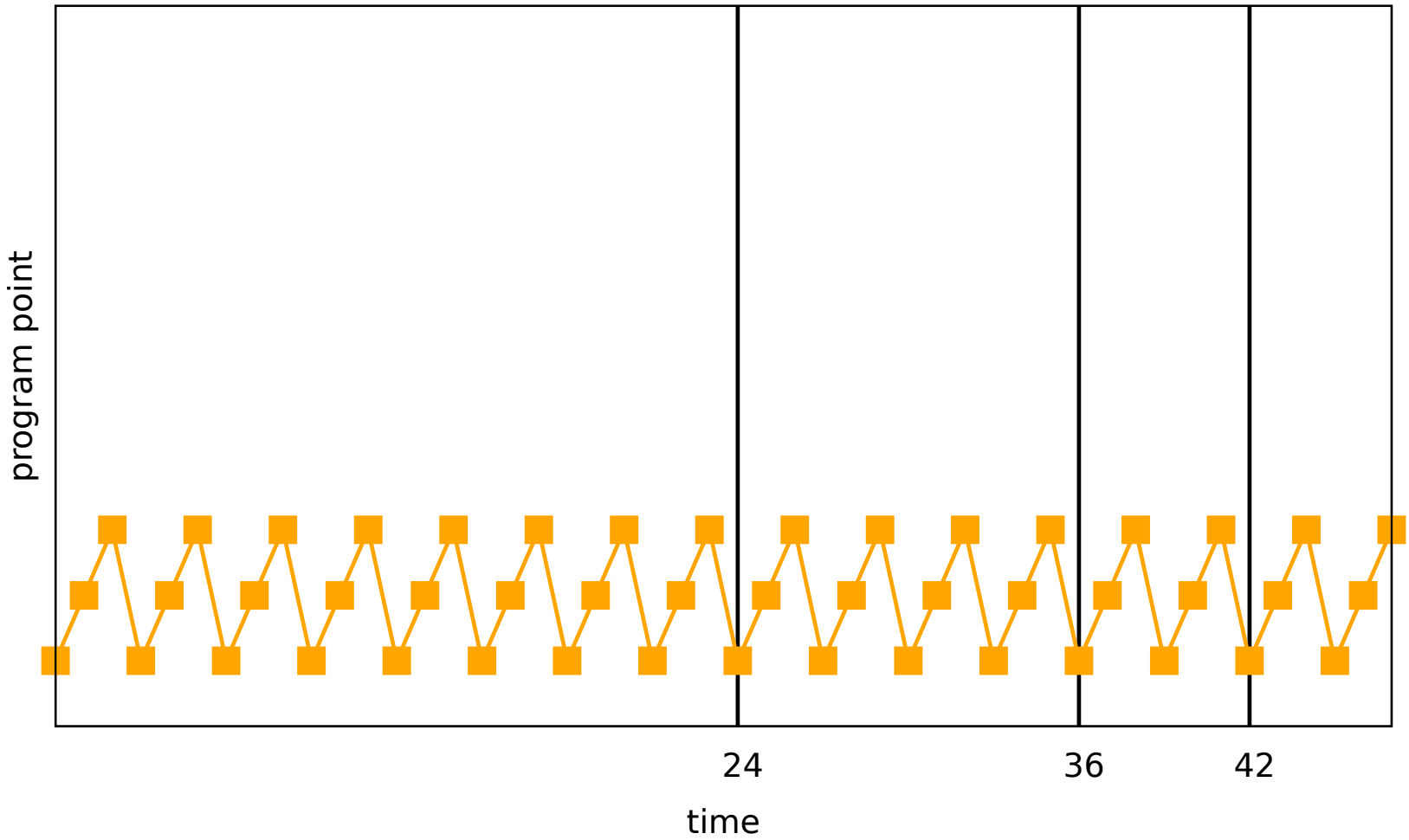

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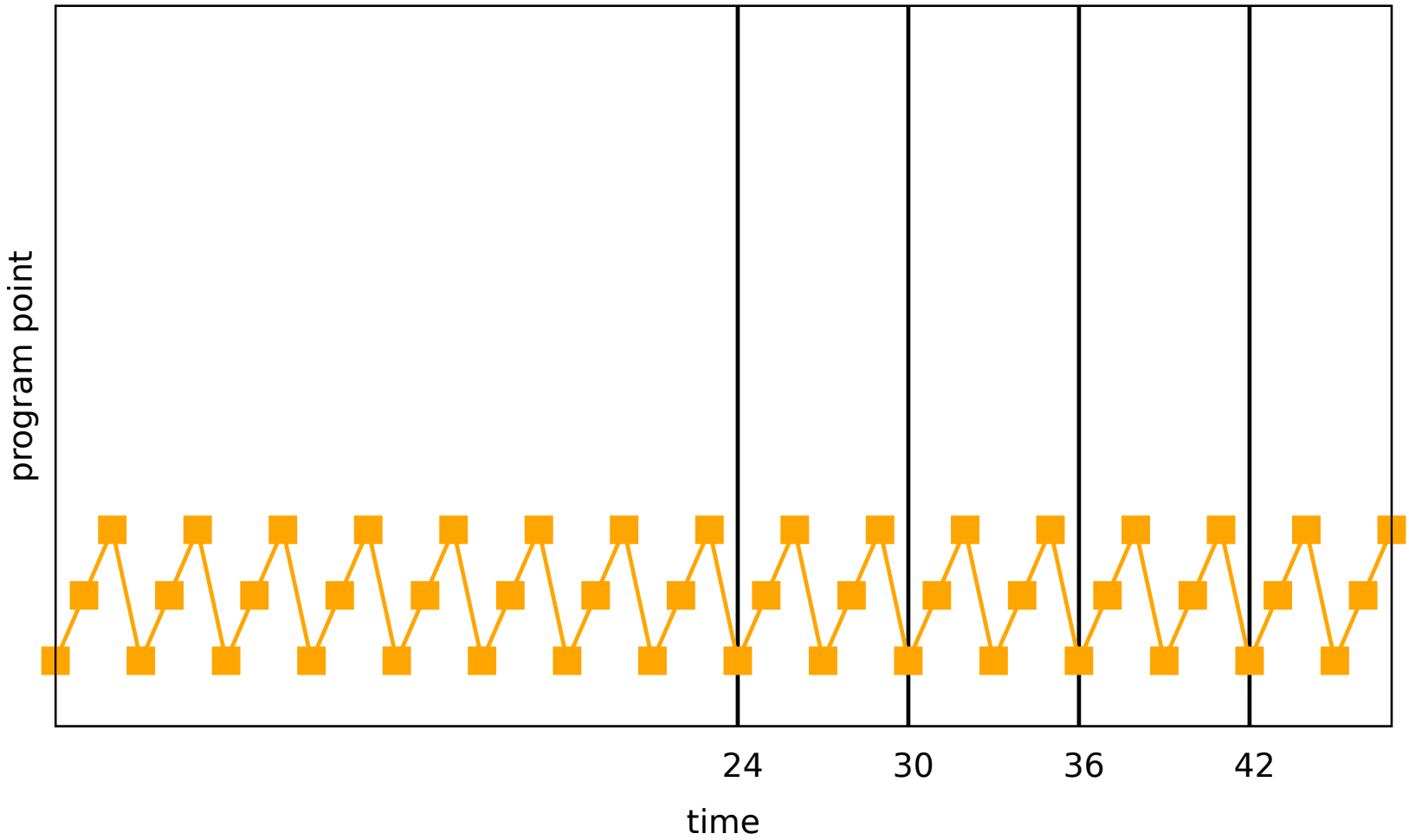

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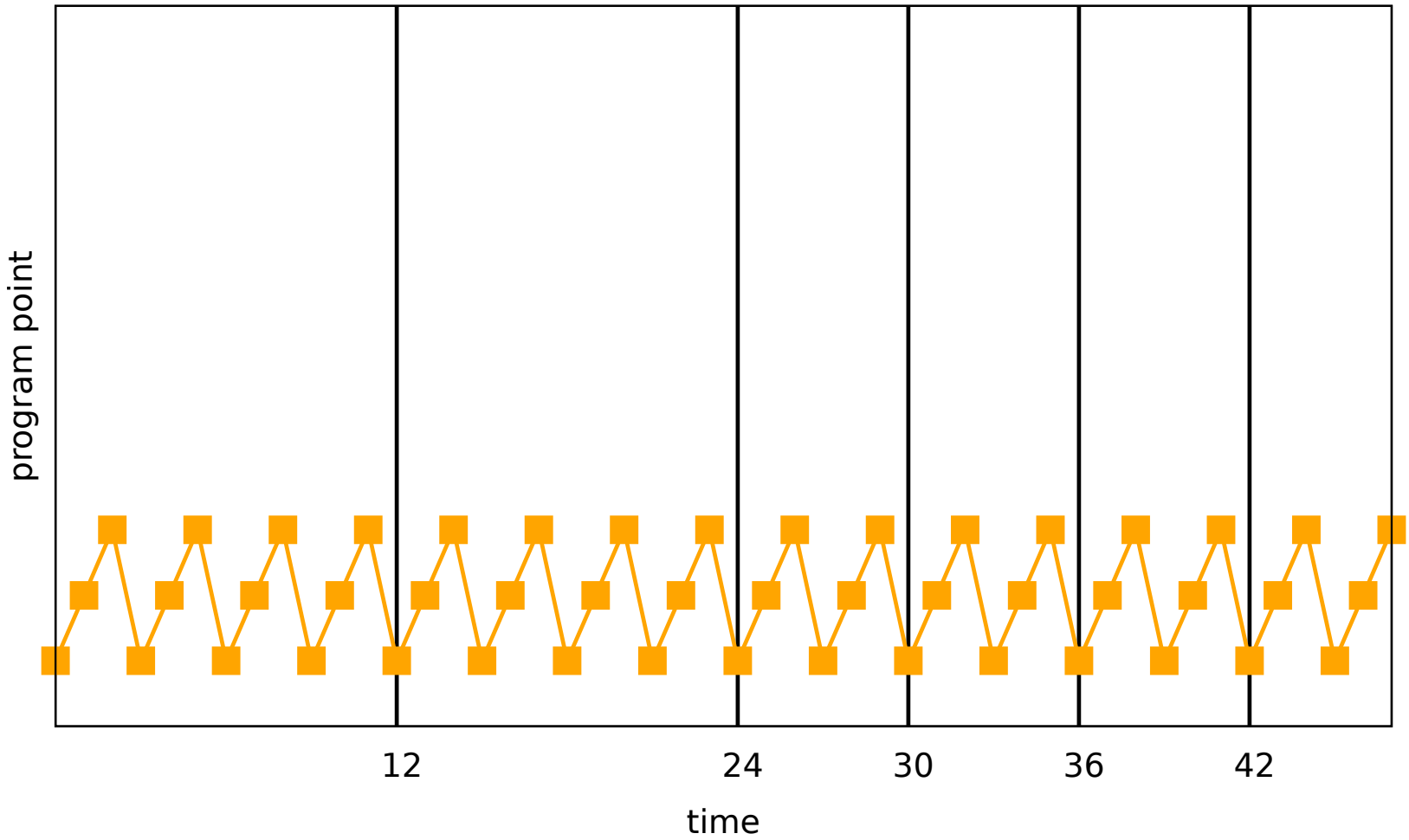

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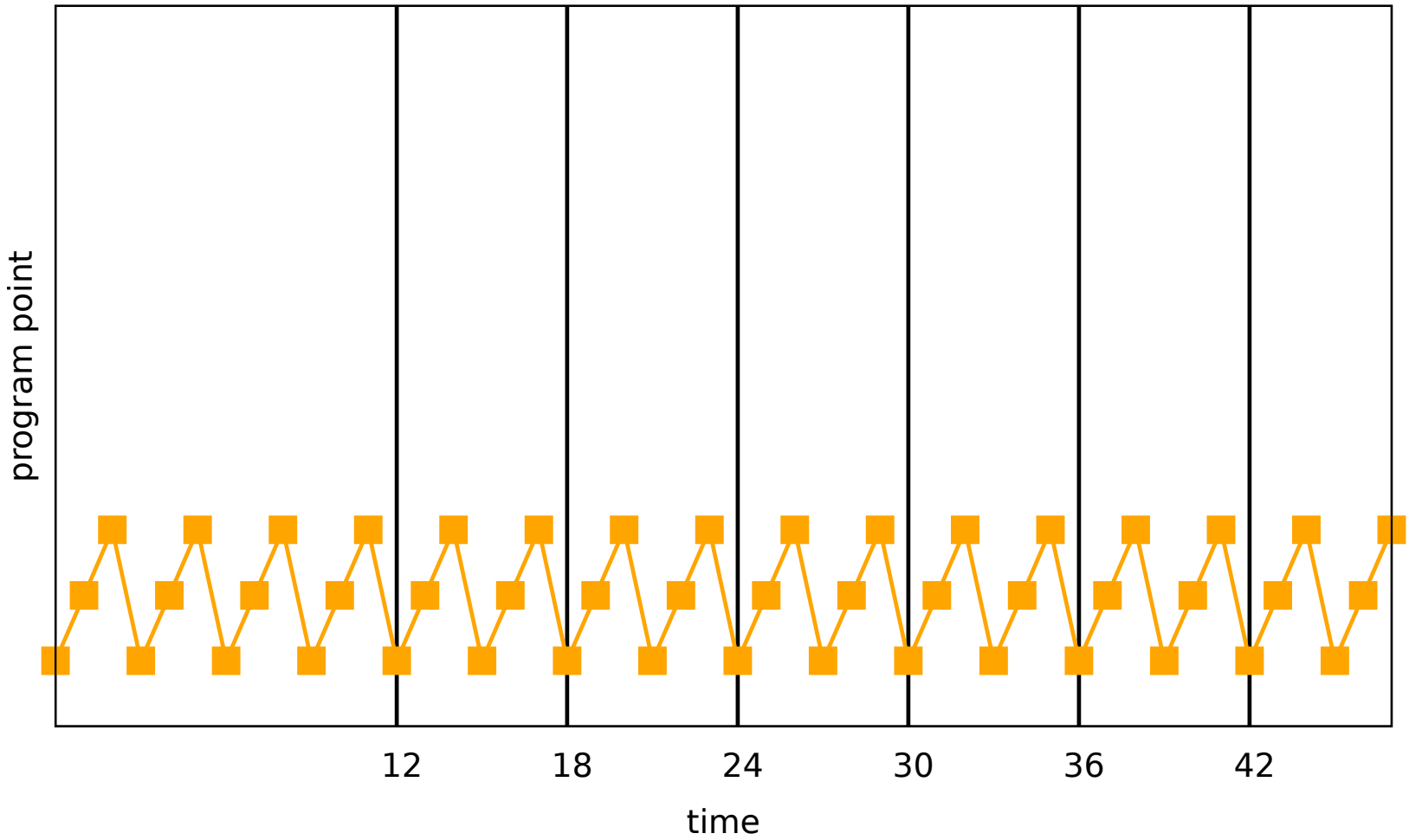

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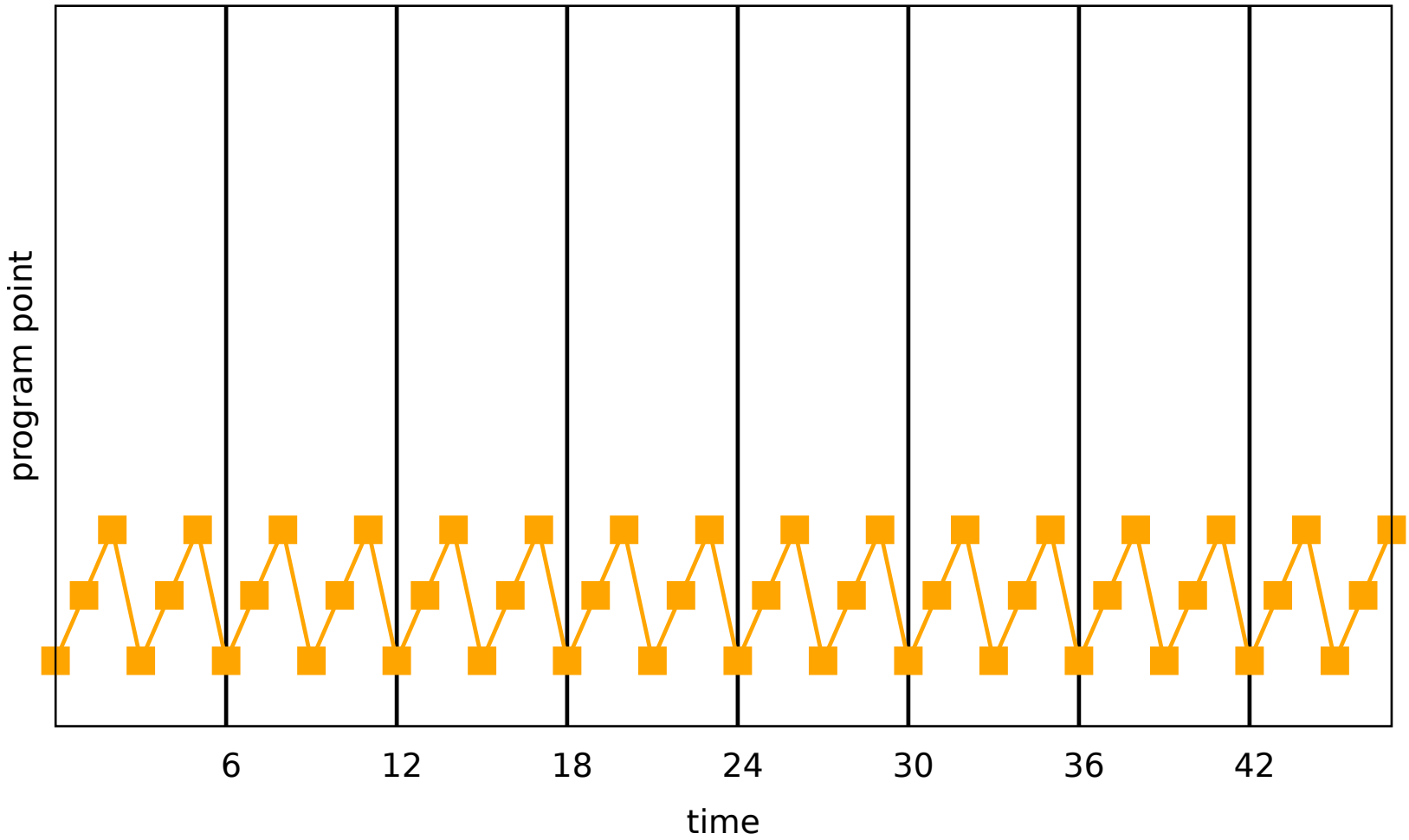
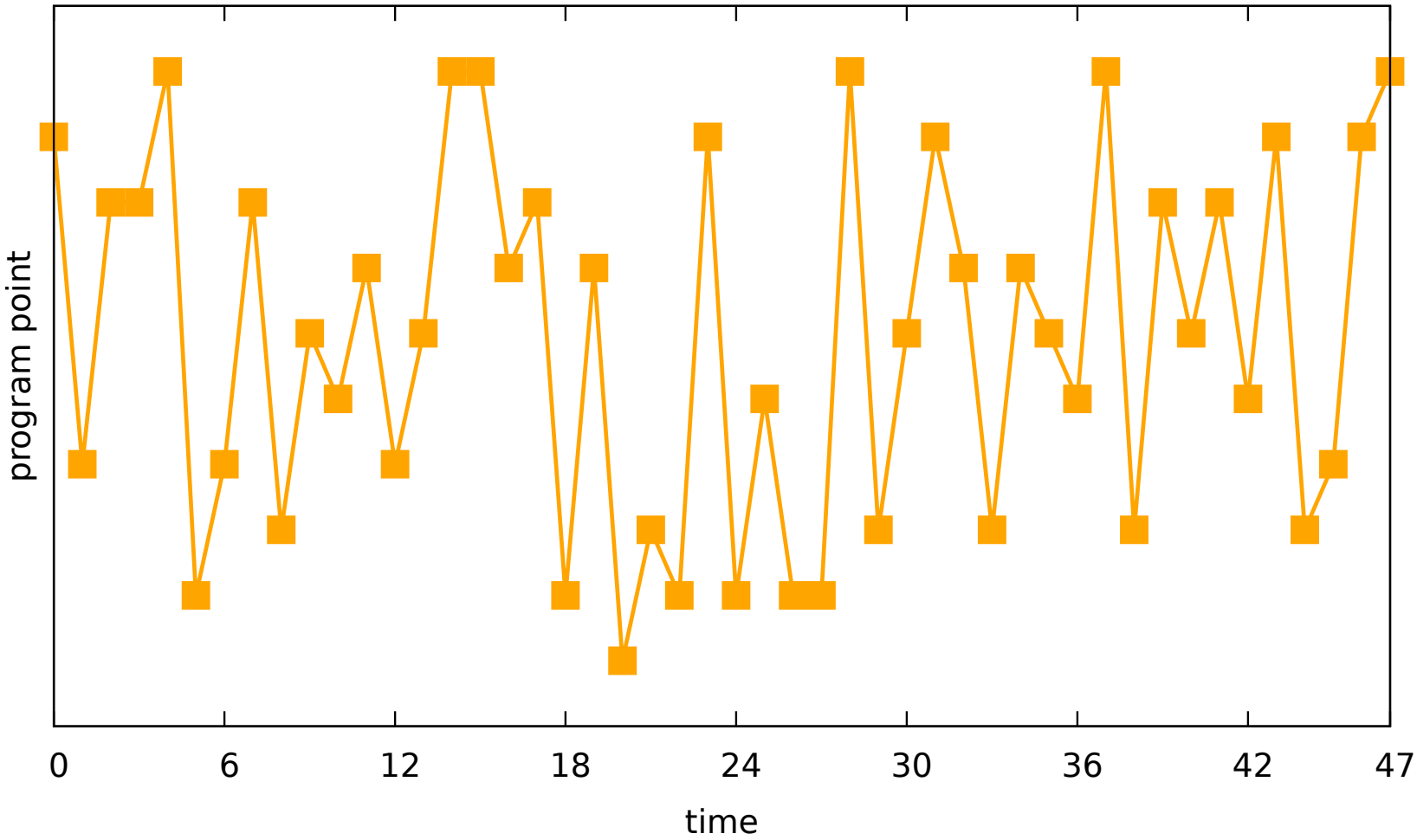


Execution Trace of Arbitrary Code



Execution Trace of Arbitrary Code

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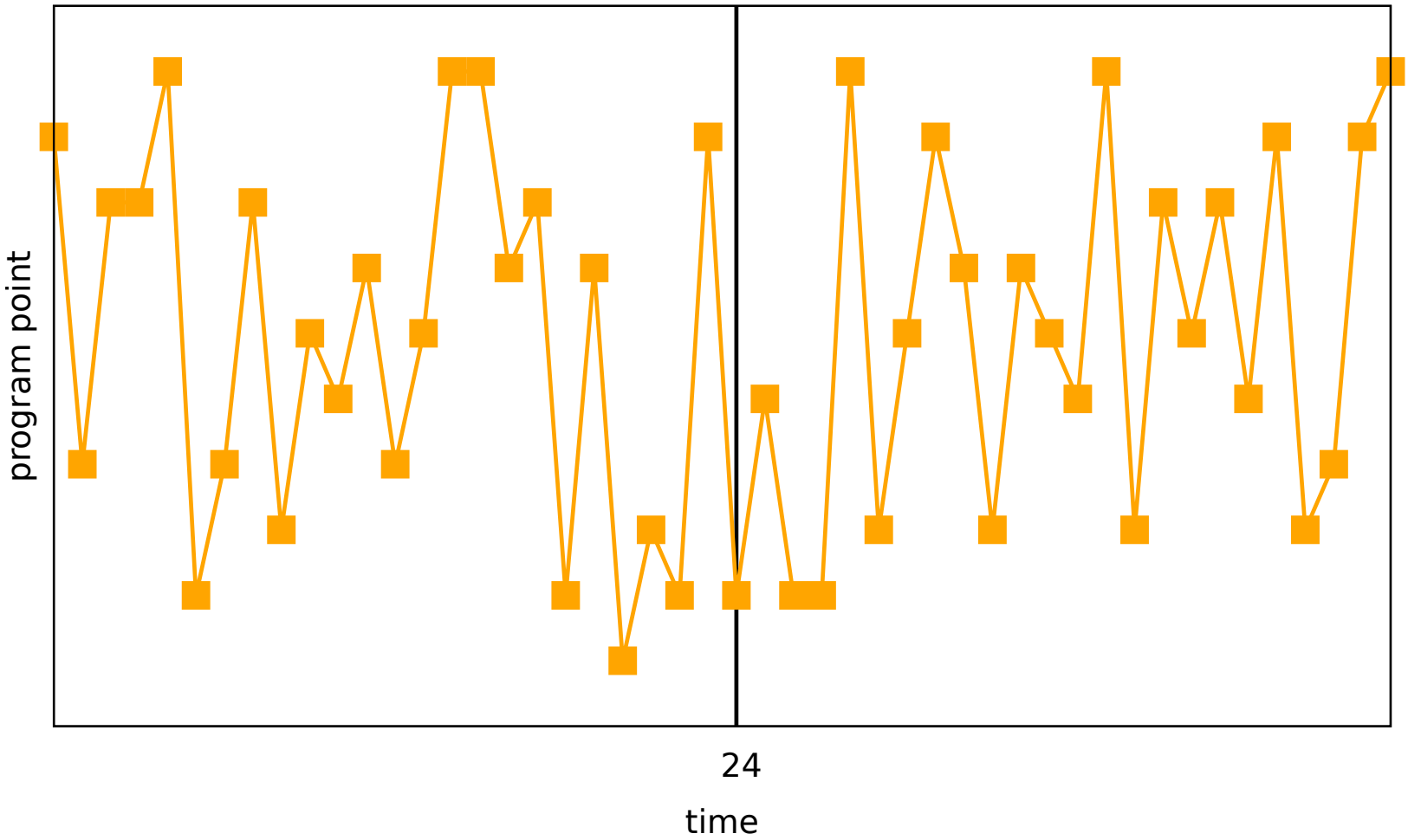

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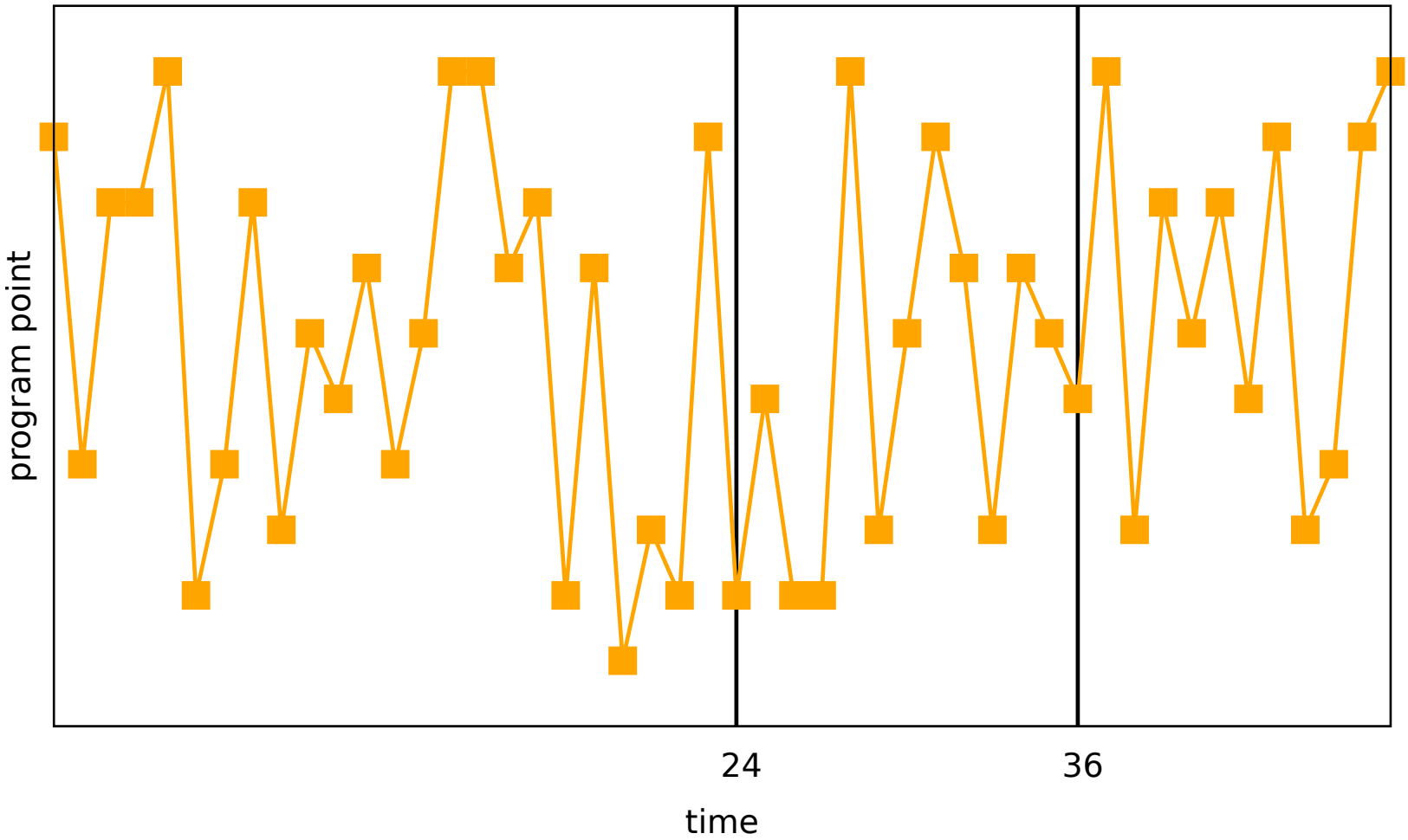

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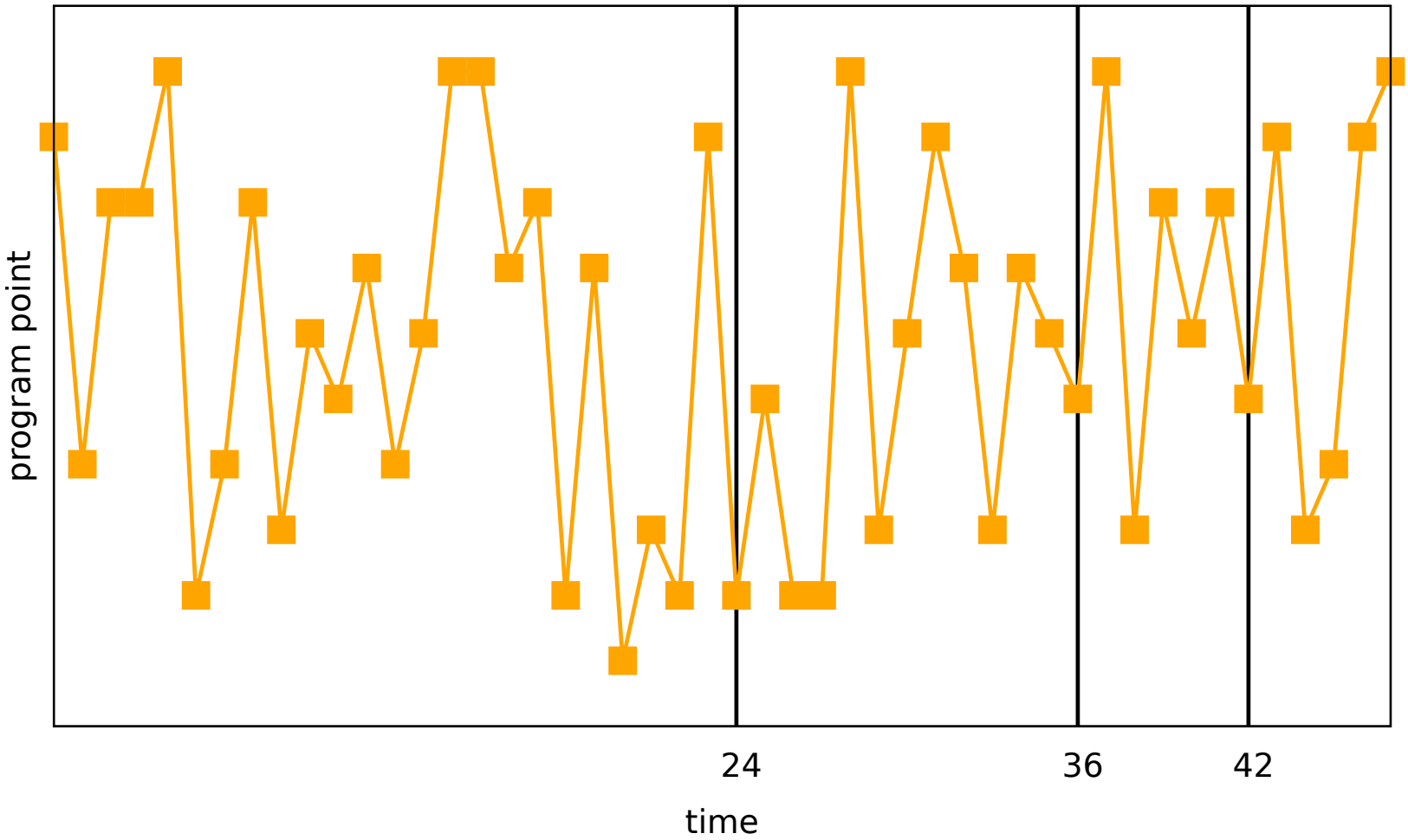

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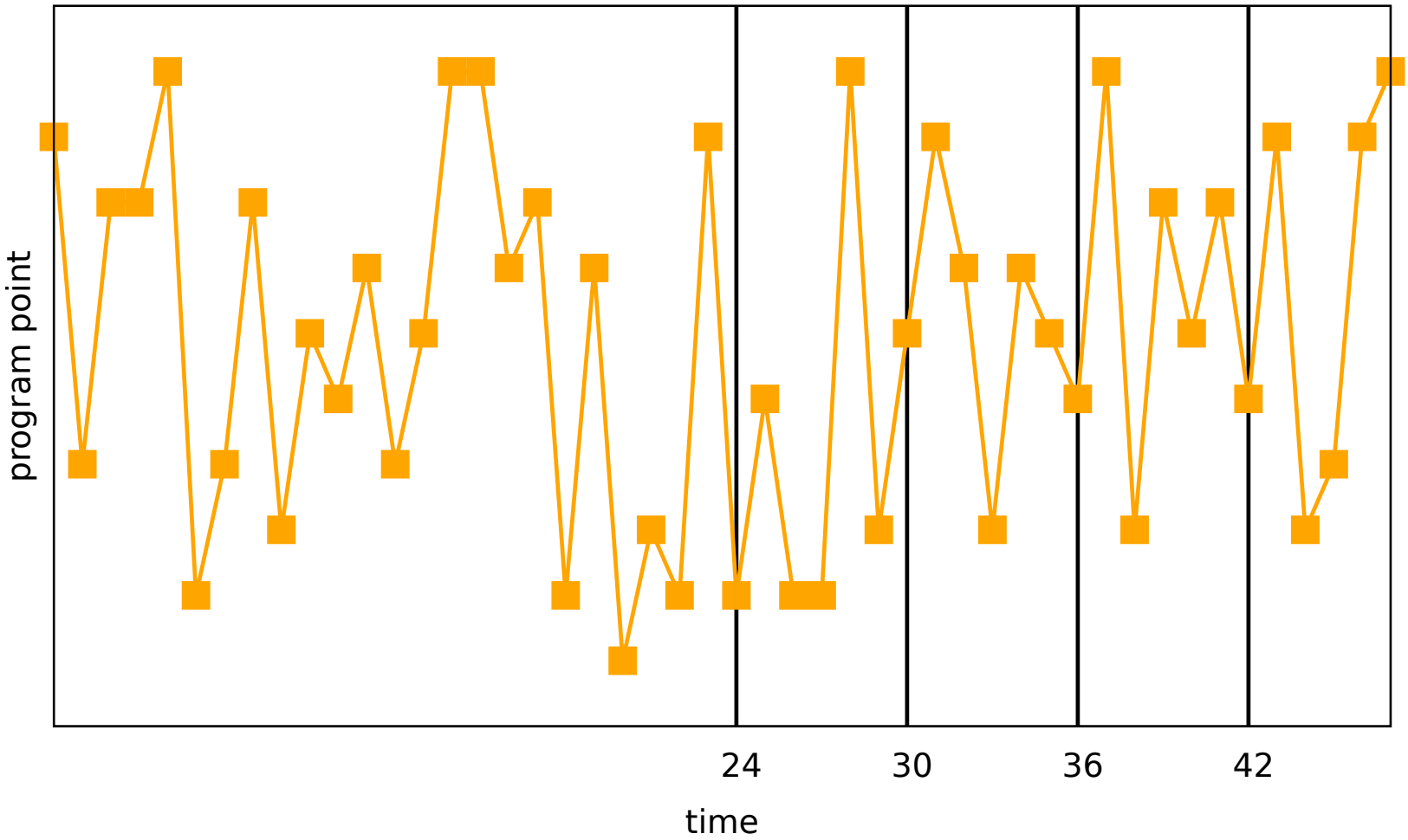

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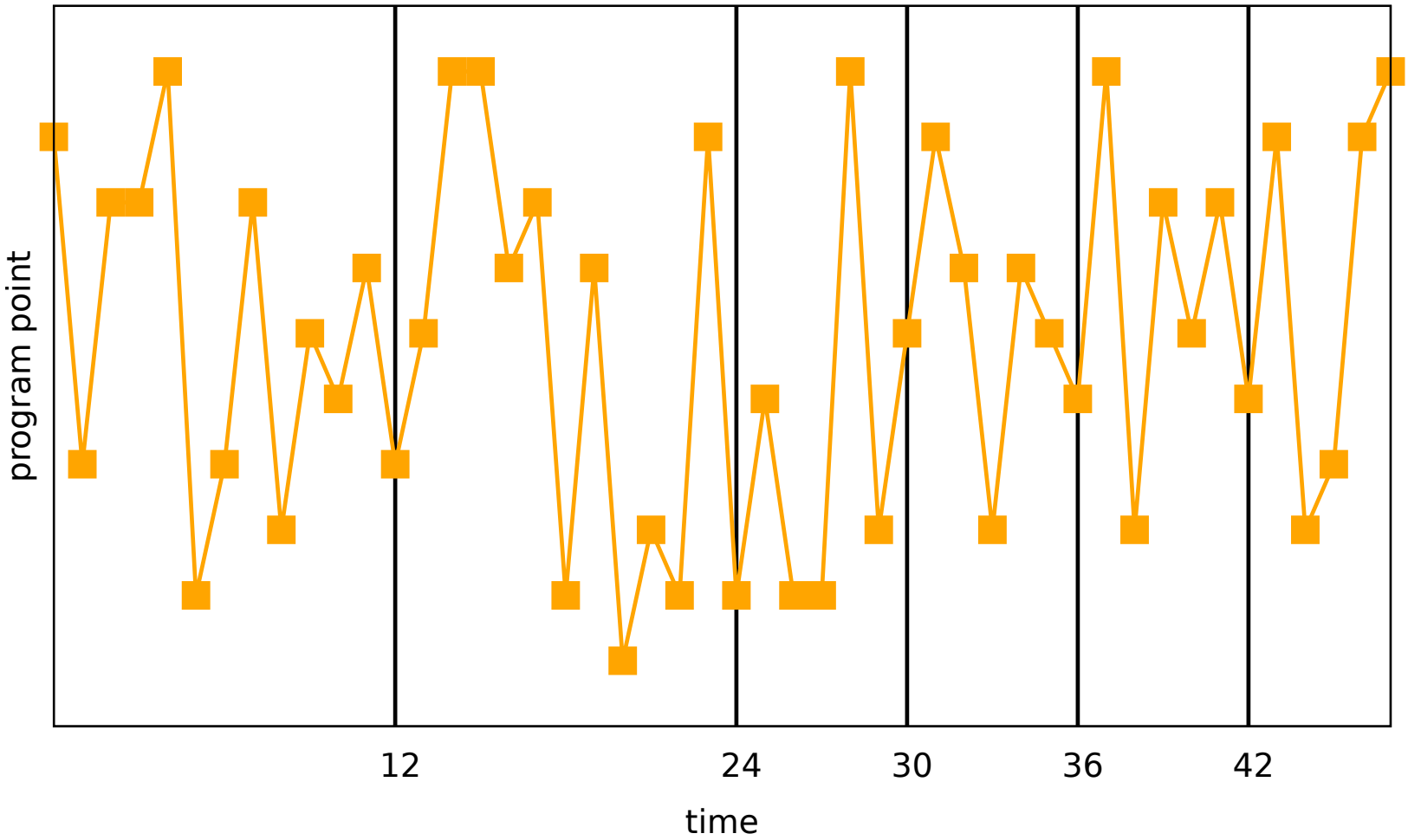

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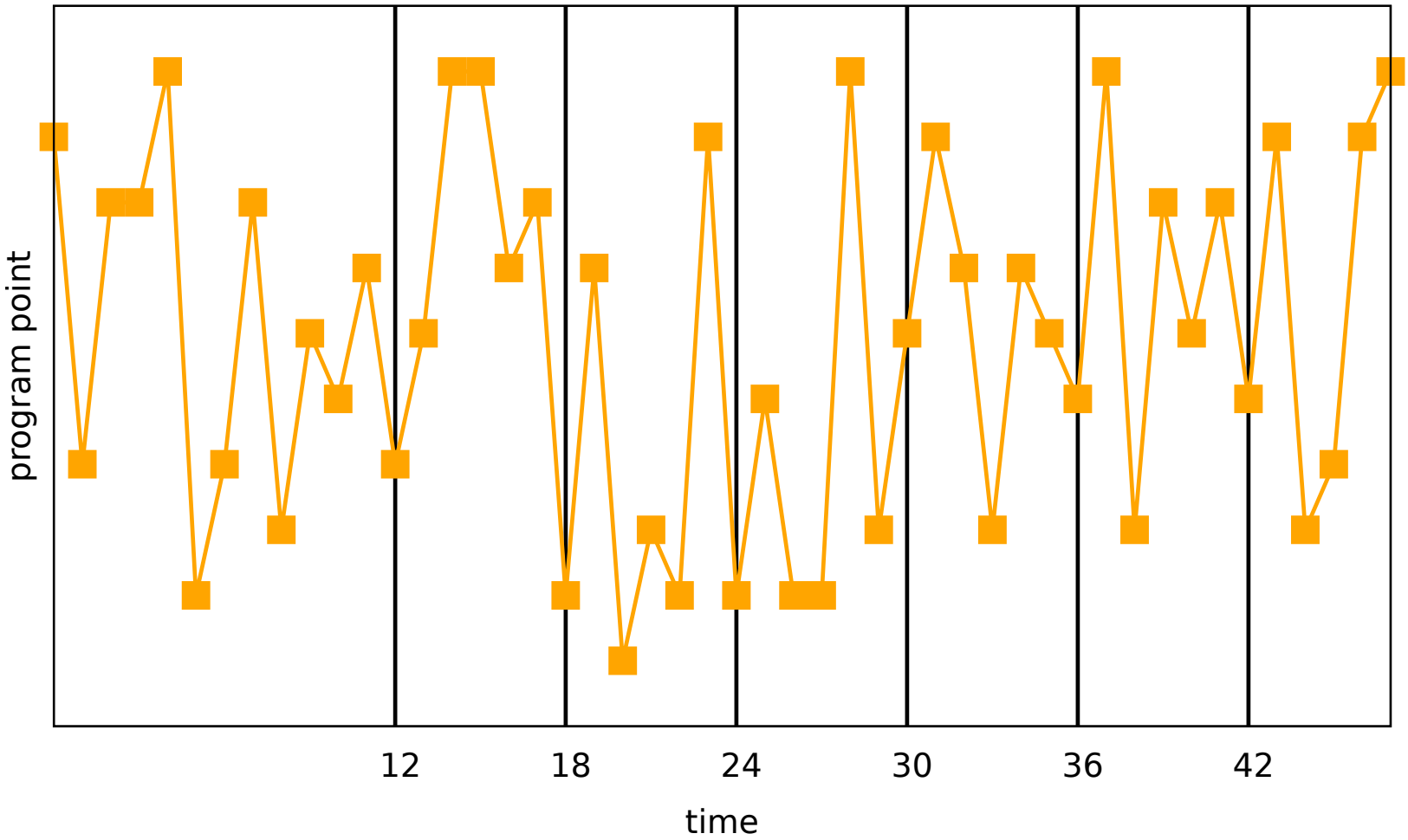

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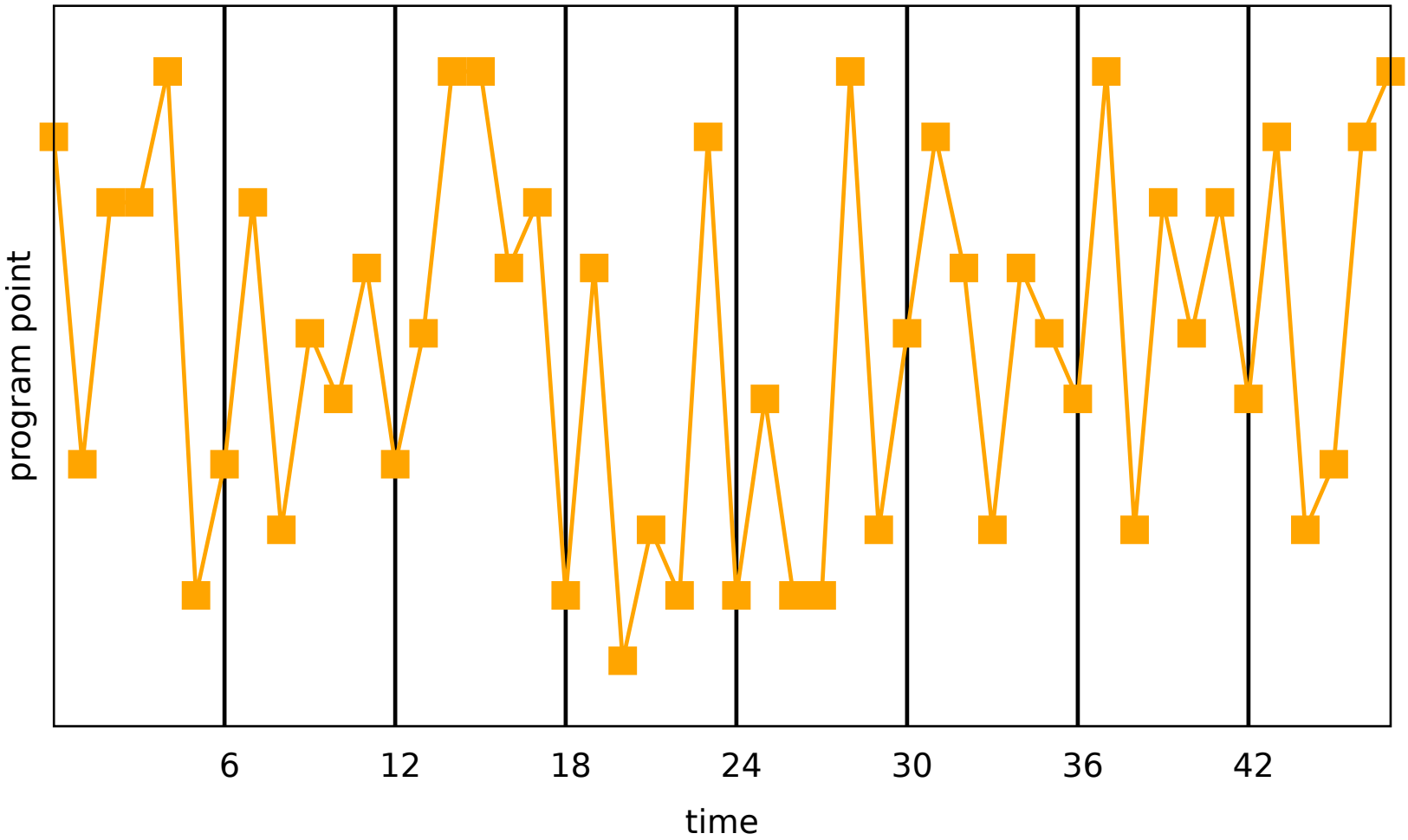

Key Challenges

Need to interleave generation of the network with forward and backward passes through the network.

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Portions of the network need to be (re)generated, and (re)evaluated with forward and backward passes, multiple times and out of order.

Key Idea

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function main(w)
    local x = f(w)
    local y = h(g(x))
    local z = p(y)
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$\rightsquigarrow$

```
function main(w)
    for i = 1, 5
    if i==1 then
        local x = f(w)
    elseif i==2 then
        local t = g(x)
    elseif i==3 then
        local y = h(t)
    elseif i==4 then
        local z = p(y)
    elseif i==5 then
        return z
    end
end
```

Core Language

$$e ::= c \mid x \mid \lambda x.e \mid e_1\ e_2 \mid \textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3 \mid \diamond e \mid e_1 \bullet e_2$$

Adding AD Operators to the Core Language

$$\overleftarrow{\mathcal{J}} : f\ x\ \grave{y} \mapsto (y, \grave{x})$$

$$\overset{\checkmark}{\mathcal{J}} : f\ x\ \grave{y} \mapsto (y, \grave{x})$$

Algorithm for Divide-and-Conquer Checkpointing

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overset{\leftarrow}{\mathcal{J}}\ f\ x\ \grave{y}$ (step 0)

inductive case:

$h \circ g = f$ (step 1)

$z = g\ x$ (step 2)

$(y, \grave{z}) = \overset{\checkmark}{\mathcal{J}}\ h\ z\ \grave{y}$ (step 3)

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General-Purpose Interruption and Resumption Interface

PRIMOPS $f\ x \mapsto l$	Return the number l of evaluation steps needed to compute $y = f(x)$.
INTERRUPT $f\ x\ l \mapsto z$	Run the first l steps of the computation of $f(x)$ and return a capsule z.
RESUME $z \mapsto y$	If $z = (\text{INTERRUPT}\ f\ x\ l)$, return $y = f(x)$.

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Algorithm for Divide-and-Conquer Checkpointing

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Example of CPS Conversion

```
function f(x)
   return q(p(g(x), h(x)))
end
```

$\rightsquigarrow$

```
function f(c, x)
   return g(function(t1)
               return h(function(t2)
                           return p(function(t3)
                                       return q(c, t3)
                                    end, t1, t2)
                        end, x)
            end, x)
end
```

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CPS Conversion as a Program Transformation

$$\lceil x|k \rceil \rightsquigarrow k\ x$$
$$\lceil (\lambda x.e)|k \rceil \rightsquigarrow k\ (\lambda k'\ x.\lceil e|k' \rceil)$$
$$\lceil (e_1\ e_2)|k \rceil \rightsquigarrow \lceil e_1|(\lambda x_1.\lceil e_2|(\lambda x_2.(x_1\ k\ x_2))\rceil)\rceil$$
$$e_0 \rightsquigarrow \lceil e_0|(\lambda x.x)\rceil$$

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$$\lceil (e_1\ e_2)|k \rceil \rightsquigarrow \lceil e_1|(\lambda x_1.\lceil e_2|(\lambda x_2.(x_1\ k\ x_2))\rceil)\rceil$$

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Implementation

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2. Thread step count and limit.
3. Translate CPS to C.
4. Combine with general-purpose interruption and resumption interface and $\check{\mathcal{J}}$ written in C.
5. Compile to machine code.

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\lceil x|k \rceil &\rightsquigarrow k\ x \\
\lceil (\lambda x.e)|k \rceil &\rightsquigarrow k\ (\lambda k\ x.\lceil e|k \rceil) \\
\lceil (e_1\ e_2)|k \rceil &\rightsquigarrow \lceil e_1|(\lambda\ x_1. \\
&\qquad \lceil e_2|(\lambda\ x_2. \\
&\qquad\quad (x_1\ k\ x_2)), \\
&\qquad \rceil), \\
&\rceil
\end{aligned}
$$

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\lceil x|k \rceil &\rightsquigarrow k\ x \\
\lceil (\lambda x.e)|k \rceil &\rightsquigarrow k\ (\lambda k\ x.\lceil e|k \rceil) \\
\lceil (e_1\ e_2)|k \rceil &\rightsquigarrow \lceil e_1|(\lambda\ x_1. \\
&\qquad \lceil e_2|(\lambda\ x_2. \\
&\qquad\quad (x_1\ k\ x_2)), \\
&\qquad \rceil), \\
&\rceil
\end{aligned}
$$

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\lceil x|k,n \rceil &\rightsquigarrow k\ (n+1)\ x \\
\lceil (\lambda x.e)|k,n \rceil &\rightsquigarrow k\ (n+1)\ (\lambda k\ n\ x.\lceil e|k,n \rceil) \\
\lceil (e_1\ e_2)|k,n \rceil &\rightsquigarrow \lceil e_1|(\lambda n\ x_1. \\
&\qquad \lceil e_2|(\lambda n\ x_2. \\
&\qquad\quad (x_1\ k\ n\ x_2)), \\
&\qquad\quad n \rceil), \\
&\qquad (n+1) \rceil
\end{aligned}
$$

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\lceil x|k,n,l\rceil &\rightsquigarrow k\ (n+1)\ l\ x \\
\lceil(\lambda x.e)|k,n,l\rceil &\rightsquigarrow k\ (n+1)\ l\ (\lambda k\ n\ l\ x.\lceil e|k,n,l\rceil) \\
\lceil(e_1\ e_2)|k,n,l\rceil &\rightsquigarrow \lceil e_1|(\lambda n\ l\ x_1. \\
&\qquad \lceil e_2|(\lambda n\ l\ x_2. \\
&\qquad\quad (x_1\ k\ n\ l\ x_2)), \\
&\qquad\quad n,l\rceil), \\
&\qquad (n+1),l\rceil
\end{aligned}
$$

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\ulcorner x|k,n,l\urcorner &\rightsquigarrow k\ (n+1)\ l\ x \\
\ulcorner(\lambda x.e)|k,n,l\urcorner &\rightsquigarrow k\ (n+1)\ l\ (\lambda k\ n\ l\ x.\ulcorner e|k,n,l\urcorner) \\
\ulcorner(e_1\ e_2)|k,n,l\urcorner &\rightsquigarrow \ulcorner e_1|(\lambda n\ l\ x_1. \\
&\qquad \ulcorner e_2|(\lambda n\ l\ x_2. \\
&\qquad\quad (x_1\ k\ n\ l\ x_2)), \\
&\qquad n,l\urcorner), \\
&\quad (n+1),l\urcorner
\end{aligned}
$$

$$
\langle\!\langle e \rangle\!\rangle_{k,n,l} \rightsquigarrow \textbf{if } n = l \textbf{ then } [\![k, \lambda k\ n\ l\ \_.e]\!] \textbf{ else } e
$$

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\ulcorner x|k,n,l\urcorner &\rightsquigarrow \langle\!\langle k\ (n+1)\ l\ x\rangle\!\rangle_{k,n,l} \\
\ulcorner(\lambda x.e)|k,n,l\urcorner &\rightsquigarrow \langle\!\langle k\ (n+1)\ l\ (\lambda k\ n\ l\ x.\ulcorner e|k,n,l\urcorner)\rangle\!\rangle_{k,n,l} \\
\ulcorner(e_1\ e_2)|k,n,l\urcorner &\rightsquigarrow \langle\!\langle \ulcorner e_1|(\lambda n\ l\ x_1.\\
&\qquad \ulcorner e_2|(\lambda n\ l\ x_2.\\
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$$

$$
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Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\ulcorner x|k,n,l\urcorner &\rightsquigarrow \langle\!\langle k\ (n+1)\ l\ x\rangle\!\rangle_{k,n,l} \\
\ulcorner(\lambda x.e)|k,n,l\urcorner &\rightsquigarrow \langle\!\langle k\ (n+1)\ l\ (\lambda k\ n\ l\ x.\ulcorner e|k,n,l\urcorner)\rangle\!\rangle_{k,n,l} \\
\ulcorner(e_1\ e_2)|k,n,l\urcorner &\rightsquigarrow \langle\!\langle \ulcorner e_1|(\lambda n\ l\ x_1. \\
&\qquad \ulcorner e_2|(\lambda n\ l\ x_2. \\
&\qquad\quad (x_1\ k\ n\ l\ x_2)), \\
&\qquad n,l\urcorner), \\
&\quad (n+1),l\urcorner\rangle\!\rangle_{k,n,l}
\end{aligned}
$$

$$\vdots$$

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Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\ f\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x$$

$$\text{INTERRUPT}\ f\ x\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x$$

$$\text{RESUME}\ [\![k,f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \bot$$

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Code Generation

$$
\begin{aligned}
\mathcal{S}\,\pi\,() &= \texttt{null\_constant} \\
\mathcal{S}\,\pi\,\textbf{true} &= \texttt{true\_constant} \\
\mathcal{S}\,\pi\,\textbf{false} &= \texttt{false\_constant} \\
\mathcal{S}\,\pi\,(c_1, c_2) &= \texttt{cons}((\mathcal{S}\,\pi\,c_1),\ (\mathcal{S}\,\pi\,c_1)) \\
\mathcal{S}\,\pi\,n & \\
\mathcal{S}\,\pi\,\text{`k'} &= \texttt{continuation} \\
\mathcal{S}\,\pi\,\text{`n'} &= \texttt{count} \\
\mathcal{S}\,\pi\,\text{`l'} &= \texttt{limit} \\
\mathcal{S}\,\pi\,\text{`x'} &= \texttt{argument} \\
\mathcal{S}\,\pi\,x &= \texttt{as\_closure(target)->environment}[\pi\,x]
\end{aligned}
$$

Code Generation

$\mathcal{S}\ \pi\ (\lambda_3 n\ l\ x.e)$ = ({

```
thing function(thing target,
               thing count,
               thing limit,
               thing argument) {
return (S (φ e) e);
}
thing lambda = (thing)GC_malloc(sizeof(struct {
  enum tag tag;
  struct {
    thing (*function)();
    unsigned n;
    thing environment[|φ e|];
  }))
}
set_closure(lambda);
as_closure(lambda)->function = &function;
as_closure(lambda)->n = |φ e|;
as_closure(lambda)->environment[0] = S π (φ e)0
⋮
as_closure(lambda)->environment[|φ e| - 1] = S π (φ e)|φ e|-1
lambda;
})
```

Code Generation

$\mathcal{S}\ \pi\ (\lambda_4 k\ n\ l\ x.e)$ = ({

```
thing function(thing target,
               thing continuation,
               thing count,
               thing limit,
               thing argument) {
return (S (φ e) e);
}
thing lambda = (thing)GC_malloc(sizeof(struct {
  enum tag tag;
  struct {
    thing (*function)();
    unsigned n;
    thing environment[|φ e|];
  }))
}
set_closure(lambda);
as_closure(lambda)->function = &function;
as_closure(lambda)->n = |φ e|;
as_closure(lambda)->environment[0] = S π (φ e)0
⋮
as_closure(lambda)->environment[|φ e| - 1] = S π (φ e)|φ e|-1
lambda;
})
```

Code Generation

$$\mathcal{S}\ \pi\ (e_1\ e_2\ e_3\ e_4) = \texttt{continuation\_apply}\,((\mathcal{S}\ \pi\ e_1),\ (\mathcal{S}\ \pi\ e_2),\ (\mathcal{S}\ \pi\ e_3),\ (\mathcal{S}\ \pi\ e_4))$$

$$\mathcal{S}\ \pi\ (e_1\ e_2\ e_3\ e_4\ e_5) = \texttt{converted\_apply}\,((\mathcal{S}\ \pi\ e_1),\ (\mathcal{S}\ \pi\ e_2),\ (\mathcal{S}\ \pi\ e_3),\ (\mathcal{S}\ \pi\ e_4),\ (\mathcal{S}\ \pi\ e_5))$$

$$\mathcal{S}\ \pi\ (\textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3) = \texttt{(!is\_false}\,((\mathcal{S}\ \pi\ e_1))\ \texttt{?}(\mathcal{S}\ \pi\ e_2)\texttt{:}(\mathcal{S}\ \pi\ e_3))$$

$$\mathcal{S}\ \pi\ (\diamond e) = (\mathcal{N}\ \diamond)\,((\mathcal{S}\ \pi\ e))$$

$$\mathcal{S}\ \pi\ (e_1 \bullet e_2) = (\mathcal{N}\ \bullet)\,((\mathcal{S}\ \pi\ e_1),\ (\mathcal{S}\ \pi\ e_2))$$

$$\mathcal{S}\ \pi\ (\overrightarrow{\mathcal{J}}\ e_1\ e_2\ e_3) = (\mathcal{N}\ \overrightarrow{\mathcal{J}})\,((\mathcal{S}\ \pi\ e_1),\ (\mathcal{S}\ \pi\ e_2),\ (\mathcal{S}\ \pi\ e_3))$$

$$\mathcal{S}\ \pi\ (\overleftarrow{\mathcal{J}}\ e_1\ e_2\ e_3) = (\mathcal{N}\ \overleftarrow{\mathcal{J}})\,((\mathcal{S}\ \pi\ e_1),\ (\mathcal{S}\ \pi\ e_2),\ (\mathcal{S}\ \pi\ e_3))$$

$$\mathcal{S}\ \pi\ (\overset{\checkmark}{\mathcal{J}}\ e_1\ e_2\ e_3) = (\mathcal{N}\ \overset{\checkmark}{\mathcal{J}})\,((\mathcal{S}\ \pi\ e_1),\ (\mathcal{S}\ \pi\ e_2),\ (\mathcal{S}\ \pi\ e_3))$$

Implementation

1. Convert source program to CPS.
2. Thread step count and limit.
3. Translate CPS to C.
4. Combine with general-purpose interruption and resumption interface and $\checkmark\mathcal{I}$ written in C.
5. Compile to machine code.

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\ f\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x$$

$$\text{INTERRUPT}\ f\ x\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x$$

$$\text{RESUME}\ [\![k,f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \perp$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

written in C

```
static thing lambda_expression_that_returns_x
(thing f, thing n, thing l, thing x) {
  return x;
}

static thing lambda_expression_that_returns_n
(thing f, thing n, thing l, thing x) {
  return n;
}

static thing lambda_expression_that_resumes
(thing f, thing continuation, thing n, thing l, thing x) {
  if (!is_interrupt(x)) internal_error();
  return converted_apply(as_interrupt(x)->closure,
                         as_interrupt(x)->continuation,
                         make_real(0.0),
                         l,
                         null_constant);
}
```

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

```
static unsigned long primops(thing f, thing x) {
  thing result = converted_apply(f,
                                 continuation_that_returns_n,
                                 make_real(0.0),
                                 make_real(HUGE_VAL),
                                 x);
  else if (is_real(result)) return (unsigned long)as_real(result);
}

static thing interrupt(thing f, thing x, thing l) {
  thing result = converted_apply(f,
                                 continuation_that_returns_x,
                                 make_real(0.0),
                                 l,
                                 x);
  if (!is_interrupt(result)) internal_error();
  return result;
}
```

Algorithm for Divide-and-Conquer Checkpointing

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overset{\leftarrow}{\mathcal{J}}\ f\ x\ \grave{y}$ (step 0)

inductive case:

$h \circ g = f$ (step 1)

$z = g\ x$ (step 2)

$(y, \grave{z}) = \overset{\checkmark}{\mathcal{J}}\ h\ z\ \grave{y}$ (step 3)

$(z, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{z}$ (step 4)

Algorithm for Divide-and-Conquer Checkpointing

written in C

```
static thing checkpoint_starj(thing f, thing x, thing y_cotangent)
{
  thing loop(thing f, thing x, thing y_cotangent, unsigned long l) {
    if (l<=base_case_duration) return ternary_starj(f, x, y_cotangent);
    else {
      thing u = interrupt(f, x, make_real(l/2));
      thing y_u_cotangent = loop(closure_that_resumes, u, y_cotangent, l-l/2);
      if (!is_pair(y_u_cotangent)) internal_error();
      thing u_x_cotangent =
        loop(make_closure_for_interrupt(f, l/2),
             x,
             as_pair(y_u_cotangent)->cdr,
             l/2);
      if (!is_pair(u_x_cotangent)) internal_error();
      return cons(as_pair(y_u_cotangent)->car,
                  as_pair(u_x_cotangent)->cdr);
    }
  }
  return loop(f, x, y_cotangent, primops(f, x));
}
```

Implementation

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Three Reference Implementations

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Same space and time complexity. Differ only in constant factors.

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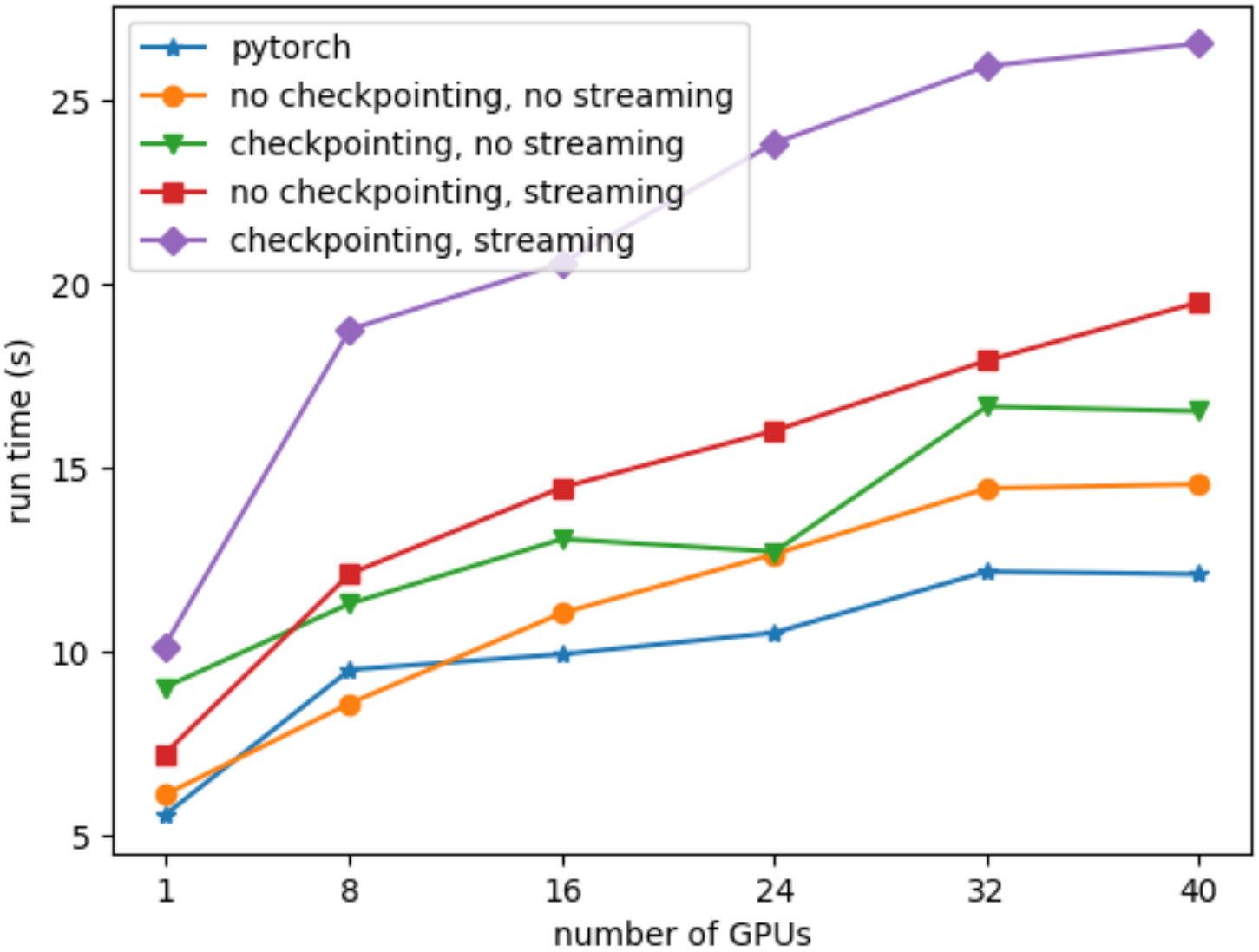
Scorch

`list->`⟨*residence*⟩`-`⟨*type*⟩`-tensor`	`fill-`⟨*residence*⟩`-`⟨*type*⟩	⟨*residence*⟩`-`⟨*type*⟩`-tensor?`
⟨*residence*⟩`-`⟨*type*⟩	`randn-`⟨*residence*⟩`-`⟨*floating type*⟩	`normal-`⟨*residence*⟩`-`⟨*floating type*⟩
`size`	`tensor->list`	`view`
`transpose`	`narrow-tensor`	`expand-tensor`
`concat-tensors`	`dot`	`sumall`
`addmv`	`addmm`	`baddbmm`
`addr`	`resize`	`pad`
`crop`	`decimate`	`interpolate`
`interpolate-to-size`	`upsample-nearest`	`downsample-nearest`
`permute`	`ReLU`	`LeakyReLU`
`GeLU`	`sigmoid`	`convolve`
`transpose-convolve`	`batch-normalization-training`	`batch-normalization-test`
`initialize-batch-normalization`	`layer-normalization`	`convolve-add-tied`
`embedding`	`max-pool`	`average-pool`
`dropout`	`dropout-planewise`	`max-value`
`index-of-max`	`cross-entropy-loss`	`softmax`
`fused-scale-mask-softmax`		

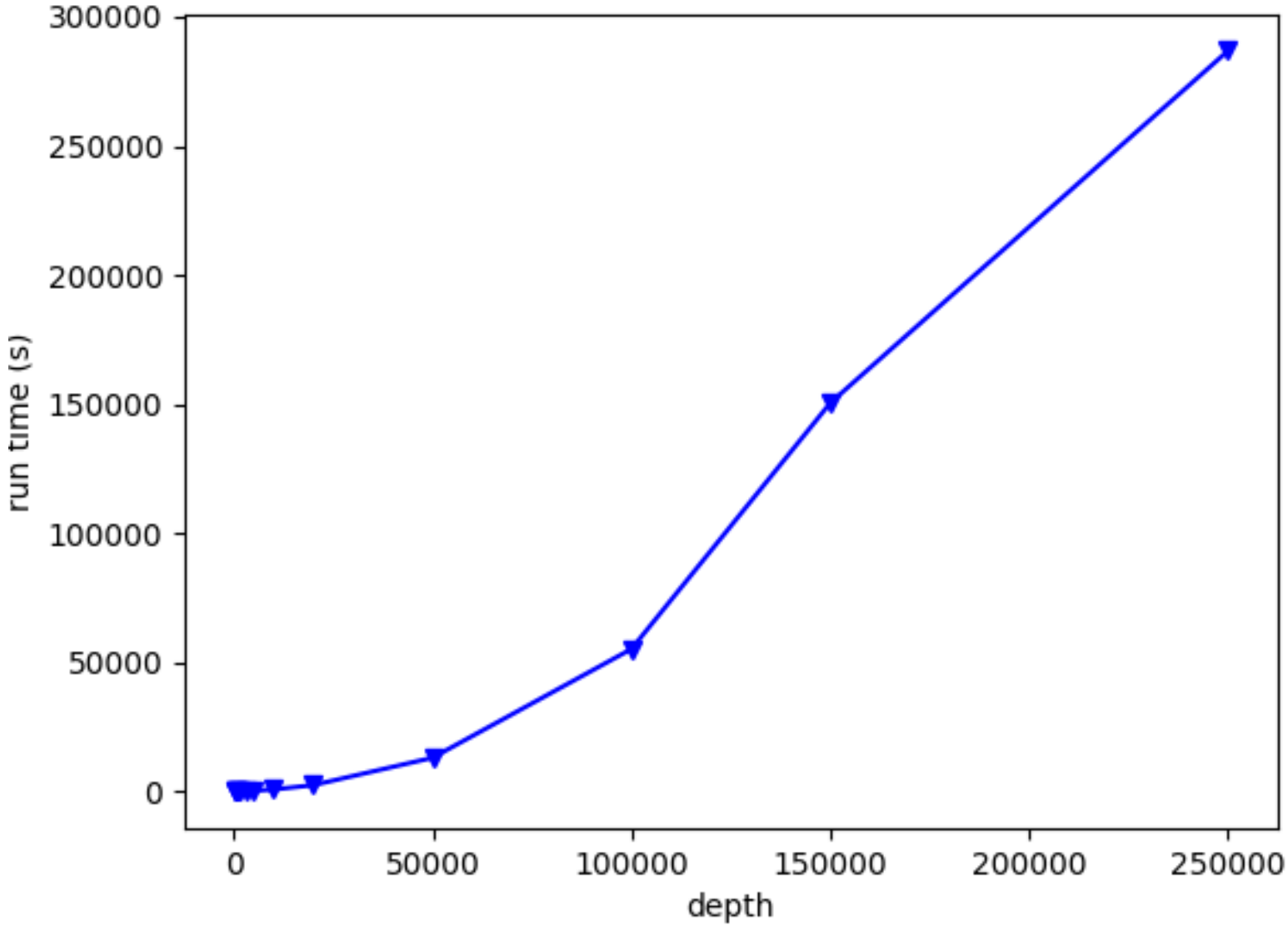
Beyond these, the standard unary basis procedures `sqrt`, `exp`, `log`, `sin`, `cos`, `zero?`, `positive?`, and `negative?` are extended pointwise to tensors and the standard binary basis procedures `+`, `-`, `*`, `/`, `max`, `min`, `atan`, `=`, `<`, `>`, `<=`, and `>=` are extended to apply to a tensor and a scalar, or a scalar and a tensor, or two tensors of the same type and dimensions, in a pointwise fashion.

SCORCH

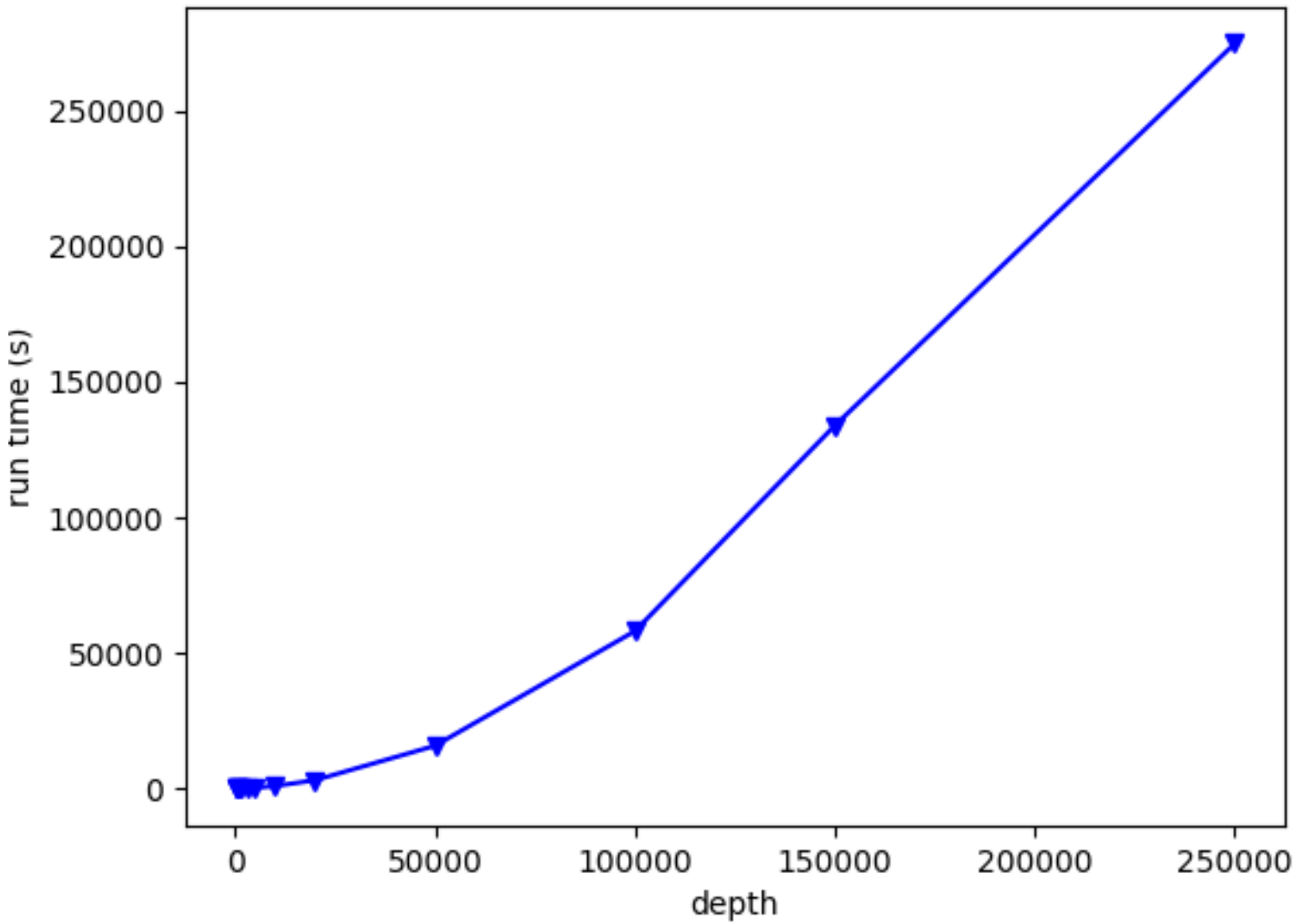
ResNet


pytorch
no checkpointing, no streaming
checkpointing, no streaming
no checkpointing, streaming
checkpointing, streaming
run time (s)
5
10
15
20
25
1
8
16
24
32
40
number of GPUs

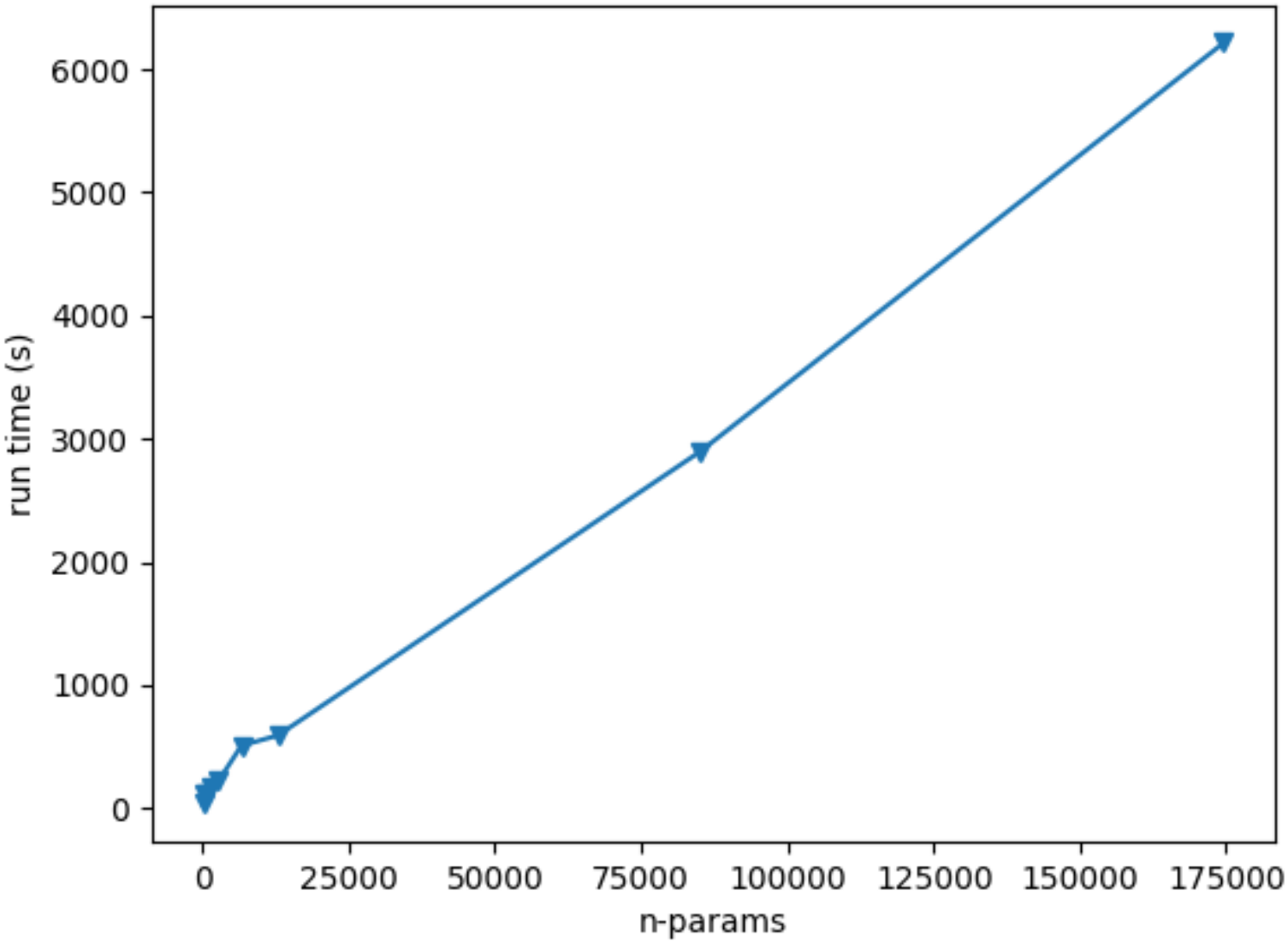
ResNet


300000
250000
200000
150000
100000
50000
0
run time (s)
0
50000
100000
150000
200000
250000
depth

DRANet


250000
200000
150000
100000
50000
0
run time (s)
0
50000
100000
150000
200000
250000
depth

GPT


6000
5000
4000
3000
2000
1000
0
run time (s)
0
25000
50000
75000
100000
125000
150000
175000
n-params

Ray Tracer

```
(map (lambda (y)
       (map (lambda (x)
              (shoot-ray (list y x))) (iota width)))
     (iota height))
```

Ray Tracer

```
(if antecedent consequent alternate)
```

Ray Tracer

```
(shoot-ray
 (map (lambda (y)
        (map (lambda (x) (list y x))
             (iota width)))
      (iota height)))
```

Ray Tracer

```
(shoot-ray
 (list
  (map (lambda (y)
         (map (lambda (x) y) (iota width)))
       (iota height))
  (map (lambda (y)
         (map (lambda (x) x) (iota width)))
       (iota height))))
```

Ray Tracer

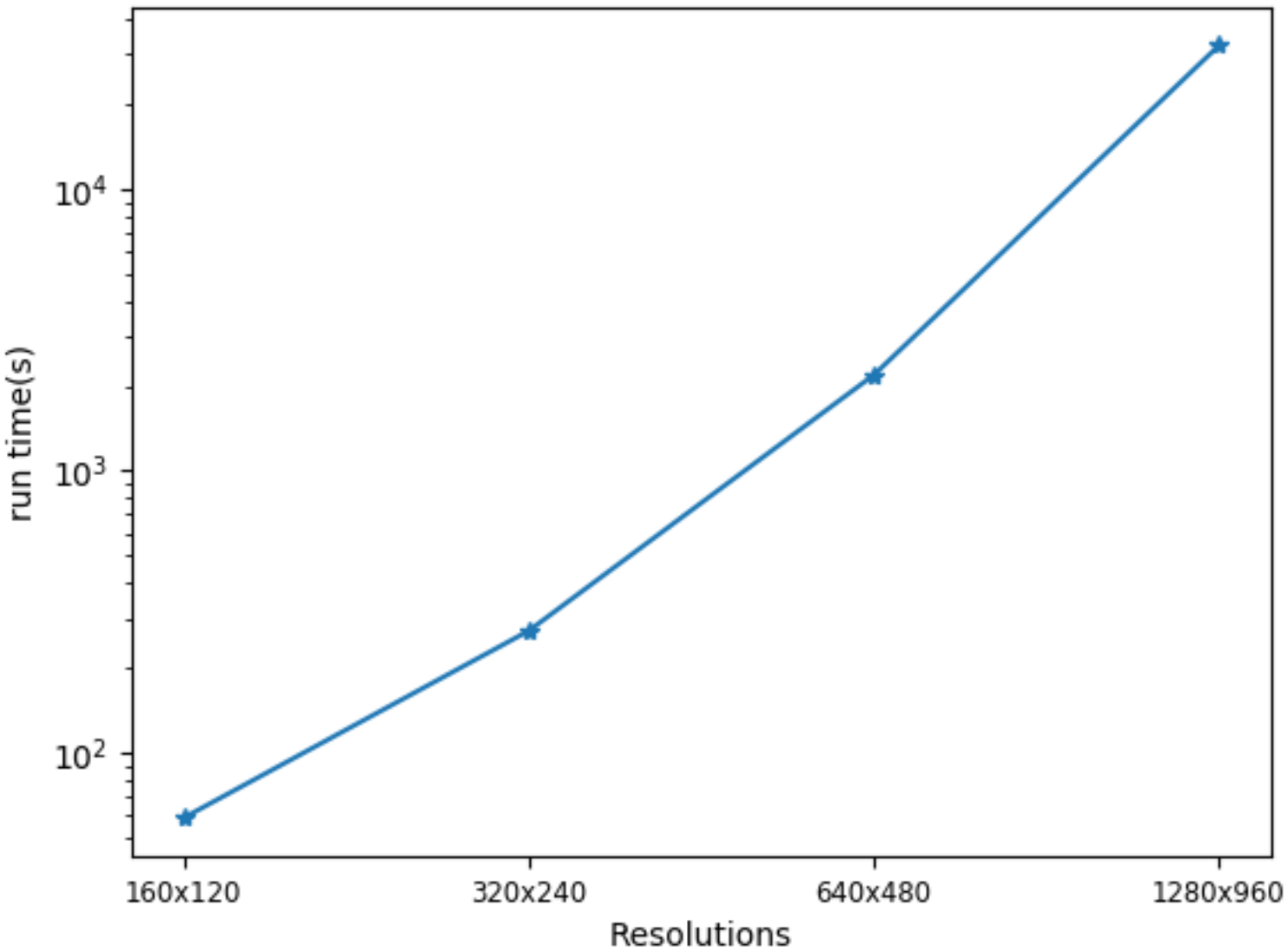
```
(map (lambda (y)
       (map (lambda (x) y)
            (iota width)))
     (iota height))
```

Ray Tracer

```
(define (if-function antecedent consequent alternate)
 (if antecedent consequent alternate))

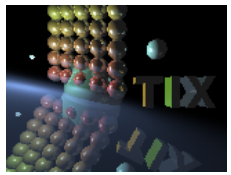
(define (if-tensor antecedent consequent alternate)
 (map (lambda (antecedent consequent alternate)
        (map (lambda (antecedent consequent alternate)
               (if-function antecedent
                            consequent
                            alternate))
             antecedent consequent alternate))
      antecedent consequent alternate))
```

Ray Tracer


10^4
10^3
10^2
run time(s)
160x120
320x240
640x480
1280x960
Resolutions

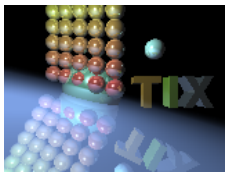
Target Scene

Light source at position `(-40 -15 60)`



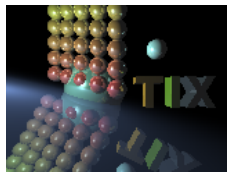
Initial Scene

Light source at position `(-20 -30 30)`



Final Scene after 111 iterations

Light source at position `(-42.7 -13.8 39.3)`



Comparison

	ResNet	GPT	Ray Tracer
TensorFlow (individual checkpointing)	608	4	280×210
PyTorch (right-branching checkpointing)	760	48	320×240
DeepSpeed	760	48	n/a
L2L	38,000	3,000	n/a
Scorch	**250,000**	**14,000**	**1280×960**

Take-Home Message

Divide-and-Conquer checkpointing

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- is traditionally formulated around loop iterations

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Divide-and-Conquer checkpointing

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metaphor: a CPU is an instruction-execution loop

Thank You