Basics of Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$X(z) = \mathcal{Z}\{x[n]\}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

$$x[n] = \mathcal{Z}^{-1}\{X(z)\}$$

Linearity: $\mathcal{Z}\{a_1 x_1[n] + a_2 x_2[n]\}$

$$= a_1 X_1(z) + a_2 X_2(z)$$

- extends for more than 3 signals

Time Shift: $x[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0} X(z)$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]\, z^{-n}$$

$n' = n - n_0 \Rightarrow n = n' + n_0$

limits still $-\infty$ to $+\infty$

$$Y(z) = \sum_{n'=-\infty}^{\infty} x[n']\, z^{-(n+n_0)}$$

$$= z^{-n_0} \sum_{n'=-\infty}^{\infty} x[n']\, z^{-n'} = z^{-n_0} X(z)$$

Convolution Property

$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{Z}} Y(z) = X(z)H(z)$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

- Take ZT of both sides and invoke linearity

$$\mathcal{Z}\{y[n]\} = \sum_{k=-\infty}^{\infty} x[k]\, \mathcal{Z}\{h[n-k]\}$$

$$Y(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}\, H(z)$$

$$= X(z)\, H(z)$$

- Thus: $H(z) = \dfrac{Y(z)}{X(z)}$ = ratio of 2 polynomials for a difference equation LTI system

Basic Z-T Pair

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a} \qquad \text{ROC: } |z|>|a|$$

Time-Shift Property dictates

$$a^{n-1} u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-1}\frac{z}{z-a} = \frac{1}{z-a}$$

$$= \frac{1}{a} a^n u[n-1] \qquad \text{ROC: same as above}$$

- Thus, one could use same rules/procedure for partial fraction expansion that you learned for the laplace Transform

$$e^{at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a} \Rightarrow a^{n-1}u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z-a}$$

Basic Z-Transform Pair

$$x[n] = \alpha^n u[n] \overset{Z}{\longleftrightarrow} X(z) = ?$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z-\alpha}$$

for $|\alpha z^{-1}| < 1 \Rightarrow |z|^{-1}|\alpha| < 1$

$\Rightarrow |z| > |\alpha|$ Region of Convergence

$$\alpha^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z-\alpha}$$

$$\alpha^{n-1} u[n-1] \overset{Z}{\longleftrightarrow} z^{-1} \frac{z}{z-\alpha} = \frac{1}{z-\alpha}$$

compare to: $e^{at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}$