Key Illustrative Example

- Simple single-pole system:

$$y[n] = a\,y[n-1] + x[n]$$

- Use Z-Transform to find output $y[n]$ when input is $x[n] = b^n u[n]$
- We already know answer from Chap. 2

$h[n] = a^n u[n]$ } recursively determined in Chap 2

And we derived:

$$b^n u[n] * a^n u[n] = \frac{a}{a-b} a^n u[n] + \frac{b}{b-a} b^n u[n]$$

- So we already know the answer.

See if Z-Transform provides same answer

- Take ZT of both sides of Difference Equation

$$Y(z) = a z^{-1} Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

- Now, our basic ZT pair is:

$$b^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-b} \qquad (\text{ROC: } |z| > |b|)$$

- <u>Thus</u>: $Y(z) = H(z)X(z)$

$$= \frac{z}{(z-a)} \frac{z}{(z-b)} = z \frac{z}{(z-a)(z-b)}$$

do partial fraction expansion of this term and multipy by z afterwards

$$\frac{z}{(z-a)(z-b)} = c_1 \frac{1}{z-a} + c_2 \frac{1}{z-b} = G(z)$$

$$c_1 = G(z)(z-a)\Big|_{z=a} \qquad c_2 = G(z)(z-b)\Big|_{z=b}$$

$$c_1 = \left.\frac{z}{z-b}\right|_{z=a} = \frac{a}{a-b}$$

$$c_2 = \left.\frac{z}{z-a}\right|_{z=b} = \frac{b}{b-a}$$

Thus: $$Y(z) = \frac{a}{a-b}\,\frac{z}{z-a} + \frac{b}{b-a}\,\frac{z}{z-b}$$

Taking Inverse ZT:

$$y[n] = \frac{a}{a-b}\,a^n u[n] + \frac{b}{b-a}\,b^n u[n]$$

This agrees with the result that we derived previously