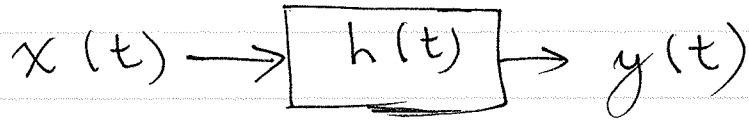


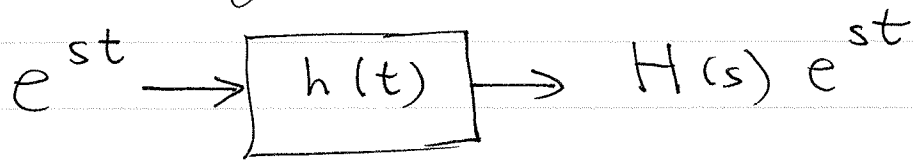
Laplace and Fourier Transforms for 1 CT Signals and Systems

• Recall, for CT LTI System



$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \end{aligned}$$

Definition of Laplace Transform (Chap. 9)
motivated by:



$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\ &= \left\{ \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \right\} e^{st} \end{aligned}$$

THUS: $h(t) \xleftrightarrow{\mathcal{L}} H(s)$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

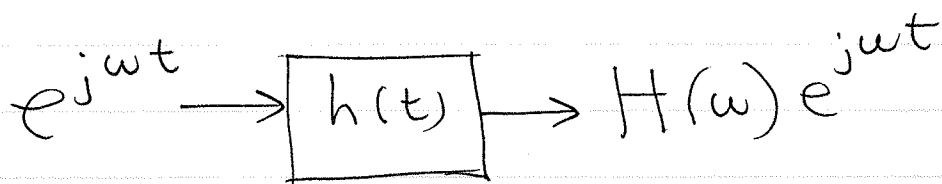
Same def'n for signal: $x(t) \xleftrightarrow{\mathcal{L}} X(s)$

Key property:

$$x(t) * h(t) \xleftrightarrow{\mathcal{L}} X(s) H(s)$$

Fourier Transform: consider $s=j\omega$
(Chap. 4)

②



$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau$$

$$= \left\{ \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right\} e^{j\omega t}$$

$$h(t) \xleftrightarrow{\mathcal{F}} H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

Same definition for signal:

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

For certain class of signals:

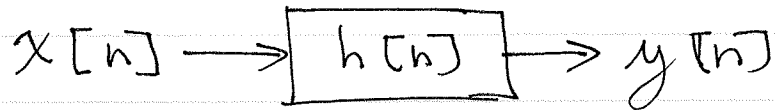
$$X(\omega) = \mathcal{L}\{x(t)\} \Big|_{s=j\omega}$$

- However, we also have "generalized" Fourier Transforms for sinewaves "turned on" for all time, sinc function, etc., for which Laplace Transform is not defined
- One of many key properties:

$$x(t) * h(t) \xleftrightarrow{\mathcal{F}} X(\omega) H(\omega)$$

Z-Transform and Discrete-Time Fourier Transform (DTFT) for DT Signals/Systems (3)

Recall, for DT LTI System:



$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

Definition of Z-Transform (Chap. 10) motivated by:



$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^{n-k}$$

$$= \left\{ \sum_{k=-\infty}^{\infty} h[k] z^{-k} \right\} z^n$$

$$\text{THUS: } h[n] \xleftrightarrow{\mathcal{Z}} H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

Same definition for signal: $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$

Key property:

$$x[n] * h[n] \xleftrightarrow{\mathcal{Z}} X(z) H(z)$$

Why did we consider inputting geometric sequence into DT LTI System, in contrast to inputting exponential signal into CT LTI System? (CT = Continuous Time)

• Because we obtain geometric sequence when we sample exponential signal:

$$x(t) = e^{st}$$

$$x[n] = x(t) \Big|_{t=nT, \substack{n \text{ integer} \\ -\infty < n < \infty}}$$

$$= e^{snT}$$

$$= (e^{sT})^n$$

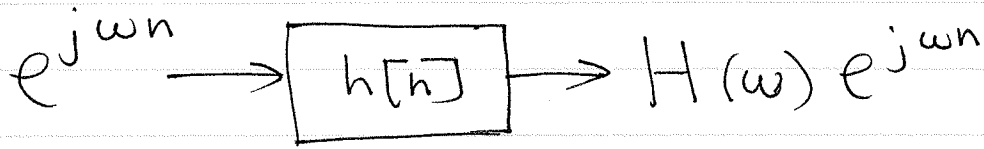
$$= z^n$$

$$z = e^{sT}$$

generally complex-valued since s is generally complex-valued

(5)

Discrete-Time Fourier Transform (Chap. 5)
 = (DTFT) definition motivated by considering
 z on unit circle, so that z^n is a
 complex sinewave $\Rightarrow z = e^{j\omega} \Rightarrow z^n = e^{j\omega n}$



$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= \left\{ \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right\} e^{j\omega n}$$

$$h[n] \xrightarrow{\text{DTFT}} H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

Same definition for signal:

$$x[n] \xrightarrow{\text{DTFT}} X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

For certain class of signals:

$$X(\omega) = \mathcal{Z}\{x[n]\} \Big|_{z=e^{j\omega}}$$

- However, we define $X(\omega)$ for infinite-length DT sinewaves, DT sinc functions, etc., for which \mathcal{ZT} not defined
- Key property: $x[n] * h[n] \xleftrightarrow{\text{DTFT}} X(\omega) H(\omega)$

Consider: (subscript a for analog)

(6)

$$X_a(\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t} dt$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

• Suppose: $x[n] = x_a(t) \Big|_{t=nT} = x_a(nT)$

• Question: How are $X(\omega)$ and $X_a(\omega)$ related?

Recall, sampling rate: $F_s = \frac{1}{T}$

(Interim) Answer: $X(\omega) = X_s(F_s \omega)$

where:

$$\sum_{n=-\infty}^{\infty} x_a(nT) \delta(t-nT) \xleftrightarrow{\mathcal{F}} X_s(\omega)$$

This is easy to see, since $\delta(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}$

$$\text{Hence, } \mathcal{F} \left\{ \sum_{n=-\infty}^{\infty} x_a(nT) \delta(t-nT) \right\} = \sum_{n=-\infty}^{\infty} x_a(nT) e^{j\omega nT}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{\omega}{F_s} n} = X_s(\omega)$$

THUS: $X(\omega) = X_s(F_s \omega)$

Now, how is $X_s(\omega)$ related to $X_a(\omega)$? (7)

We derived that previously for Sampling Theory:

Recall:

$$x_s(t) = x_a(t) p(t)$$

$$\text{where: } p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$\text{Note: } x_s(t) = x_a(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT) \delta(t-nT)$$

And we just showed:

$$x_s(t) \xleftrightarrow{+} X_s(\omega) = \sum_{n=-\infty}^{\infty} x_a(nT) e^{-j\omega nT}$$

Alternatively, when we derived Sampling Theory we derived $X_s(\omega)$ by using multiplication property

$$x_s(t) = x(t) p(t) \xleftrightarrow{+} X_s(\omega) = \frac{1}{2\pi} X_a(\omega) * P(\omega)$$

Since $p(t)$ is periodic:

$$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j k \frac{2\pi}{T} t} \xleftrightarrow{+} P(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k \frac{2\pi}{T})$$

Fourier Series coefficients for
 $\sum_n \delta(t-nT)$

TITUS:

8

$$X_s(t) = X_a(t)p(t) \xleftrightarrow{F} X_s(\omega) = \frac{1}{2\pi} X_a(\omega) * P(\omega)$$

$$X_s(\omega) = \frac{1}{2\pi} X_a(\omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k \frac{2\pi}{T})$$

$$X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\omega - k \frac{2\pi}{T})$$

Fourier Transform of $X_a(t)$

repeated every integer multiple of $\frac{2\pi}{T}$ in the frequency domain

as derived previously in textbook

Summarizing: If $x[n] = X_a(nT)$ and

$$x[n] \xleftrightarrow{\text{DTFT}} X(\omega)$$

$$X_a(t) \xleftrightarrow{F} X_a(\omega)$$

THEN: $X(\omega) = X_s(\frac{F_s \omega}{2\pi})$

$$\boxed{F_s = \frac{1}{T}}$$

where $X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\omega - k \frac{2\pi}{T})$

The compression by the sampling rate is

such that $X_s(\omega)$ periodic with period $\frac{2\pi}{T}$

is compressed so that $X(\omega)$ is periodic with period 2π

To see this, note:

$$\begin{aligned}
X(\omega) &= X_s(F_s \omega) \\
&= X_s\left(\frac{1}{T}\omega\right) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(\frac{1}{T}\omega - k \frac{2\pi}{T}\right) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(\frac{1}{T}(\omega - k 2\pi)\right) \\
&= F_s \sum_{k=-\infty}^{\infty} X_a(F_s(\omega - k 2\pi))
\end{aligned}$$

repeats every 2π so that sam is periodic with period 2π

This is consistent with the observation:

$$\begin{aligned}
X(\omega + l 2\pi) &= \sum_{n=-\infty}^{\infty} x[n] e^{j(\omega + l 2\pi)n} \quad l \text{ integer} \\
&= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \underbrace{\left(e^{-j 2\pi}\right)^{ln}}_{=1} \\
&\equiv X(\omega) \Rightarrow \text{is periodic with period } 2\pi
\end{aligned}$$

• Also consistent with Chp. 2 observation that any two complex sinewaves whose frequencies are separated by an integer multiple of 2π are the same sinewave