Laplace and Fourier Transforms for CT Signals and Systems

- Recall, for CT LTI System

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Definition of Laplace Transform (Chap. 9) motivated by:

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)\, e^{st}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$
$$= \left\{ \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \right\} e^{st}$$

THUS: $h(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s)$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

Same defn for signal: $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$

Key property:

$$x(t) * h(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)\, H(s)$$

Fourier Transform: consider $s = j\omega$
(Chap. 4)

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(\omega)\, e^{j\omega t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$= \left\{ \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau \right\} e^{j\omega t}$$

$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

Same definition for signal:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

For certain class of signals:

$$X(\omega) = \mathcal{L}\{x(t)\}\Big|_{s=j\omega}$$

- However, we also have "generalized" Fourier Transforms for sinewaves "turned on" for all time, sinc functions, etc., for which Laplace Transform is not defined
- One of many key properties:

$$x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)\, H(\omega)$$

Z-Transform and Discrete-Time Fourier Transform (DTFT) for DT Signals/Systems

- Recall for DT LTI System:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Definition of Z-Transform (Chap. 10) motivated by:

$$z^n \longrightarrow \boxed{h[n]} \longrightarrow H(z)\, z^n$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$
$$= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$
$$= \left\{ \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \right\} z^n$$

THUS: $h[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$

Same definition for signal: $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$

Key property:
$$x[n] * h[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)\, H(z)$$

Why did we consider inputting geometric sequence into DT LTI System, in contrast to inputting exponential signal into CT LTI System? (CT = Continuous Time)

- Because we obtain geometric sequence when we sample exponential signal:

$$x(t) = e^{st}$$

$$x[n] = x(t)\Big|_{t=nT}, \quad \begin{matrix} n \text{ integer} \\ -\infty < n < \infty \end{matrix}$$

$$= e^{snT}$$

$$= \left(e^{sT}\right)^n$$

$$= z^n \qquad z = e^{sT}$$

generally complex-valued since s is generally complex-valued

Discrete-Time Fourier Transform (Chap. 5)
$\Rightarrow$ (DTFT) definition motivated by considering z on unit circle, so that z^n is a complex sinewave $\Rightarrow z = e^{j\omega} \Rightarrow z^n = e^{j\omega n}$

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(\omega)\, e^{j\omega n}$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega(n-k)}$$

$$= \left\{ \sum_{k=-\infty}^{\infty} h[k]\, e^{-jk\omega} \right\} e^{j\omega n}$$

$$h[n] \overset{DTFT}{\longleftrightarrow} H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

Same definition for signal:

$$x[n] \overset{DTFT}{\longleftrightarrow} X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

For certain class of signals:

$$X(\omega) = Z\{x[n]\}\Big|_{z = e^{j\omega}}$$

- However, we define $X(\omega)$ for infinite-length DT sinewaves, DT sinc functions, etc., for which ZT not defined
- Key property: $x[n] * h[n] \overset{DTFT}{\longleftrightarrow} X(\omega) H(\omega)$

Consider: (subscript a for analog)

$$X_a(\omega) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j\omega t}\, d\omega$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- Suppose: $x[n] = x_a(t)\Big|_{t=nT} = x_a(nT)$

- <u>Question:</u> How are $X(\omega)$ and $X_a(\omega)$ related?

Recall, sampling rate: $F_s = \frac{1}{T}$

(Interim) <u>Answer:</u> $X(\omega) = X_s(F_s\,\omega)$

where:

$$\sum_{n=-\infty}^{\infty} x_a(nT)\,\delta(t-nT) \xleftrightarrow{\mathcal{F}} X_s(\omega)$$

This is easy to see, since $\delta(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}$

Hence,

$$\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x_a(nT)\,\delta(t-nT)\right\} = \sum_{n=-\infty}^{\infty} x_a(nT)\, e^{-j\omega nT}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\frac{\omega}{F_s}n} = X_s(\omega)$$

THUS: $X(\omega) = X_s(F_s\omega)$

Now, how is $X_s(\omega)$ related to $X_a(\omega)$? 7

We derived that previously for Sampling Theory.

<u>Recall:</u>

$$x_s(t) = x_a(t)\, p(t)$$

<u>where:</u> $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

<u>Note:</u> $x_s(t) = x_a(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\, \delta(t-nT)$$

And we just showed:

$$x_s(t) \xleftrightarrow{\mathcal{F}} X_s(\omega) = \sum_{n=-\infty}^{\infty} x_a(nT)\, e^{-j\omega nT}$$

Alternatively, when we derived Sampling Theory we derived $X_s(\omega)$ by using multiplication property

$$x_s(t) = x(t)p(t) \xleftrightarrow{\mathcal{F}} X_s(\omega) = \frac{1}{2\pi} X_a(\omega) * P(\omega)$$

since $p(t)$ is periodic:

$$p(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T}} e^{jk\frac{2\pi}{T}t} \xleftrightarrow{\mathcal{F}} P(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\, \delta\!\left(\omega - k\frac{2\pi}{T}\right)$$

Fourier Series coefficients for $\sum_{n} \delta(t-nT)$

Thus:

$$x_s(t) = x_a(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} X_s(\omega) = \frac{1}{2\pi} X_a(\omega) * P(\omega)$$

$$X_s(\omega) = \cancel{\frac{1}{2\pi}} X_a(\omega) * \sum_{k=-\infty}^{\infty} \frac{\cancel{2\pi}}{T} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$X_s(\omega) = \underbrace{\frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(\omega - k\frac{2\pi}{T}\right)}$$

Fourier Transform of $x_a(t)$ repeated every integer multiple of $\frac{2\pi}{T}$ in the frequency domain as derived previously in textbook

<u>Summarizing</u>: If $x[n] = x_a(nT)$ and

$$x[n] \overset{DTFT}{\longleftrightarrow} X(\omega) \qquad x_a(t) \overset{\mathcal{F}}{\longleftrightarrow} X_a(\omega)$$

THEN: $X(\omega) = X_s(F_s\omega)$ $\quad \boxed{F_s = \frac{1}{T}}$

where $X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(\omega - k\frac{2\pi}{T}\right)$

The compression by the sampling rate is such that $X_s(\omega)$ periodic with period $\frac{2\pi}{T}$ is compressed so that $X(\omega)$ is periodic with period 2π

To see this, note:

$$X(\omega) = X_s(F_s\,\omega)$$

$$= X_s\left(\frac{1}{T}\omega\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{1}{T}\omega - k\frac{2\pi}{T}\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{1}{T}(\omega - k2\pi)\right)$$

$$= F_s \underbrace{\sum_{k=-\infty}^{\infty} X_a\left(F_s(\omega - k2\pi)\right)}$$

repeats every 2π so that sum
is periodic with period 2π

This is consistent with the observation:

$$X(\omega+\ell 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+\ell 2\pi)n} \qquad \ell \text{ integer}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \underbrace{\left(e^{-j2\pi}\right)^{\ell n}}_{=1}$$

$$= X(\omega) \Rightarrow \text{is periodic with period } 2\pi$$

- Also consistent with Chap. 2 observation that any two complex sinewaves whose frequencies are separated by an integer multiple of 2π are the same sinewave