Further Results on Sampling Theory and Ideal DAC: Signal Reconstruction From Samples

- The following FT pair has been derived in both Text Chap 7 and the notes

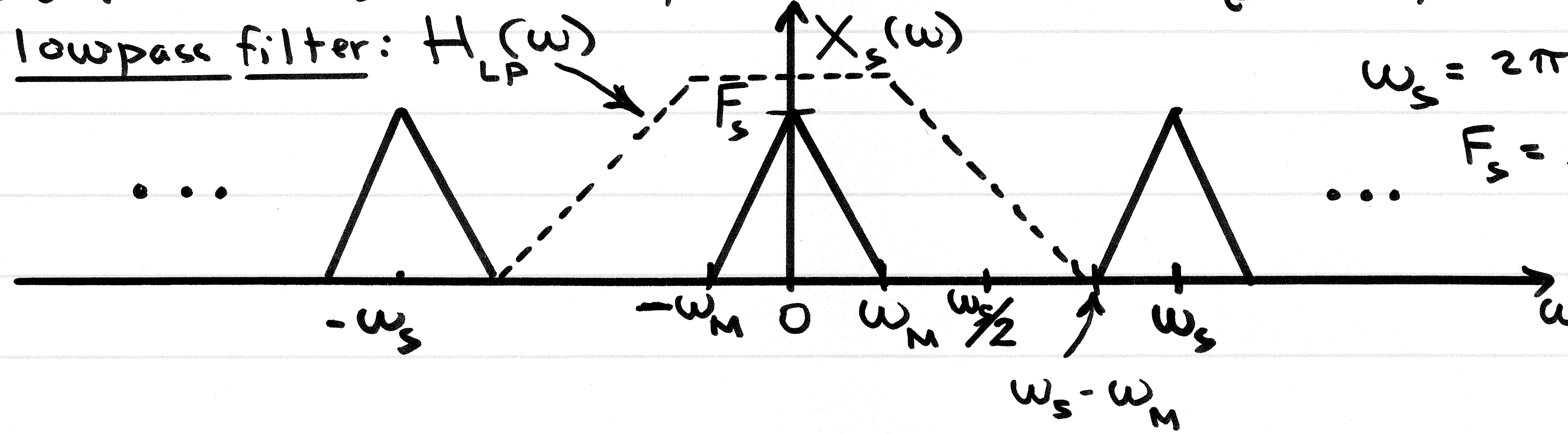
$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\delta(t - nT_s) \overset{\mathcal{F}}{\longleftrightarrow} X_s(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

where: $\omega_s = 2\pi F_s$ and $F_s = \frac{1}{T_s}$ = sampling rate

and $\quad x_a(t) \overset{\mathcal{F}}{\longleftrightarrow} X_a(\omega)$

- Sampling Theory assumes some max. frequency, ω_M for which $X(\omega) = 0$ for $|\omega| > \omega_M$
- ω_M is referred to as the bandwidth for a "baseband" signal

- We have standard picture for triangular spectrum

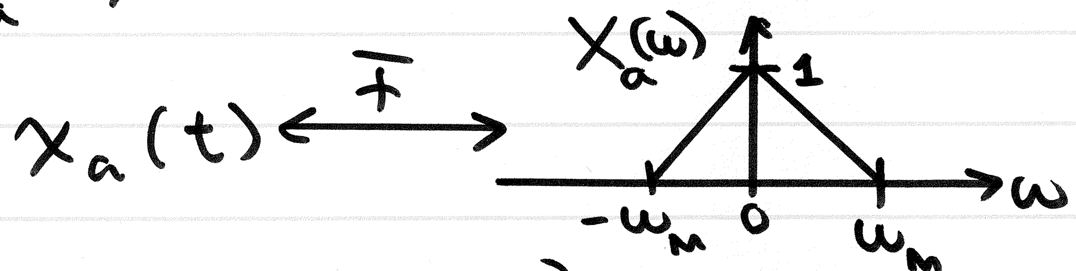


$\omega_s = 2\pi F_s$

$F_s = \frac{1}{T_s}$

assuming: $\omega_s > 2\omega_M$ $\quad (\omega_s - \omega_M > \omega_M)$

and $X_a(\omega) = 0$ for $|\omega| > \omega_M$:



- $\frac{\omega_s}{2} = \frac{1}{2}\left\{\omega_M + (\omega_s - \omega_M)\right\}$ = middle of "don't care" gap that results with greater than Nyquist rate sampling

- reconstructed signal:

$$x_r(t) = x_s(t) * h_{LP}(t)$$

$$= \left\{ \sum_{n=-\infty}^{\infty} x_a(nT_s)\, \delta(t - nT_s) \right\} * h_{LP}(t)$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT_s)\, h_{LP}(t - nT_s)$$

- assuming: $\omega_s > 2\omega_M$ <u>and</u> $H_{LP}(\omega)$ satisfies

$$H_{LP}(\omega) = \begin{cases} T_s, & |\omega| < \omega_M \\ 0, & |\omega| > \omega_s - \omega_M \end{cases}$$

- then: $x_r(t) = x_a(t) \Rightarrow$ perfect reconstruction

- so far, we've used: $h_{LP}(t) = \dfrac{\sin\left(\frac{\omega_s}{2} t\right)}{\frac{\omega_s}{2} t}$

$$\boxed{\begin{aligned} \omega_s &= 2\pi F_s \\ &= \frac{2\pi}{T_s} \end{aligned}}$$

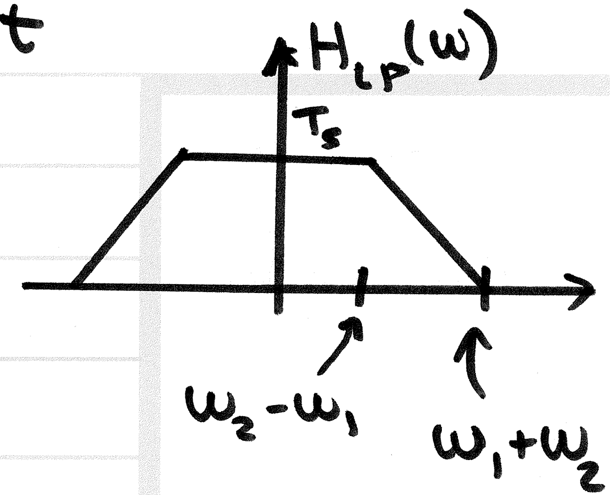
- With oversampling, there are many other possibilities for the interpolating LPF, $h_{LP}(t)$, that will yield perfect reconstruction $x_r(t) = x_a(t)$

- Examples: $h_{LP}(t) = T_s \dfrac{\sin(\omega_c t)}{\pi t}$ $\quad \omega_M < \omega_c < \omega_s - \omega_M$

- Also, consider product of 2 sinc functions:

$$h_{LP}(t) = \frac{\pi}{\omega_1} T_s \frac{\sin(\omega_1 t)}{\pi t} \frac{\sin(\omega_2 t)}{\pi t}$$

Require: $\boxed{\begin{aligned} \omega_2 - \omega_1 &= \omega_M \\ \omega_2 + \omega_1 &= \omega_s - \omega_M \end{aligned}}$



$2\omega_2 = \omega_s$ $\qquad \omega_1 = \omega_2 - \omega_M$

$\Rightarrow \omega_2 = \dfrac{\omega_s}{2}$ $\qquad \Rightarrow \omega_1 = \dfrac{\omega_s}{2} - \omega_M$

- Thus, with oversampling ($\omega_s > 2\omega_M$)

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\, h_{LP}(t - nT_s)$$

$$h_{LP}(t) = \frac{\pi}{\frac{\omega_s}{2} - \omega_M} \cdot \frac{\sin\left(\frac{\omega_s}{2} t\right)}{\frac{\omega_s}{2} t} \cdot \frac{\sin\left(\left(\frac{\omega_s}{2} - \omega_M\right) t\right)}{\pi t}$$

- See the matlab code:

SampleSincProductAboveNyquist.m

posted at the course web site to see this formula implemented to yield perfect reconstruction