Proof: Sampling above the Nyquist Rate, it does not matter where the samples "start," you just have to center the interpolating function (i.e., sinc functions) at the sample times

$$x_s(t) = x_a(t) \cdot p(t-\tau)$$

where:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \overset{\mathcal{F}}{\longleftrightarrow} P(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - 2\pi\frac{k}{T_s}\right)$$

Thus:

$$p(t-\tau) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{T_s} e^{-j\omega\tau} \sum_k \delta\left(\omega - 2\pi\frac{k}{T_s}\right)$$

$$= \sum_n \delta(t - \tau - nT_s)$$

$$= \frac{1}{T_s} \sum_k e^{-j2\pi\frac{k}{T_s}\tau} \delta(\omega - k\omega_s)$$

$$\boxed{F_s = \frac{1}{T_s}}$$

sift property of Dirac Delta

Thus:

$$x_s(t) = x_a(t)\, p(t-\tau) \overset{\mathcal{F}}{\longleftrightarrow} X_a(f) * F_s \sum_k e^{-j 2\pi k \frac{\tau}{T_s}} \delta(f - k\omega_s)$$

$$= F_s \sum_k e^{-j 2\pi k \frac{\tau}{T_s}} X_a(f - k\omega_s)$$

note:

- Original CTFT centered at $f=0$, scaled by $e^0 = 1$ corresponds to $k=0$
- other replicas are scaled by $e^{-j2\pi k \frac{\tau}{T}}$ BUT we filter them out so the ~~phase~~ scaling is inconsequential
- Assuming F_s is above Nyquist Rate

$$x_a(t) = x_s(t) * \frac{\sin\left(\frac{\pi}{T_s} t\right)}{\frac{\pi}{T_s} t}$$

<u>let:</u> $-\frac{T_s}{2} < \tau < \frac{T_s}{2}$ or $\tau = \epsilon T_s$

$-\frac{1}{2} < \epsilon < \frac{1}{2}$

$$p(t-\tau) = \sum_{n=-\infty}^{\infty} \delta(t-\tau-nT_s) = \sum_{n=-\infty}^{\infty} \delta(t-(n+\epsilon)T_s)$$

$$x_s(t) = x_a(t)\, p(t-\tau)$$

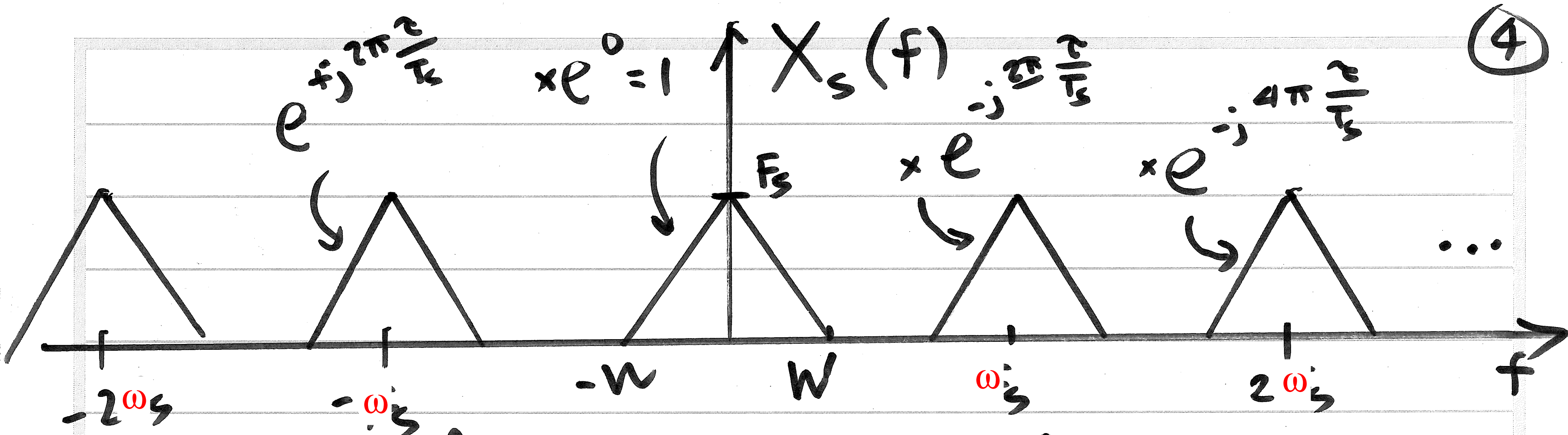
$$= \sum_{n=-\infty}^{\infty} x_a((n+\epsilon)T_s)\, \delta(t-(n+\epsilon)T_s)$$

$$x_s(t) * \frac{\sin\left(\frac{\pi}{T_s} t\right)}{\frac{\pi}{T_s} t} = \sum_{n=-\infty}^{\infty} x_a((n+\epsilon)T_s) \frac{\sin\left(\frac{\pi}{T_s}(t-(n+\epsilon)T_s)\right)}{\frac{\pi}{T_s}(t-(n+\epsilon)T_s)}$$

- So, above Nyquist Rate:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a((n+\epsilon)T_s) \left(\frac{\sin\left(\frac{\pi}{T_s}(t-(n+\epsilon)T_s)\right)}{\frac{\pi}{T_s}(t-(n+\epsilon)T_s)} \right)$$

$-.5 < \epsilon < .5$



replicas are all filtered out so the phase factors are inconsequential (original corresponding to $k=0$ has no phase factor)