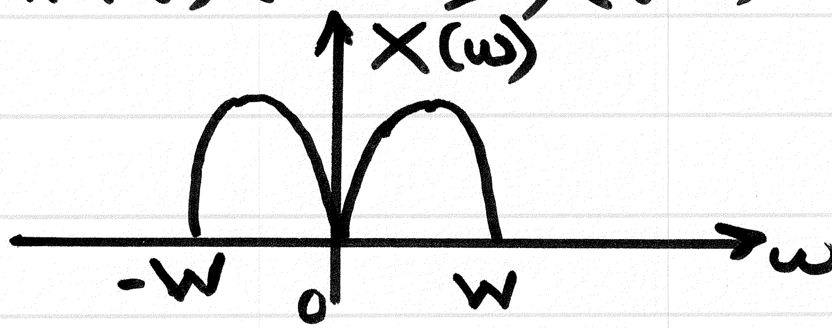
# Single Sideband Modulation

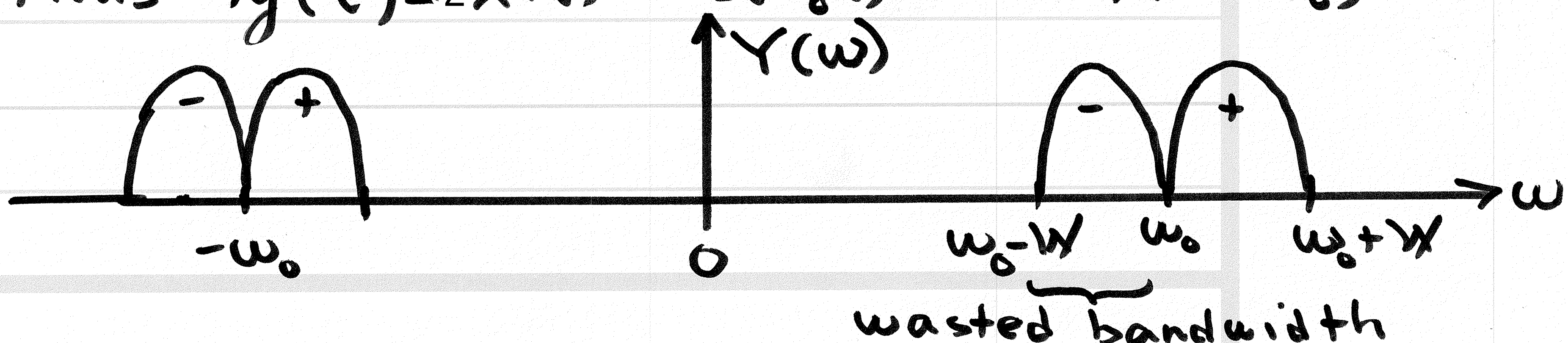
- removing the negative frequency sideband in the transmitted spectrum to save bandwidth
- Consider: $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$ where:




- recall: if $x(t)$ is real-valued: $X(-\omega) = X^*(\omega)$

$\Rightarrow |X(-\omega)| = |X(\omega)| \Rightarrow$ magnitude is symmetric about $\omega = 0 \Rightarrow$ implies negative frequency content

- Thus: $y(t) = 2x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0) + X(\omega + \omega_0)$



- To remove the negative frequency sideband, consider the following steps

- Form $\hat{x}(t)$ by passing $x(t)$ thru an LTI system whose frequency response is

$$h_{HT}(t) = \frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} H_{HT}(\omega) = \begin{cases} j, & \omega < 0 \\ -j, & \omega > 0 \end{cases}$$

$$= j\,u(-\omega) - j\,u(\omega)$$

- This LTI system is called a Hilbert Transformer

$$x(t) \longrightarrow \boxed{H_{HT}(\omega)} \longrightarrow \hat{x}(t)$$

- $\hat{x}(t) = x(t) * h_{HT}(t) \overset{\mathcal{F}}{\longleftrightarrow} \hat{X}(\omega) = H_{HT}(\omega)\,X(\omega)$

$$\hat{X}(\omega) = \begin{cases} j\,X(\omega), & \omega < 0 \\ -j\,X(\omega), & \omega > 0 \end{cases}$$

- Next, form complex-valued signal

$$\tilde{x}(t) = x(t) + j\hat{x}(t)$$

- Examine what happens in frequency domain

$$\tilde{X}(\omega) = X(\omega) + j\hat{X}(\omega)$$

$$= \begin{cases} X(\omega) + j\,(jX(\omega)), & \text{for } \omega < 0 \\ X(\omega) + j\,(-jX(\omega)), & \text{for } \omega > 0 \end{cases}$$

$$= \begin{cases} (1-1)X(\omega) = 0, & \text{for } \omega < 0 \\ (1+1)X(\omega) = 2X(\omega), & \text{for } \omega > 0 \end{cases}$$

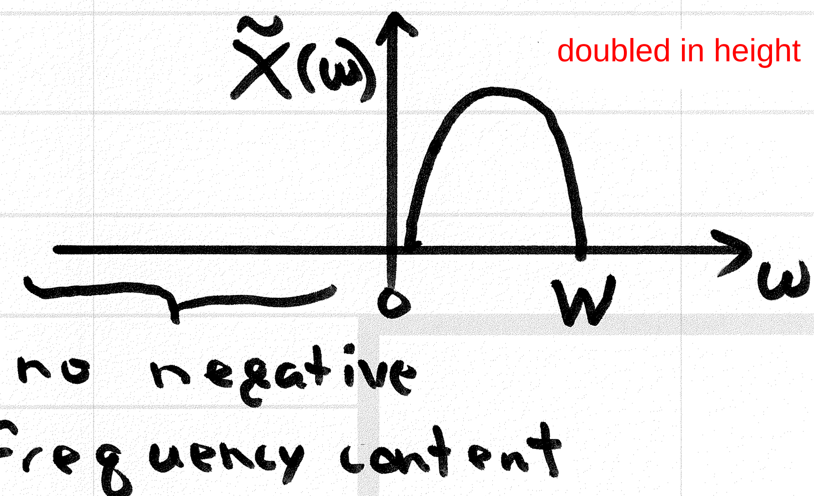
$\Rightarrow$ negative frequencies are removed

- Thus,

$$\tilde{x}(t) = x(t) + j\hat{x}(t) \qquad \tilde{X}(\omega) = X(\omega) + j\hat{X}(\omega)$$

$$= x(t) + j\,x(t) * h_{HT}(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad = X(\omega) + j\,X(\omega)H_{HT}(\omega)$$

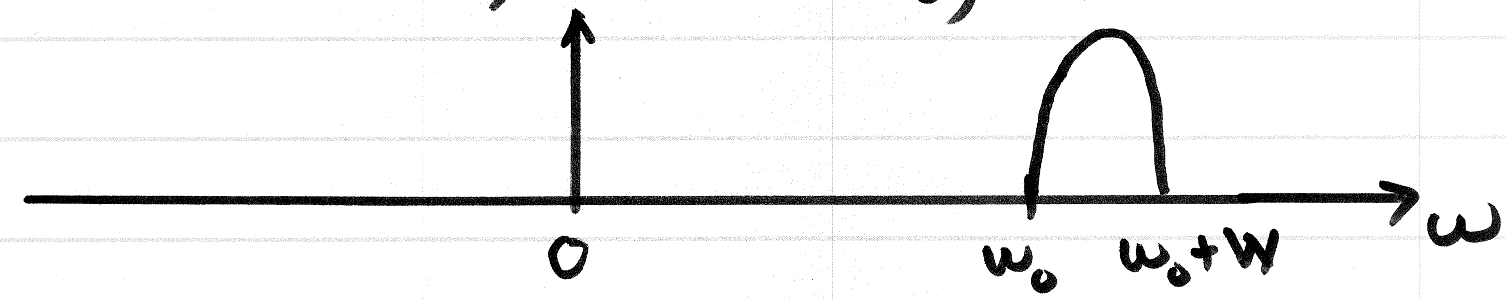
$$= x(t) * \left(\delta(t) + j h_{HT}(t)\right) \qquad = X(\omega)\left(1 + jH_{HT}(\omega)\right)$$




- Note, though, that $\tilde{x}(t)$ is complex-valued and we can only transmit real-valued signals

$$v(t) = \tilde{x}(t)\, e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} V(\omega)$$

$$V(\omega) = \tilde{X}(\omega - \omega_0)$$




- Consider transmitting real part of $v(t)$

$$y(t) = \mathrm{Re}\{v(t)\} = \tfrac{1}{2} v(t) + \tfrac{1}{2} v^*(t)$$

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- <u>Note</u>: Property of Fourier Transform

If $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$, then $x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-\omega)$

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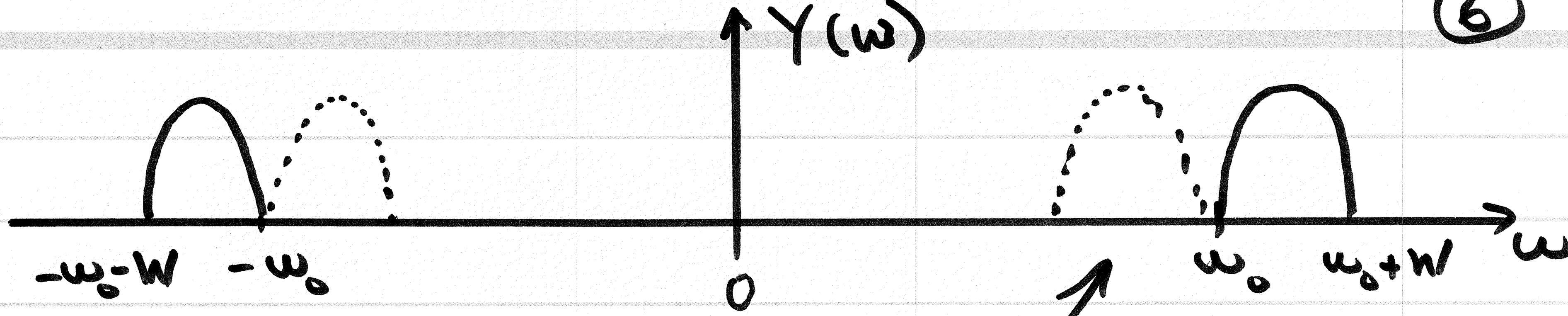
Thus, $$Y(\omega) = \tfrac{1}{2} V(\omega) + \underbrace{\tfrac{1}{2} V^*(-\omega)}_{\text{creates negative frequency content}}$$




dashed lower sideband due to negative frequency content of original signal is gone!

- What is real part of $v(t)$?

$$y(t) = \mathrm{Re}\{v(t)\} = \mathrm{Re}\{\tilde{x}(t)e^{j\omega_0 t}\}$$

$$= \mathrm{Re}\left\{\left(x(t)+j\hat{x}(t)\right)\left(\cos(\omega_0 t)+j\sin(\omega_0 t)\right)\right\}$$

$$= x(t)\cos(\omega_0 t) - \hat{x}(t)\sin(\omega_0 t)$$

This is the real-valued signal that is Transmitted

# Alternative Derivation

$$y(t) = x(t)\cos(\omega_0 t) - \hat{x}(t)\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} Y(\omega) = ?$$

$$x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)$$

$$\hat{x}(t)\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j}\hat{X}(\omega-\omega_0) - \frac{1}{2j}\hat{X}(\omega+\omega_0)$$

Substitute: $\hat{X}(\omega) = -j\,\mathrm{sgn}(\omega)X(\omega)$

$$\hat{x}(t)\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} -\frac{1}{2}\mathrm{sgn}(\omega-\omega_0)X(\omega-\omega_0) + \frac{1}{2}\mathrm{sgn}(\omega+\omega_0)X(\omega+\omega_0)$$

Thus:

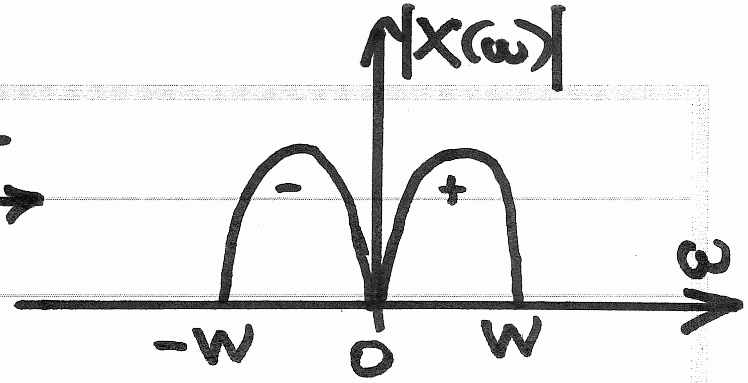
$$Y(\omega) = \frac{1}{2}\left(1+\mathrm{sgn}(\omega-\omega_0)\right)X(\omega-\omega_0) + \frac{1}{2}\left(1-\mathrm{sgn}(\omega+\omega_0)\right)X(\omega+\omega_0)$$

$$\frac{1}{2}\left(1+\mathrm{sgn}(\omega-\omega_0)\right) = \begin{cases} 1, & \omega > \omega_0 \\ 0, & \omega < \omega_0 \end{cases}$$

$$\frac{1}{2}\left(1-\mathrm{sgn}(\omega+\omega_0)\right) = \begin{cases} 1, & \omega < -\omega_0 \\ 0, & \omega > \omega_0 \end{cases}$$

**Summarizing:** Let: $x(t) \overset{\mathcal{F}}{\longleftrightarrow}$

real-valued




$$\tilde{x}(t) = x(t) + j\, x(t) * h_{HT}(t)$$

where:

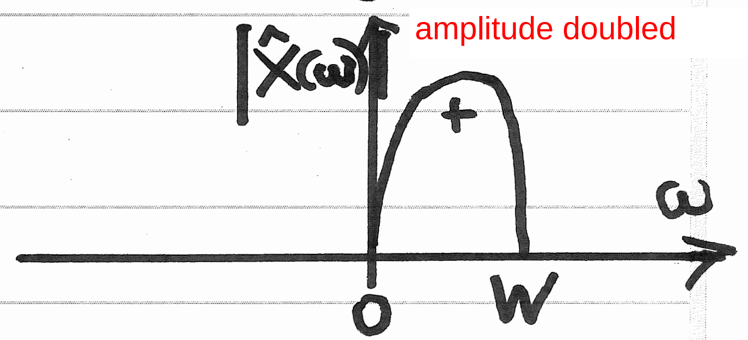
$$h_{HT}(t) = \frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} -j\,\mathrm{sgn}(\omega)$$

$$= j, \quad \omega < 0$$

$$-j, \quad \omega > 0$$

$\tilde{x}(t) \overset{\mathcal{F}}{\longleftrightarrow}$

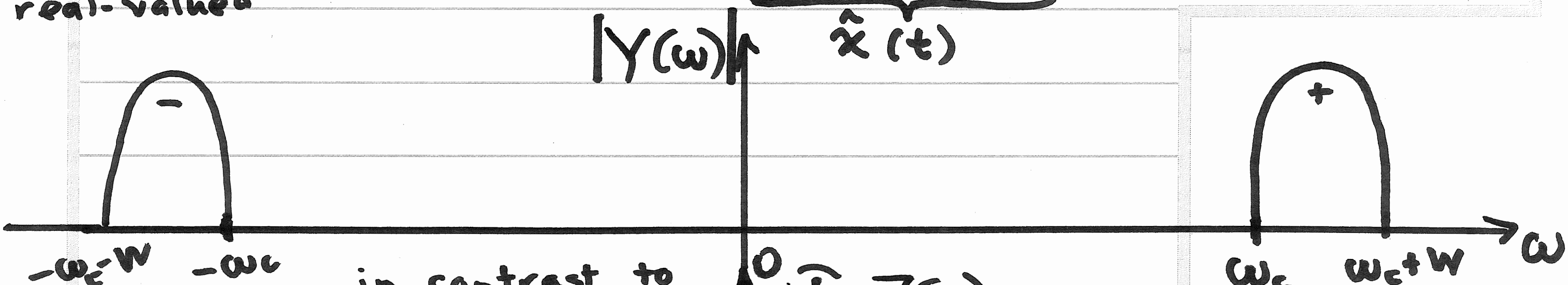
complex-valued




$$y(t) = x(t)\cos(\omega_c t) - \underbrace{\left(x(t) * h_{HT}(t)\right)}_{\hat{x}(t)}\sin(\omega_c t)$$

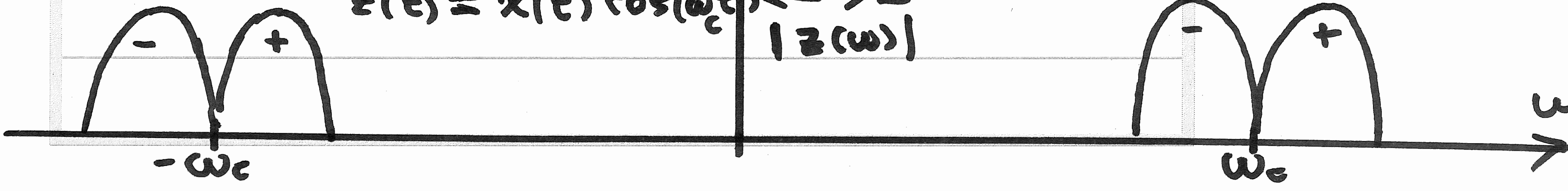
real-valued

$\omega_c \gg W$




in contrast to

$z(t) = x(t)\cos(\omega_c t) \overset{\mathcal{F}}{\longleftrightarrow} Z(\omega)$



<u>Observe:</u> First, recall two trig. identities:

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta) \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

- <u>Again:</u> $y(t) = x(t)\cos(\omega_c t) - \hat{x}(t)\sin(\omega_c t)$

$$\boxed{\hat{x}(t) = x(t) * \frac{1}{\pi t}}$$

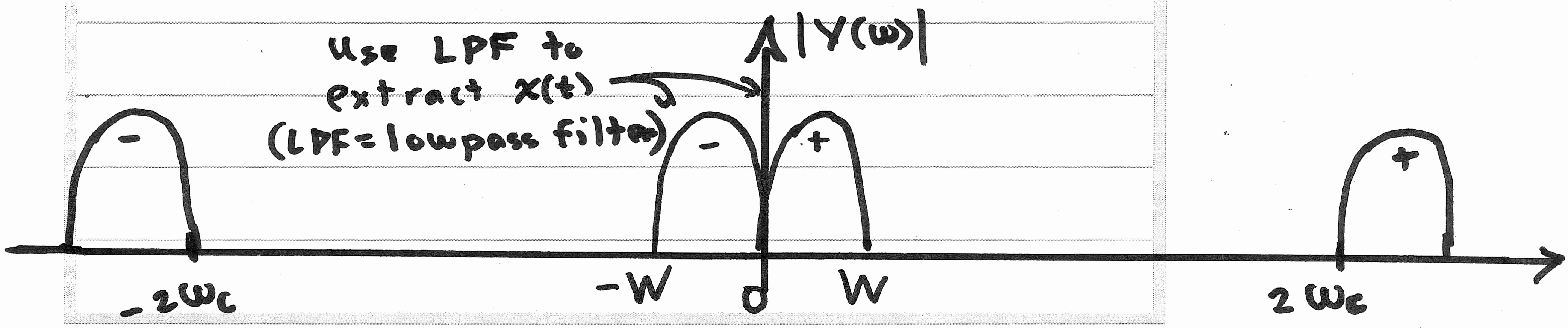
- <u>Consider:</u> $v(t) = y(t)\, 2\cos(\omega_c t)$

$$v(t) = x(t)\, 2\cos^2(\omega_c t) - \hat{x}(t)\, 2\sin(\omega_c t)\cos(\omega_c t)$$

$$= x(t)\left(1 + \cos(2\omega_c t)\right) - \hat{x}(t)\sin(2\omega_c t)$$

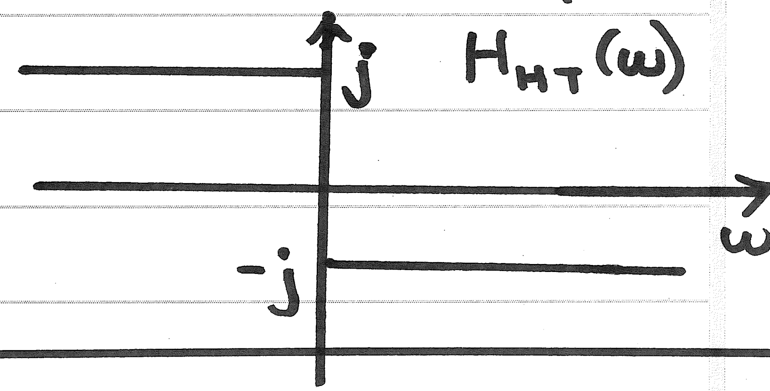
$$= x(t) + \left\{x(t)\cos(2\omega_c t) - \hat{x}(t)\sin(2\omega_c t)\right\}$$

similar to $y(t)$ on previous page
BUT at <u>double</u> the carrier frequency



## Add'l Note: how to derive the Fourier Transform Pair

$$h_{HT}(t) = \frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} -j\,\mathrm{sgn}(\omega)$$




Subscript HT = Hilbert Transformer

$$H_{HT}(\omega) = \begin{cases} j, & \omega < 0 \\ -j, & \omega > 0 \end{cases}$$

**Derivation**: Use FT pair for Unit Step Table 4.2: $u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$

- **Plus**, two FT properties from Table 4.1:
- **First**:

**If**: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ **Then**: $x_o(t) =$ odd part $\overset{\mathcal{F}}{\longleftrightarrow} j\,\mathrm{Im}\{X(\omega)\}$

$$= \frac{1}{2}\left(x(t) - x(-t)\right)$$

**Thus**: $u(t) - u(-t) \overset{\mathcal{F}}{\longleftrightarrow} j\left(\frac{1}{-\omega}\right)2 = -j\frac{2}{\omega}$

- **Second**: Duality Prop: $X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$

**Thus**: $-j\frac{2}{t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\left(-u(\omega) + u(-\omega)\right) = 2\pi\left(u(-\omega) - u(\omega)\right)$

Divide by $-j2\pi$ on both sides: $\frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} j\,u(-\omega) - j\,u(\omega) = \begin{cases} j, & \omega < 0 \\ -j, & \omega > 0 \end{cases}$

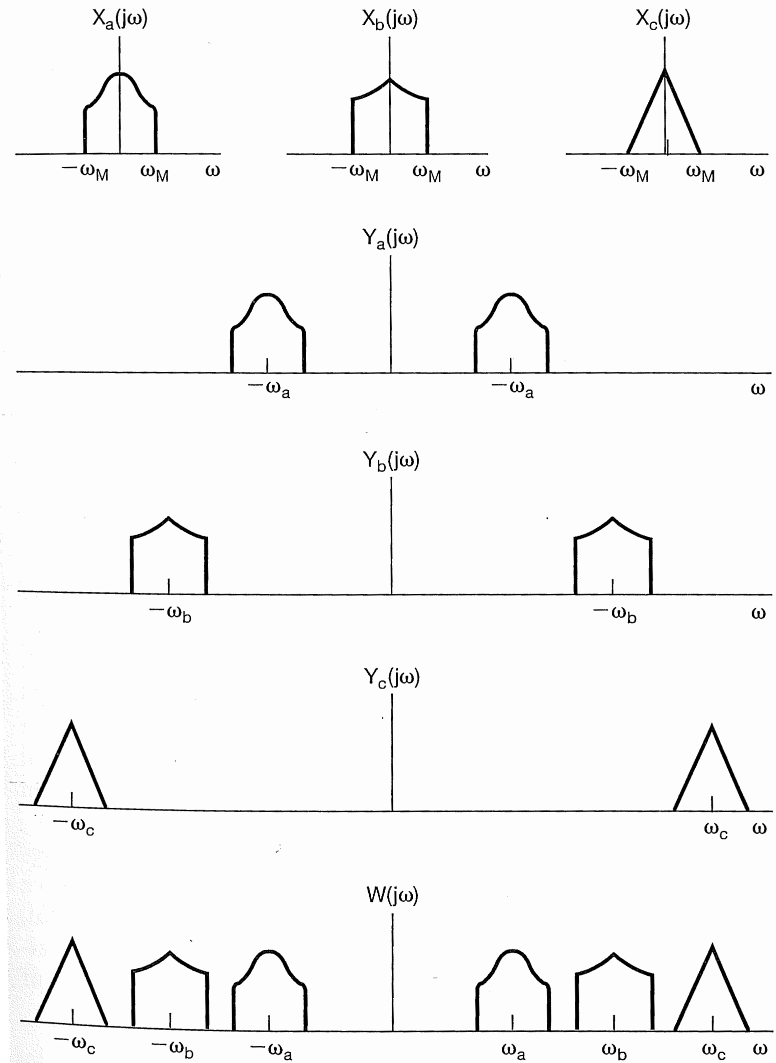



**Figure 8.16** Spectra associated with the frequency-division multiplexing system of Figure 8.15.

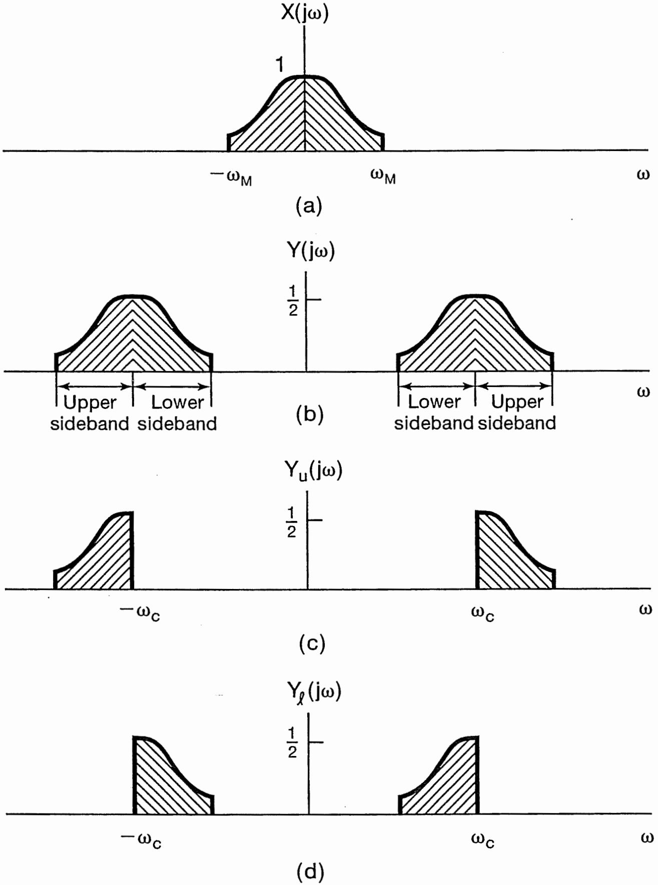



**Figure 8.19** D
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sidebands; (d) sp
lower sidebands.

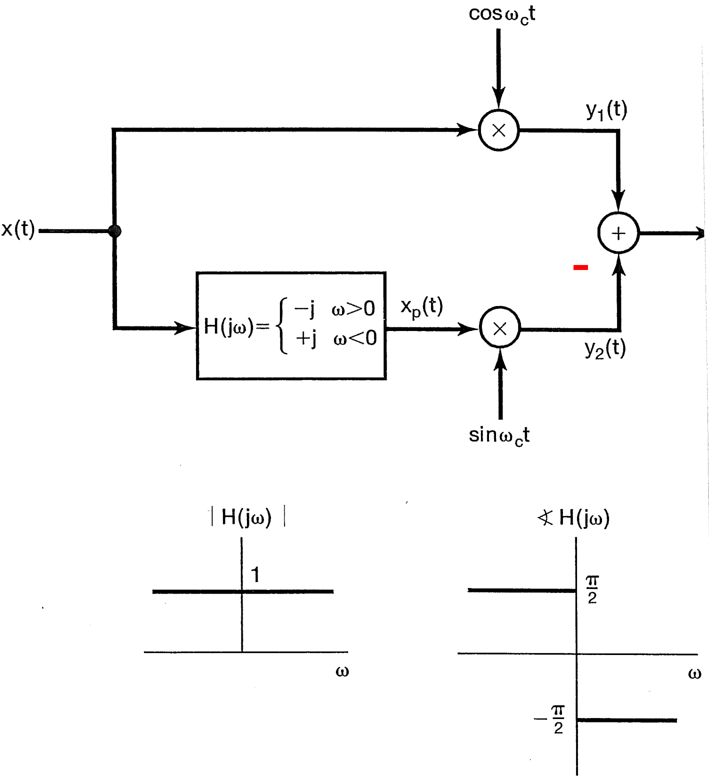
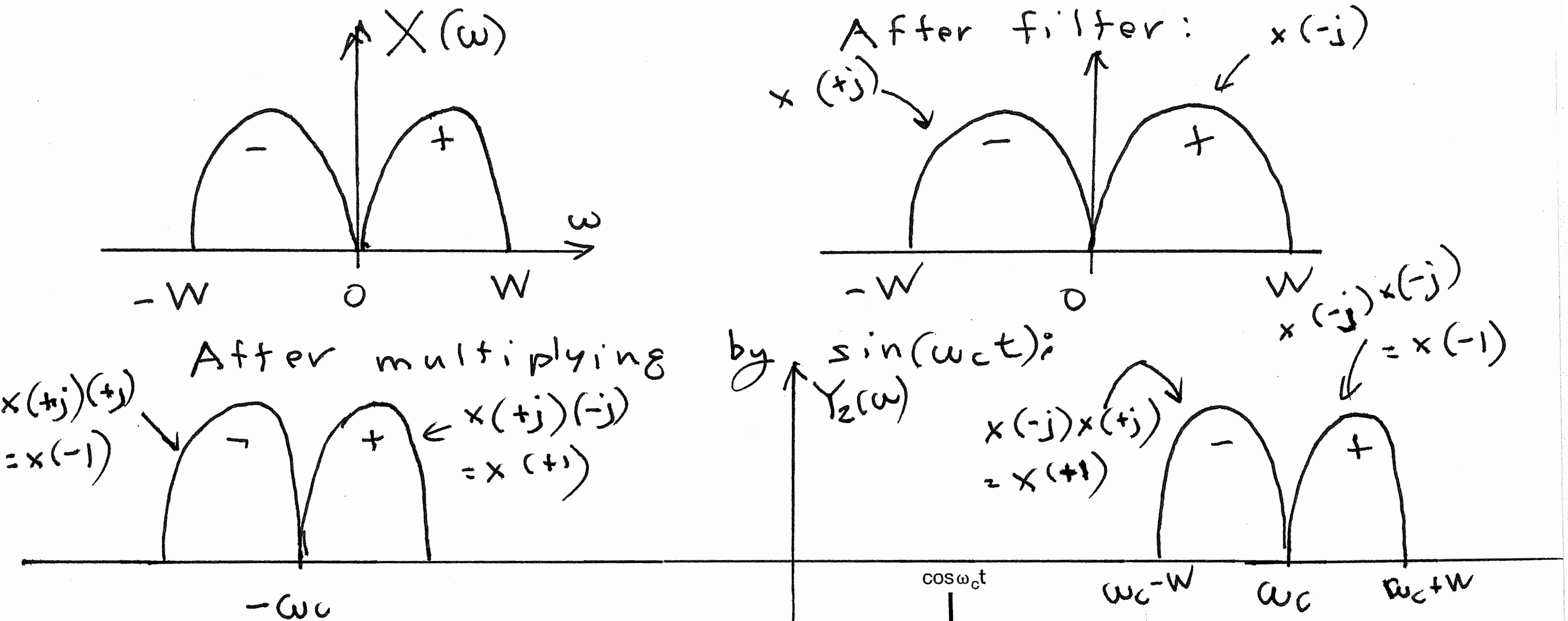



**Figure 8.21** System for single-sideband amplitude modulation, using phase-shift network, in which only upper sidebands are retained.



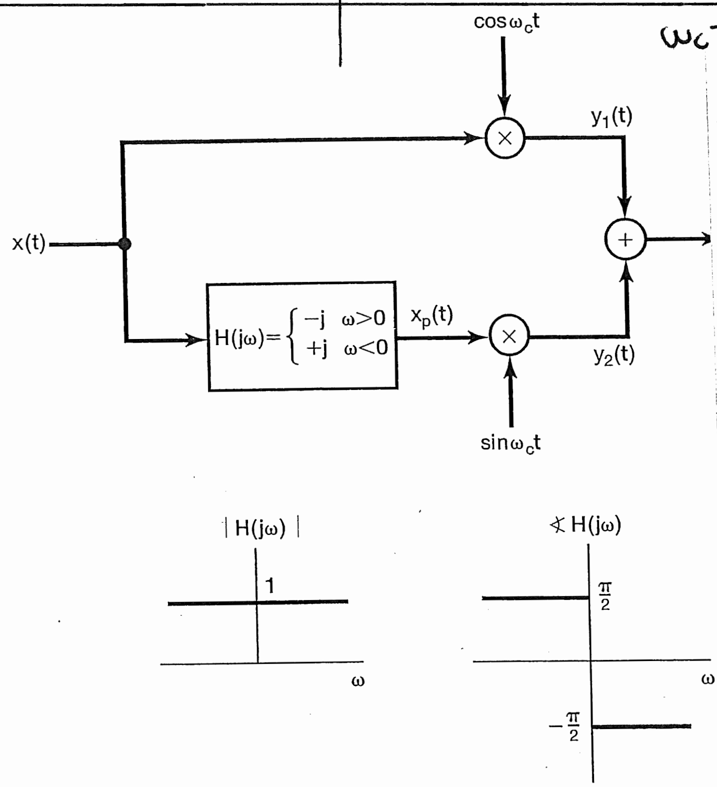




**Figure 8.21** System for single-sideband amplitude modulation, using phase-shift network, in which only the lower sidebands are retained.

$$x(t)\sin(\omega_c t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j} X(\omega-\omega_c) - \frac{1}{2j} X(\omega+\omega_c)$$

$$= \underbrace{-j\,\tfrac{1}{2} X(\omega-\omega_c)}_{\substack{\text{spectrum shifted} \\ \text{to right by } \omega_c, \\ \text{multiplied by } -j}} \; \underbrace{+j\,\tfrac{1}{2} X(\omega+\omega_c)}_{\substack{\text{spectrum shifted} \\ \text{to left by } \omega_c, \\ \text{multiplied by } +j}}$$

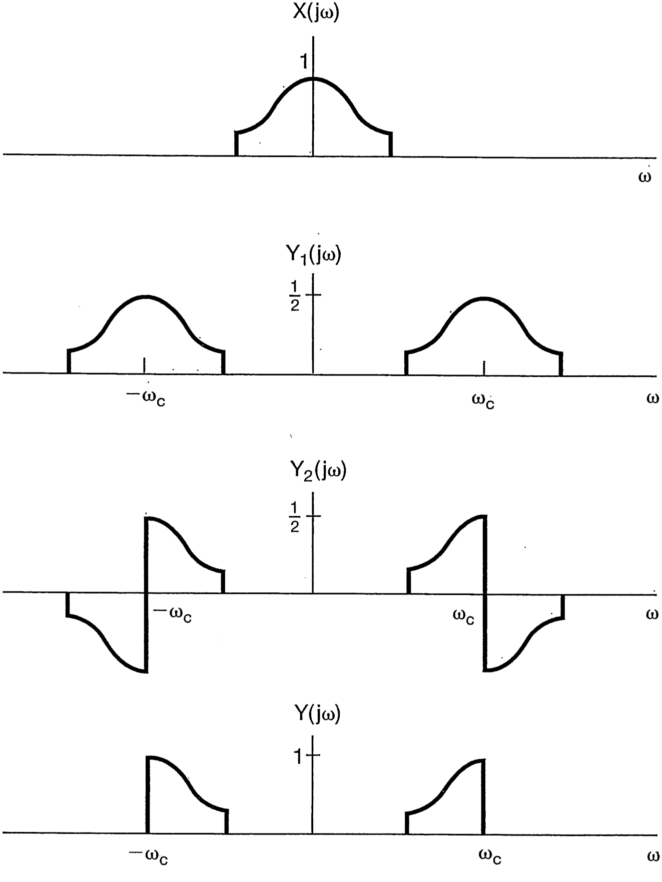



**Figure 8.22**
with the single-
Figure 8.21.

| Frequency range | Designation | Typical uses | Propagation method | Channel features |
|---|---|---|---|---|
| 30–300 Hz | ELF (extremely low frequency) | Macrowave, submarine communication | Megametric waves | Penetration of conducting earth and seawater |
| 0.3–3 kHz | VF (voice frequency) | Data terminals, telephony | Copper wire | |
| 3–30 kHz | VLF (very low frequency) | Navigation, telephone, telegraph, frequency and timing standards | Surface ducting (ground wave) | Low attenuation, little fading, extremely stable phase and frequency, large antennas |
| 30–300 kHz | LF (low frequency) | Industrial (power line) communication, aeronautical and maritime long-range navigation, radio beacons | Mostly surface ducting | Slight fading, high atmospheric pulse |
| 0.3–3 MHz | MF (medium frequency) | Mobile, AM broadcasting, amateur, public safety | Ducting and ionospheric reflection (sky wave) | Increased fading, but reliable |
| 3–30 MHz | HF (high frequency) | Military communication, aeronautical mobile, international fixed, amateur and citizen's band, industrial | Ionospheric reflecting sky wave, 50–400 km layer altitudes | Intermittent and frequency-selective fading, multipath |
| 30–300 MHz | VHF (very high frequency) | FM and TV broadcast, land transportation (taxis, buses, railroad) | Sky wave (ionospheric and tropospheric scatter) | Fading, scattering, and multipath |
| 0.3–3 GHz | UHF (ultra high frequency) | UHF TV, space telemetry, radar, military | Transhorizon tropospheric scatter and line-of-sight relaying | |
| 3–30 GHz | SHF (super high frequency) | Satellite and space communication, common carrier (CC), microwave | Line-of-sight ionosphere penetration | Ionospheric penetration, extraterrestrial noise, high directly |
| 30–300 GHz | EHF (extremely high frequency) | Experimental, government, radio astronomy | Line of sight | Water vapor and oxygen absorption |
| $10^3$–$10^7$ GHz | Infrared, visible light, ultraviolet | Optical communications | Line of sight | |

**Figure 8.18** Allocation of frequencies in the RF spectrum.